TWO ESSAYS ON THE AVERAGE BID AUCTION

A Dissertation

by

JIN HYUNG LEE

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Silvana Krasteva
Co-Chair of Committee,	Vikram Manjunath
Committee Members,	Haeshin Hwang
	Marco A. Palma
Head of Department,	Timothy Gronberg

August 2017

Major Subject: Economics

ABSTRACT

The dissertation investigates an *average bid auction*, in which the bid closest to the average wins the auction. In the first study, I investigate the equilibrium bidding strategy in the *average bid auction*, when cost constitutes a private value without the winner's curse. More specifically, in addition to the pooling equilibrium established by the existing literature, I characterize a partially pooling equilibrium with three bidders, in which the bid function is constant for small cost realizations and strictly increasing in cost for high cost realizations. Moreover, complete necessary conditions for the existence of partially pooling equilibrium are provided in the case of more than three bidders. Lastly, unlike the *first price auction*, the *average bid auction* loses efficiency, and the buyer is likely to pay the more compared to *first price auction*.

The second study investigates the characterization of the equilibrium bid function with three bidders in the *average bid auction*, considering the possibility of cost overrun and penalty. When the penalty is zero, every bidder bids an identical amount. If the penalty is large, either some bidders bid identically and the rest of bidders follow a strictly increasing bidding strategy, or all bidders place an identical bid. Finally, I compare the *average bid auction* with the *first price reverse auction* from the buyer's point of view. If the penalty is not imposed on insolvent bidders, the buyer who chooses the average bid auction could be better off. On the other hand, if the penalty is large enough to prevent all bidders from defaulting, the average bid auction is always worse. This result partly explains why the average bid auction has been widespread in Italy, and the *first price reverse auction* has been dominantly used for the procurement in the U.S.

DEDICATION

To my mother and my father.

ACKNOWLEDGMENTS

I would like to express my special appreciation to Dr. Silvana Krasteva and Dr. Vikram Manjunath. You have been tremendous mentors for me. I would like to thank you for encouraging my research and for allowing me to grow as an economist. Your advice on my research as well as on my career has been priceless. I would also like to thank Dr. Haeshin Hwang and Dr. Marco A. Palma for serving as my committee members, even at hardship. I appreciate you for letting my defense be an enjoyable moment, and for your brilliant comments and suggestions.

At the end I would like to give special thanks to my family as well. Words cannot express how grateful I am to my mother, Hyunsook Han. Your whole life has motivated me to make a ceaseless effort.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a dissertation committee consisting of Dr. Silvana Krasteva and Dr. Vikram Majunath and Dr. Haeshin Hwang of the Department of Economics and Dr. Marco A. Palma of the Department of Agricultural Economics.

All other work conducted for the dissertation was completed by the student independently.

Funding Sources

No outside funding was received for this work.

TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

1.1 Introduction

This dissertation examines the reverse auction, where many sellers compete to provide services or goods to a single buyer. A good example of a reverse auction is Priceline.com's "Name Your Own Price" system. The reverse auction has become an important acquisition method for both government and private businesses. It provides convenience to the buyer in many ways. For example, since bidders are forced to spontaneously reveal their cost types through competition, the reverse auction method resolves the asymmetric information problem between the buyer and the bidders. The reverse auction, though, has not attracted wide attention from economists, so it is difficult for buyers to find useful guidance about which auction format is best given a certain economic environment. A series of my studies deals with this issue as it relates to the reverse auction. The objective of my studies is to provide buyers with useful guidance about which reverse auction format is beneficial, given their economic environments. From a cost-savings standpoint, the best choice for the buyer seems to always be the *first price reverse auction* in which the lowest bid is the winner.

If the winning bidder shows any likelihood of defaulting, however, the *first price reverse auction* may not be the best. According to Decarolis (2009, 2013), for construction projects in Italy where insolvent bidders might default, the prevalent auction is the *average bid auction*. In this type of auction, the winning bid is the closest to the average of all submitted bids. In contrast, the prevalent auction type in the U.S. (with its strong contract enforcement system) is the *first price reverse auction*. The difference between the U.S. and Italy is intriguing and raises questions about the impact of a certain economic environment on bidding strategy and a buyer's preference of auction formats. Furthermore, the *average* *bid auction* is not well known to economists, despite its frequent use in countries such as Italy, the U.S, and China according to Decarolis (2009) and Engel (2009). Additionally, Decarolis (2013) and Spagnolo, Albano and Bianchi (2006) deal with equilibrium bidding strategies, but they focus on a pooling equilibrium. Thus, how this mechanism works remains ambiguous.

In Chapter 2, I characterize a partially pooling equilibrium with three bidders, when cost constitutes a private value without default of the winner. I also investigate the complete necessary conditions for the existence of partially pooling equilibrium with more than three bidders. Lastly, I compare the *first price reverse auction* with the *average bid auction* for the buyer. When there is not the possibility of default, the auctioned item is the most likely to be allocated to the lowest cost bidder through the *first price reverse auction*. In the *average bid auction*, in contrast, the winner is determined from among bids closest to the mean value; thus, this auction loses efficiency, and the buyer is more likely to pay more for the winner than the buyer in the *first price reverse auction*.

Chapter 3 introduces environmental uncertainty which implies the default risk of the winner. The buyer should choose the auction format that minimizes both the bid price and the default risk. Cost overruns could cause the realized cost to exceed the bid compelling the winner to default. To reduce the risk of default, the buyer could impose a penalty on a winner who breaches the contract. The winner should then consider the loss associated with carrying out the contract and adding on the default penalty, and thus being discouraged from refusing to uphold the contract. When no penalty is imposed, and the project value is high enough, the better choice for the buyer is the *average bid auction*. When the penalty is sufficiently high, the better choice for the buyer is always the *first price reverse auction*. The winning bid in the *average bid auction* is likely to be higher than that in the *first price reverse auction*; hence, the winner is likely to be rewarded with a larger profit margin and lower default risk. If the project value is sufficiently large, and the penalty is negligible, the benefit from reducing the default is greater than the cost induced by the high bid price. Thus, the *average bid auction* could be better for the buyer. However, in the case of a sufficiently large penalty with nobody defaulting, the *first price reverse auction* regains its efficiency.

2. COMPETITIVE BIDDING BEHAVIOR IN THE AVERAGE BID AUCTION

2.1 Introduction

A reverse auction is the mechanism by which sellers compete to provide goods or services to a single buyer. They are commonly used by both government and private business in order to save on procurement costs. According to the U.S. Government Accountability Office Government Accountability Office (2013), even though only seven percent of all commercial acquisitions by four agencies[1] were via reverse auction (three times as much as in 2008), they were worth $828 million. Furthermore, the Government Accountability Office (2013) reported that half of the 24 agencies involved in the Office of Management and Budget (OMB) have adopted reverse auctions for the purpose of cost savings.

Of all reverse auctions, the *first price reverse auction* (FPA) has been widely used for procurement. Since the auctioned item is acquired from the bidder with the lowest cost through FPA, it is effective in reducing procurement costs. However, FPA is not the only reverse auction that is observed in practice. An alternative is the *average bid auction* (ABA).

In an *average bid auction* ABA, the bid closest to the average of all bids is the winner. Thus, ABA does not guarantee that the lowest cost bidder wins the auction. Nonetheless, ABA is used in several countries. According to Decarolis (2009, 2013) and Decarolis and Giorgiantonio (2014, 2015), ABA was the prevalent mechanism for the public procurement in Italy before 2006. ABA has also been used in other countries such as Chile, China, Peru, and Taiwan (Engel (2009)). Even in the United States, the states of Florida and New York have used ABA (Spagnolo, Albano and Bianchi (2006)).

Despite being used in the real world, the ABA has not attracted wide attention from

[1] Army, Department of Homeland Security (DHS), Department of the Interior (DOI), Department of Veterans Affairs (VA)

economists. Since its use is mainly limited to construction projects, it has been studied in the engineering literature (Ioannou and Leu (1993)). [2] In contrast to this work, they assume that there are no tie bids, and the equilibrium bid is strictly increasing in cost. However, the average is likely to be between bid of potential bidder with the lowest cost and bid of the possible bidder with the highest cost under this assumption, and the possible bid with the lowest cost and his neighbors have incentive to deviate from their bids. Thus, there is no reason for these bidders to follow a strictly increasing bidding strategy.

Spagnolo, Albano and Bianchi (2006) have considered reasons for using ABA. They offer the explanation that ABA is effective in softening price competition as a foundation for the usage of ABA. However, they only investigate the special case of three bidders with known costs only among bidders. Additionally, they do not fully characterize the equilibrium in this special environment. Instead, they focus on one specific type of equilibrium (pooling). On the other hand, Decarolis (2009, 2013) deals with the equilibrium bid in ABA with at least three bidders, and privately known costs. However, his necessary conditions for partially pooling equilibrium is limited in that the shape of the equilibrium bid function is vague, and the possible maximum value of the function is not clearly articulated. Furthermore, he does not detect the possibility to characterize equilibrium, considering the tractability of the equilibrium bid function according to the number of bidders. [3]

In order to explore the rationale for ABA and make comparative analysis with FPA, the first step is to fully characterize the equilibria. As mentioned above, the literature concentrates on one specific type of equilibrium (pooling) and does not provide full characterization of the equilibrium set. Accordingly, the purpose of this study is to fully characterize

[2] They define the winning probability as follows: $Pr(win|b_i) = Pr(|\bar{b} - b_i| < |\bar{b} - b_j|, \forall j \neq i)$. They only use the strictly inequality instead of $|\bar{b} - b_i| \leq |\bar{b} - b_j|$. The strictly inequality $|\bar{b} - b_i| < |\bar{b} - b_j|$ implies that an equilibrium bid function is strictly increasing over cost.

[3] For three bidders, we characterize equilibrium since the differential equation of equilibrium bid function is tractable. However, for at least four bidders, we can not fully characterize equilibrium.

the equilibrium bid function in the ABA. We do so for three bidders.

First, this study provides a complete set of necessary conditions for the existence of the partially pooling equilibrium bid function [4] with at least three bidders. In particular, partially pooling equilibrium bid functions have the following shape: The bid is a constant up to a threshold cost $\widetilde{c}$, and strictly increasing after $\widetilde{c}$, with a possible jump at $\widetilde{c}$. A bidder with cost $\widetilde{c}$ is indifferent between the constant bid and the higher bid prescribed by the increasing portion of the bid function. Finally the highest bid equals the highest possible cost $\overline{c}$. The potential discontinuity in the equilibrium bid function arises for the reasons provided by Chen and Chiu (2011). There are typically multiple equilibria, one for each choice of $\widetilde{c}$ in a particular range. The only case where any bidder bids above $\overline{c}$ is in a pooling equilibrium. Thus, there are the only two types of equilibria-pooling and partially pooling. Still, for the general case with more than three bidders, a closed form solution of every possible equilibrium bid function is not tractable.

We do, however, provide a full characterization of the equilibrium bid function for three bidders. Despite being a special case, it yields an important insight about the equilibrium characterization for more than three bidders. [5]

Finally, we provide a theoretical foundation for determining that FPA yields a higher cost savings than ABA, when the ex-ante cost is equivalent to the ex-post cost.

This paper is organized as follows. In Section 2.2, we introduce the economic environment. In Section 2.3, we provide necessary conditions on the equilibrium bid functions for at least three bidders and a full characterization for three bidders. In Section 2.4, this full characterization plays a pivotal role in performing the comparative analysis between

[4] In a partially pooling equilibrium, some potential bidders adopt an identical bidding strategy, and the rest of possible bidders employ different bidding strategies according to their cost types.

[5] As you may see later, the shape of equilibrium bid function with more than three bidders is the same as the case with three bidders. Thus, at least, we can assert that the shape of equilibrium bid function is unique. If the shape is unique, bidding behaviours for at least four bidders are supposed to be similar to the case with three bidders.

FPA and ABA in terms of the expected payoff of the buyer. We conclude in Section 2.5.

2.2 The Model

A set of bidders seeks to obtain a contract to provide either a good or a service to a buyer. Let n be the number of bidders and assume that $n \geq 3$. The i^{th} bidder incurs a cost c_i for each $i \in \{1, \cdots, n\}$. While a bidder i knows c_i, he does not know c_j for any other bidder j. Assume that costs are independently and identically distributed according to a cumulative distribution function F associated with a density $f > 0$ on the interval $[\underline{c}, \overline{c}]$. We consider sealed bid auctions where each bidder i submits a bid $b_i \in \mathbb{R}_+$. Denote a profile of submitted bids by $\mathbf{b} \in \mathbb{R}^n_+$. Given $\mathbf{b}$, let $w(\mathbf{b})$ be the set of winning bidders. For a first price auction (FPA), $w(\mathbf{b})$ consists of those bidders who submit the lowest bids. For the average bid auction (ABA), it is the set of bidders who submit bids closest to the average of all bids. That is, letting $\mathbb{R}^n_+$, $\overline{b} = \frac{1}{n}\sum_{i=1}^n b_i$, $w(\mathbf{b}) = \left\{ i \in \{1, 2, \cdots, n\} \mid \text{for each } i, \text{and } \forall j \neq i, |b_i - \overline{b}| \leq |b_j - \overline{b}| \right\}$.

Since we consider auctions where the winning bidder is payed his bid, payoff for bidder i is as follows:

$$\pi_i(\mathbf{b}, c_i) = \begin{cases} \frac{1}{|w(\mathbf{b})|}(b_i - c_i) & \text{if } i \in w(b) \\ 0 & \text{otherwise} \end{cases}$$

The reservation price r is such that bidders are constrained to bid no higher than r.

2.3 Equilibrium Characterization

A strategy for bidder i is a bid function $b_i : [\underline{c}, \overline{c}] \to \mathbb{R}_+$ that determines i's bid as a function of his cost. Since bidders are ex ante symmetric, our solution concept is symmetric Bayes Nash Equilibrium (SBNE).

In the ABA, the most relevant statistic of the bid distribution is the average. We first establish that bidders may pool at any bid between $\overline{c}$ and r. If an single bidder deviates from this common bid, the average remains closer to the common bid and the deviator is not among the winners. Thus, it is impossible for a single bidder to benefit by deviating.

We show in the Appendix that every SBNE bid function is weakly increasing.

Proposition 1 (General Pooling Equilibrium)**.** *For each $\widetilde{b} \in (\overline{c}, r]$, the SBNE bid function $b : [\underline{c}, \overline{c}] \to \mathbb{R}_+$ is such that for each $c \in [\underline{c}, \overline{c}]$, $b(c) = \widetilde{b}$. Moreover, such pooling bid functions are the only SBNE bid function that takes values above $\overline{c}$.*

Proposition 1 not only says that pooling equilibria exist, but also that *only* pooling equilibrium bid functions take values above $\overline{c}$. Since every SBNE bid function is weakly increasing, if it takes values above $\overline{c}$, then the bidder with cost $\overline{c}$ bids this value. Intuitively, if a bid function is strictly increasing in the neighborhood of $\overline{c}$, the winning probability of the bidder with cost $\overline{c}$ is zero since his rivals are closer to the average. If he lowers his bid without dropping below his cost, his bid could equal that of one of his rivals and he may win the auction with positive probability. Thus, the bidder with cost $\overline{c}$ has an incentive to deviate downward. Thus, in any SBNE bid function, if the highest cost bidder bids above $\overline{c}$, then his neighbors select the same bid. Even then, unless there is high enough probability that the average bid is below this identical bid, the bidder with the highest cost still gains by reducing his bid as argued above. If, with high probability, the average bid is close to the identical bid placed by the bidder with cost $\overline{c}$ and his neighbors, then bidders with low cost have an incentive to deviate upwards. Accordingly, the remaining possibility is that any SBNE bid function that takes values above $\overline{c}$ is a pooling equilibrium.

In Proposition 1, we have only considered bid functions that take values above $\overline{c}$. We now consider bid functions that lie below $\overline{c}$. If the bid function takes a single value below $\overline{c}$ for enough of the costs, the average is likely to be close to that identical bid. This would imply a very low probability of winning for bidders with costs above the identical bid. On the other hand, bidders who place the identical bid have a higher probability of winning.

However, at least, possible bidders with costs above the identical bid are not able to submit the identical bid. Thus, if bidders whose bids are the identical amount have strictly positive profit, there ought to exist a threshold cost such that every bidder with a cost lower than this threshold bids the identical bid. Lemma 1 says that the equilibrium bid function is strictly increasing for costs above the threshold.

Lemma 1 (Shape of SBNE Bid Function). *Suppose that $r \geq \overline{c}$. If $b : [\underline{c}, \overline{c}] \to \mathbb{R}_+$ is an SBNE bid function, then there exists $\widetilde{c} \in [\underline{c}, \overline{c}]$ such that*

1. *for every $c, c' \in [\underline{c}, \widetilde{c})$, $b(c) = b(c')$. Let this common bid for costs below $\widetilde{c}$ be $\widetilde{b}$.*

2. *for every $c, c' \in [\widetilde{c}, \overline{c}]$ where $c < c'$, $b(c) < b(c') \leq \overline{c}$ and $\widetilde{b} \leq b(\widetilde{c})$.*

3. $F(\widetilde{c}) \geq \frac{n(n-2)}{(n-1)^2}$.

4. *The bidder with cost $\widetilde{c}$ is indifferent between bidding $\widetilde{b}$ and $b(\widetilde{c})$,*

When every possible bidder bids below $\overline{c}$, Lemma 1 isolates the shape of the SBNE bid functions. In particular, it says that such a bid function is constant up to the threshold $\widetilde{c}$ and strictly increasing above it. At $\widetilde{c}$, it could contain a jump and a bidder with cost $\widetilde{c}$ is indifferent between pooling with those with lower costs or the higher bid. This implies that an SBNE bid function is upper hemicontinuous in cost as would be implied by Theorem 4.1 of Reny (2011). Therefore, if an equilibrium bid function takes values below $\overline{c}$, it has the shape described in Lemma 1.

The difference between the equilibria considered in Proposition 1 and Lemma 1 is essentially in the strategy of the bidder with cost $\overline{c}$. In the former case, even the bidder with cost $\overline{c}$ has a positive profit margin. Thus, if the bid function is strictly increasing, he may increase his probability of winning by reducing his bid. Reducing his bid always gives rise to the profitable deviaton. Thus, we find that there is pooling. However, for the equilibrium bid functions considered in Lemma 1, the bidder with cost $\overline{c}$ is not able to deviate downward since his bid equals his cost. This enables the bid function to be strictly increasing near $\overline{c}$.

Proposition 1 and Lemma 1 show that there are no strictly monotone SBNE bid functions. The reasons is that under such a bid function, the average bid is almost surely between $\underline{c}$ and $\overline{c}$. Thus, bidders with cost $\underline{c}$ have an incentive to deviate upwards, while those with cost $\overline{c}$ have an incentive to deviate downwards.

While Proposition 1 and Lemma 1 provide useful information about what the equilibria look like, they fall short of a full characterization. For the sake of tractability, we restrict attention to the case of $n = 3$.[6] By providing a closed-form solution to the SBNE bid functions for $n = 3$, we make possible the comparative analysis between FPA and ABA. Additionally, Proposition 1 and Lemma 1 also confirm that for $n \geq 3$, only two types of equilibria remain (pooling above $\overline{c}$ and partial pooling). Thus, the case of $n = 3$ provides intuition about how equilibria with $n > 3$ works.

Proposition 2 is a complete characterization of SBNE bid functions for three bidders.

For this characterization, we do not assume that $r \geq \overline{c}$. When the reservation price is below $\overline{c}$, given a SBNE bid function as described in Lemma 1, each bidder's winning probability, conditional on his cost, is as follows:

$$
\begin{aligned}
&Pr(win|\, c < \widetilde{c}) \\
&= Pr(A)Pr(win|\, c < \widetilde{c}, A) + Pr(B)Pr(win|\, c < \widetilde{c}, B) + Pr(C)Pr(win|\, c < \widetilde{c}, C) \\
&= F(c^*)^2(\tfrac{1}{3}F(\widetilde{c})^2 + F(\widetilde{c})(F(c^*) - F(\widetilde{c}))) + \tfrac{1}{2}\tbinom{2}{1}F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2 \\
&Pr(win|\, \widetilde{c} \leq c \leq c^*) \\
&= Pr(A)Pr(win|\, \widetilde{c} \leq c \leq c^*, A) + Pr(B)Pr(win|\, \widetilde{c} \leq c \leq c^*, B) + \\
&\quad Pr(C)Pr(win|\, \widetilde{c} \leq c \leq c^*, C) \\
&= F(c^*)^2(2F(c)(F(c^*) - F(c))) + \tfrac{1}{2}\tbinom{2}{1}F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2
\end{aligned}
$$

where $c^* = min\{r, \overline{c}\}$, $b(c^*) = c^*$. Costs of rivals $j,\ k$ are classified as follows

[6] See the Appendix for a demonstration of the intractability of $n > 3$.

$$A = \{\,(c_j,\, c_k)\,|\, c_j\,\text{and}\, c_k \leq c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\},$$

$$B = \{\,(c_j,\, c_k)\,|\, c_j\ \text{ or }\ c_k > c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\},$$

$$C = \{\,(c_j,\, c_k)\,|\, c_j\,\text{and}\, c_k > c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\}.$$

Each bidder considers three possibilities as follows: all rivals' costs are below the reservation price, one rival's cost is below the reservation price and the other's cost is above the reservation price, all rivals' costs are above the reservation price.

Given that one rival's cost is below r and the other's cost is above r, the winning probability is $\frac{1}{2}$ since two bidders remain below r. Moreover, if all rivals' costs exceed r, then, the winning probability is exactly 1. Therefore, considering three events, the winning probability is comprised of three terms as mentioned above. When the reservation price is above the highest cost $\overline{c}$, the last two terms disappear.

Proposition 2 (Equilibrium Characterization)**.** *Let $n = 3$ and $c^* = min\,\{r,\, \overline{c}\}$. The bid function $b : [\underline{c},\, c^*] \rightarrow \mathbb{R}_+$ is a SBNE function if and only if either*

1. *$\overline{c} \leq r$ and there exists $\widetilde{b} \in (\overline{c},\, r]$ such that for $c \in [\underline{c},\, \overline{c}]$, $b(c) = \widetilde{b}$, or*

2. *there exists $\widetilde{c} \in [\underline{c},\, c^*]$ such that $p = F(\widetilde{c}) \geq \frac{3}{4}F(c^*)$ and*

$$b(c) = \begin{cases} \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})} dy & \text{if } c \in [\underline{c},\, \widetilde{c}) \\ c + \int_{c}^{c^*} \frac{G(y)}{G(c)} dy & \text{if } c \in [\widetilde{c},\, c^*] \end{cases}$$

where

$$\begin{aligned} H(\widetilde{c}) &= F(c^*)^2(\tfrac{1}{3}p^2 + p(F(c^*) - p)) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2 \\ G(y) &= F(c^*)^2 2F(y)(F(c^*) - F(y)) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2. \end{aligned}$$

Per Proposition 2, there are two types of equilibria: partially pooling and pooling. In a partially pooling equilibrium, there is a threshold cost $\widetilde{c}$ such that every bidder with a cost below $\widetilde{c}$ bids an identical amount and bidders with costs above $\widetilde{c}$ bid according to a strictly increasing function. The bidder with $\widetilde{c}$ is indifferent between bidding the identical amount and bidding the higher amount. Indeed, there is such an equilibrium for each $\widetilde{c}$ such that $F(\widetilde{c}) \geq \frac{3}{4}F(\min\{r, \overline{c}\})$. Since $F(\widetilde{c})$ is high enough, with high probability, the winning bidder has cost lower than $\widetilde{c}$. By pooling below the reservation price, bidders with cost below $\widetilde{c}$ reduce the likelihood of ties, thereby boosting their expected payoff. [7]

That all pooling equilibria are either at r or above $\overline{c}$ is by the same reasoning as Proposition 1. Neither type of equilibrium reveals the cost of each type of bidder since there is a great deal of pooling even in the partially pooling equilibrium.

[7] In a pooling equilibrium, every possible bidder has the winning chance. (an equilibrium bid function is $b(c) > \overline{c}$). If 100 bidders participate in the auction, and all of them pool bid, the winning chance is $\frac{1}{100}$. However, in a partially pooling equilibrium, all bidders except the bidder with $\overline{c}$ are likely to bid below $\overline{c}$. Thus, if 80 of 100 bidders pool bid, their winning chance is $\frac{1}{80}$. Thus, bidders have incentive to increase the likelihood of ties in a partially pooling equilibrium.

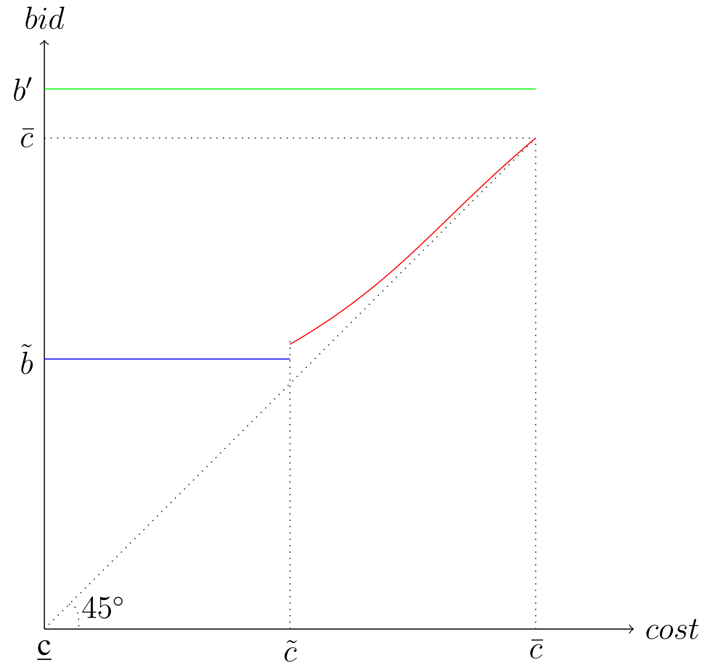

Figure 2.1: Depiction of Proposition 2

Bidders with cost below $\widetilde{c}$ bid the identical amount $\widetilde{b}$ in the partially pooling equilibrium. Bidders with cost above $\widetilde{c}$ bid according to the strictly increasing bidding strategy. $\widetilde{b}$ varies according to $\widetilde{c}$. If $\widetilde{c}$ equals $\overline{c}$, every bidder bids the identical amount $\overline{c}$. Furthermore, the pooling equilibrium, where every bidder bids an identical amount b', can also exist. Accordingly, there could exist two types of equilibria.

2.4 Comparison Between FPA and ABA

In this section, we leverage the characterization of the SBNE bid functions presented in Proposition 2 to compare FPA and ABA.

Proposition 3 (Comparison between FPA and ABA)**.** *Let $n = 3$. Every equilibrium in the ABA results in lower expected payoff for the buyer, compared to the FPA.*

In Proposition 3, we compare ABA with FPA in terms of the expected payoff for the buyer. This comparison gives the buyer good guidance about which of FPA and ABA is helpful for his cost-cutting effort. Intuitively, since the average is at least as much as the lowest bid, the winning bid in ABA is the most likely to be higher than the winning bid in FPA. Thus, ABA could be more expensive for the buyer, compared to FPA.

A more detailed explanation is as follows. By the Revenue Equivalence Theorem Myerson (1981), it is evident that FPA is the same as the Second Price Reverse Auction (SPA) in terms of the expected payment of the buyer, regardless of the reservation price. Considering the case in which three bidders bid, as depicted in Figure 2.2, the second lowest bid in SPA is always less than the second lowest bid in ABA. Since the buyer pays the amount of the second lowest bid in both auctions with three bidders, it implies that ABA costs more for the buyer, compared to SPA. Thus, FPA is cheaper than ABA for the buyer.

This result is not limited to the case where three bidders participate in the auction. When more than three bidders join ABA, the winning bid is at least as much as the second lowest bid. Moreover, the winning bid in ABA is higher than the second lowest bid in SPA. It implies that ABA results in higher expected payment for the buyer, compared to SPA. Accordingly, FPA is less expensive than ABA for the buyer.

Finally, the expected payoff for the buyer is comprised of two terms as follows: the difference between the project value and the expected payment conditional on the successful auction and the loss of the buyer conditional on the auction failure. [8] Given the reservation price, since both the chance of the auction failure and the loss of the buyer due to the failure are the same across two auction formats, the difference in expected payoff for the buyer is determined by the expected payment of the buyer. Since ABA is more expensive, ABA results in the lower expected payoff for the buyer, compared to FPA.

[8]The auction failure is the case where costs of all bidders exceed the reservation price, and nobody participates in the auction.

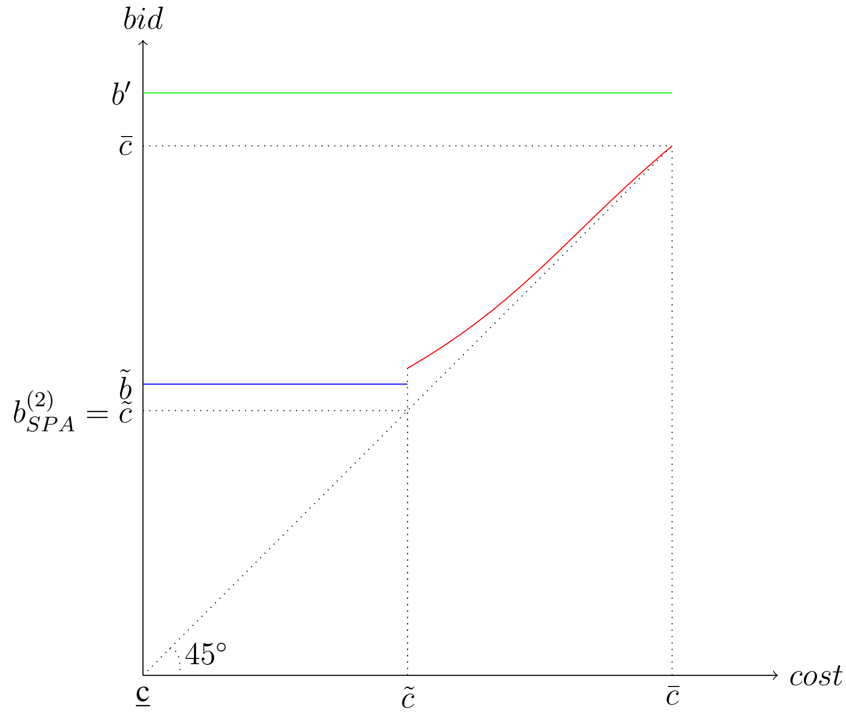

Figure 2.2: Comparison between FPA and ABA

If the second lowest cost is $\widetilde{c}$, the payment to the winning bidder in SPA is $b(\widetilde{c}) = \widetilde{c}$. At the same time, the payment to the winner in ABA is $\widetilde{b}$.

2.5 Concluding Remark

This study investigates the equilibrium in ABA. We show that partially pooling SBNE bid functions exist. For the general case of at least three bidders, we provide necessary conditions for a bid function to be an SBNE bid function and show that only pooling SBNE bid functions involve bids higher than the highest cost. A closed-form solution of equilibrium bid functions is not tractable for more than three bidders. However, for the case of three bidders, we offer a complete characterization of SBNE bid functions. These include both partial pooling and the pooling bid functions. This analysis provides good guidance to compare FPA with ABA in terms of the expected payoff of the buyer.

The result shows the buyer is better off under FPA than ABA when there is no shock to increase the ex-post cost. In further work [9], we consider cases where ex-post costs may be affected by shocks, increasing the risk of default by the bidders. In such cases, we show that there are circumstances where ABA is actually superior to FPA.

[9] Refer to Lee (2016*a*)

3. AVERAGE BID VERSUS FIRST PRICE AUCTION: TRADEOFF BETWEEN PREVENTION OF DEFAULT AND EFFICIENCY LOSS

3.1 Introduction

The reverse auction forces many sellers to compete against each other for the opportunity to contract with a buyer. Of all the reverse auction formats, one that has emerged as a popular procurement method is the first price reverse auction (FPA). The FPA is effective at allocating the auctioned item to the bidder with the lowest cost. When default concerns are absent, the FPA functions well at identifying bidders with low costs from bidders with high costs, which is helpful in reducing the buyer's payment. However, in the case of the winner's default risk, the FPA may not be the best.

Many contracts are allocated to bidders through the reverse auction, according to Spulber (1990). Since most contracts are long-term, they require the buyer and the winner to interact over a long period. Thus, a breach of contract could cause a serious loss for the buyer. Chillemi and Mezzetti (2014) point out that the buyer could not affordably replace a winner in the middle of a contract due to the sunk costs created by specialized resources. Good examples are construction projects. Calveras, Ganuza and Hauk (2004) emphasize that in the United States between 1990 and 1997, more than 80,000 contractors went bankrupt in private and public construction projects, leaving liabilities exceeding $21 billion.

The main causes of business failures in construction projects are related to cost overrun. According to Grant Thornton's survey[1] (2005, 2007), the main reasons for business failures were low profit margin, insufficient capital, and poor estimating. To lessen the risk of the winner defaulting rise, the buyer could impose a penalty [2] on an insolvent winner.

[1] Refer to Grant Thornton (2005, 2007)

[2] Another alternative is that a buyer could set up an institutional system to appraise if each bidder is eligi-

As winners would need to factor in the cost of such a penalty, their decision would be affected by the size of the penalty.

In the U.S., for a winner facing default, a penalty can be a heavy burden. According to Moody's (Emery et al. (2007)), between 1987 and 2006 in the U.S. bank loans were recovered at an average of 82 percent at resolution, and senior secured bonds were recovered at an average of 65 percent across all industries. The World Bank (2004) (WBG) noted that the recovery rate in the U.S. for secured creditors in 2004 was approximately 80 percent in the U.S. This implies that enforcing the contract in the U.S. is relatively efficient in terms of the number of procedures, days, cost, and procedural complexity.

Italy, however, has proven to be quite different from the U.S. Its penalty is relatively small. The WBG reported that the recovery rate for the secured creditors in 2004 was approximately 35 percent. Thus, contract enforcement in Italy has sacrificed more creditors than in the U.S. More importantly, an insolvent winner could be easily exempted from the liquidation process. According to Oglio (2007), the national average showed that in 2006, 47 percent of bankruptcy procedures in Italy were excluded in 2006, and in some areas of Southern Italy, almost 90 percent were free from bankruptcy procedures. Thus, since the bankruptcy law [3] is so restrictive, it is hard for the buyer to make up for default-induced

---

ble for the contract or not. If the buyer screens out insolvent bidders by himself, it could be very expensive since he needs a lot of expertise to overcome asymmetric information between bidders and the buyer. Instead, he designs the system to filter out insolvent bidders. For example, considering the monitoring system and insurance policy in the U.S, surety bond companies play a pivotal role in evaluating the eligibility of bidders. They are expert groups that evaluate bidders in terms of capability, capital, and character. According to the Miller Act in the U.S., when bidders participate in the tender process worthy of more than $150, 000 supported by the federal government, they are obliged to purchase the performance bond and payment bond. These bonds cover 100 percent of the contract value. In the case that the winner refuses to carry out the project, surety bond companies should guarantee the completion of the project, according to the contract between the surety bond company and the winner.

However, there are a few expert groups similar to surety bond companies in Italy. According to Decarolis (2009, 2013) and Decarolis and Giorgiantonio (2014, 2015), most of central and local governments except Turin were not able to screen out invalid bids. Centre d'Etudes d'Assurances Centre d'Etudes d'Assurances and Centre Scientifique et Technique du Bâtiment (2010) state that according to the Civil code, the winner is compelled to buy insurance to prevent the default, but the insurance covers only 10~15 percent of the contract value.

[3] For details, refer to Cabrillo and Depoorter (1999), Manganelli (2010), Santella and Brogi (2004),

losses through the liquidation process.

When default caused by cost overrun is a main concern in the competitive bidding, and damage could be partially covered by the contract enforcement, FPA might not be the best procurement method. In a FPA, price competition leads to low profit margins, and real costs could easily exceed the bidding price. Thus, the buyer is likely to be exposed to a default risk. If the buyer can not receive any compensation in the case of a default, they could be very reluctant to adopt the FPA.

One alternative to the FPA is to modify the auction rule. This could be why the average bid auction (ABA) is still widely used in Italy. In the ABA, the winning bid is the one closest to the average of all submitted bids. In Italy between 1998 and 2006, according to Decarolis (2009), ABA was almost the sole mechanism for public procurement. Furthermore, Decarolis, Giorgiantonio and Giovanniello (2011) point out that even though local governments have been legally allowed to choose between the ABA and the FPA since 2006, most of them continue to use ABA.

Since bidders try to bid closest to the average, ABA could alleviate price competition, which may lead to an increased profit margin. Even though cost overruns do occur, these could be covered by the increased profit margin. Therefore, buyers can easily find themselves facing a dilemma. When they use FPA, they can pay less for the auctioned contract but should worry about the risk of default. When they use ABA, they are likely to pay at least as much as they would with FPA, but they do not need to worry about the default risk.

Both the U.S. and Italy have experience with the FPA and ABA formats, yet each tends to use a different auction format. If the economic environment is the same in the two countries, the same auction format dominates in both countries. Generally, however, the U.S. uses FPA for procurement, while Italy tends to use ABA. This implies that the two countries differ in their economic environments. In fact, it is the enforcement systems that

---

Cocito (2013) and Rodano, Serrano-Velarde and Tarantino (2012) differ, with the U.S. enjoying more efficient enforcement of penalty clauses.

In other words, it costs a winner in Italy much less to default. If a contract value is not negligible for the buyer, then an important determinant for him in choosing an auction format may be the level of contract enforcement. Therefore, this study investigates how penalties affect buyers' choosing FPA or ABA, considering the default risk of the winner. That is, this study explores whether FPA or ABA is more desirable in terms of the expected payoff for the buyer when the default risk caused by cost overrun is considered.

3.2 Related Literature

Since ABA does not guarantee that the cost efficient bidder is the most likely to win the auction, it could give rise to an efficiency loss. In spite of that, ABA has been an important procurement method in many countries. Decarolis (2009, 2013) and Decarolis and Giorgiantonio (2014, 2015) observe that in Italy prior to 2006 ABA was prevalent. Engel (2009) indicates that countries such as Chile, China, Peru, and Taiwan employ ABA. Additionally, Spagnolo, Albano and Bianchi (2006) find that ABA was used in Florida and New York State.

Such usage of ABA raises the question of why such a format is preferred to the well-known and efficient FPA. The first researchers to try to answer this question were Spagnolo, Albano and Bianchi (2006). They suggest that ABA is designed to soften price competition. While their study provides a rationale for the use of ABA, they focus on the special case of three bidders with publicly known costs, which precludes default. Moreover, they provide only partial characterization since they focus only on pooling equilibria.

Chang, Chen and Salmon (2014) are also interested in why ABA is used in procurement. They assume that the procurement auction is a pure common value auction. According to their experimental results, the equilibrium bid function under the ABA strictly increases in signal which is the prediction of the true cost. Moreover, for each signal, the bid under the ABA is higher than that under the FPA. Thus, ABA successfully helps mitigate the winner's curse. The authors do not, however, provide a theoretical foundation to explain the experimental finding.

Cost overrun and penalty in the reverse auction have been considered theoretically by Spulber (1990). According to his study, when a cost overrun occurs with a FPA, and a winner is facing default, the liability of the winner is limited to a uniform penalty imposed by the buyer. Thus, insolvent bidders have an identical expected cost and are forced to bid the expected cost through a Bertrand competition. Efficient bidders with relatively small cost overruns follow a strictly increasing bidding strategy, but inefficient bidders with relatively large cost overruns bid an identical amount. At the extreme, when no penalty is imposed on insolvent bidders, all possible bidders bid the lowest expected cost. Accordingly, Spulber reveals how a penalty affects bidding behavior, but his research is limited to the FPA format. Zheng (2001) also pays attention to default risk at a private value auction. He considers default risk and heterogeneous penalties due to limited liability and different budgets across bidders at the first price auction. He concludes that more budget-constrained bidders are more aggressive in bidding due to a binding limited liability, compared to deep-pocket bidders. Therefore, the auctioneer is exposed to default risk due to budget-constrained bidders. Calveras, Ganuza and Hauk (2004) are interested in the second price reverse auction with default risk, and heterogeneous penalties are the result of limited liability and different assets across bidders. Their result is in line with that of Zheng (2001). That is, since bidders with small assets have a tendency to bid aggressively, the buyer is exposed to default risk. However, these authors also focus only on the second price reverse auction.

The first researchers to compare FPA and ABA are Decarolis (2009, 2013) and Decarolis and Giorgiantonio (2014, 2015). According to their research, buyers want to adopt FPA, but the monitoring cost to screen out insolvent bidders is so huge that buyers are reluctant to employ FPA. Thus, they use ABA without any monitoring procedure. They also provide an important foundation to detect an equilibrium bid function in ABA. However, buyers do not consider as an important determinant the impact of the penalty. In 2005 and 2006, Italy underwent a dramatic reform[4] governing its public procurement and bankruptcy policies. Before 2005, the central government forced the local governments to employ ABA. Since 2005, the local governments have been empowered to choose either FPA or ABA. Even though ABA is still widespread in Italy, since the reform the local governments have been gradually adopting FPA. At the same time, the bankruptcy law in Italy has, since 2005, accommodated the essential elements from the enforcement system in the U.S. According to Beye and Nasr (2008) and Esposito, Lanau and Pompe (2014), this reform in bankruptcy law has been so effective that the recovery rate has nearly doubled. If the monitoring cost is the only determinant that affects the buyer's preference for the auction format, this can not explain the gradual adoption of FPA. Thus, this phenomenon indicates that the penalty (contract enforcement) could seriously affect the buyer's preference over the auction format.

As noted above, the literature focuses on either the first price reverse auction or the second price reverse auction, when cost is uncertain. Moreover, even though the average bid auction is considered, the literature either concentrates only on characterizing pooling equilibria or does not provide a theoretical foundation. Additionally, the penalty has not been an important determinant in characterizing the equilibrium bidding strategy under the ABA and the buyer's preference over auction formats. The purpose of this study then is to fully characterize the equilibrium bidding behavior under the ABA. In a model with three bidders and the possibility of cost overrun, it shows that the penalty size impacts the bidding strategy in equilibrium. Moreover, it provides a rationale for the ABA by showing

[4]For further details about the reform of the procurement, see Decarolis, Giorgiantonio and Giovanniello (2011) For more detailed information about the reform of the bankruptcy law, refer to Manganelli (2010).

that the buyer is better off under ABA than FPA when the default penalty is small.

This study is organized as follows. Section 3.3 introduces the economic environment. Section 3.4, characterizes equilibrium. Section 3.5 compares the FPA with the ABA in terms of the expected payoff of the buyer. Section 3.6 presents the conclusions.

3.3 The Model

Three bidders bid for a contract to provide goods or services to a buyer. [5] Bidder i has two cost states $C^i \in \{c,\, c+\theta^i\}$ about cost where a θ^i is a common cost shock that occurs with probability $\beta \in (0,\, 1)$. [6] θ^i is private information to bidder's i and is independently and identically distributed according to a density $f\ >\ 0$ [7] on the interval $\left[0,\, \overline{\theta}\right]$. The reservation price r is $c + \beta\overline{\theta}\ \le\ r\ <\ c + \overline{\theta}$. The value of the contract to the buyer, V is $V \ge c + \overline{\theta}$. Thus, $c + \beta\overline{\theta} \le r < c + \overline{\theta} \le V$.

Denote a profile of submitted bids by $\mathbf{b}\ \in\ \mathbb{R}^n_+$. Given $\mathbf{b}$, let $w(\mathbf{b})$ be the set of the winning bidders. In the ABA, [8] $w(\mathbf{b})$ consists of bidders who submit bids closest to the average of all submitted bids. Particularly, $\bar{b}\ = \frac{1}{3}(b_1 + b_2 + b_3)$ and $w(\mathbf{b})\ =\ \left\{i \in \{1,\, 2,\, 3\}\ |\ \text{for each } i,\, \forall j \ne i,\, |b_i - \bar{b}| \le |b_j - \bar{b}|\right\}$.

Since the buyer pays the winning bidder's bid, the payoff [9] for a bidder i is as follows:

$$\pi_i(\mathbf{b},\, C^i) = \begin{cases} \frac{1}{|w(b)|}(b_i - C^i) & \text{if } i \in w(b) \\ \\ 0 & \text{otherwise} \end{cases}$$

Bidder i's strategy is a bid function $b : \left[0,\, \overline{\theta}\right] \to \mathbb{R}_+$, and bidders are ex ante symmetric. [10]

---

[5] That is, let n be the number of bidders and assume that $n = 3$

[6] In other words, every possible bidder has the identical cost c. If a shock occurs with the common probability β, cost of each bidder i heterogeneously increases from c to $c + \theta^I$.

[7] Thus, F associated with the f is a strictly increasing cumulative function

[8] For FPA, $w(\mathbf{b})$ contains bidders who submit the lowest bids.

[9] This payoff assumes that there is no default.

[10] Thus, the solution concept is symmetric Bayes Nash Equilibrium (SBNE).

The environment follows the set-up of Spulber (1990), who derives an equilibrium bid function[11] under the FPA. Reviewing the cost structure, the expected cost is made up of base cost c and cost overrun θ. If the market is competitive without shock, every possible bidder incurs the base cost. However, every potential bidder heterogeneously controls cost overruns. Thus, the expected cost is likely to vary across bidders. Law (2011) shows that, where cost overruns are frequently observed, firms have significantly heterogeneous abilities to control costs in construction projects. [12]

The assumption $r \geq c + \beta\overline{\theta}$ [13] guarantees that there is no possibility of auction failure. [14] If the reservation price is less than the highest possible expected cost, bidders with expected costs higher than the reservation price could not join the auction, which can give rise to the auction failure.

Similarly, the assumption [15] $r < c + \overline{\theta}$ avoids the other extreme in which there is no cost overrun. This assumption of Spulber (1990) plays an important role in inducing the Bertrand competition among bidders breaching the contract. Additionally, $c + \beta\overline{\theta} \leq r < c + \overline{\theta} \leq V$ implies that the buyer's budget is not sufficient, but the contract value is enough to cover all possible costs.

[11]This cost structure makes it possible to find a closed-form solution to a SBNE function. If heterogeneous base costs are introduced, a asymmetric equilibrium function should be detected, but it is well known that there is no general analytical solution for it (Krishna (2009)).

[12]According to Flyvbjerg (2006), 9 out of 10 in the construction project experienced cost overruns.

[13]The new assumption also reflects reality. According to Decarolis (2013, 2014), since the reservation price is set by formulas in Italy, the buyer does not have the full discretionary power to control the reservation price. At the same time, he said that in Italy the reservation price is established high enough to prevent the auction failure.

[14]The auction failure is the case, where costs of all participants exceed reservation price, and nobody is able to participate in the auction.

[15]Even though this assumption is not included, his equilibrium characterization in FPA is still valid. The detailed proof is not provided in this paper. However, the idea is as follows. Suppose that possible bidders are likely to breach the contract under a penalty. When a strictly increasing bidding strategy is adopted, the winning probability of the bidder with $c + \overline{\theta}$ is always zero. Thus, if his bid is above his expected cost (in pursuit of default), he has an incentive to deviate downward under any strictly increasing bidding strategy. His neighbors know that he is not following the strictly increasing bidding strategy. Thus, it leads to Bertrand competition in the neighborhood of the bidder with $c+\overline{\theta}$. Therefore, even though the assumption, $r < c+\overline{\theta}$ is not added, we have the same result as Spulber (1990).

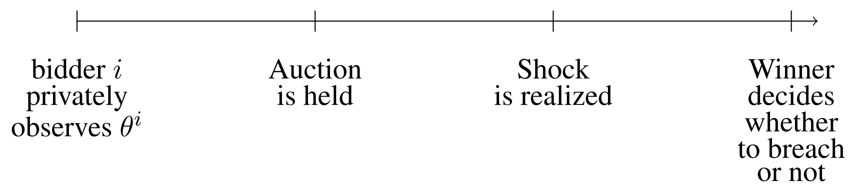

Figure 3.1: Timing of the events in the model

The timing of the game is illustrated in Figure 3.1. First, each bidder i observes his own θ^i, after which the auction is held, and the winner is determined. Subsequently, the random variable β is realized , determining the winner's cost state. Observing the cost state, the winner decides whether to perform or breach the contract.

The sequence of events is as follows [16]

1. Each bidder i observes his own c and θ^i
2. Auction is held, and a winner is determined
3. Winner observes the realized cost
4. Winner decides to perform or breach the contract according to max $\{b - c - \theta^i, \, -D\}$, where penalty $D \geq 0$
5. If the winner breaches the contract, the buyer does not pay b. Instead, the buyer imposes the penalty D on the winner. The penalty D is exogenously determined before the auction

3.4 Equilibrium Characterization in the ABA

In order to determine how the penalty impacts the equilibrium bid function in the ABA, we first consider the extreme case of no penalty. Proposition 1 states that when the buyer takes all the default risk, every bidder in ABA bids an identical amount.

[16] The sequence is depicted in Figure 3.1

Proposition 1. *Suppose that $D = 0$. The bid function $b : \left[0,\, \overline{\theta}\right] \to \mathbb{R}_+$ is a SBNE function if and only if for each $\theta \in \left[0,\, \overline{\theta}\right]$, $b(\theta) = b^* \in [c,\, r]$*

When a winner breaches the contract without any sacrifice, a strictly increasing bidding strategy for cost overrun θ is not in equilibrium. Otherwise, bidders with the lowest θ and the highest θ have an incentive to deviate upward and downward, respectively. Particularly, when the strictly increasing bidding strategy is adopted, the winning chance of the bidder with the lowest θ is zero. If he deviates upward, he increases the winning probability and bid. Thus, he has an incentive to deviate upward. Likewise, the bidder with the highest θ is sure to lose the auction under the strictly increasing bidding strategy. If he deviates downward, his winning probability is not zero. Furthermore, his profit margin is always strictly above zero. [17] Thus, he has an incentive to deviate downward. In summary, when no penalty is imposed on an insolvent winner, the average is the unique concern of all possible bidders. Thus, they intensively bid in a very narrow interval.

Next, when the penalty is high enough to deter all possible bidders from defaulting on their bids, an equilibrium bid function in ABA is detected. Since every bidder has no choice but to complete the contract due to the high penalty, the bidders take on all the risk.

Proposition 2. *Suppose that $D \geq (1-\beta)\,\overline{\theta}$.*
The bid function $b : \left[0,\, \overline{\theta}\right] \to \mathbb{R}_+$ is a SBNE function if and only if either

1. *there exists a $b^* \in (c + \beta\overline{\theta},\, r]$ such that for $\theta \in \left[0,\, \overline{\theta}\right]$, $b(\theta) = b^*$, or*

2. *there exists a $\widetilde{\theta} \in [0,\, \overline{\theta}]$ such that $\frac{3}{4} = F(\widehat{\theta}) \leq p = F(\widetilde{\theta})$, and*

[17] His expected cost is either $c + \beta\overline{\theta}$ or c. In the case that $b - c - \theta \geq 0$, his profit margin is strictly greater than zero. Even though $b - c - \theta < 0$, his profit margin is strictly above zero due to cost c.

$$b(\theta) = \begin{cases} c + \beta\widetilde{\theta} + \beta \int_{\widetilde{\theta}}^{\overline{\theta}} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2 + p(1-p)} dy & \text{if } \theta \in [0, \widetilde{\theta}\,) \\ c + \beta\theta + \beta \int_{\theta}^{\overline{\theta}} \frac{F(y)(1-F(y))}{F(\theta)(1-F(\theta))} dy & \text{if } \theta \in [\widetilde{\theta}, \overline{\theta}] \end{cases}$$

If a bidder attempts to perform the contract, his bid is at least as much as his expected cost. However, if his bid is exceeded by the realized cost, and the penalty is within the scope of difference between expected cost and realized cost, he is enticed to refuse the contract. Thus, if the penalty is imposed, exceeding the difference, the loss caused by completing the project is less than that caused by paying the penalty. Thus, a bidder is willing to carry out the contract in order to avoid the penalty. Since $(1-\beta)\overline{\theta}$ is the highest of all differences, $D \geq (1-\beta)\overline{\theta}$ implies that all potential bidders are forced to perform the contract. Thus, there is no default.

According to Proposition 2, there are two types of equilibria: partially pooling and pooling. In a partially pooling equilibrium, there is a threshold of cost overrun $\widetilde{\theta}$ such that every bidder with cost overrun below $\widetilde{\theta}$ bids an identical amount and bidders with cost overruns above $\widetilde{\theta}$ bid according to a strictly increasing function. The bidder with $\widetilde{\theta}$ is indifferent to the identical bidding strategy as well as the strictly increasing bidding strategy. Meanwhile, every possible bidder bids a constant in a pooling equilibrium.

Since nobody defaults, the possible bidders bid according to their expected costs, conditional on θ. If every possible bidders bid an identical constant above the highest cost $c + \beta\overline{\theta}$, the bidding strategy is in a pooling equilibrium. If the majority of bidders pool a bid below the highest cost, the average is highly likely to be close to the bid. Of all bidders who cannot pool, a bidder with the lower cost overrun bids closer to the pooling bid. Thus, the rest of bidders follow a strictly increasing bidding strategy.

Explored now, in Proposition 3 are equilibria between two extreme penalties. It differs from Proposition 2 in that bidders with large cost overruns are likely to breach the contract.

Since those bidders pay a uniform penalty D in the event of cost overrun, their expected costs are identical, which forces them into a Bertrand competition. Thus, they come to a pooling strategy. Accordingly, every potential bidder with cost overrun below $\widetilde{\theta}$ and every possible bidder with cost overrun above θ^* follow a pooling strategy in equilibrium.

Proposition 3. *Suppose that* $0 < D < (1-\beta)\,\overline{\theta}$.

The bid function $b : \left[0,\, \overline{\theta}\right] \to \mathbb{R}_+$ *is a SBNE function if and only if either*

1. *there exists a* $b^* \in (c + \frac{\beta}{1-\beta}D,\, r]$ *such that for each* $\theta \in \left[0,\, \overline{\theta}\right]$, $b(\theta) = b^*$, *or*

2. *there exists a* $\widetilde{\theta} \in [0,\, \overline{\theta}]$ *such that* $\frac{3}{4} = F(\widehat{\theta}) \leq p = F(\widetilde{\theta}) < F(\theta^*)$, $\theta^* = \frac{D}{1-\beta}$,

$$\textit{and } b(\theta) = \begin{cases} c + \beta\widetilde{\theta} + \beta \int_{\widetilde{\theta}}^{\theta^*} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2 + p(1-p)} dy & \text{if } \theta \in [0,\, \widetilde{\theta}\,) \\ c + \beta\theta + \beta \int_{\theta}^{\theta^*} \frac{F(y)(1-F(y))}{F(\theta)(1-F(\theta))} dy & \text{if } \theta \in [\widetilde{\theta},\, \theta^*\,) \\ c + \frac{\beta}{1-\beta} D & \text{if } \theta \in \left[\theta^*,\, \overline{\theta}\right] \end{cases}$$

Proposition 3 is depicted in Figure 3.2. One equilibrium is a partially pooling equilibrium. Efficient bidders with relatively small cost overruns (bidders with θ less than $\widetilde{\theta}$) bid identically low and could exclude bidders with θ higher than $\widetilde{\theta}$. In addition, since the majority of potential bidders ($F(\widetilde{\theta}) \geq \frac{3}{4}$) submit an identical bid $\widetilde{b}$, the average is the most likely to be close to $\widetilde{b}$. Thus, bidders with θ above $\widetilde{\theta}$ have an incentive to place their bids as close to $\widetilde{b}$ as they can. This makes bidders with θ between $\widetilde{\theta}$ and θ^* follow a strictly increasing bidding strategy. Bidders with θ above θ^* default in the event of a cost overrun. Since their base cost is c and they only pay the uniform penalty D, the expected costs for bidders with θ above θ^* are identical. Since they also want to bid as close to $\widetilde{b}$ as they can, they are enforced to enter a Bertrand competition. Accordingly, they eventually bid the expected cost.

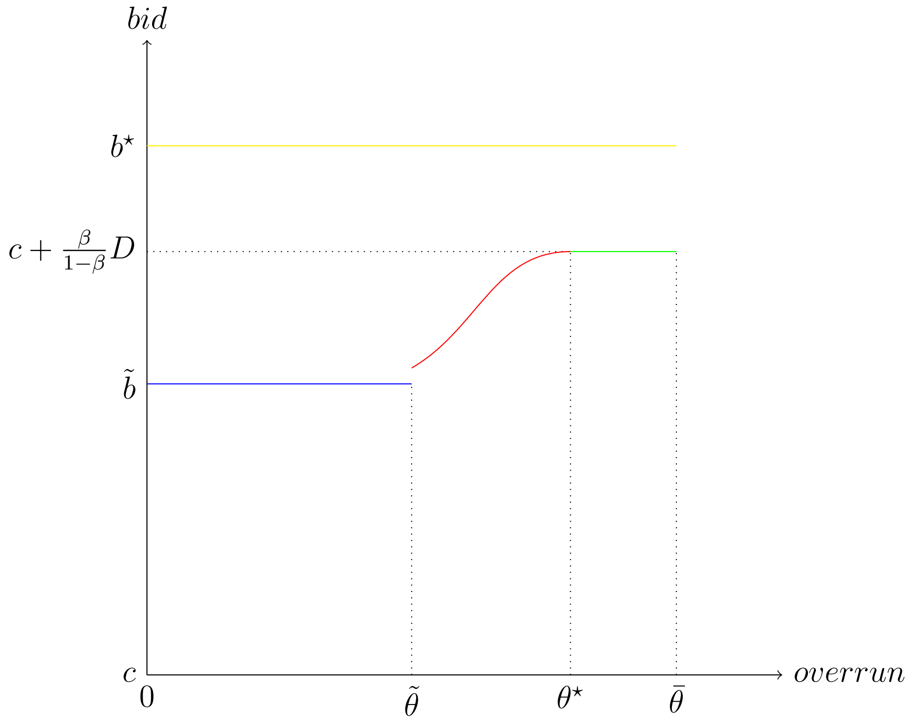

Figure 3.2: Equilibrium, when $0 < D < (1-\beta)\bar{\theta}$

b^* is also another equilibrium, where every bidder bids an identical amount b^*. When every bidder is believed to bid one specific amount b^*, equilibrium is attained since, if a bidder deviates from b^*, he surely loses the auction. Every time a cost overrun occurs, it results in a pooling equilibrium above the expected cost $c+\frac{\beta}{1-\beta}D$ in the pursuit of default.

3.5 Comparison Between FPA and ABA

In this section, FPA is compared with ABA from the buyer's point of view. In Proposition 4, a non-negative penalty is imposed on insolvent bidders. However, the penalty is not sufficient to prevent all possible bidders from defaulting. In this case, Proposition 4 indicates that every partially pooling equilibrium in the ABA is worse for the buyer than those in the FPA. All partially pooling equilibria in the ABA are ineffective at reducing default. In particular, when two bidders have θ's above θ^*, the winner in the ABA defaults in the event of cost overrun. However, a winner in the FPA carries out the contract. The ABA is thus riskier than the FPA. Moreover, a winning bid in the ABA is likely to be higher than one in the FPA. Therefore, every partially pooling equilibrium in the ABA results in a lower payoff.

Proposition 4. *Let $0 \leq D < (1-\beta)\overline{\theta}$, and $K = \pi_{FPA}$ be the expected profit of the buyer in FPA. Let b^* be such that $V = b^* + D + \frac{K-D}{1-\beta+\beta F(b^*-c+D)}$.*

1. *Every partially pooling equilibrium under ABA results in strictly lower expected payoff for the buyer than FPA.*

2. *If $V \geq r + D + \frac{K-D}{1-\beta+\beta F(r-c+D)}$, a pooling equilibrium at $b \in [b^*, r]$ in ABA results in weakly higher expected payoff for the buyer than π_{FPA} in FPA.*

3. *If $V \geq r + D + \frac{K-D}{1-\beta+\beta F(r-c+D)}$, a pooling equilibrium at $b \in [c + \frac{\beta}{1-\beta} D, b^*]$ under ABA results in weakly lower expected payoff for the buyer than π_{FPA} in FPA.*

4. *Let b' be such that $b' = argmin_b\, b + D + \frac{K-D}{1-\beta+\beta F(b-c+D)}$, and $V = \widehat{b} + D + \frac{K-D}{1-\beta+\beta F\left(\widehat{b}-c+D\right)}$ with $b^* \leq \widehat{b}$. If $V \geq b' + D + \frac{K-D}{1-\beta+\beta F(b'-c+D)}$ and $V \geq r + D + \frac{K-D}{1-\beta+\beta F(r-c+D)}$, there exists an interval $[b^*, \widehat{b}]$ such that a pooling equilibrium at $b \in [b^*, \widehat{b}]$ in ABA results in weakly higher expected payoff for the buyer than π_{FPA} in FPA.*

Thus, if there is an equilibrium such that ABA is superior to FPA, it is pooling in ABA. A pooling equilibrium near a partially pooling equilibrium in ABA is likely to be inferior to a unique equilibrium in FPA.

A higher winning bid in ABA reduces the chance of default. The higher bid, however, causes a higher cost for the buyer, and there is a trade-off between the high winning bid and the default risk. When the decrease in profit due to the high bid is cancelled by the increase in profit due to reduced default, the buyer's profit increases. Moreover, since an equilibrium in FPA is unique for a given penalty, the expected payoff for the buyer in FPA is also unique. This implies that a number of profits caused by pooling equilibria in ABA is comparable to the unique profit in FPA. Of all multiple equilibria, there exist equilibria such that when a high winning bid in ABA gives rise to a sufficient gain in reducing the chance of default, ABA is likely to be superior to FPA.

If the gain from reducing the default likelihood is sufficient to pay a high bid in ABA until the bid reaches the reservation price, there could exist a b^* such that a winning bid below b^* in ABA brings about a worse profit for the buyer, and a winning bid above b^* in ABA gives rise to a better profit for the buyer. [18] However, if the high bid is relatively more expensive than the gain before it reaches the reservation price, there might exist an interval $[b^*, \widehat{b}]$ such that a winning bid within that interval leads to a better profit for the buyer than what could be obtained via FPA.

Proposition 4 is depicted in Figure 3.3. The vertical line stands for the expected payoff for the buyer. The horizontal line implies a bid in equilibrium. The red line represents the expected payoff using ABA. The green line indicates the expected payoff using FPA. K represents the expected payoff using FPA. Since an equilibrium in FPA is unique, K is a constant. However, the expected payoff [19] under ABA varies according to the bid in equilibrium. The condition, $V \geq r + D + \frac{K-D}{1-\beta+\beta F(r-c+D)}$ means that the expected payoff under ABA is at least as much as FPA at the reservation price, r. Thus, the red line is higher than the green line at r. Since the expected payoff using ABA is less than that using FPA at the pooling equilibrium, where the winning bid is $c + \frac{\beta}{1-\beta}D$, the red line is lower than the

[18] We reach 2 and 3 of Proposition 4.

[19] As a winning bid in equilibrium increases under ABA, the risk of default decreases. However, the increase in bid implies that an increase in cost for the buyer. This implies a trade-off between the bid and default risk. Therefore, if the contract value is sufficiently large, the expected payoff function is concave.

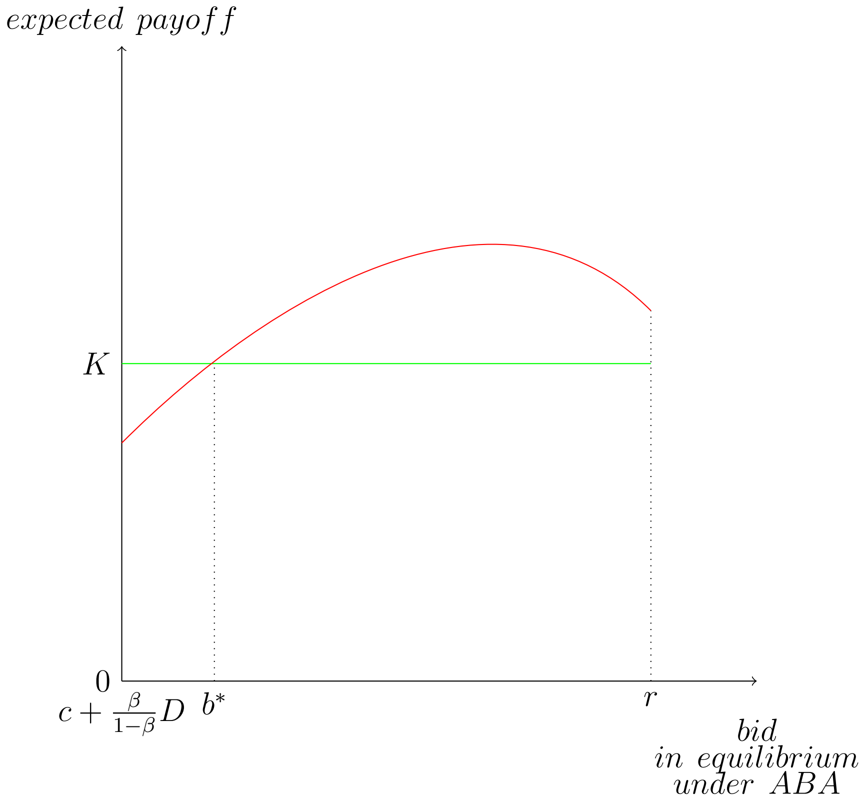

Figure 3.3: Comparison between FPA and ABA, when $0 \leq D < (1-\beta)\overline{\theta}$

green line at $c + \frac{\beta}{1-\beta}D$. The red line intersects with the green line at b^*. Therefore, every pooling equilibrium at $b \in [b^*, r]$ under ABA results in a weakly higher expected payoff for the buyer than K using FPA, and every pooling equilibrium at $b \in [c+\frac{\beta}{1-\beta}D, b^*]$ using ABA results in a weakly lower expected payoff for the buyer than K under FPA.

Finally, Corollary 1 is derived from Proposition 4. As with Proposition 4, the high winning bid reduces the risk of default. In contrast to Proposition 4, though, when a winning bid in ABA equals the base cost c, the expected payoff in ABA is the same as that in FPA.

Corollary 1. *Let $D = 0$, π_{FPA} be the expected profit of the buyer in FPA. Moreover, let $h(b) = b + \frac{\pi_{FPA}}{1-\beta+\beta F(b-c)}$ and b^* be such that $h(b^*) = V$ with $c \leq b^*$.*

1. *If* $V \geq h(r)$*, every equilibrium under ABA results in the weakly higher expected payoff for the buyer than FPA .*

2. *If* $V < h(r)$*, an equilibrium at* $b \in [c, b^*]$ *in ABA is more beneficial to the buyer than FPA.*

This implies that the equilibrium of ABA is not necessarily worse off than the neighborhood of the equilibrium of FPA. Therefore, every equilibrium in ABA could be more profitable to the buyer than those in FPA.

3.6 Concluding Remark

This study has investigated why a buyer in Italy prefers ABA to FPA, and why in the U.S. FPA is predominantly used for procurement. FPA is believed to be an efficient mechanism in which the bidder with the lowest cost is the most likely to win the contract. When default is a major concern, however, and contracts are weakly enforced, FPA is not necessarily the best choice for the buyer. Thus, ABA is still widespread in Italy perhaps because of the likelihood of cost overruns and the fact that contracts are weakly enforced. This does not necessarily imply, however, that Italy should intensify its contract enforcement. Instead, we need to consider various alternatives to enhance efficiency under those circumstances.

4. SUMMARY AND CONCLUSIONS

ABA has been an important acquisition method for construction projects. Yet because economists have paid this auction format little attention, its mechanism is not well understood. To understand the rationale of ABA, an observer must discern its bidding behavior. Therefore, Chapter 2 presents an investigation of equilibrium in ABA, when the ex-ante cost equals the ex-post cost. More specifically, when at least three bidders participate in the auction, we provide necessary conditions for a SBNE bid function. Moreover, we show that when every possible bidder bids higher than the highest cost, only pooling SBNE bid functions exist. However, a closed-form solution of partially pooling SBNE bid functions is not tractable for more than three bidders. Instead, we provide the full characterization of SBNE bid functions with three bidders. This analysis provides good guidance to compare FPA with ABA in terms of the expected payoff for the buyer. The result shows that the better choice for the buyer is FPA. This raises the question about what conditions are favorable for ABA.

An answer to this question is presented in Chapter 3, where we consider the case of insolvent bidders defaulting. When a contract is likely to have weak enforcement, we show a circumstance, where the buyer is better off using ABA. The default risk and the weak enforcement of the contract could be an important clue as to why ABA is still popular in Italy. This does not necessarily imply, however, that to use FPA we need to intensify contract enforcement.

REFERENCES

Beye, Mema, and Joanna Nasr. 2008. "Celebrating Reform 2008." 92–107. Washington, D.C.: The World Bank. Repaying creditors without imprisoning debtors.

Cabrillo, Francisco, and Ben W. F. Depoorter. 1999. "Bankruptcy proceedings." *Encyclopedia of Law and Economics*, 5: 261–272.

Calveras, Aleix, Juan-Jose Ganuza, and Esther Hauk. 2004. "Wild bids. Gambling for resurrection in procurement contracts." *Journal of Regulatory Economics*, 26(1): 41–68.

Centre d'Etudes d'Assurances and Centre Scientifique et Technique du Bâtiment. 2010. "Liability and insurance regimes in the construction sector: national schemes and guidelines to stimulate innovation and sustainability." European Liability Insurance Organization Schemes, Paris, France.

Chang, Wei-Shiun, Bo Chen, and Timothy C. Salmon. 2014. "An investigation of the average bid mechanism for procurement auctions." *Management Science*, 61: 1237 – 1254.

Chen, Bin R, and Y. Stephen Chiu. 2011. "Competitive bidding with a bid floor." *International Journal of Economic Theory*, 7(4): 351–371.

Chillemi, Ottorino, and Claudio Mezzetti. 2014. "Optimal procurement mechanisms: bidding on price and damages for breach." *Economic Theory*, 55(2): 335–355.

Cocito, Federico. 2013. "Bankruptcy law, entrepreneurship and innovation. The 2006 Italian reform as a case study." Master's Thesis, Tilburg Law School International Business Law LLM.

Decarolis, Francesco. 2009. "When the highest bidder loses the auction: theory and evidence from public procurement." Working Paper, Bank of Italy.

Decarolis, Francesco. 2013. "Comparing public procurement auctions." Working Paper, Boston University Department of Economics.

Decarolis, Francesco. 2014. "Awarding price, contract performance, and bids screening: evidence from procurement auctions." *American Economic Journal: Applied Economics*, 6(1): 108–132.

Decarolis, Francesco, and Cristina Giorgiantonio. 2014. "Favouritism and Inefficiency in Procurement: Evidence from Public Works in Italy." *Public Procurement Place in the World*, Palgrave Macmillan UK: 155–183.

Decarolis, Francesco, and Cristina Giorgiantonio. 2015. "Local public procurement regulations: The case of Italy." *International Review of Law and Economics*, 43: 209–226.

Decarolis, Francesco, Cristina Giorgiantonio, and Valentina Giovanniello. 2011. "The awarding of public works in Italy: an analysis of the mechanisms for the selection of contractors." *Questioni di Economia e Finanza, Bank of Italy*, 13(83): 235–272.

Emery, Kenneth, Richard Cantor, David Keisman, and Sharon Ou. 2007. "Moody's Ultimate Recovery Database." Moody's, New York City, NY.

Engel, Andreas R. 2009. "Procurement Auctions and the Risk of Bankrupt Bidders." *Supply Management Research*, Gabler: 31–46.

Esposito, Gianluca, Sergi Lanau, and Sebastiaan Pompe. 2014. "Judicial System Reform in Italy; A Key to Growth." Working Paper, International Monetary Fund.

Flyvbjerg, Bent. 2006. "From Nobel prize to project management: Getting risks right." *Project Management Journal*, 37(3): 5–15.

Government Accountability Office. 2013. "Reverse Auctions: Guidance Is Needed to Maximize Competition and Achieve Cost Savings." Government Accountability Office, Washington, D.C.

Grant Thornton. 2005. "2005 Surety Credit Survey for Construction Contractors: The Bond Producer's Perspective." Grant Thornton LLP, Chicago, IL.

Grant Thornton. 2007. "2007 Surety Credit Survey for Construction Contractors: The Bond Producer Perspective." Grant Thornton LLP, Chicago, IL.

Ioannou, Photios G, and Sou-Sen Leu. 1993. "Average-bid method-competitive bidding strategy." *Journal of Construction Engineering and Management*, 119(1): 131–147.

Krishna, Vijay. 2009. "Auction theory." . 2nd ed. San Diego, CA: Elsevier Inc.

Law, Mark D. 2011. "The Utilization Of Accounting Cost Controls In Conjunction With

Information Technology In The Home Building Arena." *Review of Business Information Systems (RBIS)*, 12(4).

Lee, Jin Hyung. 2016*a*. "Average Bid versus First Price Auction: Trade Off Between Prevention of Default and Efficiency Loss." Working Paper, Texas A&M University.

Lee, Jin Hyung. 2016*b*. "Competitive Bidding Behavior in the Average Bid Auction." Working Paper, Texas A&M University.

Manganelli, Paolo. 2010. "Evolution of the Italian and US Bankruptcy Systems-A Comparative Analysis." *Journal of Business & Technology Law*, 5: 237.

Myerson, Roger B. 1981. "Optimal auction design." *Mathematics of operations research*, 6(1): 58–73.

Oglio, Livia. 2007. "Italian bankruptcy law requires restyling only one year after its enactment." International Bar Association, London, United Kingdom. Insolvency and Restructuring International.

Reny, Philip J. 2011. "On the Existence of Monotone Pure Strategy Equilibria in Bayesian Games." *Econometrica*, 79(2): 499–553.

Rodano, Giacomo, Nicolas Serrano-Velarde, and Emanuele Tarantino. 2012. "Bankruptcy law and the cost of banking finance." Working Paper, Oxford University Centre for Business Taxation.

Santella, Paolo, and Riccardo Brogi. 2004. "Two New Measures of Bankruptcy Efficiency." Law and Economics Working Paper.

Spagnolo, Giancarlo, Gian Luigi Albano, and Milo Bianchi. 2006. "Bid average methods in Procurement." *Rivista Di Politica Economica*, 1(2): 41–64.

Spulber, Daniel F. 1990. "Auctions and contract enforcement." *Journal of Law, Economics, & Organization*, 6(2): 325–344.

The World Bank. 2004. "Doing business in 2004." The World Bank, Washington, D.C.

Zheng, Charles Z. 2001. "High bids and broke winners." *Journal of Economic Theory*, 100(1): 129–171.

APPENDIX A

Lemma A1 proves that the SBNE bid function in the *average bid auction* is weakly increasing over cost.

Lemma A1 (Weakly Increasing property of SBNE in the *average bid auction*)**.** *Every SBNE bid function in the average bid auction is weakly increasing over cost.*

Proof of the Lemma.

Suppose that an equilibrium bid function, $b(c)$ is not weakly increasing over cost. Then, there exists an interval, $[c_1,\, c_2]$ where the equilibrium bid function, $b(c)$ is strictly decreasing. Assume that $b(c_1) = b_1$, and $b(c_2) = b_2$ with $b_1 > b_2$.

In order to prove that there does not exist any interval, where $b(c)$ is strictly decreasing, firstly, we need to prove $b_2 > c_2$, if $b(c)$ is strictly decreasing. Otherwise, $b_2 \leq c_2$. Then, the expected payoff $\pi(b_2,\, c_2) = Pr(win \,|\, b_2)(b_2 - c_2) \leq 0$. It gives rise to a profitable deviation to b_1 with an expected payoff $\pi(b_1,\, c_2) > 0$. Therefore, $b_2 > c_2$ in equilibrium.

Secondly, given $b_1 > b_2 > c_2$, we want to show that the probability of winning is strictly decreasing in b with $Pr(win|b_1) < Pr(win|b_2)$. Suppose that $Pr(win \,|\, b_i) \geq Pr(win \,|\, b_j)$ with $b_1 > b_i > b_j > b_1$. In the average bid auction, $Pr(win \,|\, b_i) = \frac{Pr(|\bar{b}-b_i| \leq |\bar{b}-b_k|\,,\forall k\neq i)}{\sum_{j=k}^{N} \mathbf{1}(|\bar{b}-b_k| = |\bar{b}-b_i|)}$, where $\mathbf{1}(|\bar{b} - b_k| = |\bar{b} - b_i|)$ is the indicator function, and $\bar{b} = \frac{1}{N}\sum_{k=1}^{N} b_k$. Since $b(c)$ is strictly decreasing in $[c_1, c_2]$, there is no b_k such that for any b_i with $b_1 > b_i > b_2$, $b_k = b_i$ for $\forall\ k \neq i$. Moreover, since for any two random variables $|\bar{b} - b_i|$ and $|\bar{b} - b_k|$ with $k \neq i$, $Pr(|\bar{b} - b_i| = |\bar{b} - b_k|) = 0$, then $\sum_{j=k}^{N} \mathbf{1}(|\bar{b} - b_k| = |\bar{b} - b_i|) = 1$. Thus, $Pr(win \,|\, b_i) = \frac{Pr(|\bar{b}-b_i| \leq |\bar{b}-b_k|\,,\forall k\neq i)}{\sum_{k=1}^{N} \mathbf{1}(|\bar{b}-b_k| = |\bar{b}-b_i|)} = Pr(|\bar{b}-b_i| < |\bar{b}-b_k|\,, \forall k \neq i)$.

Then, for any b_i and b_j with $b_1 > b_i > b_j > b_2$, $Pr(win|b_i) \geq Pr(win|b_j)$ implies

$Pr(|\bar{b} - b_i| < |\bar{b} - b_k|, \forall k \neq i) \geq Pr(|\bar{b} - b_j| < |\bar{b} - b_k|, \forall k \neq j)$. Therefore, $|\bar{b} - b_i| \leq |\bar{b} - b_j|$. Since $b_j < b_i$, either $b_j < b_i \leq \bar{b}$ or $b_j < \bar{b} \leq b_i$. In any case, when b_j slightly deviates upward from b_j to $b_2 + \varepsilon$, b_2 gets closer to the average. Accordingly, $Pr(win \,|\, b_j + \varepsilon) > Pr(win \,|\, b_j)$. Thus, the bidder with b_j has an incentive to deviate from b_j to $b_j + \varepsilon$. It contradicts that $b(c)$ is the strictly decreasing equilibrium bid function in $[c_1, c_2]$.

Finally, we want to show that there does not exist any interval, where the equilibrium bid function, $b(c)$ is strictly decreasing. Since bidder1 chooses b_1 in equilibrium, his expected profit is maximized at b_1. Thus, $\Delta p(b - c_1) + p\Delta b > 0$ for every $b \in (c_1, b_1)$, where $\Delta b = b + \varepsilon - b$, $\varepsilon > 0$, and $\Delta p = Pr(win|b + \varepsilon) - Pr(win|b)$. Then, for every $b \in (b_2, b_1) \subset (c_2, b_1)$, $\Delta p(b - c_2) + p\Delta b > \Delta p(b - c_1) + p\Delta b > 0$, since both $\Delta p < 0$ and $(b - c_2) < (b - c_1)$.

Therefore, when switching from b_2 to b_1, the marginal profit of bidder 2 is strictly above that of bidder 1. Accordingly, bidder 2 has an incentive to deviate upward from b_2. It implies there is not any Nash equilibrium that is decreasing anywhere. Thus, the equilibrium bid is weakly increasing over cost. □

Lemma A2 is used for the proof of Proposition 2.

Lemma A2 (Shape of SBNE Bid Function). *Suppose that* $c^* = min\{r, \bar{c}\}$. *If* $b : [\underline{c}, \bar{c}] \to \mathbb{R}_+$ *is an SBNE bid function, then there exists* $\widetilde{c} \in [\underline{c}, \bar{c}]$ *such that*

1. *for every* $c, c' \in [\underline{c}, \widetilde{c})$, $b(c) = b(c')$. *Let this common bid for costs below* $\widetilde{c}$ *be* $\widetilde{b}$.

2. *for every* $c, c' \in [\widetilde{c}, \bar{c}]$ *where* $c < c'$, $b(c) < b(c') \leq \bar{c}$.

3. $F(\widetilde{c}) \geq \frac{3}{4} F(c^*)$.

4. *The bidder with cost* $\widetilde{c}$ *is indifferent between bidding* $\widetilde{b}$ *and* $b(\widetilde{c})$

Proof of the Lemma. The *Lemma* is proven in series of steps.

Step 1. *$\widetilde{c}$ exists and for every $c < \widetilde{c}$, $b(c) = \widetilde{b}$*

According to Lemma A1, the equilibrium bid function, $b(c)$ is weakly increasing in the cost c. To show that $b(c)$ is a constant $\widetilde{b}$ for every $c < \widetilde{c}$ in equilibrium, we want to show that the equilibrium bid function, $b\,(c)$ can not be strictly increasing in the neighborhood of $\underline{c}$. In order to do it, suppose by contradiction that the equilibrium bid function, $b(c)$ is strictly increasing in the neighborhood of $\underline{c}$.

Let the neighborhood be $[\underline{c},\, \underline{c} + \delta)$ for a sufficiently small $\delta > 0$. Then, i bidder with cost $c \in [\underline{c},\, \underline{c} + \delta)$ has the winning probability as follows.

$Pr(win|\, \underline{c} \leq c < \underline{c} + \delta)$

$= Pr(A)Pr(win|\, \widetilde{c} \leq c \leq \underline{c}+\delta, A)+Pr(B)Pr(win|\, \widetilde{c} \leq c \leq \underline{c}+\delta, B)+Pr(C)Pr(win|\, \widetilde{c} \leq c \leq \underline{c} + \delta, C) = F(c^*)^2(2F(c)(F(c^*) - F(c))) + \frac{1}{2}\binom{2}{1}F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

$= F(c^*)^2(2F(c)(F(c^*) - F(c))) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

where $c^* = min\{r,\, \overline{c}\}$, $b(c^*) = c^*$. Costs of rivals $j,\, k$ are classified as follows.

$A = \{\, (c_j,\, c_k)\,|\, c_j \,\text{and}\, c_k \leq c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\},$

$B = \{\, (c_j,\, c_k)\,|\, c_j \;\text{ or }\; c_k > c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\},$

$C = \{\, (c_j,\, c_k)\,|\, c_j \,\text{and}\, c_k > c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\}.$

For every $c < c' \in [\underline{c},\, \underline{c} + \delta) \cap [\underline{c},\, \ddot{c})$ with $F\,(\ddot{c}) = \frac{1}{2}F(c^*)$, $Pr\,(win\,|\,c) < Pr\,(win\,|\,c')$. Thus, the deviation gives rise to the increase in both bid and winning probability, and a bidder in the neighborhood of $\underline{c}$ has an incentive to deviate. It contradicts that $b\,(c)$ is the equilibrium bid function which is strictly increasing in $[\underline{c},\, \underline{c} + \delta)$. Therefore, there exists a $\widetilde{c}$ such that for every $c \in [\underline{c}, \widetilde{c}]$, $b(c) = \widetilde{b}$

Step 2. $p = F(\widetilde{c}) \geq \frac{3}{4}F(c^*)$

Accordingly, there is a $\widetilde{c}$ such that $\widetilde{b} = b\,(c)$, for every $c \in [\underline{c},\, \widetilde{c})$.

When a bidder with $c < \widetilde{c}$ submits $\widetilde{b}$, the winning probability is as follows

$Pr(win \,|\, \widetilde{b}) = Pr(win \,|\, \underline{c} \leq c < \widetilde{c}) = F(c^*)^2(\frac{1}{3}p^2 + p(F(c^*) - p)) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

When a bidder with $c < \widetilde{c}$ deviates from $\widetilde{b}$ to $\widetilde{b} + \varepsilon$, he pretends to be a bidder with $\widetilde{c} \leq c' \leq c^*$. Then, his winning probability is as follows

$Pr(win \,|\, \widetilde{b} + \varepsilon) = Pr(win \,|\, \widetilde{c} \leq c' \leq c^*) = F(c^*)^2 2p'(F(c^*) - p') + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$,

where $p' = F(b^{-1}(\widetilde{b} + \varepsilon)), \widetilde{c} \leq c' = b^{-1}(\widetilde{b} + \varepsilon) < c^*$.

Thus, for $c \in [\underline{c}, \widetilde{c}]$, $\pi(\widetilde{b}, c) = Pr(win \,|\, \widetilde{b})(\widetilde{b} - c)$, $\pi(\widetilde{b} + \varepsilon, c) = Pr(win \,|\, \widetilde{b} + \varepsilon)(\widetilde{b} + \varepsilon - c)$.

Now, we want to show that if $p = F(\widetilde{c}) = \frac{3}{4}F(c^*)$, bidders with $c \in [\underline{c}, \widetilde{c})$ does not have any incentive to deviate from $\widetilde{b}$. When $F(\widetilde{c}) = \frac{3}{4}F(c^*)$, $\left\{\frac{1}{3}p^2 + p(F(c^*) - p)\right\} = 2p(F(c^*) - p)$. Moreover, since for any sufficiently small $\varepsilon > 0$ and $p' = F(b^{-1}(\widetilde{b} + \varepsilon))$, then, $2p'(F(c^*) - p') < 2p(F(c^*) - p)$. Thus, $Pr(win \,|\, \widetilde{b} + \varepsilon) < Pr(win \,|\, \widetilde{b})$. Thus, $\Delta Pr(win \,|\, \widetilde{b}) = Pr(win \,|\, \widetilde{b} + \varepsilon) - Pr(win \,|\, \widetilde{b}) < 0$. $\Delta\pi(\widetilde{b}, \widetilde{c}) = \Delta Pr(win \,|\, \widetilde{b})(\widetilde{b} - \widetilde{c}) + Pr(win \,|\, \widetilde{b})\Delta b < 0$ for any small $\varepsilon > 0$ and $\Delta b = \varepsilon$. However, for $c \in [\underline{c}, \widetilde{c})$, $\Delta Pr(win \,|\, \widetilde{b})(\widetilde{b} - c) + Pr(win \,|\, \widetilde{b})\Delta b < \Delta Pr(win \,|\, \widetilde{b})(\widetilde{b} - \widetilde{c}) + Pr(win \,|\, \widetilde{b})\Delta b < 0$. Therefore, bidders with $c \in [\underline{c}, \widetilde{c})$ does not have any incentive to deviate from $\widetilde{b}$.

Next, we also want to show that if $p = F(\widetilde{c}) > \frac{3}{4}F(c^*)$, bidders with $c \in [\underline{c}, \widetilde{c})$ does not have any incentive to deviate from $\widetilde{b}$. When $F(\widetilde{c}) > \frac{3}{4}F(c^*)$, $\frac{1}{3}p^2 + p(F(c^*) - p) > 2p(F(c^*) - p)$. Then for any small $\varepsilon \geq 0$, $Pr(win \,|\, \widetilde{b}) > Pr(win \,|\, \widetilde{b} + \varepsilon)$. Therefore, there exists a sufficiently small $\varepsilon > 0$ such that for $c \in [\underline{c}, \widetilde{c})$, $Pr(win \,|\, \widetilde{b})(\widetilde{b} - c) > Pr(win \,|\, \widetilde{b} + \varepsilon)(\widetilde{b} + \varepsilon - c)$. Thus, a bidder with $c \in [\underline{c}, \widetilde{c})$ does not have any incentive to deviate from $\widetilde{b}$ to $\widetilde{b} + \varepsilon$ since there is a descret decrease in the winning probability, which makes the bidders with $c \in [\underline{c}, \widetilde{c})$ discouraged from deviating. Therefore, if $p = F(\widetilde{c}) > \frac{3}{4}F(c^*)$, bidders with $c \in [\underline{c}, \widetilde{c})$ does not have any incentive to deviate from $\widetilde{b}$.

Step 3. $b\,(c) < b\,(c')$*, for every* $c,\, c' \in [\widetilde{c},\, c^*]$ *with* $c < c'$

Suppose that there exists another identical bid b' such that for every $c \in (\dot{c},\, \ddot{c}) \subset [\widetilde{c},\, c^*]$, $b' = b(c)$. Let $q = F(\ddot{c}) - F(\dot{c})$, which implies q is the portion of bidders choosing the identical b' among all potential bidders, when all rivals' costs are below c^*.
$h = F(\dot{c})$ is the portion of bidders choosing $b < b'$, when all rivals' costs are below c^*.
$r = F(c^*) - F(\ddot{c})$ is the portion of bidders choosing $b > b'$, when all rivals' costs are below c^*.

When bidders with $c \in (\dot{c},\, \ddot{c})$ choose b', the winning probability is as follows:
$H(b') = F(c^*)^2(\frac{1}{3}q^2 + \frac{1}{2} \times 2 \times hq + \frac{1}{2} \times 2 \times qr + 2hr) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$
$= F(c^*)^2(\frac{1}{3}q^2 + q(F(c^*) - q) + 2hr) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

We want to show the bidder with $c \in (\dot{c},\, \ddot{c})$ has an incentive to deviate downward from b'. If the bidder slightly deviates downward from b' to $b' - \gamma$, the winning probability conditional on all rivals' costs below c^* is $2h'(F(c^*) - h')$, where $h' = F(b^{-1}(b' - \gamma))$. Since $[\underline{c},\, \widetilde{c}] \subset [\underline{c},\, \dot{c}]$, it implies $h > \frac{3}{4}F(c^*)$ and $\frac{1}{3}q^2 + q(F(c^*) - q) + 2hr < 2h(q + r) = 2h(F(c^*) - h) < 2h'(F(c^*) - h')$.

Then, there exists a sufficiently small $\gamma > 0$ such that $H(b')(b' - c) < G(b')(b' - \gamma - c)$, where $H(b') = F(c^*)^2(\frac{1}{3}q^2 + q(F(c^*) - q) + 2hr) + F(c^*)(1 - F(c^*)) + (1 - F(c^*)$,
$G(b') = F(c^*)^2 2h(F(c^*) - h) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

Furthermore, $G(b')(b' - \gamma - c) < G(b' - \gamma)(b' - \gamma - c)$.
where $G(b' - \gamma) = F(c^*)^2 2h'(F(c^*) - h') + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

Since $H(b')(b' - c) < G(b')(b' - \gamma - c) < G(b' - \gamma)(b' - \gamma - c)$, the bidder with $c \in (\dot{c},\, \ddot{c})$ has an incentive to deviate downward from b'. Accordingly, there cannot exist any identical b' such that for every $c \in (\dot{c},\, \ddot{c})$, $b' = b(c)$.

Step 4. *If* $r > \overline{c}$*, then* $b(\overline{c}) = \overline{c}$

Suppose by contradiction $b(\overline{c}) > \overline{c}$. Since the bid is strictly increasing over the interval

$[\widetilde{c},\, \overline{c}]$ according to the previous step, the winning probability of $\overline{c}$ bidder is zero. If he slightly deviates downward, his bid equals one of rivals' bids, and the winning probability is not zero anymore. Thus, $\overline{c}$ bidder has an incentive to deviate downward. It contradicts that $b(c)$ is an equilibrium bid function.

Step 5. *the bidder with $\widetilde{c}$ is indifferent between $\widetilde{b}$ and $b(\widetilde{c})$ in terms of his expected payoff*

Define $p = F(\widetilde{c})$, $h = F(c)$,

$Pr(win|\, c < \widetilde{c}) = H(\widetilde{c})$

$= Pr(A)Pr(win|\, c < \widetilde{c}, A) + Pr(B)Pr(win|\, c < \widetilde{c}, B) + Pr(C)Pr(win|\, c < \widetilde{c}, C)$

$= F(c^*)^2(\frac{1}{3}p^2 + p(F(c^*) - p)) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

$Pr(win|\, \widetilde{c} \leq c \leq c^*) = G(c)$

$= Pr(A)Pr(win|\, \widetilde{c} \leq c \leq c^*, A) + Pr(B)Pr(win|\, \widetilde{c} \leq c \leq c^*, B) + Pr(C)Pr(win|\, \widetilde{c} \leq c \leq c^*, C)$

$= F(c^*)^2 2h(F(c^*) - h) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

where $c^* = min\{r,\, \overline{c}\}$, $b(c^*) = c^*$. Costs of rivals $j,\, k$ are classified as follows

$A = \{\, (c_j,\, c_k)\,|\, c_j \text{ and } c_k \leq c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\},$

$B = \{\, (c_j,\, c_k)\,|\, c_j \text{ or } c_k > c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\},$

$C = \{\, (c_j,\, c_k)\,|\, c_j \text{ and } c_k > c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\}.$

Firstly, suppose that there exists an equilibrium where $H(\widetilde{c})(\widetilde{b} - \widetilde{c}) < G(\widetilde{c})(b(\widetilde{c}) - \widetilde{c})$. Then, there is a sufficiently small $\gamma > 0$ such that $H(\widetilde{b})(\widetilde{b} - \widetilde{c} + \gamma) < G(\widetilde{c})(b(\widetilde{c}) - \widetilde{c})$. Thus, a bidder with $\widetilde{c} - \gamma$ has an incentive to deviate from $\widetilde{b}$ to $b(\widetilde{c})$. It contradicts that there exists the equilibrium where $H(\widetilde{c})(\widetilde{b} - \widetilde{c}) < G(\widetilde{c})(b(\widetilde{c}) - \widetilde{c})$.

Alternatively, suppose that there exists an equilibrium where $H(\widetilde{c})(\widetilde{b} - \widetilde{c}) > G(\widetilde{c})(b(\widetilde{c}) - \widetilde{c})$. Then, there is a sufficiently small $\gamma > 0$ such that $b' = b(\widetilde{c}) + \gamma$, $c' = b^{-1}(b')$, $H(\widetilde{c})(\widetilde{b} - \widetilde{c}) > G(\widetilde{c})(b' - \widetilde{c})$. Furthermore, there exists $\kappa > 0$ such that $H(\widetilde{c})(\widetilde{b} - \widetilde{c} - \kappa) = G(\widetilde{c})(b' - \widetilde{c})$.

Now, consider $\widetilde{c} < \hat{c} < min\{\widetilde{c} + \kappa,\, c'\}$. Then, $H(\widetilde{c})(\widetilde{b} - \hat{c}) \geq G(\widetilde{c})(b' - \widetilde{c}) > G(\hat{c})(b(\hat{c}) - \hat{c})$.

Thus, a bidder with $\hat{c}$ has an incentive to deviate from $b(\hat{c})$ to $\widetilde{b}$. It contradicts that there exists the equilibrium where $H(\widetilde{c})(\widetilde{b} - \widetilde{c}) > G(\widetilde{c})(b(\widetilde{c}) - \widetilde{c})$. □

Proposition 2 (Equilibrium Characterization)**.**

Proof of the Proposition.

Step 1. *Derive a partially pooling equilibrium.*

Suppose the bid function is $b : [\underline{c},\, c^*] \to \mathbb{R}_+$. According to Lemma A2, there exists a $\widetilde{c}$ such that for every $c \in [\underline{c},\, \widetilde{c})$, $\widetilde{b} = b(c)$, and for every $c,\, c' \in [\widetilde{c},\, c^*]$ with $c < c'$, $b(c) < b(c')$ and $F(\widetilde{c}) \geq \frac{3}{4}F(c^*)$. Then, we use Lemma A2 to derive the strictly increasing bidding strategy.

Define $p = F(\widetilde{c})$, $h = F(c)$,

$Pr(win|\, c < \widetilde{c}) = H(\widetilde{c})$

$= Pr(A)Pr(win|\, c < \widetilde{c}, A) + Pr(B)Pr(win|\, c < \widetilde{c}, B) + Pr(C)Pr(win|\, c < \widetilde{c}, C)$

$= F(c^*)^2(\frac{1}{3}p^2 + p(F(c^*) - p)) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

$Pr(win|\, \widetilde{c} \leq c \leq c^*) = G(c)$

$= Pr(A)Pr(win|\, \widetilde{c} \leq c \leq c^*, A) + Pr(B)Pr(win|\, \widetilde{c} \leq c \leq c^*, B) + Pr(C)Pr(win|\, \widetilde{c} \leq c \leq c^*, C)$

$= F(c^*)^2 2h(F(c^*) - h) + F(c^*)(1 - F(c^*)) + (1 - F(c^*))^2$

where $c^* = min\{r,\, \overline{c}\}$, $b(c^*) = c^*$. Costs of rivals $j,\, k$ are classified as follows

$A = \{\, (c_j,\, c_k) \,|\, c_j \text{ and } c_k \leq c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\}$,

$B = \{\, (c_j,\, c_k) \,|\, c_j \text{ or } c_k > c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\}$,

$C = \{\, (c_j,\, c_k) \,|\, c_j \text{ and } c_k > c^*,\, \forall\, c_j,\, c_k \in [\underline{c},\, \overline{c}]\}$.

The strictly increasing bid is derived from the maximization of the expected payoff $max_y G(y)(b(y) - c)$ for y and a given $c \in [\widetilde{c},\, c^*]$.

Thus, FOC: $g(y)(b(y)-c)+G(c)b'(y)$, where $g(c)=G'(c)$

Consider, $g(y)(b(y)-c)+G(c)b'(y)=0$ at $y=c$.

This means when the bidder tells the truth about his cost type, his expected payoff is maximized. Furthermore, since $F(\widetilde{c}) \geq \frac{3}{4}F(c^*)$, there is the specific $c \in [\widetilde{c},\, c^*]$ satisfying FOC.

By the integral by parts, $[G(y)b(y)]_c^{c^*} = [yG(y)]_c^{c^*} - \int_c^{c*} G(y)dy$

Then, $b(c) = c + \int_c^{c^*} \frac{G(y)}{G(c)}dy$ at $c \in [\widetilde{c},\, c^*]$.

Next, we use Lemma A2 and the strictly increasing bidding strategy derived to caculate the identical bid $\widetilde{b}$. According to the Lemma A2, the bidder with $\widetilde{c}$ is indifferent between $\widetilde{b}$ and the strictly increasing bid $b(\widetilde{c})$ in terms of his expected payoff.

Thus, $H(\widetilde{c})(\widetilde{b}-\widetilde{c}) = G(\widetilde{c})(b(\widetilde{c})-\widetilde{c})$.

Accordingly, $\widetilde{b} = \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy$ at $c \in [\underline{c},\, \widetilde{c})$

Finally, $p = F(\widetilde{c}) \geq \frac{3}{4}F(c^*)$ and $b(c) = \begin{cases} \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy, & \text{if } c \in [\underline{c},\, \widetilde{c}) \\ c + \int_c^{c^*} \frac{G(y)}{G(c)}dy, & \text{if } c \in [\widetilde{c},\, c^*] \end{cases}$

Step 2. *If* $p = F(\widetilde{c}) \geq \frac{3}{4}F(c^*)$ *and* $b(c) = \begin{cases} \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy, & \text{if } c \in [\underline{c},\, \widetilde{c}) \\ c + \int_c^{c^*} \frac{G(y)}{G(c)}dy, & \text{if } c \in [\widetilde{c},\, c^*] \end{cases}$, $b(c)$ *constitutes Bayesian Nash Equilibria.*

First of all, consider $\widetilde{b} = \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy = \widetilde{c} + \frac{G(\widetilde{c})}{H(\widetilde{c})} \int_{\widetilde{c}}^{c^*} \frac{G(y)}{G(\widetilde{c})}dy$.

Since $2p(F(c^*)-p) < \frac{1}{3}p^2 + p(F(c^*)-p)$ at $p = F(\widetilde{c}) > \frac{3}{4}F(c^*)$, $\widetilde{b} = \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy <$ $b = \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{G(\widetilde{c})}dy$. It implies if $p = F(\widetilde{c}) > \frac{3}{4}F(c^*)$, there is a jump of bid at $\widetilde{c}$.

Next, we need to show that if $\widetilde{b} = \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy$ at $c \in [\underline{c},\, \widetilde{c})$, there is no deviation from a potential bidder with cost $c \in [\underline{c}, \widetilde{c})$. In order to do it, we need to show that if $\widetilde{b} > \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}d$ or $\widetilde{b} < \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}d$, a bidder with $c \in [\widetilde{c},\, c^*]$ has an incentive to deviate downward or a bidder with $c \in [\underline{c},\, \widetilde{c})$ has an incentive to deviate upward, respectively.

Then, suppose that for a sufficiently small $\varepsilon > 0$, $\widetilde{b} = \widetilde{c} + \varepsilon + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy$. There exists a δ such that for sufficiently small δ with $0 < \delta < \varepsilon$, $H(\widetilde{c})(\varepsilon - \delta) + \int_{\widetilde{c}}^{c^*} G(y)dy > \int_{\widetilde{c}+\delta}^{c^*} G(y)dy$.

Accordingly, the bidder with $\widetilde{c} + \delta$ has an incentive to deviate downward to $\widetilde{b}$

Moreover, for $0 < \varepsilon < \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy$, take $\widetilde{b} = \widetilde{c} - \varepsilon + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy$.

Since $0 < -H(\widetilde{c})\varepsilon + \int_{\widetilde{c}}^{c^*} G(y)dy < \int_{\widetilde{c}}^{c^*} G(y)dy$, there exists a $c \in (\widetilde{c} - \varepsilon, \widetilde{c})$ such that $0 < H(\widetilde{c})(\widetilde{c} - c) - H(\widetilde{c})\varepsilon + \int_{\widetilde{c}}^{c^*} G(y)dy < \int_{\widetilde{c}}^{c^*} G(y)dy$

Thus, $0 < H(\widetilde{c})(\widetilde{c} - c) - H(\widetilde{c})\varepsilon + \int_{\widetilde{c}}^{c^*} G(y)dy < \int_{\widetilde{c}}^{c^*} G(y)dy < G(\widetilde{c})(\widetilde{c} - c) + \int_{\widetilde{c}}^{c^*} G(y)dy$.

It implies that the bidder with $c \in (\widetilde{c} - \varepsilon, \widetilde{c})$ has an incentive to choose the strictly increasing bid $b(\widetilde{c}) = \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{G(\widetilde{c})}dy$, instead of $\widetilde{b}$.

Therefore, the equilibrium bid function for $\widetilde{b}$ is $\widetilde{b} = \widetilde{c} + \int_{\widetilde{c}}^{c^*} \frac{G(y)}{H(\widetilde{c})}dy$

Lastly, we need to show that if $b(c) = c + \int_{c}^{c^*} \frac{G(y)}{G(c)}dy$ at $c \in [\widetilde{c}, c^*]$, there is no deviation from a potential bidder with cost $c \in [\widetilde{c}, c^*]$. According to Lemma A2, bidders with $c \in [\widetilde{c}, c^*]$ choose $b(c)$ as the strictly increasing bid strategy. Thus, we want to show that the given strictly increasing bidding strategy $b(c) = c + \int_{c}^{c^*} \frac{G(y)}{G(c)}dy$ is the best response for a bidder with cost $c \in [\widetilde{c}, c^*]$. In order to do it, we want to prove that if a bidder with cost c pretends to be a bidder with cost $x \neq c$, his expected payoff is worse off.

Firstly, $x < c$

$G(x)(b(x) - c) - G(c)(b(c) - c)$

$= F(c^*)^2\{2F(x)(F(c^*) - F(x))(b(x) - c) - 2F(c)(F(c^*) - F(c))(b(c) - c)\}$

$= F(c^*)^2\{2F(x)(F(c^*) - F(x))(x - c) + 2\int_{x}^{c} F(y)(F(c^*) - F(y))dy\} < 0$

Therefore, $G(x)(b(x) - c) < G(c)(b(c) - c)$

Secondly, $x > c$

$G(x)(b(x) - c) - G(c)(b(c) - c)$

$= F(c^*)^2\{2F(x)(F(c^*) - F(x))(b(x) - c) - 2F(c)(F(c^*) - F(c))(b(c) - c)\}$

$= F(c^*)^2\{2F(x)(F(c^*) - F(x))(x - c) - 2\int_{x}^{c} F(y)(F(c^*) - F(y))dy\} < 0$

Therefore, $G(x)(b(x) - c) < G(c)(b(c) - c)$

Thus, $b(c) = c + \int_{c}^{c^*} \frac{G(y)}{G(c)} dy$ is the best response for a bidder with $c \in [\widetilde{c},\, c^*]$.

Finally, when bidders with $c \in [\underline{c},\, \widetilde{c})$ choose $\widetilde{b}$, and bidders with $c \in [\widetilde{c},\, c^*]$ choose $b(c)$ at the same time, there is no deviation. Therefore, this is the equilibrium bid function.

Step 3. *Derive a pooling equilibrium*

Suppose the equilibrium bid function is $b : [\underline{c},\, \overline{c}] \to \mathbb{R}_+$. First of all, we want to show that there exists an identical bid in the neighborhood of $\underline{c}$. Lemma A1 indicates that the bid is weakly increasing over cost. Then, there is a $\widetilde{c}_{low}$ such that for every $c,\, c' \in [\underline{c},\, \widetilde{c}_{low})$, $b(c) = b(c')$ and $F(\widetilde{c}_{low}) \geq \frac{3}{4}$ analogously to Lemma A2.

Likewise, there exists a $\widetilde{c}_{high}$ such that for every $c,\, c' \in (\widetilde{c}_{high},\, \overline{c}]$, $b(c) = b(c')$. Otherwise, $b(c)$ is strictly increasing over the interval $(\widetilde{c}_{high},\, \overline{c}]$. Bidder with $\overline{c}$ knows his bid is the most distant from the average and loses the auction. Furthermore, since $b(\overline{c}) > \overline{c}$, he does not have any reason to follow the strictly increasing bidding strategy and has an incentive to deviate downward. Analogously to Lemma A2, $p' = 1 - F(\widetilde{c}_{high})$, $p' \geq \frac{3}{4}$.

Since both $p \geq \frac{3}{4}$ and $p' \geq \frac{3}{4}$, then, $[\underline{c},\, \widetilde{c}_{low})$ and $(\widetilde{c}_{high},\, \overline{c}]$ are interconnected. Accordingly, for every $c \in [\underline{c},\, \widetilde{c}_{low})$ and every $c' \in (\widetilde{c}_{high},\, \overline{c}]$, $b(c) = b(c')$. Therefore, there exists a $\widetilde{b} \in (\overline{c}, r]$ such that for $c \in [\underline{c},\, \overline{c}]$, $b(c) = \widetilde{b}$.

Step 4. *For $c \in [\underline{c},\, \overline{c}]$, if $b(c) = \widetilde{b} > \overline{c}$, $b(c)$ constitutes Bayesian Nash Equilibria.*

For every $c,\, c' \in [\underline{c},\, \overline{c}]$, if $b(c) = b(c') = \widetilde{b}$, nobody has an incentive to deviate. Otherwise, if there is a bidder who deviates from $\widetilde{b}$, his winning probability is zero, and his expected payoff is zero. $\square$

Proposition 3 (Comparison between FPA and ABA)**.**

Proof of the Proposition.

The expected payoff of the buyer is determined as follows.

$\pi_i = (1 - (1 - F(r))^n)(V - E(b_i^{pay})) - D(1 - F(r))^n$

where $i \in \{FPA,\ ABA\}$, r: the reservation price, n: the number of bidders,

V: the project value, D : the damage caused by the auction failure,

b_i^{pay}: the payment that the buyer should pay for the winner in either FPA or ABA,

$E(b_i^{pay})$: the expected payment of the buyer in either FPA or ABA

$(1 - F(r))^n$ implies the possibility of the auction failure in the case that every possible bidder's cost turns out to exceed the reservation price. Thus, $(1 - (1 - F(r))^n)$ means the possibility of the successful auction. Actually, given the reservation price, the expected damage caused by the auction failure $D(1-F(r))^n$ is the same across two auction formats. Therefore, the expected payment of the buyer $E(b_i^{pay})$ determines which of FPA and ABA is better for the buyer in terms of the expected payoff. Therefore, we investigate which of FPA and ABA is greater than the other in terms of the expected payment of the buyer.

Step 1. *Comparison of the expected payment between FPA and ABA when three bidders bid*

According to the Revenue Equivalence Theorem between the first price reverse auction and the second price reverse auction, the expected payment for the buyer in the second price reverse auction is equivalent to the expected payment for the buyer in the first price reverse auction.

Now, $b_{SPA}^{(2)}$ is the second lowest bid in the second price reverse auction and $b_{ABA}^{(2)}$ is the second lowest bid in the average bid auction. b_{ABA}^{win} is the winning bid in the average bid auction, and $c^{(2)}$ is the second lowest cost. Given cost, $b_{ABA}^{(2)} > b_{SPA}^{(2)}$ since $b_{ABA}^{(2)} > c^{(2)} = b_{SPA}^{(2)}$.

Furthermore, considering three bidders, $b_{ABA}^{win} = b_{ABA}^{(2)}$. Accordingly, it implies the expected payment for the buyer in the average bid auction is greater than the expected payment for the buyer in the second price reverse auction.

Step 2. *Comparison of the expected payment between FPA and ABA when more than three* *bidders bid*

Consider the following case without loss of generality.

$b^{(1)}_{ABA} \leq b^{(2)}_{ABA} \leq \cdots \leq b^{(i)}_{ABA} \leq \cdots \leq b^{(n)}_{ABA}$,

where $b^{(i)}_{ABA}$: i th lowest bid in ABA, and b^{win}_{ABA}: winning bid in ABA

Clearly, $b^{(2)}_{ABA} \leq b^{win}_{ABA}$ according to the rule to determine the winner in ABA

Now, we want to show that $b^{(2)}_{SPA} = c^{(2)} < b^{(2)}_{ABA}$ in equilibrium.

In order to do it, firstly, suppose that $b^{(2)}_{ABA} = c^{(2)}$ in equilibrium. Then, we want to show that there exists a bid b' such that $Pr(win \mid b') > 0$, and $b' > b^{(2)}_{ABA} = c^{(2)}$, and $b^{(2)}_{ABA}$ has an incentive to deviate from $c^{(2)}$ to b'. Thus, for every $b' > b^{(2)}_{ABA}$, if $Pr(win \mid b') = 0$, it implies $c^{(2)} = c^{(3)} = \cdots = c^{(n)}$, which contradicts that $c^{(2)} < c^{(3)} < \cdots < c^{(n)} \leq \overline{c}$. Therefore, $b^{(2)}_{ABA}$ has an incentive to switch toward b', which contradicts $b^{(2)}_{ABA} = c^{(2)}$ is the equilibrium bid.

Since $b^{(2)}_{SPA} = c^{(2)} < b^{(2)}_{ABA} \leq b^{win}_{ABA}$, the expected payment for the buyer in the average bid auction is greater than the expected payment for the buyer in the second price reverse auction. □

Lemma 1 (Shape of SBNE Bid Function)**.**

Proof of the Lemma.

Step 1. *$\widetilde{c}$ exists and for every c with $\underline{c} \leq c < \widetilde{c}$, $b(c) = \widetilde{b}$*

According to Lemma A1, the equilibrium bid function, $b(c)$ is weakly increasing in the cost c. To show that $b(c)$ is a constant $\widetilde{b}$ for every $c < \widetilde{c}$ in equilibrium, we want to show that $b\,(c)$ can not be strictly increasing in the neighborhood of $\underline{c}$. In order to do it, suppose by contradiction that the equilibrium bid function, $b(c)$ is strictly increasing in the neighborhood of $\underline{c}$. Thus, for some $\widetilde{c}$, bid is strictly increasing in $c \in [\underline{c}, \widetilde{c})$. Since bid is increasing, for every $c \in (\underline{c}, \overline{c}]$, $b(\underline{c}) < b(c)$. Therefore, the only way to be equilibrium is that $b(\underline{c}) = \overline{b}$, where $\overline{b} = \frac{1}{n}\sum_{i=1}^{n} b_i$. Thus, for $\underline{c} = c^{(1)} < c^{(2)} < \cdots < c^{(n)}$, then,

$b(c^{(k)}) = \bar{b}$. It contradicts $b(c)$ is stricly increasing in $c \in [\underline{c}, \widetilde{c})$ in equilibrium. Therefore, $b(c)$ can not be strictly increasing in the neighborhood of $\underline{c}$. Finally, there exists $\widetilde{c}$ such that for c with $\underline{c} \le c < \widetilde{c}$, $b(c) = \widetilde{b}$.

Step 2. $F(\widetilde{c}) \ge \frac{n(n-2)}{(n-1)^2}$

Firstly, sets are defined as follows

$J \equiv \{1, 2, \cdots, n\}$, and $i \in J$

$IBid_i(b) = \{j \in \{1, \cdots, n\} \mid b_j = b, \forall j \neq i\}$,

$LBid_i(b) = \{j \in \{1, \cdots, n\} \mid b_j < b, \forall j \neq i\}$,

$UBid_i(b) = \{j \in \{1, \cdots, n\} \mid b_j > b, \forall j \neq i\}$,

$CBid_i(b) = \bigcap_{\forall j \notin \{i\} \cup IBid_i(b)} \{(b, \boldsymbol{b}_{-i}) \mid |\bar{b} - b| < |\bar{b} - b_j|\}$, where $\boldsymbol{b}_{-i} = (b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_n)$

$p = F(\widetilde{c})$, $p' = F(b^{-1}(b'))$, $m = \left[\frac{n-1}{2}\right]$, where if $l \le \frac{n-1}{2} < l + 1$, $\left[\frac{n-1}{2}\right] = l$, l: integer,

$\bar{b} = \frac{1}{n} \sum_{j=1}^{n} b_j$

Secondly, the winning probability of bidder i with cost $\underline{c} \le c < \widetilde{c}$ submitting an identical bid $\widetilde{b}$ is as follows

$Pr(win \mid b_i = \widetilde{b}) = \sum_{k=0}^{m} \frac{1}{n-k} Pr(|IBid_i(\widetilde{b})| = n-k-1\,, CBid_i(\widetilde{b}),\, |UBid_i(\widetilde{b})| = k)$

$= \frac{1}{n} p^{n-1} + \frac{1}{n-1} \binom{n-1}{n-2} p^{n-2}(1-p) + \cdots$

$+ \frac{1}{m+1} \binom{n-1}{m} p^m (1-p)^{n-m-1} Pr(CBid_i(\widetilde{b}) \mid |IBid_i(\widetilde{b})| = m,\, |UBid_i(\widetilde{b})| = n - m - 1)$

Let $b' = \widetilde{b} + \varepsilon$ for sufficiently small $\varepsilon > 0$. When a bidder i with cost $\underline{c} \le c < \widetilde{c}$ slightly deviates upward from $\widetilde{b}$ to b', the winning probability is as follows

$Pr(win \mid b_i = b') = \sum_{k=1}^{n-1} Pr(|LBid_i(b')| = k,\, |UBid_i(b')| = n - k - 1\,, CBid_i(b'))$

$= \sum_{k=1}^{n-2} \binom{n-1}{k} (p')^k (1-p')^{n-k-1} Pr(CBid_i(b') \mid |LBid_i(b')| = k,\, |UBid_i(b')| = n - k - 1)$

Now, we want to show that $p \ge \frac{n(n-2)}{(n-1)^2}$ is the necessary condition for the equilibrium. In order to do it, we show that if $p < \frac{n(n-2)}{(n-1)^2}$, there always exists a profitable deviation from $\widetilde{b}$.

If $p < \frac{n(n-2)}{(n-1)^2}$, the following inequality is also true.

$\frac{(n-1)^2}{n} p^{n-2}\{p - \frac{n(n-2)}{(n-1)^2}\} = \frac{1}{n}p^{n-1} + \frac{1}{n-1}\binom{n-1}{n-2}p^{n-2}(1-p) - \binom{n-1}{n-2}p^{n-2}(1-p) < 0$

It implies that for $n > 3$, $Pr(win \,|\, \widetilde{b}) < Pr(win \,|\, b')$. Then, for a bidder with $\underline{c} \leq c < \widetilde{c}$, $Pr(win \,|\, \widetilde{b})(\widetilde{b} - c) < Pr(win \,|\, b')(b' - c)$. It means that if $p < \frac{n(n-2)}{(n-1)^2}$, a bidder with $\underline{c} \leq c < \widetilde{c}$ has an incentive to deviate from $\widetilde{b}$ to b'. Therefore, $p \geq \frac{n(n-2)}{(n-1)^2}$ is the necessary condition for the equilibrium bid function.

Step 3. *for every* c, c' *with* $\widetilde{c} \leq c < c' \leq \overline{c}$, $b(c) < b(c') \leq \overline{c}$

As mentioned before, sets are defined as follows

$J \equiv \{1, 2, \cdots, n\}$, and $i \in J$

$IBid_i(b) = \{j \in \{1, \cdots, n\} \mid b_j = b, \, \forall j \neq i\}$,

$LBid_i(b) = \{j \in \{1, \cdots, n\} \mid b_j < b, \, \forall j \neq i\}$,

$UBid_i(b) = \{j \in \{1, \cdots, n\} \mid b_j > b, \, \forall j \neq i\}$,

$CBid_i(b) = \bigcap_{\forall j \notin \{i\} \cup IBid_i(b)} \{(b, \, \boldsymbol{b}_{-i}) \mid |\overline{b} - b| < |\overline{b} - b_j|\}$, where $\boldsymbol{b}_{-i} = (b_1, \cdots, b_{i-1}, \, b_{i+1}, \cdots, b_n)$

Suppose that there is another identical bid b' over the interval $(\dot{c}, \, \ddot{c}) \subset [\widetilde{c}, \, \overline{c}]$.

Let $p = F(\dot{c})$, $q = F(\ddot{c}) - F(\dot{c})$, and $r = 1 - F(\ddot{c})$.

Then, the winning probability of bidder i submitting the identical bid b' is as follows

$Pr(win \,|\, b_i = b') = \frac{1}{|IBid_i(b')| + 1} Pr(CBid_i(b'), \, IBid_i(b'))$

$= \frac{1}{n}q^{n-1} + \frac{1}{n-1}\binom{n-1}{1}q^{n-2}(1-q) + \cdots$

$+ \frac{1}{n-k-1}\binom{n-1}{k}q^{n-k-1}\left(p^k \beta_{k,0} + \cdots + \binom{k}{h}p^{k-h}r^h \beta_{k-h,\,h} + \cdots + r^k \beta_{0,\,k}\right) + \cdots$

$+ (p^{n-1}\beta_{n-1,\,0} + \cdots + \binom{n-1}{h}p^{n-h-1}r^h \beta_{n-h-1,\,h} + \cdots + r^{n-1}\beta_{0,\,n-1})$

where $\beta_{x,\,y} = Pr(CBid_i(b') \,|\, |LBid_i(b')| = x, \, |UBid_i(b')| = y)$

If a bidder slightly deviates downward from b', his winning probability is as follows

For a sufficiently small $\varepsilon > 0$, let $b'' = b' - \varepsilon$, and $p' = F(b^{-1}(b''))$, $q' = F(\ddot{c}) - F(b^{-1}(b''))$

$Pr(win \,|\, b_i = b'') = Pr(CBid_i(b''))$

$= \binom{n-1}{1} q'^{n-2} p' + \binom{n-1}{2} q'^{n-3} \left(p'^2 \beta_{2,0} + \binom{2}{1} p'^1 r^1 \beta_{2,2} + r^2 \beta_{2,2} \right) + \cdots + (p'^{n-1} \beta_{n-1,0} + \cdots$
$+ (p'^{n-1} \beta_{n-1,0} + \cdots + \binom{n-1}{h} p'^{n-h-1} r^h \beta_{n-h-1,h} + \cdots + r^{n-1} \beta_{0,n-1})$

Now, we want to show that a bidder placing the identical bid b' has an incentive to deviate downward from b'. In order to do it, firstly, remind that $p \geq \frac{n(n-2)}{(n-1)^2}$ according to the previous step. Furthermore, since $(\dot{c}, \ddot{c}) \subset [\widetilde{c}, \overline{c}]$, then, $q < 1 - p$. It implies that for $n > 3$ and sufficiently small $\varepsilon \geq 0$, $\frac{1}{n} q^{n-1} + \frac{1}{n-1} \binom{n-1}{1} q^{n-2} (1-q) < \binom{n-1}{1} q'^{n-2} p'$. It implies that for sufficiently small $\varepsilon \geq 0$, $Pr(win \,|\, b_i = b') < Pr(win \,|\, b_i = b'')$. Therefore, there exists a sufficiently small $\varepsilon > 0$ such that $Pr(win \,|\, b_i = b')(b' - c) < Pr(win \,|\, b_i = b'')(b'' - c)$ for $c \in (\dot{c}, \ddot{c})$. Therefore, a bidder placing the identical bid b' has an incentive to deviate downward from b'. Accordingly, there does not exist any another identical bid b'.

Step 4. $b(\overline{c}) = \overline{c}$

Suppose by contradiction that $b(\overline{c}) > \overline{c}$ in equilibrium. Since the bid is strictly increasing in the neighborhood of $\overline{c}$ according to the step above, the winning probability of $\overline{c}$ bidder is zero. If he slightly deviates downward, his bid equals one of rivals' bids, and the winning probability is not zero anymore. Thus, $\overline{c}$ bidder has an incentive to deviate downward. It contradicts that $b(\overline{c}) > \overline{c}$ in equilibrium.

Step 5. *a bidder with $\widetilde{c}$ receives the same payoff whether he bids either $\widetilde{b}$ or the strictly increasing bid $b(\widetilde{c})$, and $\widetilde{b} \leq b(\widetilde{c})$*

Firstly, the sets are defined as follows

$J \equiv \{1, 2, \cdots, n\}$, and $i \in J$

$IBid_i(b) = \{j \in \{1, \cdots, n\} \,|\, b_j = b, \forall j \neq i\}$,

$LBid_i(b) = \{j \in \{1, \cdots, n\} \,|\, b_j < b, \forall j \neq i\}$,

$UBid_i(b) = \{j \in \{1, \cdots, n\} \,|\, b_j > b, \forall j \neq i\}$,

$CBid_i(b) = \bigcap_{\forall j \notin \{i\} \cup IBid_i(b)} \{(b, \boldsymbol{b}_{-i}) \,|\, |\overline{b} - b| < |\overline{b} - b_j|\}$, where $\boldsymbol{b}_{-i} = (b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_n)$

$p = F(\widetilde{c})$, $p' = F(b^{-1}(b'))$, $m = \left[\frac{n-1}{2}\right]$, where if $l \leq \frac{n-1}{2} < l+1$, $\left[\frac{n-1}{2}\right] = l$, l: integer, $\overline{b} = \frac{1}{n}\sum_{j=1}^{n} b_j$

Secondly, the winning probability of bidder i choosing the identical bid $\widetilde{b}$ is as follows

$Pr(win \,|\, b_i = \widetilde{b}) = \sum_{k=0}^{m} \frac{1}{n-k} Pr(|IBid_i(\widetilde{b})| = n-k-1\,,\, CBid_i(\widetilde{b}),\, |UBid_i(\widetilde{b})| = k)$
$= \frac{1}{n}p^{n-1} + \frac{1}{n-1}\binom{n-1}{n-2}p^{n-2}(1-p) + \cdots$
$+ \frac{1}{m+1}\binom{n-1}{m}p^{m}(1-p)^{n-m-1} Pr(CBid_i(\widetilde{b}) \,|\, |IBid_i(\widetilde{b})| = m,\, |UBid_i(\widetilde{b})| = n-m-1)$

When a bidder i slightly deviates upward from $\widetilde{b}$ to b', the winning probability is as follows

$Pr(win \,|\, b_i = b') = \sum_{k=1}^{n-1} Pr(|LBid_i(b')| = k,\, |UBid_i(b')| = n-k-1\,,\, CBid_i(b'))$
$= \sum_{k=1}^{n-2}\binom{n-1}{k}(p')^{k}(1-p')^{n-k-1} Pr(CBid_i(b') \,|\, |LBid_i(b')| = k,\, |UBid_i(b')| = n-k-1)$

Let's define two payoff functions as follows

$\pi(\widetilde{b},\, c) = Pr(win \,|\, \widetilde{b})(\widetilde{b} - c)$, $\pi(b^*,\, c) = Pr(win \,|\, b^*)(b^* - c)$

where $b^* = argmax_{b'} Pr(win \,|\, b')(b' - c)$.

Then, $H(c) = \pi(\widetilde{b},\, c) - \pi(b^*,\, c)$, and $H(c)$ is continuous for $c \in \left[\underline{c},\, \widetilde{b}\right]$

Now, we want to show that there exists a $\widetilde{c}$ such that $H(\widetilde{c}) = 0$. Firstly, $H(\underline{c}) > 0$ since the bidder with $\underline{c}$ has no incentive to deviate from $\widetilde{b}$ to b^* in equilibrium. Furthermore, $H(\widetilde{b}) < 0$ in equilibrium. According to the intermediate value theorem, there exists a $\widetilde{c}$ such that $H(\widetilde{c}) = 0$ in equilibrium, where a bidder with $\widetilde{c}$ receives the same payoff whether he bids either $\widetilde{b}$ or the strictly increasing bid $b(\widetilde{c}) = b^*$.

Next, we want to show that $\widetilde{b} \leq b(\widetilde{c})$. If $Pr(win \,|\, \widetilde{b}) = Pr(win \,|\, b^*)$, then, $\widetilde{b} = b(\widetilde{c})$. Moreover, if $Pr(win \,|\, \widetilde{b}) > Pr(win \,|\, b^*)$, then, $\widetilde{b} < b(\widetilde{c})$. $\square$

Proposition 1 (General Pooling Equilibrium)**.**

Proof of the Proposition.

Step 1. *there exists a* $\widetilde{b} \in (\overline{c},\, r]$ *such that for every* $c \in [\underline{c},\, \overline{c}]$, $b(c) = \widetilde{b}$

Suppose the equilibrium bid function is $b : [\underline{c}, \overline{c}] \to \mathbb{R}_+$. If every bidder bids an identical amount $\widetilde{b}$, where $\overline{c} < \widetilde{b} \leq r$, it is the equilibrium bid. Otherwise, if a bidder deviates from $\widetilde{b}$, he is sure to lose the auction. Thus, every bidder has no incentive to deviate from $\widetilde{b}$.

Step 2. *there does not exist any equilibrium bid such that for every* c, c' *with* $\underline{c} \leq c < c' \leq \overline{c}$, $b(c) \leq b(c')$, $\overline{c} < b(\overline{c})$, *and* $b(\underline{c}) < b(\overline{c})$

Suppose the equilibrium bid function is $b : [\underline{c}, \overline{c}] \to \mathbb{R}_+$. First of all, we want to show that there exists an identical bid in the neighborhood of $\underline{c}$. Lemma A1 indicates that the bid is weakly increasing over cost. Then, there is an interval $[\underline{c}, \widetilde{c}_{low})$ such that for every $c \in [\underline{c}, \widetilde{c}_{low})$, $b(c) = \widetilde{b}$, and $p \geq \frac{n(n-2)}{(n-1)^2}$, where $p = F(\widetilde{c}_{low})$, analogously to Lemma 1.

Likewise, there exists a $\widetilde{c}_{high}$ such that for every c, $c' \in (\widetilde{c}_{high}, \overline{c}]$, $b(c) = b(c')$. Otherwise, if $b(c)$ is strictly increasing over the interval $(\widetilde{c}_{high}, \overline{c}]$, the bidder with $\overline{c}$ knows his bid is the most distant from the average and loses the auction. Furthermore, since $b(\overline{c}) > \overline{c}$, he has an incentive to deviate downward. Analogously to Lemma 1, $1 - q \geq \frac{n(n-2)}{(n-1)^2}$, where $q = F(\widetilde{c}_{high})$.

For $n \geq 3$, since $\frac{n(n-2)}{(n-1)^2} \geq \frac{3}{4}$, both $[\underline{c}, \widetilde{c}_{low})$ and $(\widetilde{c}_{high}, \overline{c}]$ are connected. Thus, for every $c \in [\underline{c}, \widetilde{c}_{low})$ and every $c' \in (\widetilde{c}_{high}, \overline{c}]$, $b(c) = b(c')$. Therefore, there exists the only equilibrium, where for every $c \in [\underline{c}, \overline{c}]$, $b(c) = \widetilde{b}$ $\square$

Intractability of $n > 3$

Consider $n = 3$ at first. When bid is strictly increasing over cost, there exists an one to one correspondence between bid and cost. Thus, $b_2 < b_1 < b_3$ or $b_3 < b_1 < b_2$ implies $c_2 < c_1 < c_3$ or $c_3 < c_1 < c_2$ respectively. Accordingly, the winning probability of b_1 is follows.

$Pr(win \,|\, b_1) = Pr(win \,|\, c_1) = 2F(c_1)(1 - F(c_1))$.

Thus, the winning probability is directly derived from the cost distribution, which makes it possible for the closed-form solution to be provided as in Proposition 2.

However, when $n > 3$, the winning probability can not be directly derived from the cost distribution. For instance, consider $n = 4$ and the case in which b_1 wins. Such a possible case is as follows: $b_2 < b_3 < b_1 < b_4$. If the bid is strictly increasing, there is an one to one correspondence between bid and cost, $c_2 < c_3 < c_1 < c_4$.

However, it does not imply $Pr(win \,|\, c_1) = F(c_1)^2(1-F(c_1))$, but indicate $Pr(win \,|\, c_1) = F(c_1)^2(1 - F(c_1))Pr(|\bar{b} - b(c_1)| < |\bar{b} - b(c_j)| \,|\, c_2 < c_3 < c_1 < c_4)$, where $j \in \{2, 3, 4\}$.

$Pr(|\bar{b}-b(c_1)| < |\bar{b}-b(c_j)| \,|\, c_2 < c_3 < c_1 < c_4)$ is computed by using Total Probability Theorem, but this approach does not guarantee a closed-form solution for the general probability of winning.

APPENDIX B

Proposition 1.

Proof. [Proof of Proposition 1]

Step 1. *There is no fully separating equilibrium over* $\left[0,\, \overline{\theta}\right]$

In order to prove this statement, we show that if an equilibrium bid function is fully separating, the lowest possible bid and the highest possible bid always have an incentive to deviate upward and downward, respectively.

Suppose an equilibrium bid function $b(\theta)$ is fully separating over $\left[0,\, \overline{\theta}\right]$. Since the function $b(\theta)$ is fully separating, there is one to one correspondence from θ to b. Thus, there is the unique highest bid $b(\theta_{high})$ such that for $\forall\theta \in \left[0,\, \overline{\theta}\right] \setminus \theta_{high}$, $b(\theta_{high}) > b(\theta)$. At the same time, there is the unique lowest bid $b(\theta_{low})$ such that for $\forall\theta \in \left[0,\, \overline{\theta}\right] \setminus \theta_{low}$, $b(\theta_{low}) < b(\theta)$. Both $b(\theta_{high})$ and $b(\theta_{low})$ always lose the auction. Thus, I want to show they have incentives to deviate.

Firstly, suppose $b(\theta_{high})$ deviates downward to b' such that for some $\theta \in [0,\, \overline{\theta}] \setminus \{\theta',\, \theta_{high}\}$, $b(\theta') = b' \geq b(\theta)$. Since the equilibrium bid $b(\theta)$ is fully separating, then $b' > c$. When $b(\theta_{high})$ deviates to b', either $b' \geq c + \theta_{high}$ or $b' < c + \theta_{high}$. If $b' \geq c + \theta_{high}$, $b' - c - \beta\theta_{high} > 0$.

Even though $b' < c + \theta_{high}$, $(1-\beta)(b-c) > 0$. Thus, he has an incentive to deviate downward. Likewise, assume $b(\theta_{low})$ deviates upward to b' such that for some $\theta \in \left[0,\, \overline{\theta}\right]$, $b(\theta') = b' > b(\theta)$. Since the equilibrium bid $b(\theta)$ is fully separating, then $b' > c$. When $b(\theta_{low})$ deviates to b', either $b' \geq c + \theta_{low}$ or $b' < c + \theta_{low}$. If $b' \geq c + \theta_{low}$, $b' - c - \beta\theta_{low} > 0$. Even though $b' < c + \theta_{low}$, $(1-\beta)(b-c) > 0$. Thus, he has an incentive to deviate upward.

Therefore, an equilibrium bid function is not fully separating.

Step 2. *there are no unique lowest bid and unique highest bid in equilibrium*

Suppose there exists either unique lowest bid or unique highest bid in equilibrium. Analogously to Step 1, unique lowest bid and unique highest bid have an incentive to deviate upward and downward, respectively.

Thus, there is $\left[\theta^1_{high},\, \theta^2_{high}\right] \in \left[0,\, \overline{\theta}\right]$ such that for $\forall\theta,\, \theta' \in \left[\theta^1_{high},\, \theta^2_{high}\right]$, $b(\theta') = b(\theta)$, and there is $[\theta^1_{low},\, \theta^2_{low}] \in \left[0,\, \overline{\theta}\right]$ such that for $\forall\theta,\, \theta' \in [\theta^1_{low},\, \theta^2_{low}]$, $b(\theta') = b(\theta)$.

Step 3. *If the bid function* $b : \left[0,\, \overline{\theta}\right] \to \mathbb{R}_+$ *is a SBNE function, then, for each* $\theta \in \left[0,\, \overline{\theta}\right]$, $b(\theta) = b^* \in [c,\, r]$

For $\forall\theta,\, \theta' \in \left[\theta^1_{high},\, \theta^2_{high}\right]$, let $b_{high} = b(\theta') = b(\theta)$ and $p_{high} = F(\theta^2_{high}) - F(\theta^1_{high})$. p_{high} implies the portion of bidders submitting b_{high}. Thus, $Pr(win \,|\, b_{high}) = \frac{1}{3}p^2_{high} + p_{high}\,(1 - p_{high})$. For a sufficiently small $\varepsilon > 0$, let $b' = b_{high} - \varepsilon$. That is, b' implies the downward deviation from b_{high}. Then, if $\varepsilon = 0$, $b' = b_{high}$ and $Pr(win \,|\, b') = 2p_{high}\,(1 - p_{high})$.

For a sufficiently small $\varepsilon \geq 0$, if either $Pr(win \,|\, b_{high})\,(b_{high} - c - \beta\theta) > Pr(win \,|\, b')\,(b_{high} - c - \beta\theta)$ or $Pr(win \,|\, b_{high})\,(b_{high} - c) > Pr(win \,|\, b')\,(b_{high} - c)$, then, there is no deviation from b_{high}.

Accordingly, $p_{high} > \frac{3}{4}$. Likewise, $p_{low} > \frac{3}{4}$, where $p_{low} = F(\theta^2_{low}) - F(\theta^1_{low})$, which means the portion of bidders choosing b_{low}.

Therefore, $[\theta^1_{low},\, \theta^2_{low}]$ and $\left[\theta^1_{high},\, \theta^2_{high}\right]$ are connected, which implies $b_{low} = b_{high}$. Accordingly, for each $\theta \in \left[0, \overline{\theta}\right]$, $b^* = b(\theta)$, $b^* \in [c,\, r]$.

Step 4. *For each* $\theta \in \left[0,\, \overline{\theta}\right]$, *if* $b(\theta) = b^* \in [c,\, r]$, *then, the bid function* $b : \left[0,\, \overline{\theta}\right] \to \mathbb{R}_+$ *is a SBNE function.*

For each $\theta \in \left[0, \overline{\theta}\right]$, if $b(\theta) = b^*$, it is an equilibrium. Otherwise, a bidder who deviates from b^* loses the auction. $\square$

Proposition 2.

Proof. [Proof of Proposition 2]

Step 1. *When $D \geq (1-\beta)\overline{\theta}$, nobody defaults on his bid*

In order to prove this statement, we show that for each $\theta \in [0, \overline{\theta}]$, $b - c - \theta \geq -(1-\beta)\overline{\theta} \geq -D$. Firstly, in equilibrium, when completing the contract, $b - c - \theta \geq \beta\theta - \theta = -(1-\beta)\theta > -(1-\beta)\overline{\theta}$. Secondly, in equilibrium, when defaulting on his bid in the event of cost overrun, $c + \frac{\beta}{1-\beta}D \leq b \leq c + \theta$. Then, $b - c - \theta \geq \frac{\beta}{1-\beta}D - \theta$. Since $D \geq (1-\beta)\overline{\theta}$, then, $b - c - \theta \geq \frac{\beta}{1-\beta}D - \theta \geq \beta\overline{\theta} - \theta \geq -(1-\beta)\overline{\theta}$. Thus, $b - c - \theta \geq -(1-\beta)\overline{\theta}$

For the proof of Step 2 and Step 3, refer to Proposition 2 of Lee (2016*b*). The difference is that cost for each possible bidder in Proposition 2 of Lee (2016*b*) is replaced with the expected cost $c + \beta\theta$.

Step 2. *A partially pooling equilibrium exists.*

$$\frac{3}{4} \leq p = F\left(\widetilde{\theta}\right), b\left(\theta\right) = \begin{cases} c + \beta\widetilde{\theta} + \beta\int_{\widetilde{\theta}}^{\overline{\theta}} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2 + p(1-p)} dy & \text{if } \theta \in [0, \widetilde{\theta}\,) \\ c + \beta\theta + \beta\int_{\theta}^{\overline{\theta}} \frac{F(y)(1-F(y))}{F(\theta)(1-F(\theta))} dy & \text{if } \theta \in \left[\widetilde{\theta}, \overline{\theta}\right] \end{cases}$$

Step 3. *A pooling equilibrium exists.*

For each $\theta \in \left[0, \overline{\theta}\right]$, $c + \beta\overline{\theta} < b(\theta) = b^* \leq r$ $\square$

Proposition 3.

Proof. [Proof of Proposition 3]

Step 1. *Derive a partially pooling equilibrium*

According to Proposition 2, there exists a partially pooling equilibrium, when penalty is sufficiently enough. However, when penalty is not sufficient, possible bidders with relatively large θ have an incentive to default. Since penalty is uniform, insolvent bidders have the same cost structure, which induces those bidders to bid an identical amount. Thus, we want to prove that the only bidders who bid the expected cost are insolvent.

Since $r < c + \overline{\theta}$, a bidder with $\theta \in (r + D - c, \overline{\theta}]$ always defaults on his bid, every time θ occurs. Thus, a bidder with $\theta \in (r + D - c, \overline{\theta}]$ has the expected cost $c + \frac{\beta}{1-\beta}D$ which implies that there is a threshold θ^* with $0 < \theta^* \leq r + D - c$ such that for each $\theta \in [\theta^*, \overline{\theta}]$, $b(\theta) = b^* \leq r$.

Firstly, we want to show that $b^* = c + \frac{\beta}{1-\beta}D$. Particularly, we show that if bidders with $\theta \in [\theta^*, \overline{\theta}]$ bid above the expected cost, they have an incentive to deviate downward from b^*.

When bidders bid b^*, $\pi(b^*) = (\frac{1}{3}{p^*}^2 + p^*(1-p^*))(b^* - c - \frac{\beta}{1-\beta}D)$, where $p^* = 1 - F(\theta^*)$, the portion of possible bidders who bid b^*. If $\widetilde{b} = b^*$, a pooling equilibrium exists. Thus, $\widetilde{b} < b^*$ in a partially pooling equilibrium. If $\widetilde{b} < b^*$, $p^* < \frac{1}{4}$. For a sufficiently small $\varepsilon > 0$, if $p^* < \frac{1}{4}$, $\pi(b^* - \varepsilon) > 2p^*(1 - p^*)(b^* - \varepsilon - c - \frac{\beta}{1-\beta}D)$. Furthermore, if $p^* < \frac{1}{4}$, $\pi(b^* - \varepsilon) > 2p^*(1 - p^*)(b^* - \varepsilon - c - \frac{\beta}{1-\beta}D) > \pi(b^*)$.

Accordingly, if $b^* > c + \frac{\beta}{1-\beta}D$, a bidder with $\theta \in [\theta^*, \overline{\theta}]$ has an incentive to deviate downward from b^*. Thus, $b^* = c + \frac{\beta}{1-\beta}D$.

Next, we want to show that $\theta^* = \frac{D}{1-\beta}$. In order to do it, we show that if $\theta^* \neq \frac{D}{1-\beta}$, θ^* is not the threshold to submit b^*.

First of all, if $b^* - c - \beta\theta^* > (1 - \beta)(b^* - c) - \beta D = 0$, the bidder with θ^* finishes the contract even in the event of θ^*. It contradicts that θ^* is the threshold to submit b^*.

Likewise, if $b^* - c - \beta\theta^* < (1-\beta)(b^* - c) - \beta D = 0$, there exists an $\varepsilon > 0$ such that $b^* - c - \beta\theta^* < b^* - c - \beta(\theta^* - \varepsilon) < (1-\beta)(b^* - c) - \beta D = 0$. Then, a bidder with $\theta - \varepsilon$ defaults on his bid in the event of $\theta - \varepsilon$. It also contradicts that θ^* is the threshold to submit b^*.

Thus, the bidder with θ^* is indifferent between choosing the default and finishing the contract, and $b^* - c - \beta\theta^* = (1-\beta)(b^* - c) - \beta D = 0$. Thus, $\theta^* = \frac{D}{1-\beta}$.

Considering Proposition 2, possible bidders with $\theta \in [\theta^*, \overline{\theta}]$, where $\theta^* = \frac{D}{1-\beta}$, bid $b^* = c + \frac{\beta}{1-\beta}D$. Finally, all types of equilibria are determined as follows

$$b(\theta) = \begin{cases} c + \beta\widetilde{\theta} + \beta\int_{\widetilde{\theta}}^{\theta^*} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2 + p(1-p)}dy & \text{if } \theta \in [0, \widetilde{\theta}\,) \\ c + \beta\theta + \beta\int_{\theta}^{\theta^*} \frac{F(y)(1-F(y))}{F(\theta)(1-F(\theta))}dy & \text{if } \theta \in [\widetilde{\theta}, \theta^*\,) \\ c + \frac{\beta}{1-\beta}D & \text{if } \theta \in \left[\theta^*, \overline{\theta}\right] \end{cases}$$

Step 2. *If a bid function is* $b(\theta) = \begin{cases} c + \beta\widetilde{\theta} + \beta\int_{\widetilde{\theta}}^{\theta^*} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2 + p(1-p)}dy & \text{if } \theta \in [0, \widetilde{\theta}\,) \\ c + \beta\theta + \beta\int_{\theta}^{\theta^*} \frac{F(y)(1-F(y))}{F(\theta)(1-F(\theta))}dy & \text{if } \theta \in [\widetilde{\theta}, \theta^*\,) \\ c + \frac{\beta}{1-\beta}D & \text{if } \theta \in \left[\theta^*, \overline{\theta}\right] \end{cases}$*, then, the function is a SBNE function.*

Firstly, we want to show that a bidder with $\theta \in \left[0, \widetilde{\theta}\right]$ bids $b(\theta) = c + \beta\widetilde{\theta} + \beta\int_{\widetilde{\theta}}^{\theta^*} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2 + p(1-p)}dy$ in equilibrium. In order to do it, we show that if $b(\theta) \neq c + \beta\widetilde{\theta} + \beta\int_{\widetilde{\theta}}^{\theta^*} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2 + p(1-p)}dy$, there always exists a bidder who has an incentive to deviate from his bidding strategy.

Thus, suppose for a sufficiently small $\varepsilon > 0$, $\widetilde{b} = c + \beta\widetilde{\theta} + \varepsilon + \beta\int_{\widetilde{\theta}}^{\theta^*} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2 + p(1-p)}dy$

Then, for a sufficiently large $N > 0$

$$(\tfrac{1}{3}p^2 + p(1-p))(\widetilde{b} - c - \beta\widetilde{\theta} - \tfrac{1}{N}\varepsilon) = \tfrac{N-1}{N}\varepsilon(\tfrac{1}{3}p^2 + p(1-p)) + 2\beta\int_{\widetilde{\theta}}^{\theta^*} F(y)(1-F(y))dy$$

$> 2\beta \int_{\widetilde{\theta}+\frac{1}{N}\varepsilon}^{\theta^*} F(y)(1-F(y))dy$

Thus, the bidder with $\widetilde{\theta} + \frac{1}{N}\varepsilon$ has an incentive to deviate to $\widetilde{b}$

Now, assume $\widetilde{b} = c + \beta\widetilde{\theta} - \varepsilon + \beta \int_{\widetilde{\theta}}^{\theta^*} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2+p(1-p)}dy$

Then, for a sufficiently large $N > 0$

$(\frac{1}{3}p^2 + p(1-p))(\widetilde{b} - c - \beta\widetilde{\theta} + \frac{1}{N}\varepsilon) = -\frac{N-1}{N}\varepsilon(\frac{1}{3}p^2 + p(1-p)) + 2\beta \int_{\widetilde{\theta}}^{\theta^*} F(y)(1-F(y))dy$

$> 2\beta \int_{\widetilde{\theta}}^{\theta^*} F(y)(1-F(y))dy$

Thus, the bidder with $\widetilde{\theta} - \frac{1}{N}$ has an incentive to deviate to $b(\widetilde{\theta})$

Next, we want to show that a bidder with $\theta \in \left[\widetilde{\theta},\, \theta^*\right]$ bids $b(\theta) = c + \beta\theta + \beta \int_{\theta}^{\theta^*} \frac{F(y)(1-F(y))}{F(\theta)(1-F(\theta))}dy$ in equilibrium. In order to do it, we show that for each $\theta \in \left[\widetilde{\theta},\, \theta^*\right]$, $b(\theta) = c + \beta\theta + \beta \int_{\theta}^{\theta^*} \frac{F(y)(1-F(y))}{F(\theta)(1-F(\theta))}dy$ is satisfied with profit maximization.

$2F(x)(1-F(x))(b(x) - c - \beta\theta) - 2F(\theta)(1-F(\theta))(b(\theta) - c - \beta\theta)$

$= 2\beta F(x)(1-F(x))(x-\theta) + 2\beta \int_{x}^{\theta^*} F(y)(1-F(y))dy - 2\beta \int_{\theta}^{\theta^*} F(y)(1-F(y))dy$

If $x > \theta$, $2\beta F(x)(1-F(x))(x-\theta) - 2\beta \int_{\theta}^{x} F(y)(1-F(y))dy < 0$

If $x < \theta$, $-2\beta F(x)(1-F(x))(\theta - x) + 2\beta \int_{x}^{\theta} F(y)(1-F(y))dy < 0$

Accordingly, the expected payoff is maximized at $x = \theta$

Lastly, we want to show that a bidder with $\theta \in \left[\theta^*,\, \overline{\theta}\right]$ bids $b(\theta) = c + \frac{\beta}{1-\beta}D$ in equilibrium. In order to do it, we show that a bidder with $\theta \in \left[\theta^*,\, \overline{\theta}\right]$ who bids $b(\theta) = c + \frac{\beta}{1-\beta}D$ has no incentive to deviate.

It is clear that for each $\theta \in \left[0,\, \overline{\theta}\right]$, $b(\theta) \leq c + \frac{\beta}{1-\beta}D$. Thus, any bid $b > c + \frac{\beta}{1-\beta}D$ loses the auction. It means that bidders with $c + \frac{\beta}{1-\beta}D$ have no incentive to deviate from $c + \frac{\beta}{1-\beta}D$.

Step 3. *Derive a pooling equilibrium*

For any $D > 0$ and any given b with $b \geq c$, bidders with $0 \leq \theta \leq b - c + D$ do not default on their bids. Analogously to Proposition 1, for $\theta, \theta' \in \left[0,\, \widetilde{\theta}\right] \subseteq [0,\, b - c + D]$, there exists an identical bid $b(\theta) = b(\theta') = \widetilde{b}$. Let $F(\widetilde{\theta})$ be the portion of bidders placing an identical bid. Analogously to Proposition 1, $F(\widetilde{\theta}) \geq \frac{3}{4}$

Furthermore, since $r < c + \overline{\theta}$, a bidder with $\theta \in \left(r + D - c,\, \overline{\theta}\right]$ always defaults on his bid, every time θ occurs. Since a bidder with $\theta \in \left(r + D - c,\, \overline{\theta}\right]$ has the identical expected cost $c + \frac{\beta}{1-\beta}D$, bidders with $\theta \in \left(r + D - c,\, \overline{\theta}\right]$ submit another identical bid.

If bidders with $\theta \in \left(r + D - c,\, \overline{\theta}\right]$ bid the another identical amount, there exists an θ^* such that $0 \leq \theta^* \leq r + D - c$, $1 - F(\theta^*) \geq \frac{3}{4}$, and every bidder with $\theta \in [\theta^*,\, \overline{\theta}]$ bids the another identical amount. Otherwise, a bidder with $\theta \in \left(r + D - c,\, \overline{\theta}\right]$ has an incentive to deviate from the another identical bid. Since $F(\widetilde{\theta}) > \frac{3}{4}$ and $1 - F(\theta^*) \geq \frac{3}{4}$, $\left[0,\, \widetilde{\theta}\right]$ and $[\theta^*,\, \overline{\theta}]$ are connected, whenever $\widetilde{b} \geq c + \frac{\beta}{1-\beta}D$. It implies that two identical bids are equal to each other.

Step 4. *If a bid function is that for $\theta \in \left[0,\, \overline{\theta}\right]$, $b(\theta) = b^* \in \left(c + \frac{\beta}{1-\beta}D,\, r\right]$, then, the function is a SBNE function*

Suppose every bidder bids an identical amount $b^* > c + \frac{\beta}{1-\beta}D$. When all bidders bid identically, any bidder who deviates from the identical bid b^* is sure to lose the auction. Thus, he has no incentive to deviate from b^*.

Accordingly, b^* is the equilibrium bid, where for $\forall\, \theta \in \left[0,\, \overline{\theta}\right]$, $b(\theta) = b^* \in \left(c + \frac{\beta}{1-\beta}D,\, r\right]$

□

Proposition 4.

Proof. [Proof of 1]

Step 1. *The expected profit of the buyer at a partially pooling equilibrium in ABA*

Firstly, let $b^{(k)}$ be the k-th lowest bid. Thus, $b^{(2)}$ is the median bid. In the average bid auction with three bidders, the median bid always equals the value of the winning bid. Thus, the median bid is the representative of the winning bid. Furthermore, all possible events are classified into three events according to the position of the median bid.

$A = \left\{b^{(2)} = \underline{b}\right\}, B = \left\{\underline{b} < b^{(2)} < \overline{b}\right\}, C = \left\{b^{(2)} = \overline{b}\right\}$

where $\underline{b} = c + \beta\widetilde{\theta} + \beta \int_{\widetilde{\theta}}^{\theta^*} \frac{2F(y)(1-F(y))}{\frac{1}{3}p^2+p(1-p)} dy$ and $\overline{b} = c + \frac{\beta}{1-\beta} D$

Thus, the probability of each event is as follows

$Pr(A) = F_A = F(\widetilde{\theta})^3 + 3F(\widetilde{\theta})^2(1 - F(\widetilde{\theta}))$,

$Pr(B) = F_B = 3 \times 2 \int_{\widetilde{\theta}}^{\theta^*} (F(y) - F(y)^2) f(y) dy$,

$Pr(C) = F_C = (1 - F(\theta^*))^3 + 3(1 - F(\theta^*))^2 F(\theta^*)$

Moreover, the expected value of the winning bid conditional on each event is as follows

$E\left[b^{(2)}|A\right] = \underline{b}F_A$,

$E\left[b^{(2)}|B\right] = 3 \times 2 \int_{\widetilde{\theta}}^{\theta^*} b(y)(F(y) - F(y)^2) f(y) dy$,

$E\left[b^{(2)}|C\right] = \overline{b}F_C$

Now, the expected profit of the buyer in the average bid auction is as follows

$\pi_{ABA} = Pr(No\,cost\,overrun)(V - expected\,winning\,bid)+$

$Pr(cost\,overrun)\{Pr(No\,default)\times V - expected\,winning\,bid\,given\,no\,default + Pr(default)\times$

$Penalty\}$

$= (1 - \beta(1 - F_A - F_B))V - E\left[b^{(2)}|A\right] - E\left[b^{(2)}|B\right] - (1-\beta)E\left[b^{(2)}|C\right] + \beta F_C D$

Step 2. *The expected profit of the buyer in the first price reverse auction*

Since the lowest bid is the winning bid in the first price reverse auction, the expected value of the lowest bid is the expected payment that the buyer pays. Thus, the expected profit of the buyer in the first price reverse auction is as follows

$\pi_{FPA} = (1-\beta(1-G(\theta^*)))V - E[b^{(1)}|\theta < \theta^*] - (1-G(\theta^*))\{(1-\beta)\bar{b} - \beta D\}$

where $Pr(b^{(1)} < y) = G(y) = 1-(1-F(y))^3$, $g(y) = \frac{dG(y)}{dy}$, $E[b^{(1)}|\theta < \theta^*] = \int_0^{\theta^*} b(y)g(y)dy$

Step 3. *Show that* $\pi_{FPA} > \pi_{ABA}$

$\pi_{FPA} > \pi_{ABA}$ implies

$(1-\beta(1-G(\theta^*)))V - E[b^{(1)}|\theta < \theta^*] - (1-G(\theta^*))\{(1-\beta)\bar{b} - \beta D\} >$

$(1-\beta(1-F_A-F_B))V - E\left[b^{(2)}|A\right] - E\left[b^{(2)}|B\right] - (1-\beta)E\left[b^{(2)}|C\right] + \beta F_C D$

In order to prove that the above inequality is true, we investigate if the following inequality is true at first.

$E\left[b^{(2)}|A\right] + E\left[b^{(2)}|B\right] + (1-\beta)\bar{b}F_C > E\left[b^{(1)}|\theta < \theta^*,\, A\cup B\right] + (1-\beta)E\left[b^{(1)}|\theta < \theta^*,\, C\right] +$
$(1-\beta)(1-G(\theta^*))\bar{b}$

When there is no default, Revenue Equivalence Theorem says

$E\left[b^{(2)}|A\right] + E\left[b^{(2)}|B\right] > E\left[b^{(1)}|\theta < \theta^*,\, A\cup B\right]$

Moreover, $(1-\beta)\bar{b}F_C > (1-\beta)(1-G(\theta^*))\bar{b} + (1-\beta)E\left[b^{(1)}|\theta < \theta^*,\, C\right]$

Therefore, $E\left[b^{(2)}|A\right] + E\left[b^{(2)}|B\right] + (1-\beta)\bar{b}F_C > E\left[b^{(1)}|\theta < \theta^*,\, A\cup B\right] + (1-\beta)E\left[b^{(1)}|\theta < \theta^*,\, C\right] + (1-\beta)(1-G(\theta^*))\bar{b}$

Next, we investigate if the following inequality is true or not.

$(1-\beta(1-G(\theta^*))V + \beta(1-G(\theta^*))D \geq (1-\beta(1-F_A-F_B))V + \beta F_C D$

First of all, sets A, B, C are disjoint, and $A\cup B \equiv I \setminus C$, (I: whole set).

Thus, $Pr(A) + Pr(B) = Pr(I\setminus C)$ [1] which means $F_A + F_B = 1-(1-F(\theta^*))^3 - 3(1-$

[1] Let $p_1 = F(\widetilde{\theta})$,$p_2 = F(\theta^*)$

$Pr(B) = \binom{3}{1}\binom{2}{1}\int_{\widetilde{\theta}}^{\theta^*} F(y)(1-F(y))f(y)dy = \binom{3}{1}\binom{2}{1}\left[\frac{1}{2}F(y)^2\right]_{\widetilde{\theta}}^{\theta^*} - \binom{3}{1}\binom{2}{1}\left[\frac{1}{3}F(y)^3\right]_{\widetilde{\theta}}^{\theta^*}$

$= 3(p_2^2 - p_1^2) - 2(p_2^3 - p_1^3)$

$Pr(A) + Pr(C) = p_1^3 + \binom{3}{2}p_1^2(1-p_1) + (1-p_2)^3 + \binom{3}{2}p_2(1-P_2)^2$

$F(\theta^*))^2F(\theta^*)$

Therefore, $1 - F_A - F_B = (1 - F(\theta^*))^3 + 3(1 - F(\theta^*))^2F(\theta^*)$

Now, consider the following equation.

$(1 - \beta(1 - G(\theta^*))V + \beta(1 - G(\theta^*))D - (1 - \beta(1 - F_A - F_B))V - \beta F_C D$

$= \beta(1 - F_A - F_B - (1 - F(\theta^*))^3)V - 3\beta(1 - F(\theta^*))^2F(\theta^*)D$

$= \beta((1 - F(\theta^*))^3 + 3(1 - F(\theta^*))^2F(\theta^*) - (1 - F(\theta^*))^3)V - 3\beta(1 - F(\theta^*))^2F(\theta^*)D$

$= 3\beta(1 - F(\theta^*))^2F(\theta^*)(V - D) > 0$

Then, we show that $\pi_{FPA} - \pi_{ABA} > 0$

$\pi_{FPA} - \pi_{ABA}$

$= 3\beta(1 - F(\theta^*))^2F(\theta^*)(V - D) - E\left[b^{(1)}|\theta < \theta^*\right] - (1 - \beta)(1 - G(\theta^*))\bar{b} + E\left[b^{(2)}|A\right] + E\left[b^{(2)}|B\right] + (1 - \beta)\bar{b}F_C$

$= 3\beta(1 - F(\theta^*))^2F(\theta^*)(V - D) - E\left[b^{(1)}|\theta < \theta^*\right] + E\left[b^{(2)}|A\right] + E\left[b^{(2)}|B\right] + 3(1 - \beta)(1 - F(\theta^*))^2F(\theta^*)\bar{b}$

$> 3\beta(1 - F(\theta^*))^2F(\theta^*)(V - D) - \beta E\left[b^{(1)}|\theta < \theta^*, C\right] + (1 - \beta)E\left[b^{(1)}|\theta < \theta^*, C\right] + 3(1 - \beta)(1 - F(\theta^*))^2F(\theta^*)\bar{b}$

$> 3\beta(1 - F(\theta^*))^2F(\theta^*)(V - D) - \beta E\left[b^{(1)}|\theta < \theta^*, C\right]$

$> 3\beta(1 - F(\theta^*))^2F(\theta^*)(V - \bar{b} - D) > 0$

$\square$

Proof. [Proof of 2 and 3]

$= 1 - \left\{3(p_2^2 - p_1^2) - 2(p_2^3 - p_1^3)\right\} = 1 - Pr(B)$

Thus, $Pr(A) + Pr(B) = 1 - Pr(C)$

Step 1. *the pooling equilibrium at $\overline{b} = c + \frac{\beta}{1-\beta}D$ under ABA results in the lower expected payoff than FPA.*

In this case, the default probability is $1 - F(min\{\overline{b} - c + D, \overline{\theta}\}) = 1 - F(\frac{D}{1-\beta}) = 1 - F(\theta^*)$ and $G(\theta^*) = 1 - (1 - F(\theta^*))^3$.

Now, the expected profits of the buyer in ABA and FPA are as follows

$\pi_{ABA} = (1 - \beta(1 - F(\theta^*)))(V - \overline{b}) + \beta(1 - F(\theta^*))D$, where $\theta^* = \frac{1}{1-\beta}D$

$K = \pi_{FPA} = (1 - \beta(1 - G(\theta^*)))V - E[b^{(1)}|\theta < \theta^*] - (1 - G(\theta^*))\{(1 - \beta)\overline{b} - \beta D\}$

Then, the following inequality is true.

$\pi_{FPA} - \pi_{ABA} = \beta(G(\theta^*) - F(\theta^*))(V - D) + \{(1-\beta)G(\theta^*) + \beta F(\theta^*)\}\overline{b} - E[b^{(1)}|\theta < \theta^*]$

$\geq \beta(G(\theta^*) - F(\theta^*))(V - D) + \{(1 - \beta)G(\theta^*) + \beta F(\theta^*)\}\overline{b} - G(\theta^*)\overline{b}$

$= \beta(G(\theta^*) - F(\theta^*))(V - D) - \beta\{G(\theta^*) - F(\theta^*)\}\overline{b}$

$= \beta(G(\theta^*) - F(\theta^*))(V - D - \overline{b}) = \beta(G(\theta^*) - F(\theta^*))(V - c - \theta^*) \geq 0$

Thus, the pooling equilibrium at $\overline{b} = c + \frac{\beta}{1-\beta}D$ under ABA results in the lower expected payoff, compared to FPA.

Step 2. *The expected payoff of the buyer under FPA is greater than the penalty. That is, $K - D > 0$*

Now, the expected profits of the buyer in ABA and FPA are as follows

$K = \pi_{FPA} = (1 - \beta(1 - G(\theta^*)))V - E[b^{(1)}|\theta < \theta^*] - (1 - G(\theta^*))\{(1 - \beta)\overline{b} - \beta D\}$, where $\theta^* = \frac{1}{1-\beta}D$

Then, $K - D = \pi_{FPA} - D$

$= (1 - \beta(1 - G(\theta^*)))V - E[b^{(1)}|\theta < \theta^*] - (1 - G(\theta^*))\{(1 - \beta)\overline{b} - \beta D\} - D$

$= (V - D) - \beta(1 - G(\theta^*))(V - D) - E[b^{(1)}|\theta < \theta^*] - (1 - G(\theta^*))\overline{b} + \beta(1 - G(\theta^*))\overline{b}$

$= (V - D - (1 - G(\theta^*))\overline{b} - E[b^{(1)}|\theta < \theta^*]) - \beta(1 - G(\theta^*))(V - \overline{b} - D)$

$> (V - \overline{b} - D) - \beta(1 - G(\theta^*))(V - \overline{b} - D) = (1 - \beta(1 - G(\theta^*)))(V - \overline{b} - D) > 0$

Step 3. *ABA is better than FPA or FPA is better than ABA.*

Let $h(b) = b + D + \frac{K-D}{1-\beta+\beta F(b-c+D)}$.

According to Step 1, since $\pi_{ABA} - \pi_{FPA} \leq 0$ at $\bar{b}$ in ABA, $h(\bar{b}) \geq V$.

According to Step 2, since $K - D > 0$, $h(\cdot)$ is globally convex.

Thus, if $h(r) \leq V$, there exists a b^* such that $h(b^*) = V$.

It means that $h(b) \leq V$ for $b \in [b^*, r]$.

Then,$\pi_{ABA} - \pi_{FPA}$

$= (1-\beta)(V-b) + \beta F(b-c+D)(V-b) + \beta D - \beta F(b-c+D)D - K$

$= (1-\beta+\beta F(b-c+D))(V-b-D) - (K-D) \geq 0$.

Therefore, ABA results in the weakly better payoff of the buyer than FPA.

Analogously, $\pi_{ABA} - \pi_{FPA} \leq 0$ for $b \in [c + \frac{\beta}{1-\beta}D, b^*]$. Therefore, every pooling equilibrium at $b \in [b^*, r]$ under ABA results in the weakly higher expected payoff to the buyer, and every pooling equilibrium at $b \in [c + \frac{\beta}{1-\beta}D, b^*]$ under ABA results in the weakly lower expected payoff to the buyer.

Step 4. *There exists an interval $[b^*, \widehat{b}]$ such that ABA is better than FPA.*

Let $h(b) = b + D + \frac{K-D}{1-\beta+\beta F(b-c+D)}$.

Since $\pi_{ABA} - \pi_{FPA} \leq 0$ for $b \in [c + \frac{\beta}{1-\beta}D, b^*]$ under ABA, and $h(\cdot)$ is concave, it is trivial. □

Corollary 1.

Proof. [Proof of Corollary 1]

Step 1. *If $V \geq h(r)$, every equilibrium under ABA results in the weakly higher expected payoff.*

The payoff of the buyer is as follows

$\pi_{ABA} = (1-\beta)(V-b) + \beta Pr(\theta \leq b-c)(V-b) = (1-\beta(1-F(b-c)))(V-b)$

$\pi_{FPA} = (1-\beta)(V-c)$

Notice that $\pi_{ABA} = \pi_{FPA}$ at c under ABA. Moreover, $h(\cdot)$ is concave.

Therefore, $V \geq h(r)$ implies that for every $b \in [c,\, r]$, $V \geq h(b)$.

$V \geq h(b)$ means that $\pi_{ABA} \geq \pi_{FPA}$.

Accordingly, if $V \geq h(r)$, every equilibrium under ABA results in the weakly higher expected payoff.

Step 2. *If $V < h(r)$, every equilibrium at $b \in [c,\, b^*]$ in ABA is more beneficial to the buyer.*

Since $\pi_{ABA} = \pi_{FPA}$ at c under ABA, and $h(\cdot)$ is concave, it is trivial. □