OPTIMAL DEMAND RESPONSE SCHEMES FOR RENEWABLE

INTEGRATION USING THERMAL INERTIAL LOADS

A Dissertation

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirement for the degree of

DOCTOR OF PHILOSOPHY

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August 2016

Major Subject: Electrical Engineering

ABSTRACT

We consider a smart microgrid environment where renewable power sources like wind generators are available to service the thermal inertial load along with conventional non-renewable energy sources. The flexibility in power consumption of thermal inertial loads, like air-conditioners can be used towards absorbing the fluctuations in intermittently available renewable power sources. Several optimization schemes can be used towards this goal. We discuss and analyze some of these optimization models. An optimization model which promotes renewable consumption by penalizing non-renewable consumption, but does not account for variations in the load requirements, lead to an optimal solution in which all the loads' temperatures behave in a lockstep fashion. That is, the power is allocated in such a fashion that all the temperatures are brought to a common value and they are kept the same after that time, resulting in synchronization among all the loads. We show that under a model which additionally penalizes the comfort range violation, the optimal solution is in fact of a de-synchronizing nature, where the temperatures are intentionally kept apart to avoid power surges resulting from simultaneous comfort violation across many loads.

In the sequel, we additionally take into account several other factors, such as the privacy requirements from the users of loads, architectural simplicity, and tractability of the solution. We propose a demand response architecture where no information from the end-user is required to be transferred in order to optimally co-ordinate their power consumption. We propose a simple threshold value based policy which is architecturally simple, compu-

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tationally inexpensive, and achieves optimal staggering among loads to smooth the variations in non-renewable power requirements. We show that it is possible to compute the optimal solution in a number of scenarios, and give a heuristic approach to approximate the optimal solution for the scenarios where information such as cooling/heating rates, etc. is not available.

To my parents, and my wife

ACKNOWLEDGMENTS

This thesis would not be possible without the generosity of many people. First and foremost I would like to thank my advisor, Dr. P. R. Kumar for his constant support. The countless hours of argument on both technical and philosophy are and will be the most cherished part of my journey as a PhD student. I am very grateful to him for his patience, the freedom he gave me for letting me pursue my own ideas, and giving me an opportunity to learn from my own mistakes. These last fives years were filled with struggle, anxiousness, fear, and excitements. I would like to thank my advisor for giving me direction and letting me steer through this long journey.

I would also like to thank Dr. Xie, Dr. Hong-Hou, and Dr. Howard for agreeing to be a part of my committee. Dr. Hong-Hou was a great influence to me during the start of my PhD program, his lectures in and discussion with him improved my way of thinking as a researcher. I am very grateful to Dr. Howard, who not only helped me greatly during my masters program in mathematics department, but also guided me on my reseach problem on several occasions. Dr. Xie allowed me to be a part of his team meetings, that helped me immensely in gaining a practical perspective to the power systems in general.

Lastly, I would like to thank my family and my friends who supported me throughout this process. A special thanks to Dr. Haldar, whose positive outlook, unique perspective, technical proficiency, and infectious energy is very uplifting and has deeply influenced my way of thinking.

NOMENCLATURE

TCL	Thermostatically controlled loads, air conditioners.
LSE	Load Serving Entity, the aggregator which co-ordinates the end-user's air conditioners to achieve the desired demand response
$\mathscr{M}(q_0, q_1)$	A binary Markov stochastic process, with transition probability rates q_0 and q_1.
q_0	Rate of transition from state "wind off". $\frac{1}{q_0}$ is the mean time the wind is not available for consumption.
q_1	Rate of transition from state "wind on". $\frac{1}{q_1}$ is the mean time the wind is available for consumption.
q_{ij}	For general wind model the rate of transition from wind state i to wind state j. In state i the loads can cool at rate $i\frac{c}{M}$
$\Theta_{m,i}$	The minimum temperature level of the comfortable range of temperatures specified by i-th end-user.
$\Theta_{M,i}^{(j)}$	The maximum temperature level of the comfortable range of temperatures, the j-th value of the upper temperature specified by i-th end user
$\Theta_m, \Theta_M^{(j)}$	The subscript i is dropped from $\Theta_{m,i}$, and $\Theta_{M,i}^{(j)}$ when loads are homogeneous
r_1	Rate of transition from a state with upper comfort level $\Theta_M^{(0)}$ to a state with upper comfort level $\Theta_M^{(1)}$
r_2	Rate of transition from a state with upper comfort level $\Theta_M^{(1)}$ to a state with upper comfort level $\Theta_M^{(0)}$
r_{ij}	Rate of transition from a state with upper comfort level $\Theta_M^{(i)}$ to a state with upper comfort level $\Theta_M^{(j)}$

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1   INTRODUCTION

Motivated by the higher utilization of renewable energy sources, like wind, there is significant interest in techniques to adjust to the uncertainty in the power generated by renewable sources, such as wind and solar while providing service to the loads. In addition to generation side compensation, like automatic generation control, several demand response schemes are proposed to efficiently support these renewable sources [31], [2], [16], [6], [7], [11].

In this thesis, we propose to exploit thermal inertial loads to preferentially utilize a renewable source such as wind power over non-renewable power sources. Using such a scheme leads to twofold benefits. First, it promotes reduced consumption of costly and polluting non-renewable generation, and second, the controllable thermal loads can absorb temporary fluctuations in renewable sources, thereby helping stabilize the overall power system. Some recent studies indicated that the thermal loads' consumption accounts for over 60% of the total energy consumption for an average consumer [32]. Therefore thermal loads not only provide a substantial opportunity to be used as a controllable smart load, but also are cost effective as they do not require additional infrastructure changes, e.g., adding expensive energy storage units.

The users (or consumers) of thermostatic loads typically specify a desired comfort range, within which they would like their temperature to lie. Of course, any change in comfort range may lead to temporary violation of comfort, making the current tempera-

ture fall outside the comfort range. Except for the short period when comfort range is so changed, the demand response scheme is generally required to respect the imposed temperature limits. This leads to an important design issue of what control policy design can ensure such adherence. Several optimization criteria can be used. We discuss some of these formulations, and the nature of the resulting demand response that arises from using these criteria in Chapter 2.

We consider a system where a large number of distributed loads are supported by a combination of renewable and fossil fuel generators. An issue that arises is that of "resource-pooling". In order to achieve effective demand response by absorbing fluctuating wind power, and to that end, exploiting a large number of loads, we employ an architecture that consist of a central controller. We refer to such a controller as a load serving entity (LSE) (also called "load aggregator" or simply an "aggregator"). For convenience of exposition, we suppose that the thermal loads are all air conditioning loads, and that the renewable energy source is wind.

Under a model where we need to ensure that load temperature stays within the comfort range at all times, during long periods when wind is not available, non-renewable power is required to supplement it in order to avoid overheating. This leads to an important question of how to reduce the variations in the non-renewable power requirement from such a system. Low peak to average ratio is desirable as it leads to reduced reliance on costly operating reserves. We incorporate this goal by employing a quadratic cost term in the optimization that prefers a reduced peak to average ratio.

A system where only non-renewable power is used to cool a collection of thermal loads is studied by Kundu et al. [16]. Their work shows that a collection of thermal loads which consumes the constant power results from a distribution of loads where fraction of heating and cooling loads between two temperatures is proportional respectively to the time taken to heat or cool between the two temperatures. This leads to a population which is evenly spaced in time and consumes a constant power without any variations. The presence of stochastic wind power leads to an interesting complication. When stochastic generation like wind, is a common source available for all the loads, a scenario where every load is cooled together whenever sufficient wind is available, leads to a synchronization issue. That is, the loads behave in a lockstep fashion. This is undesirable, because a sudden simultaneous change in comfort range when wind power is not sufficient will lead to a hugh surge in total non-renewable power consumption to cool the loads. To address this issue we propose a scheme which explicitly models such comfort range variations, and show that under a cost function which penalizes the variations in non-renewable power consumption, it is optimal to stagger the loads. This "symmetry breaking" by staggering hedges against coordinated comfort range change. Mathematically, this interestingly arises from a local concavity of the cost-to-go function in HJB equation.

From a load's perspective, another important issue arises, is that of the privacy of load state information. Individuals do not want to reveal their thermal states to the aggregator. Revealing such information can be linked to other activities of daily routines, working schedules etc [14]. A solution where no instantaneous temperature measurement

of load temperature is ever required to be communicated to the aggregator, is therefore desirable. This leads to an important issue of how an aggregator can influence the "collective behavior" of the ensemble, i.e., their total power consumption, without any specific information about individual temperatures.

There are also issues related to communication system requirements. A system where an aggregator sends a continuous signal to each of the loads is not desirable, as it requires great communication bandwidth, and also raises concerns related to link reliability issues. Thus, rather than seeking a centralized solution where the aggregator controls the cooling of each load, we seek a solution where each load controls its own cooling in a distributed fashion.

Consider a collection of identical air-conditioners, with each of load specifying a comfort range $[\Theta_m, \Theta_M]$, within which its temperature should lie. Suppose renewable generation, say wind, is used to cool all homes whenever it is blowing. Non-renewable power is only used when the temperature hits Θ_M. Under such a scheme, eventually all the loads will turn on and off at the same time. If due to some externality, loads change their Θ_M at the same time, this will result in a huge surge of non-renewable power, since all the loads will start consuming non-renewable power together upon hitting Θ_M. Such "synchronized" behavior is undesirable.

In Chapter 3, we propose and analyze a privacy-respecting policy to stagger the temperatures of the loads within the comfort range by assigning each home a temperature at which it should start non-renewable power consumption. By selecting these thermostat set-

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ting thresholds across the population of loads we can stagger the loads within their comfort range. The distribution of these parameters can be used to control the "collective behavior" of the ensemble of loads. We show that such optimal staggering can be analytically determined when number of homes is large. Moreover, this scheme allows each load's cooling to be controlled individually by its own thermostat, with no information transfer being required from the loads to the aggregator.

2 MODELS FOR OPTIMAL DEMAND RESPONSE, AND DESYNCHRONIZATION

We begin by considering the problem of direct load control. We address the problem of centrally controlling the thermostatically controlled loads (TCLs) in the presence of stochastic renewable sources. For specificity, we assume that the renewable energy source is wind. We will refer to the central controlling agent as a load serving entity (LSE). The LSE serves two functions. First, it estimates and allocates the available renewable power denoted W(t), along with non-renewable power to individual TCLs. Second, the LSE has a separate communication channel to monitor each TCL's state and sends the control actions to all the loads. The LSE can utilize the knowledge of the states of the TCLs as well as the renewable generation process to achieve predefined objectives. From the power distribution grid side the LSE acts as a cumulative load which co-ordinates TCLs efficiently.

The end-users are assumed to allow some flexibility in their load temperatures, specified as a continuous range of acceptable temperatures. We refer to such range as the comfort range. From the grid's perspective end-users can be incentivized to provide more margin to compensate for stochasticity in renewable power generation, and in effect reduce grid power consumption.

For simplicity we suppose that all the TCLs are air-conditioning loads. In our models we make several simplifications to focus on the central problem of interest, which is to understand the nature of the optimal demand response. First, while in practice, residential

thermal loads have a constant cooling rate and consume fixed power, we relax this and assume that loads can be run at fractional capacity. This fractional cooling rate can be approximated by fast switching (chattering) between two cooling rates. Second, we simplify analysis by assuming constant ambient heating h, resulting in the fact that the rate of change in temperature is assumed to be an affine function of power supplied P, resulting in the temperature dynamics of $\dot{x} = h - P$, which we shall hereafter refer to as constant rate dynamics. However the results can be easily extended to any linear dynamics, e.g., to $\dot{x} = ax + h - P$. Here a is the capacity (inertia) parameter of the air-conditioners.

In the sequel we will consider several end-user criteria. One possibility is to measure end-user discomfort by the square of the amount by which the temperature of the load exceeds a user specified maximum temperature Θ_M. This corresponds in an optimization framework to a cost function $[(x - \Theta_M)^+]^2$, where x is the temperature of a TCL and $z^+ :=$ $\max(z, 0)$. In contrast to such soft constraint, another possibility is to impose a hard upper bound constraint Θ_M. Then, in order to keep the temperature below Θ_M, the TCL may have to draw grid power if wind power is not available. In the simple model $\dot{x} = h - P$, if x hits Θ_M, and the wind is not blowing, the TCL will have to draw power of at least h. The comfort setting Θ_M could also be stochastic. In that case, we require that if $x > \Theta_M$, then the TCL must draw maximum grid power $M = (h + c)$, where $c > 0$ is the maximum cooling rate of the air-conditioner. We may also have a minimum temperature constraint Θ_m. Then, when x hits Θ_m, the total of grid and wind power chosen must not be greater than h to prevent over-cooling.

Concerning reducing variations in the power P_n drawn from the non-renewables, we will penalize variations in it by employing a quadratic running cost $\int_0^T P_n^2(t)dt$, where $P_n(t) := \sum_i P_{(n,i)}(t)$ is the sum of the powers drawn by the individual TCLs indexed by i. In place of the quadratic penalty, any other strictly increasing, strictly convex penalty in the total non-renewable power drawn would also penalize variations.

For the wind model, there are several choices [25], [5]. For simplicity, we will focus on a simplified wind process modeled as a two state continuous time Markov chain. States "on" and "off" have mean holding times $\frac{1}{q_1}$ and $\frac{1}{q0}$ respectively. We will refer to this process as a $\mathscr{M}(q_0, q_1)$ process. In the on state the wind can provide W watts, while there is no wind power in the off state.

By a "synchronizing policy" we will mean a policy which reduces the difference in TCLs' temperature states by allocating more cooling power to higher temperature TCL. On the contrary a de-synchronizing policy may occasionally increase the temperature difference by providing power to a lower temperature TCL and thus cooling it, while letting the higher temperature one increase. We note that a de-synchronizing policy may be a state dependent policy that sometimes drives temperatures towards one another, and sometimes apart, i.e., it need not always strive to separate temperatures.

2.1 Models for optimization

Now we describe several alternative models and optimization criteria, and the nature of demand response that results from the combination of both.

2.1.1 Contract violation probability model

One model of a contract with end users is to maintain their temperatures in a specified comfort range. This results in an objective of minimizing the probability of contract violation. Consider N TCLs with temperatures denoted $(x_1, x_2, ..., x_N)$, following the temperature dynamics $\vec{\dot{x}} = \vec{f}(\vec{P})$ as a function of supplied power $\vec{P}$. Suppose the contractual ranges are $\{[\Theta_{m,i}, \Theta_{M,i}]\}_{i=1}^{N}$. Then we obtain the following optimization problem to minimize the contract violation probability:

$$\textbf{Minimize } \sum_{i=1}^{N} \text{Prob}\left(\{x_i \notin [\Theta_{m,i}, \Theta_{M,i}]\}\right)$$

$$\textbf{Subject to } \frac{d\vec{x}}{dt} = \vec{f}(\vec{P}(t)) \tag{2.1}$$

$$\sum_{i=1}^{N} P_i(t) \leq W(t) \tag{2.2}$$

$$P_i(t) \geq 0 \text{ for } i = 1, 2, ..N \tag{2.3}$$

$$W(t) \frown \mathcal{M}(q_0, q_1). \tag{2.4}$$

For this problem, we will show in the sequel that since all loads outside the comfort zone contributes the same amount to cost, the loads which are nearer to the comfort zone are preferred by the optimization policy because they can be brought within the comfort range more quickly. The resulting scheme is not fair across loads, as the loads with maximum discomfort are provided with lesser proportion of the available power.

2.1.2 Variance minimization model

To correct the above bias we propose next a model in which the penalty function is the square of the deviation above Θ_M. The optimization problem is:

$$\textbf{Minimize } \int_0^T \left(\sum_{i=1}^N \mathbb{E}[(x_i(t) - \Theta_{M,i})^+]^2)|\vec{x}(0)] \right)$$

$$\textbf{Subject to } . \frac{d\vec{x}}{dt} = \vec{f}(\vec{P}(t))$$

$$\sum_{i=1}^N P_i(t) \leq W(t)$$

$$P_i(t) \geq 0 \text{ for } i = 1, 2, ..N$$

$$W(t) \frown \mathcal{M}(q_0, q_1).$$

Due to the convex quadratic penalty, keeping the TCLs' temperatures apart from one another costs more than keeping their temperatures the same. The result is that it is in fact optimal to bring all TCLs to the same temperature and maintain them so.

2.1.3 Hard temperature threshold model

One may note that the preceding model does not guarantee any upper bound on the maximum temperature level. Such a hard temperature constraint cannot be met without a reliable supplemental non-renewable power source. Therefore, we introduce a hard temperature upper bound in addition to a soft temperature comfort goal $\theta_{soft,i}$ mentioned earlier. We also allow for but penalize the non-renewable power required to operate within the hard bound $x \in [\Theta_{m,i}, \Theta_{M,i}]$.

This yields the optimization problem:

$$\textbf{Min. } \mathbb{E}\left[\int_0^T \left(\sum_{i=1}^{N}[(x_i(t) - \Theta_{soft,i})^+]^2 + \gamma\left(\sum_{i=1}^{N}P_{n,i}(t)\right)^2 |\vec{x}(0)\right)\right]$$

$$\textbf{Subject to } \frac{d\vec{x}}{dt} = \vec{f}(\vec{P}_w(t) + \vec{P}_n(t)) \tag{2.5}$$

$$x_i(t) \in [\Theta_{m,i}, \Theta_{M,i}] \text{ for } i = 1, 2, ..., N \tag{2.6}$$

$$\sum_{i=1}^{N}P_{w,i}(t) \leq W(t) \tag{2.7}$$

$$P_{w,i}(t) \geq 0, P_{n,i}(t) \geq 0 \text{ for } i = 1, 2, ...N \tag{2.8}$$

$$W(t) \backsim \mathscr{M}(q_0, q_1).$$

In this model, at first sight one may speculate that keeping the TCLs apart hedges against their hitting the maximum temperature at the same time, thereby reducing the term in the cost function that is quadratic in the total power. However, it turns out that driving the TCLs to the same temperature and then maintaining them at equal temperature continues to be optimal, as we show in Section 2.3. We will thereby conclude that this model does not result in the desired behavior of desynchronization.

2.1.4 Stochastic threshold variation model

We now introduce the additional feature that there could be environmental, social or other extraneous events due to which the loads may change the set-point in a coordinated fashion. To capture this effect we will assume that there are two levels $\Theta_M^{(0)}$ and $\Theta_M^{(1)}$, where $\Theta_M^{(0)} < \Theta_M^{(1)}$. All the TCLs switch between these levels at the same time instants according to a Markov process $\Theta(t) \backsim \mathscr{M}(r_1, r_2)$ with mean holding times $\frac{1}{r_1}$ and $\frac{1}{r_2}$ in the two states $\Theta_M^{(1)}$ and $\Theta_M^{(1)}$. Due to a sudden reduction in the set-point, a TCL that was previously

within the desired temperature range $[\Theta_m, \Theta_M^{(1)}]$ may suddenly be at a higher temperature than $\Theta_M^{(0)}$. When a TCL thereby violates the threshold constraint we will require that it be provided grid power at the maximum possible level M that the TCL can sustain, to cool it quickly:

$$\textbf{Min.} \quad \int_0^T \mathbb{E}\left[\left(\sum_{i=1}^N P_{n,i}(t)\right)^2 |\vec{x}(0)\right]$$

$$\textbf{s.t.} \quad (2.5, 2.7, 2.8) \text{ and}$$

$$x_i(t) \in [\Theta_m, \Theta_M^{(1)}] \text{ for } i = 1, 2, ..., N$$

$$P_{n,i} = M \ \{ \text{ if } x_i(t) > \Theta(t) \}$$

$$W(t) \backsim \mathcal{M}(q_0, q_1), Th(t) \backsim \mathcal{M}(r_1, r_2).$$

We will show that with this model the optimal power allocation results in de-synchronization of the TCL temperature states. The optimal policy does not maintain all the TCLs at the same state. When there are TCLs above a certain level it is optimal to keep their temperatures different, to hedge against the future eventuality that the thermostats are switched down to $\Theta_M^{(0)}$.

2.2 Synchronization and desynchronization properties of the optimal solutions

For simplicity, we will consider a homogeneous population of TCLs, with identical temperature ranges, dynamics, heating, and comfort range variations. We illustrate all results, for the $N = 2$ case. The results can be generalized to

Theorem 1. *The optimal response in the contract violation probability model with*

$\mathscr{M}(q_0, q_1)$ *wind process and linear dynamics is to provide all the wind power to the coolest TCL that is outside the temperature range. Therefore the optimal policy is de-synchronizing.*

Theorem 2. *Under the variance minimization model and $\mathscr{M}(q_0, q_1)$ wind process, the optimal control policy is of synchronizing nature, for any linear dynamics.*

Theorem 3. *Under the hard temperature threshold model with $\mathscr{M}(q_0, q_1)$ wind process and constant dynamics the optimal policy is of synchronizing nature.*

In the sequel we illustrate the *de-synchronizing* nature of the optimal policy for the *stochastic threshold variation* model with $\mathscr{M}(r_1, r_2)$ as the upper comfort threshold process, by numerically computing its solution.

2.3 Proofs of results

Proof of Theorem 1: For *N=2* and a wind power realization $W(t, \omega), \omega \in \Omega$, and, for initial state $\vec{x}(0) = (x_1(0), x_2(0))$ with $\Theta_M < x_1(0) < x_2(0)$, let the optimal cost be $C^*_\omega(\vec{x}(0))$, and the resulting state $\overset{\star}{x}(t, \omega)$. Now consider a policy $\hat{\Pi}$, with resulting cost $\hat{C}_\omega(\vec{x}(0))$, which gives all the power to TCL1 while $x_1 > \Theta_M$, and subsequently to TCL2 while maintaining TCL1 at Θ_M, and denote the resulting state by $\hat{x}(t, \omega)$. Due to linear dynamics we have $\overset{\star}{x}_1(t, \omega) + \overset{\star}{x}_2(t, \omega) = \hat{x}_1(t, \omega) + \hat{x}_2(t, \omega)$, therefore the event $\{\hat{x}_1, \hat{x}_2 > \Theta_M\} \subset \{\overset{\star}{x}_1, \overset{\star}{x}_2 > \Theta_M\}$. Also the probability of the event $\{\hat{x}_i > \Theta_M, \hat{x}_{!i} \leq \Theta_M\}$, where $!i$

denotes the TCL other than TCL i, is minimized by $\hat{\Pi}$ we have,

$$C_\omega^*(\vec{x}(0)) = \int_{\{\overset{\star}{x}_i > \Theta_M, \overset{\star}{x}_{!i} \leq \Theta_M\}} 1dt + \int_{\{\overset{\star}{x}_1, \overset{\star}{x}_2 > \Theta_M\}} 2dt$$

$$\geq \int_{\{\hat{x}_i > \Theta_M, \hat{x}_{!i} \leq \Theta_M\}} 1dt + \int_{\{\hat{x}_1, \hat{x}_2 > \Theta_M\}} 2dt = \hat{C}_\omega(\vec{x}(0)).$$

Taking the expectation yields the desired result. $\qquad\qquad\qquad$ □.

2.3.1 HJB equation for variance minimization model

Let the optimal cost-to-go from state (x_1, x_2) and wind condition i ($i{=}1$ in *on* state and $i{=}0$ on *off* state) be $V_i^*(\vec{x}, t)$. The state-space is $\mathscr{S} = [0, \infty) \times [0, \infty) \times \{0, 1\}$. We denote by $\mathscr{A} \subset \mathbb{R}_+^2$ be the set of state-dependent *admissible* actions

$$\mathscr{A}(\vec{x}, i) = \{(P_1, P_2) : P_1 + P_2 \leq W \text{ if (wind } i = 1),$$

$$P_1 = P_2 = 0 \text{ if (wind } i = 0),$$

$$\text{if}(x_j(t) = 0) \text{ then } f(P_j) \geq 0\}.$$

The Hamilton-Jacobi-Bellman equation is:

$$\inf_{(P_1, P_2) \in \mathscr{A}(\vec{x}, i)} \{\mathbf{1}^T[(\vec{x} - \Theta_M)^+]^2 + \nabla^T V_i^*(\vec{x}, t)\vec{f}(\vec{P}) - q_i V_i^*(\vec{x}, t)$$

$$+ q_i V_{!i}^*(\vec{x}, t)\} + \frac{\partial V_i^*}{\partial t}(\vec{x}, t) = 0 \quad \text{For } i = 0, 1. \tag{2.9}$$

Equation (2.9) involves the following minimization problem:

$$\inf_{(P_1, P_2) \in \mathscr{A}(\vec{x}, i)} \left\{ \frac{\partial V_i^*}{\partial x_1}(\vec{x}, t)f(P_1) + \frac{\partial V_i^*}{\partial x_2}(\vec{x}, t)f(P_2) \right\}. \tag{2.10}$$

Since $f(P)$ is linear, the infimum above is achieved for either $P_1 = W$ or $P_2 = W$, depending on whether $\frac{\partial V_i^*}{\partial x_1} > \frac{\partial V_i^*}{\partial x_2}$ or $\frac{\partial V_i^*}{\partial x_1} < \frac{\partial V_i^*}{\partial x_2}$. When $\frac{\partial V_i^*}{\partial x_1} = \frac{\partial V_i^*}{\partial x_2}$, any allocation $\{(P_1, P_2) : P_1 + P_2 = W\}$ is optimal.

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Lemma 1. *The optimal cost-to-go function $V^*(\vec{x}, t)$ is component-wise non-decreasing in $\vec{x}$.*

Proof. For a realization of wind power $W(t, \omega)$, where ω denotes a sample point. Let the optimal state trajectory from initial state $(x_1(0), x_2(0))$ be $(x_1(t, \omega), x_2(t, \omega))$. From initial state $y_1(0) < x_1(0)$, we provide the same power as before, except at times when temperature goes below Θ_m to obtain trajectory $(\hat{y}_1(t, \omega), x_2(t, \omega))$. Here $\hat{y}_1(t, \omega) = max\{\Theta_m, y_1(0) - x_1(0) + x_1(t, \omega)\}$. We have,

$$V_\omega^*(\vec{x}(0)) \geq \int_0^T [(\hat{y}_1(t, \omega) - \Theta_M)^+]^2 + [(x_2(t, \omega) - \Theta_M)^+]^2 dt$$

$$= \hat{V}_\omega(x_1(0), x_2(0)) \geq V_\omega^*(x_1(0), x_2(0)).$$

Taking the expectation yields the desired result. □

Lemma 2. *If $f(P)$ is a linear, then $V^*(\vec{x}, t)$ is convex in $\vec{x}$.*

Proof. For a wind realization $W(t, \omega), \omega \in \Omega$, and a pair of initial conditions $\vec{x}^{(0)}(0)$ and $\vec{x}^{(1)}(0)$, we take a convex combination of the states $\vec{x}^{(\alpha)}(0) := (1 - \alpha)\vec{x}^{(0)}(0) + \alpha\vec{x}^{(1)}(0)$ and provide the convex combination of the optimal power supplied from the former initial states, i.e., $\vec{P}^{(\alpha)} = (1 - \alpha)\vec{P}^{(0)} + \alpha\vec{P}^{(1)}$. Due to linearity we obtain the trajectory $\vec{x}^{(\alpha)}(t, \omega) = (1 - \alpha)\vec{x}^{(0)}(t, \omega) + \alpha\vec{x}^{(1)}(t, \omega)$. Since $\vec{x}^{(0)}(t, \omega), \vec{x}^{(1)}(t, \omega) \in [\Theta_m, \Theta_M]$, it follows $\vec{x}^{(\alpha)}(t, \omega) \in [\Theta_m, \Theta_M]$. Letting $V^\alpha(x^{(\alpha)}(0))$ be the cost for this allocation, we obtain,

$$(1 - \alpha)V_\omega^*(\vec{x}^{(0)}(0)) + \alpha V_\omega^*(\vec{x}^{(1)}(0))$$

$$= \int_0^T (1 - \alpha)\mathbf{1}^T (\vec{x}^{(0)}(t) - \Theta_M)^{+2} + \alpha\mathbf{1}^T (\vec{x}^{(1)}(t) - \Theta_M)^{+2} dt$$

$$\geq \int_0^T \mathbf{1}^T [(\vec{x}^{(\alpha)}(t)) - \Theta_M)^+]^2 dt = V_\omega^\alpha(\vec{x}^{(\alpha)}(0)) \geq V_\omega^*(\vec{x}^{(\alpha)}(0)).$$

Taking the expectation proves the desired result. □

Proof of Theorem 2: Notice that Lemma (1) implies that $\frac{\partial V^*}{\partial x_i} \geq 0$ for i=1,2. Equation (2.10) thus implies

$$(P_1^*, P_2^*) = \begin{cases} (W,0) & \text{if } \frac{\partial V_1^*}{\partial x_1} > \frac{\partial V_1^*}{\partial x_2} \\ \\ (0,W) & \text{if } \frac{\partial V_1^*}{\partial x_1} < \frac{\partial V_1^*}{\partial x_2} \end{cases}.$$

Also, since $V_i^*(\vec{x},t)$ is convex in $\vec{x}$, we have $\frac{\partial V_i^*(x_1,x_2)}{\partial x_i}$ increases with x_i for $i = 1,2$. So if (x_1,x_2) is a temperature state with $x_1 > x_2$, then $\frac{\partial V_i^*(\vec{x})}{\partial x_1} > \frac{\partial V_i^*(\vec{x})}{\partial x_2}$, so the minimizer in the HJB equation is $P_1^* = W, P_2^* = 0$. Therefore x_1 decreases while x_2 increases till $x_1 = x_2$. So we conclude that the optimal policy is synchronizing in nature. □

2.3.2   HJB equation for hard threshold model

Let $V_i^*(\vec{x}(0))$ be the optimal cost-to-go from $\vec{x}(0)$ when wind state is i (for $i = 1$ when wind is *on*, $i = 0$ when wind is *off*). The control variables are $\vec{P_g}$ and $\vec{P_w}$, the allocated grid and wind power vectors to the TCLs. The state-space is $\mathscr{S} = [\Theta_m,\Theta_M] \times [\Theta_m,\Theta_M] \times \{0,1\}$. Let $\widehat{\mathscr{A}}(\vec{x},i) \subset \mathbb{R}_+^4$ be the set of state dependent *admissible* actions.

$$\widehat{\mathscr{A}}(\vec{x},i) = \{(\vec{P_n},\vec{P_w}) : P_{w,1} + P_{w,2} \leq W \text{ if (Wind } i\text{=}1),$$

$$\text{if (Wind } i\text{=}0) \; P_{w,1} = P_{w,2} = 0,$$

$$\text{if } (x_j(t) = 0) \text{ then } f(P_{n,j} + P_{w,j}) \geq 0 \text{ for } j = 1,2,$$

$$\text{if } (x_j(t) = \Theta_M) \text{ then } f(P_{n,j} + P_{w,j}) \leq 0 \text{ for } j = 1,2\}.$$

The Hamilton-Jacobi-Bellman equation is,

$$\inf_{(\vec{P}_n,\vec{P}_w)\in\widehat{\mathscr{A}}(\vec{x},i)} ((\mathbf{1}^T\vec{P}_n)^2 + \nabla^T V_i^*(\vec{x},t)\vec{f}(\vec{P}_n+\vec{P}_w)) - q_iV_i^*(\vec{x},t)$$

$$+ q_iV_{!i}^*(\vec{x},t)\} + \frac{\partial V_i^*}{\partial t}(\vec{x},t) = 0 \quad \text{For } i = 0,1. \tag{2.11}$$

For homogeneous loads with the state dynamics $f(P) = S - P$ (here $S = $ heating due to ambient temperature when $P = 0$), the infimum term for control separates. So for $x_i \in (0,\Theta_M)$, the optimal wind and grid power allocations are given by

$$\vec{P}_n^*(\vec{x},i) = \underset{\vec{P}_n\geq 0}{\arg\inf}\left((P_{n,1}+P_{n,2})^2 - \frac{\partial V_i^*}{\partial x_1}P_{n,1} - \frac{\partial V_i^*}{\partial x_2}P_{n,2}\right) \tag{2.12}$$

$$\vec{P}_w^*(\vec{x}) = \underset{(P_{w,1}+P_{w,2}=W)}{\arg\inf}\left(-\frac{\partial V_1^*}{\partial x_1}P_{w,1} - \frac{\partial V_1^*}{\partial x_2}P_{w,2}\right). \tag{2.13}$$

Lemma 3. *The optimal cost-to-go function $V^*(\vec{x},t)$ is component wise non-decreasing in* $\vec{x}$.

Proof. We use the same construction as Lemma 1. Letting $(\hat{P}_1(t,\omega),P_2(t,\omega))$ be the power used for trajectory $(\hat{y}_1(t,\omega),x_2(t,\omega))$, we have

$$V_\omega^*(x_1(0),x_2(0)) = \int_0^T (\overset{\star}{P}_{n,1}+\overset{\star}{P}_{n,2})^2 ds$$

$$\geq \hat{V}_\omega(y_1(0),x_2(0)) \geq V_w^*(y_1(0),x_2(0)).$$

Taking expectations yields the result. $\qquad\square$

Lemma 4. $V^*(\vec{x},t)$ *is convex in* $\vec{x}$.

Proof. We use the same construction as Lemma 2. Noting that $\vec{x}^{(\alpha)}(t,\omega) \in [\Theta_m,\Theta_M]$ and

17

$(\vec{P}_n^{(\alpha)}, \vec{P}_w^{(\alpha)}) \in \widehat{\mathscr{A}}(\vec{x}, i)$, we have

$$(1-\alpha)V_\omega^*(\vec{x}^{(0)}) + \alpha V_\omega^*(\vec{x}^{(1)})$$

$$= (1-\alpha)\int_0^T (\mathbf{1}^T \vec{P}_n^{(0)}(t,\omega))^2 + \alpha \int_0^T (\mathbf{1}^T \vec{P}_n^{(1)}(t,\omega))^2 ds$$

$$\geq \int_0^T (\mathbf{1}^T (1-\alpha)\vec{P}_n^{(0)}(t,\omega) + \alpha \vec{P}_n^{(0)}(t,\omega)))^2 ds$$

$$= V_\omega^{(\alpha)}(x^{(\alpha)}(0)) \geq V_\omega^*(x^{(\alpha)}(0)).$$

taking expectation establishes the convexity of $V^*(\vec{x})$. $\qquad\square$

Proof of Theorem 3: Equations (2.12),(2.13) specify the optimal power allocation. Lemma (3) proves that $\frac{\partial V_i^*}{\partial x_j} \geq 0$, and from Lemma (4), $\frac{\partial V^*}{\partial x_j}(\vec{x})$ is a increasing function of x_j. Therefore when $x_1 < x_2$, $\frac{\partial V^*}{\partial x_1} \leq \frac{\partial V^*}{\partial x_2}$. Using this, the minimizers in (2.12)-(2.13) are as follows

$$(P_{w,1}^*, P_{w,2}^*) = \begin{cases} (W,0) & \text{if } \frac{\partial V_1^*}{\partial x_1} > \frac{\partial V_1^*}{\partial x_2}. \\ \\ (0,W) & \text{if } \frac{\partial V_1^*}{\partial x_1} < \frac{\partial V_1^*}{\partial x_2}. \end{cases} \qquad (2.14)$$

$$(P_{n,1}^*(\vec{x},i), P_{n,2}^*(\vec{x},i)) = \begin{cases} (\frac{1}{2}\frac{\partial V_i^*}{\partial x_1}(\vec{x}),0) & \text{if } \frac{\partial V_i^*}{\partial x_1} > \frac{\partial V_i^*}{\partial x_2} \\ \\ (0,\frac{1}{2}\frac{\partial V_i^*}{\partial x_2}(\vec{x})) & \text{if } \frac{\partial V_i^*}{\partial x_1} < \frac{\partial V_i^*}{\partial x_2} \qquad (2.15) \\ \\ (\frac{1}{2}\frac{\partial V_i^*}{\partial x_1}(\vec{x}), \frac{1}{2}\frac{\partial V_i^*}{\partial x_2}(\vec{x})) & \text{if } \frac{\partial V_i^*}{\partial x_1} = \frac{\partial V_i^*}{\partial x_2}. \end{cases}$$

Where $P_{n,j}^*(\vec{x},i)$ is the non-renewable power given to load j when wind is in state i ($i = 0$ and 1 meaning wind is *off* and *on* respectively). Thus we see that the optimal grid power is allocated so that only the higher temperature TCL is cooled, until the lower temperature TCL increases to the its temperature. Thereafter, both the TCLs are cooled at the same

rate. Therefore we conclude the optimal policy is of *synchronizing* nature. □

2.3.3   De-synchronized response under stochastic user preferences

The HJB equation in this case is the same as in the hard threshold model, but the admissible actions need to be modified when the TCLs are above the thermostat set-point:

$$\widetilde{\mathscr{A}}(\vec{x}, i, \Theta) = \{(\vec{P}_n, \vec{P}_w) \in \widehat{\mathscr{A}}(\vec{x}, i) :$$

$$\text{if } (x_j(t)) > \Theta_M^{(0)} \text{ and } \Theta(t) = \blacksquare_M^{(0)}) \text{ then } P_{n,j} = M\}.$$

Lemma 5. *If the optimal cost-to-go function is concave in the interval* $[a, b] \times [a, b] \subset [0, \Theta_M] \times [0, \Theta_M]$, *then the optimal policy is de-synchronizing in nature in* $[a, b] \times [a, b]$.

Proof. Notice that the HJB equations are the same as (2.11), with $\widehat{\mathscr{A}}$ replaced by $\widetilde{\mathscr{A}}$. When $\Theta(t)$ is $\Theta_M^{(0)}$ all the TCLs having temperature above $\Theta_M^{(0)}$ need to be necessarily provided grid power M. This is neither *synchronizing* nor *de-synchronizing*. Otherwise, the solution is given by equations (2.14) and (2.15).Now we observe that when the optimal cost-to-go is locally concave $\frac{\partial V_i^*(\vec{x})}{\partial x_1} \geq \frac{\partial V_i^*(\vec{x})}{\partial x_2}$, when $x_1 < x_2$, thus both grid and wind power are allocated to TCL1, thereby making it even cooler, while at the same time TCL2's higher temperature is rising even higher. Therefore the actions taken are *de-synchronizing*. □

2.4   Numerical computation of solution

In this section we numerically compute the solution of HJB equation for the *stochastic user preference model*, and exhibit the local concavity of the optimal cost-to-go function. Since the HJB equations (2.11) are a system of non-linear partial differential equations, an analytical solution is difficult to obtain. To compute the exact solution we use
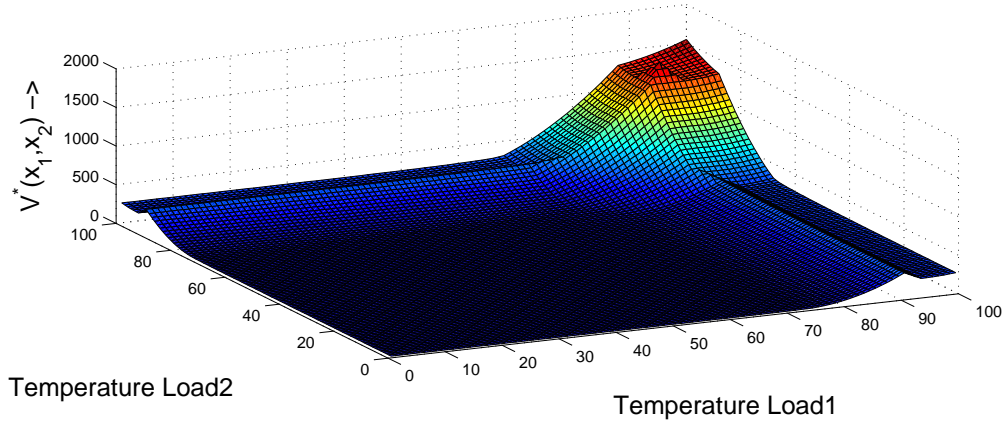
Figure 2.1: The optimal cost-to-go function is concave in the higher temperature region, but is convex in the lower temperature region.

the value-iteration method [4]. To that end, we discretize both the temperature levels and time. We also convert the continuous time Makrov processes for wind and thermostat settings into two-state discrete time Markov chains. To obtain a state space which is the set of integers, we use integer values for M and S, and optimize with respect to $\vec{P}_n, \vec{P}_w$ in the integer action space. This results in the following infinite time discounted cost problem with discount factor $\beta \in (0,1)$:

$$\text{Minimize } \sum_{t=0}^{\infty} \beta^t \mathbb{E}[(\mathbf{1}^T \vec{P}_n[t])^2]$$

$$\text{Subject to. } x_i[t+1] = x_i[t] + S - P_{n,i} - P_{w,i} \text{ for } i = 1,2$$

$$x_1, x_2 \in \{0, 1, 2..., \Theta_M\}$$

$$P_{w,1} + P_{w,2} = W[t]$$

$$W[t] \backsim \mathscr{M}[q_0, q_1], P_{g,i} \geq 0, P_{w,i} \geq 0 \text{ for } i = 1,2.$$

We use the value iteration algorithm to obtain $V_i[\vec{x}]$ for $i = 1, 2$. Figure 2.1 illustrates the local concavity in optimal cost-to-go function computed numerically. Figure 2.2 exhibits a simulation under the optimal policy, for the initial conditions $x_1(0) = x_2(0) = 95$. It shows that the optimal policy de-synchronizes the two TCLs when either of the TCLs is at high temperature, while it attempts to synchronize them when they are at low temperatures.
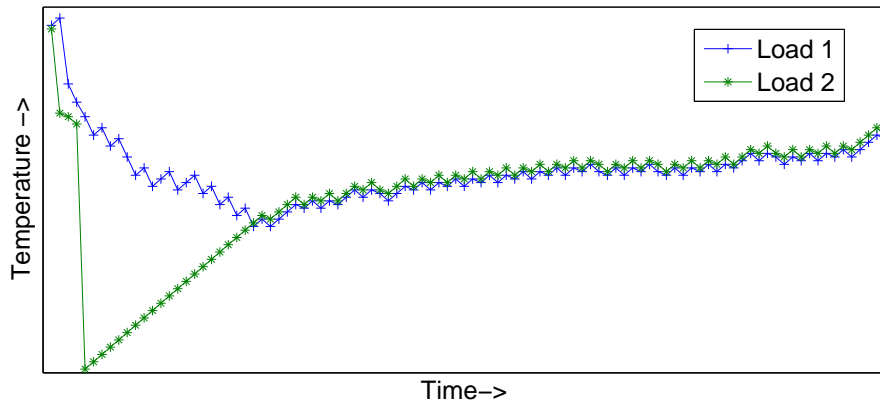

Figure 2.2: Simulation under the optimal policy. Note that when either of the TCLs is above a threshold temperature, the TCLs are separated apart, while at low temperatures, the TCLs are brought to the same temperature.

2.5 Heuristic approximation of optimal solution

We now propose a simple heuristic approximation of the optimal policy which attempts to capture its main characteristics. To do so, it is useful to plot the vector field of rate of change of the temperature state vector. Figure 2.3 indicates a threshold above
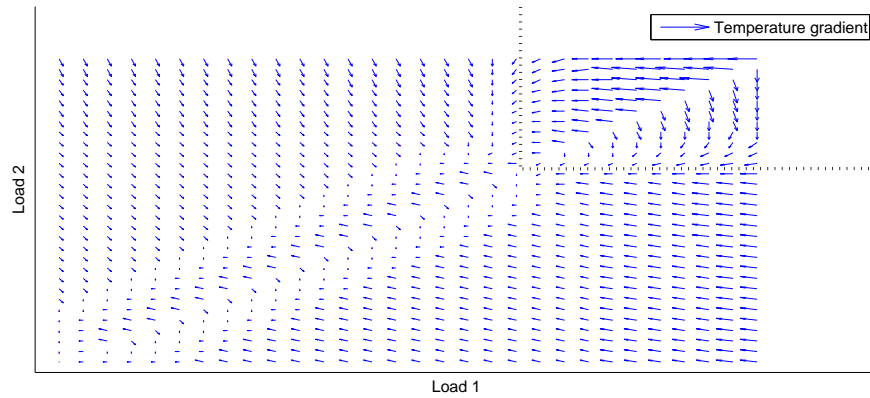
Figure 2.3: Vector field of the rate of change of temperature states under the optimal policy when wind is *off* and thermostat is *high*. At higher temperatures the TCLs are separated, while at low temperature they are brought closer together.

which the TCLs are de-synchronized; let us denote such a level by $\tau_h^{(i,j)}$ where i, j denote the wind condition and thermostat setting respectively.
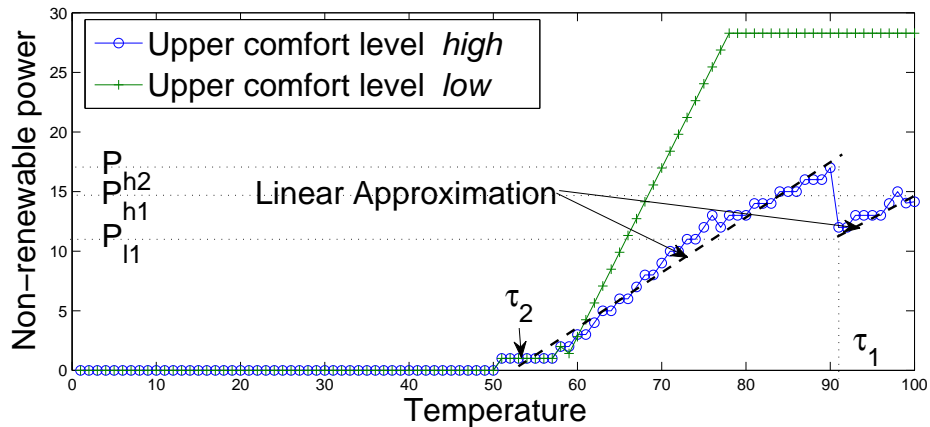

Figure 2.4: Magnitude of the total non-renewable drawn by $N = 2$ TCLs when wind is *on*, under the numerically computed optimal policy, when $x_1 = x_2$. Note that it can be linearly approximated by parameters $(P_{h1}, P_{h2}, P_{l1}, \tau_1, \tau_2)$.

Similarly we can approximate the magnitude of the grid power allocated by two linear splines, one above $\tau^{(i,j)}$ and one below, as shown in Figure 2.4. Therefore the overall heuristic policy has five parameters $(\tau_1, \tau_2, P_{h1}, P_{l1}, P_{h2})$ per state. Figure 2.5 compares the grid power drawn by the heuristic policy $\hat{\Pi}$, the numerically computed optimal synchronizing policy Π_{sync}, and a passive bang-bang policy Π_{bang} for which the TCLs are turned on and off when they hit Θ_m and Θ_M respectively. One can note that the variations in power trajectory are lesser under the heuristic policy than under the synchronized policy.
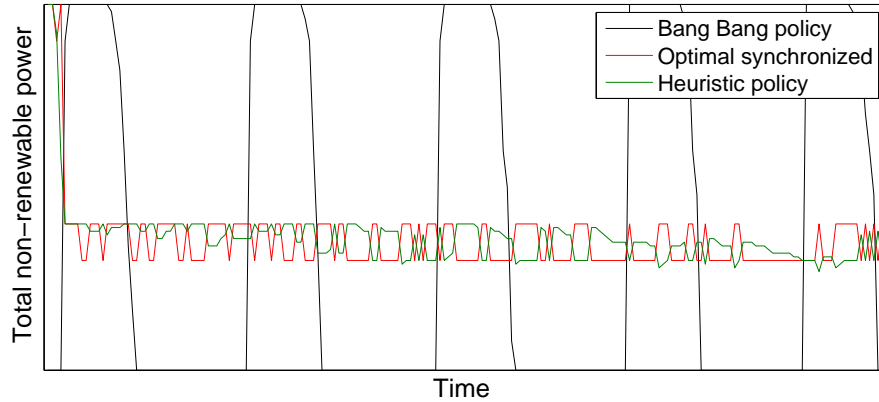

Figure 2.5: Comparison of non-renewable power drawn for different policies for $N = 10$, $\Theta_M^{(1)} = 100, \Theta_M^{(0)} = 70$. The heuristic de-synchronizing policy is smoother than the synchronizing policy.

3 PRIVACY RESPECTING ARCHITECTURE FOR OPTIMAL DEMAND RESPONSE

There are several issues with the policy proposed in the previous chapter. First for a large number of houses it becomes difficult to solve the optimization problem exactly. The HJB Equation 2.11 is nonlinear, it therefore can not be solved directly. Also when using numerical methods the complexity increases exponentially with the number of loads. Second, since the control action is a function of temperature, the loads need to send their temperature information back to the aggregator. This is both expensive with respect to communication requirements, and is intrusive due to the revelation of its state variable to the aggregator.

Motivated by these shortcomings, in this section we will introduce a threshold based set-point policy. We will illustrate that the optimal solution for this set-point policy is easy to compute. In-fact, under some conditions the policy turns out to be an explicit function when the population size becomes infinite.

The aggregator under the threshold policy provides a "wind availability" signal to the loads, which can also be regarded as akin to a "price signal". The loads choose their own comfort level set-points. Under this threshold policy, each load i, uses the wind upon availability to cool itself, unless it is already at the lowest temperature allowed in its comfort range.

When wind is not available the temperature of the loads rises. Consider a collec-

tion of identical thermal inertial loads, each of which under a simple model follows the following constant rate temperature dynamics,

$$\dot{x}_i = h - P$$

when load is supplied with power P. Here h denotes the ambient heating effect. We further assume the maximum cooling power is $h + c$, resulting in maximum cooling rate c.

Further, we assume the identical loads have a common desired comfort range $[\Theta_m, \Theta_M(t)]$, within which they would like their temperature to lie. To prevent the sudden demand increase resulting from a simultaneous comfort range, we assume that the upper comfort setting is variable. Here we observe, that whenever this comfort setting change happens, temperatures of some loads may exceed their upper comfort level. Under such a situation, we use the non-renewable power to cool at a maximum cooling rate c, to restore the user-provided comfort range specifications at the earliest.

We illustrate the threshold policy with a simple case. We assume that the upper comfort level $\Theta_M(t)$ is a piecewise constant Markov process with two levels $\Theta_M^{(0)}$ and $\Theta_M^{(1)}$ with mean holding times $\frac{1}{r_0}$ and $\frac{1}{r_1}$, respectively. Further, let wind also be a two-state Markov process with states "Blowing" and "Not Blowing", with respective mean holding times $\frac{1}{q_1}$ and $\frac{1}{q_0}$.

When wind is in "Blowing" state, each load uses wind to cool at maximum rate, unless it is already in its lowest possible state, i.e. at $x = \Theta_m$. At $x = \Theta_m$, loads use just

enough wind power to prevent over-cooling. Therefore we have

$$\dot{x}(t) = \begin{cases} -c & \text{When } x > \Theta_m \text{ and wind is blowing} \\ \\ 0 & \text{When } x = \Theta_m \text{ and wind is blowing} \end{cases}$$

When wind is in "Not Blowing" state, and the load is inside the comfort range $(\Theta_m, \Theta_M(t))$, loads heat at the rate h when they don't use non-renewable power. The loads use the non-renewable power whenever the loads are outside the comfort-range $[\Theta_m, \Theta_M(t)]$, and cool themselves at the maximum rate c in order to return to comfort range quickly. In addition, in a threshold policy, each load i is assigned with a threshold temperature Z_i. Each load is allowed to heat till it hits either its upper comfort level $\Theta_M(t)$, or the threshold temperature level Z_i. Upon hitting either, the loads draw just enough non-renewable power to maintain themselves at that temperature. This is shown in Figure 3.1.
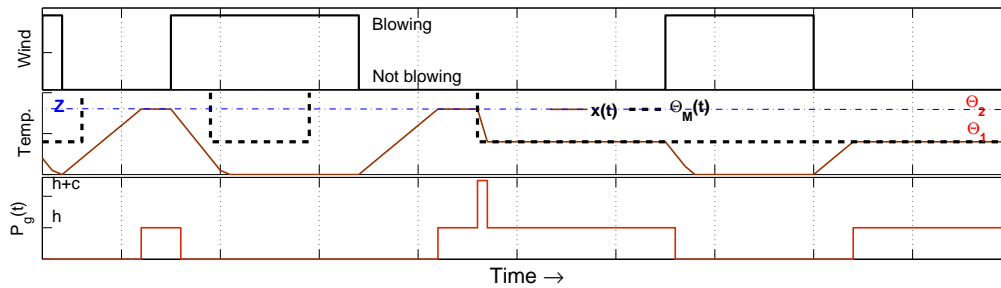

Figure 3.1: The wind process, the temperature, and non-renewable generation drawn. Wind power is used whenever it is available to stay in the comfort range. Non-renewable power is used, if it must, to maintain the upper limit (or Z-value) to prevent comfort violation.

Therefore the temperature follows

$$
\dot{x}_i(t) = \begin{cases} 0 & \text{if } x_i(t) = \min(Z_i, \Theta_M(t)) \ \& \text{ wind not blowing} \\ h & \text{if } x_i(t) < \min(Z_i, \Theta_M(t)) \ \& \text{ wind not blowing} \\ -c & \text{if } x_i(t) > \Theta_M(t) \text{ unconditionally} \end{cases}
$$

Let $P_w(t)$ and $P_n(t)$ denote the wind and non-renewable power. For the policy described above, we have

$$
P_w(t) = \begin{cases} h+c & \text{when } x(t) > \Theta_m \\ h & \text{when } x(t) = \Theta_m \end{cases},
$$

and,

$$
P_n(t) = \begin{cases} h+c & \text{When } x(t) > \Theta_M(t) \text{ and wind not blowing} \\ h & \text{When } x(t) = \min(Z, \Theta_M(t)) \end{cases}
$$

For a collection of N loads, with the i-th load using the threshold Z_i, where $\Theta_m \leq Z_i \leq \Theta_M^{(1)}$, let $\mathbf{Z} = (Z_1, Z_2, ..., Z_N)$, and let us call the overall threshold policy as "$\mathbf{Z}$-policy". Let $x_i(t)$ be the temperature of load-i at time t, and $P_{n,i}(t)$ be the non-renewable power it draws. When the user changes its comfort level, it may occasionally violate the temperature range constraint. We penalize such violations by $[(x_i(t) - \Theta_M(t))^+]^2$ ($x^+ := \max(x, 0)$), and consider the average cost of such discomfort $\lim_{T \to \infty} \frac{1}{T} \int_0^T \sum_{i=1}^N [(x_i(t) - \Theta_M(t))^+]^2 dt$.

We also require the total non-renewable power drawn $\sum_{i=1}^N P_{n,i}(t)$ to be as constant as possible. We penalize the variations in total non-renewable power drawn by imposing a cost quadratic cost function $\lim_{T \to \infty} \frac{1}{T} \int_0^T (\sum_{i=1}^N P_{n,i}(t))^2$.

We therefore consider the problem of optimally selecting $\mathbf{Z} = (Z_1, Z_2, ..., Z_N)$ to op-

timize the overall cost

$$\lim_{T \to \infty} \int_0^T \frac{1}{T} \{ (\sum_{i=1}^N P_{n,i}(t))^2 + \gamma^{(N)} \sum_{i=1}^N [(x_i(t) - \Theta_M(t))^+]^2 dt \},$$

where, $\gamma^{(N)}$ trades off discomfort against less variations in total non-renewable power consumption.

Although we have considered a square cost in both power consumption and discomfort, any other cost function of total non-renewable power $C_{power}(\sum_{i=1}^N P_{n,i})$, and discomfort to the customer $C_{discomfort}(x_i(t))$ would invoke a similar analysis.

3.1 The probability distribution of load temperatures under a threshold policy

In order to evaluate the long-term average terms in the cost function, as a first step we need to evaluate the probability distribution under a Z-policy. For two wind levels ("Blowing" and "Not blowing"), and two upper comfort limits $(\Theta_M^{(0)}, \Theta_M^{(1)})$, let $p_{ij}^z(x)$ denote the probability density functions, after employing the Z-policy for $x \in (\Theta_m, \Theta_M^{(0)}) \cup (\Theta_M^{(0)}, \Theta_M^{(1)})$, where $i = 0, 1$ denotes wind "Not blowing", and "Blowing" respectively, and $j = 0, 1$ denotes Θ_M equal to $\Theta_M^{(0)}$, and $\Theta_M^{(1)}$ respectively. Since, under the threshold policy, the load will remain at $\Theta_m, \Theta_M^{(0)}, z$ each for a non-zero fraction of time, there will be a probability mass at each of these temperatures. Denote by $\delta_{\Theta_m,j}^z$, the probability mass at $x = \Theta_m$ and comfort setting j as described above. Also let $\delta_{\Theta_M^{(0)}}^z$, and δ_z^z denote the probability mass function at temperatures $\Theta_M^{(0)}$ and z respectively. One may notice that each of the four probability mass functions $\delta_{\Theta_m,0}^z, \delta_{\Theta_m,1}^z, \delta_{\Theta_M^{(0)}}^z$, and δ_z^z, are defined such that they can only be associated with a unique wind and comfort level settings, and as a re-

sult of this it is easier to evaluate the relation between density function p_{ij} and probability mass at the boundary. One can also notice that the total probability mass at $x = \Theta_m$ is

$$\delta^z_{\Theta_m} = \delta^z_{\Theta_m,0} + \delta^z_{\Theta_m,1}.$$

Lemma 6. *The density functions* $p^z(x) := [p^z_{00}, p^z_{01}, p^z_{10}, p^z_{11}]$, *and probability mass* $\delta^z_{\Theta_m,0}, \delta^z_{\Theta_m,1}, \delta^z_{\Theta^{(0)}_M}$, *and* $\delta^z_{\bar{z}}$ *under a z-policy for binary state wind and binary comfort settings, are given by the following linear system;*

$$D(x)\frac{d}{dx}p^z(x) = Qp^z(x) \tag{3.1}$$

$$Q_1\delta^z_{\Theta^{(0)}_M} = D(\Theta^{(0)}_M-)p^z(\Theta^{(0)}_M-) - D(\Theta^{(0)}_M+)p^z(\Theta^{(0)}_M+) \tag{3.2}$$

$$Q_2\delta^z_{\bar{z}} = D(\Theta^{(1)}_M-)p^z(\Theta^{(1)}_M-) \tag{3.3}$$

$$Q_3\delta^z_{\Theta_m,1} + Q_4\delta^z_{\Theta_m,1} = D(\Theta_m+)p^z(\Theta_m+) \tag{3.4}$$

$$\int_{\Theta_m}^{z} 1^T p^z(x)dx + \delta^z_{\Theta_m,0} + \delta^z_{\Theta_m,1} + \delta^z_{\Theta^{(0)}_M} + \delta^z_{\bar{z}} = 1. \tag{3.5}$$

Here, $D(x)$ is a diagonal matrix representing the dynamics of the states, $D_{ii}(x) := \frac{dx}{dt}$ where $i = 1,2,3,4$ denote the states $ij = 00, 01, 10, 11$ respectively. Therefore, for the description of z-policy in last section, we have $(D_{22}, D_{33}, D_{44})(x) = (h, -c, -c)$ and $D_{11}(x) =$

$$\begin{cases} h & \text{for } x < \Theta^{(0)}_M \\ \\ -c & \text{for } x > \Theta^{(0)}_M \end{cases}$$

Q denotes the generator matrix for continuous time Markov process whose columns are Q_1, Q_2, Q_3, Q_4, i.e. $Q = [Q_1 \vdots Q_2 \vdots Q_3 \vdots Q_4]$. $Q_1, Q_2, Q_3,$ and Q_4 respectively denote rates of transition vectors out of states $ij = 00, 01, 10,$ and 11.

Lemma 7. *The density function $p^z(x)$ satisfies the conservation law $1^T D(x)p^z(x) = 0, \forall x \in$*

$[\Theta_m, \Theta_M^{(1)}]$.

Proof. In equation (3.1), Q is the generator matrix of a Markov process, thus $1^T Q = 0$, which yields $\frac{d}{dx} 1^T D(x) p^z(x) = 0$. Also we have $1^T D(\Theta_i) p^z(\Theta_i) = 0$, at each of the boundary points (Equations (3.2-3.4)). So we have $1^T D(x) p^z(x)$ conserved at each temperature x. $\square$

The above method can also be generalized to more states of wind levels and comfort settings. In particular when there are $\mathcal{W}$ wind states and $\mathcal{C}$ comfort setting levels below temperature threshold z, there are $\mathcal{W} \times \mathcal{C}$ number of states. The differential equation 3.1 will still be the same, and will consist of $\mathcal{C}$ linear differential equations involving $\mathcal{C}$ vectors of $\mathcal{W}\mathcal{C}$ variables, resulting in a total of $\mathcal{C}^2 \mathcal{W}$ variables.

To solve for these we will need to evaluate the boundary conditions. Let $\delta_{\Theta_m, i}^z$ denote the probability mass at $x = \Theta_m$, under wind condition $i, i \in \{1, 2, ... \mathcal{W} - 1\}$ and comfort settings $\Theta_j, j \in \{0, 1, 2, ..., \mathcal{C} - 1\}$, and let $\delta_{\Theta_M^j}^z$ denote the probability mass at $\Theta_M^j, j \in \{1, 2, ..., \mathcal{C}\}$. This leads to additional $\mathcal{W}\mathcal{C}$ variables.

To solve for these $\mathcal{C}^2 \mathcal{W} + \mathcal{W}\mathcal{C}$ variable we have the $\mathcal{C} + 1$ relations relating probability mass variables δ_θ^z at $\theta \in \{\Theta_m, \Theta_M^{(0)}, ..., \Theta_M^{(\mathcal{C}-1)}\}$ to each of $\mathcal{C}\mathcal{W}$ states, thus a total of $(\mathcal{C} + 1)\mathcal{W}\mathcal{C} = \mathcal{C}^2 \mathcal{W} + \mathcal{W}\mathcal{C}$ relations. We will still need the normalization equation , since their is a dependency due to conservation law of Lemma 7. Thus we can obtain the distribution for arbitrary wind levels and comfort level setting.

In fact, the same formulation holds for any other dynamics too. The dynamics matrix $D(x)$ captures the effect of different dynamics. In particular, the cooling dynamics $\dot{x} =$
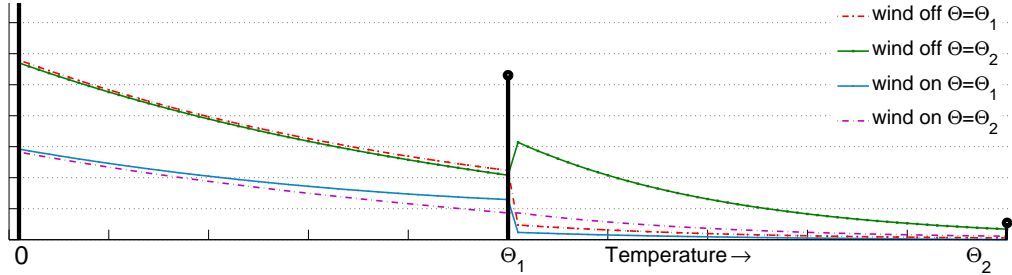
$Ax + b$, can be solved in a similar fashion.

Figure 3.2: Temperature distribution of a particular load following, the set-point policy for values $(z, \Theta_M^{(1)}, \Theta_M^{(0)}, q_0, q_1, r_0, r_1, c, h) = (100, 100, 50, 0.04, 0.04, 0.02, 0.02, 1.1, 1)$. The probability distribution consists of three point masses and four probability distributions which are continuous everywhere except at $\Theta_M^{(0)}$

3.2 Optimization for finite loads case under Z-policy

Figure 3.3 shows typical temperature trajectories for the case when $N = 3$, and $\mathbf{Z} = (60, 70, 80)$, with $\Theta_M^{(1)} = 100$. Since we are considering the average cost problem, we can assume without loss of generality that all loads start with the same initial temperature. In order to calculate the average cost, we need the distribution of $\sum_{i=1}^{N} P_{n,i}$, which entails knowing the joint probability distribution of $\{P_{n,i}\}_{i=1}^{N}$. More generally, as we consider a large number of loads, which we do in the sequel, we need the joint distribution of the grid power draws over the loads.

We resolve the dimensionality "curse" [3], by utilizing an important property of stochastic domination. For any two loads i and j with $Z_i < Z_j$ we have $x_i(t) \le x_j(t)$ for all

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t, regardless of the wind process realization. An important consequence is that whenever the j-th load hits its upper temperature limit Z_i, the i-th load has already hit its upper temperature limit too. Using ω to denote a sample point in the probability space, we have the following inclusion of events:

$$\{X_j(t,\omega) = Z_j\} \subset \{X_i(t,\omega) = Z_i\}, \quad \text{if } Z_i < Z_j. \tag{3.6}$$

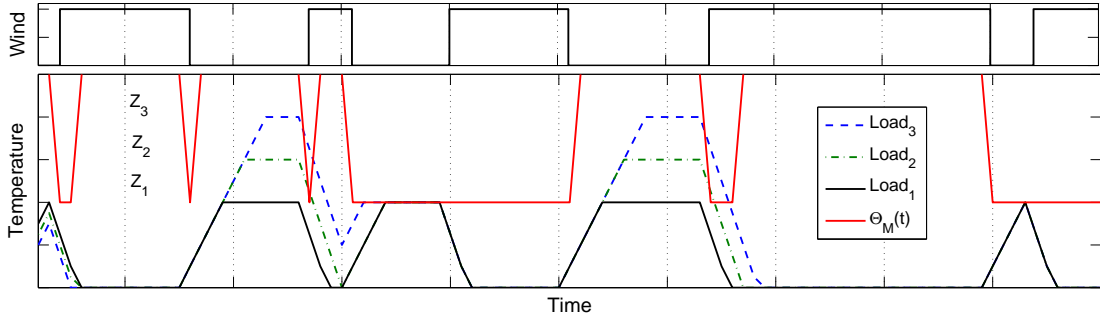Now we can calculate the expected cost of a Z-policy. First consider just one TCL. The

Figure 3.3: When the wind is blowing, loads cool at the maximum rate, subject to not going below temperature Θ_m. Load temperatures do not rise above minimum of Z and $\Theta_M(t)$

fossil fuel power drawn is h when $\Theta(t) = \Theta_M^{(1)}$ and the load temperature is at Z, while it is $h+c$ when $X(t) > \Theta(t)$. The cost due to discomfort is $\gamma^{(1)}[(x(t) - \Theta(t))^+]^2$, For any event A, denoting $\mathbb{P}^i(A) := \mathbb{P}(A \cap \{\Theta(t) = \Theta_i\})$ for $i = 1, 2$. The cost of the Z-policy is

$$C^{(1)}(Z) = h^2 \mathbb{P}^2(X = Z) + [(h+c)^2 \mathbb{P}^1(X > \Theta_M^{(0)}) + h^2 \mathbb{P}^1(X = \Theta_M^{(0)})]$$

$$+ \gamma^{(1)} \int_{\Theta_m}^{z} [(x - \Theta_M^{(0)})^+]^2 (p_{00}^z(x) + p_{10}^z(x)) dx.$$

For brevity let us denote the second term by $\Phi(z)$,

$$\Phi(z) := \int_{\Theta_m}^{z} [(x - \Theta_M^{(0)})^+]^2 (p_{00}^z + p_{10}^z)(x) dx.$$

Now we consider the case of two loads. In this case we need to consider the *total* dispatch-able fossil fuel generation drawn by the two loads. Thus we need the *joint probability distribution* of the two loads. Their marginals will not suffice, unlike the case of just one load. It is here that we exploit the above stochastic dominance which provides informa-tion on the joint-distribution of the loads. When $(\Theta_M(t) = \Theta_M^{(1)})$ with $Z_1 \leq Z_2$, the total fossil fuel power is $2h$ when $X_2 = Z_2$ because $X_2 = Z_2 \Rightarrow X_1 = Z_1$. Also, when $X_2 < Z_2$ but $X_1 = Z_1$, the fossil fuel power is h. Similarly when $\Theta_M(t) = \Theta_M^{(0)}$, total fossil fuel power is $2(h+c)$ when $X_1 > \Theta_M^{(0)}$, $(h+c)+h$ when $X_1 = \Theta_M^{(0)}$ but $X_2 > \Theta_M^{(1)}$, and $2h$ when $X_2 = \Theta_M^{(1)}$. So the total cost $C^{(2)}$ with relative weight $\gamma^{(2)} > 0$ is,

$$
\begin{aligned}
C^{(2)}(\mathbf{Z}) = {} & (2h)^2 \mathbb{P}^2(X_2 = Z_2) + h^2 \mathbb{P}^2(X_2 < Z_2 \cap X_1 = Z_1) \\
& + (2h)^2 \mathbb{P}^1(X_2 = \Theta_M^{(0)}) + (2(h+c))^2 \mathbb{P}^1(X_1 > \Theta_M^{(0)}) \\
& + ((h+c)+c)^2 \mathbb{P}^1(X_2 > \Theta_M^{(0)} \cap X_1 = \Theta_M^{(0)}) \\
& + \gamma^{(2)}(\Phi(Z_1) + \Phi(Z_2)).
\end{aligned}
$$

Also, since, $\{X_2 = Z_2\} \subset \{X_1 = Z_1\}$, we have $\mathbb{P}(\{X_1 = Z_1\}) = \mathbb{P}(\{X_1 = Z_1\} \cap \{X_2 < Z_2\}) + \mathbb{P}^2(\{X_2 = Z_2\})$, i.e.,

$$\mathbb{P}^2(\{X_1 = Z_1\} \cap \{X_2 < Z_2\}) = \delta_{Z_1}^{Z_1} - \delta_{Z_2}^{Z_2}. \tag{3.7}$$

When $\Theta(t) = \Theta_M^{(0)}$, by a similar argument we obtain,

$$\mathbb{P}^1(\{X_1 = \Theta_M^{(0)}\} \cap \{X_2 > \Theta_M^{(1)}\}) = \delta_{\Theta_M^{(0)}}^{Z_1} - \delta_{\Theta_M^{(0)}}^{Z_2}. \tag{3.8}$$

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This analysis extends to the case of N loads. For any choice $(Z_1, Z_2, ..., Z_N)$ of the set points with $Z_1 \leq Z_2 .... \leq Z_N$, we can evaluate the value of the average. It is

$$C^{(N)} = \gamma^{(N)} \sum_{k=1}^{N} \Phi(Z_i) + N^2 [h^2 (\mathbb{P}^2(X_1 = Z_1) +$$

$$\mathbb{P}^1(X_N = \Theta_M^{(0)})) + (h+c)^2 \mathbb{P}^1(X_1 > \Theta_M^{(0)})] +$$

$$\sum_{k=1}^{N-1} \Big[((N-k)h)^2 \mathbb{P}^2(X_{k+1} = Z_{k+1} \cap X_k < Z_k) +$$

$$\big((N-k)(h+c) + (kh)\big)^2 \mathbb{P}^1(X_k = \Theta_M^{(0)} \cap X_{k+1} > \Theta_M^{(0)}) \Big]$$

Now we turn to an important issue concerning the choice of the scaling parameter $\gamma^{(N)}$. $\int_0^T (\sum_{i=1}^N P_{g,i})^2 dt$ grows like $\Omega(N^2)$, but $\int_0^T \sum_{i=1}^N [(x_i(t) - \Theta(t))^+]^2 dt$ grows like $\Omega(N)$. We therefore scale $\gamma^{(N)}$ as $\gamma^{(N)} = \gamma.N$. Let $\widehat{C}^{(N)} := \frac{C^{(N)}}{N^2}$ denote the normalized cost. It evaluates to

$$\widehat{C}^{(N)} = \frac{\gamma}{N} \sum_{k=1}^{N} \Phi(Z_i) + h^2 \{\mathbb{P}^2(X_1 = Z_1) + \tag{3.9}$$

$$\mathbb{P}^1(X_N = \Theta_M^{(0)})\} + (h+c)^2 \mathbb{P}^1(X_1 > \Theta_M^{(0)}) +$$

$$\sum_{k=1}^{N-1} \Big[(1 - \frac{k}{N})^2 h^2 \mathbb{P}^2(X_{k+1} = Z_{k+1} \cap X_k < Z_k) +$$

$$\Big((1 - \frac{k}{N})(h+c) + (\frac{k}{N}h)\Big)^2 \mathbb{P}^1(X_k = \Theta_M^{(0)} \cap X_{k+1} > \Theta_M^{(0)}) \Big] \tag{3.10}$$

We thereby arrive at the following optimization problem for the case of a finite number N of TCLs:

$$\text{Minimize} \quad \widehat{C}^{(N)}(\mathbf{Z})$$

$$\text{s.t.} \quad Z_i \leq Z_{i+1} \text{ for } i = 1, 2, ..., N-1,$$

$$Z_1 \geq 0, Z_N \leq \Theta_M^{(1)}. \tag{3.11}$$

This problem is intractable because of the large number of constraints. Motivated by this we next examine its infinite load population limit.

3.3 Continuum limit for binary wind and comfort setting model

We now consider the infinite population limit where there is a continuum of loads. Let $u(z)$ denote the fraction of loads with set points no more than z. Let $\mathscr{U}$ denote the space of piecewise continuous increasing positive functions on $[0,1]$, noting that $u \in \mathscr{U}$. The resulting cost, following a similar analysis to (3.10), is

$$C^{[0,1]}(u) = h^2(\delta_{\Theta_M^{(1)}}^{\Theta_M^{(1)}} + \delta_{\Theta_M^{(0)}}^{\Theta_M^{(0)}}) + \int_{\Theta_m}^{\Theta_M^{(1)}} [(hu)^2(z)\mathbb{P}^2(\{X_z = z\} \cap \{X_{z+dz} < z+dz\})) +$$

$$(hu(z) + (h+c)(1-u(z)))^2 \times \mathbb{P}^1(\{X_z = \Theta_M^{(0)}\} \cap \{X_{z+dz} > \Theta_M^{(0)}\})] +$$

$$\gamma \int_{\Theta_m}^{\Theta_M^{(1)}} \Phi(z)u'(z)dz.$$

The first term is the cost of violation of upper temperature limit, while the second term is that due to variability in total dispatchable fossil fuel generation. Note that from (3.7),(3.8) $\mathbb{P}^2(\{X_z = z\} \cap \{X_{z+dz} < z+dz\}) = -\frac{d\delta_z^z}{dz}dz$ and $\mathbb{P}^2(\{X_z = \Theta^1\} \cap \{X_{z+dz} > \Theta^1\}) = -\frac{d\delta_{\Theta_M^{(0)}}^z}{dz}dz.$

Define $D_1(z) := -\frac{d\delta_z^z}{dz}, D_2(z) := -\frac{d\delta_{\Theta_M^{(0)}}^z}{dz}$. The above can be simplified (as in [29]) to

$$C^{[0,1]}(u) = h^2(\delta_{\Theta_M^{(1)}}^{\Theta_M^{(1)}} + \delta_{\Theta_M^{(0)}}^{\Theta_M^{(0)}}) + \left[\int_{\Theta_m}^{\Theta_M^{(1)}} [\gamma u'(z)\Phi(z) + \right.$$

$$\left. (hu(z))^2 D_1(z) + ((h+c) - cu(z))^2 D_2(z)]dz \right].$$

Ignoring the first term that does not deoend on u, we obtain the following Calculus of Variations [20] optimization problem:

$$\text{Min} \quad J[u] = \int_{\Theta_m}^{\Theta_M^{(1)}} \left[(hu(z))^2 D_1 + (h+c-cu(z))^2 D_2 + \right.$$

$$\left. \gamma \Phi(z) u'(z) \right] dz \tag{3.12}$$

$$\text{s.t.} \quad u \in \mathcal{U}, u(\Theta_m) = 0, u(\Theta_M^{(1)}) = 1. \tag{3.13}$$

If one informally uses the Euler-Lagrange equation, one obtains the resulting solution $u_{EL}(z) = \frac{\gamma \Phi'(z) + 2c(c+h)D_2(z)}{2(h^2 D_1(z) + c^2 D_2(z))}$. This however need not be a positive increasing function or satisfy the boundary condition $u_{EL}(\Theta_m) = 0$, $u_{EL}(\Theta_M^{(1)}) = 1$. Figure shows a particular solution where $u_{EL} \notin \mathcal{U}$ and $u_{EL}(\Theta_M^{(1)}) \neq 1$.
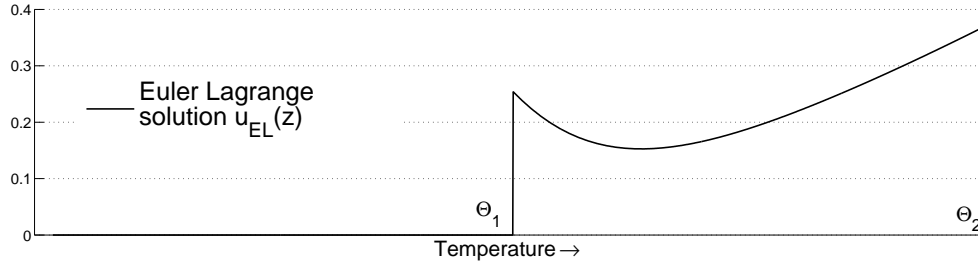

Figure 3.4: The Euler Lagrange solution obtained for the values used to find distribution in Figure 3.2.

Therefore the Euler Lagrange solution is not admissible for all the scenarios. We consider next how to solve this problem using Pontryagin's Minimum Principle [26].

3.3.1 Evaluation of optimal solution using minimum principle

We employ optimal control to solve the optimization problem. Assume that state $u(z)$ follows the dynamic system $\dot{u}(z) = f(u, v, z)$ for any admissible control $v(z) \in \mathbf{V}$, and $z \in [\Theta_m, \Theta_M^{(1)}]$. Consider the cost function is $J[u] = \int_{\Theta_m}^{\Theta_M^{(1)}} g(u, v, z) dz$.

Defining $h(u, v, \lambda, z) = g(u, v, z) + \lambda(z) f(u, v, z)$, the optimal control $v^*(z)$ which takes the state $u(z)$ from $u(\Theta_m) = 0$ to $u(\Theta_M^{(1)}) = 1$, satisfies the following two conditions [26]:

1. $\dot{\lambda}(z) = -\frac{\partial g}{\partial u}$

2. $v^*(z) \in \mathbf{V}$ is the pointwise minimizer of $h(u, v, \lambda, z)$.

Using Theorem 3.36 from [9], since $u(z) \in \mathcal{U}$ has bounded variation, and $\Phi(z)$ is continuous, we have

$$\int_{\Theta_m}^{\Theta_M^{(1)}} \Phi(z) u'(z) dz = \Phi(\Theta_M^{(1)}) - \int_{\Theta_m}^{\Theta_M^{(1)}} \Phi'(z) u(z) dz. \tag{3.14}$$

We use (3.12),(3.14) to collect terms depending on $u(z)$, and rewrite the cost in the following equivalent way:

$$J'[u] = \int_{\Theta_m}^{\Theta_M^{(1)}} u^2(z) w(z) - u(z)(\gamma \Phi'(z) + 2c(c+h) D_2(z)) dz$$

$$= \int_{\Theta_m}^{\Theta_M^{(1)}} \left[u(z) - u_{EL}(z) \right]^2 w(z)$$

$$- \frac{(\gamma \Phi'(z) + 2c(c+1) D_2(z))^2}{4(h^2 D_1(z) + c^2 D_2(z))} dz.$$

Here $w(z) := (h^2 D_1(z) + c^2 D_2(z)) > 0$. Thus, we will focus on optimizing the cost functional $J[u] = \int_{\Theta_m}^{\Theta_M^{(1)}} w(z)[u(z) - u_{EL}(z)]^2 dt$, $u(\Theta_m) = 0$ and $u(\Theta_M^{(1)}) = 1$. Without loss of generality, we can take $u_{EL}(z) \in [0, 1]$ (otherwise replace u_{EL} by $\max(0, \min(1, u_{EL}))$).

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We first consider the Euler Lagrange solution $u_{EL}(z)$ obtained in Section IV. Using the notations above, we set $\mathbf{V} = \mathbb{R}$, $f(u, v, z) = v^2$ and $g(u, v, z) = w(z)[u(z) - u_{EL}(z)]^2$.

The optimizers v^*, λ^* satisfy,

$$\dot{\lambda}^*(z) = -2(u(z) - u_{EL}(z))w(z) \tag{3.15}$$

$$\dot{u}(z) = v^{*2}(z) \tag{3.16}$$

$$v^*(t) = \arg\min_{v \in R} \left[w(z)(u(z) - u_{EL}(z))^2 + \lambda^*(z)v^2(z) \right] \tag{3.17}$$

One can rewrite (3.17) as:

$$v^*(z) = \begin{cases} 0 & \text{if } \lambda^*(z) > 0 \\ -\infty & \text{if } \lambda^*(z) < 0 \\ \text{arbitrary} & \text{if } \lambda^*(z) = 0 . \end{cases}$$

One can observe that $\lambda^*(z)$ can-not be strictly negative, as this leads to unbounded $u(z)$ from (3.16), and further results in infinite cost J. This also suggests that jumps in $u(z)$ are possible only at those points where $\lambda^*(z)$ is zero.

Also from (3.15), we observe that $\lambda^*(z)$ remains constant only on the points where $u(z) = u_{EL}(z)$. It increases when $u(z) < u_{EL}(z)$, and decreases when $u(z) > u_{EL}(z)$. From (3.16), (3.17) we observe that if $\lambda^*(z) > 0$, then $v^*(z)$ is zero, leading to a constant $u(z)$. So, $\lambda^*(z) > 0$ implies $u(z) = \text{constant}$. Next, since $u(z)$ has to be increasing from (3.16), we conclude that $\lambda^*(z)$ can-not be constant on the points where $u_{EL}(z)$ is decreasing, however $\lambda^*(z)$ can be constant when $u_{EL}(z)$ is increasing.

Now we consider $u_{EL}(z)$ obtained in Section IV. Since we require $u^*(\Theta_M^{(1)}) = 1$, there

are jumps at $\Theta_M^{(0)}$ and $\Theta_M^{(1)}$. Next if $u_{EL}(z)$ is decreasing in $[\Theta_M^{(0)}, \theta]$ then $\lambda^*(z)$ is not

constant in $[\Theta_M^{(0)}, \theta]$ and $u^*(z)$ is a constant in $[\Theta_M^{(0)}, \theta]$, let $u^*|_{z \in [\Theta_M^{(0)}, \theta]} = \kappa$. Since $\lambda(z)$

can not be negative, $\kappa = u^*(\Theta_M^{(0)}) < u_{EL}(\Theta_M^{(0)})$ and also since $\lambda(\Theta_M^{(1)})$ needs to be zero,

$u(\Theta_M^{(1)}) > \min_{z \in [\theta, \Theta_M^{(1)}]}\{u_{EL}(z)\}$, so that $\lambda(z)$ decreases when $u_{EL}(z) < \kappa$. $\lambda(z)$ decreases

to zero, and subsequently stays at zero, and $u^*(z)$ remains equal to $u_{EL}(z)$ till $\Theta_M^{(1)}$. Note

that $u^*(z)$ has a discontinuity at $\Theta_M^{(1)}$, since we require $u^*(\Theta_M^{(1)}) = 1$. This is illustrated in
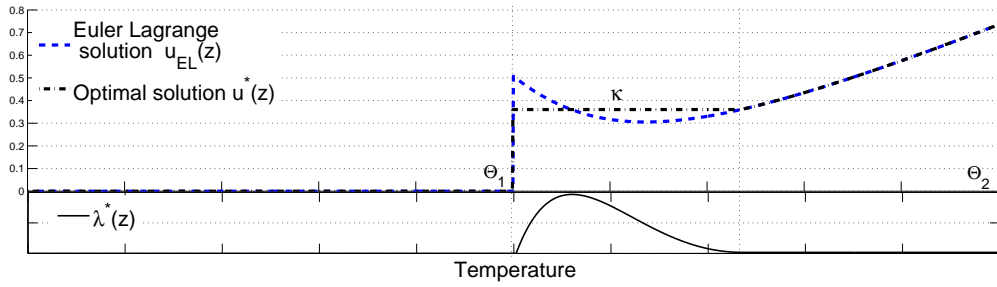
Figure 3.5.

Figure 3.5: The incorrect solution from an incorrect application of the Euler Lagrange results from Section V contrasted with the correct optimal solution obtained from the Pontryagin's minimum principle.

Notice that the above κ satisfies $\int_{y_1}^{y_2}(\kappa - u_{EL}(z))w(z)dz = 0$, for $y_1 := \sup\{z < \theta :$

$u_{EL}(z) < \kappa\}$ and $y_2 := \inf\{z > \theta : u_{EL}(z) > \kappa\}$. For a general $u_{EL}(z)$, we can find an

optimal solution $u^*(z)$ in a similar way. We first find the partition $\{\Theta_M^{(0)} = \theta_0, \theta_1, ..., \theta_N =$

$\Theta_M^{(1)}\}$, such that (WLOG) u_{EL} is increasing in $[\theta_{2i}, \theta_{2i+1}]$, and decreasing in $[\theta_{2i-1}, \theta_{2i}]$, and

finding κ_i such that $\int_{y_{2i-1}}^{y_{2i+i}}(\kappa_i - u_{EL}(z))w(z)dz = 0$, for $y_{2i-1} = \sup\{z < \theta_{2i} : u_{EL}(z) < \kappa_i\}$

and $y_{2i+1} = \inf\{z > \theta_{2i} : u_{EL}(z) > \kappa_i\}$. This is shown in Figure 3.6.

Let us denote this operator which transforms the Euler Lagrange solution $u_{EL}(z)$ to optimal solution $u^*(z)$ via the minimum principle as $\mathscr{P}[.]$, i.e. $u^*(.) = \mathscr{P}[u_{EL}(.)]$.
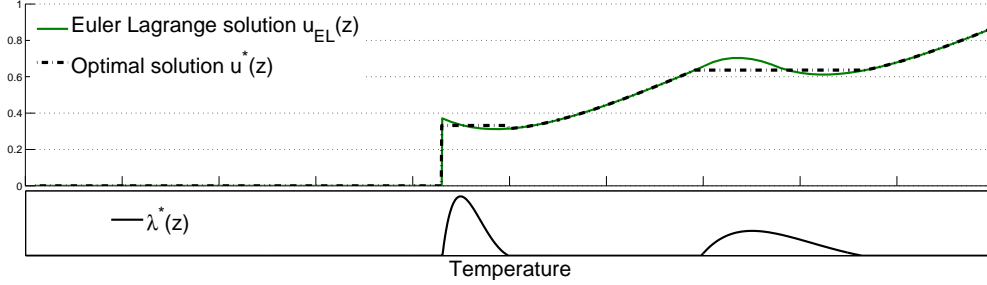

Figure 3.6: The optimal solution in the general case will remain equal to the Euler Lagrange solution in some intervals but will remain constant in other intervals.

3.4 Extension of results for general models

3.4.1 Model with more upper comfort levels

Consider three upper comfort levels $\Theta_M^{(0)}, \Theta_M^{(1)}, \Theta_M^{(2)}$. In this situation, when the comfort range for identical loads toggles to $\Theta_M^{(1)}$, all the loads whose set-points are between $\Theta_M^{(0)}$ and $\Theta_M^{(1)}$ remains unaffected. Therefore, under the event $\{\Theta_M(t) = \Theta_M^{(1)}\}$ the cost

will be a function of the fraction of set-points below $\Theta_M^{(1)}$, i.e. $u(\Theta_M^{(1)})$. In particular,

$$\text{Power cost} = \text{Cost when } \{\Theta_M = \Theta_M^{(2)}\}+$$

$$\text{Cost when } \{x \geq \Theta_M = \Theta_M^{(1)}\}+$$

$$\text{Cost when } \{x \geq \Theta_M = \Theta_M^{(0)}\}.$$

$$C_{pow}[u] = \int_0^{\Theta_M^{(2)}} D^{\Theta_M^{(1)}}(z)(hu(z)+$$

$$(h+c)(1-u(z)-u(\Theta_M^{(1)})))^2+$$

$$D^z(z)h^2u^2(z) + D^{\Theta_M^{(0)}}(z)((h+c)-u(z))^2 dz,$$

where $D^{\Theta_i}(z) := -\frac{\delta^z_{\Theta_i}}{dz} = -\frac{d}{dz}\mathbb{P}(X_z = \Theta_i)$ for $i = 1,2$, and $D^z(z) := -\frac{\delta^z_z}{dz} = -\frac{d}{dz}\mathbb{P}(X_z = z)$,

and

$$\text{Discomfort cost} = \int_0^{\Theta_M^{(2)}} \mathbb{E}[(x - \Theta_M(t))^+)]^2 u(z) dz$$

$$C_{disc}[u] = \int_0^{\Theta_M^{(2)}} \Phi_1(z)u'(z) + \Phi_2(z)u'(z) dz.$$

Where $\Phi_i(z) = \int_0^z [(x - \Theta_M^{(0)})^+]^2 (p_{1i} + p_{0i})(z) dz$, for $i = 1,2$. Total cost $C[u] = C_{Pow}[u] + \gamma C_{disc}[u]$.

After simplification the cost function can be re-written as,

$$J[u] = \int_0^{\Theta_M^{(2)}} (u(z) - u_{EL}(z, u(\Theta_M^{(1)})))^2 w(z) dz + C'(u(\Theta_M^{(1)})).$$

For $u \in \mathcal{U}$,

$$u_{EL}(z, u(\Theta_M^{(1)})) =:$$

$$\frac{\gamma\Phi'(z) + 2c(c+h)(D^{\Theta_M^{(0)}}(z) + D^{\Theta_M^{(1)}}(z)(1 - u(\Theta_M^{(1)})))}{2(h^2D^z(z) + c^2D^{\Theta_M^{(0)}}(z) + c^2D^{\Theta_M^{(1)}}(z))},$$

and $C'(u(\Theta_M^{(1)})) := \Phi(\Theta_M^{(2)}) + \int_0^{\Theta_M^{(2)}} (h + c)^2(D_1(z) + D_2(z)(1 - u(\Theta_M^{(1)}))) -$

$u_{EL}^2(z, u(\Theta_M^{(1)})) w(z) dz.$

The calculus of variations problem is difficult to solve because the functional $J[u]$ also contains point evaluations $u(\Theta_M^{(1)})$. We propose to solve this iteratively, first by replacing $u(\Theta_M^{(1)})$ with a variable v obtaining the optimizer $u^*(z)$ for a fixed v, and then updating v such that $|u^*(\Theta_M^{(1)}) - v|$ decreases in each step. Thus we can obtain an optimal solution $u^*(z)$ with $v = u^*(\Theta_M^{(1)})$.

For an initial guess v_0 of $u^*(\Theta_M^{(1)})$ the problem reduces to minimizing $J[u] = \int_0^{\Theta_M^{(2)}} (u - u_{EL}(z, v_0))^2 w(z) dz$, whose solution from Section (3.3.1) is obtained as $\mathscr{P}[u_{EL}(., v_0)](z)$.

Lemma 8. *Let v be such that $v = \mathscr{P}[u_{EL}(., v)]$. Then v lies between v_0 and $\mathscr{P}[u_{EL}(., v_0)](\Theta_M^{(1)})$.*

Proof. As $\mathscr{P}[u]$ is increasing in u, and $u_{EL}(., v)$ is decreasing in v, if $v_0 \leq v$ then $v \leq \mathscr{P}[u_{EL}(., v)]$. On the other hand, if $v_0 \geq v$, then $v \geq \mathscr{P}[u_{EL}(., v)]$, which yields the desired result. $\qquad\square$

Therefore, we define a range in which v lies, let $v_0^\uparrow := \max(v_0, \mathscr{P}(u_{EL}(., v_0))(\Theta_M^{(1)}))$, and $v_0^\downarrow := \min(v_0, \mathscr{P}(u_{EL}(., v_0))(\Theta_M^{(1)}))$.

For $(n{+}1)$-th iteration we update the values of $v_n, v_n^\uparrow, v_n^\downarrow$ as, $v_{n+1} = \frac{v_n^\uparrow + v_n^\downarrow}{2}$, $v_{n+1}^\uparrow = \min(v_n^\uparrow, \max(v_n, \mathscr{P}[u_{EL}(., v_n)](\Theta_M^{(1)})))$, and $v_{n+1}^\downarrow = \max(v_n^\downarrow, \min(v_n, \mathscr{P}[u_{EL}(., v_n)](\Theta_M^{(1)})))$.

Lemma 9. *The iteration tuples $(v_i^\downarrow, v_i^\uparrow)$ converges to the same unique value as $i \to \infty$. That is, there exist a v such that $v_i^\downarrow \to v$, and $v_i^\uparrow \to v$ as $i \to \infty$. Morever such a v is a fixed point of the function $\mathscr{P}[u_{EL}(., v)](\Theta_M^{(1)})$, i.e. it satisfies $v = \mathscr{P}[u_{EL}(., v)](\Theta_M^{(1)})$.*

Proof. It is clear by induction that for all i, $v_i^{\uparrow} \geq v_i^{\downarrow}$, since for all i, as $v_i = \frac{v_i^{\uparrow} + v_i^{\downarrow}}{2}$, we have

$\max(v_i, \mathscr{P}[u_{EL}(.,v_i)](\Theta_M^{(1)})) > v_i^{\downarrow}$ and $\min(v_i, \mathscr{P}[u_{EL}(.,v_i)](\Theta_M^{(1)})) < v_i^{\uparrow}$.

Also, the sequences $\{v_i^{\uparrow}\}$ and $\{v_i^{\downarrow}\}$ are both monotonically decreasing and increasing respectively, therefore they must converge. It remains to show that both of these sequences converges to the same point. To show that, we will prove that the range spanned by $v_i^{\uparrow}$, and $v_i^{\downarrow}$ decreases with i, in particular, $v_{n+1}^{\uparrow} - v_{n+1}^{\downarrow} \leq \frac{v_n^{\uparrow} - v_n^{\downarrow}}{2}$, which then concludes the proof, as $v \in [v_i^{\downarrow}, v_i^{\uparrow}]$.

Assume, first that $v_i \leq \mathscr{P}[u_{EL}(.,v_i)](\Theta_M^{(1)})$, then

$$v_{i+1}^{\uparrow} - v_{i+1}^{\downarrow} = \min\left(v_i^{\uparrow}, \mathscr{P}[u_{EL}(.,v_i)](\Theta_M^{(1)})\right) - v_i$$

$$\leq v_i^{\uparrow} - v_i = \frac{v_i^{\uparrow} - v_i^{\downarrow}}{2}.$$

On the other hand, if $v_i > \mathscr{P}[u_{EL}(.,v_i)](\Theta_M^{(1)})$, then

$$v_{i+1}^{\uparrow} - v_{i+1}^{\downarrow} = v_i - \max\left(v_i^{\downarrow}, \mathscr{P}[u_{EL}(.,v_i)](\Theta_M^{(1)})\right)$$

$$\leq v_i - v_i^{\downarrow} = \frac{v_i^{\uparrow} - v_i^{\downarrow}}{2}.$$

This concludes the proof to show that there is a unique point v and $v_i^{\uparrow}, v_i^{\downarrow} \to v$ as $i \to \infty$. $\square$

The same technique can be used for more than three comfort settings, In particular for $\mathscr{C}$ comfort settings, we will have $\mathscr{C} - 2$ such tuples $(v^{\uparrow}, v^{\downarrow})$, which can be iterated in a round-robin fashion to obtain the optimal solution.

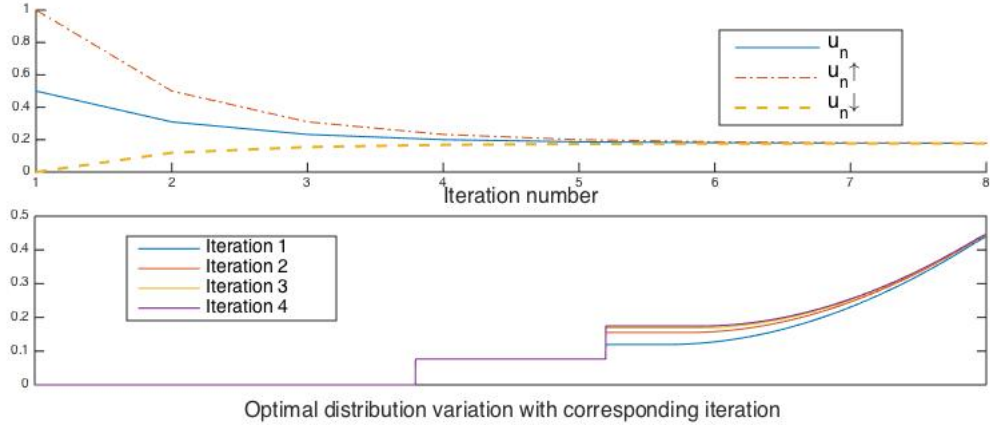Optimal distribution variation with corresponding iteration

Figure 3.7: Convergence of the parameters $v_n^{\uparrow}, v_n^{\downarrow}$, and v_n with iteration numbers along with the optimal solutions $u^*(., v_i)$ for $i = 1, 2, 3, 4$.

3.4.2 Extension to multiple wind models

Consider the case when we have a binary model for comfort setting and, ternary model for wind states, namely 0, 1, 2. In state 0, the wind is not available, so houses heat at the rate h. In state 2, all houses can cool at the maximum rate c. In addition, we have an intermediate state 1, in which all the houses can be cooled at a rate $c/2$.

This system can be solved by using the technique in Section (3.1), to obtain the density functions $p_{00}, p_{10}, p_{20}, p_{10}, p_{11}, p_{21}$, and the probability masses $\delta_{0,0}, \delta_{0,1}, \delta_{\Theta_M^{(0)},0}, \delta_{\Theta_M^{(0)}}, \delta_{\Theta_M^{(1)}}$. We notice when $x > \Theta_M$ and wind is in state 1, we can cool the loads at a maximum rate c by providing $c/2$ non-renewable power, which adds to the cost function.

Without delving further into finite loads case, we directly write the cost function for

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the case of asymptotic loads scenario. The cost function is,

$$C[u] = \int_0^{\Theta_M^{(1)}} (-\delta_z^z)' h^2 u^2(z)(-\delta_{\Theta_M^{(0)},0}^z)'(hu(z) + (h+c)(1-u(z)))^2$$

$$+ (\hat{D})(z)(\frac{c}{2}(1-u(z)))^2 dz + \gamma \int_0^{\Theta_M^{(1)}} \Phi(z)u'(z)dz,$$

where $\hat{D}(z) = -\frac{d}{dz}\mathbb{P}(X_z > \Theta_M^{(0)} \cap W = 1)$. Denoting $D_y^z = -\frac{d}{dx}\delta_y^z$ for $y = z, (\Theta_M^{(0)},0)$, and $(\Theta_M^{(0)},1)$, and simplifying the above expression we obtain the optimization problem

$$J[u] = \int_0^{\Theta_M^{(1)}} (u - u_{EL}(z))^2 w(z)dz$$

$$s.t. \quad u \in \mathscr{U}$$

$$u(0) = 0, u(\Theta_M^{(1)}) = 1,$$

where $u_{EL}(z) = \dfrac{\gamma\Phi'(z) + 2c(c+h)D_{\Theta_M^{(0)},0}^z + c\hat{D}(z)}{2(h^2 D_z^z(z) + c^2 D_{\Theta_M^{(0)}}^z(z) + (\frac{c}{2})^2\hat{D}(z))}$, and $w(z) = h^2 D_z^z(z) + c^2 D_{\Theta_M^{(0)}}^z(z) + (\frac{c}{2})^2\hat{D}(z)$.

We notice that this is the exact same problem as solved in section (3.3.1), whose solution $\mathscr{P}[u_{EL}(.)](z)$ gives the optimal distribution for set-points.

The same analysis can be extented for an arbitrary number $\mathscr{W} > 3$ wind states, where the states are numbered such that in state $w \in [0,1,...,\mathscr{W}-1]$ the available wind power is sufficient to cool all the houses at a rate $\frac{ic}{\mathscr{W}-1}$. (Therefore the available wind power in state i is $h + \frac{ic}{\mathscr{W}-1}$.) If there are $\mathscr{C}$ set-points, then the solution of the problem is given by $\mathscr{P}[u_{EL}(.,\vec{v}^*)]$, where u_{EL} is given by equation (3.18)

$$u_{EL}(z,\vec{v}) = \frac{\gamma\Phi'(z) + 2c(c+h)(D_{\Theta_M^{(0)}}^z(z) + \sum_{j=2}^{\mathscr{C}}(1-v_i)D_{\Theta_j}^z(z)) + 2c\sum_{i=1}^{\mathscr{W}-1}((1-\frac{i}{\mathscr{W}-1})\hat{D}_{\Theta_M^{(0)}}^z(z) - \sum_{j=1}^{\mathscr{C}}v_i\hat{D}_{\Theta_j}^z(z)}{2(h^2 D_z^z(z) + c^2 D_{\Theta_M^{(0)}}^z(z) + \sum_{i=1}^{\mathscr{W}-1}(1-\frac{i}{\mathscr{W}-1})^2 c^2\hat{D}(z))} \tag{3.18}$$

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and $\vec{v}^* = (v_2^*, v_3^*, ..., v_{\mathscr{C}-\infty}^*)$ is the fixed point of the equations

$$v_j^* = \mathscr{P}[u_{EL}(., \vec{v}^*)](\Theta_j)$$

for $j = (2, 3, ..., \Theta_{\mathscr{C}-\infty})$.

3.4.3 Numerical solution for non-homogeneous loads

The solution for the homogeneous loads scenario, with uniform change of set-points is not easy to generalize. In the general case, there can be many sources of in-homogeneity, the cooling and heating rates could be different for different loads, and the set-point changes could be arbitrary. All such factors could lead to non-coupled temperature trajectories. Computation of the overall power is difficult under the non-coupled temperature case, as the complexity of the state-space increases geometrically in this case.

We can use numerical simulation to get an estimate of the optimal solution for a general scenario. The idea is to calculate the optimal solution for a population N of loads, which will give us the density function $\tilde{u}(x) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{I}(x \geq z_i^*)$ as an approximate of the asymptotic optimal solution. (Where $\mathbb{I}(x)$ is an indicator function)

We used the coupling from the past algorithm to estimate the joint probability distribution $P(\vec{X} < \vec{x})$ from the perfect samples. The cost function comes from the joint distribution as

$$\tilde{u}(z) = \sum_{\substack{S \subset \{1,2,...,N\} \\ R \subset \{1,2,...,N\}-S}} \left(\frac{1}{N}\sum_{i \in S, j \in R}\left(P_i^c(x_i) + P_j^h(z_i)\right)\right)^2 \times \mathbb{P}(X_S = z_S \cap X_R > \Theta_R) +$$

$$\sum_{i=1}^{N}\gamma\frac{1}{N}\mathbb{E}[(X_i - \Theta_i)^+]^2,$$

where $P_i^h(z_i) :=$ Power required to maintain the temperature of house-i at set-point z_i, and

$P_i^c(x_i) :=$ Power required to cool the house-i at temperature x_i.

For simplicity, consider the case where the loads are identical to the previous sections, but the set-point changes are independent. The cost function simplifies to

$$\tilde{u}(z) = \sum_{\substack{S \subset \{1,2,\ldots,N\} \\ R \subset \{1,2,\ldots,N\}-S}} \left(h\frac{|S|}{N} + (h+c)\frac{|R|}{N} \right)^2 \times \mathbb{P}(X_S = z_S \cap X_R > \Theta_R)$$
$$+ \sum_{i=1}^{N} \gamma\frac{1}{N}\mathbb{E}[(X_i - \Theta_i)^+]^2,$$

where for any set U, we denote the event $X_U > Y_U$ as $\{X_i > Y_i, \forall i \in U\}$.

The solution $\tilde{u}(\{z_i\})$ resulting from the optimization may have discontinuities; there-fore using this as a distribution to generate set-points will result in an accumulation at the points of discontinuity. One may use interpolation, or smoothing the discrete distribution by any convolution kernel $g(x)$ using operator $K[u](x) = \int g(x-z)u(z)dz$ to resolve this issue. This is shown in Figure (3.8).
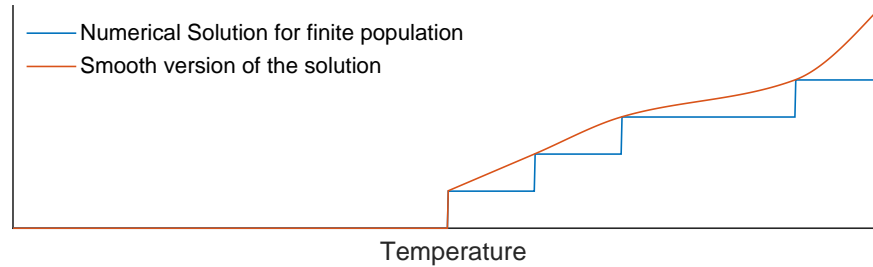

Figure 3.8: The numerical solution obtained from using coupling from the past algorithm, for a scenario with five loads when the upper comfort level is allowed to change indepen-dently.

3.5 Simple heuristic approaches for sub-optimal threshold policy

In a practical scenario where the air-conditioner parameters, comfort range values, etc., are not known, it is difficult to analyze and propose an optimal threshold policy. A suboptimal heuristic which adaptively updates the thresholds' distribution to minimize the overall cost is desired to tackle this scenario.

A general heuristic method can be proposed when the distribution is chosen from a class $\mathscr{F}_\alpha$ of distributions, where a function $\phi(.,\alpha) \in \mathscr{F}_\alpha$ is characterized by a parameter α. A sequence $\{\phi(.,\alpha_n)\} \subset \mathscr{F}_\alpha$ is chosen to achieve $\widehat{J}(\phi(.,\alpha_n)) \geq \widehat{J}(\phi(.,\alpha_{n+1}))$. Here $\widehat{J}(\phi)$ is an estimate of the average cost, when ϕ is the chosen threshold distribution. The total grid power consumed and the total discomfort cost are assumed to be accessible to the aggregator, without any loss of privacy information from the end-user. The aggregator then estimates the average cost for a long duration and adapts the heuristic distribution to achieve the low overall cost.

Several classes of distributions $\mathscr{F}_\alpha$ can be chosen. One such choice is the polynomial distribution on $[\Theta_m, \Theta_M]$ i.e.,

$$\mathscr{F}_\alpha^N([\Theta_m, \Theta_M]) = \left\{ \sum_{i=0}^N \alpha_i z^i \geq 0 : \sum_{i=1}^N i\alpha_i z^{i-1} \geq 0 \text{ for } z \in [\Theta_m, \Theta_M] \right\},$$

However, as we have shown in the previous sub-sections that there is a discontinuity at the intermediate comfort levels, a polynomial distribution is unable to fit properly.

We therefore consider a set of piecewise polynomials defined on a partition of comfort range, i.e., restriction of the overall distribution on a segment of comfort range to one

such polynomial above. The parameterized class of distributions becomes,

$$\mathcal{F}_\alpha^{(N,T)}([\Theta_m, \Theta_M]) = \{(f_1, f_2, ..f_{2^T}) : f_i \in \mathcal{F}_{\alpha_i}^N([\Theta_m + \frac{(i-1)\Delta\Theta}{2^T}, \Theta_m + \frac{i\Delta\Theta}{2^T}]),$$

$$f_i(\Theta_m + \frac{i\Delta\Theta}{2^T}) \le f_{i+1}(\Theta_m + \frac{i\Delta\Theta}{2^T})\}.$$

For large N the constraint set $\sum_{i=1}^N i\alpha_i z^{i-1} \ge 0$ becomes complex. For simplicity of implementation, assume that $N = 0$, i.e. the distribution is piecewise constant in $[\Theta_m, \Theta_M]$, i.e, we would need an adaptive rule for a set of parameters $\{\alpha_i\}_1^{2^T}$, s.t. $\alpha_i \le \alpha_{i+1}$, and $\alpha_i \ge 0$, and the distribution $f\big|_{[\Theta_m + \frac{(i-1)\Delta\Theta}{2^T}, \Theta_m + \frac{i\Delta\Theta}{2^T}]} \equiv \alpha_i$.

3.5.1 Algorithm for heuristic policy

First we assume that the distribution is piecewise constant. In our first approach we suppose that there are fixed partitions (T=constant). The adaptive law is assumed to be similar to gradient descent ($\dot{\alpha}_i = -\varepsilon \frac{\partial J}{\partial \alpha_i}$), where ε is the learning rate parameter. In discrete system we have used $\alpha_i(k+1) = \alpha_i(k) - \varepsilon \frac{\widehat{J(k)} - \widehat{J(k-1)}}{\alpha_i(k) - \alpha_i(k)}$, where k is the time index, and $\widehat{J(k)}$ is the simulated cost at time k. To make the algorithm robust, we stop the adaptation when the increase in the cost is smaller than a value Δ_J

In the successive refining approach, we double the number of partitions each time after finding an (sub)optimal distribution. Each (sub)optimal solution for one partition size is considered as the initial solution for the optimization of the next stage where we double the number of partitions.

There are several benefits of this approach over the fixed partition one. First, this algorithm reduces the cost more rapidly than fixed partition scheme, as initially when the partitions are bigger, adaptation results in a bigger change in overall cost reduction. Sec-

ond, we can stop the sub-partition after the cost reduction is not significant, thus it saves the un-necessary partitioning computation and adaptation. Third, using this approach also reduces search space requirements for adaptation, as the (sub)optimal solution at a coarser level gives a good starting point for the finer level partition.
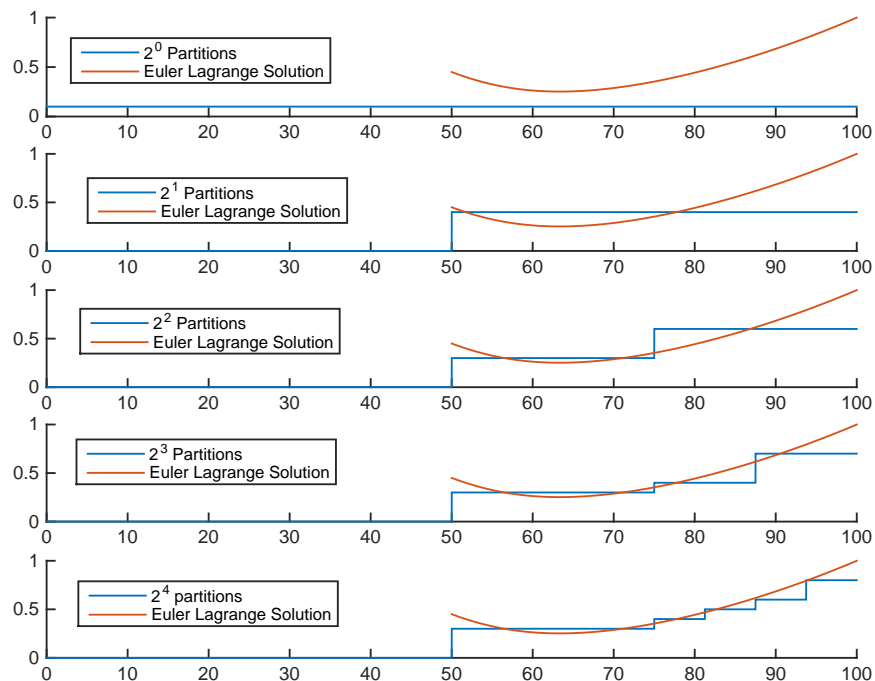

Figure 3.9: Successive refinement heuristic with piecewise constant distribution, where the distribution is optimized for a fixed partiton size. Subsequently each partition is subdivided into two and further optimization is done with previous optimal solution as the initial guess.

Using a piecewise constant distribution has an issue of discontinuity, i.e., the TCL setpoint can accumulate the points to discontinuity. To resolve this minor issue, we considered

piecewise linear distribution, where the distribution is the line-segment joining at mid-points of the partitions at levels α_i, i.e. the class

$$\mathscr{F}_\alpha^T(z) = \{\frac{z - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}} : z \in [\Theta_m + \frac{\Delta\Theta}{2^T}(i - \frac{1}{2}), \Theta_m + \frac{\Delta\Theta}{2^T}(i + \frac{1}{2})]$$

$$\text{for } 0 = \alpha_0 \le \alpha_1 \le \alpha_2 ... \le \alpha_N \le \alpha_{2^T + 1} = 1\},$$
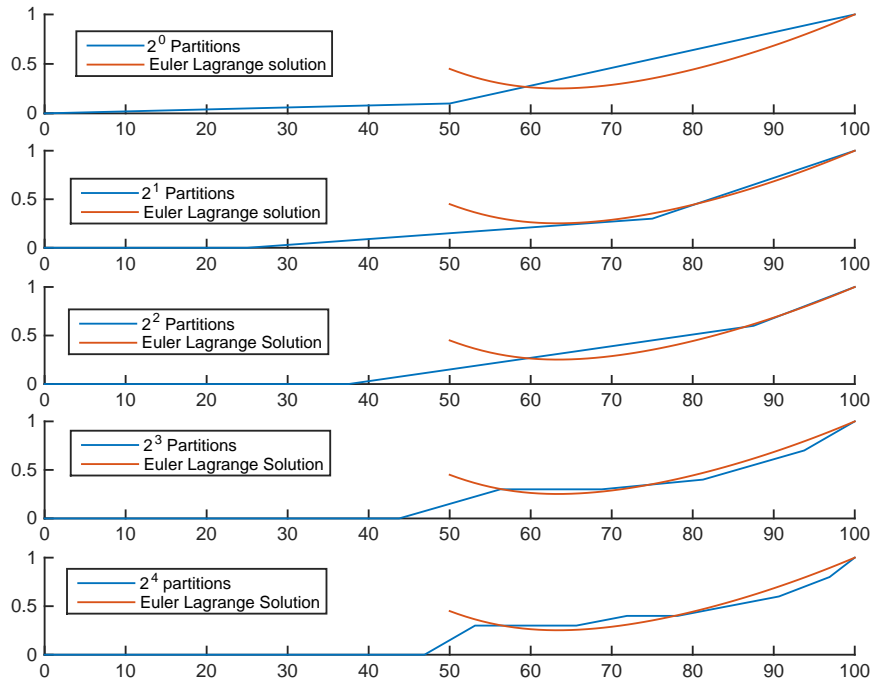
where $\Delta\Theta = \Theta_M - \Theta_m$.

Figure 3.10: Successive refinement heuristic with piecewise linear distribution.

4   SUMMARY AND CONCLUDING REMARKS

We have considered a holistic approach of utilizing an intermittent renewable energy source such as wind to support thermal inertial loads in a microgrid environment. In the first part of this thesis we have focused on the issue of reducing non-renewable power variations while adhering to the comfort specifications of end-users of the loads. We have analyzed a scenario where a common wind source is available to support a number of identical loads. We have identified a key factor of comfort range variation, due to which, even identical loads are treated differently. We have proposed an heuristic method to generalize the nature of the optimal demand response to a large number of loads, for which the exact optimal solution is difficult to obtain.

In the second part of this thesis, we have considered an additional issue of the privacy of the end-user. We have proposed a simple architecture where no information from loads is conveyed, and therefore no privacy is lost. We have calculated the optimal solution for a continuum limit of loads, which can used to control the collective behavior of the loads, without knowing their individual temperatures. We have shown that an explicit solution is analytically computable in a number of scenarios. For the cases where analytical solution is not available but parameters, such as ambient heating rate, cooling rates, are known, we have demonstrated a numerical approach to compute the solution. When such parameters are not available, we proposed and demonstrated an adaptive heuristic approach to obtain reasonable demand response from thermal loads.

Several open directions are available for further research. One direction is to generalize from the microgrid environment to the electric grid, where no information about renewable source is available, and several market structures determine the price of non-renewable consumption and utilization of renewable power sources. Another direction is to incentivize the end user to adhere to a comfort range prescription, where we allow comfort range changes but incentivize (penalize) the desired (undesired) comfort range changes.

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