LOCALIZATION OF VEHICLES USING RANGE MEASUREMENTS

A Thesis

by

SAI KRISHNA KANTH HARI

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Chair of Committee,	Swaroop Darbha
Committee Members,	Reza Langari
	Sivakumar Rathinam
Head of Department,	Andreas A. Polycarpou

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ABSTRACT

Localization is an important required task for enabling vehicle autonomy. It entails the determination of the position of center of mass and orientation of a vehicle from the available measurements. In this paper, we focus on localization by using range measurements available to a vehicle from the communication of its multiple on-board receivers with roadside beacons (acoustic beacons in the case of underwater vehicles). The model proposed for measurements assumes that the true distance between a receiver and a beacon is at most equal to a predetermined function of the range measurement. The proposed procedure for localization is as follows: Based on the range measurements specific to a receiver from the beacons, a convex optimization problem is proposed to estimate the location of the receiver. The estimate is essentially a center of the set of possible locations of the receiver. In the second step, the location estimates of the vehicle are corrected using rigid body motion constraints and the orientation of the rigid body is thus determined. Numerical examples provided at the end corroborate the procedures developed in this paper.

DEDICATION

Dedicated to my parents Dr. H V S Satyanarayana and H. Manjula for motivating and supporting me always.

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NOMENCLATURE

LP	Linear Program
SDP	Semidefinite Program
SOCP	Second Order Coninc Program
MVE	Maximum Volume Ellipsoid
OWTT	One Way Travel Time
GPS	Global Positioning System
V2I	Vehicle to Infrastructure
IMU	Intertial Measurement Unit
SOS	Sum of Squares
SO(3)	Special Orthogonal Group
AUV	Autonomous Underwater Vehicle

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1. INTRODUCTION *

Vehicles require the knowledge of their location and orientation for autonomous guidance, navigation, and control [12]. Knowledge of its location is necessary to ensure that the vehicle autonomously tracks a desired trajectory while the knowledge of its orientation is important for accounting for gravity as well as for preventing rollover in large trucks. The problem of localization deals with the estimation of location and orientation of a vehicle. Localization therefore requires sensing, and correspondingly, localization procedures are dependent on the available sensory measurements. Current vehicle platforms are equipped with IMU/GPS, gyros, altimeters, pressure sensors, communication devices or some subset of them to aid the localization [13, 10, 9, 14, 2]. Since sensory measurements are usually contaminated with noise, the problem of localization also requires filtering the noise in order to determine an accurate estimate of location and orientation.

While the present work is motivated by the the problem of infrastructure-aided localization for transportation applications, it is of independent interest to localization problems appearing in other domains such as underwater vehicles. In this scenario, we envision the infrastructure to be equipped with a set of beacons which transmit information to passing automated vehicles using their transmitters. Multiple receivers on-board an autonomous vehicle receive the transmitted signal; the one-way travel time (OWTT) between the signals can be used to determine the distance between the beacons and the receivers on-board the vehicle. In the ideal scenario of noise-free sensory measurements, one can use triangulation or multilateration to determine the location and orientation of the vehicle.

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2. LITERATURE REVIEW *

The idea of infrastructure-aided navigation and control for automated vehicles is not new and has been considered at least since California PATH's Automated Highway Systems (AHS) program. However, this idea is useful for other applications such as Advanced Traffic Management Systems (ATMS) and Advanced Traveler Information System (ATIS); one can design V2I (vehicle to infrastructure) communication protocols by which the type of vehicle is identified along with the time stamp for the communication thereby obviating the need for traffic measurement devices such as loop detectors which are error-prone; moreover, a lot of processing is required to discriminate passenger vehicles from trucks etc. using loop-detector data. However, this problem can be and, to some extent, has been obviated with V2I communication [4]. In its current form, V2I schemes for vehicle detection rely on GPS and wheel speed information from vehicles [4]. The wheel speed is used to compute the vehicle speed near the location of the V2I station.

From the point of view of vehicle control, infrastructure-aided navigation does not suffer from GPS's problem of blockage of line-of-sight with the satellites in urban canyons. Furthermore, the measurement of velocity in the ground frame and angular velocity of wheels and yaw rate is required for the purposes of computation of slip and side slip angles for accurate vehicle control; an independent measurement of velocity is currently available only through the availability of GPS information, which as we noted before can be a problem in urban canyons.

The orientation of the sprung mass of the vehicle is important for many applications ranging from rollover prevention in trucks, estimation of slope of a road, estimation of pitch angle etc., all of which find application in vehicle control and diagnostics. Earlier

*Reprinted with permission from "Estimation of Location and Orientation From Range Measurements", Sai Krishna Kanth Hari and Swaroop Darbha,DSCC 2015, Paper No. DSCC2015-9972, Copyright 2015 by ASME work by Gerdes and his coworkers [2, 4, 1] dealt with the determination of vehicle orientation and associated kinematic quantities such as sideslip angle using IMU/GPS and other sensory information using Kalman filtering. The work proposed here is an alternative method which can supplement such schemes and is of independent interest in other applications such as localization of Underwater Vehicles using acoustic beacons.

The proposed work on localization differs from the literature in the following aspects: (1) it relies only on the range information gathered by multiple on-board receivers of a vehicle from fixed beacons (transmitters) in space whose coordinates are known a priori, (2) the sensing model used in this work assumes that the radius of the sphere centered at the beacon in which the receiver is guaranteed to be placed is an increasing function of the measured distance between the beacon and the receiver, and (3) this work utilizes a novel, but computationally efficient method based on the error model to process the gathered information. The processing involves solving a semi-definite program (SDP) (or its linear programming (LP) relaxation) to arrive at an estimate of the location and orientation. A preliminary version of this work has been published as a conference paper [5].

The rest of the work is organized as follows: In sections 3 and 4, we describe the setup of physical problem and its mathematical formulation, respectively. In section 5, we present formulations and algorithms to solve the associated optimization problems for the location and orientation estimation and in section 6, we corroborate the effectiveness of our proposed approach through extensive numerical results.

3. PROBLEM SETUP *

A collection of beacons fixed in space and located at previously determined locations aid a vehicle in its localization. Let $B_i := (x_i^b, y_i^b, z_i^b)$, $i = 1, 2, \ldots, N$ be the coordinates of the N beacons and $r_j := (x_j, y_j, z_j)$, $j = 1, 2, \ldots, L$ be the estimates of the locations of the L on-board receivers of a vehicle. The problem setup is shown in figure 3.1 . Communication with the beacons provides the vehicle with an instantaneous measurement of the distance between each on-board receiver in the vehicle and each beacon. Let D_{ij} be the measured distance of a vehicle from the i^{th} beacon while d_{ij} is the true value. These measurements need not correspond it its true value; let $\phi(D_{ij})$ be an increasing function that provides a bound on the true distance, d_{ij}, *i.e.*, $d_{ij} \leq \phi(D_{ij})$. Essentially, the sensing model indicates that the j^{th} receiver will always lie inside the sphere of radius $\phi(D_{ij})$ and centered at the i^{th} beacon. The analytical form of the function is obtained using a set of range measurements obtained from the on-board receivers. The problem of determining this function $\phi(\cdot)$ is formulated as a semi-definite program in section 4.2. This sensing model has been chosen so that it is amenable to experimental corroboration and makes the subsequent formulation cleaner. Using the aforementioned notations, the problem of localization is stated as follows:

Determine the estimates (x_j, y_j, z_j), $j = 1, 2, \ldots, L$ of the location of the L on-board receivers as well as the orientation of the vehicle that is treated as a rigid body using available information.

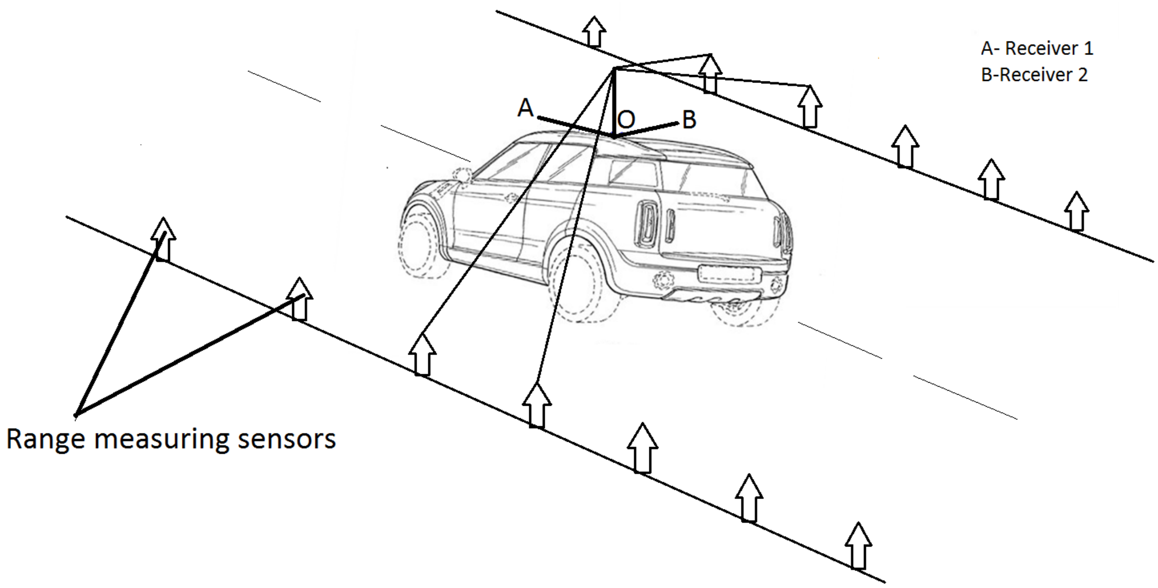

Figure 3.1: Problem setup

4. MATHEMATICAL FORMULATION*

In this section we present a mathematical formulation of the following three problems: (i) Optimal estimation of the location of the j^{th} on-board receiver, (ii) determination of the function $\phi(\cdot)$ for the sensing model, and (iii) estimation of orientation of the vehicle from the location estimates of the on-board receivers.

4.1 Optimal estimation of the location of the j^{th} on-board receiver

Since the distance measurements are available from each beacon, it is readily clear that

$$\|\boldsymbol{r}_j - \boldsymbol{B}_i\| = d_{ij} \leq \phi(D_{ij}),\ i = 1, \ldots, N. \tag{4.1}$$

The above set of N distance constraints are convex in $\boldsymbol{r}_j$; note that $\boldsymbol{B}_i$ are known a priori. Let $\mathcal{F}_j$ be the feasible values of $\boldsymbol{r}_j$ for the above set of constraints. Essentially, the feasible set is the set obtained by intersecting spheres centered at the beacons and of radii determined by the range measurements gathered by the j^{th} receiver; hence, $\mathcal{F}_j$ is a convex set. The feasible set indicates the set of all possible locations of the j^{th} receiver. The center of the set $\mathcal{F}_j$ can be considered as the best estimate of $\boldsymbol{r}_j$. We consider two notions of "center" of the set $\mathcal{F}_j$: the center of the largest inscribed disk (referred to as chebyshev center) and the center of the maximum volume inscribed ellipsoid. The former can be computed via linear programming while the latter, via semi-definite programming.

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4.1.1 Chebyshev center of $\mathcal{F}_j$

If $\boldsymbol{u}$ is any unit vector and $\boldsymbol{r}_{c,j}$ is the center of the chebyshev disk, then the Chebyshev center of $\mathcal{F}_j$ can be computed by the following optimization problem:

$$(\mathcal{L}_1) \quad \max_{l,\boldsymbol{r}_{c,j}} l, \text{ subject to:} \tag{4.2}$$

$$\|\boldsymbol{r}_{c,j} + l\boldsymbol{u} - \boldsymbol{B}_i\| \leq \phi(D_{ij}),\ i = 1, \ldots, N,\ \forall \boldsymbol{u}. \tag{4.3}$$

The constraints (4.3) are convex conic constraints [3] and describe a disk. The radius of the disk l is defined by the model for measurement and is characterized by the function ϕ. These constraints can be relaxed to linear constraints in the same manner as approximating a disk by a regular polygon circumscribing the disk. Note that for any vector $\boldsymbol{x}$, its norm $\|\boldsymbol{x}\| = \max_{\boldsymbol{v}:\|\boldsymbol{v}\|=1} \boldsymbol{v} \cdot \boldsymbol{x}$. Hence, the conic constraints can be recast as a semi-infinite set of linear inequalities [3] as:

$$\max_{\boldsymbol{v}:\|\boldsymbol{v}\|=1} (\boldsymbol{v} \cdot \boldsymbol{r}_{c,j} + l\boldsymbol{v} \cdot \boldsymbol{u} - \boldsymbol{v} \cdot \boldsymbol{B}_i) \leq \phi(D_{ij}),\ i = 1, ., N, \forall \boldsymbol{u}, j$$

The tightest inequality corresponds to $\boldsymbol{u} = \boldsymbol{v}$ and hence, the semi-infinite linear program describing the location estimation problem is:

$$(\mathcal{L}_2) \quad \max_{l,\boldsymbol{r}_{c,j}} l, \text{ subject to:}$$

$$\boldsymbol{v} \cdot \boldsymbol{r}_{c,j} + l\,\|\boldsymbol{v}\| \leq \boldsymbol{v} \cdot \boldsymbol{B}_i + \phi(D_{ij}),\ i = 1, \ldots, N, \forall \boldsymbol{v}.$$

4.1.2 *Maximum volume inscribed ellipsoid of $\mathcal{F}_j$*

The volume of the feasible set $\mathcal{F}_j$ is a measure of the uncertainty/confidence associated with the position of the j^{th} on-board receiver. Here, the "best" estimate is given by the center of the maximum volume ellipsoid $\mathcal{E}_j$ contained in $\mathcal{F}_j$. The rationale behind this approach is that the volume V_j of this ellipsoid $\mathcal{E}_j$ provides a lower bound for the volume of $\mathcal{F}_j$ while the minimum volume of an ellipsoid containing $\mathcal{F}_j$ is within $\sqrt{3}V_j$ in a three dimensional space [3]. Hence the volume V_j can be used as a proxy measure for the volume of $\mathcal{F}_j$. Moreover, the problem of determining the maximum volume ellipsoid is convex while the problem of determining the minimum volume ellipsoid containing $\mathcal{F}_j$ is intractable given the inequality constraints and the estimate of the center of this ellipsoid is invariant under affine transformations unlike the chebyshev center [3].

Any $\boldsymbol{x} \in \mathcal{E}_j$ can be written as $\boldsymbol{x} = P_j\boldsymbol{u} + \boldsymbol{r}_{c,j}$, where $P_j \succeq 0$ (symmetric and positive semi-definite matrix of appropriate dimension), $\boldsymbol{r}_{c,j}$ is the center of the $\mathcal{E}_j$ and $\|\boldsymbol{u}\|^2 \leq 1$. Using these notations, the problem of computing the maximum volume inscribed ellipsoid $\mathcal{E}_j$ of $\mathcal{F}_j$ can be formulated as a SDP:

$(\mathcal{L}_3) \quad V_j := \max \log\det\ P_j,$ subject to:

$$P_j \succeq 0 \text{ and} \tag{4.4}$$

$$\|P_j\boldsymbol{u} + \boldsymbol{r}_{c,j} - \boldsymbol{B}_i\| \leq \phi(D_{ij}),\ i = 1, \ldots, N, \qquad \forall\{\boldsymbol{u} : \|\boldsymbol{u}\|^2 \leq 1\}. \tag{4.5}$$

The constraints (4.5) are convex and represent the ellipsoid $\mathcal{E}_j$ that is constrained to lie in the intersection of the of the spheres centered at the beacon locations of radii determined by the range measurements gathered by the j^{th} receiver.

Proposition 1. For a fixed value of $i \in \{1, \ldots, N\}$, the set of infinite constraints (4.5) in formulation $\mathcal{L}_3$ is equivalent to the following set of constraints :

$$\lambda \geq 0 \text{ and } \begin{bmatrix} \phi(D_{ij}) - \lambda & (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i)^T & 0 \\ (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i) & \phi(D_{ij}) I_3 & P_j \\ 0 & P_j & \lambda I_3 \end{bmatrix} \succeq 0. \tag{4.6}$$

Proof : See Appendix 1.

Hence, an equivalent formulation for computing the center of $\mathcal{E}_j$ is given by:

$(\mathcal{L}_4) \quad V_j = \max \log \det \ P_j,$ subject to:

$$\begin{bmatrix} \phi(D_{ij}) - \lambda & (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i)^T & 0 \\ (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i) & \phi(D_{ij}) I_3 & P_j \\ 0 & P_j & \lambda I_3 \end{bmatrix} \succeq 0,$$

$P_j \succeq 0,$ and $\lambda \geq 0.$

4.2 Determination of the function $\phi(\cdot)$ for the sensing model

As stated in the aforementioned section, given that the measured distance from a beacon i to an the on-board receiver j, D_{ij}, the sensing model assumes that $\phi(D_{ij})$ is an increasing function that provides a bound on the true distance d_{ij}. The ramification of choosing an increasing function ϕ is that the range measurements that originate from the beacons sufficiently far away from the vehicle will not matter. The model also provides a bound on the true distance $d_{ij} \leq \phi(D_{ij})$. In this section, we formulate the problem of finding an increasing function ϕ using measurements and true distance data as a SDP. This problem is solved off-line before the beacons and the on-board receivers are deployed and can be considered as a part of the receiver calibration process. We consider a set of true distance data indexed by the set K. For each data point d_k, $k \in K$ multiple range measurements are obtained while placing an arbitrary beacon and an on-board receiver d_k units apart. Let $[D_k^l, D_k^u]$ denote the lower and upper bounds of the measurements corresponding to the true distance d_k, $k \in K$. Define $D^l := \min_{k \in K} D_k^l$ and $D^u := \max_{k \in K} D_k^u$. The problem now is to determine a univariate increasing function $\phi : [D^l, D^u] \to \mathbb{R}^+$ that solves the following optimization problem:

$$\min \sum_{k \in K} (\phi(D_k^u) - d_k), \text{ subject to:} \tag{4.7}$$

$$\phi'(d) \geq 0, \ \forall d \in [D^l, D^u], \text{ and} \tag{4.8}$$

$$\phi(D_k^l) \geq d_k, \ \forall k \in K, \tag{4.9}$$

The constraints (4.8) enforce the function ϕ to be an increasing function in its domain and the constraints (4.9) ensure for any range measurement D_{ij} between a beacon i and an on-board receiver j, $\phi(D_{ij})$ provides an upper bound on the true distance d_{ij}. The objective ensures that this bound obtained is the least upper bound while satisfying the other constraints. To solve the above problem, we approximate $\phi(x)$ using a univariate polynomial. The condition (4.8) is equivalent to the polynomial $\phi'(\cdot)$ being non-negative in the interval $[D^l, D^u]$. We now state two known results using which we recast the non-negativity restrictions on the univariate polynomial to a SDP.

Theorem 1. (Markov-Lukàcs theorem) Let $a < b$. Then a univariate polynomial $p(x)$ is non-negative on $[a, b]$, if and only if it can be written as

$$p(x) = \begin{cases} s(x) + (x-a)(b-x)t(x), & \text{if degree}(p) \text{ is even} \\ (x-a)s(x) + (b-x)t(x), & \text{if degree}(p) \text{ is odd} \end{cases}$$

where $s(x)$ and $t(x)$ are 'Sum of Squares' (SOS). In the first case, we have $\text{degree}(p) = 2k$, and $\text{degree}(s) \leq 2k$, $\text{degree}(t) \leq 2k-2$. In the second, $\text{degree}(p) = 2k+1$, and

$\text{degree}(s) \leq 2k$, $\text{degree}(t) \leq 2k$.

Proof : See [8].

Theorem 2. Let $p(x)$ be a univariate polynomial of degree $2k$. Then, $p(x)$ is SOS if and only if there exists a $(k+1) \times (k+1)$ positive semi-definite matrix $\boldsymbol{P}$ that satisfies

$$p(x) = [x]_k^T \boldsymbol{P} [x]_k$$

where, $[x]_k = [1\; x\; x^2 \ldots x^k]^T$.

Proof : See [7].

Let $\phi(x) = a_0 + a_1 x + \cdots + a_n x^n$ be an n^{th} degree polynomial whose coefficients are to be determined. Then, we have

$$\phi'(x) = a_1 + 2a_2 x + \ldots n a_n x^{n-1} = \sum_{i=1}^{n} i a_i x^{i-1}. \tag{4.10}$$

The constraint (4.8) requires $\phi'(x)$ to be non-negative on $[D^l, D^u]$. Suppose that the degree of ϕ is even (the case when ϕ has an odd degree has a similar reduction and hence, is not presented). Then by theorem 1, we have

$$\phi'(x) = (x - D^l)s(x) + (D^u - x)t(x) \tag{4.11}$$

where, $s(x)$ and $t(x)$ are SOS. Let $\text{degree}(\phi') = 2k+1$, then degree of $s(x)$ and $t(x)$ is at most $2k$. Then by theorem 2, we have

$$s(x) = [x]_k^T \boldsymbol{S} [x]_k,\; t(x) = [x]_k^T \boldsymbol{T} [x]_k \tag{4.12}$$

where, $\boldsymbol{S}$ and $\boldsymbol{T}$ are $(k+1) \times (k+1)$ positive semi-definite matrices. Combining (4.10), (4.11) and (4.12), we obtain:

$$\sum_{i=1}^{2k+1} ia_i x^{i-1} = (x - D^l)[x]_k^T \boldsymbol{S}[x]_k + (D^u - x)[x]_k^T \boldsymbol{T}[x]_k.$$

Indexing the rows and columns of $\boldsymbol{S}$ and $\boldsymbol{T}$ by $\{0, 1, \ldots, k\}$ and equating the coefficients of the M^{th} power of x on both sides of the above equation, we obtain a set of $(2k+1)$ linear equations relating the coefficients of the polynomial ϕ and then entries of the positive semi-definite matrices $\boldsymbol{S}$ and $\boldsymbol{T}$ as follows:

$$(M + 1)a_{M+1} = \sum_{0\leq i,j\leq k}^{i+j=M-1} (\boldsymbol{Q}_{ij} - \boldsymbol{T}_{ij}) + \sum_{0\leq i,j\leq k}^{i+j=M} \left(D^u \boldsymbol{T}_{ij} - D^l \boldsymbol{S}_{ij}\right). \quad (4.13)$$

Hence, an equivalent SDP for computing the function ϕ, which is approximated by a degree n polynomial is given by:

$$(\mathcal{L}_5) \ \min \sum_{k\in K} (\phi(D_k^u) - d_k), \text{ subject to:}$$

$$(4.13), (4.9), \text{ and } \boldsymbol{S}, \boldsymbol{T} \succeq 0.$$

Constraints (4.13) and the postive semi-definiteness of $\boldsymbol{S}, \boldsymbol{T}$ together is equivalent to enforcing the polynomial ϕ to be increasing; the constraints (4.13) and (4.9) are linear constraints.

4.3 Estimation of the orientation of the vehicle from the location estimates of the on-board receivers

Suppose $\mathcal{F}$ is a frame of reference attached to the rigid body with its origin at O and unit vectors $\hat{\imath}, \hat{\jmath}, k$, respectively. Let the coordinates of the vehicle's on-board receivers in $\mathcal{F}$ be $\boldsymbol{w}_j = (a_j, b_j, c_j)$, $j = 1, 2, \ldots, L$, respectively and its estimated location in the ground frame be $\boldsymbol{r}_{c,j} = (x_{c,j}, y_{c,j}, z_{c,j})$. Let $\boldsymbol{R}$ be the rotation matrix associated with the body describing its orientation. Let $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ denote the estimate of the location of the origin O of the body frame $\mathcal{F}$. Then, it is clear that the following rigid body motion constraints must hold when there is no estimation error in the location of the on-board receivers:

$$\boldsymbol{r}_{c,j} = \boldsymbol{r}_0 + \boldsymbol{R}\boldsymbol{w}_j, \; \forall j = 1, 2, \ldots, L. \tag{4.14}$$

Essentially, these constraints guarantee that the angles between line segments joining the receivers as inferred from the location estimates will remain the same as their true values and the distance between the receivers as inferred from their locations will remain the same as their true values. Compactly, one can rewrite the above equation as:

$$\begin{bmatrix} \boldsymbol{r}_{c,1} \cdots \boldsymbol{r}_{c,L} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_0 \cdots \boldsymbol{r}_0 \end{bmatrix} + \boldsymbol{R} \begin{bmatrix} \boldsymbol{w}_1 \cdots \boldsymbol{w}_L \end{bmatrix}. \tag{4.15}$$

However, the estimates may not satisfy the above relationship due to errors in measurements and subsequent location estimation of on-board sensors. In particular, the estimate of the distance between the on-board receivers need not equal the actual distance between them. As a consequence, the relative configuration of the on-board receivers indicated by their location estimates will not be the same as the true relative configuration of receivers. One then needs to correct these location estimates in order to ensure that the distance between the on-board receivers is its true value. Since the errors in the location estimates will be non-zero, let us define an error matrix, $\boldsymbol{E}$ as:

$$\boldsymbol{E} := \begin{bmatrix} \boldsymbol{r}_{c,1} \cdots \boldsymbol{r}_{c,L} \end{bmatrix} - \begin{bmatrix} \boldsymbol{r}_0 \cdots \boldsymbol{r}_0 \end{bmatrix} - \boldsymbol{R} \begin{bmatrix} \boldsymbol{w}_1 \cdots \boldsymbol{w}_L \end{bmatrix}. \tag{4.16}$$

The problem of localization can now be posed as

$$(\mathcal{L}_6) \quad J = \min_{\boldsymbol{r}_0, \boldsymbol{R} \in SO(3)} \mathrm{trace}(\boldsymbol{E}^T \boldsymbol{E}) \text{ subject to (4.16).}$$

In the above formulation $\mathcal{L}_6$, $\mathrm{trace}(\boldsymbol{E}^T \boldsymbol{E})$ is the square of the Frobenius norm of the error matrix $\boldsymbol{E}$, and $SO(3) = \{\boldsymbol{R} : \det(\boldsymbol{R}) = 1, \boldsymbol{R}^{-1} = \boldsymbol{R}^T\}$ is referred to as the 'Special Orthogonal Group'.

5. ALGORITHMS*

In this section, we focus on solving the relevant optimization problems from the previous section. In section 5.1, we will outline a cutting plane algorithm to solve the formulations $\mathcal{L}_2$ and $\mathcal{L}_3$, respectively. In section 5.2, we will provide a solution procedure to solve the problem of localization given by the formulation $\mathcal{L}_6$ and thereby determine the optimal orientation that minimizes the square of the Frobenius norm of the error matrix, $\boldsymbol{E}$. As for the formulation $\mathcal{L}_3$ and $\mathcal{L}_5$, they can be solved to optimality using off-the-shelf semi-definite solvers like SCS [6].

5.1 Location estimation procedure

5.1.1 Algorithm to estimate the Chebychev center

The procedure involves a relaxation of the semi-infinite LP in the formulation $\mathcal{L}_2$ to a finite LP by ignoring all but finite constraints and providing an iterative way of adding the required constraints from the dropped set of constraints. This generic procedure is referred to as a cutting plane method (see [3]).

To that end, let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M$ be the M sides of a circumscribing polygon of the feasible region, then a relaxation of $\mathcal{L}_2$ is given by

$$\bar{l}_{\max} = \max_{l, \boldsymbol{r}_{c,j}} \; l, \text{ subject to:}$$

$$\boldsymbol{v}_k \cdot \boldsymbol{r}_{c,j} + l \, \|\boldsymbol{v}_k\| \leq \boldsymbol{v}_k \cdot \boldsymbol{B}_i + \phi(D_{ij}), \qquad i = 1, \ldots, N, \; k = 1, \ldots, M.$$

Clearly, the feasible set of this LP, $\mathcal{F}^{-}{}_{j}$ contains the feasible set $\mathcal{F}_j$ of the original problem as all by finite constraints of the original semi-infinite LP have been dropped. Suppose the optimal solution, $(l^{-}{}_{\max}, \boldsymbol{r}^{-}{}_{c,j})$, of the relaxed finite LP satisfies semi-infinite constraints; then it is clear that $(l^{-}{}_{\max}, \boldsymbol{r}^{-}{}_{c,j})$ is optimal for $\mathcal{L}_2$. Otherwise, for some unit vector $\boldsymbol{v}_{M+1}$

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distinct from those considered before, and for some i, the following inequality holds:

$$\boldsymbol{v}_{M+1} \cdot \bar{\boldsymbol{r}}_{c,j} + \bar{l}_{\max} \|\boldsymbol{v}_{M+1}\| > \boldsymbol{v}_{M+1} \cdot \boldsymbol{B}_i + \phi(D_{ij}).$$

By adding the "cut"

$$\boldsymbol{v}_{M+1} \cdot \boldsymbol{r}_{c,j} + l \|\boldsymbol{v}_{M+1}\| \leq \boldsymbol{v}_{M+1} \cdot \boldsymbol{B}_i + \phi(D_{ij}),$$

which must be satisfied by the optimal solution for the semi-infinite LP and violated by the previously obtained optimal solution for the finite LP, we improve the solution. This is akin to finding another face of the polygon circumscribing the disk that cuts off a vertex of the previously obtaining polygon. This cutting plane method can be used along with an off-the shelf LP solver to solve the semi-infinite LP to arbitrary accuracy.

5.1.2 *Algorithm to estimate the center of the maximum volume inscribed ellipsoid*

A cutting plane algorithm similar to the one developed for computing the Chebychev center in section 5.1.1 is developed to estimate the center of the maximum volume inscribed ellipsoid using the formulation $\mathcal{L}_3$. In this case, we relax the conic constraints in (4.5) by ignoring all but finite constraints. We solve the initial conic SDP for a finite number of $\boldsymbol{u}$ with $\|\boldsymbol{u}\|^2 \leq 1$ and add violated constraints iteratively.

To that end, let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_M$ be M vectors such that $\|\boldsymbol{u}_k\|^2 \leq 1$, $k = 1, \ldots, M$. Then a relaxation of $\mathcal{L}_3$ is given by

$$\bar{V}_j = \max \log\det\ P_j,\ \text{subject to: } P_j \succeq 0,$$

$$\|P_j \boldsymbol{u} + \boldsymbol{r}_{c,j} - \boldsymbol{B}_i\| \leq \phi(D_{ij}),\ i = 1, \ldots, N,\ k = 1, \ldots, M.$$

Off-the-shelf primal dual interior point solvers can be used to solve the above conic SDP to obtain an optimal solution $(\bar{P}_j, \bar{r}_{c,j})$. If the solution to this relaxed problem is satisfies all the constraints in (4.5), then it is optimal for $\mathcal{L}_3$. Otherwise, for a vector $\boldsymbol{u}_{M+1}$, with $\|\boldsymbol{u}_{M+1}\| \leq 1$, distinct from the vectors considered before, and for some i, the following inequality holds:

$$\left\|\bar{P}_j \boldsymbol{u}_{M+1} + \bar{\boldsymbol{r}}_{c,j} - \boldsymbol{B}_i\right\| > \phi(D_{ij}).$$

This produces the "cut"

$$\|P_j \boldsymbol{u}_{M+1} + \boldsymbol{r}_{c,j} - \boldsymbol{B}_i\| \leq \phi(D_{ij}),$$

which was violated by the previously obtained optimal solution. The cut is added to the relaxation and the problem is resolved to compute an improved optimal solution. This process is iterated till the optimal solution satisfies all the constraints in (4.5).

5.2 Procedure for orientation estimation and correcting the location estimates taking rigid body constraints into account

The location estimates of the onboard receivers have been obtained without regard to the rigid body motion constraints between them. Since receivers are attached to the rigid body, the distance between any pair of them is pre-specified. The estimates may not satisfy the distance constraints and even the angle between line segments joining the receivers computed from their location estimates may not correspond to their true values. For this reason, a correction procedure for the location estimates is required. Fortunately, this pursuit involves the estimation of orientation of the body.

Let,

$$\boldsymbol{e}_j := \boldsymbol{r}_{c,j} - \boldsymbol{r}_0 - \boldsymbol{R}\boldsymbol{w}_j,\ j = 1, \ldots, L.$$

The term $\boldsymbol{e}_j$ describes the error in the estimate of the location to maintain a rigid body constraint for the j^{th} receiver and is a function of the the location $\boldsymbol{r}_0$ of the origin of the body frame and the rotation matrix $\boldsymbol{R}$ that describes the orientation of the rigid body. One can observe that $\mathrm{trace}(\boldsymbol{E}^T\boldsymbol{E}) = \sum_{j=1}^{L} \boldsymbol{e}_j^T \boldsymbol{e}_j$ and $\boldsymbol{e}_j$ is a linear function of $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ and $\boldsymbol{R}$. Hence, minimization over $\boldsymbol{r}_0, \boldsymbol{R}$ can be performed sequentially since there are no constraints if we explicitly express $\mathrm{trace}(\boldsymbol{E}^T\boldsymbol{E})$ as a function of $\boldsymbol{r}_0, \boldsymbol{R}$. Let

$$\bar{\boldsymbol{w}} = \frac{1}{L}\sum_{j=1}^{L} \boldsymbol{w}_j,\ \bar{\boldsymbol{r}} = \frac{1}{L}\sum_{j=1}^{L} \boldsymbol{r}_{c,j}.$$

Minimization of $\mathrm{trace}(\boldsymbol{E}^T\boldsymbol{E})$ with respect to $\boldsymbol{r}_0$ yields

$$\boldsymbol{r}_0 = \frac{1}{L}\sum_{j=1}^{L} [\boldsymbol{r}_{c,j} - \boldsymbol{R}\boldsymbol{w}_j] = \bar{\boldsymbol{r}} - \boldsymbol{R}\bar{\boldsymbol{w}}.$$

Define for $j = 1, \ldots, L$

$$\tilde{\boldsymbol{r}}_{c,j} := \boldsymbol{r}_{c,j} - \bar{\boldsymbol{r}},\ \tilde{\boldsymbol{w}}_j := \boldsymbol{w}_j - \bar{\boldsymbol{w}}.$$

With these definitions and the optimizing value of $\boldsymbol{r}_0$,

$$\boldsymbol{e}_j = \tilde{\boldsymbol{r}}_{c,j} - \boldsymbol{R}\tilde{\boldsymbol{w}}_j,\ j = 1, \ldots, L.$$

Correspondingly,

$$\begin{aligned}
&\mathrm{trace}(\boldsymbol{E}^T\boldsymbol{E}) = \sum_{j=1}^{L} (\tilde{\boldsymbol{r}}_{c,j} - \boldsymbol{R}\tilde{\boldsymbol{w}}_j)^T (\tilde{\boldsymbol{r}}_{c,j} - \boldsymbol{R}\tilde{\boldsymbol{w}}_j) \\
&= \sum_{j=1}^{L} \left(\tilde{\boldsymbol{r}}_{c,j}^T \tilde{\boldsymbol{r}}_{c,j} + \tilde{\boldsymbol{w}}_j^T \tilde{\boldsymbol{w}}_j\right) - 2\,\mathrm{trace}\left(\left(\sum_{j=1}^{L} \tilde{\boldsymbol{w}}_j \tilde{\boldsymbol{r}}_{c,j}^T\right) \boldsymbol{R}\right).
\end{aligned}$$

Define $\boldsymbol{W} := \sum_{j=1}^{L} \tilde{\boldsymbol{w}}_j \tilde{\boldsymbol{r}}_{c,j}^T$ so that

$$\boldsymbol{R}^* = \underset{\boldsymbol{R} \in SO(3)}{\arg\min}\, \mathrm{trace}(\boldsymbol{E}^T \boldsymbol{E}) = \underset{\boldsymbol{R} \in SO(3)}{\arg\max}\, \mathrm{trace}(\boldsymbol{W}\boldsymbol{R}).$$

This stems from the other terms being independent of $\boldsymbol{R}$.

Let the singular value decomposition of $W = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ where $\boldsymbol{U}$, $\boldsymbol{V}$ are the left and right singular vectors of $\boldsymbol{W}$, respectively and $\boldsymbol{\Sigma}$ is a 3×3 diagonal matrix consisting of its singular values. The problem of maximizing $\mathrm{trace}(\boldsymbol{W}\boldsymbol{R})$ over the set of all rotation matrices is referred to as the "Orthogonal Procrustes problem" (see [11]).

Theorem 3. $\boldsymbol{R}^* = \boldsymbol{V}\boldsymbol{U}^T$ maximizes $\mathrm{trace}(\boldsymbol{W}\boldsymbol{R})$ over the set of all proper rotation matrices.

Proof : We have due to the property of trace of a product of matrices:

$$\mathrm{trace}(\boldsymbol{W}\boldsymbol{R}) = \mathrm{trace}(\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{R}) = \mathrm{trace}(\boldsymbol{U}\boldsymbol{V}^T\boldsymbol{R}\boldsymbol{\Sigma}).$$

Note that $\boldsymbol{V}$, $\boldsymbol{R}$ and $\boldsymbol{U}$ are all orthonormal matrices, so $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^T\boldsymbol{R}$ is also an orthonormal matrix. Suppose x_{ii}, $i = 1, 2, 3$ denotes the diagonal elements of $\boldsymbol{X}$, then

$$\mathrm{trace}(\boldsymbol{W}\boldsymbol{R}) = \mathrm{trace}(\boldsymbol{X}\boldsymbol{\Sigma}) = \sum_{i=1}^{3} \sigma_i x_{ii} \leq \sum_{i=1}^{3} \sigma_i$$

where, σ_i, $i = 1, 2, 3$ are the singular values of $\boldsymbol{W}$. The maximum value in the above equation is achieved when $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^T\boldsymbol{R} = \boldsymbol{I}$ *i.e.*, $\boldsymbol{R}^* = \boldsymbol{V}\boldsymbol{U}^T$.

Remark 1. The minimum value of

$$\mathrm{trace}(\boldsymbol{E}^T\boldsymbol{E}) = \sum_{j=1}^{L} \left(\tilde{\boldsymbol{r}}_{c,j}^T \tilde{\boldsymbol{r}}_{c,j} + \tilde{\boldsymbol{w}}_j^T \tilde{\boldsymbol{w}}_j \right) - 2(\sigma_1 + \sigma_2 + \sigma_3).$$

Remark 2. The location estimate of the origin of the body frame is $\boldsymbol{r}_0 = \bar{\boldsymbol{r}} - \boldsymbol{R}^* \bar{\boldsymbol{w}}$. If one places the onboard receivers in such a way that the "center of mass" of the rigid body (vehicle) coincides with the center of mass of the receivers, then the estimate of $\boldsymbol{r}_0$ is the "corrected" estimate of the location of the center of mass. The error in localization (both in the location estimation and orientation) from the "best" estimates of the locations of the receiver is given by the sum of the singular values of $\boldsymbol{W}$, a norm referred to as the nuclear norm of $\boldsymbol{W}$.

Remark 3. The updated estimate of the j^{th} receiver's location will be given by:

$$\boldsymbol{r}_{c,j} = \boldsymbol{r}_0 + \boldsymbol{R}^* \boldsymbol{w}_j.$$

6. NUMERICAL EXAMPLES AND SIMULATION RESULTS

The effectiveness of the procedures developed in this paper, and its implementation are illustrated using 2-D and 3-D examples in this section. All the simulations were performed on a Dell Precision Workstation with 12 GB RAM. The problem setup for the 3-D problem is as follows:

Assume that a set of range measuring beacons are fixed underwater at a few predetermined locations, such that it engulfs an area in which an AUV is supposed to be localized.(i.e., the AUV travels in such a way that it always lies in the convex hull of the sensor network). The coordinates of the sensors with respect to an inertial frame of reference are known. Say 8 sensors are placed at (-2,-2,-2) ,(-2,-2,12),(-2,2,-2), (-2,2,12),(-2,6,-2),(-2,6,12),(12,-2,-2),(12,-2,12),(12,2,-2),(12,2,12),(12,6,-2),(12,6,12) with respect to the inertial reference frame . The vehicle is equipped with a few on-board receivers which are fixed with respect to the vehicle reference frame. The coordinates of these receivers in the body reference frame are O (0,0,0) , A (1,0,0) , B (0,1,0) and C (0,0,1) . Given the range measurements of these on-board receivers by the acoustic beacons at various time instances, determine the location (of the Center of Mass) and orientation of the vehicle.

6.1 Simulation of range measurements

In reality, the range measurements are available from the One Way Travel Time (OWTT) of pulses from the beacons to receivers. In the current example, these range values are numerically simulated. To do so, the vehicle is assumed to follow a particular trajectory in which the coordinates of its center of mass at each time instant are given by the parametric equation r(t) = (2.5+ 2.5 cos(t) , 2.5 sin(t) , 5 sin(t/2)) ,where t is the parameter. 1000 discretization points are uniformly generated from the interval $t \in [0, \pi]$,which correspond to the center of mass of the vehicle, and the principal axes directions of the vehicle at each point are given by the tangent, normal and bi-normal directions of the curve at that particualr point These directions are calculated using Frenet - Serret formulas. Assume that the vehicle is equipped with four on-board receivers, one of which is located at the center of mass of the vehicle, and the other three are positioned one unit away from the center of mass, along each of the three principal directions of the vehicle. Once the sensor network is setup and the vehicle trajectory is decided, the actual distance (true distance) between the on-board receivers and the beacons can be computed directly using the Euclidean distance formula for 3-D Cartesian coordinates. In general, the range measurements given by the beacons are always noisy due to numerous reasons. This is simulated by adding a White Gaussian Noise N(0,0.25) to the true distances computed. Once the range measurements are available, the algorithms discussed in the paper can be implemented.

6.2 Calibrating function

The model assumes that the receiver always lies in a sphere centered at the beacon from which the range measurement is obtained. To ensure this, the radius of each sphere should be greater than the corresponding true range value. Thus, a fitting function which engineers the measured range values to form radii of these spheres is required. As discussed in the paper above, this function should be a non-negative increasing polynomial.

For the computation of function coefficients, data sets containing plausible true range values (chosen based on the application) and their corresponding range measurements (measured multiple times for each true distance considered) are required. Experimentally these data sets can be obtained by placing the receivers at different distances and noting the beacon range measurements multiple times. In this example, numerical simulations were performed to obtain the data.

6.2.1 Data generation for calibrating function

The permissible true distance values are restricted to a certain interval due to constraints like dimensions of the vehicle and location of the receivers on the vehicle, which act as a lower bound to the permissible distance between the sensors and the receivers, and the time permitted for the pulses to reach the receivers, which acts as an upper bound to the distance between the sensors and the receivers. This also helps in neglecting the sensors that are far away which are less reliable due to several reasons. Considering such aspects, the true ranges were restricted to the interval [2, 18] meters. A set of 25 values uniformly distributed in this interval are picked to form the set of true range values, and for each element in the set , 100 corresponding range measurements are generated by adding white Gaussian noise to it, in order to mimic the error present in actuality.Once the data set is available, the coefficients of the fitting polynomial can be obtained as a solution of an SDP, as discussed in section 4.2

6.2.2 Computation of the coefficients of calibrating function

Let us choose a function of degree 4 $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 (f'(x) = a_1 + a_2x + a_3x^2 + a_4x^3)$. Using Markov-Lukàcs and SOS theorems, the constraint (4.8) can be re-written as :

$(A(x))^2(x - D^l) + (B(x))^2(D^u - x) \geq 0$, where $A(x) = [x]_k^T \boldsymbol{P}[x]_k, B(x) = [x]_k^T \boldsymbol{Q}[x]_k$, $[x]_k = [1\ x\ x^2 \ldots x^k]^T, \boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \succeq 0$ and $\boldsymbol{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \succeq 0.$

This upon simplification gives the following equations :

$$a_1 = -D^l p_{11} + D^u q_{11}$$

$$2a_2 = p_{11} - 2D^l p_{12} + 2D^u q_{12} - q_{11}$$

$$3a_3 = 2P_{12} - D^l p_{22} + D^u q_{22} - 2q_{12}$$

$$4a_4 = p_{22} - q_{22}.$$

Constraint (4.9) can be written as

$$a_0 + a_1 D_k^l + a_2(D_k^l)^2 + a_3(D_k^l)^3 + a_4(D_k^l)^4 \geq d_k, \forall k \in K,$$

and the objective is to minimize the function :

$$\sum_{k \in K}(a_0 + a_1 D_k^u + a_2(D_k^u)^2 + a_3(D_k^u)^3 + a_4(D_k^u)^4 - d_k),$$

The solution of this SDP is $a_0 = 0.4639$, $a_1 = 1.4362$, $a_2 = -0.1011$, $a_3 = 0.0083$, $a_4 = -0.0002$ and it forms the calibrating function

$$f(x) = 0.4639 + 1.4362x - 0.1011x^2 + 0.0083x^3 - 0.0002x^4$$

6.3 Estimation of vehicle/receiver's location

This function enables the conversion of range measurements of each sensor-receiver pair to the radii of the spheres centered on the beacons and containing the receivers The intersection of such spheres corresponding to a particular receiver forms the feasible region for its location. The center of this region can be considered as an estimate of the receiver's location. There are two notions of center 1) Chebyshev center 2) MVE (Maximum Volume Ellipsoid) center, which are discussed below.

6.3.1 Chebyshev center

It is the center of the sphere of maximum volume that can be inscribed in the feasible region. Since the beacon locations are known apriori, and the calibrating function has been calculated from the range measurements, the Chebyshev center can now be obtained as the solution of an LP relaxation as formulated in section 4.1.1, using a cutting plane algorithm to solve the LP as discussed in section 5.1.1 ,for each discretization of the trajectory. The root mean square error in the location estimate (of receiver O) was found to be 0.39 units , and the maximum error was 1.10 units.

6.3.2 Maximum Volume Ellipsoid (MVE) center

Similar to Chebyshev center, the MVE center is the center of the ellipsoid of maximum volume that can be inscribed in the feasible region. This notion of center is better than the Chebyshev center in that it is invariant to affine transformations, and it also provides an upper bound on the volume of uncertainty in the form of a corresponding minimum volume ellipsoid inscribing the feasible region, the volume of which is $\sqrt{3}$ (incase of a 3-D problem) times the volume of the maximum volume ellipsoid inscribed in the region. This problem has been formulated as a second order conic program, and solved using two methods. It can either be converted to an equivalent SDP formulation using S-lemma and Schur complement as discussed in section 4.1.2 and Appendix 1,or a cutting plane algorithm can be implemented on the conic problem as explained in section 5.1.2. Even though the cutting plane method is expected yield faster results due to its computational simplicity, both the methods took the same time in yielding the optimal solution. This is due to the advancement of the SDP solvers over time. The root mean square error in the location estimate of receiver O was found to be 0.33 units and the maximum error was 0.92 units.

6.4 Estimating the orientation and correcting the location estimates

The receiver locations obtained from either the Chebyshev or MVE estimates help in locating the center of mass of the vehicle. But these estimates need not conform to the rigid body norms. Hence there is a need to correct these estimates such that the initial distance between the receivers and their relative orientations are preserved. This correction also enables the estimation of the vehicle orientation (as a solution of the Orthogonal Procrustes problem) in the inertial reference frame. At each time step, the receiver locations are updated and the vehicle orientation is obtained as explained in section 5.2. The initial orientation of the vehicle, the relative orientation of the receivers after the intermediate(location estimate) step using MVE center, and the final updated are shown in figure 6.1 . Since the receivers are chosen in such a way that the vectors joining them form the direction cosines of the body, their relative orientation directly represents the vehicle orientation.

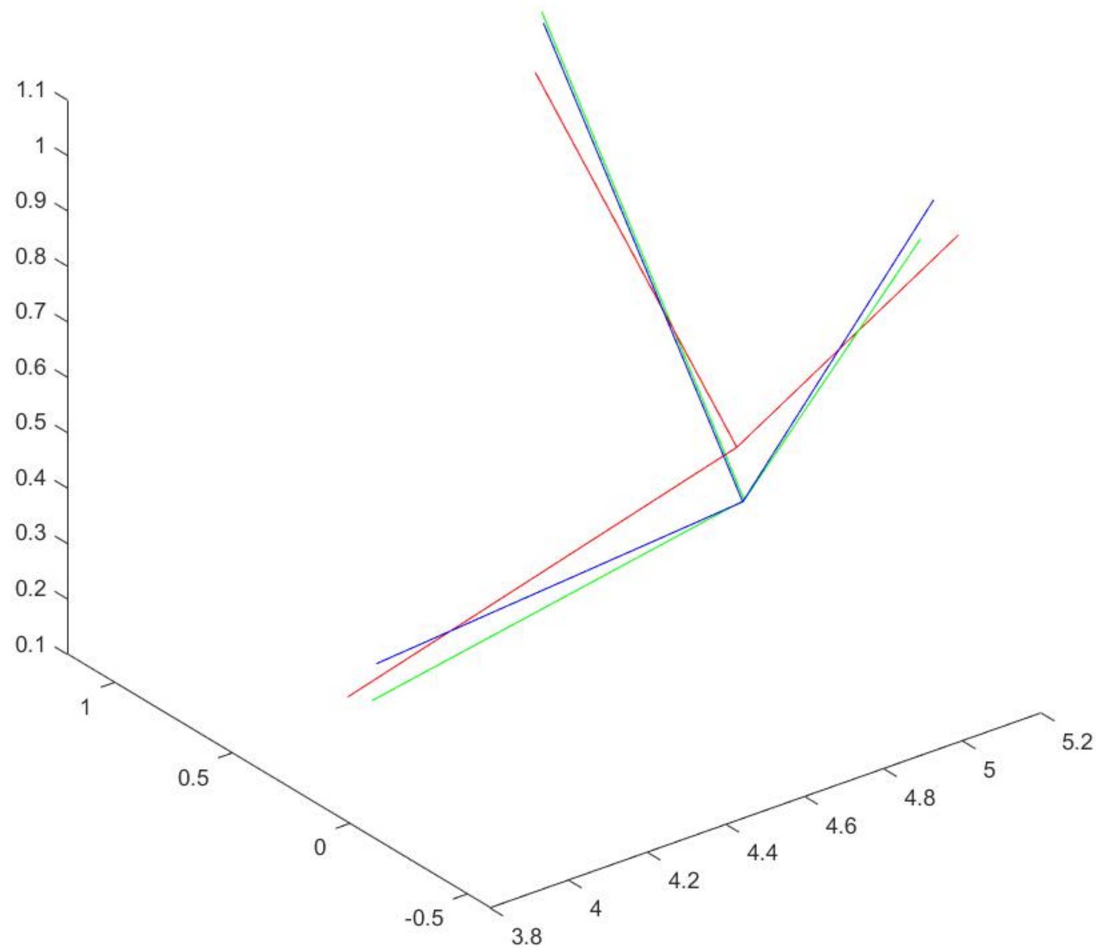

Figure 6.1: Relative orientation of on-board receivers - Actual (blue) After 1st step (red) After rigid body corrections (green)

7. SUMMARY

The problem of localization, which is very important for vehicle autonomy has been solved using a 2-step procedure. Only range measurements were used to estimate the location and orientation of the vehicles. The lack of GPS availability in the case of underwater vehicles and its lack of desired accuracy in case of roadside vehicles has been tackled in this work. A good error model, which assumes that the true distance between the receiver and the beacons is at most equal to a predetermined function of the range measurement was used to circumvent the errors in communication. The first step of estimation involves the estimation of receiver locations from a set of feasible locations formed by the intersection of spheres centered at the beacons, with radius equal to the predetermined function of the measured ranges to the receiver. The second step involves the correction of these estimates so that they conform to rigid body principles , and simultaneously estimating the orientation of the vehicle. A 3-D numerical example has been used to illustrate the efficacy of the procedures developed above.

REFERENCES

[1] Hong S. Bae, Jihan Ryu, and J. Christian Gerdes. Road grade and vehicle parameter estimation for longitudinal control using GPS. In *Proceedings of the IEEE Conference on Intelligent Transportation Systems*, pages 25–29, 2001.

[2] David M. Bevly, J. Christian Gerdes, Christopher Wilson, and Gengsheng Zhang. The use of GPS based velocity measurements for improved vehicle state estimation. In *American Control Conference, 2000. Proceedings of the 2000*, volume 4, pages 2538–2542. IEEE, 2000.

[3] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge University Press, New York, 2004.

[4] Q. Doan, Tahar Berradia, and Joseph Mouzna. Vehicle speed and volume measurement using V2i communication. In *Proc. of 9th WSEAS Int. Conf. on Applied Informatics and Communications (AIC)*, pages 366–372, 2009.

[5] Sai Krishna Kanth Hari and Swaroop Darbha. Estimation of Location and Orientation From Range Measurements. In *ASME 2015 Dynamic Systems and Control Conference*, pages V002T31A005–V002T31A005. American Society of Mechanical Engineers, 2015.

[6] Brendan OâĂŹDonoghue, Eric Chu, Neal Parikh, and Stephen Boyd. Conic optimization via operator splitting and homogeneous self-dual embedding. *Journal of Optimization Theory and Applications*, pages 1–27, 2016.

[7] Pablo A Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical programming*, 96(2):293–320, 2003.

[8] G Pólya and G Szegö. *Problems and Theorems in Analysis II*. Springer-Verlag, New York, 1976.

[9] Shahram Rezaei and Raja Sengupta. Kalman filter-based integration of DGPS and vehicle sensors for localization. *Control Systems Technology, IEEE Transactions on*, 15(6):1080–1088, 2007.

[10] Jihan Ryu and J. Christian Gerdes. Integrating inertial sensors with global positioning system (GPS) for vehicle dynamics control. *Journal of Dynamic Systems, Measurement, and Control*, 126(2):243–254, 2004.

[11] Peter H. Schönemann. A generalized solution of the orthogonal Procrustes problem. *Psychometrika*, 31(1):1–10, 1966.

[12] Steven E. Shladover and Swe-Kuang Tan. Analysis of vehicle positioning accuracy requirements for communication-based cooperative collision warning. *Journal of Intelligent Transportation Systems*, 10(3):131–140, 2006.

[13] Richard Tranfield. INS/GPS navigation systems for land applications. In *Position Location and Navigation Symposium, 1996., IEEE 1996*, pages 391–398. IEEE, 1996.

[14] Chang-Sun Yoo and Iee-Ki Ahn. Low cost GPS/INS sensor fusion system for UAV navigation. In *Digital Avionics Systems Conference, 2003. DASC'03. The 22nd*, volume 2, pages 8–A. IEEE, 2003.

APPENDIX A

PROOF OF PROPOSITION 1

For a fixed value of $i \in \{1, \ldots, N\}$, (4.5) is given by

$$\|P_j \boldsymbol{u} + \boldsymbol{r}_{c,j} - \boldsymbol{B}_i\| \leq \phi(D_{ij}), \ \forall \{\boldsymbol{u} : \|\boldsymbol{u}\|^2 \leq 1\}.$$

We can convert the above constraint to an equivalent semi-definite constraint using Schur Complement as follows:

$$\begin{bmatrix} \phi(D_{ij}) & (P_j \boldsymbol{u} + \boldsymbol{r}_{c,j} - \boldsymbol{B}_i)^T \\ P_j \boldsymbol{u} + \boldsymbol{r}_{c,j} - \boldsymbol{B}_i & \phi(D_{ij}) I_3 \end{bmatrix} \succeq 0, \quad \forall \{\boldsymbol{u} : \|\boldsymbol{u}\| \leq 1\}.$$

Equivalently,

$$\Leftrightarrow x^2 \phi(D_{ij}) + 2x\boldsymbol{y}^T (P_j \boldsymbol{u} + \boldsymbol{r}_{c,j} - \boldsymbol{B}_i) + \phi(D_{ij}) \boldsymbol{y}^T \boldsymbol{y} \geq 0, \quad \forall [x; \boldsymbol{y}], \ \{\boldsymbol{u} : \|\boldsymbol{u}\| \leq 1\},$$

$$\Leftrightarrow x^2 \phi(D_{ij}) + \min_{\boldsymbol{u}:\|\boldsymbol{u}\| \leq 1} 2x\boldsymbol{y}^T P_j \boldsymbol{u} + 2x\boldsymbol{y}^T (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i) + \phi(D_{ij}) \boldsymbol{y}^T \boldsymbol{y} \geq 0, \ \forall [x; \boldsymbol{y}],$$

$$\Leftrightarrow x^2 \phi(D_{ij}) - 2x \|P_j \boldsymbol{y}\| + 2x\boldsymbol{y}^T (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i) + \phi(D_{ij}) \boldsymbol{y}^T \boldsymbol{y} \geq 0, \ \forall [x; \boldsymbol{y}],$$

$$\Leftrightarrow x^2 \phi(D_{ij}) + 2\boldsymbol{y}^T P_j \boldsymbol{\xi} + 2x\boldsymbol{y}^T (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i) + \phi(D_{ij}) \boldsymbol{y}^T \boldsymbol{y} \geq 0, \quad \forall \{(x, \boldsymbol{y}, \boldsymbol{\xi}) : \boldsymbol{\xi}^T \boldsymbol{\xi} \leq x^2\},$$

$$\Leftrightarrow \quad \exists\lambda \quad \geq \quad 0 \quad : \quad \begin{bmatrix} \phi(D_{ij}) - \lambda & (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i)^T & 0 \\ (\boldsymbol{r}_{c,j} - \boldsymbol{B}_i) & \phi(D_{ij}) I_3 & P_j \\ 0 & P_j & \lambda I_3 \end{bmatrix} \quad \succeq \quad 0.$$

The last two equivalences follow from Cauchy-Schwarz inequality and the $\mathcal{S}$-lemma [3], respectively.