INVERSE PROBLEMS FOR FRACTIONAL DIFFUSION EQUATIONS

A Dissertation

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2017

Major Subject: Mathematics

ABSTRACT

By Fick's laws of diffusion, in the classical diffusion process, the mean square path $\langle x^2 \rangle$ is proportional to the time t as $t \to \infty$. However, in practice, some anomalous diffusion processes may occur, in which the relation $\langle x^2 \rangle \propto t^\alpha$, $\alpha \neq 1$ holds. To describe such processes, we need to add the fractional derivative on the time t, which forms the fractional diffusion equation, and we call it FDE for short.

This dissertation contains some inverse problems in FDEs. Specifically, the recovery of unknown conditions of coefficients from additional data on the solution u will be considered. The results of fractional inverse problems are totally different from the ones of the classical case. For instance, the degree of ill-posedness. This is due to the polynomial asymptotic behavior of the Mittag-Leffler function, which consists of the fundamental solution of FDE. This difference leads to new physics and we can ask a question that do similar things always occur? The short answer is not always and the slightly longer version is the analysis is always more complex. This makes the research on inverse problems in FDEs both challenging and interesting.

For each inverse problem in this dissertation, at first it was necessary to extend existing results about the direct problem, namely the situation where all parameters in the equation are known and we must recover $u(x, t)$. This includes the existence, uniqueness and regularity estimates of the solution. Then for the inverse problem, the initial step in many of these situations is to use the equation structure to obtain an operator K one of whose fixed points is the unknown function we seek. With this K, the key step is proving the monotonicity of the operator in a suitable partially ordered space and then showing uniqueness of its fixed points. In conclusion, the monotonicity property and the domain of the operator K will lead to an iterative reconstruction algorithm and some numerical

results are reproduced to verify the theoretical conclusions.

DEDICATION

I dedicate this dissertation to my parents for bringing me up. This work is the gift in return for their selfless love.

ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. William Rundell, for introducing me to the world of inverse problems. I am indebted to him for his guidance, encouragement and support during my Ph.D career. I also appreciate Dr. Bangti Jin, Dr. Zhi Zhou and Dr. Yikan Liu for their sincere assistance in my research and life.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a thesis committee consisting of Dr. William Rundell [advisor], Dr. Peter Howard and Dr. Jianxin Zhou of the Department of Mathematics, and Dr. Partha Mukherjee of the Department of Mechanical Engineering.

Section 3 is a joint work with Dr. Zhi Zhou; section 4 is a joint work with Dr. William Rundell. All other work conducted for the thesis was completed by the student independently.

Funding Sources

Graduate study was supported by a teaching assistantship from Texas A&M University and a research assistantship from NSF-DMS 1620138.

NOMENCLATURE

FDE	Fractional diffusion equation
DDE	Distributed differential equation
ODE	Ordinary differential equation

TABLE OF CONTENTS

Page

LIST OF FIGURES

FIGURE Page

LIST OF TABLES

1. INTRODUCTION

Classical Brownian motion as formulated in Einstein's 1905 paper [1] can be viewed as a random walk in which the dynamics are governed by an uncorrelated, Markovian, Gaussian stochastic process. The key assumption is that a change in the direction of motion of a particle is random and that the mean-squared displacement over many changes is proportional to time $\langle x^2 \rangle = Ct$. This easily leads to the derivation of the underlying differential equation being the heat equation.

In fact we can generalize this situation to the case of a continuous time random walk (CTRW) where the length of a given jump, as well as the waiting time elapsing between two successive jumps follow a given probability density function. In one spatial dimension, the picture is as follows: a walker moves along the x-axis, starting at a position x_0 at time $t_0 = 0$. At time t_1, the walker jumps to x_1, then at time t_2 jumps to x_2, and so on. We assume that the temporal and spatial increments $\Delta t_n = t_n - t_{n-1}$, $\Delta x_n = x_n - x_{n-1}$ are independent, identically distributed random variables, following probability density functions $\psi(t)$ and $\lambda(x)$, respectively, which is known as the waiting time distribution and jump length distribution, respectively. Namely, the probability of Δt_n lying in any interval $[a, b] \subset (0, \infty)$ is $P(a < \Delta t_n < b) = \int_a^b \psi(t)\, dt$ and the probability of Δx_n lying in any interval $[a, b] \subset \mathbb{R}$ is $P(a < \Delta x_n < b) = \int_a^b \lambda(x)\, dx$. For given ψ and λ, the position x of the walker can be regarded as a step function of t.

It it easily shown using the Central Limit Theorem that provided the first moment, or characteristic waiting time T, defined by $T = \mu_1(\psi) = \int_0^\infty t\psi(t)\, dt$ and the second moment, or jump length variance Σ, $\mu_2(\lambda) = \int_{-\infty}^{\infty} x^2\lambda(t)\, dt$ are finite, then the long-time limit again corresponds to Brownian motion,

On the other hand, when the random walk involves correlations, non-Gaussian statistics or a non-Markovian process (for example, due to "memory" effects) the diffusion equation will fail to describe the macroscopic limit. For example, if we retain the assumption that Σ is finite but relax the condition on a finite characteristic waiting time so that for large t $\psi(t)A/t^{1+\alpha}$ as $t \to \infty$ where $0 < \alpha \leq 1$, then we get very different results. Such probability density functions are often referred to as a "heavy-tailed." If in fact we take

$$\psi(t) = \frac{A_\alpha}{B_\alpha + t^{1+\alpha}} \tag{1.1}$$

then again it can be shown, [2, 3], that the effect is to modify the Einstein formulation $\langle x^2 \rangle = Ct$ to $\langle x^2 \rangle = Ct^\alpha$.

This above leads to a *subdiffusive* process and, importantly provides a tractable model where the partial differential equation is replaced by one with a fractional derivative in time of order α. Such objects have been a steady source of investigation over the last almost 200 years beginning in the 1820s with the work of Abel and continuing first by Liouville then by Riemann.

There are many ways to formulate a fractional derivative but the most useful versions start from the Abel integral operator ${}_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} u(\tau)\, \mathrm{d}\tau$. With this one can define a derivative of f in one of two ways; first take the fractional integral then differentiate the result - or reverse this order. These are the Riemann-Liouville and Djrbashyan-Caputo fractional derivatives respectively.

$$
\begin{aligned}
{}_a^R D_t^\alpha u(t) &= \frac{d^n}{dt^n} I_t^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} u(\tau) \mathrm{d}\tau, \\
{}_a^C D_t^\alpha u(t) &= I_t^{n-\alpha} \frac{d^n}{dt^n} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) \mathrm{d}\tau,
\end{aligned}
$$

where n is the nearest integer larger than α. The Džrbašjan-Caputo derivative tends to be more favored by practitioners since it allows the specification of initial conditions in the usual way. Nonetheless, the Riemann-Liouville derivative enjoys certain analytic advantages, including being defined for a wider class of functions and possessing a semigroup property.

Thus the fractional-anomalous diffusion model gives rise to the fractional differential equation

$$\partial_t^\alpha u - \mathcal{L}u = f(x,t), \qquad x \in \Omega,\ t \in (0,T) \tag{1.2}$$

where $\mathcal{L}$ is a uniformly elliptic differential operator on an open domain $\Omega \subset \mathbb{R}^d$ and usually $\partial_t^\alpha = {}^C_aD_t^\alpha$. There are two special functions; the Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$, and the Wright function are key components. The first generalizes the exponential (${}^C_0D^\alpha f = \lambda f \Rightarrow f(t) = E_{1,\alpha}(-\lambda t^\alpha)$) and the fundamental solution of FDE is given in terms of a Wright function. In the classical case, $\alpha = 1$, the fundamental solution is the Gaussian $\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$ and is analytic in x and t for $t > 0$. For the typical examples described here we have $0 < \alpha \leq 1$ and β a positive real number although further generalization is certainly possible. See, for example, [4].

The FDEs are known to capture well the dynamics of subdiffusion processes, in which the mean square variance grows at a rate slower than that in a Gaussian process, and has found a number of applications. For example, subdiffusion has been successfully used to describe thermal diffusion in media with fractal geometry [5], highly heterogeneous aquifer [6] and underground environmental problem [7]. At a microscopic level, the particle motion can be described by a continuous time random walk (CTRW), in which the waiting time of the particle motion follows a heavy tailed distribution, as opposed to a Gaussian process, which is characteristic of the normal diffusion equation. The macroscopic counterpart is a diffusion equation with a Caputo fractional derivative in time.

Due to unprecedented modeling capability of the FDE model, the analytical study of the direct problem for FDE has received much attention in recent years, e.g., [8, 9].

However, in practice, the parameters in the equation are often unknown and have to be estimated from experimental data, which leads to a wide variety of inverse problems in FDEs. Even though the study on inverse problems remains very scarce, nonetheless, some interesting works have been displayed. [10] provided the first mathematical study on the inverse problems for fractional diffusion, and established the uniqueness in determining the diffusion coefficient and the fractional order α from the lateral Cauchy data; see also [11] for further results. [12] used a Carleman estimate to deduce the conditional stability in determining a zeroth-order coefficient with one half order Caputo derivative. For Carleman estimate in time fractional diffusion, we also refer [13, 14, 15]. [16] numerically observed that one single spectrum uniquely determines the potential in a fractional Sturm-Liouville problem with a Caputo fractional derivative in space. The unique determination of a nonlinear boundary condition from overposed boundary data was studied in [17]; see also [18] for the determination of the nonlinear source term from boundary data and [11] for the unique determination of the spatial coefficient and/or the fractional order. The fractional backward problem were analyzed in [9] theoretically and in [19, 20] from a numerical point of view. [21] developed an optimal perturbation algorithm to simultaneously recover the diffusion coefficient and fractional order in a time fractional diffusion equation. We refer to [22] for an updated overview on inverse problems in anomalous diffusion.

However, in the subdiffusive process, such a specific form for $\psi(t)$ as given by (1.1) is rather restrictive as it assumes a quite specific scaling factor between space and time distributions and there is no reason to expect nature is so kind to only require a single value for α.

One approach around this is to take a finite sum of such terms each corresponding to a different value of α. This leads to a model where the time derivative is replaced by a finite sum of fractional derivatives of orders α_j and by analogy leads to the law $\langle x^2 \rangle = g(t, \alpha)$ where g is a finite sum of fractional powers. This formulation replaces the single value fractional derivative by a finite sum $\sum_1^m q_j \partial_t^{\alpha_j} u$ where a linear combination of m fractional powers has been taken. Physically this represents a fractional diffusion model that assumes diffusion takes place in a medium in which there is no single scaling exponent; for example, a medium in which there are memory effects over multiple time scales.

This seemingly simple device leads to considerable complications. For one, we have to use the so-called multi-index Mittag-Leffler function $E_{\alpha_1,\dots\alpha_m,\beta_1,\dots\beta_m}(z)$ in place of the two parameter $E_{\alpha,\beta}(z)$ and this adds complexity not only notationally but in proving required regularity results for the basic forwards problem of knowing Ω, $\mathcal{L}$, f, u_0 and recovering $u(x,t)$. see [23, 24] and the references within.

It is also possible to generalize beyond the finite sum by taking the so-called distributed fractional derivative,

$$\partial_t^{(\mu)} u(t) = \int_0^1 \mu(\alpha) \partial_t^\alpha u(t)\, d\alpha. \tag{1.3}$$

Thus the finite sum derivative can be obtained by taking $\mu(\alpha) = \sum_{j=1}^m q_j \delta(\alpha - \alpha_j)$. See [25, 26, 27, 28, 29], for several studies incorporating this extension. This in turn allows a more general function probability density distribution function ψ in (1.1) and hence a more general value for $g(t, \alpha)$. Furthermore, if we replace ∂_t^α in (1.2) by $\partial_t^{(\mu)} u(t)$, then the distributed differential equation (DDE) can be obtained

$$\partial_t^{(\mu)} u - \mathcal{L}u = f(x,t), \qquad x \in \Omega,\ t \in (0,T).$$

With the FDE and DDE model, there are some natural questions: what is the value of the order α or the derivative component $\mu(\alpha)$ and how to recover them? These questions lead to an interesting variety of inverse problems in FDEs and DDEs. Needless to say there has been much work done on this; experiments have been set up to collect additional information that allows a best fit for α in a given setting. One of the earliest works here is from 1975, [30] and in part was based on the Montroll-Weiss random walk model [2]. See also [7]. Mathematically the recovery in models with a single value for α turns out to relatively straightforward provided we are able to choose the type of data being measured. This would be chosen to allow us to rely on the known asymptotic behavior of the Mittag-Leffler function for both small and large arguments. An exception here is when we also have to determine α as well as an unknown coefficient in which case the combination problem can be decidedly much more complex. See, for example, [10, 21, 17]. Amongst the first papers in this direction with a rigorous existence and uniqueness analysis is [31]. By the way, the multi-term case, although similar in concept, is quite nontrivial but has been shown in [23, 24]. In these papers the authors were able to prove an important uniqueness theorem: if given the additional data consisting of the value of the normal derivative $\frac{\partial u}{\partial \nu}$ at a fixed point $x_0 \in \partial\Omega$ for all t then the sequence pair $\{q_j, \alpha_j\}_{j=1}^{m}$ can be uniquely recovered.

This thesis paper concerns two FDE models and one DDE model. Both direct problem and inverse problem works are exhibited and can be seen in the following chapters.

2. THE FRACTIONAL INVERSE PROBLEM WITH UNKNOWN DIFFUSIVITY

2.1 Introduction

This chapter considers the FDE with a continuous and positive coefficient function $a(t)$:

$$\begin{cases} {}^C D_t^\alpha u(x,t) - a(t)\mathcal{L}u(x,t) = F(x,t), & x \in \Omega,\ t \in (0,T]; \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T]; \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded and smooth subset of $R^n, n = 1, 2, 3$, $-\mathcal{L}$ is a symmetric uniformly elliptic operator defined as

$$-\mathcal{L}u = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + c(x)u$$

with conditions

$$a^{ij}, c \in C^2(\overline{\Omega})\ (i,j = 1, \ldots, n),\ \partial\Omega \text{ is } C^3, \tag{2.2}$$

and ${}^C D_t^\alpha$ is the left-sided Djrbashian–Caputo α-th order derivative with respect to time t. This work is an extension of [32] from a simple space domain Ω to $\mathbb{R}^n$, considers the more general analysis for the direct problem and contains an existence argument for the inverse problem of recovering $a(t)$.

This chapter consists of two parts; the direct problem and the inverse problem. For the direct problem, we build the spectral representation of the weak solution $u(x,t;a)$. The notation $u(x,t;a)$ is used for displaying the dependence of the solution u on the diffusivity $a(t)$. Then the existence, uniqueness and regularity results are proved with several assumptions on the coefficient function $a(t)$. Unlike [32], the right hand side function

$F(x,t)$ is not of the form $f(x)g(t)$, so that the proof of regularity is more delicate. For the inverse problem, we use the single point flux data

$$a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a) = g(t),\ x_0 \in \partial\Omega$$

to recover the coefficient $a(t)$ (We choose the data $a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a) = g(t)$ instead of the classical flux $\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a)$ because in practice, $a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a)$ is usually measured as the flux). For the reconstruction, we only consider to recover a continuous and positive $a(t)$ to match the assumptions set in the direct problem. Acting a flux data, we introduce an operator K one of whose fixed points is the coefficient $a(t)$. Using the weak maximum principle [33], we establish the monotonicity and uniqueness of the fixed points of operator K, and the proof of uniqueness leads to a numerical reconstruction algorithm. Since we consider a multidimensional domain Ω here, the Sobolev Embedding Theorem yields that we need to add the condition (2.2) on the operator $-\mathcal{L}$ to ensure the C^1-regularity of the series representation of u. Then the operator K is well-defined, where the proofs can be seen in section 4. This is a significant difference from [32]. Furthermore, an existence argument of the fixed points of K is included by this paper, which [32] does not contain.

The rest of this chapter follows the following structure. In section 2, we collect some preliminary results about fractional calculus and the eigensystem of $-\mathcal{L}$. The direct problem is discussed in section 3, i.e. we establish the existence, uniqueness and some regularity results of the weak solution for FDE (2.1). Then section 4 deals with the inverse problem of recovering $a(t)$. Specifically, an operator K is introduced at the beginning of this section, then its monotonicity and uniqueness of its fixed points give an algorithm to recover the coefficient $a(t)$. In particular, the existence argument of the fixed points of K is included by this section. In section 5, some numerical results are presented to illustrate the theoretical basis.

2.2 Preliminary material

2.2.1 Mittag-Leffler function

In this part, we describe the Mittag-Leffler function which plays an important role in fractional diffusion equations. This is a two-parameter function defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},\ z \in \mathbb{C}.$$

It generalizes the natural exponential function in the sense that $E_{1,1}(z) = e^z$. We list some important properties of the Mittag-Leffler function for future use.

Lemma 2.2.1. *Let* $0 < \alpha < 2$ *and* $\beta \in \mathbb{R}$ *be arbitrary, and* $\frac{\alpha\pi}{2} < \mu < \min(\pi, \alpha\pi)$*. Then there exists a constant* $C = C(\alpha, \beta, \mu) > 0$ *such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|}, \quad \mu \le |\arg(z)| \le \pi.$$

Proof. This proof can be found in [34]. □

Lemma 2.2.2. *For* $\lambda > 0,\ \alpha > 0$ *and* $n \in \mathbb{N}^+$*, we have*

$$\frac{d^n}{dt^n} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda t^\alpha),\ t > 0.$$

In particular, if we set $n = 1$*, then there holds*

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha),\ t > 0.$$

Proof. This is [9, Lemma 3.2]. □

Lemma 2.2.3. *If* $0 < \alpha < 1$ *and* $z > 0$*, then* $E_{\alpha,\alpha}(-z) \ge 0$.

Proof. This proof can be found in [35, 36, 37]. □

Lemma 2.2.4. *For* $0 < \alpha < 1$, $E_{\alpha,1}(-t^\alpha)$ *is completely monotonic, that is,*

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,1}(-t^\alpha) \geq 0, \ for\ t > 0\ and\ n = 0, 1, 2, \cdots.$$

Proof. See [38]. □

2.2.2 Fractional calculus

In this part, we collect some results of fractional calculus. The next lemma states the extremal principle of ${}^C D_t^\alpha$.

Lemma 2.2.5. *Fix* $0 < \alpha < 1$ *and given* $f(t) \in C[0,T]$ *with* ${}^C D_t^\alpha f \in C[0,T]$. *If* f *attains its maximum (minimum) over the interval* $[0,T]$ *at the point* $t = t_0,\ t_0 \in (0,T]$, *then* ${}^C D_{t_0}^\alpha f \geq (\leq) 0$.

Proof. Even though the conditions are different from the ones of [33, Theorem 1], the maximum case can be proved following the proof of [33, Theorem 1]. For the minimum case, we only need to set $\overline{f} = -f$. □

The following lemma about the composition between ${}^C D_t^\alpha$ and the fractional integral I_t^α is presented in [39].

Lemma 2.2.6. *Define the Riemann-âĂŞLiouville* α*-th order integral* I_t^α *as*

$$I_t^\alpha u = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) \mathrm{d}\tau.$$

For $0 < \alpha < 1$, $u(t), {}^C D_t^\alpha u \in C[0,T]$, *we have*

$$({}^C D_t^\alpha \circ I_t^\alpha u)(t) = u(t), \quad (I_t^\alpha \circ {}^C D_t^\alpha u)(t) = u(t) - u(0),\ t \in [0,T].$$

2.2.3 Eigensystem of $-\mathcal{L}$

Since $-\mathcal{L}$ is a symmetric uniformly elliptic operator, we denote the eigensystem of $-\mathcal{L}$ by $\{(\lambda_n, \phi_n) : n \in \mathbb{N}^+\}$. Then we have $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ where finite multiplicity is possible, $\lambda_n \to \infty$ and $\{\phi_n : n \in \mathbb{N}^+\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ forms an orthonormal basis of $L^2(\Omega)$.

Moreover, with the condition (2.2), for each $n \in \mathbb{N}^+$, it holds that $\phi_n \in H^3(\Omega)$ [40]. Then by the Sobolev Embedding Theorem, we have $\phi_n \in C^1(\overline{\Omega})$ and $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ is well-defined for each $n \in \mathbb{N}^+$. Hence, without loss of generality, we can suppose

$$\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq 0, \text{ for each } n \in \mathbb{N}^+. \tag{2.3}$$

Otherwise, if $\frac{\partial \phi_k}{\partial \overrightarrow{\mathbf{n}}}(x_0) < 0$ for some $k \in \mathbb{N}^+$, we can replace ϕ_k by $-\phi_k$. $-\phi_k$ satisfies all the properties we need, such as it is an eigenfunction of $-\mathcal{L}$ corresponding to the eigenvalue λ_k, composes an orthonormal basis of $L^2(\Omega)$ together with $\{\phi_n : n \in \mathbb{N}^+, n \neq k\}$ and $\frac{\partial(-\phi_k)}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq 0$. The assumption (2.3) will be used in Section 4.

2.3 Direct problem–existence, uniqueness and regularity

Throughout this section, we suppose $a(t)$, $u_0(x)$ and $F(x,t)$ satisfy the following assumptions:

Assumption 2.3.1.

(a) $a(t) \in C^+[0,T] := \{\psi \in C[0,T] : \psi(t) > 0,\ t \in [0,T]\}$;

(b) $F(x,t) \in C([0,T]; L^2(\Omega))$;

(c) $u_0(x) \in H_0^1(\Omega)$.

2.3.1 Spectral representation

Definition 2.3.2. *We call $u(x,t;a)$ a weak solution of FDE* (2.1) *in $L^2(\Omega)$ corresponding to the coefficient $a(t)$ if $u(\cdot,t;a) \in H_0^1(\Omega)$ for $t \in (0,T]$ and for any $\psi(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, it holds*

$$({}^C D_t^\alpha u(x,t;a), \psi(x)) - (a(t)\mathcal{L}u(x,t;a), \psi(x)) = (F(x,t), \psi(x)),\ t \in (0,T];$$
$$(u(x,0;a), \psi(x)) = (u_0(x), \psi(x)),$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$.

With the above definition, we give a spectral representation for the weak solution in the following lemma.

Lemma 2.3.1. *Define $b_n := (u_0(x), \phi_n(x)), F_n(t) = (F(x,t), \phi_n(x)),\ n \in \mathbb{N}^+$. The spectral representation of the weak solution of FDE* (2.1) *is*

$$u(x,t;a) = \sum_{n=1}^{\infty} u_n(t;a)\phi_n(x),\ (x,t) \in \Omega \times [0,T], \tag{2.4}$$

where $u_n(t;a)$ satisfies the fractional ODE

$${}^C D_t^\alpha u_n(t;a) + \lambda_n a(t) u_n(t;a) = F_n(t),\ u_n(0;a) = b_n,\ n \in \mathbb{N}^+. \tag{2.5}$$

Proof. For each $n \in \mathbb{N}^+$, multiplying $\phi_n(x)$ on both sides of FDE (2.1) and integrating it on x over Ω allow us to deduce that

$${}^C D_t^\alpha (u(x,t;a), \phi_n(x)) + \lambda_n a(t)(u(x,t;a), \phi_n(x)) = F_n(t), \tag{2.6}$$

where $(-\mathcal{L}u(x,t;a), \phi_n(x)) = (u(x,t;a), -\mathcal{L}\phi_n(x)) = \lambda_n(u(x,t;a), \phi_n(x))$ follows from the symmetricity of $-\mathcal{L}$. Set $u_n(t;a) = (u(x,t;a), \phi_n(x))$ and define $u(x,t;a) = \sum_{n=1}^{\infty} u_n(t;a)\phi_n(x)$. Then (2.6) and the completeness of $\{\phi_n(x) : n \in \mathbb{N}^+\}$ lead to the desired result. ☐

2.3.2 Existence and uniqueness

In order to show the existence and uniqueness of the weak solution (2.4), we state the following lemma [34, Theorem 3.25].

Lemma 2.3.2. *For the Cauchy-type problem*

$$^C D_t^\alpha y = f(y,t),\ y(0) = c_0,$$

if for any continuous $y(t)$, $f(y,t) \in C[0,T]$, $\exists A > 0$ which is independent of $y \in C[0,T]$ and $t \in [0,T]$ s.t. $|f(t,y_1) - f(t,y_2)| \le A|y_1 - y_2|$, then there exists a unique solution $y(t)$ for the Cauchy-type problem, which satisfies $^C D_t^\alpha y \in C[0,T]$.

The theorem of existence and uniqueness for $u(x,t;a)$ follows from Lemma 2.3.2.

Theorem 2.3.3 (Existence and Uniqueness)**.** *Suppose Assumption 2.3.1 holds. Under Definition 2.3.2, there exists a unique weak solution $u(x,t;a)$ of FDE* (2.1) *with the spectral representation* (2.4) *and for each $n \in \mathbb{N}^+$, $u_n(t;a) \in C[0,T]$ is the unique solution of the fractional ODE* (2.5) *with $^C D_t^\alpha u_n(t;a) \in C[0,T]$.*

Proof. From the spectral representation (2.4), it suffices to show the existence and uniqueness of $u_n(t;a), n \in \mathbb{N}^+$. Fix $n \in \mathbb{N}^+$, Assumption 2.3.1 (a) and (b) yield that the fractional ODE (2.5) satisfies the conditions of Lemma 2.3.2. Hence the existence and uniqueness for $u_n(t;a)$ hold. ☐

2.3.3 Sign of $u_n(t;a)$

In this part, we state two properties of $u_n(t;a)$ which play important roles in building the regularity of $u(x,t;a)$.

Lemma 2.3.3. *Given $h \in C^+[0,T]$, $f \in C[0,T]$ with ${}^C D_t^\alpha f \in C[0,T]$, if $f(0) \le (\ge) 0$ and ${}^C D_t^\alpha f + h(t)f(t) \le (\ge) 0$, then $f \le (\ge) 0$ on $[0,T]$.*

Proof. Since $f(t) \in C[0,T]$, $f(t)$ attains its maximum over $[0,T]$ at some point $t_0 \in [0,T]$. If $t_0 = 0$, then $f(t) \le f(0) \le 0$. If $t_0 \in (0,T]$, with Lemma 2.2.5, we have ${}^C D_t^\alpha f(t_0) \ge 0$, which yields $h(t_0)f(t_0) \le 0$, i.e. $f(t_0) \le 0$ due to $h > 0$ on $[0,T]$. The definition of t_0 assures $f \le 0$.

For the case of "≥ 0", let $\overline{f}(t) = -f(t)$, then the above proof gives $\overline{f} \le 0$, i.e. $f \ge 0$. □

The following corollary, which concerns the sign of $u_n(t;a)$, follows from Lemma 2.3.3 directly.

Corollary 2.3.1. *Set $u_n(t;a)$ be the unique solution of the fractional ODE* (2.5)*. Then ${}^C D_t^\alpha u_n(t;a) + \lambda_n a(t) u_n(t;a) \le (\ge) 0$ on $[0,T]$ and $u_n(0;a) \le (\ge) 0$ imply $u_n(t;a) \le (\ge) 0$ on $[0,T],\ n \in \mathbb{N}^+$.*

Proof. Assumption 2.3.1 gives that $\lambda_n a(t) \in C^+[0,T]$. Then the proof is completed by applying Lemma 2.3.3 to the fractional ODE (2.5). □

2.3.4 Regularity

In this part, we establish the regularity of $u(x,t;a)$. To this end, we split FDE (2.1) into

$$\begin{cases} {}^C D_t^\alpha u(x,t) - a(t)\mathcal{L}u(x,t) = F(x,t), & x \in \Omega,\ t \in (0,T]; \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T]; \\ u(x,0) = 0, & x \in \Omega, \end{cases} \tag{2.7}$$

and

$$\begin{cases} {}^C D_t^\alpha u(x,t) - a(t)\mathcal{L}u(x,t) = 0, & x \in \Omega,\ t \in (0,T]; \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T]; \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.8}$$

Denote the weak solutions of FDEs (2.7) and (2.8) by $u^r(x,t;a)$ and $u^i(x,t;a)$, respectively (“r" and “i" denote the initials of “right-hand side" and “initial condition"). The following lemma about $u^r(x,t;a)$ and $u^i(x,t;a)$ follows from Lemma 2.3.1 and Theorem 2.3.3.

Lemma 2.3.4. *Suppose Assumption 2.3.1 holds. Then $u^r(x,t;a)$ and $u^i(x,t;a)$ are the unique solutions for FDEs* (2.7) *and* (2.8)*, respectively, with the spectral representations as*

$$u^r(x,t;a) = \sum_{n=1}^{\infty} u_n^r(t;a)\phi_n(x),\ u^i(x,t;a) = \sum_{n=1}^{\infty} u_n^i(t;a)\phi_n(x), \tag{2.9}$$

where $u_n^r(t;a),\ u_n^i(t;a)$ satisfy the following fractional ODEs

$${}^C D_t^\alpha u_n^r(t;a) + \lambda_n a(t) u_n^r(t;a) = F_n(t),\ u_n^r(0;a) = 0,\ n \in \mathbb{N}^+; \tag{2.10}$$

$${}^C D_t^\alpha u_n^i(t;a) + \lambda_n a(t) u_n^i(t;a) = 0,\ u_n^i(0;a) = b_n,\ n \in \mathbb{N}^+. \tag{2.11}$$

Moreover, Theorem 2.3.3 ensures the weak solution $u(x,t;a)$ of FDE (2.1) *can be written as $u(x,t;a) = u^r(x,t;a) + u^i(x,t;a)$, i.e. $u_n(t;a) = u_n^r(t;a) + u_n^i(t;a),\ n \in \mathbb{N}^+$.*

2.3.5 Regularity of u^r

For each $n \in \mathbb{N}^+$, define

$$F_n^+(t) = \begin{cases} F_n(t), & \text{if } F_n(t) \geq 0; \\ 0, & \text{if } F_n(t) < 0, \end{cases} \quad F_n^-(t) = \begin{cases} F_n(t), & \text{if } F_n(t) < 0; \\ 0, & \text{if } F_n(t) \geq 0. \end{cases} \tag{2.12}$$

It is obvious that $F_n = F_n^+ + F_n^-$, the supports of F_n^+ and F_n^- are disjoint and $F_n^+, F_n^- \in C[0,T]$ which follows from $F_n \in C[0,T]$. Split $u_n^r(t;a)$ as $u_n^r(t;a) = u_n^{r,+}(t;a) + u_n^{r,-}(t;a)$, where $u_n^{r,+}(t;a),\ u_n^{r,-}(t;a)$ satisfy

$$ {}^C D_t^\alpha u_n^{r,+}(t;a) + \lambda_n a(t) u_n^{r,+}(t;a) = F_n^+(t),\ u_n^{r,+}(0;a) = 0,\ n \in \mathbb{N}^+; \tag{2.13} $$

$$ {}^C D_t^\alpha u_n^{r,-}(t;a) + \lambda_n a(t) u_n^{r,-}(t;a) = F_n^-(t),\ u_n^{r,-}(0;a) = 0,\ n \in \mathbb{N}^+, \tag{2.14} $$

respectively. The existence and uniqueness of $u_n^{r,+}(t;a)$ and $u_n^{r,-}(t;a)$ hold due to Lemma 2.3.2 and we can write

$$ u^r(x,t;a) = u^{r,+}(x,t;a) + u^{r,-}(x,t;a), \tag{2.15} $$

where

$$ u^{r,+}(x,t;a) = \sum_{n=1}^{\infty} u_n^{r,+}(t;a)\phi_n(x),\ u^{r,-}(x,t;a) = \sum_{n=1}^{\infty} u_n^{r,-}(t;a)\phi_n(x). \tag{2.16} $$

Then we state some properties of $u_n^{r,+}(t;a)$ and $u_n^{r,-}(t;a)$.

Lemma 2.3.5. *For any* $n \in \mathbb{N}^+$, $u_n^{r,+}(t;a) \geq 0$ *and* $u_n^{r,-}(t;a) \leq 0$ *on* $[0,T]$.

Proof. This proof follows from Corollary 2.3.1 directly. □

Lemma 2.3.6. *Given* $a_1(t), a_2(t) \in C^+[0,T]$ *with* $a_1(t) \leq a_2(t)$ *on* $[0,T]$*, we have*

$$ 0 \leq u_n^{r,+}(t;a_2) \leq u_n^{r,+}(t;a_1),\ u_n^{r,-}(t;a_1) \leq u_n^{r,-}(t;a_2) \leq 0,\ t \in [0,T],\ n \in \mathbb{N}^+. $$

Proof. Pick $n \in \mathbb{N}^+$, $u_n^{r,+}(t;a_1)$ and $u_n^{r,+}(t;a_2)$ satisfy the following system:

$$\begin{cases} {}^C D_t^\alpha u_n^{r,+}(t;a_1) + \lambda_n a_1(t) u_n^{r,+}(t;a_1) = F_n^+(t); \\ {}^C D_t^\alpha u_n^{r,+}(t;a_2) + \lambda_n a_2(t) u_n^{r,+}(t;a_2) = F_n^+(t); \\ u_n^{r,+}(0;a_1) = u_n^{r,+}(0;a_2) = 0, \end{cases}$$

which leads to

$${}^C D_t^\alpha w + \lambda_n a_1(t) w(t) = \lambda_n u_n^{r,+}(t;a_2)(a_2(t) - a_1(t)) \geq 0, \ w(0) = 0,$$

where $w(t) = u_n^{r,+}(t;a_1) - u_n^{r,+}(t;a_2)$ and the last inequality follows from Lemma 2.3.5 and $a_1 \leq a_2$. Hence, Corollary 2.3.1 shows that $w(t) \geq 0$, i.e. $u_n^{r,+}(t;a_2) \leq u_n^{r,+}(t;a_1)$ and Lemma 2.3.5 gives $0 \leq u_n^{r,+}(t;a_2) \leq u_n^{r,+}(t;a_1), \ t \in [0,T]$.

Similarly, we have $u_n^{r,-}(t;a_1) \leq u_n^{r,-}(t;a_2) \leq 0, \ t \in [0,T]$, completing the proof. □

Assumption 2.3.1 (a) implies there exists constants q_a, Q_a s.t.

$$0 < q_a < a(t) < Q_a \text{ on } [0,T]. \tag{2.17}$$

From Lemma 2.3.6, we obtain

$$|u_n^{r,+}(t;a)| \leq |u_n^{r,+}(t;q_a)|, \ |u_n^{r,-}(t;a)| \leq |u_n^{r,-}(t;q_a)| \text{ on } t \in [0,T], \ n \in \mathbb{N}^+, \tag{2.18}$$

where $u_n^{r,+}(t;q_a), u_n^{r,-}(t;q_a)$ are the unique solutions of fractional ODEs (2.13) and (2.14) respectively with $a(t) \equiv q_a$ on $[0,T]$. The next two lemmas concern the regularity of $u^{r,+}(x,t;a)$ and ${}^C D_t^\alpha u^{r,+}(x,t;a)$, respectively.

Lemma 2.3.7.

$$\|u^{r,+}\|_{L^2(0,T;H^2(\Omega))} \le C\|F\|_{L^2([0,T]\times\Omega)}.$$

Proof. Calculating $\|u^{r,+}(x,t;a)\|^2_{L^2(0,T;H^2(\Omega))}$ directly yields

$$\begin{aligned}\|u^{r,+}(x,t;a)\|^2_{L^2(0,T;H^2(\Omega))} &= \int_0^T \|u^{r,+}(x,t;a)\|^2_{H^2(\Omega))}\mathrm{d}t \le \int_0^T C\|(-\mathcal{L}u^{r,+})(x,t;a)\|^2_{L^2(\Omega)}\mathrm{d}t \\ &= C\int_0^T \|\sum_{n=1}^{\infty}\lambda_n u_n^{r,+}(t;a)\phi_n(x)\|^2_{L^2(\Omega)}\mathrm{d}t \\ &= C\int_0^T \sum_{n=1}^{\infty}\lambda_n^2|u_n^{r,+}(t;a)|^2\mathrm{d}t \le C\int_0^T \sum_{n=1}^{\infty}\lambda_n^2|u_n^{r,+}(t;q_a)|^2\mathrm{d}t,\end{aligned}$$

where the last inequality is obtained from (2.18). By the Monotone Convergence Theorem, we have

$$\|u^{r,+}(x,t;a)\|^2_{L^2(0,T;H^2(\Omega))} \le C\int_0^T \sum_{n=1}^{\infty}\lambda_n^2|u_n^{r,+}(t;q_a)|^2\mathrm{d}t = C\sum_{n=1}^{\infty}\int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2\mathrm{d}t. \tag{2.19}$$

For each $n \in \mathbb{N}^+$, [9] gives the explicit representation of $u_n^{r,+}(t;q_a)$

$$u_n^{r,+}(t;q_a) = \int_0^t F_n^+(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau,$$

which together with Young's inequality leads to

$$\begin{aligned}\int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2\mathrm{d}t &= \|F_n^+(t) * (\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a t^\alpha))\|^2_{L^2[0,T]} \\ &\le \|F_n^+\|^2_{L^2[0,T]}\|\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a t^\alpha)\|^2_{L^1[0,T]}.\end{aligned}$$

Lemmas 2.2.2, 2.2.3 and 2.2.4 give the bound of $\|\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n q_a t^\alpha)\|_{L^1[0,T]}$

$$\begin{aligned}\|\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n q_a t^\alpha)\|_{L^1[0,T]} &= \int_0^T \left|\lambda_n \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n q_a \tau^\alpha)\right| \mathrm{d}\tau \\ &= \int_0^T \lambda_n \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n q_a \tau^\alpha) \mathrm{d}\tau \\ &= -q_a^{-1} \int_0^T \frac{d}{d\tau} E_{\alpha,1}(-\lambda_n q_a \tau^\alpha) \mathrm{d}\tau \\ &= q_a^{-1}(1 - E_{\alpha,1}(-\lambda_n q_a T^\alpha)) \le q_a^{-1};\end{aligned}$$

while the definition (2.12) provides the bound of $\|F_n^+\|_{L^2[0,T]}$ as $\|F_n^+\|_{L^2[0,T]} \le \|F_n\|_{L^2[0,T]}$. Consequently, it holds $\int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2 \mathrm{d}t \le q_a^{-2} \|F_n\|^2_{L^2[0,T]},\ n \in \mathbb{N}^+$, i.e.

$$\sum_{n=1}^\infty \int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2 \mathrm{d}t \le q_a^{-2} \sum_{n=1}^\infty \|F_n\|^2_{L^2[0,T]},$$

which together with (2.19) and the completeness of $\{\phi_n(x) : n \in \mathbb{N}^+\}$ in $L^2(\Omega)$ gives

$$\begin{aligned}\|u^{r,+}(x,t;a)\|^2_{L^2(0,T;H^2(\Omega))} &\le C \sum_{n=1}^\infty \int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2 \mathrm{d}t \\ &\le C \sum_{n=1}^\infty \|F_n\|^2_{L^2[0,T]} = C\|F\|^2_{L^2([0,T]\times\Omega)},\end{aligned}$$

where the constant C only depends on $a(t)$. This completes the proof. □

Lemma 2.3.8.

$$\|{}^C D_t^\alpha u^{r,+}\|_{L^2([0,T]\times\Omega)} \le C\|F\|_{L^2([0,T]\times\Omega)}.$$

Proof. (2.13), (2.16), definition (2.12) and the Monotone Convergence Theorem give

$$\begin{aligned}\|{}^{C}D_t^{\alpha}u^{r,+}\|^2_{L^2([0,T]\times\Omega)} &= \int_0^T \|\sum_{n=1}^{\infty} {}^{C}D_t^{\alpha}u_n^{r,+}(\cdot;a)\phi_n(x)\|^2_{L^2(\Omega)}\mathrm{d}t = \sum_{n=1}^{\infty}\int_0^T |{}^{C}D_t^{\alpha}u_n^{r,+}(\cdot;a)|^2\mathrm{d}t \\ &\le \sum_{n=1}^{\infty}\int_0^T \left(2|\lambda_n a(t)u_n^{r,+}(t;a)|^2 + 2|F_n^{+}(t)|^2\right)\mathrm{d}t \\ &\le 2\sum_{n=1}^{\infty}\int_0^T |\lambda_n a(t)u_n^{r,+}(t;a)|^2\mathrm{d}t + 2\sum_{n=1}^{\infty}\int_0^T |F_n(t)|^2\mathrm{d}t.\end{aligned} \tag{2.20}$$

The estimate of $\sum\limits_{n=1}^{\infty}\int_0^T |\lambda_n a(t)u_n^{r,+}(t;a)|^2\mathrm{d}t$ follows from (2.17), (2.18) and the proof of Lemma 2.3.7

$$\sum_{n=1}^{\infty}\int_0^T |\lambda_n a(t)u_n^{r,+}(t;a)|^2\mathrm{d}t \le Q_a\sum_{n=1}^{\infty}\int_0^T |\lambda_n u_n^{r,+}(t;q_a)|^2\mathrm{d}t \le C\|F\|^2_{L^2([0,T]\times\Omega)};$$

while the completeness of $\{\phi_n(x) : n \in \mathbb{N}^+\}$ gives $\sum\limits_{n=1}^{\infty}\int_0^T |F_n(t)|^2\mathrm{d}t = \|F\|^2_{L^2([0,T]\times\Omega)}$. Hence, (2.20) develops $\|{}^{C}D_t^{\alpha}u^{r,+}\|^2_{L^2([0,T]\times\Omega)} \le C\|F\|^2_{L^2([0,T]\times\Omega)}$, which implies the indicated conclusion. □

The following corollary follows immediately from the proofs of Lemmas 2.3.7 and 2.3.8.

Corollary 2.3.2.

$$\|u^{r,-}\|_{L^2(0,T;H^2(\Omega))} \le C\|F\|_{L^2([0,T]\times\Omega)},\ \|{}^{C}D_t^{\alpha}u^{r,-}\|_{L^2([0,T]\times\Omega)} \le C\|F\|_{L^2([0,T]\times\Omega)}.$$

From Lemmas 2.3.7, 2.3.8, Corollary 2.3.2 and (2.15), we are able to deduce the regularity for $u^r(x,t;a)$ and ${}^{C}D_t^{\alpha}u^r(x,t;a)$.

Lemma 2.3.9 (Regularity of u^r)**.**

$$\|u^r\|_{L^2(0,T;H^2(\Omega))} + \|^C D_t^\alpha u^r\|_{L^2([0,T]\times\Omega)} \leq C\|F\|_{L^2([0,T]\times\Omega)}.$$

Proof. (2.15) gives $u^r(x,t;a) = u^{r,+}(x,t;a) + u^{r,-}(x,t;a)$, which leads to

$$\begin{aligned}
&\|u^r\|_{L^2(0,T;H^2(\Omega))} + \|^C D_t^\alpha u^r\|_{L^2([0,T]\times\Omega)} \\
\leq\ &\|u^{r,+}\|_{L^2(0,T;H^2(\Omega))} + \|u^{r,-}\|_{L^2(0,T;H^2(\Omega))} \\
&+ \|^C D_t^\alpha u^{r,+}\|_{L^2([0,T]\times\Omega)} + \|^C D_t^\alpha u^{r,-}\|_{L^2([0,T]\times\Omega)} \\
\leq\ &C\|F\|_{L^2([0,T]\times\Omega)}.
\end{aligned}$$

□

If we impose a higher regularity on F, we can obtain the regularity estimate of $\|u^r\|_{C([0,T];H^2(\Omega))}$.

Corollary 2.3.3. *Under Assumption 2.3.1, if* $F \in C^\theta([0,T];L^2(\Omega)),\ 0 < \theta < 1$, *then*

$$\|u^r\|_{C([0,T];H^2(\Omega))} + \|^C D_t^\alpha u^r\|_{C([0,T];L^2(\Omega))} \leq C\|F\|_{C^\theta([0,T];L^2(\Omega))},$$

where C *depends on* Ω, $-\mathcal{L}$ *and* $a(t)$.

Proof. For each $t \in [0,T]$, we have

$$\begin{aligned}
\|u^{r,+}(x,t;a)\|^2_{H^2(\Omega)} &\le C\|-\mathcal{L}u^{r,+}\|^2_{L^2(\Omega)} \le C\sum_{n=1}^{\infty}|\lambda_n u_n^{r,+}(t;a)|^2 \\
&\le C\sum_{n=1}^{\infty}\left|\lambda_n\int_0^t F_n^+(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau\right|^2 \\
&\le C\sum_{n=1}^{\infty}\left|\lambda_n\int_0^t |F_n^+(\tau)-F_n^+(t)|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau\right|^2 \\
&\quad + C\sum_{n=1}^{\infty}\left|F_n^+(t)\int_0^t \lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau\right|^2.
\end{aligned}$$

The definition of $F_n^+(t)$ yields that $|F_n^+(\tau)-F_n^+(t)| \le |F_n(\tau)-F_n(t)|$; Lemma 2.2.2 gives

$$0 < \int_0^t \lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau = q_a^{-1}(1-E_{\alpha,1}(-\lambda_n q_a t^\alpha)) < q_a^{-1}.$$

Hence,

$$\begin{aligned}
\|u^{r,+}(x,t;a)\|^2_{H^2(\Omega)} &\le C\sum_{n=1}^{\infty}\left|\lambda_n\int_0^t |F_n(\tau)-F_n(t)|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_a(t-\tau)^\alpha)\mathrm{d}\tau\right|^2 \\
&\quad + C\sum_{n=1}^{\infty}|F_n(t)|^2.
\end{aligned}$$

By [9, Lemma 3.4], we have

$$\|u^{r,+}(x,t;a)\|^2_{H^2(\Omega)} \le C\|F\|^2_{C^\theta([0,T];L^2(\Omega))} + C\|F(\cdot,t)\|^2_{L^2(\Omega)},\ t\in[0,T],$$

which gives

$$\|u^{r,+}\|_{C([0,T];H^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))},$$

and the constant C depends on Ω, $-\mathcal{L}$ and $a(t)$. Similarly, we can show

$\|u^{r,-}\|_{C([0,T];H^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))}$.

For ${}^C D_t^\alpha u^r$, by (2.10), we have ${}^C D_t^\alpha u^{r,+} = \sum_{n=1}^\infty [-\lambda_n a(t) u_n^{r,+}(t;a) + F_n^+(t)]\phi_n(x)$. Then for each $t \in [0,T]$,

$$\begin{aligned}\|{}^C D_t^\alpha u^{r,+}\|_{L^2(\Omega)}^2 &\le C\sum_{n=1}^\infty Q_a^2 |\lambda_n u_n^{r,+}(t;a)|^2 + C\sum_{n=1}^\infty |F_n(t)|^2 \\ &\le C\sum_{n=1}^\infty |\lambda_n u_n^{r,+}(t;a)|^2 + C\|F(\cdot,t)\|_{L^2(\Omega)}^2.\end{aligned}$$

From the above proof for $\|u^{r,+}\|_{H^2(\Omega)}^2$, it holds

$$\|{}^C D_t^\alpha u^{r,+}\|_{L^2(\Omega)}^2 \le C\|F\|_{C^\theta([0,T];L^2(\Omega))}^2 + C\|F(\cdot,t)\|_{L^2(\Omega)}^2,\ t \in [0,T],$$

which gives

$$\|{}^C D_t^\alpha u^{r,+}\|_{C([0,T];L^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))}.$$

Analogously, we can show $\|{}^C D_t^\alpha u^{r,-}\|_{C([0,T];L^2(\Omega))} \le C\|F\|_{C^\theta([0,T];L^2(\Omega))}$.

The estimates of $u^{r,+},\ u^{r,-},\ {}^C D_t^\alpha u^{r,+}$ and ${}^C D_t^\alpha u^{r,-}$ yield the desired result and complete this proof. □

2.3.6 Regularity of u^i

In this part we consider the regularity of u^i. Just as in the regularity results for u^r, we first state two lemmas which concern the positivity and monotonicity of u^i, respectively.

Lemma 2.3.10. *With the representation* (2.9) *and the fractional ODE* (2.11)*, for each* $n \in \mathbb{N}^+$, $b_n \le (\ge) 0$ *implies that* $u_n^i(t;a) \le (\ge) 0$ *on* $[0,T]$.

Proof. This is a directly result of Corollary 2.3.1. □

Lemma 2.3.11. *Given* $a_1, a_2 \in C^+[0,T]$ *with* $a_1 \le a_2$ *on* $[0,T]$*, for each* $n \in \mathbb{N}^+$*, we* *have*

$$\begin{cases} 0 \le u_n^i(t;a_2) \le u_n^i(t;a_1), \ \textit{if}\, b_n \ge 0; \\ u_n^i(t;a_1) \le u_n^i(t;a_2) \le 0, \ \textit{if}\, b_n \le 0. \end{cases}$$

Proof. Fix $n \in \mathbb{N}^+$, from the fractional ODE (2.11), the functions $u_n^i(t;a_1)$ and $u_n^i(t;a_2)$ satisfy the following system

$$\begin{cases} {}^C D_t^\alpha u_n^i(t;a_1) + \lambda_n a_1(t) u_n^i(t;a_1) = 0; \\ {}^C D_t^\alpha u_n^i(t;a_2) + \lambda_n a_2(t) u_n^i(t;a_2) = 0; \\ u_n^i(0;a_1) = u_n^i(0;a_2) = b_n. \end{cases}$$

This gives

$$ {}^C D_t^\alpha w + \lambda_n a_1(t) w(t) = \lambda_n u_n^i(t;a_2)(a_2(t) - a_1(t)), \ w(0) = 0, \tag{2.21}$$

where $w(t) = u_n^i(t;a_1) - u_n^i(t;a_2)$.

If $b_n \ge 0$, Corollary 2.3.1 shows that $u_n^i(t;a_1), u_n^i(t;a_2) \ge 0$. Also, Lemma 2.3.10 and $a_1 \le a_2$ ensures the right side of (2.21) is nonnegative, which together with Corollary 2.3.1 implies $w \ge 0$, i.e. $0 \le u_n^i(t;a_2) \le u_n^i(t;a_1)$. The similar argument yields $u_n^i(t;a_1) \le u_n^i(t;a_2) \le 0$ for the case $b_n \le 0$. □

Lemma 2.3.12 (Regularity for u^i)**.**

$$\|u^i\|_{L^2(0,T;H^2(\Omega))} + \|{}^C D_t^\alpha u^i\|_{L^2([0,T]\times\Omega)} \le C T^{\frac{1-\alpha}{2}} \|u_0\|_{H^1(\Omega)}.$$

Proof. Given $t \in [0, T]$, the direct calculation and Lemma 2.3.11 yield that

$$\|u^i(x,t;a)\|^2_{H^2(\Omega)} \le C\| - \mathcal{L}u^i(x,t;a)\|^2_{L^2(\Omega)} = C\|\sum_{n=1}^{\infty} \lambda_n u_n^i(t;a)\phi_n(x)\|^2_{L^2(\Omega)}$$
$$= C\sum_{n=1}^{\infty} |\lambda_n u_n^i(t;a)|^2 \le C\sum_{n=1}^{\infty} |\lambda_n u_n^i(t;q_a)|^2.$$

Recall that [9] established the representation as $u_n^i(t;q_a) = b_n E_{\alpha,1}(-\lambda_n q_a t^\alpha),\ n \in \mathbb{N}^+$. Hence, by Lemma 2.2.1,

$$\begin{aligned}\|u^i(x,t;a)\|^2_{H^2(\Omega)} &\le C\| - \mathcal{L}u^i(x,t;a)\|^2_{L^2(\Omega)} \le C\sum_{n=1}^{\infty} |\lambda_n b_n E_{\alpha,1}(-\lambda_n q_a t^\alpha)|^2\\ &\le C\sum_{n=1}^{\infty} |\frac{1}{1+\lambda_n q_a t^\alpha}|^2 \lambda_n^2 b_n^2 = C\sum_{n=1}^{\infty} |\frac{(\lambda_n q_a t^\alpha)^{\frac{1}{2}}}{1+\lambda_n q_a t^\alpha}|^2 t^{-\alpha} q_a^{-1}\lambda_n b_n^2\\ &\le Ct^{-\alpha}\sum_{n=1}^{\infty}((-\mathcal{L})^{\frac{1}{2}}u_0,\phi_n)^2 \le Ct^{-\alpha}\|u_0\|^2_{H^1(\Omega)},\end{aligned} \tag{2.22}$$

which leads to $\|u^i\|^2_{L^2(0,T;H^2(\Omega))} \le C\int_0^T t^{-\alpha}\|u_0\|^2_{H^1(\Omega)}\mathrm{d}t = CT^{1-\alpha}\|u_0\|^2_{H^1(\Omega)}$, i.e.

$$\|u^i\|_{L^2(0,T;H^2(\Omega))} \le CT^{\frac{1-\alpha}{2}}\|u_0\|_{H^1(\Omega)}. \tag{2.23}$$

For the estimate of ${}^C D_t^\alpha u^i(x,t;a)$, (2.9) and (2.11) yield

$${}^C D_t^\alpha u^i(x,t;a) = \sum_{n=1}^{\infty} {}^C D_t^\alpha u_n^i(t;a)\phi_n(x) = -\sum_{n=1}^{\infty} \lambda_n a(t) u_n^i(t;a)\phi_n(x),$$

which together with (2.17) gives

$$\begin{aligned}\|{}^C D_t^\alpha u^i(x,t;a)\|^2_{L^2(\Omega)} &\le Q_a^2 \sum_{n=1}^{\infty} |\lambda_n u_n^i(t;a)|^2\\ &= Q_a^2\| - \mathcal{L}u^i(x,t;a)\|^2_{L^2(\Omega)} \le Ct^{-\alpha}\|u_0\|^2_{H^1(\Omega)},\ t \in [0,T],\end{aligned}$$

where the last inequality follows from (2.22). This result implies that

$$\|^C D_t^\alpha u^i(x,t;a)\|_{L^2([0,T]\times\Omega)}^2 = \int_0^T \|^C D_t^\alpha u^i(x,t;a)\|_{L^2(\Omega)}^2 \mathrm{d}t \le CT^{1-\alpha}\|u_0\|_{H^1(\Omega)}^2,$$

i.e. $\|^C D_t^\alpha u^i\|_{L^2([0,T]\times\Omega)} \le CT^{\frac{1-\alpha}{2}}\|u_0\|_{H^1(\Omega)}$, which together with (2.23) completes the proof. □

Moreover, with a stronger condition on u_0, such as assuming $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, we can deduce the C-regularity estimate of u^i.

Corollary 2.3.4. *With Assumption 2.3.1 and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then*

$$\|u^i\|_{C([0,T];H^2(\Omega))} + \|^C D_t^\alpha u^i\|_{C([0,T];L^2(\Omega))} \le C\|u_0\|_{H^2(\Omega)}.$$

Proof. Lemma 2.2.1 yields that

$$\sum_{n=1}^\infty |\lambda_n b_n E_{\alpha,1}(-\lambda_n q_a t^\alpha)|^2 \le C\sum_{n=1}^\infty |\lambda_n b_n|^2 = C\|-\mathcal{L}u_0\|_{L^2(\Omega)}^2 \le C\|u_0\|_{H^2(\Omega)}^2,\ t\in[0,T];$$

meanwhile, the following estimates have been shown in the proof of Theorem 2.3.12

$$\begin{cases} \|u^i(x,t;a)\|_{H^2(\Omega)}^2 \le C\|-\mathcal{L}u^i(x,t;a)\|_{L^2(\Omega)}^2 \le C\sum_{n=1}^\infty |\lambda_n b_n E_{\alpha,1}(-\lambda_n q_a t^\alpha)|^2, \\ \|^C D_t^\alpha u^i(x,t;a)\|_{L^2(\Omega)}^2 \le Q_a^2 \sum_{n=1}^\infty |\lambda_n u_n^i(t;a)|^2 = C\|-\mathcal{L}u^i(x,t;a)\|_{L^2(\Omega)}^2. \end{cases}$$

Hence, it holds that

$$\|u^i(x,t;a)\|_{H^2(\Omega)} + \|^C D_t^\alpha u^i(x,t;a)\|_{L^2(\Omega)} \le C\|u_0\|_{H^2(\Omega)},\ t\in[0,T],$$

which leads to the claimed result. □

2.3.7 Main theorem for the direct problem

The main theorem for the direct problem follows from Theorem 2.3.3, Lemmas 2.3.9 and 2.3.12, Corollaries 2.3.3 and 2.3.4, and the relation $u(x,t;a) = u^r(x,t;a) + u^i(x,t;a)$.

Theorem 2.3.4 (Main theorem for the direct problem)**.** *Let Assumption 2.3.1 be valid, then under Definition 2.3.2, there exists a unique weak solution* $u(x,t;a)$ *of FDE* (2.1) *with the spectral representation* (2.4) *and the following regularity estimates:*

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|{}^C D_t^\alpha u\|_{L^2([0,T]\times\Omega)} \le C(\|F\|_{L^2([0,T]\times\Omega)} + T^{\frac{1-\alpha}{2}}\|u_0\|_{H^1(\Omega)}).$$

Moreover, if the conditions $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $F \in C^\theta([0,T]; L^2(\Omega)),\ 0 < \theta < 1$ *are added, we have:*

$$\|u\|_{C([0,T];H^2(\Omega))} + \|{}^C D_t^\alpha u\|_{C([0,T];L^2(\Omega))} \le C(\|F\|_{C^\theta([0,T];L^2(\Omega))} + \|u_0\|_{H^2(\Omega)}).$$

2.4 Inverse problem–reconstruction of the diffusion coefficient $a(t)$

In this section, we discuss how to recover the coefficient $a(t)$ through the output flux data

$$a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a) = g(t),\ x_0 \in \partial\Omega.$$

All cross the inverse problem work, the operator $-\mathcal{L}$ is assumed to satisfy the condition (2.2), then the expression $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ makes sense. We only consider this reconstruction in the space $C^+[0,T]$, which can be regarded as the admissible set for $a(t)$. To this end, we introduce an operator K, which will be shown to have a fixed point consisting of the desired coefficient $a(t)$.

2.4.1 Operator K

The operator K is defined as

$$K\psi(t) := \frac{g(t)}{\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;\psi)} = \frac{g(t)}{\sum\limits_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)},\ t\in[0,T]$$

with domain

$$\mathcal{D}(K) := \{\psi \in C^+[0,T] : \psi(t) \geq g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1},\ t\in[0,T]\}.$$

To analyze K, we make the following assumptions.

Assumption 2.4.1. *u_0, F and g should satisfy the following restrictions:*

(a) $u_0 \in H^3(\Omega)\cap H_0^1(\Omega)$ *with* $b_n := (u_0,\phi_n) \geq 0,\ n\in\mathbb{N}^+$;

(b) $\exists\theta\in(0,1)$ *s.t.* $F(x,t)\in C^\theta([0,T];H^3(\Omega)\cap H_0^1(\Omega))$ *with* $F_n(t) := (F(\cdot,t),\phi_n) \geq 0$ *on* $[0,T]$ *for each* $n\in\mathbb{N}^+$;

(c) $\exists N\in\mathbb{N}^+$ *s.t.* $\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$, $b_N > 0$ *and* $F_N(t) > 0$ *on* $[0,T]$;

(d) $g\in C^+[0,T]$.

The next remark shows that the equality in the definition of K is valid.

Remark 2.4.1. Given $\psi\in C^+[0,T]$ and for each $t\in[0,T]$, by the proofs of Corollaries

2.3.3 and 2.3.4, we have

$$\begin{aligned}
&\|u^{r,+}(x,t;\psi)\|^2_{H^3(\Omega)}\\
&\le C\|(-\mathcal{L})^{3/2}u^{r,+}\|^2_{L^2(\Omega)} \le C\sum_{n=1}^{\infty}|\lambda_n^{3/2}u_n^{r,+}(t;\psi)|^2\\
&\le C\sum_{n=1}^{\infty}\left|\lambda_n^{3/2}\int_0^t F_n^+(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_\psi(t-\tau)^\alpha)\mathrm{d}\tau\right|^2\\
&\le C\sum_{n=1}^{\infty}\left|\lambda_n\int_0^t \lambda_n^{1/2}|F_n(\tau)-F_n(t)|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n q_\psi(t-\tau)^\alpha)\mathrm{d}\tau\right|^2\\
&\quad + C\sum_{n=1}^{\infty}\left|\lambda_n^{1/2}F_n(t)(1-E_{\alpha,1}(-\lambda_n q_\psi t^\alpha))\right|^2\\
&\le C\|(-\mathcal{L})^{1/2}F\|^2_{C^\theta([0,T];L^2(\Omega))} + C\|(-\mathcal{L})^{1/2}F(\cdot,t)\|^2_{L^2(\Omega)}\\
&\le C\|F\|^2_{C^\theta([0,T];H^1(\Omega))} + C\|F(\cdot,t)\|^2_{H^1(\Omega)}
\end{aligned}$$

and

$$\|u^{r,-}(x,t;\psi)\|^2_{H^3(\Omega)} \le C\|F\|^2_{C^\theta([0,T];H^1(\Omega))} + C\|F(\cdot,t)\|^2_{H^1(\Omega)},$$

which give $\|u^r\|_{C([0,T];H^3(\Omega))} \le C\|F\|_{C^\theta([0,T];H^1(\Omega))}$;

$$\begin{aligned}
\|u^i(x,t;\psi)\|^2_{H^3(\Omega)} &\le C\|(-\mathcal{L})^{3/2}u^i\|^2_{L^2(\Omega)} \le C\|\sum_{n=1}^{\infty}\lambda_n^{3/2}u_n^i(t;\psi)\phi_n(x)\|^2_{L^2(\Omega)}\\
&\le C\sum_{n=1}^{\infty}|\lambda_n^{3/2}b_nE_{\alpha,1}(-\lambda_n q_\psi t^\alpha)|^2 \le C\sum_{n=1}^{\infty}|\lambda_n^{3/2}b_n|^2\\
&= C\|(-\mathcal{L})^{3/2}u_0\|^2_{L^2(\Omega)} \le C\|u_0\|^2_{H^3(\Omega)},
\end{aligned}$$

which gives $\|u^i\|_{C([0,T];H^3(\Omega))} \le C\|u_0\|_{H^3(\Omega)}$. Combining the above two results yields that

$$\|u\|_{C([0,T];H^3(\Omega))} \le C(\|F\|_{C^\theta([0,T];H^1(\Omega))} + \|u_0\|_{H^3(\Omega)}) < \infty,$$

which means for each $t\in[0,T]$, $\|u\|_{H^3(\Omega)}<\infty$. Recall that $\Omega\subset R^n, n=1,2,3$, then the

Sobolev Embedding Theorem gives

$$u(x,t;\psi) = \sum_{n=1}^{\infty} u_n(t;\psi)\phi_n(x) \in C^1(\overline{\Omega}) \text{ for each } t \in [0,T].$$

Hence, $\sum\limits_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ is well-defined and

$$\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;\psi) = \sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0), \quad t \in [0,T].$$

The following two remarks will explain the reasonableness and reason for Assumption 2.4.1.

Remark 2.4.2. For the inverse problem, the right-hand side function $F(x,t)$ and the initial condition $u_0(x)$ are input data, which, at least in some circumstance, can be assumed to be controlled. Even though Assumption 2.4.1 (a), (b) and (c) appear restrictive, it is not hard to construct functions that satisfy them. For example, in (a) if $u_0 = c\phi_k$ for some $c > 0$, then Assumption 2.4.1 (a) will be satisfied. This will also be true if $u_0 = \sum_{k=1}^{M} c_k\phi_k$ with all $c_k > 0$. Similarly, (b) is satisfied if $F(x,t)$ is also a linear combination of $\{\phi_n : n \in \mathbb{N}^+\}$ with positive coefficients. For (c), by the completeness of $\{\phi_n : n \in \mathbb{N}^+\}$ in $L^2(\Omega)$, there should exist $N \in \mathbb{N}^+$ s.t. $\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$. Otherwise, for each $\psi \in H^3(\Omega) \subset L^2(\Omega)$, $\frac{\partial \psi}{\partial \overrightarrow{\mathbf{n}}}(x_0) = 0$ and obviously it is incorrect. Then for this N, we only need to set the coefficients of u_0 and F upon ϕ_N be strictly positive.

The output flux data $g(t)$, it is not under our control. However, if there exists $a \in C^+[0,T]$ s.t. $a(t)\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a) = g(t)$, Assumption 2.4.1 (a), (b) and Corollary 2.3.1 yield that $u_n(t;a) \geq 0$; (2.3) gives $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq 0, n \in \mathbb{N}^+$; Assumption 2.4.1 (c) ensures $\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$ and $u_N(t;a) > 0$ on $[0,T]$, where the proof can be seen in Lemma 2.4.1.

Consequently,

$$\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0, t; a) = \sum_{n=1}^{\infty} u_n(t; a)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq u_N(t; a)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0, \ t \in [0, T].$$

This together with $a \in C^+[0, T]$ gives that $g > 0$. The continuity of g follows from the ones of a and $u_n(t; a), \ n \in \mathbb{N}^+$, which are derived from the admissible set $C^+[0, T]$ and Theorem 2.3.3, respectively. Therefore, Assumption 2.4.1 (d) is reasonable and can be attained.

Remark 2.4.3. The well-definedness of the domain $\mathcal{D}(K)$ is guaranteed by Assumption 2.4.1 (a), (b), (c) and (d) in the sense that the H^3-regularity of u_0, F and the Sobolev Embedding Theorem support that $\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ and $\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0, t)$ are well defined, and the dominator of the lower bound of $\mathcal{D}(K)$

$$\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0, t)] = \sum_{n=1}^{\infty}(b_n + I_t^\alpha F_n)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq (b_N + I_t^\alpha F_N)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$$

on $[0, T]$. Recall that the numerator $g > 0$, so that the lower bound $g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0, t)]\Big]^{-1} > 0$, which gives that $\mathcal{D}(K)$ is a subspace of $C^+[0, T]$. Also, $F(x, t) \in C^\theta([0, T]; H^3(\Omega) \cap H_0^1(\Omega))$ yields that $F_N(t)$ is continuous on $[0, T]$, so is $(b_N + I_t^\alpha F_N)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0)$. Then $\exists C > 0$ s.t. $(b_N + I_t^\alpha F_N)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > C > 0$, which leads to the dominator

$$\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0, t)] > C > 0 \text{ on } [0, T].$$

The strict positivity of the dominator avoids $\mathcal{D}(K)$ degenerating to an empty set.

In order to show the well-definedness of K, Assumption 2.4.1 (a), (b) and (c) will be used. Furthermore, Assumption 2.4.1 (a) and (b) are crucial to build the monotonicity of operator K; meanwhile, Assumption 2.4.1 (c) is stated for the uniqueness of fixed points of K.

For the operator K, we have the following lemmas.

Lemma 2.4.1. *The operator K is well-defined.*

Proof. For each $\psi \in \mathcal{D}(K)$, Theorem 2.3.3 ensures that there exists a unique $u_n(t;\psi)$ for $n \in \mathbb{N}^+$, which implies the existence and uniqueness of $K\psi$.

Then it is suffice to show the dominator $\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$ on $[0,T]$. With (2.5), Lemma 2.3.1 and Assumption 2.4.1 (a) and (b), we have $u_n(t;\psi) \geq 0$ on $[0,T]$, which together with $\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq 0$ gives $\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq u_N(t;\psi)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0)$. Due to the assumption $\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$, we claim that $u_N(t;\psi) > 0$. Assume not, i.e. $\exists t_0 \in [0,T]$ s.t. $u_N(t_0;\psi) \leq 0$. The result $u_N(t;\psi) \geq 0$ yields that $u_N(t_0;\psi) = 0$ so that $u_N(t;\psi)$ attains its minimum at $t = t_0$. $u_N(0;\psi) = b_N > 0$ implies $t_0 \neq 0$, i.e. $t_0 \in (0,T]$. Then Lemma 2.2.5, $u_N(t_0;\psi) = 0$ and the ODE (2.5) show that ${}^C D_t^{\alpha} u_N(t_0;\psi) = F_N(t_0) \leq 0$, which contradicts with Assumption 2.4.1 (c) and confirms the claim. Hence,

$$\sum_{n=1}^{\infty} u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq u_N(t;\psi)\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0,$$

which completes the proof. □

Lemma 2.4.2. *K maps $\mathcal{D}(K)$ into $\mathcal{D}(K)$.*

Proof. Given $\psi \in \mathcal{D}(K)$. The continuity of $K\psi$ follows from the continuity of $u_n(t;\psi)$ for each $n \in \mathbb{N}^+$ and the continuity of g, which are established by Theorem 2.3.3 and Assumption 2.4.1 (d) respectively.

For each $n \in \mathbb{N}^+$, (2.5) ensures $u_n(t;\psi)$ satisfies

$${}^C D_t^{\alpha} u_n(t;\psi) + \lambda_n \psi(t) u_n(t;\psi) = F_n(t),\ u_n(0;\psi) = b_n.$$

Taking I_t^α on both sides of the above ODE and using Lemma 2.2.6 yield that

$$u_n(t;\psi)+\lambda_n I_t^\alpha[\psi(t)u_n(t;\psi)]=I_t^\alpha F_n+b_n.$$

From the proof of Lemma 2.4.1, we have $u_n(t;\psi)\geq 0$ on $[0,T]$, which together with $\lambda_n>0$, the positivity of ψ and the definition of I_t^α yields that $\lambda_n I_t^\alpha[\psi(t)u_n(t;\psi)]\geq 0$. Since $u_n(t;\psi)\geq 0$ and $\lambda_n I_t^\alpha[\psi(t)u_n(t;\psi)]\geq 0$, we deduce that $0\leq u_n(t;\psi)\leq I_t^\alpha F_n+b_n$ on $[0,T]$. Hence, with $\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)\geq 0$ and the smoothness assumptions $u_0\in H^3(\Omega)\cap H_0^1(\Omega),\ F\in C^\theta([0,T];H^3(\Omega)\cap H_0^1(\Omega))$ stated in Assumption 2.4.1 (a) and (b) respectively, the following inequality holds

$$\sum_{n=1}^\infty u_n(t;\psi)\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)\leq\sum_{n=1}^\infty(I_t^\alpha F_n+b_n)\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)=\frac{\partial u_0}{\partial\overrightarrow{\mathbf{n}}}(x_0)+I_t^\alpha[\frac{\partial F}{\partial\overrightarrow{\mathbf{n}}}(x_0,t)],$$

which together with $g>0$ yields that

$$K\psi(t)=\frac{g(t)}{\sum\limits_{n=1}^\infty u_n(t;\psi)\frac{\partial\phi_n}{\partial\overrightarrow{\mathbf{n}}}(x_0)}\geq g(t)\Big[\frac{\partial u_0}{\partial\overrightarrow{\mathbf{n}}}(x_0)+I_t^\alpha[\frac{\partial F}{\partial\overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1}>0,\ t\in[0,T],$$

where the last inequality follows from Remark 2.4.3. The above result and the continuity of $K\psi$ lead to $K\psi\in\mathcal{D}(K)$, which is the expected result. □

2.4.2 Monotonicity

In this part, we show the monotonicity of the operator K.

Theorem 2.4.2 (Monotonicity). *Given $a_1,a_2\in\mathcal{D}(K)$ with $a_1\leq a_2$, then $Ka_1\leq Ka_2$ on* $[0,T]$.

Proof. Pick $n \in \mathbb{N}^+$, due to (2.5), $u_n(t; a_1)$ and $u_n(t; a_2)$ satisfy

$$\begin{cases} {}^C D_t^\alpha u_n(t; a_1) + \lambda_n a_1(t) u_n(t; a_1) = F_n(t), \ u_n(0; a_1) = b_n; \\ {}^C D_t^\alpha u_n(t; a_2) + \lambda_n a_2(t) u_n(t; a_2) = F_n(t), \ u_n(0; a_2) = b_n, \end{cases}$$

which together with $a_1 \le a_2$ and Lemma 2.3.3 yields

$$ {}^C D_t^\alpha w + \lambda_n a_1(t) w(t) = \lambda_n u_n(t; a_2)(a_2(t) - a_1(t)) \ge 0, \ w(0) = 0, \tag{2.24}$$

where $w(t) = u_n(t; a_1) - u_n(t; a_2)$. Applying Lemma 2.3.3 to the above ODE yields that $w \ge 0$, i.e. $u_n(t; a_1) \ge u_n(t; a_2) \ge 0$, which together with assumption (2.3) leads to

$$\sum_{n=1}^{\infty} u_n(t; a_1) \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \ge \sum_{n=1}^{\infty} u_n(t; a_2) \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0, \ t \in [0, T].$$

Therefore, with the condition $g > 0$ stated in Assumption 2.4.1 (d),

$$K a_1(t) = \frac{g(t)}{\sum_{n=1}^{\infty} u_n(t; a_1) \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} \le \frac{g(t)}{\sum_{n=1}^{\infty} u_n(t; a_2) \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} = K a_2(t), \ t \in [0, T],$$

which completes this proof. □

2.4.3 Uniqueness

In order to show the uniqueness, we state two lemmas.

Lemma 2.4.3. *If $a_1, a_2 \in \mathcal{D}(K)$ are both fixed points of K with $a_1 \le a_2$, then $a_1 \equiv a_2$.*

Proof. Pick a fixed point $a(t)$, then

$$a(t) \sum_{n=1}^{\infty} u_n(t; a) \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = \sum_{n=1}^{\infty} a(t) u_n(t; a) \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = g(t),$$

which gives

$$\sum_{n=1}^{\infty} I_t^{\alpha}[a(t)u_n(t;a)]\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = I_t^{\alpha} g \tag{2.25}$$

by taking I_t^{α} on both sides. Similarly, taking I_t^{α} on the both sides of (2.5) and applying Lemma 2.2.6 yield that

$$I_t^{\alpha}[a(t)u_n(t;a)] = \lambda_n^{-1} I_t^{\alpha} F_n + \lambda_n^{-1} b_n - \lambda_n^{-1} u_n(t;a),\ n \in \mathbb{N}^+,$$

which together with (2.25) generates

$$\sum_{n=1}^{\infty} \lambda_n^{-1} u_n(t;a)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) = \sum_{n=1}^{\infty} \lambda_n^{-1}(I_t^{\alpha} F_n + b_n)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) - I_t^{\alpha} g. \tag{2.26}$$

In (2.26), the convergence of the two series in $C[0,T]$ is supported by Assumption 2.4.1, Remark 2.4.1 and the fact that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

Given two fixed points a_1, a_2 with $a_1 \leq a_2$, then a_1 and a_2 should satisfy (2.26) simultaneously, which gives

$$\sum_{n=1}^{\infty} \lambda_n^{-1} \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)(u_n(t;a_1) - u_n(t;a_2)) = 0. \tag{2.27}$$

In the proof of Theorem 3.3.2, we have shown that $u_n(t;a_1) \geq u_n(t;a_2) \geq 0$. Also recall that $\lambda_n^{-1}\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0) \geq 0,\ n \in \mathbb{N}^+$, then $\lambda_n^{-1}\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)(u_n(t;a_1) - u_n(t;a_2)) \geq 0$ on $[0,T]$ for $n \in \mathbb{N}^+$. Hence, (2.27) implies that

$$\lambda_n^{-1} \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)(u_n(t;a_1) - u_n(t;a_2)) = 0,\ t \in [0,T],\ n \in \mathbb{N}^+.$$

Let $n = N$, $\lambda_N^{-1}\frac{\partial \phi_N}{\partial \overrightarrow{\mathbf{n}}}(x_0) > 0$ gives $u_N(t;a_1) \equiv u_N(t;a_2)$ on $[0,T]$. Set $w(t) = u_N(t;a_1) -$

$u_N(t; a_2) = 0$. Then (2.24) yields that

$$0 = {}^C D_t^\alpha w + \lambda_N a_1(t) w(t) = \lambda_N u_N(t; a_2)(a_2(t) - a_1(t)),$$

i.e. $u_N(t; a_2)(a_2(t) - a_1(t)) \equiv 0$ on $[0, T]$; while the proof of Lemma 2.4.1 yields that $u_N(t; a_2) > 0$. Hence, we have $a_1 = a_2$ on $[0, T]$, which completes the proof. $\square$

Before showing uniqueness, we introduce a successive iteration procedure which will generate a sequence converging to a fixed point if it exists. Set

$$\overline{a}_0(t) = g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha\big[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0, t)\big]\Big]^{-1},\ \overline{a}_{n+1} = K\overline{a}_n,\ n \in \mathbb{N}.$$

Then this iteration reproduces a sequence $\{\overline{a}_n : n \in \mathbb{N}\}$ which is contained by $\mathcal{D}(K)$ due to Lemma 2.4.2.

Lemma 2.4.4. *If there exists a fixed point $a(t) \in \mathcal{D}(K)$ of operator K, then the sequence $\{\overline{a}_n : n \in \mathbb{N}\}$ will converge to $a(t)$.*

Proof. $\overline{a}_0$ is the lower bound of $\mathcal{D}(K)$ and $\{\overline{a}_n : n \in \mathbb{N}\} \subset \mathcal{D}(K)$ yield that $\overline{a}_0 \le \overline{a}_1$. Using Theorem 3.3.2, we have $\overline{a}_1 = K\overline{a}_0 \le K\overline{a}_1 = \overline{a}_2$, i.e. $\overline{a}_1 \le \overline{a}_2$. The same argument gives $\overline{a}_2 = K\overline{a}_1 \le K\overline{a}_2 = \overline{a}_3$. Continue this process, we can deduce $\overline{a}_0 \le \overline{a}_1 \le \overline{a}_2 \le \dots$, which means $\{\overline{a}_n : n \in \mathbb{N}\}$ is increasing. Since the results that $\overline{a}_0$ is the lower bound of $\mathcal{D}(K)$ and $a(t) \in \mathcal{D}(K)$, it holds $\overline{a}_0 \le a$. Applying Theorem 3.3.2 to this inequality, we obtain $\overline{a}_1 = K\overline{a}_0 \le Ka = a$, i.e. $\overline{a}_1 \le a$. This argument generates $\overline{a}_n \le a,\ n \in \mathbb{N}$, which means $a(t)$ is an upper bound of $\{\overline{a}_n : n \in \mathbb{N}\}$.

We have proved $\{\overline{a}_n : n \in \mathbb{N}\}$ is an increasing sequence in $\mathcal{D}(K)$ with an upper bound $a(t)$, which leads to $\{\overline{a}_n : n \in \mathbb{N}\}$ is convergent in $\mathcal{D}(K)$ and the limit is smaller than $a(t)$. Denote the limit of $\{\overline{a}_n : n \in \mathbb{N}\}$ by $\overline{a}$. We have $\overline{a} \in \mathcal{D}(K)$, $\overline{a} \le a$ and $\overline{a}$ is a fixed point of K in $\mathcal{D}(K)$. Hence, Lemma 2.4.3 yields $\overline{a} = a$, which is the desired result. $\square$

Now, we are able to prove the uniqueness of fixed points of K.

Theorem 2.4.3 (Uniqueness)**.** *There is at most one fixed point of* K *in* $\mathcal{D}(K)$.

Proof. Let $a_1, a_2 \in \mathcal{D}(K)$ be both fixed points of K. Lemma 2.4.4 implies that $\overline{a}_n \to a_1$ and $\overline{a}_n \to a_2$, which leads to $a_1 = a_2$ and completes this proof. $\square$

2.4.4 Existence

Assumption 2.4.1 is not sufficient to deduce the existence of the fixed points of K since $\mathcal{D}(K)$ has no upper bound so that an increasing sequence in $\mathcal{D}(K)$ may not be convergent. In this part, we discuss the existence of fixed points, by providing some extra conditions.

Assumption 2.4.4. *Additional assumptions on* u_0*,* F *and* g*:*

(a) $-\mathcal{L}u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$;

(b) $F(x,t) = -\mathcal{L}u_0(x) \cdot f(t)$ *s.t.* $f \in C^\theta[0,T], 0 < \theta < 1$ *and* $f(t) \geq g(t)\left[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\right]^{-1}$ *on* $[0,T]$.

Remark 2.4.4. Assumption 2.4.4 is set up to make sure that $F(x,t) = -\mathcal{L}u_0(x) \cdot f(t) \in C^\theta([0,T]; H^3(\Omega) \cap H_0^1(\Omega))$, so that $F(x,t)$ also satisfies Assumption 2.4.1.

Fix u_0 and f, if the measured data g does not satisfy Assumption 2.4.4 (b), then we can modify u_0 by increasing the value of u_0 in a very small neighborhood of the point x_0 so that the value of $\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)$ becomes larger. Meanwhile, since u_0 is changed in a small domain, the coefficients $\{b_n : n \in \mathbb{N}^+\}$ only vary slightly, so do $u_n(t;a)$ and $u(x,t;a)$. Hence, $\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(x_0,t;a)$ and $g(t)$ will not appear a significant change that can violate Assumption 2.4.4 (b).

Define the subspace $\mathcal{D}(K)'$ of $\mathcal{D}(K)$ as

$$\mathcal{D}(K)' := \Big\{\psi \in C^+[0,T] : g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha\big[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)\big]\Big]^{-1} \leq \psi(t) \leq g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\Big]^{-1},\ t \in [0,T]\Big\}.$$

We have proved the lower bound of $\mathcal{D}(K)'$ is positive in Remark 2.4.3 and clearly the upper bound of $\mathcal{D}(K)'$ is larger than the lower bound. Consequently, $\mathcal{D}(K)'$ is well-defined.

The next lemma concerns the range of K with domain $\mathcal{D}(K)'$.

Lemma 2.4.5. *With Assumptions 2.4.1 and 2.4.4, K maps $\mathcal{D}(K)'$ into $\mathcal{D}(K)'$.*

Proof. Given $\psi \in \mathcal{D}(K)'$, we have proved $K\psi \in C^+[0,T]$ and

$$K\psi(t) \geq g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1},\ t \in [0,T]$$

in the proof of Lemma 2.4.2, so that it is sufficient to show $K\psi \leq g(t)\big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\big]^{-1}$ on $[0,T]$.

For each $n \in \mathbb{N}^+$, let $w_n(t;\psi) = u_n(t;\psi) - b_n$, (2.5) yields the following ODE by direct calculation

$${}^C D_t^\alpha w_n(t;\psi) + \lambda_n \psi(t) w_n(t;\psi) = \lambda_n b_n (f(t) - \psi(t)) \geq 0,\ w_n(0,\psi) = 0,$$

where $\lambda_n b_n(f(t) - \psi(t)) \geq 0$ follows from the fact $\psi(t) \leq g(t)\big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\big]^{-1}$ and Assumption 2.4.4 (b). Applying Corollary 2.3.1 to the above ODE gives $w_n(t;\psi) \geq 0$, i.e. $u_n(t;\psi) \geq b_n \geq 0$ on $[0,T]$. Hence,

$$K\psi(t) = \frac{g(t)}{\sum_{n=1}^\infty u_n(t;\psi)\frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} \leq \frac{g(t)}{\sum_{n=1}^\infty b_n \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)} = g(t)\big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\big]^{-1}$$

and this proof is complete. □

The existence conclusion is derived from Lemmas 2.4.4 and 2.4.5.

Theorem 2.4.5 (Existence)**.** *Suppose Assumptions 2.4.1 and 2.4.4 be valid, then there exists a fixed point of K in $\mathcal{D}(K)'$.*

Proof. Lemma 2.4.4 yields the sequence $\{\overline{a}_n : n \in \mathbb{N}\}$ is increasing, while Lemma 2.4.5 gives $\{\overline{a}_n : n \in \mathbb{N}\} \subset \mathcal{D}(K)'$. Then $\{\overline{a}_n : n \in \mathbb{N}\}$ is an increasing sequence with an upper bound $g(t)\left[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0)\right]^{-1}$, which implies the convergence of $\{\overline{a}_n : n \in \mathbb{N}\}$. Denote the limit by $\overline{a}$, clearly $\overline{a}$ is a fixed point of K. Also, the closedness of $\mathcal{D}(K)'$ yields that $\overline{a} \in \mathcal{D}(K)'$. Therefore, $\overline{a}$ is a fixed point of K in $\mathcal{D}(K)'$, which confirms the existence. □

2.4.5 Main theorem for the inverse problem and reconstruction algorithm

Lemma 2.4.4, Theorems 2.4.3 and 2.4.5 allow us to deduce the main theorem for this inverse problem.

Theorem 2.4.6 (Main theorem for the inverse problem)**.** *Suppose Assumption 2.4.1 holds.*

(a) If there exists a fixed point of K in $\mathcal{D}(K)$, then it is unique and coincides with the limit of $\{\overline{a}_n : n \in \mathbb{N}\}$;

(b) If Assumption 2.4.4 is also valid, then there exists a unique fixed point of K in $\mathcal{D}(K)'$, which is the limit of $\{\overline{a}_n : n \in \mathbb{N}\}$.

The following reconstruction algorithm for $a(t)$ is based on Theorem 2.4.6.

Table 2.1: Numerical Algorithm

Iteration algorithm to recover the coefficient $a(t)$
1: Set up the right-hand side function $F(x,t)$ and the initial condition $u_0(x)$, then measure the output flux data $g(t)$. F, u_0 and g should satisfy Assumption 2.4.1;
2: Set the initial guess as $\overline{a}_0(t) = g(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1}$;
3: **for k = 1,...,N do**
4: Using the L1 time-stepping [41] to compute $u(x,t;\overline{a}_{k-1})$, which is the weak solution of FDE (2.1) with coefficient function $\overline{a}_{k-1}$;
5: Update the coefficient $\overline{a}_{k-1}$ by $\overline{a}_k = K\overline{a}_{k-1}$;
6: Check stopping criterion $\|\overline{a}_k - \overline{a}_{k-1}\|_{L^2[0,T]} \le \epsilon_0$ for some $\epsilon_0 > 0$;
7: **end for**
8: **output** the approximate coefficient function $\overline{a}_N$.

2.5 Numerical results for inverse problem

2.5.1 L1 time-stepping of ${}^C D_t^\alpha$

The fourth step of Table 2.1 includes solving the direct problem of FDE (2.1) numerically. To this end, we choose L1 time stepping [41, 42] to discretize the term ${}^C D_t^\alpha u(x,t)$:

$$\begin{aligned}
{}^C D_t^\alpha u(x,t_N) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{\partial u(x,s)}{\partial s}(t_N - s)^{-\alpha}\, ds \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{N-1} \frac{u(x,t_{j+1}) - u(x,t_j)}{\tau} \int_{t_j}^{t_{j+1}} (t_N - s)^{-\alpha} ds \\
&= \sum_{j=0}^{N-1} b_j \frac{u(x,t_{N-j}) - u(x,t_{N-j-1})}{\tau^\alpha} \\
&= \tau^{-\alpha}[b_0 u(x,t_N) - b_{N-1}u(x,t_0) + \sum_{j=1}^{N-1}(b_j - b_{j-1})u(x,t_{N-j})],
\end{aligned}$$

where

$$b_j = ((j+1)^{1-\alpha} - j^{1-\alpha})/\Gamma(2-\alpha),\ j = 0,1,\ldots,N-1.$$

2.5.2 Numerical results for noise free data

In this part, we set $\Omega = (0,1),\ x_0 = 0,\ T = 1,\ \mathcal{L}u = u_{xx}$, pick $u_0(x) = -\sin \pi x$, $F(x,t) = -(t+1)\sin \pi x$ and consider the following two coefficients:

(a1) smooth coefficient: $a(t) = \sin 5\pi t + 1.3$;

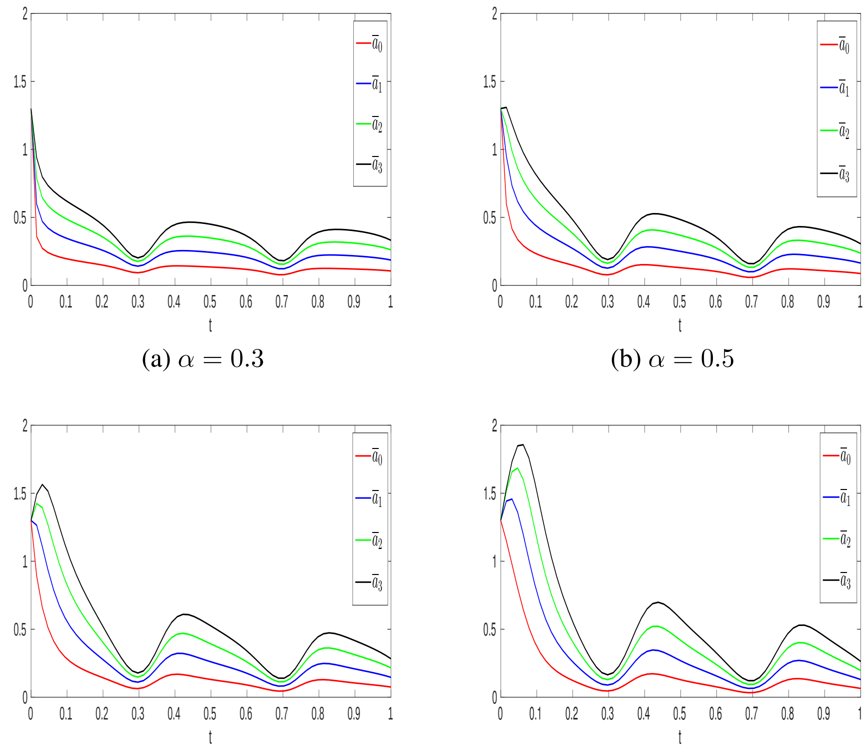
(a2) nonsmooth coefficient ("smile" function):

$$a(t) = [0.8 \sin 3\pi t + 1.5]\chi_{[0,1/3]} + [-0.5 \sin(3\pi t - \pi) + 0.6]\chi_{(1/3,2/3)} \\ + [0.8 \sin(3\pi t - 2\pi) + 1.5]\chi_{[2/3,1]}.$$

In experiment (a1), the exact coefficient we pick is a smooth function. Figure 2.1 shows the initial guess and the first three iterations, while Figure 2.2 presents the exact and approximate coefficients. From these two figures, we observe that $\{\overline{a}_n : n \in \mathbb{N}\}$ converges to $a(t)$ monotonically, which illustrates Theorems 3.3.2 and 2.4.6. Moreover, the L^2 error of the approximation in Figure 2.2 is $\|a - \overline{a}_N\|_{L^2[0,T]} = 1.04 \times 10^{-6}$, which implies us the L^2 error of this approximation may be bounded by the stopping criterion number ϵ_0. This guess is confirmed by Figure 2.4 and can be expressed as

$$\|a - \overline{a}_N\|_{L^2[0,T]} = O(\epsilon_0).$$

Several attempts of experiment (a1) for different $\alpha \in (0,1)$ are taken to find the dependence of the convergence rate of Table 2.1 on the fractional order α, which is shown in Figure 2.3. This figure shows the amounts of iterations required, i.e. N, corresponding to different α, which imply that restricted $\alpha \in (0,1)$, the larger α is, the faster the convergence rate of Table 2.1 is. This phenomenon is explained in [43] by a property of the Mittag-Leffler function; for $\alpha \in (0,1)$, the larger α is, the faster the decay rate of $E_{\alpha,1}(-z)$ is as $z \to \infty$.



(a) $\alpha = 0.3$ (b) $\alpha = 0.5$

(c) $\alpha = 0.7$ (d) $\alpha = 0.9$

Figure 2.1: Experiment (a1): the initial guess and first three iterations

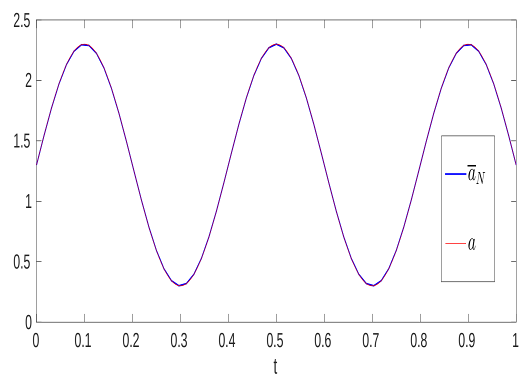


Figure 2.2: Experiment (a1): the exact and approximate coefficients for $\alpha = 0.9$ and $\epsilon_0 = 10^{-6}$

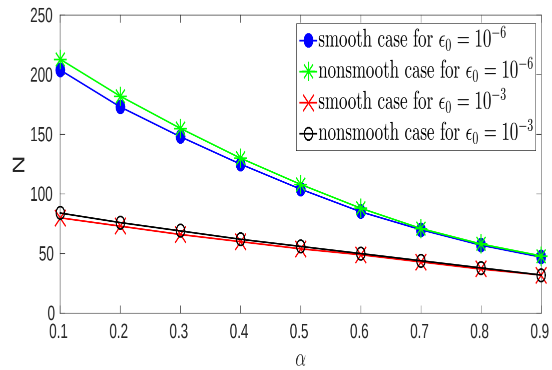


Figure 2.3: The amounts of iterations N for different α

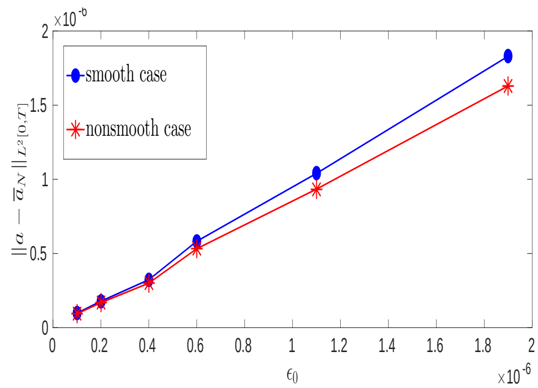
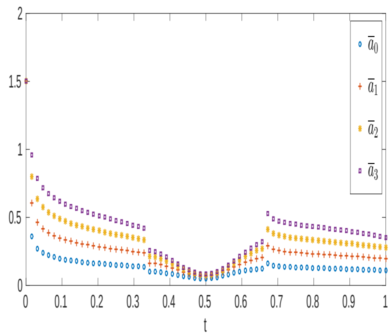


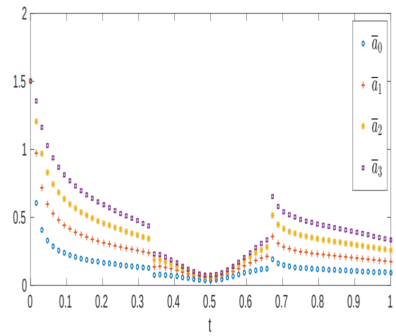
Figure 2.4: $\|a - \overline{a}_N\|_{L^2[0,T]}$ for different ϵ_0 under $\alpha = 0.9$

The definition of $\mathcal{D}(K)$ restricts the coefficient $a(t)$ in the space $C^+[0,T]$, however, the results of experiment (a2) indicate that Table 2.1 still works for nonsmooth $a(t)$, which means the numerical restriction on $a(t)$ can possibly be extended from $a(t) \in C^+[0,T]$ to $a(t) \in L^\infty[0,T]$. For discontinuous $a(t)$, Figures 2.5 and 2.6 explain that Theorems 3.3.2 and 2.4.6 still hold, while Figures 2.3 and 2.4 illustrate the similar conclusions as the larger α is, the faster the convergence rate of Table 2.1 is, and

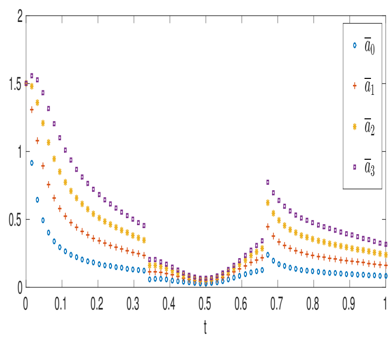
$$\|a - \overline{a}_N\|_{L^2[0,T]} = O(\epsilon_0).$$



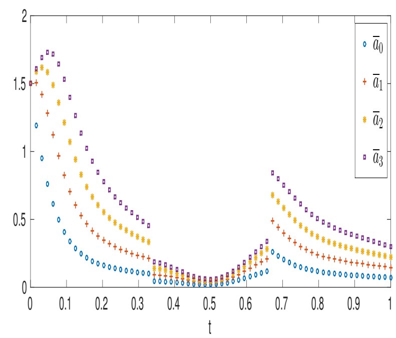
(a) $\alpha = 0.3$



(b) $\alpha = 0.5$



(c) $\alpha = 0.7$



(d) $\alpha = 0.9$

Figure 2.5: Experiment (a2): the initial guess and first three iterations

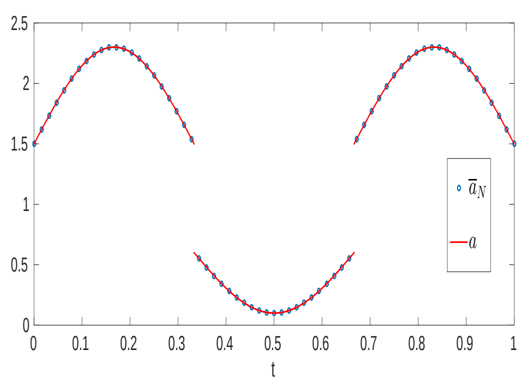


Figure 2.6: Experiment (a2): the exact and approximate coefficients for $\alpha = 0.9$ and $\epsilon_0 = 10^{-6}$

2.5.3 Numerical results for noisy data

In this subsection, we will consider data polluted by noise. Set g be the exact data and denote the noisy data by g_δ with relative noise level δ, i.e. $\|(g - g_\delta)/g\|_{L^\infty[0,T]} \le \delta$. Then the perturbed operator K_δ is

$$K_\delta \psi(t) = \frac{g_\delta(t)}{\sum_{n=1}^{\infty} u_n(t;\psi) \frac{\partial \phi_n}{\partial \overrightarrow{\mathbf{n}}}(x_0)}$$

with domain

$$\mathcal{D}(K_\delta) := \{\psi \in C^+[0,T] : g_\delta(t)\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1} \le \psi(t),\ t \in [0,T]\}.$$

Also, the sequence $\{\overline{a}_{\delta,n} : n \in \mathbb{N}\}$ can be obtained from the iteration

$$\overline{a}_{\delta,0} = g_\delta\Big[\frac{\partial u_0}{\partial \overrightarrow{\mathbf{n}}}(x_0) + I_t^\alpha[\frac{\partial F}{\partial \overrightarrow{\mathbf{n}}}(x_0,t)]\Big]^{-1},\ \overline{a}_{\delta,n+1} = K_\delta \overline{a}_{\delta,n},\ n \in \mathbb{N}.$$

Since δ is a small positive number and g is a strictly positive function, we can assume g_δ is still positive, which means Theorem 2.4.6 still holds for K_δ. Hence, if there exists a fixed point $a_\delta \in \mathcal{D}(K_\delta)$, the sequence $\{\overline{a}_{\delta,n} : n \in \mathbb{N}\}$ will converge to a_δ monotonically and we denote the limit by $\overline{a}_\delta$. Table 2.1 is still able to be used to recover $\overline{a}_\delta$ after a slightly modification$-$replacing g and K by g_δ and K_δ, respectively.

We take the experiments (a1) and (a2) with noise level $\delta > 0$. Figures 2.7 and 2.8 present the exact and approximate coefficients under $\delta = 3\%$ for experiments (a1) and (a2) respectively. From figures 2.7 and 2.8, we observe that the smaller $|a(t)|$ is, the better the approximation is. This can be explained by δ means the relatively noise level, i.e. we pick $g_\delta = (1+\zeta\delta)g$ in the codes, where ζ follows a uniform distribution on $[-1,1]$. Figure

2.9 illustrates that

$$\|a - \overline{a}_{\delta,N}\|_{L^2[0,T]}/\|a\|_{L^2[0,T]} = O(\delta),$$

showing the domination of the noise level δ in relatively L^2 error with the reason that $\epsilon_0 \ll \delta$.

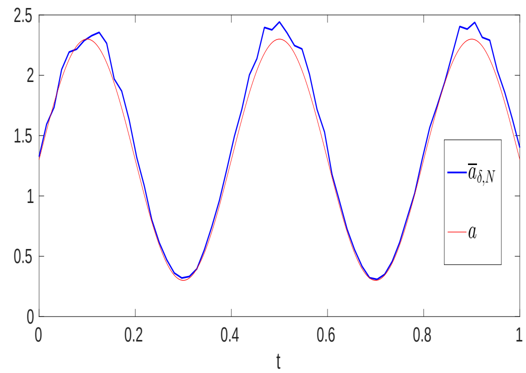


Figure 2.7: Experiment (a1): the exact and approximate coefficients with $\alpha = 0.9$, $\epsilon_0 = 10^{-6}$ and $\delta = 3\%$

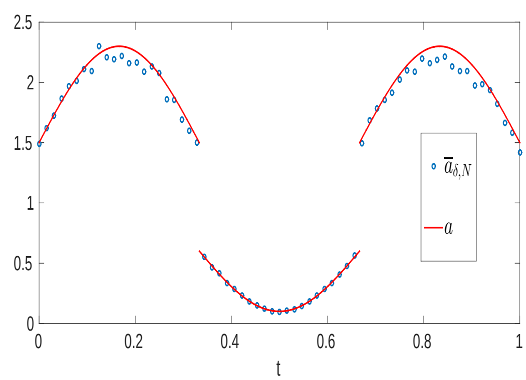


Figure 2.8: Experiment (a2): the exact and approximate coefficients with $\alpha = 0.9$, $\epsilon_0 = 10^{-6}$ and $\delta = 3\%$

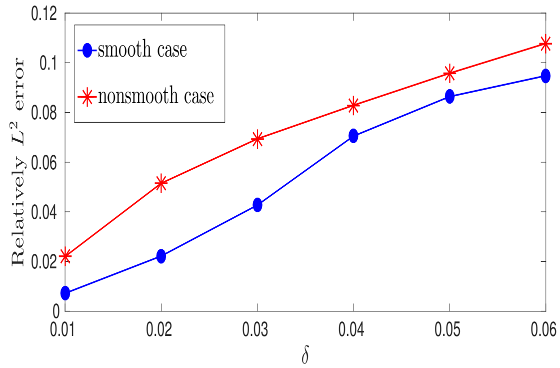
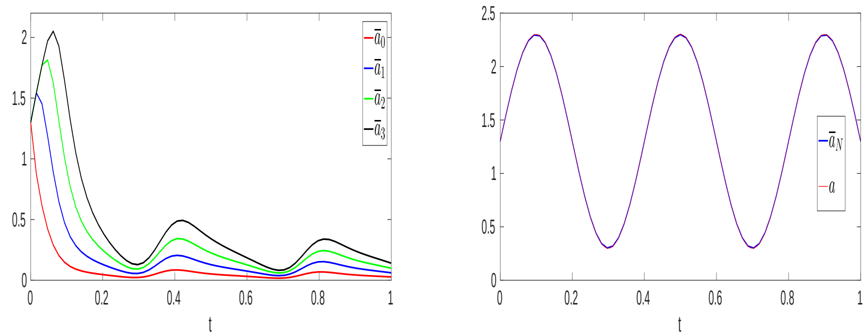


Figure 2.9: $\|a - \overline{a}_{\delta,N}\|_{L^2[0,T]}/\|a\|_{L^2[0,T]}$ for different δ under $\alpha = 0.9$ and $\epsilon_0 = 10^{-6}$

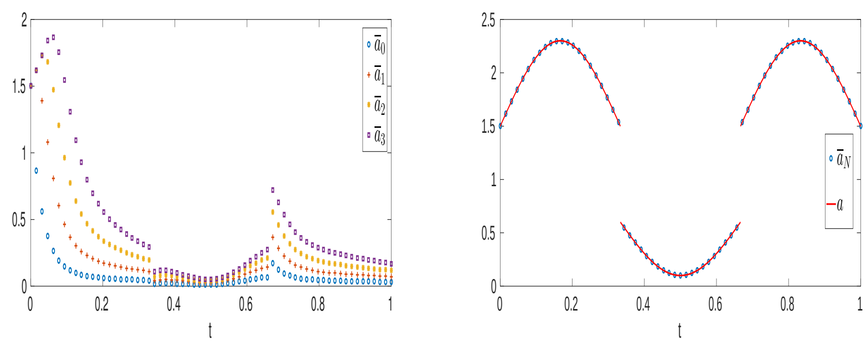
2.5.4 Numerical results in two dimensional case

In this part, the numerical experiments on a two dimensional domain will be considered. We set $\alpha = 0.9,\ \epsilon_0 = 10^{-6},\ \Omega = (0,1)^2,\ x_0 = (0, 1/2),\ T = 1,\ \mathcal{L}u = \triangle u$, choose $u_0(x,y) = -\sin[\pi xy(1-x)(1-y)]$, $F(x,y) = -(t+1)\cdot\sin[\pi xy(1-x)(1-y)]$, and consider experiments (a1) and (a2). Figures 2.10 and 2.11 confirm the theoretical conclusions in section 4.



(a) Initial guess and first three iterations (b) Exact and approximate coefficients

Figure 2.10: Experiment (a1) in two dimensional case



(a) Initial guess and first three iterations (b) Exact and approximate coefficients

Figure 2.11: Experiment (a2) in two dimensional case

3. THE FRACTIONAL POTENTIAL PROBLEM

3.1 Introduction

Here, we consider an inverse problem for the following one-dimensional time-fractional diffusion equation:

$$\begin{cases} {}^C D_t^\alpha u(x,t) - u_{xx}(x,t) + q(x)u(x,t) = f(x), & \text{in } D \times (0,T], \\ -u_x(0,t) + Hu(0,t) = u_x(L,t) + Hu(L,t) = 0, & T \geq t > 0, \\ u(x,0) = 0, & \text{in } D, \end{cases} \tag{3.1}$$

where $D = (0, L)$, $H, T > 0$ are fixed constants and f is a given source term which depends only on x. The solution to system (3.1) will be denoted by $u(x, t; q)$ in order to indicate its dependence on the potential $q \geq 0$ belonging to $L^\infty(D)$.

In this work, we are interested in the inverse problem of recovering the potential $q(x)$ in the model (3.1) from the final data

$$u(x,T) = g(x) \quad \text{for all } x \in D.$$

Physically, it represents a spatially dependent source/sink term in a bar, and hence the inverse problem arises in some physical applications. For the special case $\alpha = 1$, which corresponds to the standard parabolic equation, the inverse problem has been extensively studied [44, 45]. Specifically, Isakov [45] proved the nearly well-posedness of this determination in the Hardamard sense in some suitable Hölder spaces, say $q \in C^\lambda(\bar{D})$ with $\lambda \in (0, 1)$, using the strong maximum principle, under assumptions that $g \in C^{2+\lambda}(\bar{D})$

*Zhidong Zhang, Zhi Zhou, Recovering the potential term in a fractional diffusion equation, IMA Journal of Applied Mathematics, 2017, 82, 3, 579-600, by permission of Oxford University Press.

satisfies boundary conditions and $-g''(x)+u_t(x,T;0)\leq 0$. Besides, using the smoothing properties of the heat equation, the parabolic maximum principle and the implicit function theorem, Choulli and Yamamoto [44] showed that the problem is locally well-posed in L^2-sense, provided that u_0 is in $\dot{H}^s(D)$ with $s\in(7/2,4)$ and $q\in L^2(D)$ is assumed to be a priori supported in some suitable subset. However, the inverse problem in the fractional case has not been studied.

The goal of this chapter is to establish the unique recovery, and to design a stable and efficient numerical algorithm, via a fixed point reformulation of the inverse problems in suitable Banach spaces. To the best of our knowledge, it represents the first theoretical work on the potential inverse problem for equation (3.1). Our main contributions are as follows. First, we develop a reconstruction operator by

$$(K\psi)(x)=\frac{g''(x)-{}^C D_t^\alpha u(x,T;\psi)+f(x)}{g(x)},$$

in the admissible set $\mathcal{A}$, and then establish the unique recovery using the contractivity of the operator K for large T

$$\|Kq_1-Kq_2\|_{L^2(D)}\leq CT^{-\alpha}\|q_1-q_2\|_{L^2(D)}.$$

Second, we develop a monotone iterative reconstruction algorithm, using the monotonicity of the operator K, i.e., $Kq_1\leq Kq_2$ for $q_1,q_2\in\mathcal{A}$ and $q_1\leq q_2$. With an initial guess $q_0=(g''+f)g^{-1}\geq q$ (Corollary 3.2.1), the algorithm provides a sequence $\{K^n q_0\}$ such that

$$\|K^n q_0-q\|_{L^2(D)}\to 0\quad\text{as }\ n\to\infty$$

and $q_0\geq Kq_0\geq K^2q_0\geq ...\geq q$. Last, in the case of noisy data with level δ, we propose a regularization method using mollification and show the following error estimate for the approximate potential $q_{\epsilon,\delta}$

$$\|q - q_{\epsilon,\delta}\|_{L^2(D)} \leq C(\epsilon + \delta + \delta\epsilon^{-2}),$$

where ϵ denotes the regularization parameter. In particular, with the choice $\epsilon = \delta^{1/3}$, it gives a convergence rate $O(\delta^{1/3})$.

The rest of the chapter is organized as follows. In Section 3.2 we collect preliminary results on Mittag-Leffler function and the fractional diffusion model (3.1). In Section 3.3, we introduce a fixed point iteration and show its monotonicity and contractivity, which yields the unique determination. Then a practical algorithm is developed for noisy data based on mollification, and error estimates are given in Section 3.4. Finally in Section 3.5, numerical results for several examples are provided to illustrate the convergence theory. Throughout, the notation C, with or without a subscript, denotes a generic constant which may differ at different occurrences, but it is always independent of time T and noise level δ.

3.2 Preliminaries

In this section, we collect a useful fact on the Mittag-Leffler function and regularity results for problem (3.1).

3.2.1 An estimate on the Mittag-Leffler function

Here we state an estimate on the Mittag-Leffler function.

Lemma 3.2.1. *Let $\lambda_1, \lambda_2, T > 0$ and $\alpha \in (0,1)$, then it holds that*

$$|E_{\alpha,1}(-\lambda_1 T^\alpha) - E_{\alpha,1}(-\lambda_2 T^\alpha)| \leq C|\lambda_1 - \lambda_2|\lambda_*^{-2}T^{-\alpha},$$

where C only depends on α and λ_* denotes some constant between λ_1 and λ_2.

Proof. Applying the mean value theorem, one may find $\lambda_* \in [\lambda_1, \lambda_2]$ such that

$$|E_{\alpha,1}(-\lambda_1 T^\alpha) - E_{\alpha,1}(-\lambda_2 T^\alpha)| = |\lambda_1 T^\alpha - \lambda_2 T^\alpha||E'_{\alpha,1}(-\lambda_* T^\alpha)|.$$

Then a direct computation and Lemma 2.2.1 give

$$|E'_{\alpha,1}(-z)| = \left|\frac{1}{\alpha}\frac{1}{-z}\sum_{k=1}^{\infty}\frac{(-z)^{(k)}}{\Gamma(\alpha k)}\right| = \left|\frac{1}{\alpha z}E_{\alpha,0}(-z)\right| \le C|z|^{-2},$$

which completes the proof. □

3.2.2 Smoothing properties of the direct problem

Now we describe smoothing properties of the model (3.1). To this end, we recall the following classical Sturm-Liouville problem

$$L(q)u := -u''(x) + q(x)u(x), \quad -u'(0) + Hu(0) = u'(L) + Hu(L) = 0 \tag{3.2}$$

with $q \ge 0$ belonging to $L^\infty(D)$ and $H > 0$. Clearly, $L(q)$ is self-adjoint and positive definite. Here we denote $\{\lambda_j(q)\}_{j=1}^\infty$ and $\{\phi_j(x;q)\}_{j=1}^\infty$ to be the eigenvalues and the $L^2(D)$-orthonormal eigenfunctions of the operator $L(q)$ on D with the homogeneous Robin boundary condition. Then $\{\phi_j(x;q)\}_{j=1}^\infty$ forms an orthonormal basis in $L^2(D)$. Further we have the following useful estimates in [46].

Lemma 3.2.2. *Given nonnegative $q_i \in L^\infty(D)$, the eigenvalues $\{\lambda_n(q_i)\}$ and eigenfunctions $\{\phi_n(x;q_i)\}$ to the Sturm-Liouville problem* (3.2)*, $i = 1, 2$, satisfy for all $n \in \mathbb{N}$*

$$|\lambda_n(q_1) - \lambda_n(q_2)| \le C\|q_1 - q_2\|_{L^2(D)},$$
$$\|\phi_n(x;q_1) - \phi_n(x;q_2)\|_{L^2(D)} \le Cn^{-1}\|q_1 - q_2\|_{L^2(D)},$$

where the constant C only depends on the domain D.

Using the spectral decomposition, we define a Hilbert space $\dot{H}^s(D)$ induced by the norm:

$$\|v\|^2_{\dot{H}^s(D)} = \sum_{j=1}^{\infty} \lambda_j(q)^s (v(\cdot), \phi_j(\cdot\,;q))^2, \quad \text{for } s \geq 0.$$

To establish smoothing properties, we represent the solution of problem (3.1) using the eigenpairs $\{(\lambda_j(q), \phi_j(x;q))\}$ and introduce the operator $\bar{E}_q(t)$:

$$\bar{E}_q(t)\chi = \sum_{j=1}^{\infty} \frac{1}{\lambda_j(q)} \left(1 - E_{\alpha,1}(-\lambda_j(q)t^\alpha)\right) (\chi(\cdot), \phi_j(\cdot\,;q))\phi_j(x;q). \tag{3.3}$$

The operator $\bar{E}_q(t)$ is used to represent the solution $u(x,t;q)$ of (3.1), following from separation of variables [9]:

$$u(x,t;q) = \bar{E}_q(t)f, \tag{3.4}$$

where f is independent on time. It was shown in [9, Theorem 2.2] that for the source $f(x,t) \in L^2((0,T);L^2(D))$ and the initial data $u(0) \in L^2(D)$, there exists a unique solution in $L^2((0,T);H^2(D))$. The next result gives the solution representation and related regularity results, which are essentially established in [9, Theorem 2.1], and slightly extended in [47, 48].

Theorem 3.2.1. *Let $f \in L^2(D)$, then there exists a unique solution $u(x,t) \in L^2((0,T);H^2(D))$ for the fractional diffusion equation* (3.1) *which can be represented by* (3.4)*. Further, the solution satisfies the following smoothing property*

$$\|u\|_{C([0,T];\dot{H}^{p+2-\epsilon}(D))} + \|{}^C D_t^\alpha u\|_{C([0,T];\dot{H}^{p-\epsilon}(D))} \leq C\epsilon^{-1}T^{\epsilon\alpha/2}\|f\|_{\dot{H}^p(D)},$$

for any small $\epsilon > 0$.

3.2.3 Assumptions and general settings

To reconstruct the potential term q in (3.1) from the final measurement $u(T) = g$, we need to make suitable assumptions. To this end, we first recall a nonnegativity preservation property of problem (3.1), which follows directly from the weak maximum principle [8, Theorem 2.2].

Lemma 3.2.3. *Let $u(0) \in \dot{H}^2(D)$, $f \in \dot{H}^\epsilon(D)$ with any $\epsilon > 0$, be nonnegative. Then the solution of (3.1) satisfies that $u \geq 0$ on $\bar{D} \times [0, T]$.*

Then the following two results are direct corollaries of Lemma 3.2.3.

Corollary 3.2.1. *Assume that $f \in \dot{H}^2(D)$ is positive. Then we have ${}^C D_t^\alpha u(x, 0) > 0$ and ${}^C D_t^\alpha u(x, t) \geq 0$ on $\bar{D} \times [0, T]$.*

Proof. By Theorem 3.2.1, the solution to (3.1) satisfies $u \in C([0, T]; \dot{H}^2(D))$ and ${}^C D_t^\alpha u \in C([0, T]; L^2(D))$. Hence letting t approach zero yields $({}^C D_t^\alpha u(x, t; q))_{t=0} = f > 0$. Then taking the α-th derivative of (3.1) we obtain

$$\begin{cases} {}^C D_t^\alpha ({}^C D_t^\alpha u(x, t; q)) - ({}^C D_t^\alpha u(x, t; q))_{xx} + q(x)({}^C D_t^\alpha u(x, t; q)) = 0; \\ -({}^C D_t^\alpha u(0, t; q))_x + H({}^C D_t^\alpha u(0, t; q)) = ({}^C D_t^\alpha u(L, t; q))_x + H({}^C D_t^\alpha u(L, t; q)) = 0; \\ ({}^C D_t^\alpha u(x, t; q))_{t=0} = f(x) > 0. \end{cases}$$

Then the desired nonnegativity result follows from Lemma 3.2.3. □

Corollary 3.2.2. *Let $f \in \dot{H}^2(D)$ be positive. Then we have $u(x, T; q) > 0$ and $f(x) + u_{xx}(x, T; q) \geq 0$ for all $x \in \bar{D}$.*

Proof. First recall the Riemann fractional integral I_t^α defined by

$$I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} u(\tau) \mathrm{d}\tau.$$

Recalling the initial condition in (3.1), a direct calculation gives the following identity

$$(I_t^\alpha \circ^C D_t^\alpha)u(T) = (I_t^\alpha I_t^{1-\alpha})u'(T) = I_t^1 u'(T) = u(T) - u(0) = u(T).$$

Since ${}^C D_t^\alpha u(x,t;q) \in C([0,T]; L^2(D))$, ${}^C D_t^\alpha u(x,t;q) \geq 0$ on $\bar{D}\times[0,T]$, and ${}^C D_t^\alpha u(x,0;q) > 0$. These facts together with the definition of I_t^α give $u(x,T;q) > 0$. Further, by Lemma 3.2.3 and Corollary 3.2.1, we deduce

$$f(x) + u_{xx}(x,T;q) = {}^C D_t^\alpha u(x,T;q) + q(x)u(x,T;q) \geq 0,$$

which completes the proof. □

Now we introduce the standing assumptions on the model (3.1).

Assumption 3.2.2. *Let the data g and f satisfy the following assumptions*

(a) $g > 0$ is twice continuously differentiable on $\bar{D}$;

(b) $f > 0$ and $g''(x) + f(x) \geq 0$ for all $x \in \bar{D}$.

The smoothness requirement of g in Assumption 3.2.2 (a) follows from the regularity pickup stated in Theorem 3.2.1, while the positivity preservation properties in (a) and (b) are ensured by Corollary 3.2.2.

Last, we define the following operator K

$$(K\psi)(x) = \frac{g''(x) - {}^C D_t^\alpha u(x,T;\psi) + f(x)}{g(x)}, \quad \text{for } \psi \in \mathcal{A}, \tag{3.5}$$

where $u(x,T;\psi)$ is the solution of (3.1) with the potential ψ at $t = T$, and the admissible set $\mathcal{A}$ is defined by

$$\mathcal{A} = \{\psi \in L^\infty(D) \text{ such that } 0 \leq \psi \leq (g'' + f)/g\}.$$

This operator will play a crucial role below. Corollary 3.2.2 and Assumption 3.2.2 ensure that $g > 0$ and the operator K is well-defined. Obviously, q is a fixed point of the operator K, and by Corollary 3.2.1, the true potential q belongs to $\mathcal{A}$.

3.3 Recovery of the potential term

As indicated in Section 4.1, in this section, we show that the fixed point of K is unique, provided that Assumption 3.2.2 holds true. Further, the contractivity and monotonicity of the operator enable developing an iterative algorithm to recover the potential term q from the terminal data g.

3.3.1 Uniqueness of the fixed point

For the uniqueness, we need the following lemma, whose proof relies on properties of the Mittag-Leffler function stated in Lemmas 2.2.1 and 3.2.1.

Lemma 3.3.1. *Let q_1, $q_2 \in \mathcal{A}$, and the operator $\bar{E}_q(t)$ be defined in* (3.3). *Then there is a constant $C > 0$ independent of T such that*

$$\|{}^C D_t^\alpha (\bar{E}_{q_1} - \bar{E}_{q_2})(T) f\|_{L^2(D)} \leq C T^{-\alpha} \|q_1 - q_2\|_{L^2(D)}.$$

Proof. By the definition of $\bar{E}_q(t)$ in (3.3), we have for $q \in \mathcal{A}$

$${}^C D_t^\alpha (\bar{E}_q(t) f) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n(q) t^\alpha)(f, \phi_n(x;q))\phi_n(x;q),$$

and the following splitting for $A :=^C D_t^\alpha(\bar{E}_{q_1}(T)f - \bar{E}_{q_2}(T)f)$

$$
\begin{aligned}
A = &\sum_{n=1}^{\infty} \left(E_{\alpha,1}(-\lambda_n(q_1)T^\alpha) - E_{\alpha,1}(-\lambda_n(q_2)T^\alpha)\right)(f, \phi_n(x;q_1))\phi_n(x;q_1) \\
&+ \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n(q_2)T^\alpha)(f, \phi_n(x;q_1) - \phi_n(x;q_2))\phi_n(x;q_1) \\
&+ \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n(q_2)T^\alpha)(f, \phi_n(x;q_2))(\phi_n(x;q_1) - \phi_n(x;q_2)) := I_1 + I_2 + I_3.
\end{aligned}
$$

Then the estimate for I_1 follows from Lemma 3.2.1

$$
\begin{aligned}
\|I_1\|_{L^2(D)}^2 &= \sum_{n=1}^{\infty} \left|E_{\alpha,1}(-\lambda_n(q_1)T^\alpha) - E_{\alpha,1}(-\lambda_n(q_2)T^\alpha)\right|^2 (f, \phi_n(x;q_1))^2 \\
&\leq CT^{-2\alpha} \sum_{n=1}^{\infty} \left|\lambda_n(q_1) - \lambda_n(q_2)\right|^2 (\lambda_n^*)^{-4} (f, \phi_n(x;q_1))^2
\end{aligned}
$$

where λ_n^* is between $\lambda_n(q_1)$ and $\lambda_n(q_2)$. Note the fact that

$$
\lambda_n(q) \geq \lambda_n(0) \geq \lambda_1(0) > 0, \quad \text{for all } \; n \in \mathbb{N}^+, \;\; q \in \mathcal{A}. \tag{3.6}
$$

Since $\lambda_1(0)$ is only dependent on H, one may bound I_1 by

$$
\begin{aligned}
\|I_1\|_{L^2(D)}^2 &\leq CT^{-2\alpha} \|q_1 - q_2\|_{L^2(D)}^2 \sum_{n=1}^{\infty} n^{-2} \lambda_1(0)^{-4} (f, \phi_n(x;q_1))^2 \\
&\leq CT^{-2\alpha} \|q_1 - q_2\|_{L^2(D)}^2.
\end{aligned}
$$

Similarly, the second term I_2 can be bounded by Lemmas 2.2.1 and 3.2.2 and (3.6)

$$\begin{aligned}\|I_2\|^2_{L^2(D)} &= \sum_{n=1}^{\infty} |E_{\alpha,1}(-\lambda_n(q_2)T^\alpha)|^2 (f, \phi_n(x;q_1) - \phi_n(x;q_2))^2 \\ &\leq CT^{-2\alpha} \sum_{n=1}^{\infty} \lambda_n(q_2)^{-2} (f, \phi_n(x;q_1) - \phi_n(x;q_2))^2 \\ &\leq CT^{-2\alpha} \sum_{n=1}^{\infty} \lambda_1(0)^{-2} n^{-2} \|f\|^2_{L^2(D)} \|q_1 - q_2\|^2_{L^2(D)} \leq CT^{-2\alpha} \|q_1 - q_2\|^2_{L^2(D)}.\end{aligned}$$

Finally, the similar argument gives the bound for I_3

$$\begin{aligned}\|I_3\|_{L^2(D)} &= \sum_{n=1}^{\infty} \big|E_{\alpha,1}(-\lambda_n(q_2)T^\alpha)(f, \phi_n(x;q_2))\big| \|\phi_n(x;q_1) - \phi_n(x;q_2)\|_{L^2(D)} \\ &\leq CT^{-\alpha} \|q_1 - q_2\|_{L^2(D)} \sum_{n=1}^{\infty} \lambda_n(q_2)^{-1} n^{-1} \big|(f, \phi_n(x;q_2))\big| \\ &\leq CT^{-\alpha} \|q_1 - q_2\|_{L^2(D)} \left(\sum_{n=1}^{\infty} n^{-2}\right)^{1/2} \left(\sum_{n=1}^{\infty} \big|(f, \phi_n(x;q_2))\big|^2\right)^{1/2} \\ &\leq CT^{-\alpha} \|q_1 - q_2\|_{L^2(D)}.\end{aligned}$$

These three bounds together complete the proof of the lemma. □

Now we can give the uniqueness of the fixed point, which is a direct result of Lemma 3.3.1.

Theorem 3.3.1. *Let Assumption 3.2.2 hold. Then for a sufficiently large T, the operator K defined in* (3.5) *has at most one fixed point in the set $\mathcal{A}$.*

Proof. By the definition of the operator K, we have for any $q_1, q_2 \in \mathcal{A}$

$$\|Kq_1 - Kq_2\|_{L^2(D)} \leq \frac{\|{}^C D_t^\alpha (\bar{E}_{q_1} - \bar{E}_{q_2})(T) f\|_{L^2(D)}}{\min_{x\in[0,1]} |g(x)|}$$

where the operator $\bar{E}_q$ is defined in (3.3). Now Assumption 3.2.2 (a) and Lemma 3.3.1

yield the following contractive property

$$\|Kq_1 - Kq_2\|_{L^2(D)} \leq CT^{-\alpha}\|q_1 - q_2\|_{L^2(D)},$$

where the constant $C > 0$ is independent of T. In particular, it implies the uniqueness of the fixed point of the operator K in the set $\mathcal{A}$. □

3.3.2 Monotonicity of the operator K

The goal of this part is to prove that the operator K is monotone, i.e., $Kq_1 \leq Kq_2$ for q_1, $q_2 \in \mathcal{A}$ such that $q_1 \leq q_2$. This property is crucial to develop an efficient iterative algorithm. Specifically, by the monotonicity and Theorem 3.3.1, if there exists a fixed point $q \in \mathcal{A}$ such that $Kq = q$, then for the initial guess $q_0 = g^{-1}(g'' + f) \geq q$, we have $K^n q_0 \to q$ in sense of L^2-norm and $q_0 \geq Kq_0 \geq K^2 q_0 \geq ... \geq q$. Further, the monotone convergence is highly desirable in practical computations.

Lemma 3.3.2. *Let Assumption 3.2.2 hold and* $u(x,t;q)$ *be the solution of* (3.1) *with some* $q \in \mathcal{A}$*. Then* ${}^C D_t^\alpha u(x,t;q) \geq 0$.

Proof. First, since ${}^C D_t^\alpha u(x,t;q) \in C([0,T]; L^2(D))$, we deduce that for $q \in \mathcal{A}$

$${}^C D_t^\alpha u(x,0;q) ={}^C D_t^\alpha u(x,t;q)\big|_{t\to 0} = f > 0.$$

Now taking the α-th derivative ${}^C D_t^\alpha$ on (3.1) gives

$$\begin{cases} {}^C D_t^\alpha({}^C D_t^\alpha u(x,t;q)) - ({}^C D_t^\alpha u(x,t;q))_{xx} + q(x)({}^C D_t^\alpha u(x,t;q)) = 0; \\ -({}^C D_t^\alpha u(0,t;q))_x + H({}^C D_t^\alpha u(0,t;q)) = ({}^C D_t^\alpha u(L,t;q))_x + H({}^C D_t^\alpha u(L,t;q)) = 0, \end{cases}$$

with the positive initial data ${}^C D_t^\alpha u(x,0;q) > 0$. Then Lemma 3.2.3 yields the desired result. □

The next theorem shows the monotonicity of the operator K, i.e.,

$$Kq_1 \le Kq_2 \quad \text{for all} \quad q_1, q_2 \in \mathcal{A} \ \text{ and } \ q_1 \le q_2.$$

Theorem 3.3.2. *Let Assumption 3.2.2 hold. Then the operator K defined in* (3.5) *is monotone in $\mathcal{A}$.*

Proof. For $q_1, q_2 \in \mathcal{A}$, $q_1 \le q_2$. Let $u(x,t;q_1)$ and $u(x,t;q_2)$ be solutions of (3.1) with potentials q_1 and q_2, respectively. Then $w(x,t) = u(x,t;q_1) - u(x,t;q_2)$ satisfies

$$\begin{cases} {}^C D_t^\alpha w(x,t) - w_{xx}(x,t) + q_1(x) w(x,t) = u(x,t;q_2)(q_2(x) - q_1(x)), \\ -w_x(0,t) + Hw(0,t) = w_x(L,t) + Hw(L,t) = 0, \\ w(x,0) = 0. \end{cases}$$

Now taking the α-th order derivative on the system yields

$$\begin{cases} {}^C D_t^\alpha ({}^C D_t^\alpha w) - ({}^C D_t^\alpha w)_{xx} + q_1 ({}^C D_t^\alpha w) = ({}^C D_t^\alpha u(x,t;q_2))(q_2(x) - q_1(x)), \\ -({}^C D_t^\alpha w(0,t))_x + H({}^C D_t^\alpha w(0,t)) = ({}^C D_t^\alpha w(L,t))_x + H {}^C D_t^\alpha w(L,t) = 0, \\ {}^C D_t^\alpha w(x,0) = 0, \end{cases}$$

where the right hand side $({}^C D_t^\alpha u(x,t;q_2))(q_2(x) - q_1(x)) \ge 0$ by Lemma 3.3.2. Then the maximum principle [8, Theorem 3] yields that

$$ {}^C D_t^\alpha w(x,t) \ge 0 \Rightarrow {}^C D_t^\alpha u(x,t;q_1) \ge {}^C D_t^\alpha u(x,t;q_2) \ \text{ for all } \ x \in D, \ \ t \ge 0. \qquad (3.7)$$

Consequently, we have

$$Kq_1 = \frac{g'' - ^C D_t^\alpha u(T; q_1) + f}{g} \leq \frac{g'' - ^C D_t^\alpha u(T; q_2) + f}{g} = Kq_2,$$

from which the desired monotonicity follows. $\square$

Remark 3.3.1. All the discussion presented here can be easily extended to the case that $u(0) > 0$, provided an additional assumption that $f > \frac{u_0 g'' - u_0'' g}{g - u_0}$.

3.3.3 An iterative reconstruction algorithm

Next we describe an iterative algorithm for finding the potential in the model (3.1). It is based on the observation that the potential q is a fixed point of the operator K. The complete procedure is given in Algorithm 1. The initial guess is chosen by $q_0 = (g'' + f)g^{-1} \geq q$ (by Assumption 3.2.2 and Corollary 3.2.1), which is the upper bound of $\mathcal{A}$, and the first iteration gives $q_1 = Kq_0 \leq q_0$ by Lemma 3.3.2. Hence using Theorems 3.3.1 and 3.3.2, Algorithm 1 generates a decreasing series $\{q_n : n \in \mathbb{N}\}$ such that $q_n = K^n q_0 \to q$ in sense of L^2-norm. At each iteration, the algorithm invokes solving one forward problem with the potential q_{k-1}, which is the dominant computational expense.

Algorithm 1. An iterative algorithm for recovering the potential $q(x)$

1: **input** f and g which satisfy Assumption 3.2.2;
2: Set the initial guess $q_0 = \frac{g''(x)+f(x)}{g(x)}$;
3: **for** $k = 1, ..., N$ **do**
4: Compute $u(x, t; q_{k-1})$, the solution of (3.1) with potential q_{k-1};
5: Update the potential by
$$q_k(x) = (Kq_{k-1})(x) = \frac{g''(x) - ^C D_t^\alpha u(x, T; q_{k-1}) + f(x)}{g(x)};$$
6: Check stopping criterion $\|q_{k-1} - q_k\|_{L^2(D)} \leq \epsilon_0$ for some $\epsilon_0 > 0$;
7: **end for**
8: **output** the approximated potential q_N.

3.4 Regularized reconstruction scheme

We observe that Algorithm 1 suffers from numerical instability for noisy data since it involves a second-order numerical differentiation on the data g. Hence, we propose a strategy to numerically stabilize the scheme. There are two predominant regularization methods: Tikhonov regularization [49, 50, 51, 52] and mollification [53, 54, 55]. Here we shall follow the second route and establish the error estimate rigorously. For Tikhonov method, such an estimate will be more technical due to the involving extra penalty term.

Specifically, let g_δ be the perturbed terminal data. Further, we assume that g, $g_\delta \in C^2(D)$, g'' is Lipschitz continuous on $\bar{D} = [0, L]$ and $\|g(x) - g_\delta(x)\|_{C(\bar{D})} \leq \delta$. Then for any $g \in C^2(\bar{D})$, we may define a smooth extension $\widetilde{g}$ onto the interval $[-\epsilon, L+\epsilon]$ such that $\widetilde{g}, \widetilde{g_\delta} \in C^2[-\epsilon, L+\epsilon]$ and

$$\|\widetilde{g} - \widetilde{g_\delta}\|_{C[-\epsilon, L+\epsilon]} \leq C\|g - g_\delta\|_{C(\bar{D})} \leq C\delta. \tag{3.8}$$

We remark that one possible smooth extension is given by

$$\widetilde{\psi}(x) = \begin{cases} \psi(-x)(x^3 - 3) + \psi(-2x) + 3\psi(0), x \in [-\epsilon, 0]; \\ \psi(x), x \in D; \\ \psi(2-L)((x-L)^3 - 3) + \psi(-2x + 3L) + 3\psi(L), x \in [L, L+\epsilon]. \end{cases}$$

Further, for $\psi \in C[a, b]$, we may define the following mollification [54] by

$$G_\epsilon(\psi)(x) = \int_{x-\epsilon}^{x+\epsilon} \rho_\epsilon(x - \tau)\psi(\tau)\mathrm{d}\tau,$$

where the kernel function ρ_ϵ is defined by

$$\rho_\epsilon(x) = \begin{cases} 16(1-x/\epsilon)^2(1+x/\epsilon)^2/(15\epsilon), & |x| \le \epsilon; \\ 0, & |x| > \epsilon. \end{cases}$$

Note that $\int_{\mathbb{R}} \rho_\epsilon(x) = 1$ and $\rho_\epsilon(x) \ge 0$, as a result, $G_\epsilon(\psi) \in C[a+\epsilon, b-\epsilon]$ satisfies

$$\|G_\epsilon(\psi)\|_{C[a+\epsilon,b-\epsilon]} \le \|\psi\|_{C[a,b]}. \tag{3.9}$$

Now the regularized terminal data and the iteration operator are defined by

$$g_{\delta,\epsilon}(x) = G_\epsilon(\widetilde{g_\delta})(x) \quad \text{and} \quad K_{\delta,\epsilon} q = \frac{g''_{\delta,\epsilon}(x) - {}^C D_t^\alpha u(x,T;q) + f(x)}{g_{\delta,\epsilon}(x)}, \tag{3.10}$$

respectively. Then $g_{\delta,\epsilon}$ is well-defined on $\bar{D}$, and for small ϵ, δ we may suppose that Assumption 3.2.2 holds true. Next we shall show that the iteration $K^n_{\delta,\epsilon} q^0_{\delta,\epsilon}$ for the initial guess

$$q^0_{\delta,\epsilon} = (g''_{\delta,\epsilon} + f) g^{-1}_{\delta,\epsilon}$$

approaches an approximate data $q_{\delta,\epsilon}$ which is close to the exact potential q. To this end, we present some properties of such an extension and mollification.

Lemma 3.4.1. *Let g be Lipschitz continuous on $\bar{D}$. Then there holds*

$$\|g - G_\epsilon \widetilde{g}\|_{L^\infty(D)} \le C\epsilon,$$

where the constant C only depends on the Lipschitz constant of g.

Proof. For all $x \in D$, by the definition of the extension and Lipschitz continuity

$$\begin{aligned} |g(x) - G_\epsilon \widetilde{g}(x)| &= \left| \int_{x-\epsilon}^{x+\epsilon} \rho_\epsilon(x-y)(g(x) - \widetilde{g}(y)) \mathrm{d}y \right| \\ &\leq C \int_{x-\epsilon}^{x+\epsilon} \rho_\epsilon(x-y)|x-y| \, dy = C(16\epsilon/45) \end{aligned}$$

and the proof is complete. □

Lemma 3.4.2. *Assume that* $g(x), g_\delta(x) \in C^2(D)$, $g''(x)$ *is Lipschitz continuous on* $\bar{D}$ *and* $\|g(x) - g_\delta(x)\|_{C(\bar{D})} \leq \delta$. *Then there holds*

$$\|g'' - g''_{\delta,\epsilon}\|_{L^\infty(D)} \leq C \left(\epsilon + \delta \epsilon^{-2} \right),$$

where $g_{\delta,\epsilon} = G_\epsilon(G_\epsilon(\widetilde{g}_\delta))$ *and the constant* C *is independent of* ϵ *and* δ.

Proof. It is easy to see that $g''_{\delta,\epsilon}(x) = (G_\epsilon \widetilde{g}_\delta)'' = G_\epsilon(\widetilde{g}_\delta{}'')$. Then

$$\|g'' - g''_{\delta,\epsilon}\|_{L^\infty(D)} \leq \|g'' - G_\epsilon \widetilde{g}''\|_{L^\infty(D)} + \|G_\epsilon \widetilde{g}'' - G_\epsilon \widetilde{g}_\delta{}''\|_{L^\infty(D)}. \tag{3.11}$$

The first term can be bounded using Lemma 3.4.1

$$\|g'' - G_\epsilon \widetilde{g}''\|_{L^\infty(D)} \leq C\epsilon. \tag{3.12}$$

Now it suffices to consider the second term. In fact, we have for $x \in \bar{D}$

$$\begin{aligned} I(x) :&= (G_\epsilon \widetilde{g}'')(x) - (G_\epsilon \widetilde{g}_\delta{}'')(x) = (G_\epsilon(\widetilde{g}'' - \widetilde{g}_\delta{}''))(x) \\ &= \int_{x-\epsilon}^{x+\epsilon} \rho_\epsilon(x-\tau)(\widetilde{g}''(\tau) - \widetilde{g}_\delta{}''(\tau)) \, d\tau. \end{aligned}$$

Then integration by parts yields that

$$
\begin{aligned}
I(x) &= \rho_\epsilon(-\epsilon)(\widetilde{g}'(x+\epsilon) - \widetilde{g_\delta}'(x+\epsilon)) - \rho_\epsilon(\epsilon)(\widetilde{g}'(x-\epsilon) - \widetilde{g_\delta}'(x-\epsilon)) \\
&\quad + \frac{64}{15\epsilon^3}\int_{x-\epsilon}^{x+\epsilon} (\widetilde{g}'(\tau) - \widetilde{g_\delta}'(\tau))(x-\tau)[(x-\tau)^2/\epsilon^2 - 1]\mathrm{d}\tau \\
&= \frac{64}{15\epsilon^3}\int_{x-\epsilon}^{x+\epsilon} (\widetilde{g}'(\tau) - \widetilde{g_\delta}'(\tau))(x-\tau)[(x-\tau)^2/\epsilon^2 - 1]\mathrm{d}\tau
\end{aligned}
$$

since $\rho_\epsilon(-\epsilon) = \rho_\epsilon(\epsilon) = 0$. Now integrating by parts one more time and using (3.8)

$$
\begin{aligned}
|I(x)| &= \frac{64}{15\epsilon^3}\left|\int_{x-\epsilon}^{x+\epsilon} (\widetilde{g}(\tau) - \widetilde{g_\delta}(\tau))[3(x-\tau)^2/\epsilon^2 - 1]\mathrm{d}\tau\right| \\
&\le \frac{64}{15\epsilon^3}\int_{x-\epsilon}^{x+\epsilon} |\widetilde{g}(\tau) - \widetilde{g_\delta}(\tau)| \cdot |3(x-\tau)^2/\epsilon^2 - 1|\mathrm{d}\tau \\
&\le \frac{64}{15\epsilon^2}\|\widetilde{g}(x) - \widetilde{g_\delta}(x)\|_{C[-2\epsilon, L+2\epsilon]}\int_{-1}^{1} |3s^2 - 1|\mathrm{d}s \le C\delta\epsilon^{-2},
\end{aligned}
$$

which together with (3.11) and (3.12) completes the proof. □

The next two lemmas provide a bound for the error $\|q - q_{\delta,\epsilon}\|_{L^2(D)}$.

Lemma 3.4.3. *Let q and $q_{\delta,\epsilon}$ be defined as before. Then there holds for large T*

$$
\|q - q_{\delta,\epsilon}\|_{L^2(D)} \le \frac{1}{1 - CT^{-\alpha}}E,
$$

where the constant E is defined by

$$
E := \|\frac{g''}{g} - \frac{g''_{\delta,\epsilon}}{g_{\delta,\epsilon}}\|_{L^2(D)} + \|\frac{1}{g_{\delta,\epsilon}} - \frac{1}{g}\|_{L^2(D)}\left(\|{}^C D_t^\alpha u(\cdot, T; 0)\|_{L^\infty(D)} + \|f\|_{L^\infty(D)}\right). \quad (3.13)
$$

Proof. Using the initial guess $q^0 = (g'' + f)g^{-1}$, $q^0_{\delta,\epsilon} = (g''_{\delta,\epsilon} + f)g^{-1}_{\delta,\epsilon}$ and the triangle inequality, we have

$$\|q^0(x) - q^0_{\delta,\epsilon}(x)\|_{L^2(D)} \le \|\frac{g''}{g} - \frac{g''_{\delta,\epsilon}}{g_{\delta,\epsilon}}\|_{L^2(D)} + \|\frac{1}{g_{\delta,\epsilon}} - \frac{1}{g}\|_{L^2(D)}\|f\|_{L^\infty(D)} \le E.$$

Then for the first iteration we may deduce that

$$\begin{aligned}
&\|q^1 - q^1_{\delta,\epsilon}\|_{L^2(D)} \\
\le&\|\frac{g''}{g} - \frac{g''_{\delta,\epsilon}}{g_{\delta,\epsilon}}\|_{L^2(D)} + \|\frac{-{}^C D_t^\alpha u(\cdot,T;q^0) + f}{g} - \frac{-{}^C D_t^\alpha u(\cdot,T;q^0_{\delta,\epsilon}) + f}{g_{\delta,\epsilon}}\|_{L^2(D)} \\
\le&\|\frac{g''}{g} - \frac{g''_{\delta,\epsilon}}{g_{\delta,\epsilon}}\|_{L^2(D)} + \|\frac{1}{g_{\delta,\epsilon}} - \frac{1}{g}\|_{L^2(D)}\|f\|_{L^\infty(D)} \\
&+ \|\frac{1}{g} - \frac{1}{g_{\delta,\epsilon}}\|_{L^2(D)}\|{}^C D_t^\alpha u(\cdot,T;q^0)\|_{L^\infty(D)} \\
&+ \|\frac{1}{g_{\delta,\epsilon}}({}^C D_t^\alpha u(\cdot,T;q^0) - {}^C D_t^\alpha u(\cdot,T;q^0_{\delta,\epsilon}))\|_{L^2(D)}.
\end{aligned}$$

Since $q^0(x) > 0$ and Assumption 3.2.2 holds true, by Lemma 3.3.2 and (3.7) we have

$${}^C D_t^\alpha u(x,T;0) \ge {}^C D_t^\alpha u(x,T;q^0) \ge 0.$$

Further the bound for the last term follows from Theorem 3.3.1,

$$\|\frac{1}{g_{\delta,\epsilon}}({}^C D_t^\alpha u(\cdot,T;q^0) - {}^C D_t^\alpha u(\cdot,T;q^0_{\delta,\epsilon}))\|_{L^2(D)} \le CT^{-\alpha}\|q^0 - q^0_{\delta,\epsilon}\|_{L^2(D)} \le CT^{-\alpha}E.$$

Consequently, $\|q^1 - q^1_{\delta,\epsilon}\|_{L^2(D)} \le (1 + CT^{-\alpha})E$. Now the similar argument yields that

$$\begin{aligned}
\|q^2 - q^2_{\delta,\epsilon}\|_{L^2(D)} = \|Kq^1 - K_{\delta,\epsilon}q^1_{\delta,\epsilon}\|_{L^2(D)} &\le E + CT^{-\alpha}\|q^1 - q^1_{\delta,\epsilon}\|_{L^2(D)} \\
&\le E + CT^{-\alpha}(1 + CT^{-\alpha})E = (1 + CT^{-\alpha} + C^2T^{-2\alpha})E.
\end{aligned}$$

Continuing this iteration and the same argument shows that for T large enough

$$\|q - q_{\delta,\epsilon}\|_{L^2(D)} = \lim_{n\to\infty} \|q^n - q^n_{\delta,\epsilon}\|_{L^2(D)} \le \lim_{n\to\infty} \big(\sum_{k=0}^{n} C^k T^{-k\alpha}\big)E = \frac{E}{1 - CT^{-\alpha}}.$$

This completes the proof of the lemma. □

Lemma 3.4.4. *Assume that* $g(x), g_\delta(x) \in C^2(D)$*,* $g''(x)$ *is Lipschitz continuous on* $\bar{D}$ *and* $\|g(x) - g_\delta(x)\|_{C(\bar{D})} \le \delta$*. Let* E *be defined in* (3.13)*, then it holds that*

$$E \le C(\delta + \epsilon + \delta\epsilon^{-2}).$$

Proof. By the definition of E in (3.13), we deduce that

$$\begin{aligned} E &\le C\big(\|\frac{g''(g - g_{\delta,\epsilon})}{g g_{\delta,\epsilon}}\|_{L^2(D)} + \|\frac{g'' - g''_{\delta,\epsilon}}{g_{\delta,\epsilon}}\|_{L^2(D)} + \|\frac{g - g_{\delta,\epsilon}}{g g_{\delta,\epsilon}}\|_{L^2(D)}\big) \\ &\le C\big(\|g - g_{\delta,\epsilon}\|_{L^2(D)} + \|g'' - g''_{\delta,\epsilon}\|_{L^2(D)}\big). \end{aligned}$$

The bound of $\|g - g_{\delta,\epsilon}\|_{L^2(D)}$ follows from (3.8), (3.9) and Lemma 3.4.1

$$\begin{aligned} \|g - g_{\delta,\epsilon}\|_{L^\infty(D)} \le & \|g - G_\epsilon \widetilde{g}\|_{L^\infty(D)} + \|G_\epsilon(\widetilde{g} - G_\epsilon(\widetilde{g}))\|_{L^\infty(D)} \\ & + \|G_\epsilon(G_\epsilon(\widetilde{g} - \widetilde{g}_\delta))\|_{L^\infty(D)} \le C(\epsilon + \delta), \end{aligned}$$

which yields the estimate for the first term. This together with Lemma 3.4.2 completes the proof of the lemma. □

Now we state the main theorem which evaluates $\|q - q_{\delta,\epsilon}\|_{L^2(D)}$ and is a direct result of Lemmas 3.4.3 and 3.4.4.

Theorem 3.4.1. *Assume that* $g(x), g_\delta(x) \in C^2(D)$*,* $g''(x)$ *is Lipschitz continuous on* $\bar{D}$*,*

$\|g - g_\delta\|_{C(\bar{D})} \leq \delta$. *Then it holds for sufficiently large T that*

$$\|q - q_{\delta,\epsilon}\|_{L^2(D)} \leq \frac{C}{1 - CT^{-\alpha}}(\delta + \epsilon + \delta\epsilon^{-2}).$$

By Theorem 3.4.1, we deduce the following corollary immediately.

Corollary 3.4.1. *Assume that $g(x), g_\delta(x) \in C^2(D)$, $g''(x)$ is Lipschitz continuous on $\bar{D}$, $\|g - g_\delta\|_{C(\bar{D})} \leq \delta$ and T is sufficiently large. Then with the choice $\epsilon = \delta^{1/3}$, we obtain the optimal error that*

$$\|q - q_{\delta,\epsilon}\|_{L^2(D)} \leq \frac{C}{1 - CT^{-\alpha}}\delta^{1/3}.$$

3.5 Numerical confirmation

In this section, we present some numerical results to illustrate the theories established in Sections 3.3 and 3.4.

3.5.1 Numerical algorithm

The exact solutions for the forward problem are not available in closed form, and hence we compute the solution using the finite element method ([48, 47]) and L1 time stepping ([42, 41]). Specifically, we divide the unit interval $D = (0, L)$ into M equally spaced subintervals with a mesh size $h = L/M$. Likewise, we fix the time step size τ at T/N, where T is the time of interest. To solve the forward problem, we use the L1 scheme to discretize the Caputo fractional derivative

$$
\begin{aligned}
{}^C D_t^\alpha u(x,t_n) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{\partial u(x,s)}{\partial s}(t_n - s)^{-\alpha}\, ds \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \frac{u(x,t_{j+1}) - u(x,t_j)}{\tau} \int_{t_j}^{t_{j+1}} (t_n - s)^{-\alpha} ds \\
&= \sum_{j=0}^{n-1} b_j \frac{u(x,t_{n-j}) - u(x,t_{n-j-1})}{\tau^\alpha} \\
&= \tau^{-\alpha}[b_0 u(x,t_n) - b_{n-1} u(x,t_0) + \sum_{j=1}^{n-1} (b_j - b_{j-1}) u(x,t_{n-j})],
\end{aligned}
$$

where the weights b_j are given by

$$
b_j = ((j+1)^{1-\alpha} - j^{1-\alpha})/\Gamma(2-\alpha),\ j = 0,1,\ldots,N-1.
$$

This together with the Galerkin finite element discretization in space gives the fully discrete scheme for the forward problem (3.1): to find U^n, $n = 1,2,...,N$ such that for all $\varphi_h \in V_h$

$$
\begin{aligned}
(U_h^n, \varphi_h) + \tau^\alpha((U_h^n)', \varphi_h') + \tau^\alpha(qU_h^n, \varphi_h) + \tau^\alpha H(U_h^n(0)\varphi_h(0) + U_h^n(1)\varphi_h(1)) \\
= b_{n-1} U_h^0 + \sum_{j=1}^{n-1} (b_{j-1} - b_j)(U_h^{n-j}, \varphi_h) + \tau^\alpha (f, \varphi_h),
\end{aligned}
$$

with $U_h^0 = P_h v$ and V_h containing continuous piecewise linear functions.

To find the potential term q, we apply Algorithm 1 with an initial guess $q_0 = \frac{g''+f}{g} > q$ (by Assumption 3.2.2 and Corollary 3.2.1). In case that the data is noisy, we exploit the regularized terminal data $g_{\delta,\epsilon}$ and the iteration operator $K_{\delta,\epsilon}$ given by (3.10), and then apply Algorithm 2. Then the theoretical argument in the previous section yields the convergence such that $q_n \to q_{\delta,\epsilon}$ and $\|q_{\delta,\epsilon} - q\| \le c\delta^{1/3}$. In our experiments, we set our

Algorithm 2. An iterative algorithm for recovering the potential $q(x)$ from a noisy data g_δ

```
Let $\epsilon = \delta^{1/3}$, and exploit the regularized data $g_{\delta,\epsilon}$;
Set the initial guess $q_0 = \frac{g''_{\delta,\epsilon}(x)+f(x)}{g_{\delta,\epsilon}(x)}$;
for $k = 1, ..., N$ do
Compute $u(x,t;q_{k-1})$, the solution of (3.1) with potential $q_{k-1}$;
Update the potential by
    $q_k(x) = (K_{\delta,\epsilon}q_{k-1})(x) = \frac{g''_{\delta,\epsilon}(x)-{}^C D_t^\alpha u(x,T;q_{k-1})+f(x)}{g_{\delta,\epsilon}(x)}$;
Check stopping criterion $\|q_{k-1} - q_k\|_{L^2(D)} \le \epsilon_0$ for some $\epsilon_0 > 0$;
end for
output the approximated potential $q_N$.
```

stopping criterion by $\epsilon_0 = 10^{-12}$.

3.5.2 Numerical results for noise free data

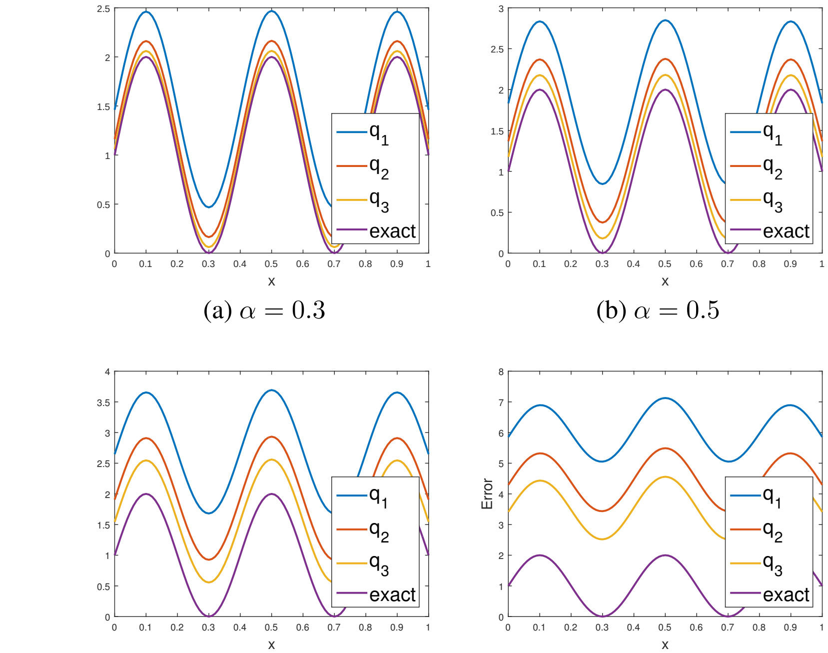
In this part, we consider the following two potentials:

(a1) smooth potential: $q_1(x) = 1 + \sin(5\pi x)$;

(a2) nonsmooth potential: $q_2(x) = \chi_{[0.1,0.3]} + \chi_{[0.6,0.9]}$.

For both examples, we take $L = 1$, and $f = 10$.

In Fig. 3.1, we present the numerical results of first three iterations from the initial guess $q_0 = \frac{g''+f}{g} > q$ in case of $\alpha = 0.3$, 0.5, 0.7 and 1. It is observed that the operator K defined in (3.5) is monotone such that $q_1 \ge q_2 \ge q_3 \ge ... \ge q$, which illustrates Theorem 3.3.2. The dependence of the convergence rate on parameters α and T are shown in Fig. 3.2. The larger is the time T and the smaller is α, the faster is the convergence of the fixed point scheme. Although not presented here, the same phenomena can be observed for $u(0) \ne 0$ and $f \equiv 0$. In this case, the solution u approaches the steady state solution as $T \to \infty$, for which the algorithm converges in one step. Further, as α approaches zero, the solution u decays faster around $t = 0$ (although it decays slower for large time), i.e., the fractional diffusion (3.1) can reach a "quasi-steady state" faster and hence the scheme

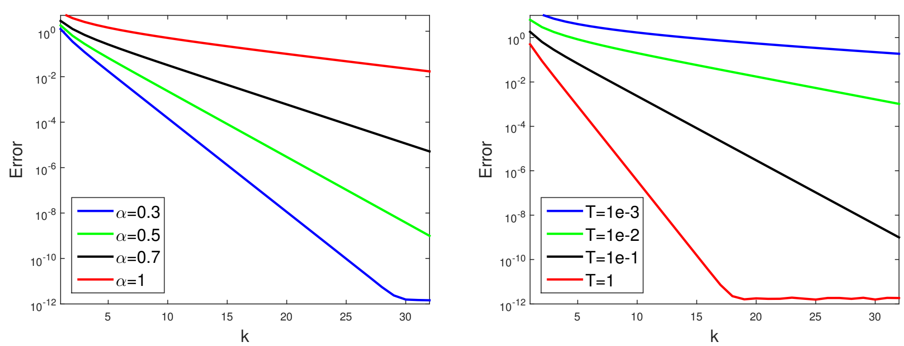


(a) $\alpha = 0.3$ (b) $\alpha = 0.5$

(c) $\alpha = 0.7$ (d) $\alpha = 1$

Figure 3.1: Example (a1): numerical results of the first three iterations at $T = 0.1$.

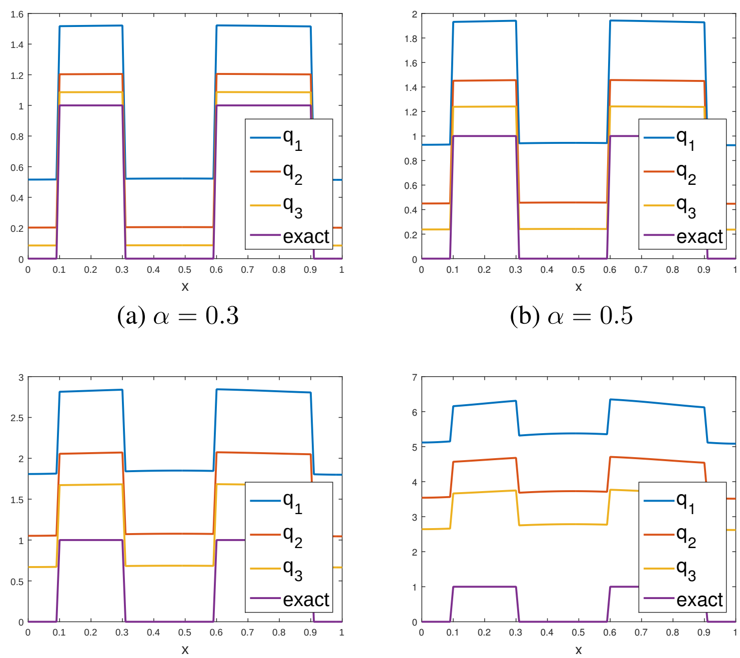
converges faster.



(a) $T = 0.1$ (b) $\alpha = 0.5$

Figure 3.2: Example (a1): error plot of the iteration with different α or T.

In Figs. 3.3 and 3.4, we present numerical results for the case of a discontinuous potential, i.e., example (a2). The monotonicity remains valid and the convergence rates agree with the preceding results in the smooth case: at a fixed time T, the smaller is the fractional order α or (respectively, at a fixed α, the larger is the time T), the faster is the convergence.



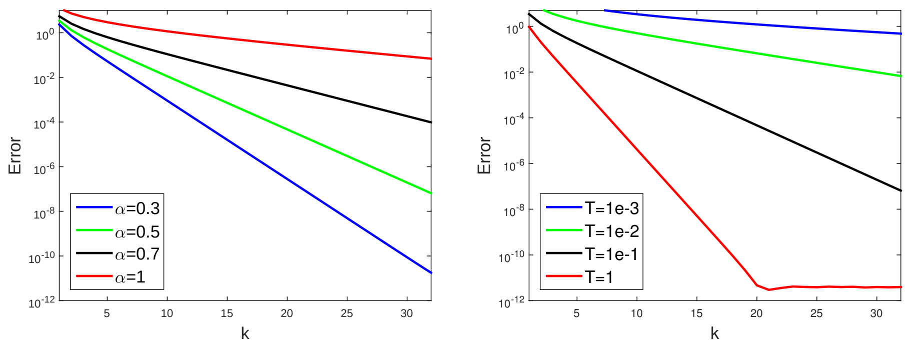
(a) $\alpha = 0.3$ (b) $\alpha = 0.5$

(c) $\alpha = 0.7$ (d) $\alpha = 1$

Figure 3.3: Example (a2): numerical results of the first three iterations at $T = 0.1$.

3.5.3 Numerical results for noisy data

Now we present numerical results for noisy data g_δ, which is generated by adding pointwise random perturbation of level δ, i.e., $g_\delta = g[1 + \delta\zeta]$, where ζ follows a uniform distribution over $[-1, 1]$. To illustrate our theory, we set $L = 10$ and $f = 1$, and consider the following two potentials:



(a) $T = 0.1$ (b) $\alpha = 0.5$

Figure 3.4: Example (a2): error plot of the iteration with different α or T.

(b1) smooth potential: $q_1(x) = 2 + \sin(0.5\pi x)$;

(b2) nonsmooth potential: $q_2(x) = 1 + \chi_{[2,4]} + \chi_{[6,8]}$.

For a fixed noise level δ, we choose the regularization parameter ϵ a priori by $\epsilon = \delta^{1/3}$ and compute the relative error $e_r = \|q - \widetilde{q}\|_{L^2(D)}/\|q\|_{L^2(D)}$. We observe a convergence rate $\delta^{1/3}$, cf. Fig. 3.5, which fully explains Corollary 3.4.1. The reconstructed potentials with $\delta = 0.02$ and 0.005 are shown in Fig. 3.6, which verifies the stability and efficiency of the algorithm.

3.6 Conclusion

In this work, we have developed a practical iterative algorithm to reconstruct the potential in the fractional diffusion equation (3.1) from the terminal data. We have shown the contractivity and monotonicity of the iteration operator, and hence the unique recovery. For a noise level δ, a regularized scheme has been developed and a convergence rate of the order $O(\delta^{1/3})$ is shown. Numerical examples show that this numerical algorithm is effective and stable.

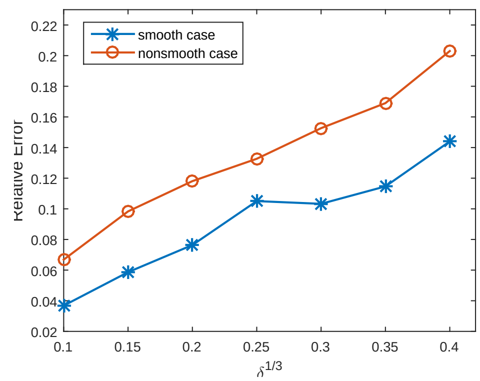
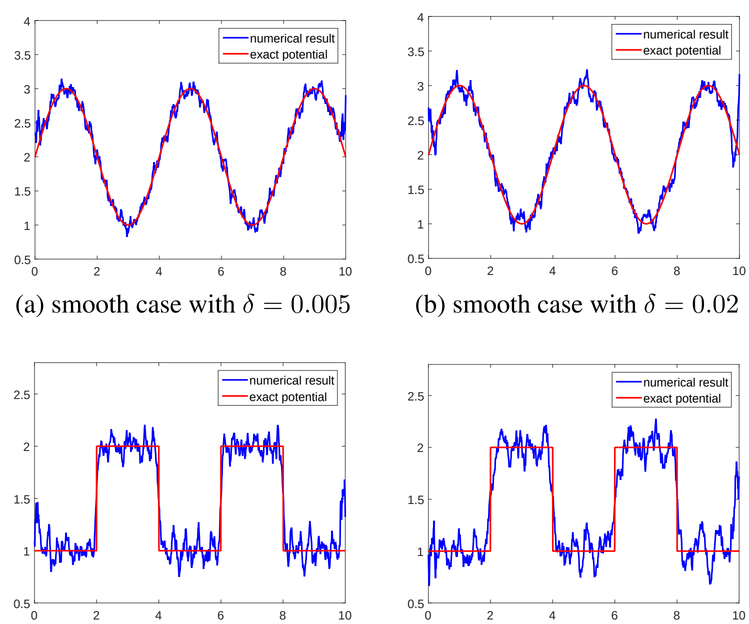


Figure 3.5: Plot of the relative error $\|q - q_{\delta,\epsilon}\|_{L^2(0,L)}/\|q\|_{L^2(0,L)}$, with $\alpha = 0.5$, $L = 10$ and $f = 1$.



(a) smooth case with $\delta = 0.005$ (b) smooth case with $\delta = 0.02$

(c) nonsmooth case with $\delta = 0.005$ (d) nonsmooth case with $\delta = 0.02$

Figure 3.6: Numerical results for different noise level δ.

4. DISTRIBUTED DIFFERENTIAL EQUATION

4.1 Introduction

In this chapter, we consider the DDE model as

$$\begin{cases} D^{(\mu)}u(x,t) - \mathcal{L}u(x,t) = f(x,t), & x \in \Omega, \quad t \in (0,T); \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T); \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where we use the Djrbashian-Caputo version for $D^{(\mu)}$: $D^{(\mu)}u = \int_0^1 \mu(\alpha)\, {}^C\!D_t^\alpha u\, \mathrm{d}\alpha$ and

$$D^{(\mu)}u = \int_0^t \left[\int_0^1 \frac{\mu(\alpha)}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha}\mathrm{d}\alpha\right] \frac{d}{d\tau}u(x,\tau)\mathrm{d}\tau := \int_0^t \eta(t-\tau)\frac{d}{d\tau}u(x,\tau)\mathrm{d}\tau, \tag{4.2}$$

with

$$\eta(s) = \int_0^1 \frac{\mu(\alpha)}{\Gamma(1-\alpha)} s^{-\alpha}\mathrm{d}\alpha. \tag{4.3}$$

The rest content is organized as follows. First, we demonstrate existence, uniqueness and regularity results for the solution of the distributed fractional derivative model on a cylindrical region in space-time $\Omega \times [0,T]$ where Ω is a bounded, open set in $\mathbb{R}^d$. Second, in the case of one spatial variable, $d = 1$, we set up representation theorems for the solution analogous to that for the heat equation itself, [56], and extended to the case of a single fractional derivative in [17].

Section 4.2 looks at the assumptions to be made on the various terms in (4.1) and utilizes these to show existence, uniqueness and regularity results for the direct problem;

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namely, to be given Ω, $\mathcal{L}$, f, u_0 and the function $\mu = \mu(\alpha)$, then to solve (4.1) for $u(x,t)$. Section 4.4 will derive several representation theorems for this solution and these will be used in the final section to formulate and prove a uniqueness result for the associated inverse problem to be discussed below. The main result of the current paper in this direction is in Section 4.5 where we show that the uniqueness results of [23, 24] can be extended to recover a suitably defined exponent function $\mu(\alpha)$.

4.2 Preliminary material

Let Ω be an open bounded domain in $\mathbb{R}^d$ with a smooth (C^2 will be more than sufficient) boundary $\partial\Omega$ and let $T > 0$ be a fixed constant.

$\mathcal{L}$ is a strongly elliptic, self-adjoint operator with smooth coefficients defined on Ω,

$$\mathcal{L}u = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x)u$$

where $a_{ij}(x) \in C^1(\overline{\Omega})$, $c(x) \in C(\overline{\Omega})$, $a_{ij}(x) = a_{ji}(x)$ and $\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq \delta \sum_{i=1}^{d} \xi^2$ for some $\delta > 0$, all $x \in \overline{\Omega}$ and all $\xi = (\xi_1, \ldots \xi_d) \in \mathbb{R}^d$.

To avoid unnecessary complications for the main theme we will make the assumption of homogeneous Dirichlet boundary conditions on $\partial\Omega$ so that the natural domain for $\mathcal{L}$ is $H^2(\Omega) \cap H_0^1(\Omega)$. Then $-\mathcal{L}$ has a complete, orthonormal system of eigenfunctions $\{\psi_n\}_1^\infty$ in $L^2(\Omega)$ with $\psi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ and with corresponding eigenvalues $\{\lambda_n\}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$ as $n \to \infty$.

The nonhomogeneous term will be taken to satisfy $f(x,t) \in C(0,T; H^2(\Omega))$. This can be weakened to assume only L^p regularity in time, but as shown in [24] this requires more delicate analysis. The initial value $u_0(x) \in H^2(\Omega)$. We will use $\langle \cdot, \cdot \rangle$ to denote the inner product in $L^2(\Omega)$.

Throughout this paper we will, by following [26], make the assumptions on the distributed derivative parameter μ.

Assumption 4.2.1.

$$\mu \in C^1[0,1],\ \mu(\alpha) \geq 0,\ \mu(1) \neq 0.$$

Remark 4.2.1. From these conditions it follows that there exists a constant $C_\mu > 0$ and an interval $(\beta_0, \beta) \subset (0,1)$ such that $\mu(\alpha) \geq C_\mu$ on (β_0, β). This will be needed in our proof of the representation theorem in Section 4.4.

4.2.1 A distributional ODE

Our first task is to analyze the ordinary distributed fractional order equation

$$D^{(\mu)} v(t) = -\lambda v(t),\ v(0) = 1,\ t \in (0,T) \tag{4.4}$$

and to show there exists a unique solution. We will need some preliminary analysis to determine the integral operator that serves as the inverse for $D^{(\mu)}$ in analogy with the Riemann-Liouville derivative being inverted by the Abel operator. If we now take the Laplace transform of η in (4.3) then we have

$$(\mathcal{L}\eta)(z) = \frac{\Phi(z)}{z}, \quad \text{where } \Phi(z) = \int_0^1 \mu(\alpha) z^\alpha \mathrm{d}\alpha. \tag{4.5}$$

The next lemma introduces an operator $I^{(\mu)}$ to analyze the distributed ODE (4.4).

Lemma 4.2.1. *Define the operator $I^{(\mu)}$ as*

$$I^{(\mu)}\phi(t) = \int_0^t \kappa(t-s)\phi(s)\mathrm{d}s, \quad \textit{where } \kappa(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{zt}}{\Phi(z)}\mathrm{d}z.$$

Then the following conclusions hold:

(1) $D^{(\mu)} I^{(\mu)} \phi(t) = \phi(t),\ I^{(\mu)} D^{(\mu)} \phi(t) = \phi(t) - \phi(0)$ *for* $\phi \in C^1(0,T)$;

(2) $\kappa(t) \in C^{\infty}(0, \infty)$ *and*

$$\kappa(t) = |\kappa(t)| \le C \ln \frac{1}{t} \text{ for sufficiently small } t > 0. \tag{4.6}$$

Proof. This is [26, Proposition 3.2]. We remark that the result in this paper include further estimates on κ that require additional regularity on μ. However, for the bound (4.6) only C^1 regularity on μ is needed. □

Remark 4.2.2. In [26, Proposition 3.2], if the condition either $\mu(0) \ne 0$ or $\mu(\alpha) \sim a\alpha^v,\ a > 0,\ v > 0$ is added, then κ is completely monotone. This property is not explicitly used in this paper, however as we remark after the uniqueness result, this condition on κ could be a useful basis for a reconstruction algorithm.

With $I^{(\mu)}$, we have the following results.

Lemma 4.2.2. *For each* $\lambda > 0$ *there exists a unique* $u(t)$ *which satisfies* (4.4).

Proof. Lemma 4.2.1 implies that (4.4) is equivalent to

$$u(t) = -\lambda I^{(\mu)} u(t) + 1 =: A_1 u.$$

Now the asymptotic and smoothness results of $\kappa(t)$ in Lemma 4.2.1 give $\kappa \in L^1(0, T)$, that is, there exists $t_1 \in (0, T)$ such that

$$\|\kappa\|_{L^1(0,t_1)} < \frac{1}{\lambda}.$$

Hence, given $\phi_1, \phi_2 \in L^1(0, t_1)$,

$$\begin{aligned}\|A_1(\phi_1) - A_1(\phi_2)\|_{L^1(0,t_1)} &\leq \lambda \int_0^{t_1} \int_0^t |\kappa(t-s)| \cdot |\phi_1(s) - \phi_2(s)| \, dsdt \\ &= \lambda \int_0^{t_1} |\phi_1(s) - \phi_2(s)| \int_s^{t_1} |\kappa(t-s)| \, dtds \\ &\leq \lambda \int_0^{t_1} |\phi_1(s) - \phi_2(s)| \cdot \|\kappa\|_{L^1(0,t_1)} ds \\ &= \lambda \|\kappa\|_{L^1(0,t_1)} \cdot \|\phi_1 - \phi_2\|_{L^1(0,t_1)}.\end{aligned}$$

From the fact that $0 < \lambda\|\kappa\|_{L^1(0,t_1)} < 1$, A_1 is a contraction map on $L^1(0, t_1)$ and so by the Banach fixed point theorem, there exists a unique $u_1(t) \in L^1(0, t_1)$ that satisfies $u_1 = A_1 u_1$.

For each $t \in (t_1, 2t_1)$, we have

$$u(t) = 1 - \lambda I^{(\mu)} u(t) = 1 - \lambda \int_{t_1}^t \kappa(t-s) u(s) \, ds - \lambda \int_0^{t_1} \kappa(t-s) u(s) \, ds.$$

Since $u = u_1$ on $(0, t_1)$ which is now known, then

$$u(t) = -\lambda \int_{t_1}^t \kappa(t-s) u(s) \, ds + 1 - \lambda \int_0^{t_1} \kappa(t-s) u_1(s) \, ds := A_2 u$$

for each $t \in (t_1, 2t_1)$. Given $\phi_1, \phi_2 \in L^1(t_1, 2t_1)$, it holds

$$\begin{aligned}\|A_2(\phi_1) - A_2(\phi_2)\|_{L^1(t_1,2t_1)} &\leq \lambda \int_{t_1}^{2t_1} \int_{t_1}^t |\kappa(t-s)| \cdot |\phi_1(s) - \phi_2(s)| dsdt \\ &= \lambda \int_{t_1}^{2t_1} |\phi_1(s) - \phi_2(s)| \int_s^{2t_1} |\kappa(t-s)| \mathrm{d}t\mathrm{d}s \\ &\leq \lambda \int_{t_1}^{2t_1} |\phi_1(s) - \phi_2(s)| \cdot \|\kappa\|_{L^1(0,t_1)} \mathrm{d}s \\ &= \lambda \|\kappa\|_{L^1(0,t_1)} \cdot \|\phi_1 - \phi_2\|_{L^1(t_1,2t_1)}.\end{aligned}$$

Hence, A_2 is also a contraction map on $L^1(t_1, 2t_1)$, which yields and shows that there exists a unique $u_2(t) \in L^1(t_1, 2t_1)$ such that $u_2 = A_2 u_2$.

Repeating this argument yields that there exists a unique solution $u \in L^1(0, T)$ of the distributed ODE (4.4), which completes the proof. $\square$

Lemma 4.2.3. *$u(t) \in C^\infty(0, T)$ is completely monotone, which gives $0 \leq u(t) \leq 1$ on $[0, T]$.*

Proof. This lemma is a special case of [26, Theorem 2.3]. $\square$

4.3 Existence, uniqueness and regularity

4.3.1 Existence and uniqueness of weak solution for DDE (4.1)

We state the definition of the weak solution as

Definition 4.3.1. *$u(x,t)$ is a weak solution to DDE (4.1) in $L^2(\Omega)$ if $u(\cdot, t) \in H_0^1(\Omega)$ for $t \in (0, T)$ and for any $\psi(x) \in H^2(\Omega) \cap H_0^1(\Omega)$,*

$$\begin{aligned} &\langle D^{(\mu)} u(x,t), \psi(x) \rangle - \langle \mathcal{L} u(x,t;a), \psi(x) \rangle = \langle f(x,t), \psi(x) \rangle,\ t \in (0,T); \\ &\langle u(x,0), \psi(x) \rangle = \langle u_0(x), \psi(x) \rangle. \end{aligned}$$

Then Lemma 4.2.2 gives the following corollary.

Corollary 4.3.1. *There exists a unique weak solution $u^*(x,t)$ of DDE (4.1) and the representation of $u^*(x,t)$ is*

$$\begin{aligned} u^*(x,t) = &\sum_{n=1}^{\infty} \Big[\langle u_0, \psi_n \rangle u_n(t) + \langle f(\cdot, 0), \psi_n \rangle I^{(\mu)} u_n(t) \\ &+ \int_0^t \langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \rangle I^{(\mu)} u_n(t-\tau) \mathrm{d}\tau \Big] \psi_n(x), \end{aligned} \tag{4.7}$$

where $u_n(t)$ is the unique solution of the distributed ODE (4.4) with $\lambda = \lambda_n$.

Proof. Completeness of $\{\psi_n(x) : n \in \mathbb{N}^+\}$ in $L^2(\Omega)$ and direct calculation show that the representation (4.7) is a weak solution of DDE (4.1); while the uniqueness of u^* follows from Lemma 4.2.2. $\square$

4.3.2 Regularity

The next two lemmas concern the regularity of u^* and $D^{(\mu)}u^*$.

Lemma 4.3.1.

$$\|u^*(x,t)\|_{C([0,T];H^2(\Omega))} \le C\big(\|u_0\|_{H^2(\Omega)} + \|f(\cdot,0)\|_{H^2(\Omega)} + T^{1/2}|f|_{H^1([0,T];H^2(\Omega))}\big)$$

where $C > 0$ depends on μ, $\mathcal{L}$ and Ω, and $|f|_{H^1([0,T];H^2(\Omega))} = \|\frac{\partial f}{\partial t}\|_{L^2([0,T];H^2(\Omega))}$.

Proof. Fix $t \in (0,T)$,

$$\begin{aligned}
\|u^*(x,t)\|_{H^2(\Omega)} \le &\big\|\sum_{n=1}^{\infty}\langle u_0,\psi_n\rangle u_n(t)\psi_n(x)\big\|_{H^2(\Omega)} && := I_1\\
&+\big\|\sum_{n=1}^{\infty}\langle f(\cdot,0),\psi_n\rangle I^{(\mu)}u_n(t)\psi_n(x)\big\|_{H^2(\Omega)} && := I_2\\
&+\big\|\sum_{n=1}^{\infty}\int_0^t\langle\frac{\partial}{\partial t}f(\cdot,\tau),\psi_n\rangle I^{(\mu)}u_n(t-\tau)\mathrm{d}\tau\ \psi_n(x)\big\|_{H^2(\Omega)} && := I_3.
\end{aligned}$$

We estimate each of I_1, I_2, and I_3 in turn using Lemmas 4.2.1 and 4.2.3 where in each case $C > 0$ is a generic constant that depends only on μ, $\mathcal{L}$ and Ω.

$$\begin{aligned}
I_1^2 &= \big\|\sum_{n=1}^{\infty}\langle u_0,\psi_n\rangle u_n(t)\psi_n(x)\big\|^2_{H^2(\Omega)} \le C\big\|\mathcal{L}\big(\sum_{n=1}^{\infty}\langle u_0,\psi_n\rangle u_n(t)\psi_n(x)\big)\big\|^2_{L^2(\Omega)}\\
&= C\big\|\sum_{n=1}^{\infty}\lambda_n\langle u_0,\psi_n\rangle u_n(t)\psi_n(x)\big\|^2_{L^2(\Omega)} = C\sum_{n=1}^{\infty}\lambda_n^2\langle u_0,\psi_n\rangle^2 u_n^2(t)\\
&\le C\sum_{n=1}^{\infty}\lambda_n^2\langle u_0,\psi_n\rangle^2 = C\big\|\mathcal{L}u_0\big\|^2_{L^2(\Omega)} \le C\|u_0\|^2_{H^2(\Omega)}.
\end{aligned}$$

$$
\begin{aligned}
I_2^2 &= \Big\| \sum_{n=1}^{\infty} \langle f(\cdot,0),\psi_n\rangle I^{(\mu)} u_n(t)\psi_n(x)\Big\|_{H^2(\Omega)}^2 \le C\sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot,0),\psi_n\rangle^2 (I^{(\mu)}u_n(t))^2 \\
&\le C\sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot,0),\psi_n\rangle^2 \Big(\int_0^t |\kappa(\tau)|\cdot|u_n(t-\tau)|\mathrm{d}\tau\Big)^2 \\
&\le C\sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot,0),\psi_n\rangle^2 \Big(\int_0^t |\kappa(\tau)|\mathrm{d}\tau\Big)^2 \\
&\le C\sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot,0),\psi_n\rangle^2 \|\kappa\|_{L^1(0,T)}^2 \le C\|\kappa\|_{L^1(0,T)}^2 \|f(\cdot,0)\|_{H^2(\Omega)}^2 .
\end{aligned}
$$

$$
\begin{aligned}
I_3^2 &= \Big\| \sum_{n=1}^{\infty} \int_0^t \langle \frac{\partial}{\partial t} f(\cdot,\tau),\psi_n\rangle I^{(\mu)} u_n(t-\tau)\mathrm{d}\tau\ \psi_n(x)\Big\|_{H^2(\Omega)}^2 \\
&\le C\sum_{n=1}^{\infty} \Big[\int_0^t \lambda_n \langle \frac{\partial}{\partial t} f(\cdot,\tau),\psi_n\rangle I^{(\mu)} u_n(t-\tau)\mathrm{d}\tau\Big]^2 \\
&\le C\sum_{n=1}^{\infty} \Big[\int_0^t \lambda_n |\langle \frac{\partial}{\partial t} f(\cdot,\tau),\psi_n\rangle| \cdot |I^{(\mu)} u_n(t-\tau)|\mathrm{d}\tau\Big]^2 \\
&\le C\|\kappa\|_{L^1(0,T)}^2 \sum_{n=1}^{\infty} \int_0^t \lambda_n^2 |\langle \frac{\partial}{\partial t} f(\cdot,\tau),\psi_n\rangle|^2 \mathrm{d}\tau \cdot \int_0^t 1^2 \mathrm{d}\tau \\
&\le CT\|\kappa\|_{L^1(0,T)}^2 \int_0^T \sum_{n=1}^{\infty} \lambda_n^2 |\langle \frac{\partial}{\partial t} f(\cdot,\tau),\psi_n\rangle|^2 \mathrm{d}\tau \\
&\le CT\|\kappa\|_{L^1(0,T)}^2 \int_0^T \Big\|\frac{\partial}{\partial t} f(\cdot,\tau)\Big\|_{H^2(\Omega)}^2 \mathrm{d}\tau = CT\|\kappa\|_{L^1(0,T)}^2 |f|_{H^1([0,T];H^2(\Omega))}^2 .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|u^*(x,t)\|_{C([0,T];H^2(\Omega))} &\le C\|u_0\|_{H^2(\Omega)} + C\|\kappa\|_{L^1(0,T)}\|f(\cdot,0)\|_{H^2(\Omega)} \\
&\qquad + CT^{1/2}\|\kappa\|_{L^1(0,T)}|f|_{H^1([0,T];H^2(\Omega))} \\
&\le C\big(\|u_0\|_{H^2(\Omega)} + \|f(\cdot,0)\|_{H^2(\Omega)} + T^{1/2}|f|_{H^1([0,T];H^2(\Omega))}\big).
\end{aligned}
$$

Due to the fact that κ is determined by μ, the constant C above only depends on μ, $\mathcal{L}$ and Ω. □

Lemma 4.3.2.

$$\|D^{(\mu)}u^*\|_{C([0,T];L^2(\Omega))} \leq C\left(\|u_0\|_{H^2(\Omega)} + T^{1/2}|f|_{H^1([0,T];H^2(\Omega))} + \|f\|_{C([0,T];H^2(\Omega))}\right),$$

where $C > 0$ only depends on μ, $\mathcal{L}$ and Ω.

Proof. For each $t \in (0, T)$,

$$\begin{aligned} D^{(\mu)}u^*(x,t) = &-\sum_{n=1}^{\infty} \lambda_n \langle u_0, \psi_n\rangle u_n(t)\psi_n(x) - \sum_{n=1}^{\infty} \lambda_n \langle f(\cdot,0), \psi_n\rangle I^{(\mu)}u_n(t)\psi_n(x) \\ &- \sum_{n=1}^{\infty} \lambda_n \int_0^t \langle \frac{\partial}{\partial t} f(\cdot,\tau), \psi_n\rangle I^{(\mu)}u_n(t-\tau)\mathrm{d}\tau\ \psi_n(x) + f(x,t), \end{aligned}$$

which implies

$$\begin{aligned} &\|D^{(\mu)}u^*\|_{L^2(\Omega)} \\ \leq &\|\sum_{n=1}^{\infty} \lambda_n \langle u_0, \psi_n\rangle u_n(t)\psi_n(x)\|_{L^2(\Omega)} + \|\sum_{n=1}^{\infty} \lambda_n \langle f(\cdot,0), \psi_n\rangle I^{(\mu)}u_n(t)\psi_n(x)\|_{L^2(\Omega)} \\ +&\|\sum_{n=1}^{\infty} \lambda_n \int_0^t \langle \frac{\partial}{\partial t} f(\cdot,\tau), \psi_n\rangle I^{(\mu)}u_n(t-\tau)\mathrm{d}\tau\ \psi_n(x)\|_{L^2(\Omega)} + \|f(\cdot,t)\|_{L^2(\Omega)}. \end{aligned}$$

Combining the estimates for I_1, I_2 and I_3 we obtain

$$\|\sum_{n=1}^{\infty} \lambda_n \langle u_0, \psi_n\rangle u_n(t)\psi_n(x)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n^2 \langle u_0, \psi_n\rangle^2 u_n^2(t) \leq C\|u_0\|_{H^2(\Omega)}^2,$$

$$\begin{aligned} \|\sum_{n=1}^{\infty} \lambda_n \langle f(\cdot,0), \psi_n\rangle I^{(\mu)}u_n(t)\psi_n(x)\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot,0), \psi_n\rangle^2 (I^{(\mu)}u_n(t))^2 \\ &\leq C\|\kappa\|_{L^1(0,T)}^2 \|f(\cdot,0)\|_{H^2(\Omega)}^2 \\ &\leq C\|\kappa\|_{L^1(0,T)}^2 \|f\|_{C([0,T];H^2(\Omega))}^2 \end{aligned}$$

and

$$
\begin{aligned}
&\|\sum_{n=1}^{\infty}\lambda_n\int_0^t\langle\frac{\partial}{\partial t}f(\cdot,\tau),\psi_n\rangle I^{(\mu)}u_n(t-\tau)\mathrm{d}\tau\ \psi_n(x)\|^2_{L^2(\Omega)}\\
=&\sum_{n=1}^{\infty}\left[\int_0^t\lambda_n\langle\frac{\partial}{\partial t}f(\cdot,\tau),\psi_n\rangle I^{(\mu)}u_n(t-\tau)\mathrm{d}\tau\right]^2\le CT\|\kappa\|^2_{L^1(0,T)}|f|^2_{H^1([0,T];H^2(\Omega))}.
\end{aligned}
$$

Therefore,

$$
\|D^{(\mu)}u^*\|_{C([0,T];L^2(\Omega))}\le C\left(\|u_0\|_{H^2(\Omega)}+T^{1/2}|f|_{H^1([0,T];H^2(\Omega))}+\|f\|_{C([0,T];H^2(\Omega))}\right),
$$

where C is dependent only on μ, $\mathcal{L}$ and Ω. □

The main theorem of this section follows from Corollary 4.3.1, Lemmas 4.3.1 and 4.3.2.

Theorem 4.3.2 (Main theorem for the direct problem)**.** *There exists a unique weak solution $u^*(x,t)$ in $L^2(\Omega)$ of the DDE* (4.1) *with the representation* (4.7) *and the following regularity estimate*

$$
\begin{aligned}
&\|u^*\|_{C([0,T];H^2(\Omega))}+\|D^{(\mu)}u^*\|_{C([0,T];L^2(\Omega))}\\
&\qquad\le C\Big(\|u_0\|_{H^2(\Omega)}+T^{1/2}|f|_{H^1([0,T];H^2(\Omega))}+\|f\|_{C([0,T];H^2(\Omega))}\Big),
\end{aligned}
$$

where $C>0$ depends only on μ, $\mathcal{L}$ and Ω.

4.4 Representation of the DDE solution for one spatial variable

In this section, we will establish a representation result for the special case $\Omega = (0,1)$, $\mathcal{L}u = u_{xx}$ in (4.1)

$$\begin{cases} D^{(\mu)}u - u_{xx} = f(x,t),\ 0 < x < 1,\ 0 < t < \infty; \\ u(x,0) = u_0(x),\ 0 < x < 1; \\ u(0,t) = g_0(t),\ 0 \le t < \infty; \\ u(1,t) = g_1(t),\ 0 \le t < \infty, \end{cases} \tag{4.8}$$

where $g_0, g_1 \in L^2(0,\infty)$ and $f(x,\cdot) \in L^1(0,\infty)$ for each $x \in (0,1)$.

We can obtain the fundamental solution by Laplace and Fourier transforms. First, we extend the finite domain to an infinite one and impose a homogeneous right-hand side, i.e. we consider the following model

$$\begin{cases} D^{(\mu)}u - u_{xx} = 0,\ -\infty < x < \infty,\ 0 < t < \infty; \\ u(x,0) = u_0(x),\ -\infty < x < \infty. \end{cases}$$

Next we take the Fourier transform $\mathcal{F}$ with respect to x and denote $(\mathcal{F}u)(\xi,t)$ by $\tilde{u}(\xi,t)$,

$$D^{(\mu)}\tilde{u}(\xi,t) + \xi^2\tilde{u}(\xi,t) = 0.$$

Then by taking the Laplace transform $\mathcal{L}$ with respect to t and denote $(\mathcal{L}\tilde{u})(\xi,z)$ by $\hat{\tilde{u}}(\xi,z)$, we obtain

$$\int_0^1 \mu(\alpha)\left(z^\alpha\hat{\tilde{u}}(\xi,z) - z^{\alpha-1}\tilde{u}_0(\xi)\right)\,d\alpha + \xi^2\hat{\tilde{u}}(\xi,z) = 0,$$

that is,

$$\hat{\tilde{u}}(\xi, z) = \frac{\Phi(z)/z}{\Phi(z) + \xi^2}\tilde{u}_0(\xi),$$

where $\Phi(z)$ comes from (4.5).

Then we have

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1} \circ \mathcal{L}^{-1}(\hat{\tilde{u}}(\xi,z)) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\tilde{u}_0(\xi)\,dz\,d\xi \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}\int_{-\infty}^{+\infty}\frac{1}{2\pi}e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\tilde{u}_0(\xi)\,d\xi\,dz \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}\big(\mathcal{F}^{-1}(\frac{\Phi(z)/z}{\Phi(z)+\xi^2}) * u_0\big)(x)\,dz, \end{aligned}$$

where the integral above is the usual Bromwich path, that is, a line in the complex plane parallel to the imaginary axis $z = \gamma + it$, $-\infty < t < \infty$, see [57]. The last equality follows from the Fourier transform formula on convolutions and γ can be an arbitrary positive number due to the fact that $z = 0$ is a singular point of the function $\frac{\Phi(z)/z}{\Phi(z)+\xi^2}$. Throughout the remainder of this paper we will use γ to denote a strictly positive constant which is larger than $e^{1/\beta}$. The number $e^{1/\beta}$ will be seen in the proof of Lemma 4.4.3. We shall assume the angle of variation z for the Laplace transforms is from $-\pi$ to π, that is $z \in \Lambda := \{z \in \mathbb{C} : arg(z) \in (-\pi, \pi]\}$.

For $\Phi(z)$, we have the following result which will be central to the rest of the paper. It can be shown by using the Cauchy-Riemann equations in polar form.

Lemma 4.4.1. $\Phi(z)$ *is analytic on* $\mathbb{C} \setminus \{0\}$.

In the next two lemmas, we obtain important properties of $\Phi(z)$.

Lemma 4.4.2. $\mathrm{Re}(\Phi^{1/2}(z)) \geq \dfrac{\sqrt{2}}{2}|\Phi^{1/2}(z)|,\ \mathrm{Re}\, z = \gamma > 0.$

Proof. $\gamma > 0$ implies that $\mathrm{Re}\, z > 0$, i.e. $arg(z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which together with $0 < \alpha < 1$ and $\mu(\alpha) \geq 0$ yields $\mathrm{Re}\,\Phi(z) \geq 0$, i.e. $arg(\Phi(z)) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This gives $arg(\Phi^{1/2}(z)) \in$

$(-\frac{\pi}{4}, \frac{\pi}{4})$. Hence,

$$\mathrm{Re}(\Phi^{1/2}(z)) = \cos(arg(\Phi^{1/2}(z)))|\Phi^{1/2}(z)| \geq \frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|,$$

which completes the proof. □

Lemma 4.4.3.

$$C_{\mu,\beta}\frac{\gamma^{\beta} - \gamma^{\beta_0}}{\ln \gamma} \leq C_{\mu,\beta}\frac{|z|^{\beta} - |z|^{\beta_0}}{\ln |z|} \leq |\Phi(z)| \leq C\frac{|z| - 1}{\ln |z|},$$

for z *such that* $\mathrm{Re}\, z = \gamma > e^{1/\beta} > 0$.

Proof. For the right-hand side of the inequality, $\mu(\alpha) \in C^1[0,1]$ obviously implies that there exists a $C > 0$ such that $|\mu(\alpha)| \leq C$ on $[0,1]$. Hence,

$$|\Phi(z)| \leq \int_0^1 |\mu(\alpha)| \cdot |z|^{\alpha}\, d\alpha \leq C\int_0^1 |z|^{\alpha}\, d\alpha = C\frac{|z| - 1}{\ln |z|}.$$

For the left-hand side, write $z = re^{i\theta}$. Since $\mathrm{Re}\, z = \gamma > 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then

$$\begin{aligned}
|\Phi(z)| \geq \mathrm{Re}(\phi(z)) &= \int_0^1 \mu(\alpha) r^{\alpha} \cos(\theta\alpha)\, d\alpha \\
&\geq C_{\mu}\int_{\beta_0}^{\beta} r^{\alpha}\cos(\theta\alpha)\, d\alpha \geq C_{\mu}\cos(\beta\theta)\int_{\beta_0}^{\beta} r^{\alpha}\, d\alpha \\
&\geq C_{\mu}\cos(\frac{\beta\pi}{2})\int_{\beta_0}^{\beta} |z|^{\alpha}\, d\alpha = C_{\mu,\beta}\frac{|z|^{\beta} - |z|^{\beta_0}}{\ln |z|}.
\end{aligned}$$

Recall $|z| \geq \gamma > e^{1/\beta}$, we have $\frac{|z|^{\beta} - |z|^{\beta_0}}{\ln |z|} \geq \frac{\gamma^{\beta} - \gamma^{\beta_0}}{\ln \gamma}$ due to the function $\frac{x^{\beta} - x^{\beta_0}}{\ln x}$ being increasing on the interval $(e^{1/\beta}, +\infty)$. □

Now we are in a position to calculate the complex integral $\mathcal{F}^{-1}\big(\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\big)$.

Lemma 4.4.4. $\mathcal{F}^{-1}(\dfrac{\Phi(z)/z}{\Phi(z) + \xi^2}) = \dfrac{\Phi^{1/2}(z)}{2z}e^{-\Phi^{1/2}(z)|x|}.$

Proof. From the inverse Fourier transform formula we have

$$\mathcal{F}^{-1}\Big(\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\Big) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi.$$

We denote the contour from $-R$ to R by C_0, the semicircle with radius R in the upper and lower half plane by C_{R^+} and C_{R^-}, respectively. Also, let C_+, C_- be the closed contours which consist of C_0, C_{R^+} and C_0, C_{R^-} respectively.

For the case of $x > 0$, working on the closed contour C_+, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi \\
&= \lim_{R\to\infty}\frac{1}{2\pi}\oint_{C_+} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi - \lim_{R\to\infty}\frac{1}{2\pi}\int_{C_R^+} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi \\
&= \lim_{R\to\infty}\frac{1}{2\pi}\oint_{C_+} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi,
\end{aligned}$$

where the second limit is 0 as follows from Jordan's Lemma. Since to $0 < \alpha < 1, \gamma > 0$, by our assumptions we have $\mathrm{Re}(\Phi(z)) \geq 0$, which in turn leads to $\mathrm{Re}(\Phi^{1/2}(z)) \geq 0$. Then there is only one singular point $\xi = i\Phi^{1/2}(z)$ in C_+ which is contained by the upper half plane. By the residue theorem [57], we have

$$\lim_{R\to\infty}\frac{1}{2\pi}\oint_{C_+} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi = \lim_{R\to\infty} 2\pi i\frac{1}{2\pi}e^{ixi\Phi^{1/2}(z)}\frac{\Phi(z)/z}{2i\Phi^{1/2}(z)} = \frac{\Phi^{1/2}(z)}{2z}e^{-\Phi^{1/2}(z)x}.$$

For the case of $x < 0$, we choose the closed contour C_-. Since $\mathrm{Re}(\Phi^{1/2}(z)) \geq 0$, it follows that $\xi = -i\Phi^{1/2}(z)$ is the unique singular point in C_-. Then a similar calculation gives

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi \\
&= -\lim_{R\to\infty}\frac{1}{2\pi}\oint_{C_-} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi + \lim_{R\to\infty}\frac{1}{2\pi}\int_{C_R^-} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi \\
&= -\lim_{R\to\infty}\frac{1}{2\pi}\oint_{C_-} e^{ix\xi}\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\,d\xi \\
&= \lim_{R\to\infty}\frac{\Phi^{1/2}(z)}{2z}e^{\Phi^{1/2}(z)x} = \frac{\Phi^{1/2}(z)}{2z}e^{\Phi^{1/2}(z)x}.
\end{aligned}
$$

Therefore,

$$
\mathcal{F}^{-1}\Big(\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\Big) = \frac{\Phi^{1/2}(z)}{2z}e^{-\Phi^{1/2}(z)|x|},
$$

which completes the proof. □

4.4.1 The fundamental solution $G_\mu(x,t)$

With the above lemma, we have

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}\int_{-\infty}^{+\infty}\frac{\Phi^{1/2}(z)}{2z}e^{-\Phi^{1/2}(z)|x-y|}u_0(y)\,dy\,dz \\
&= \int_{-\infty}^{+\infty}\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Phi^{1/2}(z)}{2z}e^{zt-\Phi^{1/2}(z)|x-y|}\,dz\right]u_0(y)\,dy.
\end{aligned}
$$

Then we can define the fundamental solution $G_{(\mu)}(x,t)$ as

$$
G_{(\mu)}(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Phi^{1/2}(z)}{2z}e^{zt-\Phi^{1/2}(z)|x|}\,dz. \tag{4.9}
$$

The following three lemmas provide some important properties of $G_{(\mu)}(x,t)$.

Lemma 4.4.5. *The integral for $G_{(\mu)}(x,t)$ is convergent for each $(x,t)\in(0,\infty)\times(0,\infty)$.*

Proof. Given $(x,t) \in (0,\infty) \times (0,\infty)$, with Lemmas 4.4.2 and 4.4.3, we have

$$\begin{aligned}
|G_{(\mu)}(x,t)| &\leq \frac{1}{4\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} |\frac{\Phi^{1/2}(z)}{z}| \cdot |e^{zt}| \cdot |e^{-\Phi^{1/2}(z)|x|}|\, dz \\
&= \frac{1}{4\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{|\Phi^{1/2}(z)|}{|z|} e^{\gamma t} e^{-\operatorname{Re}(\Phi^{1/2}(z)|x|)}\, dz \\
&\leq \frac{1}{4\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{|\Phi^{1/2}(z)|}{|z|} e^{\gamma t} e^{-\frac{\sqrt{2}}{2}|x||\Phi^{1/2}(z)|}\, dz \\
&\leq \frac{Ce^{\gamma t}}{4\pi(\ln\gamma)^{1/2}}\int_{\gamma-i\infty}^{\gamma+i\infty} |z|^{-1/2} e^{-C_{\mu,\beta}|x|(\frac{C|z|^{\beta}}{\ln|z|})^{1/2}}\, dz < \infty.
\end{aligned}$$

□

Lemma 4.4.6. $G_{(\mu)}(x,t) \in C^{\infty}((0,\infty)\times(0,\infty))$.

Proof. Fix $(x,t) \in (0,\infty)\times(0,\infty)$. Then for small $|\epsilon_x|, |\epsilon_t|$ we have

$$\begin{aligned}
|G_{(\mu)}(x+\epsilon_x, t+\epsilon_t) - G_{(\mu)}(x,t)| &\leq |G_{(\mu)}(x+\epsilon_x, t+\epsilon_t) - G_{(\mu)}(x, t+\epsilon_t)| \\
&\quad + |G_{(\mu)}(x, t+\epsilon_t) - G_{(\mu)}(x,t)|.
\end{aligned}$$

For $|G_{(\mu)}(x+\epsilon_x, t+\epsilon_t) - G_{(\mu)}(x, t+\epsilon_t)|$, the following holds

$$\begin{aligned}
&|G_{(\mu)}(x+\epsilon_x, t+\epsilon_t) - G_{(\mu)}(x, t+\epsilon_t)| \\
\leq\ &\frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} |\frac{\Phi^{1/2}(z)}{2z}| \cdot |e^{zt+z\epsilon_t}| \cdot |e^{-\Phi^{1/2}(z)|x/2|}| \cdot |e^{-\Phi^{1/2}(z)(\frac{x}{2}+\epsilon_x)} - e^{-\Phi^{1/2}(z)(x/2)}|\ dz.
\end{aligned}$$

From the proof of Lemma 4.4.5, we have

$$\begin{aligned}
|e^{-\Phi^{1/2}(z)(\frac{x}{2}+\epsilon_x)} - e^{-\Phi^{1/2}(z)(x/2)}| &\leq |e^{-\Phi^{1/2}(z)(\frac{x}{2}+\epsilon_x)}| + |e^{-\Phi^{1/2}(z)(x/2)}| \\
&\leq e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(\frac{x}{2}+\epsilon_x)} + e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(x/2)} \leq 2,
\end{aligned}$$

and

$$\frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} |\frac{\Phi^{1/2}(z)}{2z}| \cdot |e^{zt+z\epsilon_t}| \cdot |e^{-\Phi^{1/2}(z)|x/2|}|\ dz < \infty.$$

Hence, after setting $e_1(z,\epsilon_x) = |e^{-\Phi^{1/2}(z)(\frac{x}{2}+\epsilon_x)} - e^{-\Phi^{1/2}(z)(x/2)}|$, we can apply Lebesgue's dominated convergent theorem to deduce that

$$\begin{aligned}
&\lim_{\epsilon_x\to 0} |G_{(\mu)}(x+\epsilon_x, t+\epsilon_t) - G_{(\mu)}(x, t+\epsilon_t)| \\
\leq &\lim_{\epsilon_x\to 0} \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} |\frac{\Phi^{1/2}(z)}{2z}| \cdot |e^{zt+z\epsilon_t}| \cdot |e^{-\Phi^{1/2}(z)|x/2|}| \cdot e_1(z,\epsilon_x)\ dz \\
= &\frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} |\frac{\Phi^{1/2}(z)}{2z}| \cdot |e^{zt+z\epsilon_t}| \cdot |e^{-\Phi^{1/2}(z)|x/2|}| \cdot \lim_{\epsilon_x\to 0} e_1(z,\epsilon_x)\ dz = 0.
\end{aligned}$$

A similar argument also shows that $\lim_{\epsilon_t\to 0} |G_{(\mu)}(x, t+\epsilon_t) - G_{(\mu)}(x,t)| = 0$. From this we deduce that $\lim_{\epsilon_x,\ \epsilon_t\to 0} |G_{(\mu)}(x+\epsilon_x, t+\epsilon_t) - G_{(\mu)}(x,t)| = 0$, which shows that $G_{(\mu)}(x,t) \in C((0,\infty)\times(0,\infty))$.

Similarly, following from the proof of Lemma 4.4.5 and the above limiting argument, we obtain

$$G_{(\mu)}(x,t) \in C^n((0,\infty)\times(0,\infty)),\ n \in \mathbb{N}^+,$$

which leads to $G_{(\mu)}(x,t) \in C^\infty((0,\infty)\times(0,\infty))$ and this completes the proof. □

Lemma 4.4.7.

$$\lim_{t\to 0} G_{(\mu)}(x,t) = \delta(x).$$

Proof. Fix $x \neq 0$, for each $t \in (0,\infty)$,

$$\left|\frac{\Phi^{1/2}(z)}{2z}\right| \cdot |e^{zt-\Phi^{1/2}(z)|x|}| \leq e^{\gamma t}\left|\frac{\Phi^{1/2}(z)}{2z}\right| \cdot |e^{-\Phi^{1/2}(z)|x|}|.$$

The proof of Lemma 4.4.5 shows that

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \left|\frac{\Phi^{1/2}(z)}{2z}\right| \cdot |e^{-\Phi^{1/2}(z)|x|}| < \infty,$$

then by dominated convergence theorem, we can deduce that

$$\begin{aligned}\lim_{t\to 0} G_{(\mu)}(x,t) &= \lim_{t\to 0} \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{zt-\Phi^{1/2}(z)|x|}\, dz \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi^{1/2}(z)}{2z} \lim_{t\to 0} e^{zt-\Phi^{1/2}(z)|x|}\, dz \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)|x|}\, dz,\end{aligned} \tag{4.10}$$

for each $x \neq 0$. Let $z = \gamma + mi$, we have

$$\lim_{t\to 0} G_{(\mu)}(x,t) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{\Phi^{1/2}(\gamma+mi)}{\gamma+mi} e^{-\Phi^{1/2}(\gamma+mi)|x|}\, dm. \tag{4.11}$$

Recalling the definition of the closed contour C_- and the proof of Lemma 4.4.4, we see the function $\frac{\Phi^{1/2}(\gamma+mi)}{\gamma+mi} e^{-\Phi^{1/2}(\gamma+mi)|x|}$ is analytic in C_-. Then

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{\Phi^{1/2}(\gamma+mi)}{\gamma+mi} e^{-\Phi^{1/2}(\gamma+mi)|x|}\, dm \\ &= \lim_{R\to\infty}\int_{C_{R-}} \frac{\Phi^{1/2}(\gamma+mi)}{\gamma+mi} e^{-\Phi^{1/2}(\gamma+mi)|x|}\, dm \\ &= \lim_{R\to\infty}\int_{-\pi}^{0} Rie^{i\theta} \frac{\Phi^{1/2}(\gamma+Rie^{i\theta})}{\gamma+Rie^{i\theta}} e^{-\Phi^{1/2}(\gamma+Rie^{i\theta})|x|}\, d\theta,\end{aligned}$$

where $m = Re^{i\theta}$. Since $\mathrm{Re}(\gamma + Rie^{i\theta}) = \gamma - R\sin\theta \geq 0$, following from the proofs of Lemmas 4.4.2 and 4.4.3, we can deduce that

$$\begin{aligned}\mathrm{Re}(\Phi^{1/2}(\gamma+Rie^{i\theta})) &\geq \frac{\sqrt{2}}{2}|\Phi^{1/2}(\gamma+Rie^{i\theta})| \\ &\geq C_{\mu,\beta}\frac{|\gamma+Rie^{i\theta}|^{\beta}-|\gamma+Rie^{i\theta}|^{\beta_0}}{\ln|\gamma+Rie^{i\theta}|} \geq C\frac{R^{\beta}-R^{\beta_0}}{\ln R},\end{aligned}$$

and

$$|\Phi^{1/2}(\gamma+Rie^{i\theta})| \leq C\frac{|\gamma+Rie^{i\theta}|-1}{\ln|\gamma+Rie^{i\theta}|} \leq C\frac{|R|-1}{\ln|R|}$$

for large R. Hence, as $R \to \infty$,

$$\begin{aligned}\Big|Rie^{i\theta}\frac{\Phi^{1/2}(\gamma+Rie^{i\theta})}{\gamma+Rie^{i\theta}}e^{-\Phi^{1/2}(\gamma+Rie^{i\theta})|x|}\Big| \\ &\le |\frac{Rie^{i\theta}}{\gamma+Rie^{i\theta}}|\cdot|\Phi^{1/2}(\gamma+Rie^{i\theta})|\cdot|e^{-\Phi^{1/2}(\gamma+Rie^{i\theta})|x|}| \\ &\le C\frac{|R|-1}{\ln|R|}\cdot e^{-C\frac{R^{\beta}-R^{\beta_0}}{\ln R}|x|}\to 0,\end{aligned}$$

which implies

$$\left|\int_{-\infty}^{+\infty}\frac{\Phi^{1/2}(\gamma+mi)}{\gamma+mi}e^{-\Phi^{1/2}(\gamma+mi)|x|}\,dm\right| \le \pi\cdot C\frac{|R|-1}{\ln|R|}\cdot e^{-C\frac{R^{\beta}-R^{\beta_0}}{\ln R}|x|}\to 0.$$

The above result and (4.11) show that

$$\lim_{t\to 0}G_{(\mu)}(x,t)=0 \text{ for } x\neq 0. \tag{4.12}$$

Now, we are in the position to calculate $\int_{-\infty}^{\infty}\lim_{t\to 0}G_{(\mu)}(x,t)\,dx$. Equation (4.10) gives

$$\begin{aligned}\int_{-\infty}^{\infty}\lim_{t\to 0}G_{(\mu)}(x,t)\,dx &= \int_{-\infty}^{0}\lim_{t\to 0}G_{(\mu)}(x,t)\,dx+\int_{0}^{\infty}\lim_{t\to 0}G_{(\mu)}(x,t)\,dx \\ &= \int_{-\infty}^{0}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Phi^{1/2}(z)}{2z}e^{-\Phi^{1/2}(z)|x|}\,dz\,dx \\ &\quad +\int_{0}^{\infty}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Phi^{1/2}(z)}{2z}e^{-\Phi^{1/2}(z)|x|}\,dz\,dx \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{-\infty}^{0}\frac{\Phi^{1/2}(z)}{2z}e^{\Phi^{1/2}(z)x}\,dx\,dz \\ &\quad +\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{0}^{\infty}\frac{\Phi^{1/2}(z)}{2z}e^{-\Phi^{1/2}(z)x}\,dx\,dz.\end{aligned}$$

Now Lemma 4.4.2 and the fact that $\operatorname{Re} z = \gamma > 0$ shows that

$$\int_{-\infty}^{0} \frac{\Phi^{1/2}(z)}{2z} e^{\Phi^{1/2}(z)x}\, dx = \frac{e^{\Phi^{1/2}(z)x}}{2z}\Big|_{-\infty}^{0} = \frac{1}{2z},$$
$$\int_{0}^{\infty} \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)x}\, dx = \frac{e^{-\Phi^{1/2}(z)x}}{2z}\Big|_{\infty}^{0} = \frac{1}{2z}.$$

Therefore, $\displaystyle\int_{-\infty}^{\infty} \lim_{t\to 0} G_{(\mu)}(x,t)\, dx = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{2z}\cdot 2\, dz = 1$, which together with (4.12) yields the conclusion. □

Lemma 4.4.7 allows us to make the definition

$$G_{(\mu)}(x,0) = \lim_{t\to 0} G_{(\mu)}(x,t) = \delta(x). \tag{4.13}$$

4.4.2 The theta functions: $\theta_\mu(x,t)$ and $\overline{\theta}_\mu(x,t)$

One very useful way to represent solutions to initial value problems for a parabolic equation is through the $\theta-$function, [56]. For the case of the heat equation if we let $K(x,t)$ denote the fundamental solution, then set $\theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m,t)$. The value of this function lies in the following result. If $u_t - u_{xx} = 0$, $u(0,t) = f_0(t)$, $u(1,t) = f_1(t)$, $u(x,0) = u_0(x)$, then $u(x,t)$ has the representation

$$\begin{aligned} u(x,t) = &\int_0^1 [\theta(x-\xi,t) - \theta(x+\xi,t)]u_0(\xi)\, d\xi \\ &- 2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau) f_0(\tau)\, d\tau + 2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau) f_1(\tau)\, d\tau. \end{aligned} \tag{4.14}$$

A generalization to the case of the fractional equation $D_t^\alpha - u_{xx} = 0$ for a fixed α, $0 < \alpha \leq 1$ can be found in [17]. Our aim is to extend this representation result to the distributed fractional order case.

Definition 4.4.1. *We define for each $\mu(\alpha)$ which satisfies Assumption 4.2.1,*

$$\theta_{(\mu)}(x,t) = \sum_{m=-\infty}^{\infty} G_{(\mu)}(x+2m,t).$$

The uniform convergence and smoothness property of $\theta_{(\mu)}(x,t)$ are established by the next lemma.

Lemma 4.4.8. *$\theta_{(\mu)}(x,t)$ is an even function on x and uniformly convergent on $(0,2)\times(0,T)$ for any positive T. Then $\theta_{(\mu)}(x,t) \in C^{\infty}((0,2)\times(0,\infty))$.*

Proof. The even symmetric property follows from the definitions of $G_{(\mu)}(x,t)$ and $\theta_{(\mu)}(x,t)$ directly.

Given a positive T, fix $(x,t)\in(0,2)\times(0,T)$, by Lemma 4.4.2 we have

$$\begin{aligned}
\sum_{|m|>N} |G_{(\mu)}(x+2m,t)| &\le \left| \frac{1}{2\pi i} \sum_{|m|>N} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{zt-\Phi^{1/2}(z)|x+2m|}\, dz \right| \\
&= \left| \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi^{1/2}(z)}{2z} \sum_{|m|>N} e^{zt-\Phi^{1/2}(z)|x+2m|}\, dz \right| \\
&\le \frac{1}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left|\frac{\Phi^{1/2}(z)}{2z}\right| e^{\gamma t} \sum_{|m|>N} e^{-\operatorname{Re}(\Phi^{1/2}(z))|x+2m|}\, dz \\
&\le \frac{1}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left|\frac{\Phi^{1/2}(z)}{2z}\right| e^{\gamma t} \sum_{|m|>N} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)||x+2m|}\, dz.
\end{aligned} \tag{4.15}$$

For the series $\sum_{|m|>N} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)||x+2m|}$, Lemma 4.4.3 shows that

$$
\begin{aligned}
&\sum_{|m|>N} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)||x+2m|} \\
&= (1-e^{-\sqrt{2}|\Phi^{1/2}(z)|})^{-1}(e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(2N+2+x)} + e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(2N+2-x)}) \\
&= \frac{e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(2N-2)}}{1-e^{-\sqrt{2}|\Phi^{1/2}(z)|}} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|}(e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(3+x)} + e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(3-x)}) \\
&\leq 2(1-e^{-\sqrt{2}(C_{\mu,\beta}\frac{\gamma^\beta-\gamma^{\beta_0}}{\ln\gamma})^{1/2}})^{-1}(e^{-\frac{\sqrt{2}}{2}(C_{\mu,\beta}\frac{\gamma^\beta-\gamma^{\beta_0}}{\ln\gamma})^{1/2}})^{2N-2} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|} \\
&\leq A_\gamma C_\gamma^{2N-2} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|}
\end{aligned}
$$

where

$$
A_\gamma = 2(1-e^{-\sqrt{2}(C_{\mu,\beta}\frac{\gamma^\beta-\gamma^{\beta_0}}{\ln\gamma})^{1/2}})^{-1}, \quad 0 < C_\gamma = e^{-\frac{\sqrt{2}}{2}(C_{\mu,\beta}\frac{\gamma^\beta-\gamma^{\beta_0}}{\ln\gamma})^{1/2}} < 1
$$

only depend on $\gamma > 0$. Inserting the above result into (4.15) yields

$$
\sum_{|m|>N} |G_{(\mu)}(x+2m,t)| \leq \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \Big|\frac{\Phi^{1/2}(z)}{2z}\Big| e^{\gamma t} A_\gamma C_\gamma^{2N-2} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|}\, dz.
$$

Meanwhile, from the proof of Lemma 4.4.5, we have

$$
\int_{\gamma-i\infty}^{\gamma+i\infty} \Big|\frac{\Phi^{1/2}(z)}{2z}\Big| e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|}\, dz < \infty.
$$

Therefore,

$$
\sum_{|m|>N} |G_{(\mu)}(x+2m,t)| \leq C C_\gamma^{2N-2}
$$

where the constant C only depends on T, γ and $0 < C_\gamma < 1$ only depends on γ. We conclude from this that for each $\epsilon > 0$, $\exists$ sufficiently large $N \in \mathbb{N}$ independent of x, t such that

$$\sum_{|m|>N} |G_{(\mu)}(x+2m,t)| < \epsilon \text{ for each } (x,t) \in (0,2)\times(0,T),$$

which implies the uniform convergence of the series. Then the smoothness results follow from Lemma 4.4.6 and the uniform convergence. □

Now we introduce the definition of $\overline{\theta}_{(\mu)}(x,t)$ and state some of its properties.

Definition 4.4.2.

$$\overline{\theta}_{(\mu)}(x,t) = \left(I^{(\mu)}\frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}\right)(x,t),\ (x,t)\in(0,2)\times(0,\infty).$$

Lemma 4.4.9. $D^{(\mu)}\theta_{(\mu)}(x,t) = (\theta_{(\mu)}(x,t))_{xx}, \quad D^{(\mu)}\overline{\theta}_{(\mu)}(x,t) = (\overline{\theta}_{(\mu)}(x,t))_{xx}$.

Proof. The first equality follows from the fact $D^{(\mu)}G_{(\mu)}(x,t) = (G_{(\mu)}(x,t))_{xx}$ and the uniform convergence of the series representation.

For the second equality, Lemma 4.2.1 yields $D^{(\mu)}\overline{\theta}_{(\mu)} = D^{(\mu)}I^{(\mu)}\frac{\partial^2\theta_{(\mu)}}{\partial t\partial x} = \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}$ and this together with the first equality and Lemma 4.4.8 then gives

$$\begin{aligned}(\overline{\theta}_{(\mu)})_{xx} &= I^{(\mu)}\frac{\partial^2}{\partial t\partial x}(\frac{\partial^2\theta_{(\mu)}}{\partial x^2}) = I^{(\mu)}\frac{\partial^2}{\partial t\partial x}D^{(\mu)}\theta_{(\mu)} = I^{(\mu)}\frac{\partial}{\partial t}D^{(\mu)}(\frac{\partial\theta_{(\mu)}}{\partial x})\\ &= \kappa * \frac{\partial}{\partial t}[\eta * \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}] = \kappa*\eta*\frac{\partial^3\theta_{(\mu)}}{\partial t^2\partial x} + \kappa*\eta\cdot\frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x,0)\\ &= \int_0^t \frac{\partial^3\theta_{(\mu)}}{\partial t^2\partial x}\,dt + \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x,0)\\ &= \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x,t) - \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x,0) + \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x,0) = \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x},\end{aligned}$$

which shows that the second equality holds. □

Lemma 4.4.10. *For each $\psi(t) \in L^2(0,\infty)$, we have*

$$\int_0^t \overline{\theta}_{(\mu)}(0+, t-s)\psi(s)\mathrm{d}s = -\frac{1}{2}\psi(t), \quad \int_0^t \overline{\theta}_{(\mu)}(1-, t-s)\psi(s)\,ds = 0,$$
$$\int_0^t \overline{\theta}_{(\mu)}(0-, t-s)\psi(s)\mathrm{d}s = \frac{1}{2}\psi(t), \quad \int_0^t \overline{\theta}_{(\mu)}(-1+, t-s)\psi(s)\,ds = 0, \quad t \in (0,\infty).$$

Proof. Fix $(x,t) \in (0,1) \times (0,\infty)$, then computing the Laplace transform yields

$$\begin{aligned}
\mathcal{L}(\overline{\theta}_{(\mu)}(x,t)) &= \mathcal{L}\Big[\kappa * \Big(\frac{\partial^2}{\partial t \partial x} \sum_{m=-\infty}^{+\infty} G_{(\mu)}(x,t)\Big)\Big] \\
&= \mathcal{L}(\kappa) \cdot \mathcal{L}\Big(\sum_{m=-1}^{-\infty} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi(z)}{2} e^{zt + \Phi^{1/2}(z)(x+2m)}\,dz \\
&\quad - \sum_{m=0}^{+\infty} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi(z)}{2} e^{zt - \Phi^{1/2}(z)(x+2m)}\,dz \Big) \\
&= \frac{1}{\Phi(z)} \Big(\sum_{m=-1}^{-\infty} \frac{\Phi(z)}{2} e^{\Phi^{1/2}(z)(x+2m)} - \sum_{m=0}^{+\infty} \frac{\Phi(z)}{2} e^{-\Phi^{1/2}(z)(x+2m)} \Big) \\
&= \frac{e^{(x-2)\Phi^{1/2}(z)} - e^{-x\Phi^{1/2}(z)}}{2(1 - e^{-2\Phi^{1/2}(z)})},
\end{aligned} \tag{4.16}$$

where the last equality follows from the fact $\mathrm{Re}(\Phi^{1/2}(z)) > 0$ which is in turn ensured by Lemma 4.4.2. Therefore,

$$\mathcal{L}\left(\int_0^t \overline{\theta}_{(\mu)}(0+, t-s)\psi(s)\,ds \right) = \mathcal{L}(\overline{\theta}_{(\mu)}(0+,t))\mathcal{L}(\psi(t)) = -\frac{1}{2}\mathcal{L}(\psi(t));$$
$$\mathcal{L}\left(\int_0^t \overline{\theta}_{(\mu)}(1-, t-s)\psi(s)\,ds \right) = \mathcal{L}(\overline{\theta}_{(\mu)}(1-,t))\mathcal{L}(\psi(t)) = 0.$$

For $(x,t) \in (-1,0) \times (0,\infty)$, we have

$$\begin{aligned}
\mathcal{L}(\overline{\theta}_{(\mu)}(x,t)) &= \frac{1}{\Phi(z)} \Big(\sum_{m=0}^{-\infty} \frac{\Phi(z)}{2} e^{\Phi^{1/2}(z)(x+2m)} - \sum_{m=1}^{+\infty} \frac{\Phi(z)}{2} e^{-\Phi^{1/2}(z)(x+2m)} \Big) \\
&= \frac{e^{x\Phi^{1/2}(z)} - e^{-(x+2)\Phi^{1/2}(z)}}{2(1 - e^{-2\Phi^{1/2}(z)})},
\end{aligned}$$

which gives $\mathcal{L}(\overline{\theta}_{(\mu)}(0-,t)) = \frac{1}{2}$ and $\mathcal{L}(\overline{\theta}_{(\mu)}(-1+,t)) = 0$, and completes the proof. $\square$

4.4.3 Representation of the solution to the initial-boundary value problem

We will build the representation of the solution in this subsection from four representations in terms of the theta functions; the initial condition, the values of u at each boundary $x = 0$, $x = 1$, and the nonhomogeneous term f.

Definition 4.4.3.

$$\begin{aligned}
u_1(x,t) &= \int_0^1 (\theta_{(\mu)}(x-y,t) - \theta_{(\mu)}(x+y,t))u_0(y)\,dy;\\
u_2(x,t) &= -2\int_0^t \overline{\theta}_{(\mu)}(x,t-s)g_0(s)\,ds;\\
u_3(x,t) &= 2\int_0^t \overline{\theta}_{(\mu)}(x-1,t-s)g_1(s)\,ds;\\
u_4(x,t) &= \int_0^1\int_0^t [\theta_{(\mu)}(x-y,t-s) - \theta_{(\mu)}(x+y,t-s)]\cdot[\frac{\partial}{\partial t}I^{(\mu)}f(y,s)]\,ds\,dy.
\end{aligned}$$

The following four lemmas give some properties of $u_j,\ j = 1,2,3,4$.

Lemma 4.4.11. $D^{(\mu)}u_j = \dfrac{\partial^2 u_j}{\partial x^2},\ j = 1,2,3,$ $D^{(\mu)}u_4 = \dfrac{\partial^2 u_4}{\partial x^2} + f(x,t)$, *where* $(x,t) \in (0,1)\times(0,\infty)$.

Proof. For u_1, by Lemma 4.4.9, we have

$$\begin{aligned}
D^{(\mu)}u_1 &= \int_0^1 (D^{(\mu)}\theta_{(\mu)}(x-y,t) - D^{(\mu)}\theta_{(\mu)}(x+y,t))u_0(y)\,dy \\
&= \int_0^x (D^{(\mu)}\theta_{(\mu)}(x-y,t) - D^{(\mu)}\theta_{(\mu)}(x+y,t))u_0(y)\,dy \\
&\quad + \int_x^1 (D^{(\mu)}\theta_{(\mu)}(x-y,t) - D^{(\mu)}\theta_{(\mu)}(x+y,t))u_0(y)\,dy \\
&= \int_0^x \Big[\theta_{(\mu)}(x-y,t) - \theta_{(\mu)}(x+y,t)\Big]_{xx} u_0(y)\,dy \\
&\quad + \int_x^1 \Big[\theta_{(\mu)}(x-y,t) - \theta_{(\mu)}(x+y,t)\Big]_{xx} u_0(y)\,dy \\
&= \int_0^1 \Big[\theta_{(\mu)}(x-y,t) - \theta_{(\mu)}(x+y,t)\Big]_{xx} u_0(y)\,dy = \frac{\partial^2 u_1}{\partial x^2}.
\end{aligned}$$

For u_2,

$$\begin{aligned}
D^{(\mu)}u_2 &= \eta * \frac{\partial u_2}{\partial t} = -2\eta * \frac{\partial}{\partial t}(\overline{\theta}_{(\mu)} * g_0) = -2\eta * (\frac{\partial}{\partial t}\overline{\theta}_{(\mu)}) * g_0 - 2(\eta * g_0)\cdot \overline{\theta}_{(\mu)}(x,0) \\
&= -2D^{(\mu)}\overline{\theta}_{(\mu)} * g_0 = -2(\overline{\theta}_{(\mu)})_{xx} * g_0 = (-2\overline{\theta}_{(\mu)} * g_0)_{xx} = (u_2)_{xx}.
\end{aligned}$$

In an analogous fashion to the above argument, we deduce that $D^{(\mu)}u_3 = (u_3)_{xx}$.

For u_4, using Lemmas 4.4.7, 4.2.1 and 4.4.8 we obtain

$$\begin{aligned}
D^{(\mu)}u_4 &= \eta * \frac{\partial u_4}{\partial t} = \eta * \frac{\partial}{\partial t}\Big(\int_0^1 [\theta_{(\mu)}(x-y,\cdot) - \theta_{(\mu)}(x+y,\cdot)] * [\frac{\partial}{\partial t}I^{(\mu)}f(y,\cdot)]\,dy\Big) \\
&= \int_0^1 D^{(\mu)}[\theta_{(\mu)}(x-y,\cdot) - \theta_{(\mu)}(x+y,\cdot)] * [\frac{\partial}{\partial t}I^{(\mu)}f(y,\cdot)]\,dy + \eta * \frac{\partial}{\partial t}I^{(\mu)}f(x,t) \\
&= \int_0^1 [\theta_{(\mu)}(x-y,\cdot) - \theta_{(\mu)}(x+y,\cdot)]_{xx} * [\frac{\partial}{\partial t}I^{(\mu)}f(y,\cdot)]\,dy + D^{(\mu)}I^{(\mu)}f(x,t) \\
&= (u_4)_{xx} + f(x,t).
\end{aligned}$$

□

Lemma 4.4.12. $\lim_{t\to 0} u_1(x,t) = u_0(x)$, $\lim_{t\to 0} u_j(x,t) = 0$ *for* $j = 2,3,4$, $x \in (0,1)$.

Proof. For each $x \in (0,1)$, Lemmas 4.4.8 and 4.13 yield that

$$\begin{aligned}\lim_{t\to 0} u_1 &= \int_0^1 (\theta_{(\mu)}(x-y,0) - \theta_{(\mu)}(x+y,0))u_0(y)\,dy \\ &= \int_0^1 \sum_{m=-\infty}^{\infty} (\delta(x-y+2m) - \delta(x+y+2m))u_0(y)\,dy \\ &= \int_0^1 \delta(x-y)u_0(y)\,dy = u_0(x).\end{aligned}$$

The other result follows directly from the definitions of u_2, u_3 and u_4. □

Lemma 4.4.13. *$u_j(0,t) = u_j(1,t) = 0$, for $j = 1,4$ and $t \in (0,\infty)$.*

Proof. Since $\theta_{(\mu)}(x,t)$ is even on x which is stated in Lemma 4.4.8, then

$$u_1(0,t) = \int_0^1 (\theta_{(\mu)}(0-y,t) - \theta_{(\mu)}(0+y,t))u_0(y)\,dy = 0.$$

We also have

$$\begin{aligned}u_1(1,t) &= \int_0^1 (\theta_{(\mu)}(1-y,t) - \theta_{(\mu)}(1+y,t))u_0(y)\,dy \\ &= \int_0^1 (\theta_{(\mu)}(y-1,t) - \theta_{(\mu)}(1+y,t))u_0(y)\,dy \\ &= \int_0^1 \Big[\sum_{m=-\infty}^{\infty} G_{(\mu)}(y-1+2m,t) - \sum_{m=-\infty}^{\infty} G_{(\mu)}(y+1+2m,t)\Big]u_0(y)\,dy \\ &= \int_0^1 \Big[\sum_{q=-\infty}^{\infty} G_{(\mu)}(y+1+2q,t) - \sum_{m=-\infty}^{\infty} G_{(\mu)}(y+1+2m,t)\Big]u_0(y)\,dy = 0,\end{aligned}$$

where $q = m-1$.

Following from the above proof, we obtain the conclusion for u_4. □

Lemma 4.4.14. *$u_2(0,t) = g_0(t)$, $u_2(1,t) = 0$, $u_3(0,t) = 0$, $u_3(1,t) = g_1(t)$, for $t \in (0,\infty)$.*

Proof. The proof follows from Lemma 4.4.10 directly. □

Now we can state the representation theorem.

Theorem 4.4.4 (Representation theorem)**.** *There exists a unique solution $u(x,t)$ of Equations* (4.8)*, which has the representation* $u(x,t) = \sum_{j=1}^{4} u_j$.

Proof. The existence follows from Lemmas 4.4.11, 4.4.12, 4.4.13 and 4.4.14; while the uniqueness is ensured by Corollary 4.3.1. □

4.5 Determining the distributed coefficient $\mu(\alpha)$

In this section we state and prove two uniqueness theorems for the recovery of the distributed derivative μ. We show that by measuring the solution along a time trace from a fixed location x_0 one can use this data to uniquely recover $\mu(\alpha)$. This time trace can be one where the sampling point is located within the interior of $\Omega = (0,1)$ and we measure $u(x_0,t)$, or we measure the flux at $x^\star$; $u_x(x^\star,t)$ where $0 < x^\star \leq 1$. This latter case therefore includes measuring the flux on the right-hand boundary $x = 1$.

First we give the definition of the admissible set Ψ according to Assumption 4.2.1.

Definition 4.5.1. *Define the set* Ψ *by*

$$\Psi := \{\mu \in C^1[0,1] : \ \mu \geq 0, \ \mu(1) \neq 0, \ \mu(\alpha) \geq C_\Psi > 0 \text{ on } (\beta_0, \beta_1)\},$$

where the constant $C_\Psi > 0$ *and the interval* $(\beta_0, \beta_1) \subset (0,1)$ *only depend on* Ψ.

We introduce the functions $F(y;x_0)$ and $F_f(y;x^\star)$ in the next two lemmas.

Lemma 4.5.1. *Define the function* $F(y;x_0) \in C^1((0,\infty),\mathbb{R})$ *as*

$$F(y;x_0) = \frac{e^{(x_0-2)y} - e^{-x_0 y}}{2(1-e^{-2y})},$$

where $x_0 \in (0,1)$ *is a constant. Then the function* $F(y;x_0)$ *is strictly increasing on the interval* $(\frac{\ln(2-x_0) - \ln x_0}{2(1-x_0)}, \infty) \subset (0,\infty)$.

Proof. Since $x_0 \in (0,1)$, $e^{(x_0-2)y} - e^{-x_0 y} < 0$ and $2(1-e^{-2y}) > 0$ on $(0,\infty)$. A direct calculation now yields

$$\frac{d}{dy}(e^{(x_0-2)y} - e^{-x_0 y}) = (x_0-2)e^{(x_0-2)y} + x_0 e^{-x_0 y} > 0$$

for $y \in (\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty)$. Then we have $e^{(x_0-2)y} - e^{-x_0 y} < 0$ and strictly increasing on $(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty)$. The function $2(1-e^{-2y})$ is obviously both positive and strictly increasing on $(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty)$. Hence the function $F(y;x_0)$ is also strictly increasing on $(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty)$, which completes the proof. □

Lemma 4.5.2. *For the inverse problem with flux data, define the function* $F_f(y;x^\star) \in C^1((0,\infty),\mathbb{R})$ *as*

$$F_f(y;x^\star) = \frac{ye^{(x^\star-2)y} + ye^{-x^\star y}}{2(1-e^{-2y})},$$

where $x^\star \in (0,1]$ *is a constant. Then the function* $F_f(y;x^\star)$ *is strictly decreasing on the interval* $(1/x^\star,\infty) \subset (0,\infty)$.

Proof.

$$\begin{aligned}\frac{\partial F_f}{\partial y}(y;x^\star) = &\frac{((x^\star-2)y+1)e^{(x^\star-2)y} + (1-x^\star y)e^{-x^\star y}}{2(1-e^{-2y})^2} \\ &+ \frac{(-x^\star y-1)e^{(x^\star-4)y} + ((x^\star-2)y-1)e^{(-x^\star-2)y}}{2(1-e^{-2y})^2},\end{aligned}$$

hence $\frac{\partial F_f}{\partial y}(y;x^\star) < 0$ if $y \in (1/x^\star,\infty)$ and the proof is complete. □

For the important lemmas to follow, we need the Stone–Weierstrass and the Müntz–Szász Theorems. See the appendix for statements and references for these results.

The next result shows that the set $\{(nr)^x : n \in \mathbb{N}^+\}$ is complete in $L^2[0,1]$ for any positive integer r. We give two proofs of this important lemma.

Lemma 4.5.3. *For each $r \in \mathbb{N}^+$, the vector space consisting with the set of functions $\{(nr)^x : n \in \mathbb{N}^+\}$ is dense in the space $L^2[0,1]$, i.e.*

$$\overline{span\{(nr)^x : n \in \mathbb{N}^+\}} = L^2[0,1]$$

w.r.t L^2 norm. In other words, the set $\{(nr)^x : n \in \mathbb{N}^+\}$ is complete in $L^2[0,1]$.

Proof. Clearly, $span\{(nr)^x : n \in \mathbb{N}^+\}$ satisfies all the conditions of the Stone–Weierstrass Theorem, so that the closure of $span\{(nr)^x : n \in \mathbb{N}^+\}$ w.r.t the continuous norm is either $C[0,1]$ or $\{f \in C[0,1] : f(x_0) = 0, x_0 \in [0,1]\}$. The two alternatives both yield that $span\{(nr)^x : n \in \mathbb{N}^+\}$ is dense in $C[0,1]$ with respect to the L^2 norm, which together with the fact $C[0,1]$ is dense in $L^2[0,1]$ gives $span\{(nr)^x : n \in \mathbb{N}^+\}$ is dense in $L^2[0,1]$ and completes the proof.

As a second proof, if for some $h \in C[0,1]$, $\int_0^1 (nr)^x h(x)\,dx = 0$ for all $n \in \mathbb{N}^+$ then $\int_0^1 e^{x\log(rn)} h(x)\,dx = 0$ and with the change of variables $y = e^x$ this becomes $\int_1^e y^{\log(rn)} \tilde{h}(y)\,dy = 0$ for all $n \in \mathbb{N}^+$ where $\tilde{h}(y) = h(\log(y))/y$. Since $\sum_{n=1}^{\infty} 1/\log(rn)$ diverges, the Müntz-Szász theorem shows that $\tilde{h} = 0$ and hence $h(x) = 0$. □

We now have the main result of this paper.

Theorem 4.5.2 (Uniqueness theorem for the inverse problem)**.** *In the DDE* (4.8)*, set $u_0 = g_1 = f = 0$ and let g_0 satisfy the following condition*

$$(\mathcal{L}g_0)(z) \neq 0 \textit{ for } z \in (0,\infty).$$

Given μ_1, $\mu_2 \in \Psi$, denote the two weak solutions with respect to μ_1 and μ_2 by $u(x,t;\mu_1)$ and $u(x,t;\mu_2)$ respectively. Then for any $x_0 \in (0,1)$ and $x^\star \in (0,1]$, either

$$u(x_0,t;\mu_1) = u(x_0,t;\mu_2)$$

or

$$\frac{\partial u}{\partial x}(x^\star, t; \mu_1) = \frac{\partial u}{\partial x}(x^\star, t; \mu_2),\ t \in (0, \infty)$$

implies $\mu_1 = \mu_2$ *on* $[0, 1]$.

Proof. For the first case of $u(x_0, t; \mu_1) = u(x_0, t; \mu_2)$, fix $x_0 \in (0, 1)$, Theorem 4.4.4 yields for $k = 1,\ 2$:

$$u(x_0, t; \mu_k) = -2 \int_0^t \overline{\theta}_{(\mu_k)}(x_0, t - s) g_0(s)\, ds, \qquad k = 1,\ 2$$

which implies

$$\int_0^t \overline{\theta}_{(\mu_1)}(x_0, t - s) g_0(s)\, ds = \int_0^t \overline{\theta}_{(\mu_2)}(x_0, t - s) g_0(s)\, ds.$$

Taking the Laplace transform in t on both sides of the above equality gives

$$\Big(\mathcal{L}(\overline{\theta}_{(\mu_1)}(x_0, \cdot))\Big)(z) \cdot (\mathcal{L} g_0)(z) = \Big(\mathcal{L}(\overline{\theta}_{(\mu_2)}(x_0, \cdot))\Big)(z) \cdot (\mathcal{L} g_0)(z).$$

Since $(\mathcal{L} g_0)(z) \neq 0$ on $(0, \infty)$, so that

$$\Big(\mathcal{L}(\overline{\theta}_{(\mu_1)}(x_0, \cdot))\Big)(z) = \Big(\mathcal{L}(\overline{\theta}_{(\mu_2)}(x_0, \cdot))\Big)(z),\ \text{for } z \in (0, \infty).$$

This result and (4.16) then give

$$\frac{e^{(x_0 - 2)\Phi_1^{1/2}(z)} - e^{-x_0 \Phi_1^{1/2}(z)}}{2(1 - e^{-2\Phi_1^{1/2}(z)})} = \frac{e^{(x_0 - 2)\Phi_2^{1/2}(z)} - e^{-x_0 \Phi_2^{1/2}(z)}}{2(1 - e^{-2\Phi_2^{1/2}(z)})},\ z \in (0, \infty),$$

where

$$\Phi_j(z) = \int_0^1 \mu_j(\alpha) z^\alpha \mathrm{d}\alpha, \quad j = 1, 2.$$

The definition of Ψ and the fact $z \in (0,\infty)$ yield $\Phi_j^{1/2}(z) \in (0,\infty)$ and hence we can rewrite the above equality as

$$F(\Phi_1^{1/2}(z); x_0) = F(\Phi_2^{1/2}(z); x_0),\ z \in (0,\infty), \tag{4.17}$$

where the function F comes from Lemma 4.5.1.

Since $x_0 \in (0,1)$, it is obvious that $\dfrac{\ln(2-x_0)-\ln x_0}{2(1-x_0)} > 0$. Then we can pick a large $N^* \in \mathbb{N}^+$ such that

$$\int_{\beta_0}^{\beta_1} C_\Psi \cdot (N^*)^\alpha \mathrm{d}\alpha > \left(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}\right)^2,$$

which together with the definition of Ψ gives that for each $z \in (0,\infty)$ with $z \geq N^*$, $\Phi_j(z) \in (0,\infty)$ and

$$\Phi_j^{1/2}(z) > \frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \quad j = 1,2.$$

This result means that

$$\Phi_j^{1/2}(nN^*) > \frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)},\ j = 1,2,\ n \in \mathbb{N}^+. \tag{4.18}$$

Lemma 4.5.1 shows that $F(\cdot; x_0)$ is strictly increasing on the interval $\left(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty\right)$, which together with (4.17) and (4.18) yields

$$\Phi_1^{1/2}(nN^*) = \Phi_2^{1/2}(nN^*),\ n \in \mathbb{N}^+,$$

that is $\Phi_1(nN^*) = \Phi_2(nN^*),\ n \in \mathbb{N}^+$, sequentially, we have

$$\int_0^1 (\mu_1(\alpha) - \mu_2(\alpha))(nN^*)^\alpha \mathrm{d}\alpha = 0,\ n \in \mathbb{N}^+.$$

We can rewrite the above result as $\langle \mu_1(\alpha) - \mu_2(\alpha), (nN^*)^\alpha \rangle = 0$ for $n \in \mathbb{N}^+$. From the completeness of $\{(nN^*)^\alpha : n \in \mathbb{N}^+\}$ in $L^2[0,1]$ which is ensured by Lemma 4.5.3, we have $\mu_1 - \mu_2 = 0$ in $L^2[0,1]$, that is, $\|\mu_1 - \mu_2\|_{L^2[0,1]} = 0$, which together with the continuity of μ_1 and μ_2 shows that $\mu_1 = \mu_2$ on $[0,1]$.

For the case of $\frac{\partial u}{\partial x}(x^\star, t; \mu_1) = \frac{\partial u}{\partial x}(x^\star, t; \mu_2)$, following (4.16) we have

$$\begin{aligned}
&\mathcal{L}\left(\frac{\partial \bar{\theta}_{(\mu)}}{\partial x}(x,t)\right) = \mathcal{L}\left[\kappa * \left(\frac{\partial^3}{\partial t \partial x^2} \sum_{m=-\infty}^{\infty} G_{(\mu)}(x,t)\right)\right] \\
&= \mathcal{L}\left[\kappa * \mathcal{L}^{-1}\left(\sum_{m=-1}^{-\infty} \frac{\Phi^{3/2}(z)}{2} e^{\Phi^{1/2}(z)(x+2m)} \mathrm{d}z + \sum_{m=0}^{\infty} \frac{\Phi^{3/2}(z)}{2} e^{-\Phi^{1/2}(z)(x+2m)}\right)\right] \\
&= \frac{1}{\Phi(z)}\left(\sum_{m=-1}^{-\infty} \frac{\Phi^{3/2}(z)}{2} e^{\Phi^{1/2}(z)(x+2m)} + \sum_{m=0}^{\infty} \frac{\Phi^{3/2}(z)}{2} e^{-\Phi^{1/2}(z)(x+2m)}\right) \\
&= \frac{\Phi^{1/2}(z) e^{(x-2)\Phi^{1/2}(z)} + \Phi^{1/2}(z) e^{-x\Phi^{1/2}(z)}}{2(1 - e^{-2\Phi^{1/2}(z)})}.
\end{aligned}$$

Following the proof for the case $u(x_0, t; \mu_1) = u(x_0, t; \mu_2)$, we can deduce $\mu_1 = \mu_2$ from the above result and Lemmas 4.5.2 and 4.5.3. □

Remark 4.5.1. In this paper we have considered only the uniqueness question for the function $\mu(\alpha)$. Certainly, one would like to know under what conditions this function can be effectively recovered from the given data. Clearly this is an important question, but we caution there are many difficulties, especially with a mathematical analysis of the stability issue of μ in terms of the overposed data either $u(x_0, t)$ or $\frac{\partial u}{\partial x}(x^\star, t)$. One can certainly employ the representation result of section 4.4 to obtain a nonlinear integral equation for μ but the analysis of this is unclear. An alternative approach would be restrict the function

μ as in Lemma 4.2.1 to ensure that κ is completely monotone and hence use Bernstein's theorem to obtain an integral representation for this function. We hope to address some of these questions in subsequent work.

5. SUMMARY AND CONCLUSIONS

5.1 Challenges

To analyze the classic diffusion equations, we need to use some basic tools in calculus, such as product rule, chain rule and integration by parts. However, theses tools do not work in the fractional case. That means we can not just follow the popular methods from the classical case, but need to create some new approaches to obtain the desired results. This is the main challenge I met in the research process of fractional diffusion equations.

5.2 Further study

5.2.1 Identification of a discontinuous source

The model we consider is

$$\begin{cases} {}^C D_t^\alpha u - \triangle u = \chi_{_D}, & (x,t) \in \Omega \times (0,T); \\ u(x,0) = 0, & x \in \Omega; \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{5.1}$$

where D is a domain contained in $\Omega \subseteq \mathbb{R}^2$. The inverse problem we consider is to recover the domain D with $\overline{D} \subseteq \Omega$ from knowing finite flux data

$$\frac{\partial u}{\partial \overrightarrow{\mathbf{n}}}(z_l, t) = g_l(t),\ t \in [0,T],\ l = 1, \dots, m,$$

where u is the solution of FDE (5.1). The case for $\alpha = 1$ has been done in [**?**]. In fractional case, I will show show D is uniquely determined by some specific finite flux data at first, then use the regularized Newton-type methods to reconstruct the domain D numerically.

5.2.2 Inverse source problems in the space-fractional differential equation

In addition, I will work on the following equation

$$({}^{C}D_t^{\alpha}u - {}_0^{C}D_x^{\beta})u = F(x,t,u), \quad 0 \le x \le 1,\ t > 0,\ 0 < \alpha < 1,\ 1 < \beta < 2.$$

Results, including the existence, uniqueness and regularity of the solutions, for the direct problems, are still incomplete. There are considerable difficulties to be faced here. Our goal is to obtain sufficient results to attempt similar inverse problems as noted previously for the case $\beta = 2$.

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APPENDIX A

The uniqueness proof in section 4.5 requires results on the density of a certain subset of functions and we give two ways to look at this through different formulations; namely the Stone-Weierstrass and Müntz-Szász theorems. We give the statements of these results below.

The Stone-Weierstrass theorem is a generalization of Weierstrass' result of 1885 that the polynomials are dense in $C[0,1]$ and was proved by Stone some 50 years later, [58]. If X is a compact Hausdorff space and $C(X)$ those real-valued continuous functions on X, with the topology of uniform convergence, then the question is when is a subalgebra $A(X)$ dense? A crucial notion is that of separation of points; a set A of functions defined on X is said to *separate points* if, for every $x, y \in X$, $x \neq y$, there exists a function $f \in A$ such that $f(x) \neq f(y)$. Then we have

Theorem 5.2.1. *(Stone–Weierstrass). Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X)$ which contains a non-zero constant function. Then A is dense in $C(X)$ if and only if it separates points.*

The proof can be found in standard references, for example, [59, Theorem 4.45].

The Müntz-Szász theorem, (1914-1916) is also a generalization of the Weierstrass approximation theorem; it gives a condition under which one can "thin out" the polynomials and still maintain a dense set.

Theorem 5.2.2. *(Müntz–Szász) Let $\Lambda := \{\lambda_j\}_1^\infty$ be a sequence of real positive numbers. Then the span of $\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0,1]$ if and only if $\sum_1^\infty \frac{1}{\lambda_j} = \infty$.*

This result can be generalized to the $L^p[0,1]$ spaces for $1 \leq p \leq \infty$, see [60].