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MODEL ERROR IN FLOOD FREQUENCY ESTIMATION

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Abstract

Asymptotic bias in large quantiles and moments for three parameter estimation methods, including the maximum likelihood method (MLM), moments method (MOM) and linear moments method (LMM), is derived when a probability distribution function (PDF) is falsely assumed. It is illustrated using an alternative set of PDFs consisting of five two-parameter PDFs that are lower-bounded at zero, i.e., Log-Gumbel (*LG*), Log-logistic (*LL*), Log-normal (*LN*), Linear Diffusion (*LD*) and Gamma (*Ga*) distribution functions. The stress is put on applicability of LG and LL in the real conditions, where the hypothetical distribution (*H*) differs from the true one (*T*). Therefore, the following cases are considered: *H=LG; T=LL, LN, LD* and *Ga*, and *H=LL, LN, LD* and *Ga, T=LG*. It is shown that for every pair (*H; T*) and for every method, the relative bias (*RB*) of moments and quantiles corresponding to the upper tail is an increasing function of the true value of the coefficient of variation (c_v), except that *RB* of moments for MOM is zero. The value of *RB* is smallest for MOM and the largest for MLM. The bias of LMM occupies an intermediate position. Since MLM used as the approximation method is irreversible, the asymptotic bias of the MLM-estimate of any statistical characteristic is not asymmetric as is for the MOM and LMM. MLM turns out to be the worst method if the assumed *LG* or

LL distribution is not the true one. It produces a huge bias of upper quantiles, which is at least one order higher than that of the other two methods. However, the reverse case, i.e., acceptance of *LN*, *LD* or *Ga* as a hypothetical distribution while *LG* or *LL* as the true one, gives the MLM-bias of reasonable magnitude in upper quantiles. Therefore, one should be highly reluctant in choosing the *LG* and *LL* in flood frequency analysis, especially if MLM is to be applied.

Key words: flood frequency analysis, Log-Gumbel, Log-logistic distribution, quantiles, model error, asymptotic bias, method of moments, linear moments, maximum likelihood.

1. INTRODUCTION

Flood frequency analysis (FFA) entails estimation of the upper tail of a probability distribution function (PDF) of peak flows obtained from either the annual duration series or partial duration series, although the upper part of the PDF may usually be out of the range of observations. The usual empirical approach is to fit an *a priori* assumed PDF to the peak flow data, where the fitting involves estimating the parameters of the PDF, which, in turn, requires the knowledge of the PDF. Thus, one tries to find and use the most robust method of parameter estimation for a given sample size. Unfortunately, the true PDF is not known and even if it were known it might, in all probability, contain too many parameters. These parameters cannot possibly be estimated reliably and efficiently from a hydrological sample, which is of relatively small size, meaning that strictly speaking such a PDF cannot be applied. Therefore, the task of FFA reduces to (1) choosing the PDF which can be derived either by "at site" or "regional" analysis; and (2) finding and using the most robust method of parameter estimation which produces the smallest mean square error (*MSE*) and bias in moments as well as in quantiles of interest for a given sample size and the chosen distribution.

The possibility of correct identification of PDF in case of normal hydrological size of samples is small even in the ideal case when the set of alternative PDFs contains the true (*T*) distribution function. Therefore, in reality, one deals with the hypothetical PDF (*H*), called here the false distribution function (*F*), which differs more or less from the true one. This will result in a model error in any statistical characteristic of the distribution. Its magnitude for a given characteristic depends not only on how closely is *F* to *T* but on the estimation method as well. This is the objective of the present study. Although the ultimate interest of FFA is the estimation of upper tail quantiles, the model error of the two first moments is analysed as well.

For choosing an estimation method, the approach used in FFA follows the findings based on the case of a known distribution form, where the robustness of the methods is considered, i.e., as soon as the model has been chosen it is considered as the true (*T*) one. A robust method performs well over a range of situations and is able to withstand a certain amount of abuse without breaking down. It is not necessarily the best estimation method for any one model, and is characterised in terms of stability and consistency of parameter estimates. Stable estimates are characterised by small estimator dispersion or variance, while consistency implies estimates converge in probability to the "true" value of the parameters as the number of observation becomes large.

Several studies have compared methods of parameter estimation using the standard error of estimate as a criterion. There are numerous hydrologic studies dealing with comparison of the accuracy of various methods of parameter estimation for various distributions and Monte-Carlo simulated sample sizes.

A number of 2- and 3-parameter PDFs have been discussed in the literature for hydrologic FFA (Hosking and Wallis, 1997; Singh, 1998; Rao and Hamed, 2000). Likewise, several parameter estimation methods for these PDFs have been developed (Singh, 1988; Rao and Hamed, 2000). Among others, the Log-Gumbel (*LG*) and the Log-logistic (*LL*) distributions have been recently used in FFA (ref. Rowiński *et al.*, 2001). They were critically examined by Rowiński *et al.* (2001) with respect to their applicability to hydrological data and the drawbacks resulting from their mathematical properties were pointed out. Both distributions are obtained by applying the logarithmic transformation to popular Fisher-Tippett type I (Gumbel) and logistic probability density functions, respectively. Their most significant feature is the existence of the statistical moments of *LG* and *LL* for a very limited range of parameters. For these parameters, a very rapid increase of the skewness coefficient as a function of the coefficient of variation is observed (especially for the Log-Gumbel distribution) which is seldom observed in hydrologic sciences.

Since the statistics used in every estimation method differ from each other, a method of fitting a theoretical distribution to an empirical one depends on the estimation method itself and in case of MLE on the distribution function as well. The differences in fitting may become crucial if an assumed PDF differs from the true one while for practical reasons the interest is in high accuracy of estimation in a certain range of variability, i.e., in the upper tail of the distribution. The MLM is considered as the most theoretically correct method in the sense that it produces the most efficient parameter estimates. The secret of the high efficiency of the maximum likelihood method (MLM) lies in its ability to extract greater amount of information both from the sample and from the assumed distribution function, which is required for the use of MLM.

Because *LG* and *LL* differ considerably from other FF distributions, it seems reasonable to take them as illustration of the model error problem and to examine the problem of their parameter estimation from the point of view of real conditions where the hypothetical PDF differs from the true one. This study is focused on the model error of upper quantiles and moments. Applicability of the two PDFs is critically examined in the real condition of unknown true distribution and recommendations are given for the estimation method under the assumption that *LG* and *LL* are either *F* or *T*. At the same time, the study is to supply the directions for using the various estimation methods in real conditions faced in FFA.

To get rid of sample size, the asymptotic case is considered. It enables to employ the analytical approach for asymptotic bias derivation (Strupczewski *et al.*, 2002) and extend it for the case of non-existing moments. The analytical approach has the advantage over the simulation one as it produces the exact solution of the asymptotic bias problem and the solution is expressed as a function of the *T*-model's parameters while Monte-Carlo experiments provide approximate result for any given set of parameter values separately. However, the algebraic difficulties to get analytical solution are growing fast with the number of PDF parameters and then one must be satisfied with an approximate result. In particular, this concerns the application of MLM as an approximation method.

2. ASYMPTOTIC BIAS OF ESTIMATION METHODS CAUSED BY THE ASSUMPTION OF FALSE PROBABILITY DISTRIBUTION

A study of asymptotic bias can serve as a basis to assess its magnitude and give an idea about the bias for any sample size, and whether the difference in the bias due to various parameter estimation methods can be counterbalanced by their efficiency of estimation. It would also be useful to verify the correctness of Monte Carlo experiments. The theoretical background for the asymptotic bias derivation for various estimation methods when a PDF is falsely assumed is given by Strupczewski *et al.* (2002). To present a unified treatment for various distributions, the original set of parameters of every distribution was expressed as functions of moments. The estimation methods are used as methods of approximation of one distribution function by another. The study has been illustrated using the lognormal and gamma distributions forming an alternative set of PDFs (*APDF*). This time *APDF* consists of five PDFs, giving a chance for some generalisation.

However, because for *LG* and *LL* the original parameters cannot be explicitly expressed by the first two moments, the original parameters are used in this study. Furthermore, its scope is limited to the investigation of the three methods, i.e., the method of moments (MOM), the linear moments method (LMM) and the maximum likelihood method (MLM) while the least-squares methods (LSM) of approximation (Strupczewski *et al.*, 2002) are omitted.

MOM, used as an approximation method, reduces to the moment matching of *F* to *T* distribution:

$$\int_0^\infty x^r f^{(F)}(x;\mathbf{g})\,\mathrm{d}x = \int_0^\infty x^r f^{(T)}(x;\mathbf{h})\,\mathrm{d}x, \qquad r = 1,2,..,R, \tag{1}$$

where R is the number of parameters of *False* distribution, i.e., the dimension of the g-vector.

Similarly, LMM as an approximation method, reduces to the L-moments or the probability weighted moments matching:

$$\int_0^1 x\left(F^{(F)}\right)\left[F^{(F)}\right]^r = \int_0^1 x\left(F^{(T)}\right)\left[F^{(T)}\right]^r \mathrm{d}F^{(T)}, \qquad r = 1,2,..,R, \tag{2}$$

where $F^{(\cdot)}$ is the cumulative probability distribution.

In order to apply MLM as an approximation method of one distribution (T) by another (F), one has to consider the asymptotic sample of T-distribution. Then, the ML-equations have the form

$$\lim_{N\to\infty} \mathrm{E}\left(\frac{\partial \log L^{(F)}\left(X;\mathbf{g}|T\right)}{\partial \mathbf{g}}\right) = \int_0^\infty \frac{\partial \log L^{(F)}\left(x;\mathbf{g}\right)}{\partial \mathbf{g}} f^{(T)}\left(x;\mathbf{h}\right)\mathrm{d}x = 0\,, \tag{3}$$

where $f^{(T)}(x;\mathbf{h})$ is the probability density function of the *True* distribution and $\mathbf{h}$ is the vector of its parameters while $\mathbf{g}$ is the vector of parameters of the *False* distribution $f^{(F)}(x;\mathbf{g})$. Log $L^{(F)}$ denotes the log-likelihood function for the *False* distribution.

The mean-square error (MSE) of any statistical characteristic, Z, can be expressed as

$$MSE(Z) = \mathrm{var}(Z) + \left[Bias(Z)\right]^2, \tag{4}$$

where var(Z) is the variance of Z and $Bias(Z)$ is the bias of Z. For a given sample size, the ratio of the two terms in relation (4) depends on both the PDF model and the parameter estimation method. An increase in the number of model parameters (degrees of freedom) increases the first term and decreases the second one. For large samples, the standard deviation of the Z estimate becomes small in comparison to the bias caused by the wrong distribution choice (i.e., by the model error) and therefore MSE approaches the square of the asymptotic bias $B(Z) = \lim_{N\to\infty} Bias(Z)$.

In order to derive the asymptotic bias (B) of Z caused by the false (F) choice of the distributional hypothesis (H), the knowledge of the true distribution (T) together with the value of its parameters is necessary. Then, the problem is defined as an approximation of the T-function by the F-function and it, therefore, remains no longer a statistical estimation problem. Having approximated T by F, one can find for any characteristic Z both the value of z of the approximated function, i.e., $z(H{=}T)$, and the corresponding value of z of the approximating function, i.e., $z(H{=}F\,|T)$. Thus, the asymptotic bias of any statistical characteristic Z, $B(Z)$, is defined as

$$B(Z) = z(H = F|T) - z(H = T) \tag{5}$$

and the relative asymptotic bias, *RB(Z)*, as

$$RB(Z)=\frac{z(H=F|T)-z(H=T)}{z(H=T)}, \tag{6}$$

where *H, F* and *T* stand for hypothetical, false and true distributions, respectively.

MLM, used as an approximation method, is irreversible, i.e., assuming the same dimension of vectors of parameters **g** and **h**, if (**H=a**)⇒(**g=b**) then (**g=b**)⇏(***H*=a**). The asymptotic bias of the MLM-approximation of any statistical characteristic *Z* is not asymmetric as is for the MOM and LMM, i.e., $B^{MLM}(z(H=\varphi\,|\,T=f)) \neq -B^{MLM}(z(H=f\,|\,T=\varphi))$, where *f* and φ denote PDFs. Furthermore, to apply MLM as an approximation method, the approximate distribution (*H*) must have the same range as the true distribution or its domain must cover that of the true distribution. For MOM and LMM, there are no constraints with respect to the range of both distributions but the overlapping range enables the fit of the first moment.

3. SET OF ALTERNATIVE PDFs

The set of alternative PDFs (*APDF*) taken for the study consists of the following two -parameter distributions with zero lower bound: Log-Gumbel (*LG*), Log-logistic (*LL*), Log-normal (*LN*), Linear Diffusion Analogy (*LD*) and Gamma (*Ga*). Although three-parameter distributions are often recommended for flood frequency analysis, two-parameter distributions were chosen in this study for two reasons. First, the constraints in respect to the number of parameters are very rigid for normal hydrological sample sizes. Second, the objective is to show the significance of bias using simpler models. To extend the results to three-parameter lower bounded distributions, it would suffice to start with the matching of the lower bounds.

Our presentation is focused on the *LG* investigation considering it both as a false and true distribution. However, all our conclusions remain valid for the case when *LL* approximates other distributions or is approximated by them. Therefore, the two following cases were analysed in the paper:

(1) $(H = LG;\; T = LL, LN, LD, Ga)$,

(2) $(H = LL, LN, LD, Ga;\; T = LG)$.

The density functions of these distributions are given below. The formulae for moments, L-moments and quantiles can be easily found in the statistical literature (see, e.g., Hosking and Wallis, 1997; and Kaczmarek, 1977), while for *LD* are given by Strupczewski *et al.* (2001).

Log-Gumbel (*LG*) distribution

$$f(x;\alpha_{LG},\xi) = \frac{\xi}{\alpha_{LG}} x^{-1/\alpha_{LG}-1} \exp\left(-\xi\, x^{-1/\alpha_{LG}}\right), \quad x, \alpha_{LG}, \xi > 0 . \tag{7}$$

Log-logistic (*LL*) distribution

$$f(x;\alpha_{LL},\kappa) = \frac{(x/\alpha_{LL})^{\frac{1}{\kappa}}}{\kappa x\left(1+(x/\alpha_{L})^{\frac{1}{\kappa}}\right)^2}, \quad \alpha_{LL},\kappa,x > 0 . \tag{8}$$

In order to distinguish between the α parameters of *LG* and *LL* the subscripts "*LL*" and "*LG*" are used.

Log-normal (*LN*) distribution

$$f(x;\mu,\sigma) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(\ln x-\mu)^2\right], \quad x,\sigma > 0 . \tag{9}$$

Linear diffusion analogy (*LD*) distribution

$$f(x;\alpha_{LD},\beta_{LD}) = \frac{\alpha_{LD}}{\sqrt{\pi x^3}} \exp\left[-\frac{(\alpha_{LD}-\beta_{LD}x)^2}{x}\right], \quad x,\alpha_{LD},\beta_{LD} > 0 . \tag{10}$$

Gamma (*Ga*) distribution

$$f(x;\alpha_{Ga},\lambda) = \frac{\alpha_{Ga}^{\lambda}}{\Gamma(\lambda)} x^{\lambda-1} \exp(-\alpha_{Ga}x), \quad x,\alpha_{Ga},\lambda > 0 . \tag{11}$$

4. LOG-GUMBEL AS HYPOTHETICAL DISTRIBUTION

LL as true distribution

The *LL* distribution is the closest one to the *LG* distribution of all *APDF* in respect to the relation of the coefficient of skewness (c_s) to the coefficient of variation (c_v).

The asymptotic relative bias (*RB*) of the mean (*m*), variance (var) and quantile (x_p) expressed in terms of parameters of both distributions is, respectively, as follows:

$$RB(m) = \frac{m_{LG}}{m_{LL}} - 1 = \frac{\xi^{\alpha_{LG}}\Gamma(1-\alpha_{LG})}{\alpha_{LL}B(1+\kappa,1-\kappa)} - 1, \quad \alpha_{LG},\kappa < 1 , \tag{12}$$

$$RB(\text{var}) = \frac{\text{var}_{LG}}{\text{var}_{LL}} - 1 = \frac{\xi^{2\alpha_{LG}}\left[\Gamma(1-2\alpha_{LG}) - \Gamma^2(1-\alpha_{LG})\right]}{\alpha_{LL}^2\left[B(1+2\kappa,1-2\kappa) - B^2(1+\kappa,1-\kappa)\right]} - 1, \quad \alpha_{LG},\kappa < \frac{1}{2}, \tag{13}$$

$$RB(x_p) = \frac{x_{pLG}}{x_{pLL}} - 1 = \frac{\left(-\frac{1}{\xi}\ln(1-p)\right)^{-\alpha_{LG}}}{\alpha_{LL}\left(\frac{1}{p}-1\right)^{\kappa}} - 1, \tag{14}$$

where $\Gamma(\ )$ and $B(\ )$ are the Euler gamma and beta functions, respectively.

The form of the relationship of *LG*-parameters (ξ,α_{LG}) with *LL*-parameters (κ,α_{LL}) depends on the approximation method.

MOM approximation. Matching the first moments about the origin and the second central moments of both distributions, respectively, i.e., $E_{LG}X = E_{LL}X$ and $\text{var}_{LG}X = \text{var}_{LL}X$, we get *LG*-parameters in terms of *LL*-parameters for the range of the first two moments of the *LL* existence, i.e., for $\kappa < 0.5$. Then, *RB* in eqs. (12) and (13) turns to be zero. Hence, the equality of the coefficients of variation, i.e., $c_{vLG} = c_{vLL}$, takes the form

$$c_v^2 = \frac{B(1+2\kappa,1-2\kappa)}{B^2(1+\kappa,1-\kappa)} - 1 = \frac{\Gamma(1-2\alpha_{LG})}{\Gamma^2(1-\alpha_{LG})} - 1, \tag{15}$$

which (through given c_v) relates the parameters $\kappa < 0.5$ and $\alpha_{LG} < 0.5$. The relevant graph 'MOM' is shown in Fig. 1. The MOM-relationship $\alpha_{LG}(\kappa)$ is slightly nonlinear for values of κ corresponding to all c_v-values.

Substituting into eq. (14) the MOM relation for the mean values, we get the asymptotic relative bias of MOM (and MLM) estimate of quantile:

$$RB^{(MOM)}(x_p; F{=}LG,\ T{=}LL) = \frac{B(1+\kappa,1-\kappa)}{\Gamma(1-\alpha_{LG})\left[-\ln(1-p)\right]^{\alpha_{LG}}\left(\frac{1}{p}-1\right)^{\kappa}} - 1, \tag{16}$$

where κ and α_{LG} are the functions of eq. (15) of the true coefficient of variation (c_{vLL}). The function in eq. (16) is presented in Fig. 2 for some selected values of c_{vLL}. Figure 2 shows a decreasing range of the $RB(x_p)$ variability with the true coefficient of variation performed mainly through the decreasing lower-probability branch of x_p.

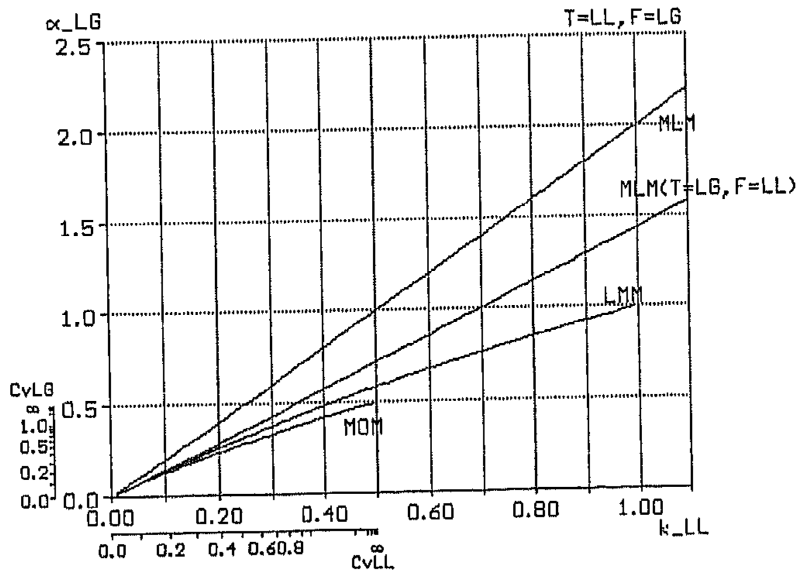

Fig. 1. Relationship between parameters κ and α_{LG} for MOM, LMM and MLM. Unlike the first two methods, MLM produces two relationships depending on which of the distributions (*LG* or *LL*) is true. Additional axes for true coefficients of variations c_v are given for comparison purposes. Axis c_{vLG} is valid for MOM and $T = LG$ while the c_{vLL} axis is valid for all cases except for $T = LG$.

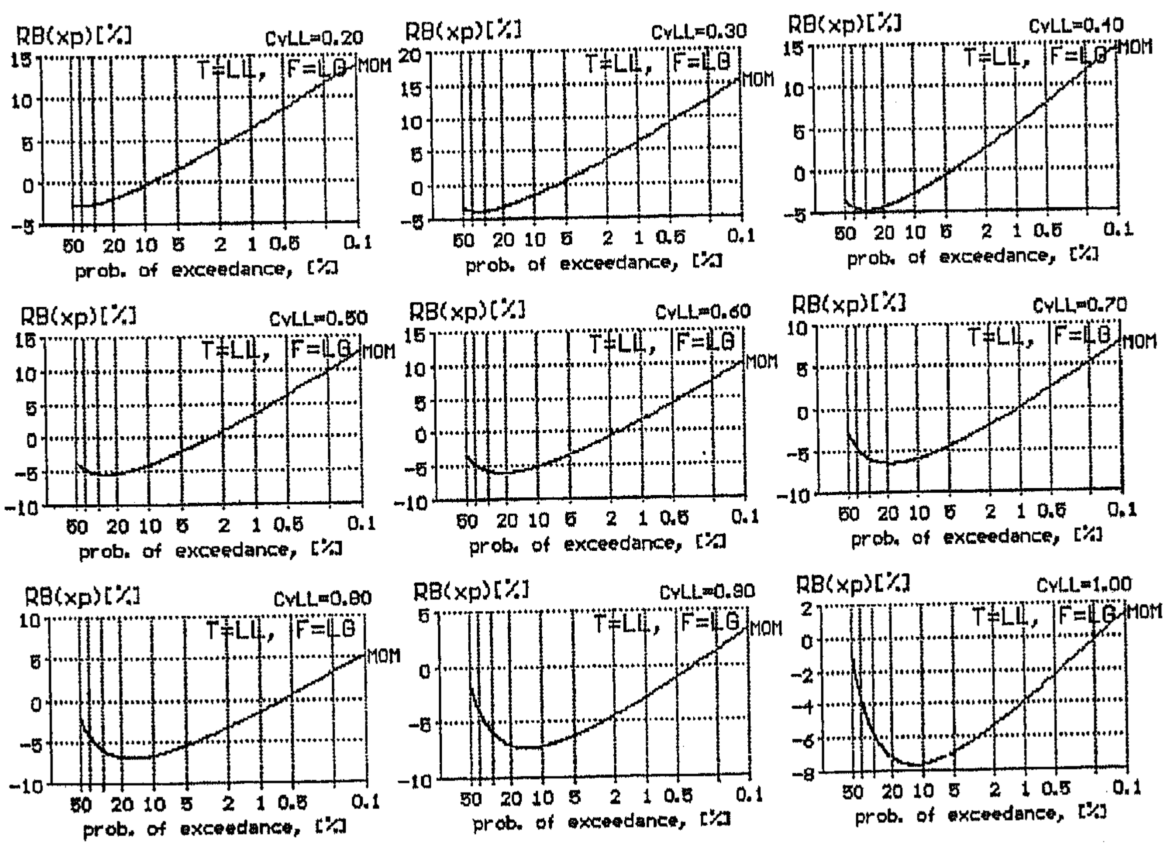

Fig. 2. Asymptotical bias of *LG* MOM-estimated quantile *vs*. probability of exceedance for some selected values of the true coefficient of variation c_{vLL}.

LMM approximation. Since the first L-moment is equivalent to the first moment about the origin, the match of the first L-moments $(\lambda_{1LG} = \lambda_{1LL})$ is equivalent to the match of the first moments $(m_{LG} = m_{LL})$ of both distributions (cf., eqs. 12). Matching of the second L-moments (λ_2) gives

$$\xi^{\alpha_{LG}} \Gamma(1-\alpha_{LG})(2^{\alpha_{LG}}-1) = \alpha_{LL}\, \kappa\, \Gamma(1+\kappa)\Gamma(1-\kappa), \quad \alpha_{LG}, \kappa < 1. \tag{17}$$

Hence, we get equality of the L-coefficients of variation as

$$\kappa = 2^{\alpha_{LG}} - 1, \tag{18}$$

where, because of eq. (17), κ, $\alpha_{LG} < 1$, which is twice the ranges permitted by MOM. A graph of eq. (18) denoted 'LMM' is shown in Fig. 1. Therefore, the mean is unbiased, while the variance and quantiles are biased. Using eq. (15) one can express κ and α_{LG} in terms of c_{vLL} and c_{vLG}, respectively. Substituting them into eq. (18) one gets the LM-relationship of both coefficients of variation, i.e., $c_{vLG} = \varphi^{LMM}(c_{vLL})$. Hence, the asymptotic relative bias of the variance given by eq. (13) is as follows:

$$RB^{(LMM)}(\text{var}; F{=}LG, T{=}LL) = \left(\frac{c_{vLG}}{c_{vLL}}\right)^2 - 1 = \frac{\Gamma(1-2\alpha_{LG})/\Gamma^2(1-\alpha_{LG})-1}{B(1+2\kappa,1-2\kappa)/B^2(1+\kappa,1-\kappa)-1} - 1, \tag{19}$$

($\kappa, \alpha_{LG} < 0.5$), which is displayed in Fig. 3 (the LMM curve) as the function of the variation coefficient of the *LL*-distribution (c_{vLL}).

Note from eq. (13) that for the *LL*-variance existence, $\kappa < 0.5$, which corresponds to $\alpha_{LG} < \ln 1.5/\ln 2 \approx 0.5850$ in eq. (18), while to have both *LL* and *LG* variances finite, both κ and α_{LG} shall not exceed 0.5. This leads (through eq. 18) to a new limit to κ, namely, $\kappa < 2^{0.5}-1 \approx 0.414$, which corresponds to $c_{vL} = 1.3349$ in eq. (15). It is a stronger constraint than for MOM-approximation, where κ, $\alpha_{LG} < 0.5$ (and for LMM estimation, where κ, $\alpha_{LG} < 1.0$, is sufficient to get finite τ).

The bias of the MOM-estimated quantiles (see Fig. 4) is defined by eq. (16), where α_{LG} is related to κ by eq. (18) but not by eq. (13) and κ is the function given by eq. (14) of c_{vLL}. The variability pattern of $RB^{(LMM)}(x_p)$ with c_{vLL}, shown also in Fig. 4, differs from the pattern of $RB^{(MOM)}(x_p)$: unlike $RB^{(MOM)}(x_p)$, the range of $RB^{(LMM)}(x_p)$ increases with c_v. This increase is moderate and the growing difference between $RB^{(LMM)}(x_p)$ and $RB^{(MOM)}(x_p)$ is mainly due to the behaviour of the MOM estimate.

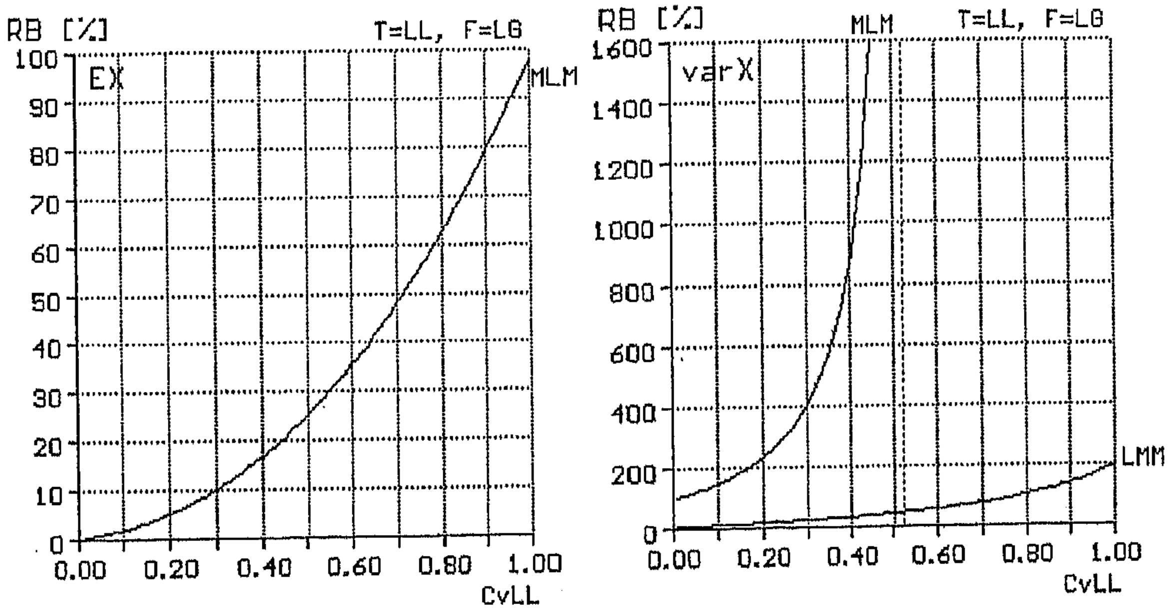

Fig. 3. Asymptotical bias of *LG* MLM-estimated mean (left), and *LG* MLM and *LG* LMM -estimated variance (right) *vs*. true coefficient of variation c_{vLL}.

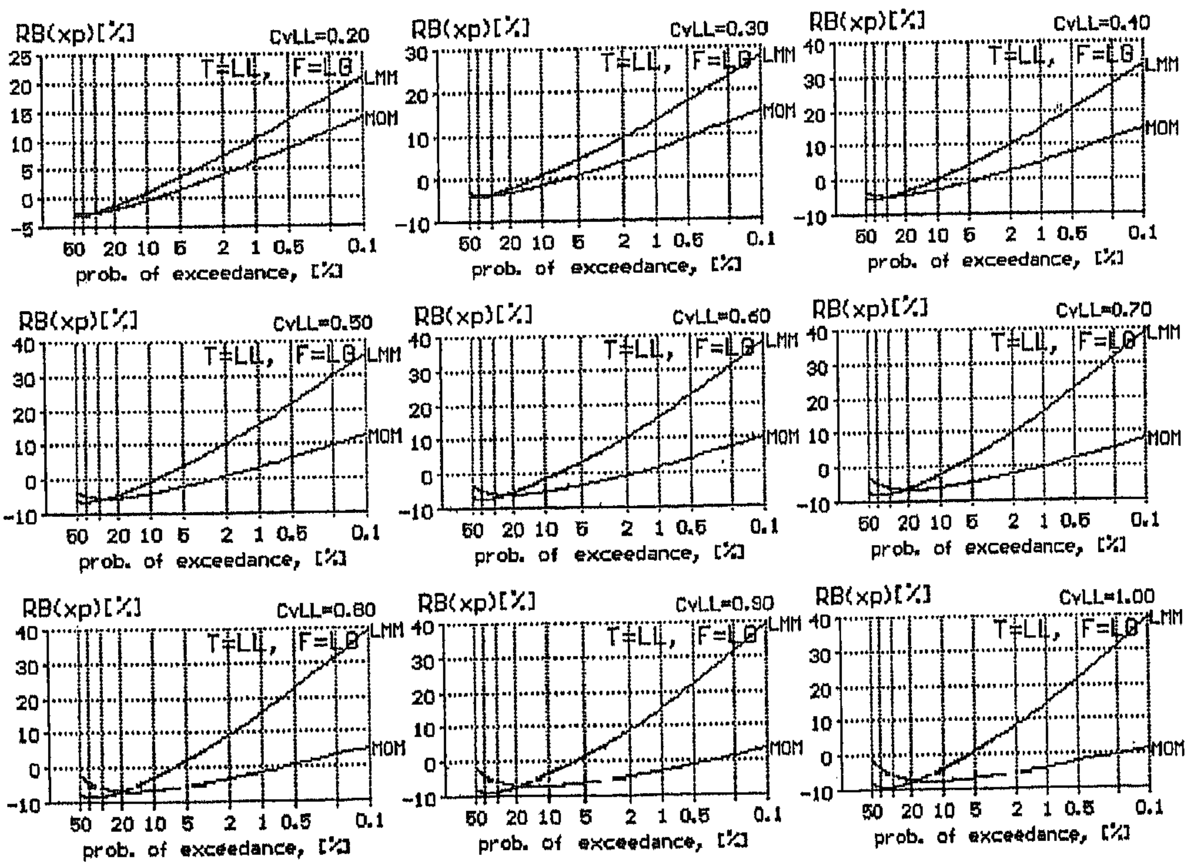

Fig. 4. Asymptotical bias of *LG* MOM and *LG* LMM-estimated quantile *vs*. probability of exceedance for some selected values of the true coefficient of variation c_{vLL}.

MLM approximation. The MLM system of equations

$$\int_0^\infty \frac{\partial \log L^{(F=LG)}(x;\xi,\alpha_{LG})}{\partial \xi} f^{(T=LL)}(x;\alpha_{LL},\kappa)\mathrm{d}x = 0, \tag{20}$$

$$\int_0^\infty \frac{\partial \log L^{(F=LG)}(x;\xi,\alpha_{LG})}{\partial \alpha_{LG}} f^{(T=LL)}(x;\alpha_{LL},\kappa)\mathrm{d}x = 0 \tag{21}$$

leads to the following equations relating to one another the parameters of the *LL* and *LG* distributions:

$$\frac{1}{\xi} - \frac{1}{(a_{LL})^{1/\alpha_{LG}}} \Gamma\left(1-\frac{\kappa}{\alpha_{LG}}\right)\Gamma\left(1+\frac{\kappa}{\alpha_{LG}}\right) = 0, \tag{22}$$

$$\frac{\alpha_{LG}}{\kappa} + \psi\left(1+\frac{\kappa}{\alpha_{LG}}\right) - \psi\left(1-\frac{\kappa}{\alpha_{LG}}\right) = 0. \tag{23}$$

Solving eq. (23) for α_{LG}/κ we get that

$$\frac{\alpha_{LG}}{\kappa} = 2. \tag{24}$$

The relationship in eq. (24) is shown in Fig. 1 as the 'MLM' graph together with the corresponding graphs for MOM and LMM. The MOM and LMM curves run relatively close to each other, the MLM one is visibly separate.

The using of eq. (24) simplifies eq. (22) to

$$\frac{1}{\xi} = \frac{1}{(\alpha_{LL})^{1/\alpha_{LG}}} \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{\pi}{2(\alpha_{LL})^{1/\alpha_{LG}}}. \tag{25}$$

Asymptotic relative bias of the mean. Substituting α_{LL} from eq. (25) into eq. (12), after simple algebraic transformation, we finally get

$$RB^{(MLM)}(m; H{=}LG,\ T{=}LL) = \left(\frac{\pi}{2}\right)^{-\alpha_{LG}} \frac{\Gamma(1-\alpha_{LG})}{B(1+\kappa,1-\kappa)} - 1, \tag{26}$$

where κ is the function given by eq. (15) of the coefficient of variation (c_{vLL}).

Proceeding in a similar way with eqs. (13) and (14), the asymptotic relative bias of variance (Fig. 3) and of quantile (Fig. 5) has been derived:

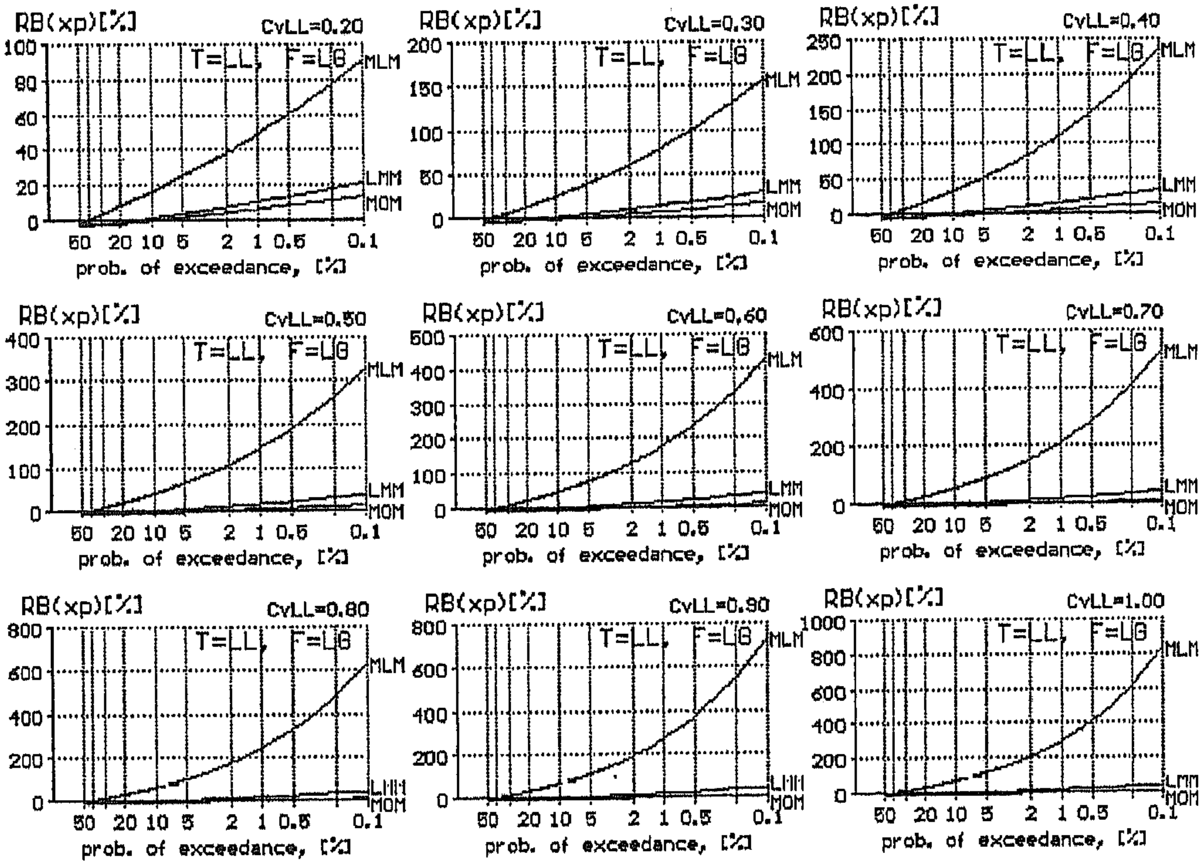

Fig. 5. Asymptotical bias of *LG* MOM, *LG* LMM and *LG* MLM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLL}.

$$RB^{(MLM)}(\text{var}; H{=}LG,\ T{=}LL) = \frac{\left[\Gamma(1-2\alpha_{LG})-\Gamma^2(1-\alpha_{LG})\right]\left(\frac{\pi}{2}\right)^{-2\alpha_{LG}}}{B(1+2\kappa,1-2\kappa)-B^2(1+\kappa,1-\kappa)}-1, \tag{27}$$

$$RB^{(MLM)}(x_p; H{=}LG,\ T{=}LL) = \frac{\left[-\ln(1-p)\right]^{-\alpha_{LG}}}{\left(\frac{1}{p}-1\right)^{\kappa}\left(\frac{\pi}{2}\right)^{\alpha_{LG}}}-1. \tag{28}$$

To get the bias of variance given by eq. (27) finite, κ must be less than 1/4, which corresponds to eq. (15) with $c_{vLL} < 0.5227$, while for LMM we have $\kappa < 2^{0.5}-1 \approx 0.414$, which is equivalent to $c_{vLL} < 1.3349$. These constraints are not reflected in graphs in Fig. 5 as the relative bias of the MLM quantiles given by eq. (28) does not impose any restrictions on parameters (beside $\kappa < 1/2$ as required by eq. 15). The asymptotic relative bias of MLM quantile (Fig. 5) is at least by one order greater than that for MOM and LMM and ranges for $x_{0.1\%}$ from about 100% for $c_{vLL} = 0.2$ to above 800% for $c_{vLL} = 1.0$, which is a rather discouraging feature of MLM. The poor fit of the upper tail is compensated for in the lower branch (Fig. 6) where MLM-approximation is much better than either of the two other methods. It is the common feature for all lower bounded PDFs, i.e., for the PDFs feasible for FF-modelling.

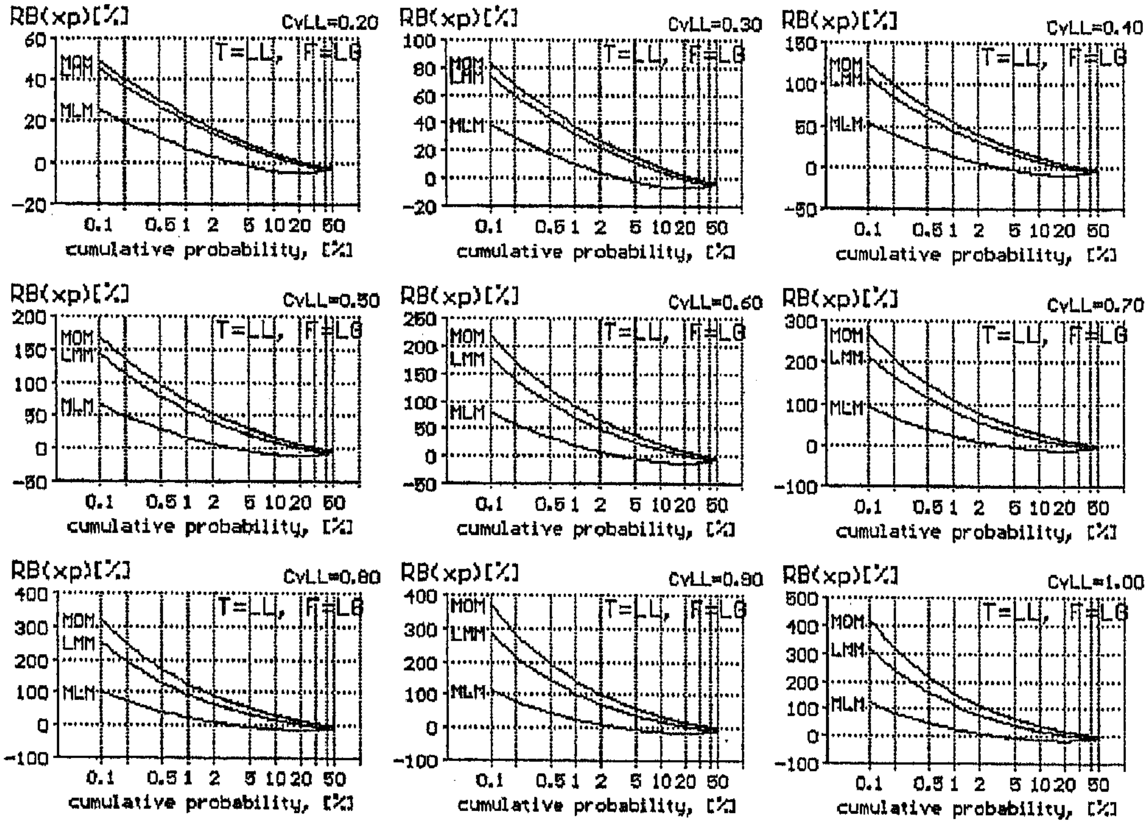

Fig. 6. Asymptotical bias of *LG* MOM, *LG* LMM and *LG* MLM-estimated quantile *vs.* cumulative probability for some selected values of the true coefficient of variation c_{vLL}.

LN as true distribution

The asymptotic relative bias of the mean (m), variance (var) and quantile (x_p) expressed in terms of parameters of both distributions is, respectively, as follows:

$$RB(m) = \frac{m_{LG}}{m_{LN}} - 1 = \frac{\xi^{\alpha_{LG}}\,\Gamma(1-\alpha_{LG})}{\exp(\mu+\sigma^2/2)} - 1, \qquad \alpha_{LG} < 1, \tag{29}$$

$$RB(\mathrm{var}) = \frac{\mathrm{var}_{LG}}{\mathrm{var}_{LN}} - 1 = \frac{\xi^{2\alpha_{LG}}\left[\Gamma(1-2\alpha_{LG}) - \Gamma^2(1-\alpha_{LG})\right]}{\exp(2\mu+\sigma^2)\left[\exp(\sigma^2)-1\right]} - 1, \qquad \alpha_{LG} < \frac{1}{2}, \tag{30}$$

$$RB(x_p) = \frac{x_{pLG}}{x_{pLN}} - 1 = \frac{\left(-\frac{1}{\xi}\ln(1-p)\right)^{-\alpha_{LG}}}{\exp(\mu+\sigma t_p)} - 1. \tag{31}$$

The form of relationship of *LG*-parameters (ξ,α_{LG}) with *LN*-parameters (μ,σ) depends on the approximation method.

MOM approximation. The condition of the first moments equality takes the form

$$\xi^{\alpha_{LG}}\,\Gamma(1-\alpha_{LG}) = \exp\left(\mu + \frac{\sigma^2}{2}\right), \qquad \alpha_{LG} < 1. \tag{32}$$

Matching the coefficients of variation, i.e., $c_{vLG} = c_{vLN}$, gives

$$c_v^2 = \frac{\Gamma(1-2\alpha_{LG})}{\Gamma^2(1-\alpha_{LG})} - 1 = \exp(\sigma^2) - 1, \tag{33}$$

which (through given c_{vLN}) relates the parameters $0 < \sigma < \infty$ and $0 < \alpha_{LG} < 0.5$. The relevant relation $\alpha_{LG}(\sigma)$ is plotted in Fig. 7 as the 'MOM' curve.

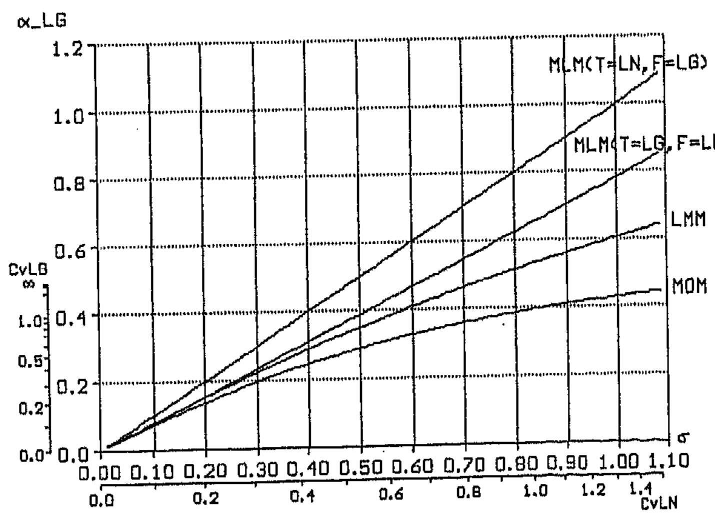

Fig. 7. Relationship between parameters α_{LG} and σ for MOM, LMM and MLM. Unlike the first two methods, MLM produces two relationships depending on which of the distributions (*LG* or *LN*) is true. Additional axes for true coefficients of variations c_v are given for comparison purposes. Axis c_{vLN} is valid for MOM and $T = LG$ while the c_{vLL} axis is valid for all cases except for $T = LG$.

From eqs. (31) and (32) we have the relative bias of the x_p quantile

$$RB^{(MOM)}\left(x_p\right) = \frac{\exp\left(\sigma^2/2-\sigma t_p\right)\left(-\ln(1-p)\right)^{-\alpha_{LG}}}{\Gamma(1-\alpha_{LG})} - 1, \tag{34}$$

where parameters α_{LG} and σ are interrelated through eq. (33). Equation (34) is illustrated in Fig. 8. $RB^{(MOM)}(x_p)$ remains within about a 30–50% variability range for all c_{vLN} with $RB^{(MOM)}(x_{0.1\%})$ increasing with c_{vLN} up to $c_{vLN} \approx 0.4$ and then decreasing with c_{vLN}. The minimum of $RB^{(MOM)}(x_p)$ deepends and shifts towards low p.

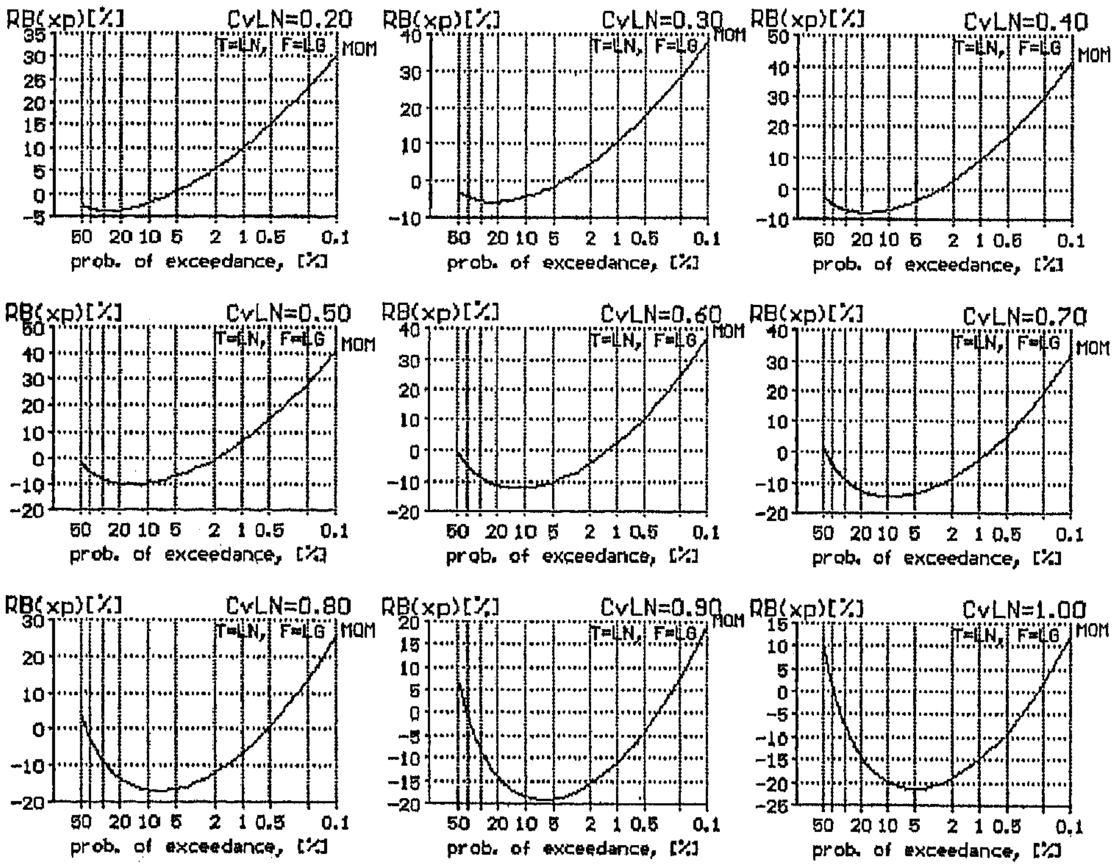

Fig. 8. Asymptotical bias of *LG* MOM-estimated quantile *vs*. probability of exceedance for some selected values of the true coefficient of variation c_{vLN}.

LMM approximation. The match of the *L*-coefficients of variation gives

$$2^{\alpha_{LG}} - 1 = 1 - 2p\left(\frac{\sigma}{\sqrt{2}}\right), \tag{35}$$

where

$$p(w)=\int_{w}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z\,. \tag{36}$$

The range (0,1) for α_{LG} corresponds to $(0,\ +\infty)$ for σ. The relationship of these two parameters is displayed in Fig. 7 as the 'LMM' curve.

Using eq. (33) one can express σ and α_{LG} by c_{vLN} and c_{vLG}, respectively. Substituting them into eq. (35) one gets the LM-relationship of both coefficients of variation, i.e., $c_{vLG}=\varphi^{LMM}(c_{vLN})$. Hence, the asymptotic relative bias of the variance given by eq. (30) is

$$RB^{(LMM)}(\text{var};F{=}LG,\ T{=}LN)=\left(\frac{c_{vLG}}{c_{vLN}}\right)^2-1\,. \tag{37}$$

The bias is displayed in Fig. 9 as a function of the true variation coefficient of *LN*-distribution, c_{vLN}. Because $\alpha_{LG}<0.5$ (see eq. 30), from eq. (35) we have that $\sigma<\sqrt{2}\,t_{1/\sqrt{2}}$, where t_p is the p–$N(0,1)$ quantile, which gives an upper limit to c_{vLN} of 0.9007. The LMM bias of variance quickly increases with c_{vLN} and exceeds 100% at $c_{vLN}\approx 0.5$.

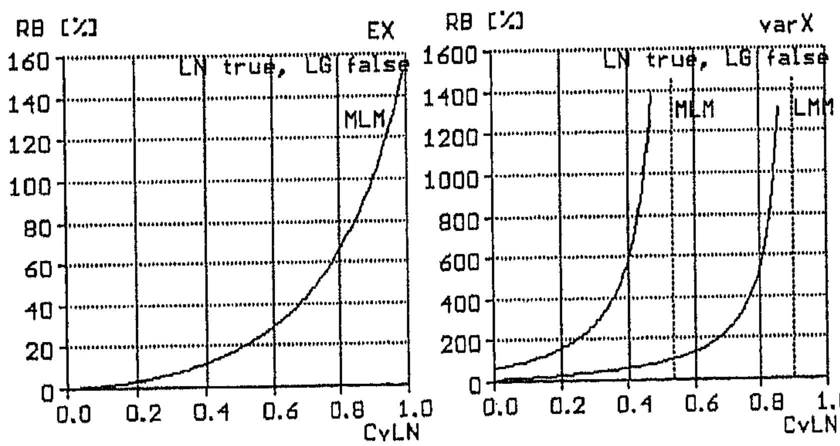

Fig. 9. Asymptotical bias of *LG* MLM-estimated mean (left, $c_{vLN}=1.3108$), and *LG* MLM and *LG* LMM-estimated variance (right) *versus* true coefficient of variation c_{vLN} ($c_{vLN}=0.5329$, $c_{vLN}=0.9007$).

The relative bias of the quantile x_p for LMM can be expressed with eq. (34) where α_{LG} and σ are related through eq. (35). The results are shown in Fig. 10. The difference between the $RB^{(LMM)}(x_p)$ and $RB^{(MOM)}(x_p)$, small for small c_{vLN}, (less then 15% for $c_{vLN}=0.2$) increases with c_v, reaching more than 100% for $x_{0.1\%}$ and $c_{vLN}=1.0$.

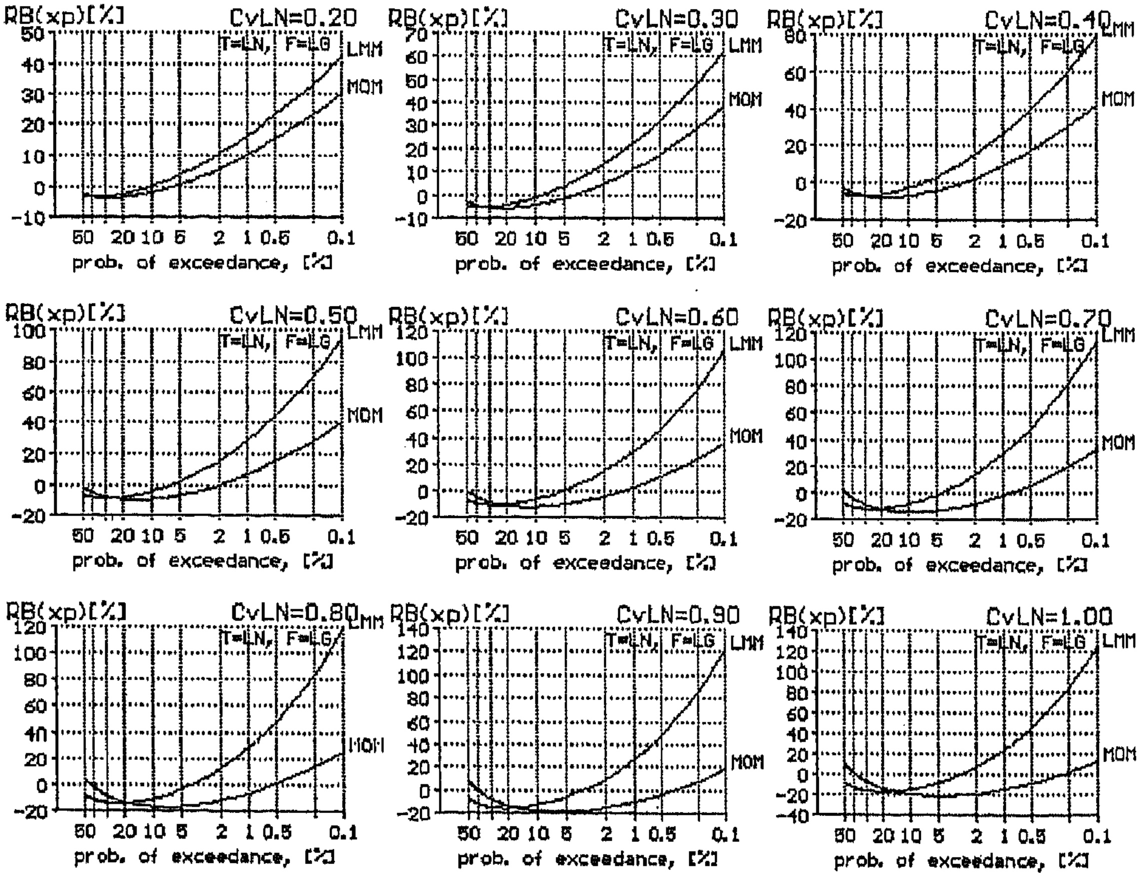

Fig. 10. Asymptotical bias of *LG* MOM and *LG* LMM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLN}.

MLM approximation. Application of the MLM method results in the following equalities:

$$\left(\frac{\sigma}{\alpha_{LG}}\right)^2 = 1, \tag{38}$$

$$\xi = \exp\left(-1/2 + \mu/\sigma\right). \tag{39}$$

The graph of $\alpha_{LG}(\sigma)$ from eq. (38) is presented in Fig. 7 as the 'MLM' curve.

The bias of mean and variance is

$$RB^{(MLM)}(m) = \frac{\Gamma\left(1-\alpha_{LG}\right)}{\exp(\alpha_{LG}/2 + \alpha_{LG}^2/2)} - 1, \quad \alpha_{LG} < 1, \tag{40}$$

$$RB^{(MLM)}(\mathrm{var}) = \frac{\Gamma(1-2\alpha_{LG}) - \Gamma^2(1-\alpha_{LG})}{\exp(\alpha_{LG}^2+\alpha_{LG})\left(\exp(\alpha_{LG}^2)-1\right)} - 1, \quad \alpha_{LG} < \frac{1}{2}. \tag{41}$$

The conditions for α_{LG} in eqs. (40) and (41) combined with eq. (38) give $\sigma < 1$ and $\sigma < 0.5$, respectively, which gives the corresponding upper limit values of c_{vLN} for the relative bias of the mean and variance: $c_{vLN} = 1.3108$ and 0.5329 (Fig. 9). The MLM relative bias of the variance starting point at $c_{vLN} = 0$ is not zero as for LMM: the limit of $RB^{(MLM)}(\mathrm{var})$ given by eq. (41) ($\alpha_{LG} \to 0$) is $\psi'(1) - 1 = \pi^2/6 - 1 \approx 0.64493 \approx 64.5\%$, where the ψ' is the trigamma function. Then the bias quickly tends to infinity.

The MLM relative bias of the quantile can be calculated using eqs. (38) and (39) combined with eq. (31):

$$RB^{(MLM)}(x_p) = \exp\left[-(0.5+t_p)\alpha_{LG}\right]\left[-\ln(1-p)\right]^{-\alpha_{LG}} - 1. \tag{42}$$

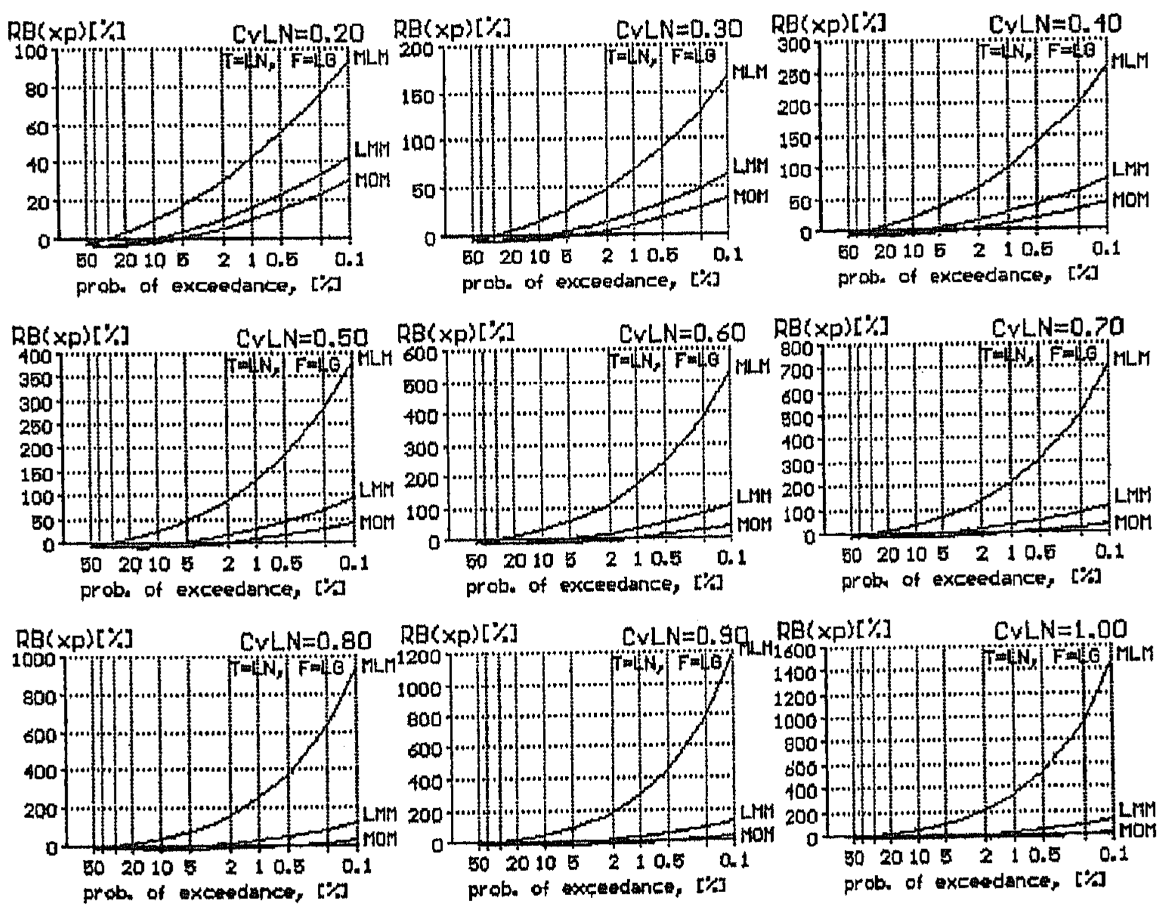

Fig. 11. Asymptotical bias of *LG* MOM, *LG* LMM and *LG* MLM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLN}.

The MLM *LG*-approximation of *LN* is similar to MLM *LG*-approximation of *LL* presented in Fig. 5 although the MLM values in Fig. 11 are greater.

LD as true distribution

In fact, *LN* is closer to *LG* in terms of the relation of the skewness coefficient to the coefficient of variation than the recently introduced *LD* distribution, as $c_s = 3c_v + c_v^3$ for *LN*, while $c_s = 3c_v$ for *LD* (Strupczewski *et al.*, 2001). The reason to include it is its attractive property when it serves as the hypothetical distribution and the MLM for parameter estimation is applied. The *L*-moments have not been derived for *LD*.

MOM approximation. The equality of first two moments of *LD* and *LG* gives the following equations:

$$\frac{\alpha_{LD}}{\beta_{LD}} = \xi^{\alpha_{LG}} \Gamma(1-\alpha_{LG}), \qquad \alpha_{LG} < 1, \tag{43}$$

$$\frac{\alpha_{LD}}{\beta_{LD}^3} = \beta^{2\alpha_{LG}} \left[\Gamma(1-2\alpha_{LG}) - \Gamma^2(1-\alpha_{LG}) \right], \qquad \alpha_{LG} < \frac{1}{2}. \tag{44}$$

The coefficients of variations are equal:

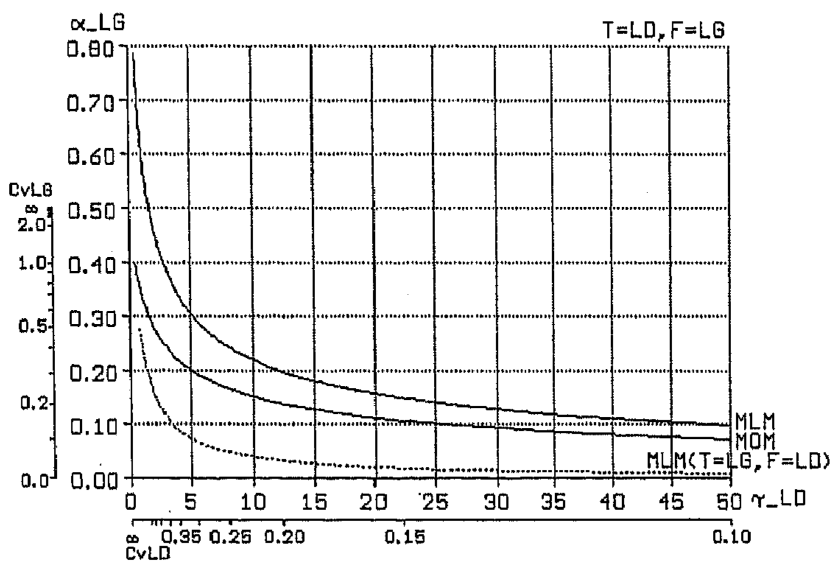

Fig. 12. Relationship between parameters α_{LG} and γ_{LD} for MOM and MLM. Unlike MOM, MLM produces two relationships depending on which of the distributions (*LG* or *LD*) is true. Additional axes for true coefficients of variations c_v are given for comparison purposes. Axis c_{vLG} is valid for MOM and $T = LG$ while the c_{vLD} axis is valid for all cases except for $T = LG$.

$$C_v^2 = \frac{1}{2\alpha_{LD}\beta_{LD}} = \frac{\Gamma(1-2\alpha_{LG})}{\Gamma^2(1-\alpha_{LG})} - 1, \qquad \alpha_{LG} < \frac{1}{2}. \tag{45}$$

Equation (45) relates the product ($\alpha_{LD}\,\beta_{LD}$), called hereinafter γ_{LD} :

$$\gamma_{LD} = \alpha_{LD}\,\beta_{LD}, \tag{46}$$

and α_{LG} (graph MOM in Fig. 12). Using eq. (43) we get the bias of the MOM quantile

$$RB^{(MOM)}(x_p) = \frac{\left[t_p^{LD}(\gamma_{LD})\right]^2}{\gamma_{LD}\,\Gamma(1-\alpha_{LG})\left[-\ln(1-p)\right]^{\alpha_{LG}}} - 1 \tag{47}$$

(see Fig. 13). The MOM LG-approximation of LD follows the pattern of the MOM LG-approximation of LN (Fig. 8) with a slightly greater amplitude.

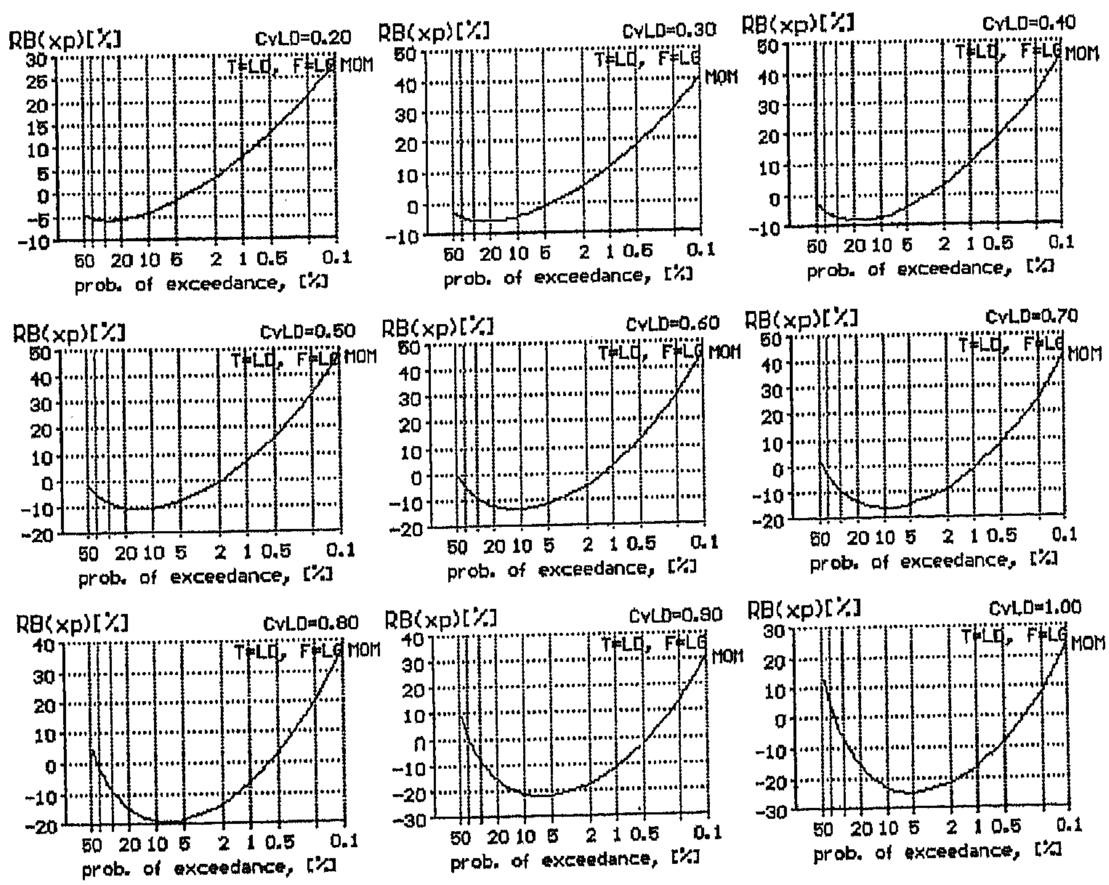

Fig. 13. Asymptotical bias of LG MOM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLD}.

MLM approximation. MLM for *LD* produces finally the following system of equations from which ML-estimates of *LG*-parameters can be obtained:

$$\frac{1}{\xi} = A\beta_{LG}^{2/\alpha_{LG}} G_{1/2+1/\alpha_{LG}}\,, \qquad \alpha_{LG} + AG'_{1/2} - \frac{G'_{1/2+1/\alpha_{LG}}}{G_{1/2+1/\alpha_{LG}}} = 0\,, \tag{48}$$

where A, G_z and G'_z in eq. (48) are functions of γ_{LD} and are defined as follows:

$$A(\gamma_{LD}) = \frac{\gamma_{LD}\exp(2\gamma_{LD})}{\sqrt{\pi}}, \tag{49}$$

$$G_z(a) = \int_0^\infty u^{-z-1}\exp\left(\frac{-u-a^2}{u}\right)du = 2a^{-z}K_z(2a)\,, \tag{50}$$

$$G'_z(a) = \frac{\partial G_z(a)}{\partial z} = -\int_0^\infty \ln(u\,u^{-z-1})\exp\left(\frac{-u-a^2}{u}\right)du \tag{51}$$

$K_z(x)$ in eq. (50) is the MacDonald function (Andrews, 1992). The second equation in (48) is an implicit function α_{LG} of γ_{LD} that enables α_{LG} ML to be calculated for a given γ_{LD}. This function is shown in Fig. 12.

The relative bias of the mean and variance is as follows (see also Fig. 14):

$$RB^{(MLM)}(m) = \frac{m_{LG}}{m_{LD}} - 1 = \frac{\Gamma(1-\alpha_{LG})}{\gamma_{LD}\left(AG_{1/2+1/\alpha_{LG}}\right)^{\alpha_{LG}}} - 1\,, \tag{52}$$

$$RB^{(MLM)}(\text{var}) = \frac{\text{var}_{LG}}{\text{var}_{LD}} - 1 = \frac{\Gamma(1-2\alpha_{LG}) - \Gamma^2(1-\alpha_{LG})}{(\gamma_{LD}/2)\left(AG_{1/2+1/\alpha_{LG}}\right)^{2\alpha_{LG}}} - 1\,. \tag{53}$$

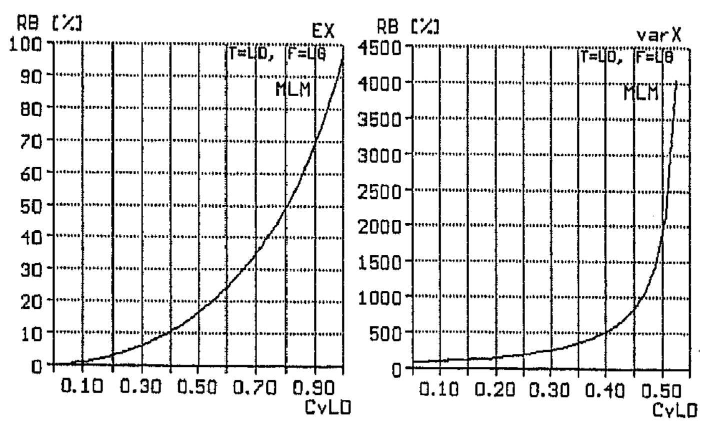

Fig. 14. Asymptotical bias of *LG* MOM-estimated mean (left) and *LG* MLM-estimated variance (right) *vs*. true coefficient of variation c_{vLD}.

The relative bias $RB^{(MLM)}(x_p)$ of quantile resulting from adopting Log-Gumbel and using MLM is

$$RB^{(MLM)}(x_p) = \frac{x_{pLG}}{x_{pLD}} - 1 = \frac{\left(t_{pLD}(\gamma_{LD})\right)^2}{(\gamma_{LD})^2\left(AG_{1/2+1/\alpha_{LG}}\right)^{\alpha_{LG}}\left(-\ln(1-p)\right)^{\alpha_{LG}}} - 1. \qquad (54)$$

The respective graphs are presented in Fig. 15. The MLM *LG*-approximation of *LD* lies in between the MLM *LG*-approximation of *LL* presented in Fig. 5 and the MLM *LG*-approximation of *LN* in Fig. 11.

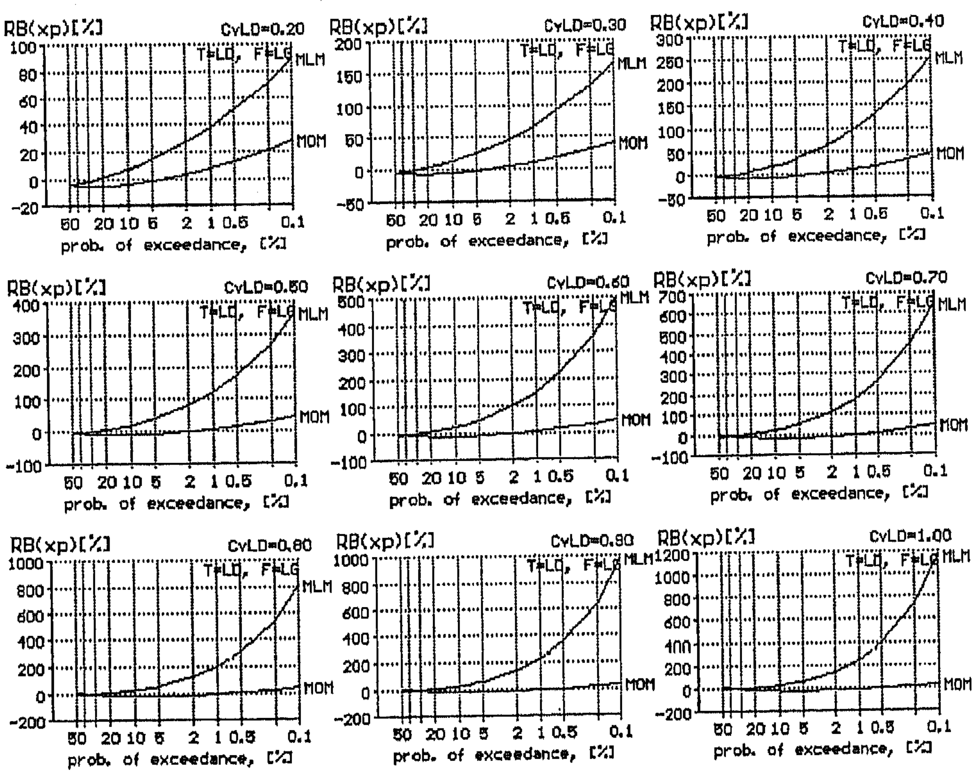

Fig. 15. Asymptotical bias of *LG* MOM and *LG* MLM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLD}.

Gamma as true distribution

The asymptotic relative bias of the mean (m), variance (var) and quantile (x_p) expressed in terms of parameters of both distributions is, respectively, as follows:

$$RB(m) = \frac{m_{LG}}{m_{Ga}} - 1 = \frac{\xi^{\alpha_{LG}} \Gamma(1-\alpha_{LG})}{\lambda / \alpha_{Ga}} - 1, \qquad \alpha_{LG} < 1, \tag{55}$$

$$RB(\mathrm{var}) = \frac{\mathrm{var}_{LG}}{\mathrm{var}_{Ga}} - 1 = \frac{\xi^{2\alpha_{LG}} \left[\Gamma(1-2\alpha_{LG}) - \Gamma^2(1-\alpha_{LG})\right]}{\lambda / \alpha_{Ga}^2} - 1, \qquad \alpha_{LG} < \frac{1}{2}, \tag{56}$$

$$RB(x_p) = \frac{x_{pLG}}{x_{pGa}} - 1 = \frac{\left(-\frac{1}{\xi}\ln(1-p)\right)^{-\alpha_{LG}}}{t_{pGa}(\lambda)/\alpha_{Ga}} - 1. \tag{57}$$

The form of relationship of *LG*-parameters (ξ, α_{LG}) with *Ga*-parameters (λ, α_{Ga}) depends on the approximation method.

MOM approximation. Employing MOM gives the following system of equations:

$$m = \xi^{\alpha_{LG}} \Gamma(1-\alpha_{LG}) = \frac{\lambda}{\alpha_{Ga}}, \quad \alpha_{LG} < 1, \tag{58}$$

$$c_v^2 = \frac{\Gamma(1-2\alpha_{LG})}{\Gamma^2(1-\alpha_{LG})} - 1 = \frac{1}{\lambda}, \quad \alpha_{LG} < \frac{1}{2}. \tag{59}$$

Equation (59) relates α_{LG} to λ as is shown in Fig. 16.

Using eqs. (57) and (58), we get the MOM relative bias of quantile

$$RB^{(MOM)}(x_p) = \frac{\lambda\left(-\ln(1-p)\right)^{-\alpha_{LG}}}{\Gamma(1-\alpha_{LG})\, t_{pGa}(\lambda)} - 1 \tag{60}$$

presented in Fig. 17. The variability pattern of $RB^{(MOM)}(x_p)$ is similar to that for *LN* $RB^{(MOM)}(x_p)$.

LMM approximation. Matching of the first two linear moments secures equality of the *L*-coefficients of variation of the both distributions, i.e.,

$$2^{\alpha_{LG}} - 1 = \frac{\Gamma(\lambda+0.5)}{\sqrt{\pi}\,\Gamma(\lambda+1)}. \tag{61}$$

The range (0, 1) for α_{LG} corresponds to (0, $+\infty$) for λ. The relationship of these two parameters is displayed in Fig. 16. Using eq. (61) one can express in eq. (56) α_{LG} and λ by the coefficient of variation of *LG* and *Ga*, respectively, hence the relative bias of the variance is

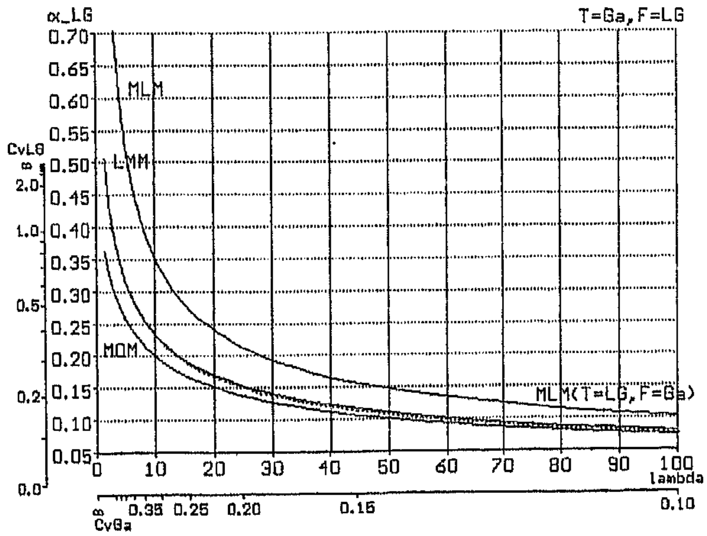

Fig. 16. Relationship between parameters α_{LG} and λ for MOM, LMM and MLM. Unlike the first two methods, MLM produces two relationships depending on which of distributions (*LG* or *Ga*) is true. Additional axes for true coefficients of variations c_v are given for comparison purposes. Axis c_{vLG} is valid for MOM and $T = LG$ (broken line) while the c_{vGa} axis is valid for all cases except for $T = LG$.

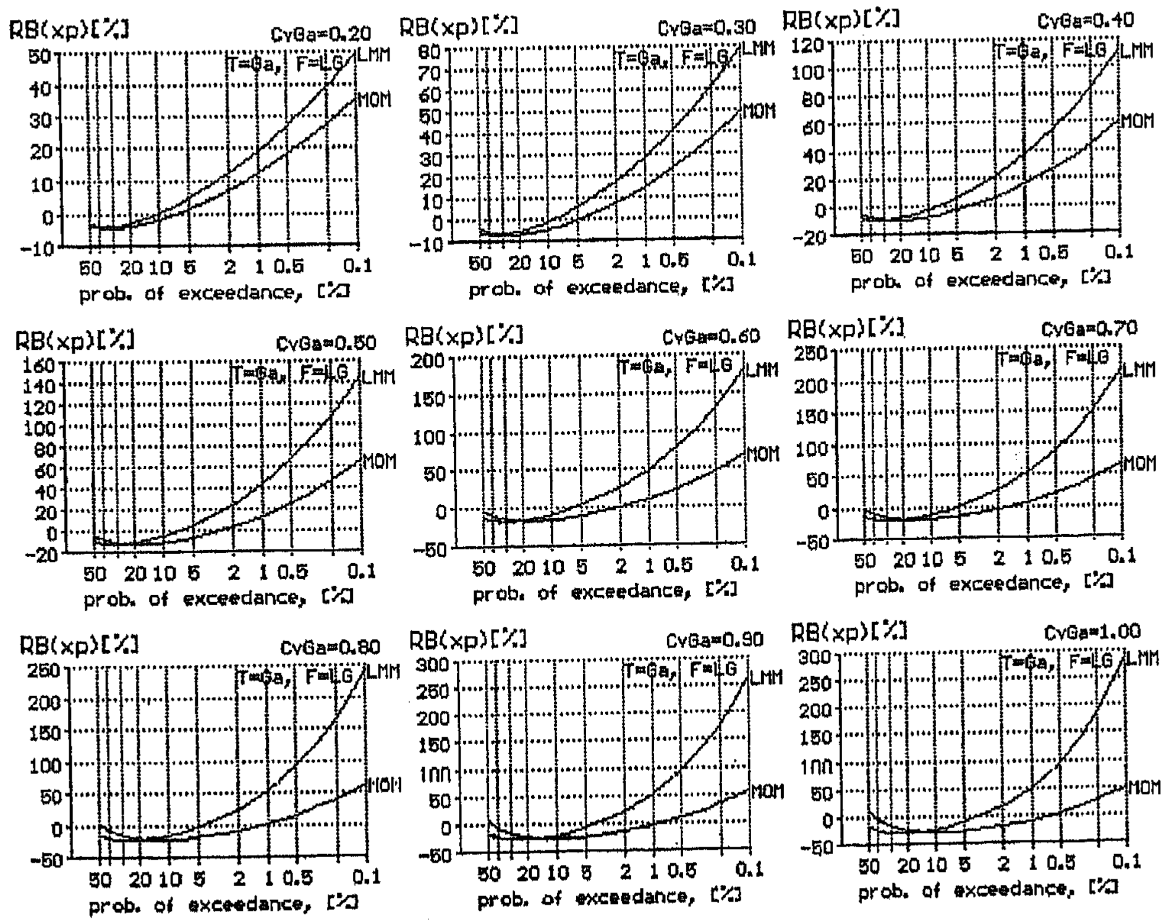

Fig. 17. Asymptotical bias of *LG* MOM and *LG* LMM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vGa}.

$$RB^{(LMM)}(\mathrm{var}) = \lambda\left[\frac{\left[\Gamma(1-2\alpha_{LG})\right]}{\Gamma^2(1-\alpha_{LG})}-1\right]-1, \quad \alpha_{LG}<\frac{1}{2}. \tag{62}$$

The graph of eq. (62) is shown in Fig. 18. The relative bias $RB^{(LMM)}(x_p)$ of quantile resulting from adopting Log-Gumbel and using LMM is

$$RB^{(LMM)}(x_p) = \frac{\lambda\left[-\ln(1-p)\right]^{-\alpha_{LG}}}{\Gamma(1-\alpha_{LG})t_{pGa}(\lambda)} - 1. \tag{63}$$

Both MOM and LMM biases, shown in Fig. 17, are significantly higher than for other true distributions.

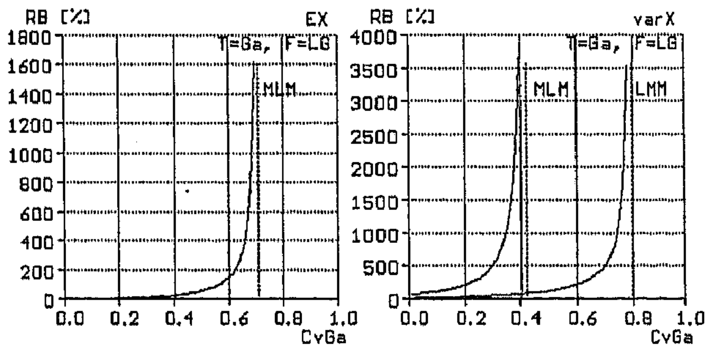

Fig. 18. Asymptotical bias of *LG* MLM-estimated mean (left), and *LG* LMM- and *LG* MLM-estimated variance (right) *vs.* true coefficient of variation c_{vGa}.

MLM approximation. The Log-Gumbel MLM applied to *Ga* gives the following system of equations:

$$\alpha_{LG} - \psi(\lambda) + \ldots\psi(\lambda-1/\alpha_{LG}) = 0, \tag{64}$$

$$\alpha_{Ga}\xi^{\alpha_{LG}} = \left[\frac{\Gamma(\lambda)}{\Gamma(\lambda-1/\alpha_{LG})}\right]^{\alpha_{LG}}. \tag{65}$$

Equation (64) defines an implicit MLM relationship between α_{LG} and λ shown in Fig. 16 together with similar relationships for MOM and LMM.

Eliminating ξ in eq. (55) by eq. (65) we get the relative bias of mean as

$$RB^{(MLM)}(m) = \frac{\Gamma(1-\alpha_{LG})}{\lambda}\left(\frac{\Gamma(\lambda)}{\Gamma(\lambda-1/\alpha_{LG})}\right)^{\alpha_{LG}} - 1. \tag{66}$$

This equation holds for $\lambda > 1/\alpha_{LG} > 1$. As both parameters are interrelated through eq. (64), we have $1/\alpha_{LG} > 1$ and $\lambda > 2$, which gives the upper limit to the coefficient of variation: $c_v < 1/\sqrt{\lambda} = 0.707$ (see Fig. 18).

The relative bias of the variance is

$$RB^{(MLM)}(\text{var}) = \frac{\Gamma(1-2\alpha_{LG}) - \Gamma^2(1-\alpha_{LG})}{\lambda}\left(\frac{\Gamma(\lambda)}{\Gamma(\lambda - 1/\alpha_{LG})}\right)^{2\alpha_{LG}} - 1, \tag{67}$$

where $\lambda > 1/\alpha_{LG} > 2$ from which we get the limiting values for α_{LG} and λ: $\alpha_{LG} < 0.5$ and $\lambda > 5.56153$, and the upper limit to the coefficient of variation (c_{vGa}) is 0.42404. The pattern followed in Fig. 18 is very similar quantitatively to that for T=LN (Fig. 9) with vertical asymptotes shifted to lower c_v values.

The relative bias $RB^{(MLM)}(x_p)$ of quantile is

$$RB^{(MLM)}(x_p) = \frac{[-\ln(1-p)]^{-\alpha_{LG}}}{t_{pGa}(\lambda)}\left(\frac{\Gamma(\lambda)}{\Gamma(\lambda - 1/\alpha_{LG})}\right)^{\alpha_{LG}} - 1. \tag{68}$$

The MLM bias in Fig. 19 is huge and for higher values of coefficient of variation is by several orders higher than MLM biases estimated before.

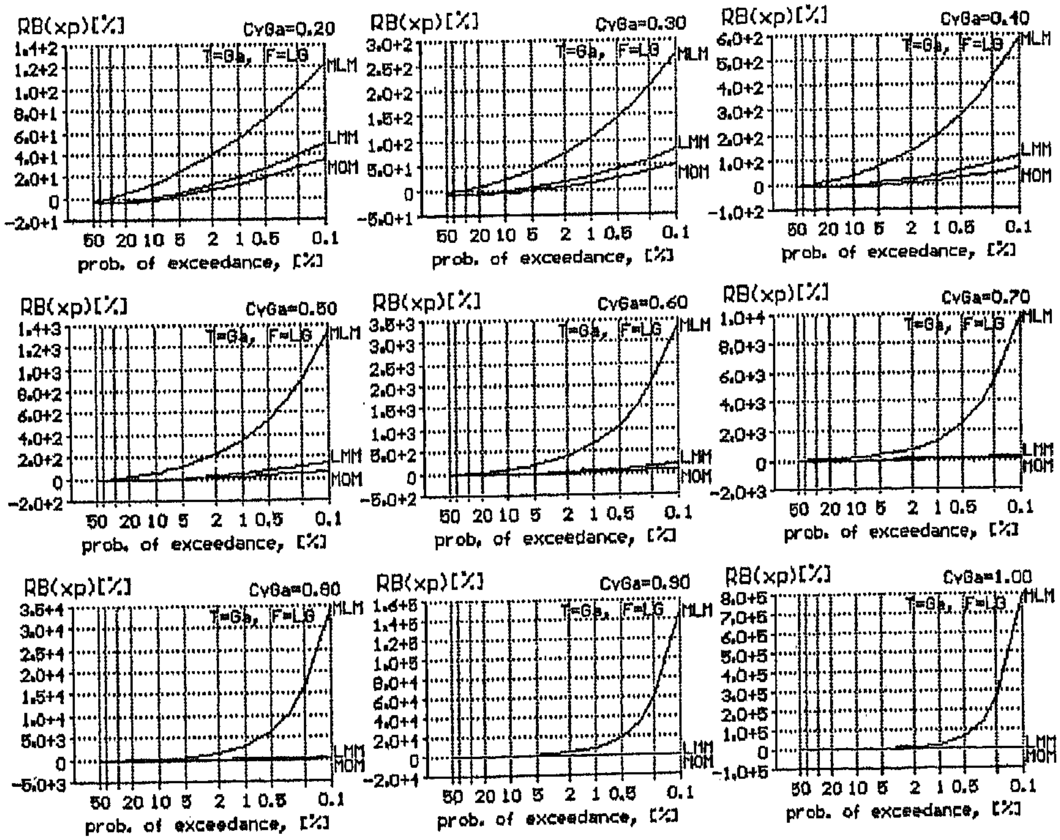

Fig. 19. Asymptotical bias of LG MOM, LG LMM and LG MLM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vGa}.

5. LOG-GUMBEL AS THE TRUE DISTRIBUTION

MOM and LMM as approximation methods are reversible, i.e., for any statistical characteristic z the relative asymptotic bias holds the asymmetric property

$$B(z; H{=}A; T{=}B) = -B(z; H{=}B; T{=}A) . \tag{69}$$

Therefore, for the relative asymptotic bias we have

$$RB(z;\ H{=}B;\ T{=}A) = \left[RB(z;\ H{=}A;\ T{=}B)+1\right]^{-1} - 1 \tag{70}$$

and all algebra presented in Section 4 for MOM- and LMM-matching and *(H=LG; T=A)* applies to the opposite case, i.e., to *(H=A; T=LG)*. Thus, proceeding in a similar way to that presented in Section 4, one can derive the asymptotic relative bias if Log -Gumbel is assumed to be the true distribution and the numerical results are obtained here by the transformation of eq. (69) of the results got in Section 4. Hence, the problem of this section is the MLM-approximation of the *LG* distribution by other lower bounded PDFs which are used in FFA. One should remember looking at all figures that *RB* is always referred to the true value of the coefficient of variation.

LL as hypothetical distribution

MLM approximation

$$\begin{aligned} &E\left(\frac{(X/\alpha_{LL})^{1/\kappa}}{1+(X/\alpha_{LL})^{1/\kappa}}\right)=\frac{1}{2} \\ &\kappa + E\ln\left(X/\alpha_{LL}\right) - 2E\left(\frac{(X/\alpha_{LL})^{1/\kappa}\ln(X/\alpha_{LL})}{1+(X/\alpha_{LL})^{1/\kappa}}\right)=0 . \end{aligned} \tag{71}$$

An application of Log-Gumbel gives the details as

$$E\ln X = \alpha_{LG}\left(C+\ln\xi\right) \tag{72}$$

$$\begin{aligned} E\left(\frac{(X/\alpha_{LL})^{1/\kappa}}{1+(X/\alpha_{LL})^{1/\kappa}}\right) &= \int_0^\infty \frac{(x/\alpha_{LL})^{1/\kappa}}{1+(x/\alpha_{LL})^{1/\kappa}}(\xi/\alpha_{LG})x^{-1/\alpha_{LG}-1}\exp\left(-\xi x^{-1/\alpha_{LG}}\right)\mathrm{d}x \\ &= ab\int_0^\infty \frac{z^{-a}}{1+z}\exp\left(-bz^{-a}\right)\mathrm{d}z = abC_1(a,b) , \end{aligned} \tag{73}$$

where $a = \kappa/\alpha_{LG}$ and $b = \xi/(\alpha_{LL})^{1/\alpha_{LG}}$. Similarly

$$E\left(\frac{(X/\alpha_{LL})^{1/\kappa}\ln(X/\alpha_{LL})}{1+(X/\alpha_{LL})^{1/\kappa}}\right) = \int_0^\infty \frac{\ln(x/\alpha_{LL})(x/\alpha_{LL})^{1/\kappa}}{1+(x/\alpha_{LL})^{1/\kappa}}\frac{\xi}{\alpha_{LG}}x^{-1/\alpha_{LG}-1}\exp\left(-\xi x^{-1/\alpha_{LG}}\right)\mathrm{d}x$$

$$= ab\kappa \int_0^{\infty} \frac{\ln z \;\; z^{-a}}{1+z} \exp\left[-bz^{-a}\right] dz = ab\kappa D_1(a,b) . \qquad (74)$$

Substituting eqs. (72), (73) and (74) into eq. (71) we get after some manipulations the system of equation:

$$\begin{aligned} abC_1(a,b) - \frac{1}{2} &= 0 \\ 1 + \frac{C}{a} + \frac{\ln b}{a} - 2abD_1(a,b) &= 0 . \end{aligned} \qquad (75)$$

Numerical solution of the system gives $(a_0, b_0) = (0.69565, 0.63032)$, which is equivalent to the equalities

$$\frac{\kappa}{\alpha_{LG}} = a_0 = 0.69565 \; , \qquad (76)$$

$$\frac{\xi}{(\alpha_{LL})^{1/\alpha_{LG}}} = b_0 = 0.63032 \; . \qquad (77)$$

The line $\kappa/\alpha_{LG} = 0.69565$ is presented in Fig. 1. The graph lies between LMM and MLM graphs for $(T = LL, F = LG)$.

Equations (76) and (77) combined with formulas for the asymptotic relative bias of mean, variance and quantile (cf., eqs. 12–14) allows to compare the respective MLM biases with those of MOM and LMM (see Figs. 20 and Fig. 21). The MLM relative bias of mean (Fig. 20) is small and does not exceed 8% for $c_{vLG} \leq 1$. The same figure shows that the LMM and MLM biases of variance are by one order greater than $RB^{(MOM)}(m)$. The relative biases of quantiles shown in Fig. 21 differ essentially from all previous respective figures (cf., Figs. 5, 11, 15 and 19) in that the MLM-estimated quantiles are not so dramatically greater than the MOM and LMM-estimated ones.

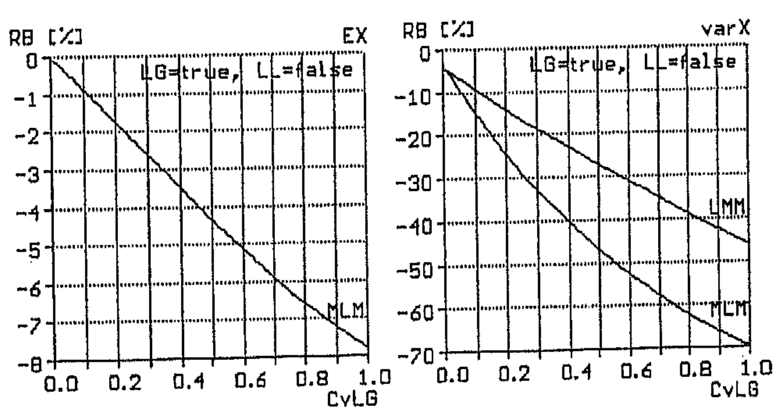

Fig. 20. Asymptotical bias of *LL* MLM-estimated mean (left), and *LL* LMM- and *LL* MLM -estimated variance (right) *vs*. true coefficient of variation c_{vLG}.

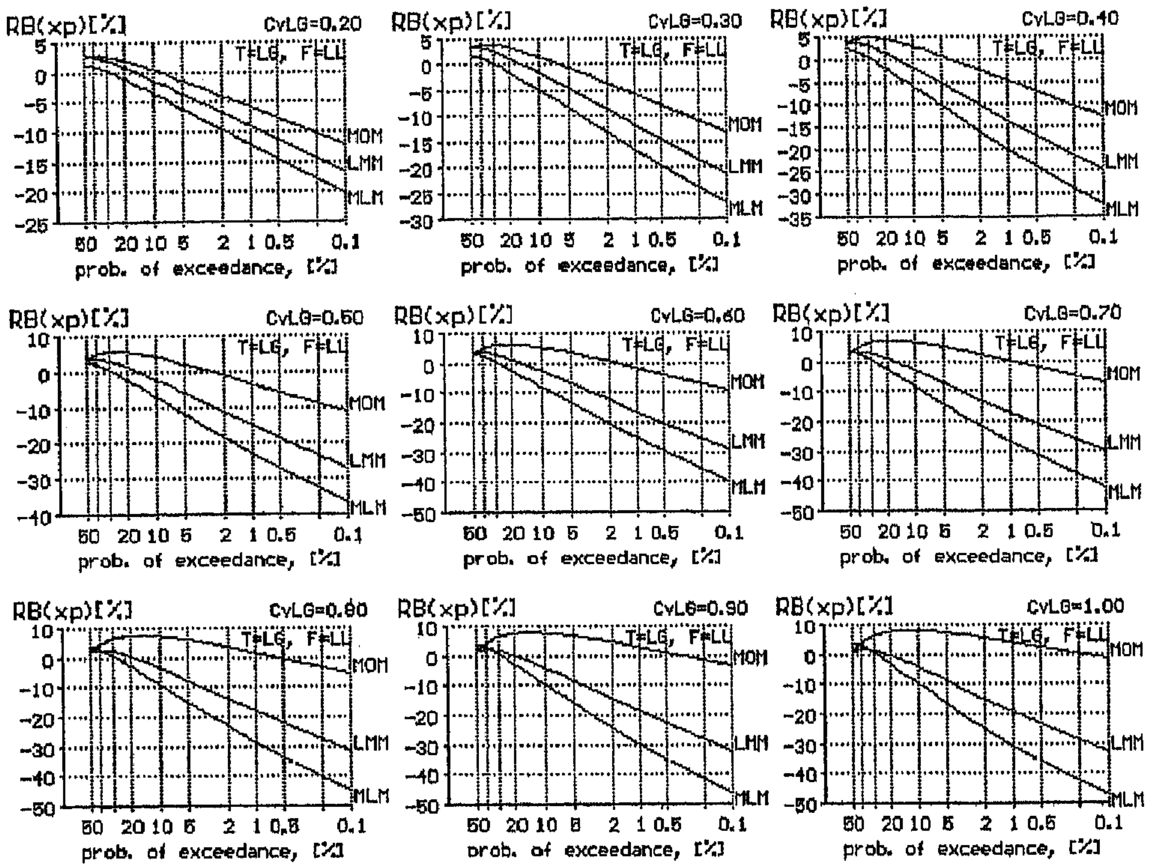

Fig. 21. Asymptotical bias of *LL* MOM, *LL* LMM and *LL* MLM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLG}.

LN as hypothetical distribution

MLM approximation. The ML estimates of the Log-normal distribution parameters are

$$\mu = \alpha_{LG}(C + \ln\xi)\,, \tag{78}$$

$$\sigma^2 = \alpha_{LG}^2\,\psi'(1)\,. \tag{79}$$

Note in Fig. 7 that the relation given by eq. (79) is closer to the MOM and LMM relations than the relation in eq. (37) derived for the opposite case, i.e., when $LN = T$ and $LG = F$.

The relative bias of the mean

$$RB^{(MLM)}(m) = \frac{\exp(\sigma^2/2 - C\alpha_{LG})}{\Gamma\left(1-\alpha_{LG}\right)} - 1, \tag{80}$$

where $\alpha_{LG} < 1$, and from eq. (79) we get that

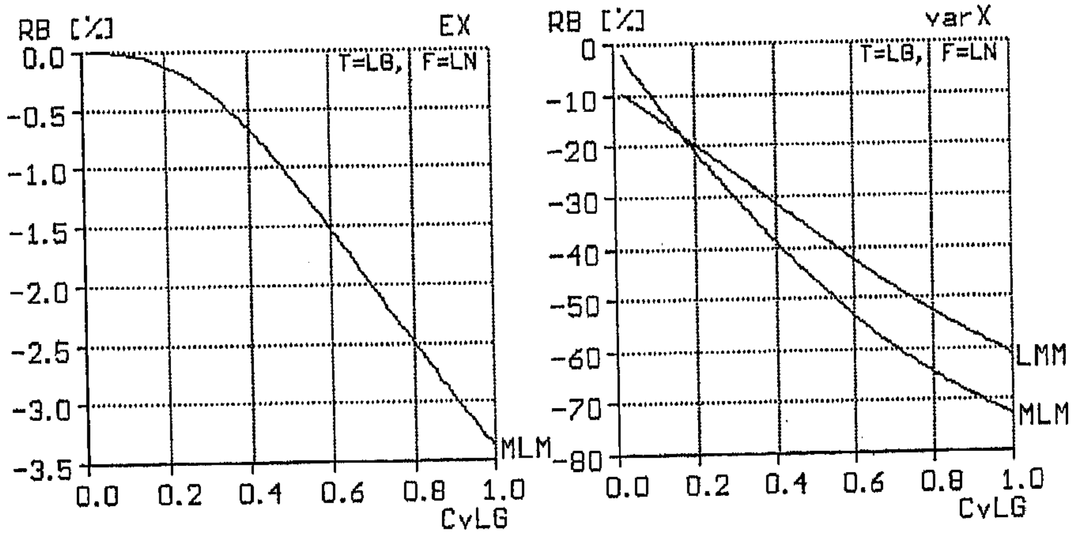

Fig. 22. Asymptotical bias of *LN* MLM-estimated mean (left), and *LN* LMM- and *LN* MLM -estimated variance (right) *vs.* true coefficient of variation c_{vLG}.

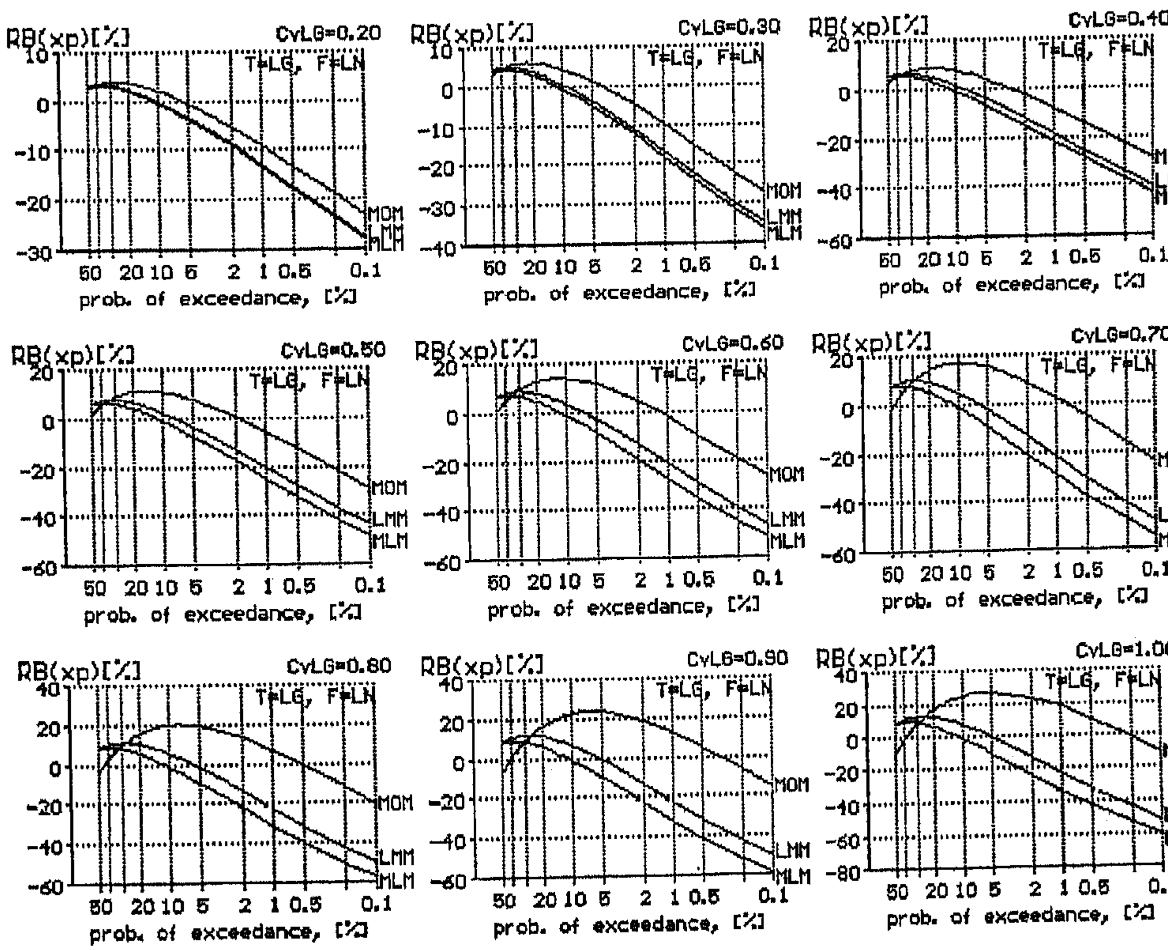

Fig. 23. Asymptotical bias of *LN* MOM, *LN* LMM and *LN* MLM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLG}.

$$\sigma < \sqrt{\psi'(1)} = 1.2825 . \tag{81}$$

The relative bias of the variance is

$$RB^{(MLM)}(\text{var}) = \frac{\exp(\sigma^2 + 2C\alpha_{LG})\left[\exp(\sigma^2)-1\right]}{\Gamma(1-2\alpha_{LG}) - \Gamma^2(1-\alpha_{LG})} - 1 , \tag{82}$$

(see Fig. 22), where $\alpha_{LG} < 0.5$, and from eq. (79) we get that σ is

$$\sigma < \frac{\sqrt{\psi'(1)}}{2} = 0.64127 . \tag{83}$$

The relative bias of quantile is given by

$$RB^{(MLM)}(x_p) = \frac{x_{pLN}}{x_{pLG}} - 1 = \frac{\exp(\mu + \sigma t_p)}{\left(-\frac{1}{\xi}\ln(1-p)\right)^{-\alpha_{LG}}} - 1 = \frac{\exp(\alpha_{LG} C + \sigma t_p)}{\left(-\ln(1-p)\right)^{-\alpha_{LG}}} - 1 \tag{84}$$

and is presented in Fig. 23 where it is compared with MOM and LMM approximations.

LD as hypothetical distribution

MLM approximation. MLM gives the following system of equations:

$$\frac{1}{2\alpha_{LD}\beta_{LD}} \equiv \frac{1}{2\gamma_{LD}} = \Gamma(1-\alpha_{LG})\Gamma(1+\alpha_{LG}) - 1 , \tag{85}$$

$$\beta_{LD} = \frac{\alpha_{LD}\xi^{-\alpha_{LG}}}{\Gamma(1-\alpha_{LG})} . \tag{86}$$

The $\gamma_{LD}(\alpha_{LG})$ relationship is shown in Fig. 12. The $\gamma_{LD}(\alpha_{LG})$ function is similar to the $\alpha_{LG}(\gamma_{LD})$ functions for the ($T = LD$, $F = LG$) case while its graph lies closest the γ_{LD} axis. Equation (86) is identical with eq.(43), which points out that the MLM estimate of the mean remains unbiased if LD serves as the assumed distribution. Note that for ($T = LD$; $F = LD$) RB of the mean is an increasing function of c_{vLD} (Fig. 14) reaching 96% for $c_{vLD} = 1$.

The relative bias of the variance

$$RB^{(MLM)}(\text{var}) = \left(\frac{1}{2\gamma_{LD}}\right)\frac{1}{\Gamma(1-2\alpha_{LG})/\Gamma^2(1-\alpha_{LG}) - 1} - 1 \tag{87}$$

is displayed in Fig. 24.

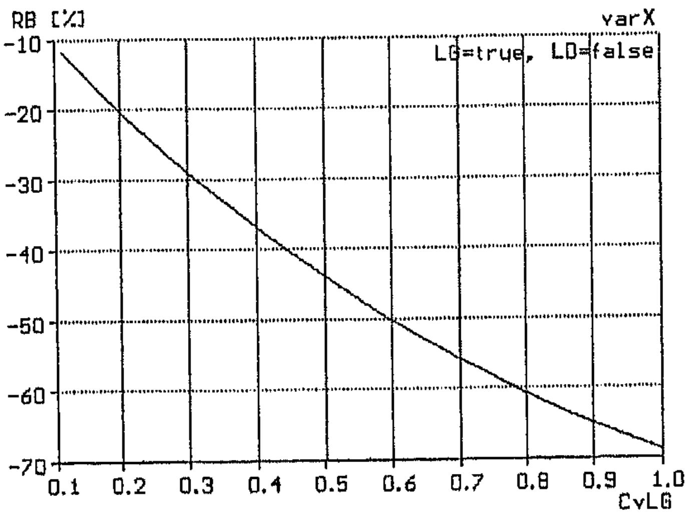

Fig. 24. Asymptotical bias of *LN* MLM -estimated variance *vs.* true coefficient of variation c_{vLG}.

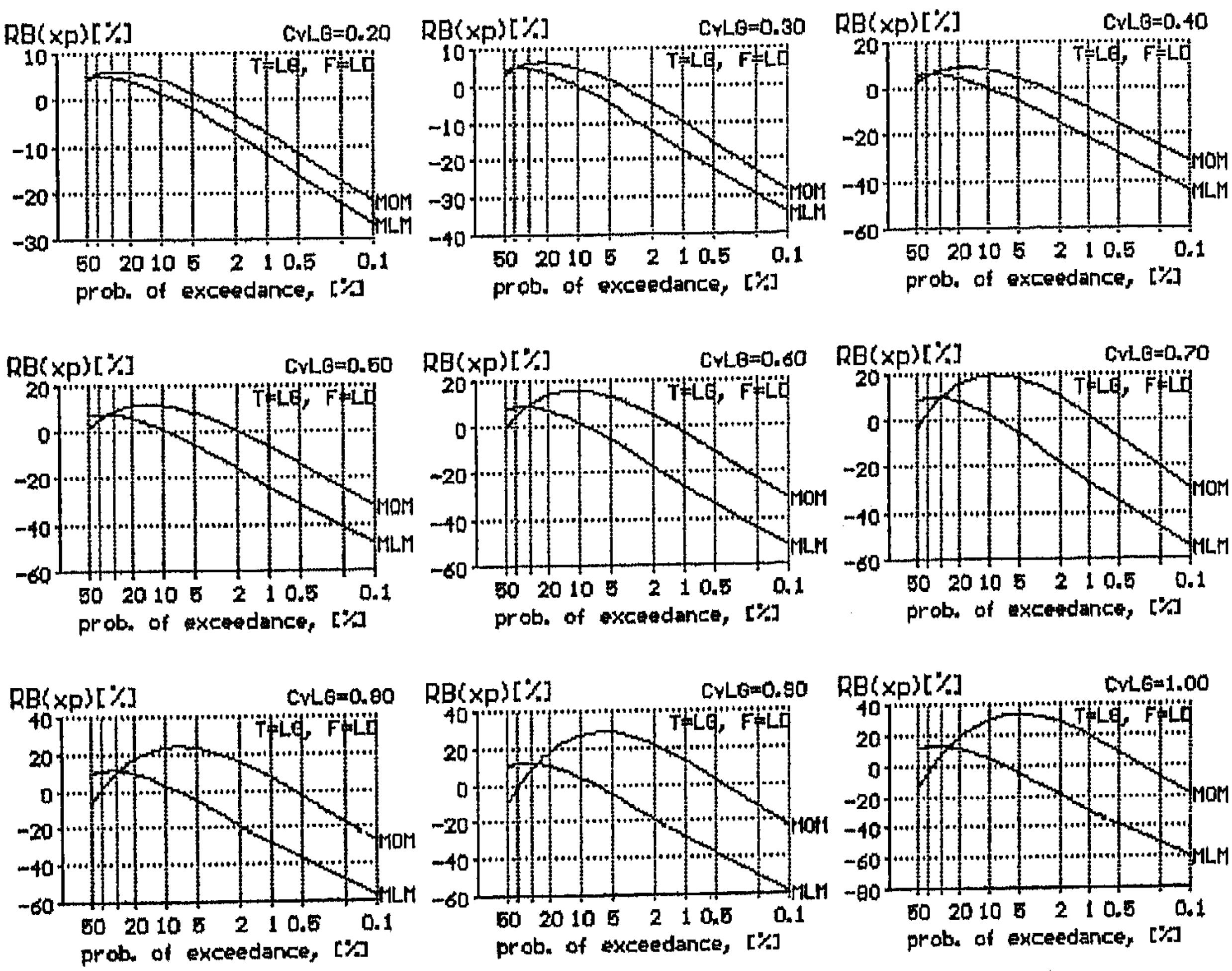

Fig. 25. Asymptotical bias of *LD* MOM and *LD* MLM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLG}.

The relative bias of quantile resulting from adopting *LD* and MLM is

$$RB^{(MLM)}(x_p) = \frac{x_{pLD}}{x_{pLG}} - 1 = \frac{(\alpha_{LD}\beta_{LD})\;\Gamma(1-\alpha_{LG})\;[-\ln(1-p)]^{\alpha_{LG}}}{\left[t_{pLD}(\gamma_{LD})\right]^2} - 1 \qquad (88)$$

is displayed in Fig. 25 together with one of the MOMs. Comparing Figs. 25 and 15, one can see that the ML-bias is much smaller for ($T = LG;\ F = LD$) than in the opposite case, i.e. ($T = LD;\ F = LG$).

Gamma as hypothetical distribution

MLM gives the following system of equations:

$$\frac{\lambda}{\alpha_{Ga}} = \xi^{\alpha_{LG}}\,\Gamma(1-\alpha_{LG}), \qquad (89)$$

$$\ln\lambda - \psi(\lambda) = \ln\Gamma(1-\alpha_{LG}) - \alpha_{LG}C. \qquad (90)$$

Equation (89) is identical with eq. (58), reminding that the MLM estimate of the mean is unbiased if Gamma serves as the hypothetical distribution. The $\lambda(\alpha_{LG})$ relationship resulting from eq. (90) is shown in Fig. 16.

The relative bias of variance

$$RB^{(MLM)}(\mathrm{var}) = \frac{1}{\lambda\left[\Gamma(1-2\alpha_{LG})/\Gamma^2(1-\alpha_{LG})-1\right]} - 1 \qquad (91)$$

is displayed in Fig. 26 together with one of LMMs. Both of them are about two orders lower than those of the opposite case (Fig. 18), i.e., when $Ga = T$ and $LG = F$.

The relative bias of quantiles resulting from adopting the Gamma distribution and MLM to *LG*

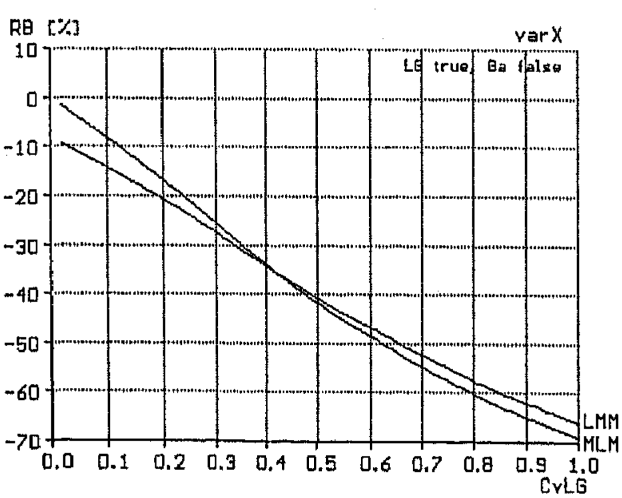

Fig. 26. Asymptotical bias of Ga LMM and Ga MLM-estimated variance *vs.* true coefficient of variation c_{vLG}.

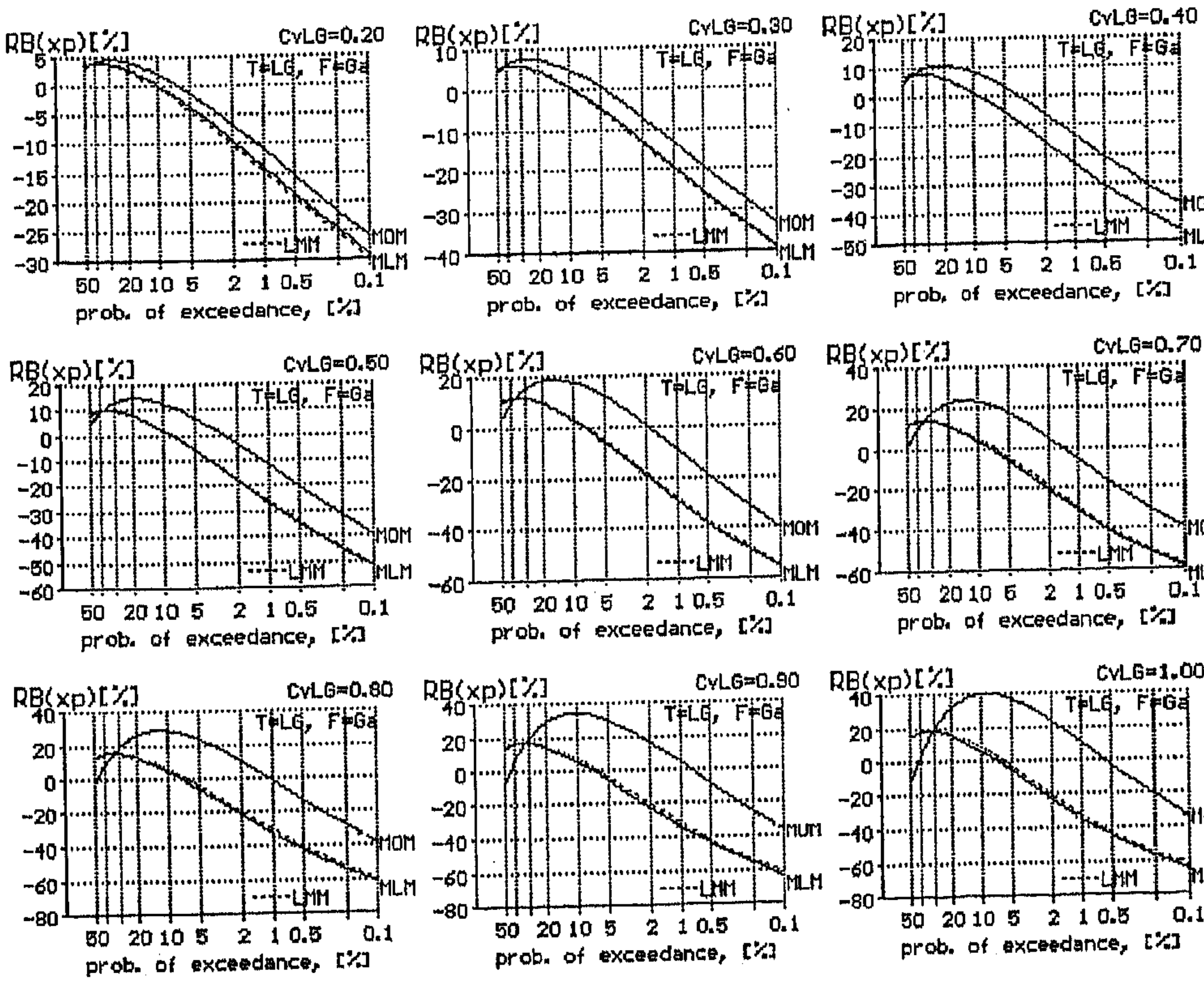

Fig. 27. Asymptotical bias of Ga MOM, Ga LMM and Ga MLM-estimated quantile *vs.* probability of exceedance for some selected values of the true coefficient of variation c_{vLG}.

$$RB^{(MLM)}(x_p) = \frac{t_{pGa}(\lambda)\ \Gamma(1-\alpha_{LG})}{\lambda[-\ln(1-p)]^{-\alpha_{LG}}} - 1 \tag{92}$$

is shown in Fig. 27. Comparing Fig. 27 and Fig. 19, one can see that the MLM-bias is much smaller when Gamma is taken as hypothetical distribution than in the opposite case.

6. DISCUSSION OF RESULTS

Tables 1, 2 and 3 illustrate in a concise form the results obtained for two cases: when *LG* is the true distribution and other PDFs are false and when the *LG* PDF is used for approximation of other PDFs. They refer always to the *true* value of the coefficient of variation.

Table 1

Asymptotic relative bias (%) of mean by the MLM approximation

True c_v	Probability Distribution Function (PDF)							
	Log-logistic		Log-normal		Linear diffusion		Gamma	
	True	*False*	*True*	*False*	*True*	*False*	*True*	*False*
0.2	5.2	−1.7	3.2	−0.1	3.1	0	4.3	0
0.4	16.6	−3.5	11.3	−0.7	10.6	0	23.5	0
0.6	35.6	−5.1	28.5	−1.5	24.7	0	152.1	0
0.8	62.6	−6.6	64.2	−2.5	49.3	0	NE	0
1.0	97.8	−7.8	150.9	−3.4	96.4	0	NE	0

NE = not existing

Table 2

Asymptotic relative bias (%) of variance by the MLM approximation

True c_v	Probability Distribution Function (PDF)							
	Log-logistic		Log-normal		Linear diffusion		Gamma	
	True	*False*	*True*	*False*	*True*	*False*	*True*	*False*
0.2	231	−24.4	156	−21.0	153	−20.3	215	−16.7
0.4	863	−40.2	577	−38.9	517	−36.8	3794	−33.8
0.6	NE	−52.6	NE	−53.4	NE	−50.1	NE	−48.6
0.8	NE	−62.3	NE	−64.5	NE	−60.7	NE	−60.3
1.0	NE	−69.8	NE	−72.6	NE	−68.8	NE	−69.0

NE = not existing

It can be seen from Table 1 that all *LG* MLM-approximations of the used PDFs produce much greater relative asymptotic bias than in the reverse case when approximated by other PDFs. The use of *LG* as an approximation of another PDF results even in infinite or indefinite bias of the mean as is for the *Ga* PDF. In both cases, the bias is growing with the value of the variation coefficient. The situation is even more dramatic for the MLM relative bias of the variance presented in Table 2, from which it follows that the use of *LG* as an approximation gives very high bias for c_v, rising fast to infinity in between $c_v = 0.4$ and 0.6.

Table 3

Asymptotic relative bias (%) of variance by the MLM approximation

True c_v	Probability Distribution Function (PDF)							
	Log-logistic		Log-normal		Linear diffusion		Gamma	
	True	*False*	*True*	*False*	*True*	*False*	*True*	*False*
0.2	17.5	−14.1	26.6	−19.7	NA	NA	28.9	−20.6
0.4	34.3	−22.9	59.3	−31.3	NA	NA	73.6	−34.2
0.6	59.2	−31.1	138.3	−42.5	NA	NA	217.9	−46.8
0.8	103.1	−38.8	541.0	−52.5	NA	NA	NE	−57.5
1.0	201.3	−46.1	NE	−61.0	NA	NA	NE	−66.0

NE = not existing, NA = not available

Compared to MLM, LMM (Table 3) extends the range of c_v for which asymptotic relative bias of variance exists and lowers the difference between the relative biases of variance for ($T = LG$, $H = non\text{–}LG$) and ($T = non\text{–}LG$, $H = LG$) cases. As expected, absolute values of bias are lower than corresponding values of MLM bias in Table 2.

The very first glimpse at Tables 4a, b, c confirms our previous findings in respect to magnitude of bias of various estimation methods (Strupczewski *et al.*, 2001). The value of *RB* of the upper quantiles is smallest for MOM and largest for MLM. The bias of LMM occupies an intermediate position. Furthermore, the relative asymptotic bias (*RB*) of quantiles corresponding to the upper tail is an increasing function of the true value of the coefficient of variation (c_v).

The asymptotic relative bias of the MOM and LMM quantiles for $p = 10\%$ seems to not essentially discern the ($T = LG$, $H = non\text{–}LG$) case from ($T = non\text{–}LG$, $H = LG$) for low c_v values because Table 4a shows almost the same values in columns *True* and *False*. For higher c_v, the *LL* column still exhibits the pattern for low c_v, while the differences in the bias in *True* and *False* columns for other PDFs increase, especially for LMM. As a rule, the MLM bias for ($T = non\text{–}LG$, $H = LG$) is much greater than in the opposite case, i.e. ($T = LG$, $H = non\text{–}LG$).

From among the PDFs used, *LL* seems to be the best approximation for *LG* as the absolute values of *LL* MOM and LMM bias are the lowest. Using MLM destroys the symmetry exhibited by MOM and LMM: the absolute MLM values for the ($T = non\text{–}LG$, $H = LG$) case are about one order or more greater than those for the ($T = LG$, $H = non\text{–}LG$) case. The bias of the *Ga* approximation by *LG* exhibits the greatest differences growing fast with true c_v and reaching ca. 550% for $c_v = 1$, compared to 6% for ($T = Ga$, $H = LG$).

Table 4a

Asymptotic relative bias (%) of quantile with probability of exceedance p (%) obtained by the various approximation methods (p = 10%)

True c_v	Method	Probability Distribution Function							
		Log-logistic		Log-normal		Linear diffusion		Gamma	
		True	*False*	*True*	*False*	*True*	*False*	*True*	*False*
0.2	MOM	–0.4	0.4	–2.0	2.1	–4.1	4.3	–2.0	2.0
	LMM	1.0	–1.3	0.0	–0.2	NA	NA	0.2	–0.3
	MLM	16.3	–3.5	9.7	–0.4	7.2	1.5	13.0	0.1
0.4	MOM	–2.8	2.9	–7.0	7.5	–7.4	8.0	–7.8	8.4
	LMM	0.1	–1.8	–2.5	1.5	NA	NA	–2.6	1.2
	MLM	32.7	–6.0	19.8	–0.7	17.7	0.5	38.1	1.3
0.6	MOM	–5.1	5.4	–12.3	14.0	–13.5	15.6	–15.1	17.8
	LMM	–1.4	–2.5	–6.5	3.7	NA	NA	–8.2	3.6
	MLM	47.2	–7.8	29.7	–0.9	23.3	1.8	91.0	2.9
0.8	MOM	–6.6	7.1	–16.7	20.0	–18.9	23.4	–22.3	28.7
	LMM	–2.9	–3.4	–10.8	5.8	NA	NA	–15.7	6.0
	MLM	59.0	–9.0	39.1	–1.1	25.9	3.3	216.4	4.6
1.0	MOM	–7.6	8.2	–19.9	24.9	–23.2	30.2	–28.5	39.9
	LMM	–4.1	–4.3	–14.9	7.5	NA	NA	–24.2	8.2
	MLM	68.1	–9.8	47.7	–1.2	25.9	4.7	548.2	6.1

The remarks concerning Table 4a refer essentially to the results presented in Table 4b and 4c. The differences lie generally in greater absolute values of the corresponding biases, faster increase of bias with the true coefficient of variation, c_v, and clearly higher LMM biases compared with MOM.

The 0.1% quantile bias for MOM and LMM does not exceed ca. 60% (in absolute values) while the difference between the MLM bias for ($T = LG$, $H = non\text{–}LG$) and ($T = non\text{–}LG$, $H = LG$) cases rapidly increases, reaching ca. 8×10^5% for ($T = Ga$, $H = LG$).

Table 4b

Asymptotic relative bias (%) of quantile with probability of exceedance p (%) obtained by the various approximation methods (p = 1.0%)

True c_v	Method	Probability Distribution Function							
		Log-logistic		Log-normal		Linear diffusion		Gamma	
		True	*False*	*True*	*False*	*True*	*False*	*True*	*False*
0.2	MOM	6.3	−6.0	10.1	−9.2	7.9	−7.3	12.3	−11.0
	LMM	10.4	−9.2	16.3	−13.0	NA	NA	19.2	−14.6
	MLM	49.0	−12.2	42.1	−13.3	38.8	−11.7	54.6	−13.9
0.4	MOM	5.2	−4.9	9.2	−8.4	9.8	−8.9	15.2	−13.2
	LMM	15.1	−13.7	26.0	−19.0	NA	NA	36.5	−23.3
	MLM	111.2	−20.2	98.0	−22.0	93.7	−21.3	193.5	−23.2
0.6	MOM	1.7	−1.7	2.2	−2.1	2.8	−2.7	10.2	−9.2
	LMM	16.2	−16.4	29.2	−21.6	NA	NA	48.4	−28.2
	MLM	178.1	−25.4	167.4	−27.7	150	−26.0	658.5	−29.0
0.8	MOM	−1.5	1.5	−6.7	7.1	−6.8	7.3	0.9	−0.9
	LMM	15.7	−18.2	27.9	−22.5	NA	NA	52.9	−30.8
	MLM	240.9	−28.8	248.2	−31.3	201.5	−28.5	2935	−32.5
1.0	MOM	−3.9	4.1	−15.0	17.7	−16.4	19.6	−9.5	10.5
	LMM	14.8	−19.5	23.9	−22.6	NA	NA	49.9	−32.3
	MLM	294.9	−31.1	337.8	−33.7	242.6	−30.0	20287	−34.7

7. CONCLUDING REMARKS

An analytical method for evaluation of the resistance of the estimates of moments and quantiles obtained by various estimation methods with respect to the distribution choice has been presented and illustrated using five two-parameter distribution functions. It is shown that the bias caused by the wrong distribution choice cannot be disregarded in evaluation of the efficiency of estimation methods in FFA. The relative asymptotic bias of the MLM-estimate of moments can be considerable and grows rapidly with increasing value of the coefficient of variation, while the MOM estimates of the two first moments are asymptotically bias-free. Similarly, the MOM estimate of

Table 4c

Asymptotic relative bias (%) of quantile with probability of exceedance p (%) obtained by the various approximation methods (p = 0.1%)

True c_v	Method	Probability Distribution Function							
		Log-logistic		Log-normal		Linear diffusion		Gamma	
		True	*False*	*True*	*False*	*True*	*False*	*True*	*False*
0.2	MOM	13.9	−12.2	30.0	−23.1	27.9	−21.8	35.6	−26.3
	LMM	20.9	−16.7	42.1	−27.5	NA	NA	49.3	−30.2
	MLM	90.9	−20.2	92.9	−27.8	88.7	−26.6	122.0	−29.4
0.4	MOM	14.6	−12.7	41.5	−29.3	45.5	−31.2	60.1	−37.5
	LMM	33.1	−24.7	79.3	−40.4	NA	NA	112.2	−46.6
	MLM	236.3	−32.5	258.9	−43.3	254.4	−43.4	586.9	−46.5
0.6	MOM	10.2	−9.3	37.2	−27.2	45.6	−31.4	68.0	−40.5
	LMM	38.2	−28.9	105.2	−46.8	NA	NA	180.4	−55.5
	MLM	425.4	−40.0	529.2	−52.2	498.5	−52.1	3366	−56.2
0.8	MOM	5.4	−5.1	25.5	−20.3	36.8	−26.9	62.6	−38.5
	LMM	39.5	−31.4	118.8	−50.0	NA	NA	241.8	−60.6
	MLM	631.1	−44.7	930.8	−57.4	808.9	−57.0	34714	−61.8
1.0	MOM	1.5	−1.4	12.0	−10.7	24.2	−19.4	50.2	−33.4
	LMM	39.2	−33.0	122.8	−51.8	NA	NA	285.3	−63.6
	MLM	327.9	−47.7	1482	−60.8	1151	−60.0	792883	−65.2

the quantiles of upper tails is more resistant to distribution choice than is the MLM estimate. The bias of LMM estimates lies between these two. Since the MLM used as the approximation method is irreversible, the asymptotic bias of the MLM-estimate of any statistical characteristic is not asymmetric as is for the MOM and LMM.

It is shown numerically that employing the Log-Gumbel distribution to some selected non-Log-Gumbel true distributions results in an asymptotical bias of quantiles and, depending on the estimation method used, in asymptotical bias of mean and variance. While asymptotical biases of MOM and LMM-estimated quantiles lie relatively close to each other, the MLM-estimated bias of quantiles is in most cases by at least one order higher, reaching as high a value as ca. 800,000% for $x_{0.1\%}$ for (T = *Ga*,

$H = LG$). These findings, illustrated with values in Tables 4a, b, and c, essentially diminish the practical usefulness of MLM for the Log-Gumbel as the hypothetical distribution because its efficiency does not compensate for the (frequent) huge bias produced by the assumption of a false PDF in the region of small exceedance probability quantiles the user is often interested in. The bias produced by the other methods (MOM and LMM) is limited to several tens per cent.

Comparing the results of Sections 4 and 5 allows to answer the question of how to choose a PDF when each of *LG* (or *LL*) and *LN*, *LD* and *Ga* PDFs with MLM estimation is taken into consideration. If we accept *LN*, *LD*, or *Ga* as a hypothetical distribution, we get the MLM bias of reasonable magnitude in upper quantiles of more than one order less than the bias obtained in the case of *LG* or *LL* as a hypothetical distribution. It is hoped that this study provides sufficient rationale for rather careful application of *LG* and *LL* in hydrology, especially in FFA.

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