SUPERGRAVITY, ADS/CFT CORRESPONDENCE AND CONFORMAL BOOTSTRAP

A Dissertation

by

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ABSTRACT

We study the $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ sector of both the SO(8) and ISO(7) gauged $\mathcal{N} = 8$ supergravity in four dimensions. By uplifting an $\mathcal{N} = 3$ critical point of the latter theory, we get a wrapped $AdS_4 \times M_6$ in ten dimensional massive type IIA supergravity. The dual three dimensional superconformal Chern-Simons theory is identified through AdS/CFT correspondence. The conjecture is tested by comparing the spin-2 Kaluza-Klein modes and spin-2 operators in the dual CFT, and by comparing the Gravitional Euclidean Action of the gravitational solution and Free Energy of the dual CFT. We review the non-perturbative method to study conformal field theory called "Conformal Bootstrap" and apply it to study CFT's with $\mathrm{F}_4/\mathrm{SU}(3)$ flavor symmetry in $6-\epsilon$ dimensions. The possibility of applying conformal bootstrap to study AdS/CFT is also discussed.

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NOMENCLATURE

ABJM	Aharony-Bergman-Jafferis-Maldacena
AdS	Anti de-Sitter Space
CFT	Conformal Field Theory
IR	Infrared
OPE	Operator Product Expansion
RG flow	Renormalization Group Flow
SCFT	Superconformal Field Theory
UV	Ultraviolet

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1. INTRODUCTION

Since its formulation [1], AdS/CFT correspondence has been playing a very important role in theoretical physics. This duality not only improves our understanding of string theory, but also introduce a new non-perturbative method to study superconformal field theories. The supergravity limit of string theory is of special interest. Let's take the duality between $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills as an example. Type IIB supergravity has a solution describing N coincident D3-branes [2]:

$$\begin{aligned} ds_{10}^2 &= \frac{1}{\sqrt{f(r)}}\eta_{\mu\nu} + \sqrt{f(r)}(dr^2 + r^2 d\Omega_5^2), \\ f(r) &= 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi\alpha'^2 g_s N, \end{aligned} \tag{1.1}$$

whose near horizon ($r \to 0$) limit is $AdS_5 \times S^5$:

$$ds_{10}^2 = \frac{L^2 dz^2}{z^2} + \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu) + L^2 d\Omega^5. \tag{1.2}$$

Roughly speaking, supergravity is a good approximation to string theory when the AdS_5 radius is much larger than string length scale $L^2 \gg \alpha'$ (which is equivalent to saying that α' correction is negligible). Since $L^4 = 4\pi g_s \alpha'^2 N$, the condition is equivalent to:

$$g_s N \gg 1. \tag{1.3}$$

Also, to suppress quantum effect so that classical supergravity is valid, the string coupling constant need to be small

$$g_s \ll 1. \tag{1.4}$$

According to AdS/CFT dictionary, string coupling is dual to Yang-Mills coupling squared, $g_s = g_{YM}^2/4\pi$, and D3 brane number N is dual to the rank of gauge group $SU(N)$. The conditions (1.3) and (1.4) then tell us that the dual CFT should be in the following limit [1]:

$$\lambda = g_{YM}^2 N \gg 1, \quad N/\lambda = 1/g_{YM}^2 \gg 1. \tag{1.5}$$

Clearly $N \gg 1$ is implied. For CFT at large N limit, the t'Hooft parameter λ plays the role of coupling constant. Supergravity is therefore dual to the strongly coupled conformal field theory. On one hand, this is good since one could use supergravity as a non-perturbative method to study superconformal field theory. On the other hand, however, it makes the testing of AdS/CFT correspondence difficult, since the strongly coupled limit of superconformal field theory is complicated. The standard examples of AdS/CFT correspondence involve maximally supersymmetric solutions and conformal field theories [1]:

$$\begin{aligned} AdS_5 \times S^5 &\quad \leftrightarrow \quad \text{Low energy effective action on D3 branes} \\ AdS_4 \times S^7 &\quad \leftrightarrow \quad \text{Low energy effective action on M2 branes} \\ AdS_7 \times S^4 &\quad \leftrightarrow \quad \text{Low energy effective action on M5 branes} \end{aligned} \tag{1.6}$$

The low energy effective action on D2 branes was long known to be the $\mathcal{N} = 4$ Super Yang-Mills theory. The low energy effective on M2 branes, on the other hand, remains mysterious for many years, it was until in [3] that people constructed an Lagrangian with six out of the eight supersymmetry manifest, which is normally referred ABJM (Aharony-Bergman-Jafferis-Maldacena) theory. As denoted by the quiver diagram in Figure 1.1. The theory contains two gauge group U(N) $\times$ U(N),

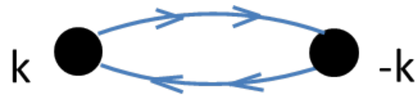

Figure 1.1: A quiver diagram for the ABJM theory.

whose kinetic terms are Chern-Simons terms whose Chern-Simons levels sum to zero. Chiral multiplets transforms as bi-fundamentals of the two gauge group. The dual solution of this theory is the $AdS_4 \times \mathbf{CP}_3$ solution of type IIA string theory, or $AdS_4 \times S^7/Z_k$ solution of M-theory [4, 5].

Let's compare the isometry of the gravational solution solution and flavor symmetries of the dual ABJM theory. Follow the discussion in [3]. In M theory, the solution is given by

$$\begin{aligned} ds^2 &= \frac{L^2}{4} ds^2_{AdS_4} + L^2 d\Omega^2_{S^7} \\ F_4 &\sim N vol(4), \qquad \frac{L}{l_p} = (32\pi^2 N)^{1/6} \end{aligned} \tag{1.7}$$

where the internal space S^7 clearly has 8 killing spinors, and therefore the solution is maximal supersymmetric. One could write S^7 as

$$ds^2_{S^7} = (d\psi' + \omega)^2 + ds^2_{\mathbf{CP3}} \tag{1.8}$$

where

$$\begin{aligned} ds^2_{\mathbf{CP3}} &= \frac{\sum_{i=1}^4 dz_i d\bar{z}_i}{\sum_{i=1}^4 z\bar{z}} - \frac{|\sum_{i=1}^4 z_i d\bar{z}_i|^2}{(\sum_{i=1}^4 z\bar{z})^2} \\ d\psi' + \omega &= \frac{\mathrm{i}}{(\sum_{i=1}^4 z\bar{z})^2}(z_i d\bar{z}_i - \bar{z}_i dz_i) \end{aligned} \tag{1.9}$$

It is then possible to perform a $\mathbf{Z}_k$ quotient, which is the Hopf fibration explained in [4], to get

$$ds^2_{S^7/\mathbf{Z}_k} = \frac{1}{k^2}(d\psi + k\omega) + ds^2_{\mathbf{CP}3} \tag{1.10}$$

where $\psi = k\psi'$. Preforming the reduction to type IIA supergravity in ten dimensional, we get a solution (in string frame)

$$\begin{aligned} ds^2_{10} &= \frac{L^3}{k}(\frac{1}{4}ds^2_{AdS_4} + ds^2_{\mathbf{CP}3}) \\ e^{2\phi} &= \frac{L^3}{k^3} \\ F_{(4)} &= \frac{3}{8}L^3 vol(4), \quad F_{(2)} = kd\omega \end{aligned} \tag{1.11}$$

Under the branching of SO(8) $\rightarrow$ SU(4) $\times$ U(1):

$$\begin{aligned} &\mathbf{8}_s \rightarrow \mathbf{6}_0 + \mathbf{1}_2 + \mathbf{1}_{-2} \\ &\mathbf{8}_v \rightarrow \mathbf{4}_1 + \bar{\mathbf{4}}_{-1} \end{aligned} \tag{1.12}$$

CP3 has only six killing spinors, which are the U(1) neutral ones. To get ABJM theory, let's start from the non-abelian $\mathcal{N} = 2$ Chern-Simons Lagrangian [6]:

$$S^{\mathcal{N}=2}_{CS} = \int \frac{k}{2\pi}\left(\int_0^1 \mathrm{Tr}[V\bar{D}^\alpha(e^{-tV}D_\alpha e^{tV})]\right) \tag{1.13}$$

the vector superfield consists of the gauge field A_μ, an auxiliary scalar field σ, and a Dirac spinor χ, and scalar D. In components, the action is simply [7]:

$$S^{\mathcal{N}=2}_{CS} = \frac{k}{4\pi}\int \mathrm{Tr}(A_{(1)} \wedge dA_{(1)} = \frac{2}{3}A_{(1)} \wedge A_{(1)} \wedge A_{(1)} - \bar{\chi}\chi + 2D\sigma). \tag{1.14}$$

where the trace are taken in the fudamental representation of the gauge group.

Introduce a few more terms, one could get a theory where the supersymmetry is enhanced to $\mathcal{N}=3$ [6,8]:

$$S^{\mathcal{N}=3}=S_{CS}^{\mathcal{N}=2}+\int d^4\theta(\bar{Q}e^VQ+\tilde{Q}e^{-V}\overline{\tilde{Q}})+\int d\theta^2(-\frac{k}{4\pi}\mathrm{Tr}\Phi^2+\tilde{Q}\Phi Q). \qquad (1.15)$$

Notice that Φ has no kinetic term, and could be integrated out to get the superpotential

$$\mathcal{W}=\frac{2\pi}{k}(\tilde{Q}T^aQ)(\tilde{Q}T^aQ) \qquad (1.16)$$

where T^a is the generator of the gauge group. The theory has an SU(2) flavor symmetry.

The ABJM theory, on the other hand, consists of two copy of (1.15) with opposite Chern-Simons level, coupled to two copy of chiral superfields through [3]:

$$\mathcal{L}_{int}=\frac{k}{8\pi}Tr[\Phi_{(2)}^2-\Phi_{(1)}^2]+Tr(B_i\Phi_{(1)}A_i)+Tr(A_i\Phi_{(2)}B_i) \qquad (1.17)$$

One could integrated out the auxilary superfield $\Phi_{(1)}$ and Φ_2 and get the superpotential

$$\mathcal{W}=\frac{2\pi}{k}\epsilon^{ab}\epsilon^{cd}\mathrm{Tr}[A_aB_bA_cB_d] \qquad (1.18)$$

In $\mathcal{N}=2$ language, the action has an explicit SU(2)$\times$ SU(2) symmetry, acting separately on A_i and B_i. If one write the action in components, the extra SU(2)$_{\mathcal{R}}$ symmetry intervenes A_i and B_i and the symmetry group is enhance to SU(4)$_{\mathcal{R}}$. Notice in (1.17) A_i and B_i plays the role of $\tilde{Q}$ and Q in (1.16), hence they form doublets of SU(2)$_{\mathcal{R}}$. SU(4) is exactly the isometry of internal space $\mathbf{CP}_3$ of the dual gravational solution.

In this thesis, we will study a specific example of AdS/CFT correspondence—the

$\mathcal{N} = 3$ theory (1.16) and its dual AdS solution. It was realized in [9] that Romans term [10] induce a net Chern-Simons level to the dual conformal field theory, meaning that the sum of Chern-Simons levels for all gauge group is proportion to the Romans mass parameter m. (Notice the ABJM theory has zero net Chern-Simons level.) Our solution would be a warped $AdS_4 \times M_6$ solution of Romans deformed type IIA supergravity in ten dimensions. The internal space M_6 has S^6 topology.

A common way to search for AdS solutions in ten dimensional type IIA/IIB or eleven dimensional M-theory is to search for critical points (vacuum solutions) with positive cosmological constants in lower dimensional gauged supergravity theories [11–16]. If the lower dimensional supergravity is a consistent truncation of the corresponding higher dimensional theory, we could in principle be able to uplift the solution to higher dimensions. Four dimensional SO(8) gauged $\mathcal{N} = 8$ supergravity is known famously as an consist reduction of eleven dimensional supergravity [17]. Recently, the dyonic ISO(7) gauged $\mathcal{N} = 8$ supergravity is shown to be a consist reduction of ten dimensional Romans deformed type IIA supergravity [18, 19]. The dual solution of (1.16) shows itself as an $\mathcal{N} = 3$ critical point of such a four dimensional maximal supergravity theory. In Chapter 2, we will first classifies the critical points in an $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ sector of $N = 8$ SO(8)/ISO(7) gauged supergravity [20, 21], and then uplift the $\mathcal{N} = 3$ solution found in ISO(7) gauged theory to a solution of massive type IIA supergravity in ten dimensions [21]. In Chapter 3, we identify the dual three dimensional superconformal field theory in three dimensions, and talk about various tests of the conjectured AdS/CFT correspondence [22].

We mentioned that one of the attractive features of AdS/CFT correspondence is that supergravity is dual to strongly coupled limited the corresponding CFT. In Chapter 4, we will introduce another non-perturbative method for conformal field theory call "Conformal Bootstrap". The idea of conformal bootstrap was introduced in the 1970s by Alexander Polyakov [23] and also by Sergio Ferrara, Raoul Gatto and Aurelio Grillo [24]. Its later application to two dimensional conformal field theories lead to the famous work of Alexander Belavin, Alexander Polyakov and Alexander Zamolodchikov [25], where two dimensional minimal models was classified. In higher dimensions, it was re-introduced in 2008 to study the particle physics model Conformal Technicolor, and later became extreme successful in the study of models such as three dimensional Ising model [26, 27], O(N) vector models [28, 29], Gross-Neveu(-Yukawa) models [30], and various superconformal theories [31–34]. We will apply this method to study CFT with $\mathrm{F}_4/\mathrm{SU}(3)$ flavor symmetry in $6-\epsilon$ dimension. We will also discuss the possibility of applying conformal bootstrap to test AdS/CFT correspondence.

Chapter 5 presents our conclusion and some discussion.

2. FOUR DIMENSIONAL SUPERGRAVITY AND CRITICAL POINTS*

The field content of four dimensional ungauged maximal supergravity is listed in Table 2.1 [35]. The theory could be derived from eleven dimensional supergravity

Fields	$g_{\mu\nu}$	$_{\mu}$	A_μ	χ	ϕ
multiplicity	1	8	28	56	70

Table 2.1: Field content of four dimensional $\mathcal{N} = 8$ supergravity.

by a seven torus reduction. The 35 complex (70 real) scalars parametrized the coset $E_{7(7)}/SU(8)$. $E_{7(7)}$ is the global symmetry, and SU(8) is the $\mathcal{R}$ symmetry group—a local symmetry whose gauge connection are composite fields made of scalars. It is a general feature of supergravity theories that bosonic fields transform as representations of global symmetry group, while fermonic fields transform as representations of $\mathcal{R}$-symmetry group.

It is possible to gauge some of the generators of the global symmetry group to get a gauged theory, for example, the famous de Witt-Nicolai theory [36] is the SO(8)$\subset$ $E_{7(7)}$ gauged version of the theory constructed in [35]. Recent years, it was noticed that the possible gauging of [35] could be classified by the so called Embedding Tensor formalism.

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*Part of the result reported in this chapter is reprinted *Holographic RG flow in a new $SO(3) \times SO(3)$ sector of ω-deformed $SO(8)$ gauged $= 8$ supergravity* by Yi Pang, C. N. Pope and Junchen Rong, published in *JHEP*, 08:122, 2015, Copyright [2015] by The Authors, [DOI:10.1007/JHEP08(2015)122]. JHEP articles are published on open access terms, with Creative Commons 4.0 (CC BY 4.0) license [https://creativecommons.org/licenses/by/4.0/] and the copyright is retained by the authors. Part of the result reported in this chapter is also reprinted with permission from *$\mathcal{N}$=3 solution in dyonic ISO(7) gauged maximal supergravity and its uplift to massive type IIA supergravity* by Yi Pang and Junchen Rong, published in Phys. Rev. D **92**, no. 8, 085037 (2015), Copyright [2015] by American Physical Society, [DOI:10.1103/PhysRevD.92.085037].

2.1 Embedding Tensor Formalism

The 4D maximal supergravity is characterised by the embedding tensor $\Theta_M{}^\alpha$ which completely specifies the gauging [37]. Embedding tensor enters the Lagrangian through the quantity

$$X_M = \Theta_M{}^\alpha t_\alpha, \quad [X_M, X_N] = -X_{MN}{}^P X_P. \tag{2.1}$$

The second equation above implies that the gauged generators form a closed algebra. Indices $M, N \ldots$ transform as **56** of $\mathrm{E}_{7(7)}$ which decompose into $\mathbf{28} \oplus \overline{\mathbf{28}}$ of SU(8) or $\mathbf{28} \oplus \mathbf{28}'$ of SL(8). The decompositions of **56** of $E_{7(7)}$ under SU(8) or SL(8) suggest two different bases in which the $\mathrm{E}_{7(7)}$ covariant quantities can be formulated.

In terms of the SL(8) basis in which $V_M = \{V_{[AB]}, V^{[AB]}\}$, the pure scalar sector of the 4D gauged maximal supergravity is given as

$$\begin{aligned} e^{-1}\mathcal{L} = & \frac{1}{8}\mathrm{Tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}) \\ & -\frac{1}{672}(X_{MN}{}^R X_{PQ}{}^S \mathcal{M}^{MP}\mathcal{M}^{NQ}\mathcal{M}_{RS} + 7X_{MN}{}^Q X_{PQ}{}^N \mathcal{M}^{MP}), \end{aligned} \tag{2.2}$$

where $\mathcal{M}^{MN}$ is the inverse of $\mathcal{M}_{MN}$ constructed from the bilinear of the 56-bein as

$$\mathcal{M}_{MN} = (LL^\dagger)_{MN}, \quad L(\phi)_M{}^{\underline{N}} = S^\dagger_M{}^{\underline{P}} L(\phi)_{\underline{P}}{}^{\underline{N}}, \tag{2.3}$$

with

$$S_{\underline{M}}{}^N = \frac{1}{4\sqrt{2}} \begin{pmatrix} \Gamma_{ij}{}^{AB} & \mathrm{i}\Gamma_{ijAB} \\ \Gamma^{ijAB} & -\mathrm{i}\Gamma^{ij}{}_{AB} \end{pmatrix}, \quad L_{\underline{M}}{}^{\underline{N}} = \exp\begin{pmatrix} 0 & \phi_{ijkl} \\ \phi^{ijkl} & 0 \end{pmatrix}. \tag{2.4}$$

The indices $\underline{M}, \underline{N}$ label the **56** irreps realized in the SU(8) basis, and therefore the unitary matrix S converts the SU(8) basis to the SL(8) basis [15]. $A, B, \ldots$ denote the fundamental representation of SL(8) while $i, j, \ldots$ denote the **8** representation of SU(8). The 2-gamma matrices ${\Gamma_{ij}}^{AB}$ comprise the generators of SO(8) in the chiral spinor representation. The position of the index is not crucial.

When the gauge group is a subgroup of $SL(8) \subset E_{7(7)}$, the embedding tensor could be expressed in the following general form:

$$X_{ABM}{}^{N} = \begin{pmatrix} -f_{ABCD}{}^{EF} & \\ & f_{ABEF}{}^{CD} \end{pmatrix}, \quad X^{AB}{}_{M}{}^{N} = \begin{pmatrix} -g^{AB}{}_{CD}{}^{EF} & \\ & g^{AB}{}_{EF}{}^{CD} \end{pmatrix}, \tag{2.5}$$

where $f_{ABCD}{}^{EF} = 2\sqrt{2}\delta^{[E}_{[A}\theta_{B][C}\delta^{F]}_{D]}$ and $g^{AB}{}_{EF}{}^{CD} = 2\sqrt{2}\delta^{[A}_{[E}\xi^{B][C}\delta^{D]}_{F]}$. θ_{BC} and ξ_{BC} are 8×8 matrix, the constraints on embedding tensor require them to satisfies the following relations [38]:

$$\theta_{AC}\chi^{CB} \propto \delta^{C}_{A}, \quad \text{or} \quad \theta_{AC}\chi^{CB} = 0 \tag{2.6}$$

The solution [39]

$$\theta = \cos(\omega)\mathbb{I}_8, \quad \theta = \sin(\omega)\mathbb{I}_8 \tag{2.7}$$

correspondes to the SO(8) gauging, the existence of the parameter ω, hence the existence of a whole one parameter family of SO(8) gauged $\mathcal{N} = 8$ supergravity in four dimensions surprised a lot of people, as the $\omega = 0$ theory, or de Witt-Nicolai theory was long assumed to be unique.

Another solution is also interesting:

$$\theta = g \cdot \mathrm{diag}(\mathbb{I}_7, 0), \quad \xi = m \cdot \mathrm{diag}(0_7, 1). \tag{2.8}$$

which corresponds to the ISO(7) gauging. ISO(7) group is defined as the group of 8 by 8 matrix that preserves the metric

$$\eta = \mathrm{diag}(\mathbb{I}_7, 0) \tag{2.9}$$

which could also be constructed from SO(8) group by Inönü-Wigner contraction. Take the $so(3)$ algebra as an example

$$[R^1, R^2] = R^3, \quad [R^2, R^3] = R^1, \quad [R^3, R^1] = R^2, \tag{2.10}$$

scale the generators as

$$R^1 \to \lambda R^1, R^2 \to \lambda R^2, \tag{2.11}$$

and take the singular limit $\lambda \to \infty$, the algebra becomes the Euclidean algebra e_2. In a similar way SO(8) could be contracted into ISO(7).

It was explained in [38] that the $SL(8)$ Cartan generator

$$\Lambda_{\mathrm{red}} = \begin{pmatrix} \mathbb{I}_7 & \\ & -7 \end{pmatrix}, \tag{2.12}$$

acts on embedding tensor and rescales the electric and magnetic coupling constant (g, m) separately. This rescaling can be compensated by a non-linear field redefinition. Consequently, the only two inequivalent choices correspond to $m = 0$ and $m \neq 0$. We will see later that this redefinition is actually inherited from ten dimensional massive type IIA supergravity.

2.2 $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ sector of $\mathcal{N}=8$ supergravity

2.2.1 Branching Rules and invariant four form

In our convention, we take the 70 scalars of $\mathcal{N}=8$ supergravity to transform as $\mathbf{35}_v$ and $\mathbf{35}_c$ representation of SO(8), while the eight supersymmetry generators transforms as $\mathbf{8}_s$ of SO(8). Denoting the $\mathbf{35}_v$ ($\mathbf{35}_c$) representation of SO(8) as (anti) self-dual invariant four forms:

$$\Phi = \frac{1}{4!}\,\phi_{ijkl} dx^i \wedge dx^j \wedge dx^k \wedge dx^l \quad \{i,j,k,l\} \in \mathbf{8}_s. \tag{2.13}$$

Finding scalars invariant under a certain subgroup of the gauge group is then equivalent to finding invariant four forms of the gauge group. We are going to follow the following branching rules

$$\begin{aligned} \mathrm{SO}(8) \subset \quad & \mathrm{SO}(3)_V \times \mathrm{SO}(3)_R \times \mathrm{SO}(3)_L \quad \subset [\mathrm{SO}(3)_V \times \mathrm{SO}(3)_R]_D \times \mathrm{SO}(3)_L \\ \mathbf{8}_s \to (\mathbf{3},\mathbf{1},\mathbf{1}) &+ (\mathbf{1},\mathbf{2},\mathbf{2}) + +(\mathbf{1},\mathbf{1},\mathbf{1}) \to (\mathbf{3},\mathbf{1}) + (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1}) \end{aligned} \tag{2.14}$$

The subscript "D" means the diagonal subgroup of $\mathrm{SO}(3)_V \times \mathrm{SO}(3)_R$. Since we have only one singlet from $\mathbf{8}_s$ after the branching, the truncated theory, consisting of all the field invariant under $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ subgroup of SO(8), would be an $\mathcal{N}=1$ supergravity. We will now work out explicitly the scalar sector of this theory. First we have to find all the scalar invariant under such a group. Branching rules tells us there are two singlets following from the branching of $\mathbf{35}_v$ representation of SO(8), and another two singlets following from the branching of $\mathbf{35}_c$. All we need to do is to construct two four form invariant under $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_D$ which are not (anti) self-dual to each other and take the self-dual and anti-self dual parts of them to get the four singlets. Starting with $\mathrm{SO}(3) \times \mathrm{SO}(4)$ subgroup, we have the following four forms:

$$
\begin{aligned}
& dx^1\wedge dx^2\wedge dx^3\wedge dx^8 \in (\mathbf{1}_{\mathrm{SO}(3)_V},\mathbf{1}_{\mathrm{SO}(4)}) \\
& dx^i\wedge dx^j\wedge dx^a\wedge dx^b \in (\mathbf{3}_{\mathrm{SO}(3)_\mathrm{V}},\mathbf{6}_{\mathrm{SO}(4)}) \text{ with } \quad i=1,2,3\in \mathrm{SO}(3) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and} \quad a\, b=4\ldots 7\in \mathrm{SO}(4) \\
& \ldots
\end{aligned}
$$

or equivalently, $\mathrm{SO}(3)_V\times\mathrm{SO}(3)_L\times\mathrm{SO}(3)_R$:

$$
\begin{aligned}
& dx^1\wedge dx^2\wedge dx^3\wedge dx^8 && \in (\mathbf{1}_{\mathrm{SO}(3)_\mathrm{V}},\mathbf{1}_{\mathrm{SO}(3)_R},\mathbf{1}_{\mathrm{SO}(3)_L}) \\
& \epsilon^i_{jk}\eta^{\hat{i}}_{ab}dx^i\wedge dx^j\wedge dx^a\wedge dx^b && \in (\mathbf{1}_{\mathrm{SO}(3)_\mathrm{V}},\mathbf{3}_{\mathrm{SO}(3)_R},\mathbf{3}_{\mathrm{SO}(3)_L}) \text{ with } \hat{i}=1,2,3\in\mathrm{SO}(3)_L \\
& \ldots
\end{aligned}
\tag{2.15}
$$

$\eta^{\hat{i}}_{ab}$ is the t'Hooft matrix. Under the $\mathrm{SO}(3)_D\times\mathrm{SO}(3)_L$ subgroup, they contains the following singlets:

$$
\begin{aligned}
& dx^1\wedge dx^2\wedge dx^3\wedge dx^8 && \in (\mathbf{1}_{\mathrm{SO}(3)_D},\mathbf{1}_{\mathrm{SO}(3)_L}) \\
& \epsilon^i_{jk}\eta^i_{ab}dx^i\wedge dx^j\wedge dx^a\wedge dx^b && \in (\mathbf{1}_{\mathrm{SO}(3)_D},\mathbf{1}_{\mathrm{SO}(3)_L}) \text{ with } i=1,2,3\in\mathrm{SO}(3)_D \\
& \ldots
\end{aligned}
\tag{2.16}
$$

Notice i is now taken to be $\mathrm{SO}(3)_D$ index, and has been contracted. The dual form are also invariant scalar to be kept, actually, they could be written as the following

$$
\begin{aligned}
& \eta^i_{ab}\eta^i_{cd}dx^a\wedge dx^b\wedge dx^c\wedge dx^d = dx^4\wedge dx^5\wedge dx^6\wedge dx^7 \\
& \eta^i_{ab}dx^i\wedge dx^a\wedge dx^b\wedge dx^8 \\
& \ldots
\end{aligned}
\tag{2.17}
$$

where we have used the t'Hooft matrix:

$$\eta^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \eta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \eta^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{2.18}$$

which transforms as **3** of $\mathrm{SO}(3)_L$, whose generators are chosen to be

$$\begin{aligned} B1 &= -\frac{1}{2}(R^{01} - R^{23}) \\ B2 &= -\frac{1}{2}(R^{02} - R^{31}) \\ B3 &= -\frac{1}{2}(R^{03} - R^{12}). \end{aligned} \tag{2.19}$$

t'Hooft matrix are invariant under $\mathrm{SO}(3)_R$, whose generators are chosen to be

$$\begin{aligned} A1 &= \frac{1}{2}(R^{01} + R^{23}) \\ A2 &= \frac{1}{2}(R^{02} + R^{31}) \\ A3 &= \frac{1}{2}(R^{03} + R^{12}). \end{aligned} \tag{2.20}$$

R^{ij}'s are $so(4)$ generators, with $(R^{ij})_{ij} = -(R^{ij})_{ji} = 1$, and other elements equal to zero.

In summary, the branching rules tell us that there are two complex scalar invariant under the $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ subgroup of SO(8) (or ISO(7)), and they could be written as

$$
\begin{aligned}
\Psi_1 &= \psi_1 dx^1\wedge dx^2\wedge dx^3\wedge dx^8 + \bar{\psi}_1 dx^4\wedge dx^5\wedge dx^6\wedge dx^7 \ , \\
\Psi_2 &= \psi_2(-dx^1\wedge dx^2\wedge dx^4\wedge dx^7 + dx^1\wedge dx^2\wedge dx^5\wedge dx^6 + dx^1\wedge dx^3\wedge dx^4\wedge dx^6 \\
&\quad + dx^1\wedge dx^3\wedge dx^5\wedge dx^7 - dx^2\wedge dx^3\wedge dx^4\wedge dx^5 + dx^2\wedge dx^3\wedge dx^6\wedge dx^7) \\
&\quad + \bar{\psi}_2(dx^1\wedge dx^4\wedge dx^5\wedge dx^8 - dx^1\wedge dx^6\wedge dx^7\wedge dx^8 + dx^2\wedge dx^4\wedge dx^6\wedge dx^8 \\
&\quad + dx^2\wedge dx^5\wedge dx^7\wedge dx^8 + dx^3\wedge dx^4\wedge dx^7\wedge dx^8 - dx^3\wedge dx^5\wedge dx^6\wedge dx^8)
\end{aligned}
\tag{2.21}
$$

To plug such an Ansatz into (2.4), we first need to specify our gamma matrix notation.

2.2.2 Gamma matrix Notation for SO(8)

SO(8) gamma matrices admit a real representation:

$$
\begin{aligned}
&\Gamma^1 = \sigma_2\otimes\sigma_2\otimes\sigma_1\otimes\sigma_0, \qquad \Gamma^2 = \sigma_2\otimes\sigma_3\otimes\sigma_0\otimes\sigma_2, \\
&\Gamma^3 = \sigma_2\otimes\sigma_0\otimes\sigma_2\otimes\sigma_3, \qquad \Gamma^4 = \sigma_2\otimes\sigma_0\otimes\sigma_2\otimes\sigma_1, \\
&\Gamma^5 = \sigma_2\otimes\sigma_1\otimes\sigma_0\otimes\sigma_2, \qquad \Gamma^6 = \sigma_2\otimes\sigma_2\otimes\sigma_3\otimes\sigma_0, \\
&\Gamma^7 = -\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2, \quad \Gamma^8 = \sigma_1\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0, \\
&\Gamma^9 = \Gamma^1\ldots\Gamma^8 = -\sigma_3\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0.
\end{aligned}
\tag{2.22}
$$

All of them are block off-diagonal:

$$
\Gamma^i = \begin{pmatrix} 0 & \hat{\Gamma}^i_{I\alpha} \\ (\hat{\Gamma}^i)^{\mathrm{T}}_{\alpha I} & 0 \end{pmatrix}. \tag{2.23}
$$

The 8 by 8 matrix $\hat{\Gamma}^{i}_{I\alpha}$ is an invariant tensor carrying all three fundamental $\{\mathbf{8}_s, \mathbf{8}_c, \mathbf{8}_v\}$ index of SO(8), such a property are normally call the "triality" of the corresponding lie algebra so(8), which could also be easily seen from the structure of the dykin diagram in Figure 2.1. Notice that since $\hat{\Gamma}^{8}_{I\alpha} = \delta_{I\alpha}$, I and α index makes

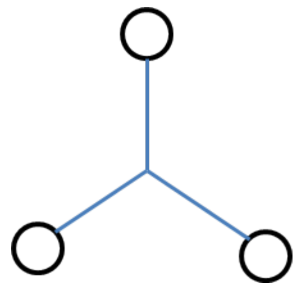


Figure 2.1: Dynkin diagram of lie algebra so(8)=D$_4$.

no difference under $SO(7)$. The SO(7)$_+$ invariant tensor $C_{+\alpha\beta\rho\lambda}$ is proportional to $\sum_{i,j=1,7}\hat{\Gamma}^{ij\mathrm{T}}_{[\alpha\beta}\hat{\Gamma}^{ij\mathrm{T}}_{\rho\lambda]}$, whilst The SO(7)$_-$ invariant tensor C_{-IJKL} is proportional to $\sum_{i,j=1,7}\hat{\Gamma}^{ij}_{[IJ}\hat{\Gamma}^{ij}_{KL]}$.

2.2.3 Scalar potential in the SO(3)$_D \times$ SO(3)$_L$ *invariant sector of SO(8) gauged supergravity*

Plug in the ansatz (2.21) into (2.4), (2.3) and then (2.2), we get the SO(3)$_D \times$ SO(3)$_L$ scalar sector in $\mathcal{N} = 8$ SO(8) gauged supergravity.

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}\Big((\partial\phi_1)^2 + \sinh^2\phi_1(\partial\sigma_1)^2\Big) - 3\Big((\partial\phi_2)^2 + \sinh^2\phi_2(\partial\sigma_2)^2\Big) - V. \quad (2.24)$$

where we have redefined the scalars as

$$\psi_1 = \tfrac{1}{2}\phi_1 e^{\mathrm{i}\sigma_1}, \qquad \psi_2 = \tfrac{1}{2}\phi_2 e^{\mathrm{i}\sigma_2}\ , \quad (2.25)$$

The potential can be expressed in terms of the superpotential W theory as

$$V = g^2 \cdot 2\Big(4|\frac{\partial W}{\partial \phi_1}|^2 + \frac{2}{3}|\frac{\partial W}{\partial \phi_2}|^2 - 3|W|^2\Big), \tag{2.26}$$

In terms of ζ_1, ζ_2 defined as

$$\zeta_1 = \tanh \tfrac{1}{2}\phi_1 e^{-\mathrm{i}\sigma_1}, \qquad \zeta_2 = \tanh \tfrac{1}{2}\phi_2 e^{\mathrm{i}\sigma_2} \ , \tag{2.27}$$

the superpotential W can be expressed as

$$W = -e^{-\mathrm{i}\omega}(1-|\zeta_1|^2)^{-\frac{1}{2}}(1-|\zeta_2|^2)^{-3}\Big[4\zeta_2^3 e^{2\mathrm{i}\omega} - 3\zeta_2^4 - 1 + \zeta_1\zeta_2^6 e^{2\mathrm{i}\omega} + 3\zeta_1\zeta_2^2 e^{2\mathrm{i}\omega} - 4\zeta_1\zeta_2^3\Big]. \tag{2.28}$$

The potential appears to be more complicated

$$\begin{aligned} 64\frac{V}{g^2} &= -256\cosh^4\phi_2 \\ &\quad +2\cosh\phi_1\cosh^2\phi_2(-57 - 20\cosh 2\phi_2 + 13\cosh 4\phi_2 + 24\sinh^4\phi_2\cos 4\sigma_2) \\ &\quad +8\sinh\phi_1\sinh^3 2\phi_2\,(\cos(\sigma_1 - 3\sigma_2) + 3\cos(\sigma_1+\sigma_2)) \\ &\quad +4\sinh^3\phi_2\,\Big\{16(\cos(2\omega - 3\sigma_2) + 3\cos(2\omega+\sigma_2))(1-\cosh\phi_1\cosh^3\phi_2) \\ &\quad +\sinh\phi_1\sinh^3\phi_2\,(6\sin(2\omega+\sigma_1)\sin 2\sigma_2 - 2\cos(2\omega+\sigma_1-6\sigma_2) \\ &\quad -8\sinh\phi_1(3\sinh\phi_2 + \sinh 3\phi_2)\ \cos(2\omega-\sigma_1) \\ &\quad -\tfrac{3}{2}\sinh\phi_1(17\sinh\phi_2 + 5\sinh 3\phi_2)\ \cos(2\omega+\sigma_1)\cos 2\sigma_2\Big\}. \end{aligned} \tag{2.29}$$

Notice the potential is invariant under the following transformation:

$$\omega \to \omega + \pi/4, \quad \sigma_1 \to \sigma_1 - \pi/2, \quad \sigma_2 \to \sigma_2 + \pi/2 \tag{2.30}$$

$$\omega \to -\omega, \quad \sigma_1 \to -\sigma_1, \quad \sigma_2 \to -\sigma_2 \tag{2.31}$$

Therefore, for inequivalent theory, ω lives in $[0, \pi/8]$.

It is useful to compare our sector with other sectors that has been studied. The symmetry group $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ could be embedded in $SO(8)$ along the following chain

$$\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L \subset \mathrm{G}_2 \subset \mathrm{SO}(7) \subset \mathrm{SO}(8). \tag{2.32}$$

On the other hand the SU(3) invariant sector of $\mathcal{N} = 8$ supergravity has been throughly studied both in the original de Wit and Nicolai theory [11, 12, 40] and in the ω deformed case [41], with the group embedding

$$\mathrm{SU}(3) \subset \mathrm{G}_2 \subset \mathrm{SO}(7) \subset \mathrm{SO}(8). \tag{2.33}$$

Using Newton-Raphson method, given the potential (2.29), we scanned for its critical points. All the critical points with G_2 or SO(7) symmetry are found, and they agree with previous results. We also found two critical points with $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ symmetry. One of them preserving $\mathcal{N} = 3$ supersymmetry was first discovered by [42]. For this critical point, the evolution of the two complex scalars and the associated cosmological constant with ω are displayed in Figure 2.2. This is the first $\mathcal{N} = 3$ vacuum in SO(8) gauged $\mathcal{N} = 8$ supergravity. The mass spectrum of the fluctuations around this vacuum is given by [42]:

$$\begin{aligned} m^2 L_0^2 : \; & 1 \times (3(1+\sqrt{3})); 6 \times (1+\sqrt{3}); 1 \times (3(1-\sqrt{3})); 6 \times (1-\sqrt{3}); \\ & 4 \times (-\frac{9}{4}); 18 \times (-2); 12 \times (-\frac{5}{4}); 22 \times 0 \; . \end{aligned} \tag{2.34}$$

By virtue of supersymmetry, the Breitenlohner-Freedman bound [43] is respected.

The other $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ critical point is non-supersymmetric. However, it turns out to be stable against fluctuations. The mass spectrum of the perturbations around this vacuum depends on ω. For $\omega = \pi/8$, it is listed below

$$
\begin{aligned}
m^2 L_0^2 : \; & 1 \times (6.72079); 1 \times (5.29013); 4 \times (-1.96647); 9 \times (-1.73861); \\
& 9 \times (-1.60284); 1 \times (-1.59124); 8 \times (-1.18046); 5 \times (-0.98076); \\
& 4 \times (-0.73134); 5 \times (0.61746); 1 \times (0.58185); 22 \times 0 \; .
\end{aligned}
\tag{2.35}
$$

Notice again that the Breitenlohner-Freedman bound is not violated at this critical point. For this critical point, the evolution of the two complex scalars and the associated cosmological constant with ω are displayed in Figure 2.3. It should be noticed that both critical points cease to exist when ω goes to 0 as the values of cosmological constant diverge. For completeness, in Table 2.2, we list all the critical points contained in the $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_R$ invariant sector. Three remarks need to be mentioned:

- a) The two transformations (2.30) and (2.31) combine into a symmetry for $\omega = \pi/8$:

  $$\sigma_1 \to -\sigma_1 - \pi/2, \quad \sigma_2 \to -\sigma_2 + \pi/2 \tag{2.36}$$

  which is explicit in the table.

- b) Points related by

  $$\phi_i \to -\phi_i, \quad \sigma_i \to \sigma_i + \pi$$

  have the same location in the complex plain, hence should not be treated as different points.

- c) It is interesting to see that there are two critical points with the same cosmological constant, but with different residue symmetry, one with G_2, another with $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$.

symmetry	ϕ_1	σ_1	ϕ_2	σ_2	$V(g=1)$	stability
$\mathcal{N}=8$ $SO(8)$	0	$-$	0	$-$	-6	$\surd$
$\mathcal{N}=0$ $SO(7)_-$	0.4195	$\frac{\pi}{2}$	0.4195	$-\frac{\pi}{2}$	-6.7482	$\times$
$\mathcal{N}=0$ $SO(7)_-$	0.6406	$-\frac{\pi}{2}$	0.6406	$\frac{\pi}{2}$	-7.7705	$\times$
$\mathcal{N}=0$ $SO(7)_+$	0.4195	π	0.4195	π	-6.7482	$\times$
$^*\mathcal{N}=0$ $SO(7)_+$	0.6406	0	0.6406	0	-7.7705	$\times$
$\mathcal{N}=1$ G_2	0.4840	$\frac{3\pi}{4}$	0.4195	$\frac{3\pi}{4}$	-7.0397	$\surd$
$\mathcal{N}=1$ G_2	0.6579	-1.9693	0.6579	1.9693	-7.9430	$\surd$
$^*\mathcal{N}=1$ G_2	0.6579	0.3985	0.6579	-0.3985	-7.9430	$\surd$
$^*\mathcal{N}=0$ G_2	$\log\vartheta$	$-\frac{\pi}{4}$	$\log\vartheta$	$\frac{\pi}{4}$	-4ϑ	$\times$
$^*\mathcal{N}=3$ $SO(3)_D\times SO(3)_L$	$\log\frac{\vartheta}{3}$	$\frac{3\pi}{4}$	$\log\vartheta$	$\frac{\pi}{4}$	-4ϑ	$\surd$
$^*\mathcal{N}=0$ $SO(3)_D\times SO(3)_L$	0.3114	$-\frac{\pi}{4}$	0.9914	$\frac{\pi}{4}$	-10.271	$\surd$

Table 2.2: Critical points in the $\omega=\pi/8$ theory of $\mathcal{N}=8$ SO(8) gauged supergavity. (We use ϑ to denote the number $\sqrt{3+2\sqrt{3}}$. The mass spectrum of fluctuation around the critical points are independent of ω, except for the last one. $\omega\to 0$, points with marked with "*" disappear, while the two points with SO(7)_ symmetry merge into one become degenerate in energy.) Table reprinted with permission from [20].

We listed all the known critical points in $\mathrm{SO}(3)_D\times\mathrm{SO}(3)_L$ sector of SO(8) gauged supergravity in Table 2.2.

2.2.4 Scalar potential of the $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ *invariant sector in ISO(7) gauged supergravity*

Plug in the ansatz (2.21) into (2.4) and (2.3), using (2.8) in (2.2), we get the $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ scalar sector in $\mathcal{N}=8$ dyonic ISO(7) gauged supergravity.

$$
\begin{aligned}
V = \ & \frac{1}{64} g^2 (\cosh\phi_2 - \cos\sigma_2 \sinh\phi_2)^2 (\cos\sigma_2 \sinh\phi_2 + \cosh\phi_2) \\
& \times \Big(196 \cos(\sigma_1 + 3\sigma_2) \sinh\phi_1 \sinh^3\phi_2 + 4\cos(\sigma_1 - 3\sigma_2) \sinh\phi_1 \sinh^3\phi_2 \\
& + 35 \cos(\sigma_1 - \sigma_2) \sinh\phi_1 \sinh\phi_2 + 467 \cos(\sigma_1 + \sigma_2) \sinh\phi_1 \sinh\phi_2 \\
& + 47 \cos(\sigma_1 - \sigma_2) \sinh\phi_1 \sinh 3\phi_2 - 97 \cos(\sigma_1 + \sigma_2) \sinh\phi_1 \sinh 3\phi_2 \\
& + 4 \cos\sigma_1 \sinh\phi_1 \cosh\phi_2 (28 \cos 2\sigma_2 \sinh^2\phi_2 + 47) \\
& - 2\cosh\phi_1 (28 \cos 3\sigma_2 \sinh^3\phi_2 + 101 \cos\sigma_2 \sinh\phi_2 - 7\cos\sigma_2 \sinh 3\phi_2 \\
& + \cosh\phi_2 (200 \cos 2\sigma_2 \sinh^2\phi_2 + 226) - 50 \cosh 3\phi_2) \\
& - 28 \cos\sigma_1 \sinh\phi_1 \cosh 3\phi_2 - 768 \Big) \\
& + g m \sin^2\sigma_2 \sinh^2\phi_2 (\cos\sigma_2 \sinh\phi_2 - \cosh\phi_2)^3 \times \Big(3 \sin\sigma_1 \sinh\phi_1 \cosh\phi_2 \\
& + \sinh\phi_2 \left(4 \sin\sigma_2 \cosh\phi_1 - \sinh\phi_1 \left(3 \sin\sigma_1 \cos\sigma_2 + 4 \sin\sigma_2 \cos\sigma_1\right)\right) \Big) \\
& + \frac{1}{2} m^2 \left(\cosh\phi_1 - \cos\sigma_1 \sinh\phi_1\right) \left(\cosh\phi_2 - \cos\sigma_2 \sinh\phi_2\right)^6 .
\end{aligned} \tag{2.37}
$$

the potential can be expressed in terms of a superpotential W, with

$$
V = 2\Big(4 \big|\frac{\partial W}{\partial \phi_1}\big|^2 + \frac{2}{3} \big|\frac{\partial W}{\partial \phi_2}\big|^2 - 3|W|^2 \Big), \tag{2.38}
$$

where

$$
\begin{aligned}
W =\ & \frac{g}{4}(\cosh\frac{\phi_2}{2} + e^{\mathrm{i}\sigma_2}\sinh\frac{\phi_2}{2})^2(\cosh\frac{\phi_2}{2} - e^{\mathrm{i}\sigma_2}\sinh\frac{\phi_2}{2})^3 \\
& \times\Big(\cosh\frac{\phi_2}{2}(7\cosh\frac{\phi_1}{2} - e^{\mathrm{i}\sigma_1}\sinh\frac{\phi_1}{2}) + e^{\mathrm{i}\sigma_2}(\cosh\frac{\phi_1}{2} - 7e^{\mathrm{i}\sigma_1}\sinh\frac{\phi_1}{2})\sinh\frac{\phi_2}{2}\Big) \\
& + \frac{\mathrm{i}m}{4}(\cosh\frac{\phi_1}{2} - e^{\mathrm{i}\sigma_1}\sinh\frac{\phi_1}{2})(\cosh\frac{\phi_2}{2} - e^{\mathrm{i}\sigma_2}\sinh\frac{\phi_2}{2})^6. \qquad (2.39)
\end{aligned}
$$

The location of the critical point depends on m/g, however, as mentioned before, the effects due to different m/g can be compensated by a nonlinear field redefinition. Therefore we can choose this ratio to be the one most convenient for our purpose. In finding the locations of the critical points, we choose

$$
\frac{m}{g} = 2. \qquad (2.40)
$$

The scalar potential (2.37) possesses an $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_D$ invariant stationary point preserving $\mathcal{N} = 3$ supersymmetry of the original $\mathcal{N} = 8$ theory which lies outside the residual $\mathcal{N} = 1$ supersymmetry of the truncated theory. In terms of the complexified fields

$$
\xi_1 = \tanh\phi_1 e^{\mathrm{i}\sigma_1}, \quad \xi_2 = \tanh\phi_2 e^{\mathrm{i}\sigma_2}, \qquad (2.41)
$$

the $\mathcal{N} = 3$ point is given by

$$
\xi_1 = \frac{3}{5} - \frac{2\mathrm{i}}{5}, \quad \xi_2 = \frac{\mathrm{i}}{2}, \qquad (2.42)
$$

The mass spectrum of the fluctuations around this vacuum has been obtained previously in [42] by a group theoretic method without referring to the detailed position of the critical point:

$$m^2 L_0^2 : \quad 1 \times (3(1+\sqrt{3})); \quad 6 \times (1+\sqrt{3}); \quad 1 \times (3(1-\sqrt{3})); \quad 6 \times (1-\sqrt{3}); \\ 4 \times (-\frac{9}{4}); \quad 18 \times (-2); \quad 12 \times (-\frac{5}{4}); \quad 22 \times 0, \tag{2.43}$$

where the AdS radius squared $L_0^2 = -6/V$, $V(g=1, m=2) = -32/\sqrt{3}$. (The integer to the left of the multiplication sign indicates the degeneracy of the mass eigenvalue, while the number to the right indicates the corresponding mass squared.) There is another $\mathcal{N}=1$ critical point with G_2 global symmetry:

$$\xi_1 = \xi_2 = -\frac{\mathrm{i}}{4}. \tag{2.44}$$

To obtain this critical point, we have combined the Newton-Raphson method with the "inverse Symbolic Calculator" technique. The mass spectrum of the scalar fluctuations around this vacuum is given by

$$m^2 L_0^2 : \quad 1 \times (4 \pm \sqrt{6}); \quad 14 \times 0; \quad 27 \times (-\tfrac{1}{6}(11-\sqrt{6})), \tag{2.45}$$

where the AdS radius squared $L_0^2 = -6/V$, $V(g=1, m=2) = -512\sqrt{3}/(25\sqrt{5})$. Besides the supersymmetric critical points, there are three more nonsupersymmetric critical points, preserving $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_D$, G_2 and $\mathrm{SO}(7)_{\mathrm{v}}$ global symmetries respectively. The $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_D$ invariant critical point is located at

$$\xi_1 = 0.353669 + 0.0552267\mathrm{i}, \quad \xi_2 = -0.0293804 + 0.534729\mathrm{i}, \tag{2.46}$$

with the mass spectrum given by

$$
\begin{aligned}
m^2L_0^2: \quad & 1\times(6.72740); \quad 1\times(5.28662); \quad 4\times(-1.96422); \quad 9\times(-1.75110);\\
& 9\times(-1.58816); \quad 1\times(-1.58552); \quad 8\times(-1.17591); \quad 5\times(-0.98271);\\
& 4\times(-0.72962); \quad 5\times(0.62977); \quad 1\times(0.58436); \quad 22\times 0, \qquad (2.47)
\end{aligned}
$$

where the AdS radius squared $L_0^2 = -6/V$, $V(g=1, m=2) = -18.662034$. The G_2 invariant point is given by

$$\xi_1 = \xi_2 = \frac{\mathrm{i}}{2}. \qquad (2.48)$$

The associated mass spectrum has the simple structure

$$m^2L_0^2: \quad 2\times 6; \quad 14\times 0; \quad 54\times(-1), \qquad (2.49)$$

where the AdS radius squared $L_0^2 = -6/V$, $V(g=1, m=2) = -32/\sqrt{3}$. It should be emphasized that the spectra associated with the nonsupersymmetric $\mathrm{SO}(3)_R\times\mathrm{SO}(3)_D$-and G_2-invariant critical points lie above the Breitenlohner-Freedman (BF) bound. The last $\mathrm{SO}(7)_{\mathrm{v}}$-invariant critical point is located at

$$\phi_1 = \phi_2 = -\tfrac{1}{6}\log\tfrac{5}{4}, \quad \sigma_1 = \sigma_2 = 0. \qquad (2.50)$$

The mass spectrum of the $\mathrm{SO}(7)_{\mathrm{v}}$-invariant critical point reads

$$m^2L_0^2: \quad 1\times 6; \quad 7\times 0; \quad 35\times(-\tfrac{6}{5}); \quad 27\times(-\tfrac{12}{5}), \qquad (2.51)$$

where the AdS radius squared $L_0^2 = -6/V$, $V(g=1, m=2) = -15\times 5^{\frac{1}{6}}/2^{\frac{1}{3}}$. This point is unstable against fluctuations as the mass squared of some scalar modes is

symmetry	ξ_1	ξ_2	$V(g=1, m=2)$	stability
$\mathcal{N}=0$ SO(7)$_-$	$-\frac{1}{6}\log\frac{5}{4}$	$-\frac{1}{6}\log\frac{5}{4}$	$-15\times 5^{\frac{1}{6}}/2^{\frac{1}{3}}$	$\times$
$\mathcal{N}=1$ G_2	$-i/4$	$-i/4$	$-512\sqrt{3}/(25\sqrt{5})$	$\surd$
$\mathcal{N}=0$ G_2	$i/2$	$i/2$	$-32/\sqrt{3}$	$\surd$
$\mathcal{N}=3$ $SO(3)_D\times SO(3)_L$	$\frac{3}{5}-\frac{2i}{5}$	$i/2$	$-32/\sqrt{3}$	$\surd$
$\mathcal{N}=0$ $SO(3)_D\times SO(3)_L$	0.353 +0.0552i	-0.0293 +0.534i	-18.6	$\surd$

Table 2.3: Critical points in the dyonic $ISO(7)$ gauged $\mathcal{N}=8$ supergavity.

below the BF bound. The scalar mass spectra associated with the two G_2-invariant critical points and the single $SO(7)_v$-invariant point coincide with those given in [16], where the G_2- and $SO(7)_v$-invariant critical points in $D=4$ maximal supergravities with all gaugings are analyzed.

Details of all the critical points in the $SO(3)_R\times SO(3)_D$-invariant sector are summarized in Table 2.3.

2.3 Uplifting critical point into solution of massive type IIA supergravity

2.3.1 Supersymmetric $SO(3)_D\times SO(3)_L$ invariant solution in massive IIA

It was realized in [18] that the four dimensional dyonic ISO(7) gauged supergavity is a consistent truncation of ten dimension massive type IIA supergravity. An solution of the lower dimensional theory would then automatically be a solution of the higher dimensional theory. In this and the following subsection, we will lift the $\mathcal{N}=3$ $SO(3)_D\times SO(3)_L$ and the two G_2 critical points to 10D.

In terms of the auxiliary coordinates on S^6

$$\begin{aligned}
\mu^1 &= \sin\xi\cos\theta_1\cos\chi_1, \quad \mu^2 = \sin\xi\cos\theta_1\sin\chi_1, \\
\mu^3 &= \sin\xi\sin\theta_1\cos\psi, \quad \mu^4 = \sin\xi\sin\theta_1\sin\psi, \\
\nu^1 &= \cos\xi\cos\theta_2, \quad \nu^2 = \cos\xi\sin\theta_2\cos\chi_2, \\
\nu^3 &= \cos\xi\sin\theta_2\sin\chi_2,
\end{aligned} \tag{2.52}$$

which satisfy $\sum_{A=1}^4 \mu^A\mu^A + \sum_{i=1}^3 \nu^i\nu^i = 1$, the metric on the round S^6 takes the form

$$\begin{aligned}
ds^2_{S^6} &= d\xi^2 + \sin^2\xi d\Omega_3^2 + \cos^2\xi d\Omega_2^2 \\
&= d\xi^2 + \sin^2\xi\left(d\theta_1^2 + \cos^2\theta_1 d\chi_1^2 + \sin^2\theta_1 d\psi^2\right) \\
&\quad + \cos^2\xi(d\theta_2^2 + \cos^2\theta_2 d\chi_2^2).
\end{aligned} \tag{2.53}$$

To lift the solution of the 4D dyonic ISO(7) gauged supergravity to that in the 10D massive type IIA supergravity, we utilize the uplift formulas given in [18], in which the internal components of the 10D metric, the dilaton, and various form fields are constructed in terms of the SL(7)-covariant blocks of the $D = 4$ scalar matrix $\mathcal{M}_{MN}$:

$$\begin{aligned}
g^{mn} &= \frac{1}{4}g^2\Delta K^m_{IJ}K^n_{KL}\mathcal{M}^{IJ,KL}, \\
e^{-\frac{3}{2}\hat{\phi}} &= -g^{mn}\hat{A}_m\hat{A}_n + \Delta x_I x_J \mathcal{M}^{I8J8}, \\
\hat{A}_m &= \frac{1}{2}g\Delta g_{mn}K^n_{IJ}x_K\mathcal{M}^{IJK8}, \\
\hat{A}_{mn} &= -\frac{1}{2}\Delta g_{pm}K^p_{IJ}\partial_n x^K \mathcal{M}^{IJ}{}_{K8}, \\
\hat{A}_{mnp} &= \hat{A}_m\hat{A}_{np} + \frac{1}{8}g\Delta g_{mq}K^q_{IJ}K^{KL}_{np}\mathcal{M}^{IJ}{}_{KL},
\end{aligned} \tag{2.54}$$

where $K^m_{IJ} = 2g^{-2}\mathring{g}^{mn}x_{[I}\partial_n x_{J]}$, $K^{IJ}_{mn} = 4g^{-2}\partial_{[m}x_I\partial_{n]}x_J$. Due to our gamma matrix notation, $\{x^I\}$ are related to $\{\mu^A, \nu^i\}$ by the similarity transformation

$$\mathcal{S} = \mathrm{diag}(1,1,1,1,-1,-1,1), \tag{2.55}$$

which brings the SL(7)-covariant blocks of the $D = 4$ scalar matrix $\mathcal{M}_{MN}$ into a form invariant under the standard $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ transformation given in [20]. The convention of $10D$ massive type IIA supergravity, is adapted to be in accord with [44], where the Lagrangian is take to be

$$\begin{aligned}\mathcal{L}_{10} = R*\mathbf{1} &- \frac{1}{2}*d\phi\wedge d\phi - \frac{1}{2}e^{\frac{3}{2}\phi}*F_{(2)}\wedge F_{(2)} - \frac{1}{2}e^{-\phi}*F_{(3)}\wedge F_{(3)}\\ &-\frac{1}{2}e^{-\frac{1}{2}\phi}*F_{(4)}\wedge F_{(4)} - \frac{1}{2}dA_{(3)}\wedge dA_{(3)}\wedge A_{(2)}\\ &-\frac{1}{6}m dA_{(3)}\wedge(A_{(2)})^3 - \frac{1}{40}m^2(A_{(2)})^5 - \frac{1}{2}m^2 e^{\frac{5}{2}\phi}*\mathbf{1}\end{aligned} \tag{2.56}$$

Define various field strength

$$\begin{aligned}F_{(2)} &= dA_{(1)} + mA_{(2)}, \quad F_{(3)} = dA_{(2)}\\ F_{(4)} &= dA_{(3)} + A_{(1)}\wedge dA_{(2)} + \frac{1}{m}A_{(2)}\wedge A_{(2)},\end{aligned} \tag{2.57}$$

and the equation of motions and Bianchi identities becomes

$$\begin{aligned}&d(e^{\frac{1}{2}\phi}*F_{(4)}) = -F_{(3)}\wedge F_{(4)}, \quad d(e^{\frac{3}{2}\phi}*F_{(2)}) = -e^{\frac{1}{2}\phi}*F_{(4)}\wedge F_{(3)},\\ &d(e^{-\phi}*F_{(3)}) = -\frac{1}{2}F_{(4)}\wedge F_{(4)} - me^{\frac{3}{2}\phi}*F_{(2)} - e^{\frac{1}{2}\phi}*F_{(4)}\wedge F_{(2)},\\ &d*d(\phi) = -\frac{5}{4}e^{\frac{5}{2}\phi}vol_{(10)} - \frac{3}{4}e^{\frac{3}{2}\phi}*F_{(2)}\wedge F_{(2)} + \frac{1}{2}e^{-\phi}*F_{(3)}\wedge F_{(3)}\\ &\qquad\qquad - \frac{1}{4}e^{\frac{1}{2}\phi}*F_{(4)}\wedge F_{(4)},\\ &dF_{(4)} = F_{(2)}\wedge F_{(3)}, \quad dF_{(3)} = 0, \quad dF_{(2)} = mF_{(3)}.\end{aligned} \tag{2.58}$$

the following scalings leave the Lagrangian and equation of motions invariant:

$$\begin{aligned}
&\hat{A}_{(1)} \to \tau \hat{A}_{(1)}, \quad \hat{A}_{(2)} \to \tau \hat{A}_{(2)}, \quad \hat{A}_{(3)} \to \tau^2 \hat{A}_{(3)}, \quad d\hat{s}_{10}^2 \to \tau^{\frac{5}{4}} d\hat{s}_{10}^2, \quad e^{\hat{\phi}} \to \tau^{-\frac{1}{2}} e^{\hat{\phi}}, \\
&\hat{A}_{(1)} \to \kappa \hat{A}_{(1)}, \quad \hat{A}_{(2)} \to \kappa^2 \hat{A}_{(2)}, \quad \hat{A}_{(3)} \to \kappa^3 \hat{A}_{(3)}, \quad d\hat{s}_{10}^2 \to \kappa^2 d\hat{s}_{10}^2, \quad m \to \frac{m}{\kappa}.
\end{aligned} \tag{2.59}$$

Notice that the second scaling is merely based on the dimensionality. The scaling symmetry of the 4D theory (2.12) reflects itself in the 10D theory as a combination of the above two scalings with $\tau = \lambda^{20}, \kappa = \lambda^{-14}$, where λ is identified as the parameter of Λ_{red} in (2.12).

Notice also that the gauge transformations

$$\begin{aligned}
A_{(1)} &\to A_{(1)} - d\Lambda_{(0)} - m\Lambda_{(1)}, \quad A_{(2)} \to A_{(2)} + d\Lambda_{(1)}, \\
A_{(3)} &\to A_{(3)} + d\Lambda_{(3)} - d\Lambda_{(0)} \wedge A_{(2)} - m\Lambda_{(1)} \wedge A_{(2)} - \frac{m}{2}\Lambda_{(1)} \wedge d\Lambda_{(1)}.
\end{aligned} \tag{2.60}$$

leave invariant the Lagrangian and equation of motion. The Romans parameter m, induce a Stueckelberg type of transformation, where a two form could "eat" a one form, and become massive.

The 10D solution corresponding to the $\mathcal{N} = 3$ critical point is then obtained as follows:

$$L^{-2} d\hat{s}_{10}^2 = \Delta^{-1}(\frac{3\sqrt{3}}{16} ds^2_{AdS_4}) + g_{mn} dy^m dy^n, \tag{2.61}$$

where

$$\Delta = 3^{\frac{9}{8}} 2^{-\frac{3}{4}} (\cos 2\xi + 3)^{-\frac{1}{8}} \Xi^{-\frac{1}{4}}, \qquad \Xi = (24\cos 2\xi + 3\cos 4\xi + 37), \tag{2.62}$$

and the internal metric on the deformed S^6 is given as

$$
\begin{aligned}
g_{mn}dy^m dy^n = \tfrac{3\sqrt{3}}{4}(\Delta\Xi)^{-1}\Big[& -\sin^2 2\xi d\xi^2 \\
& +8(\cos 2\xi+3)d\mu\cdot d\mu + 4(\cos 2\xi+3)d\nu\cdot d\nu \\
& +16\mu^A\eta^i_{AB}d\mu^B\epsilon^{ijk}\nu^j d\nu^k - \frac{16}{\cos 2\xi+3}(d\mu^A\eta^i_{AB}\mu^B\nu^i)^2\Big],
\end{aligned}
\tag{2.63}
$$

where η^i's are the generators of $\mathrm{SO}(3)_R$ embedded in $\mathrm{SO}(4)\simeq\mathrm{SO}(3)_R\times\mathrm{SO}(3)_L$,

$$
\eta^1=\begin{pmatrix} 0&0&0&-1\\ 0&0&-1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix},\quad
\eta^2=\begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ -1&0&0&0\\ 0&1&0&0 \end{pmatrix},\quad
\eta^3=\begin{pmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{pmatrix}.
\tag{2.64}
$$

Denote $\mathcal{K}^i\equiv\mu^A\eta^i_{AB}d\mu^B$, various p-form fields can then be expressed as

$$
\begin{aligned}
L^{-1}e^{\frac{3}{4}\phi_0}\hat{A}_{(1)} &= \mathcal{K}^i\nu^i\times\frac{2}{\cos 2\xi+3}, \\
L^{-2}e^{-\frac{1}{2}\phi_0}\hat{A}_{(2)} &= -\Xi^{-1}\Big[-8d\nu^i\wedge\mathcal{K}^i + 6\sin 2\xi d\xi\wedge\nu^i\mathcal{K}^i \\
&\quad +2(3\cos 2\xi+5)\nu^i\eta^i_{AB}d\mu^A\wedge d\mu^B \\
&\quad -(3\cos 2\xi+5)\epsilon^{ijk}\nu^i d\nu^j\wedge d\nu^k\Big],
\end{aligned}
\tag{2.65}
$$

$$
\begin{aligned}
L^{-3}e^{\frac{1}{4}\phi_0}\hat{A}_{(3)} &= -\Xi^{-1}\Big[6\sin 2\xi\epsilon^{ijk}d\xi\wedge\mathcal{K}^i\nu^j\wedge d\nu^k \\
&\quad +2(3\cos 2\xi+5)\epsilon^{ijk}\nu^i d\nu^j\wedge d\mu^A\wedge\eta^k_{AB}d\mu^B \\
&\quad +4\epsilon^{ijk}\mathcal{K}^i\wedge d\nu^j\wedge d\nu^k + \\
&\quad \frac{8}{3}\csc^2\xi\epsilon^{ijk}\mathcal{K}^i\wedge\mathcal{K}^j\wedge\mathcal{K}^k\Big] + \frac{3\sqrt{3}}{8}\Omega_{(3)}.
\end{aligned}
\tag{2.66}
$$

where $d\Omega_{(3)} = \mathrm{vol}(AdS_4)$ which is the volume element of the "unit" AdS_4. Finally, the 10D dilaton is given by

$$e^{-\frac{3}{2}\hat{\phi}} = e^{-\frac{3}{2}\phi_0} \frac{\Delta \Xi}{3\sqrt{3}(\cos 2\xi + 3)}. \tag{2.67}$$

Notice that everything is written in terms of $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_D$ invariant quantities (any function of ξ is invariant as $\mu \cdot \mu = \sin^2 \xi$ is an invariant quantity). Here we have introduced two constants $L^2 = 2^{-\frac{1}{12}} g^{-25/12} m^{1/12}$ and $e^{\phi_0} = 2^{\frac{5}{6}} g^{\frac{5}{6}} m^{-\frac{5}{6}}$ using the scaling symmetries (2.59). It is easy to checked that our solution satisfies the equations of motion of massive type IIA supergravity.

2.3.2 G_2*-invariant solutions in massive type IIA*

In our notation, we can write down the almost complex structure on unit S^6 as

$$J_{(2)} = \frac{1}{2!} J_{mn} dy^m \wedge dy^n = \mathcal{K}^i \wedge d\nu^i + \frac{1}{2} \nu^i \eta^i_{AB} d\mu^A \wedge d\mu^B + \frac{1}{2} \epsilon^{ijk} \nu^i d\nu^j \wedge d\nu^k, \tag{2.68}$$

which satisfies $J_{mn} J^{nl} = -\delta^l_m$, and also $-\frac{1}{2} J_{(2)} \wedge J_{(2)} = *_6 J_{(2)}$. The parallel torsion of $J_{(2)}$ is

$$G_{(3)} = -\frac{1}{3} dJ_{(2)}. \tag{2.69}$$

Then $H_{(3)} \equiv *_6 G_{(3)}$ satisfies the relation

$$dH_{(3)} = 2 J_{(2)} \wedge J_{(2)}, \tag{2.70}$$

where "$*_6$" is the Hodge dual defined with respect to the S^6 metric. The uplift of the $\mathcal{N}=1$ G_2-invariant critical point gives rise to the 10D solution

$$
\begin{aligned}
L^{-2}d\hat{s}^2_{10} &= \alpha^{-3}\big(\frac{25\sqrt{15}}{256}ds^2_{AdS_4}\big) + \alpha ds^2_{S^6}, \quad \alpha = \frac{15^{3/8}}{2\sqrt{2}},\\
e^{-\frac{3}{2}\hat{\phi}} &= e^{-\frac{3}{2}\phi_0}\alpha^{-1}, \qquad L^{-1}e^{\frac{3}{4}\phi_0}\hat{A}_{(1)} = 0,\\
L^{-2}e^{-\frac{1}{2}\phi_0}\hat{A}_{(2)} &= \frac{1}{4}J_{(2)}, \qquad L^{-3}e^{\frac{1}{4}\phi_0}\hat{A}_{(3)} = \frac{1}{4}H_{(3)} + \frac{25\sqrt{15}}{128}\Omega_{(3)},
\end{aligned}
\tag{2.71}
$$

where $ds^2_{S^6}$ is the metric of the unit S^6 given in (2.53). Again, we introduced $L^2 = 2^{-\frac{1}{12}}g^{-25/12}m^{1/12}$ and $e^{\phi_0} = 2^{\frac{5}{6}}g^{\frac{5}{6}}m^{-\frac{5}{6}}$ using the scaling symmetries. It is recalled that in [45], general $\mathcal{N}=1$[1] flux compactification of massive type IIA string has been analyzed. Our $\mathcal{N}=1$, G_2-invariant solution is a special case of their result.

The stable nonsupersymmetric G_2-invariant solution is obtained by uplifting the nonsupersymmetric G_2-invariant critical point of the $D=4$ theory and the result is

$$
\begin{aligned}
L^{-2}d\hat{s}^2_{10} &= \alpha^{-3}\big(\frac{3\sqrt{3}}{16}ds^2_{AdS_4}\big) + \alpha ds^2_{S^6}, \quad \alpha = \frac{3^{3/8}}{2^{3/4}},\\
e^{-\frac{3}{2}\hat{\phi}} &= e^{-\frac{3}{2}\phi_0}\alpha^{-1}, \qquad L^{-1}e^{\frac{3}{4}\phi_0}\hat{A}_{(1)} = 0,\\
L^{-2}e^{-\frac{1}{2}\phi_0}\hat{A}_{(2)} &= -\frac{1}{2}J_{(2)}, \qquad L^{-3}e^{\frac{1}{4}\phi_0}\hat{A}_{(3)} = -\frac{1}{2}H_{(3)} + \frac{3\sqrt{3}}{8}\Omega_{(3)}.
\end{aligned}
\tag{2.72}
$$

[1]Here $\mathcal{N}=1$ means four real supercharges.

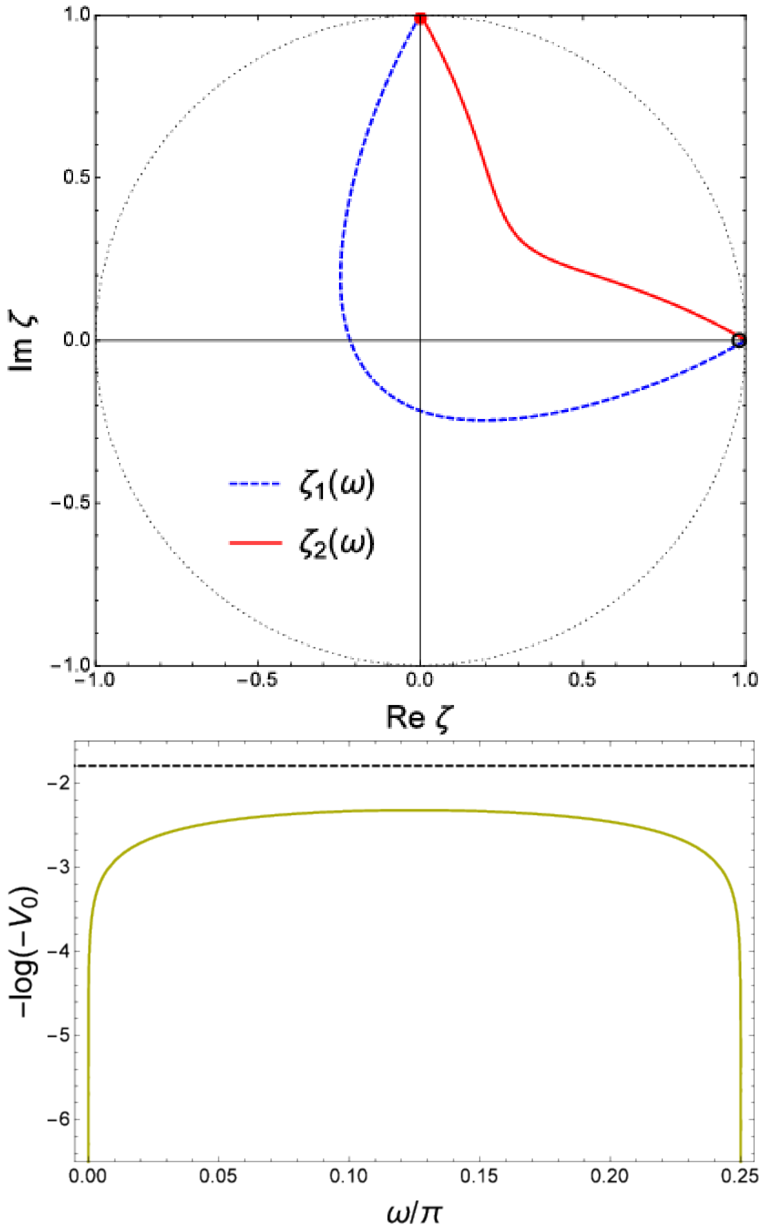


Figure 2.2: The ω dependence of $\mathcal{N}=3$ $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ critical point. (Red dot: $\omega = 0$. Black dot: $\omega = \pi/4$. Dashed line of the bottom plot corresponds to $V_0 = -6$.) Figure reprinted with permission from [20].

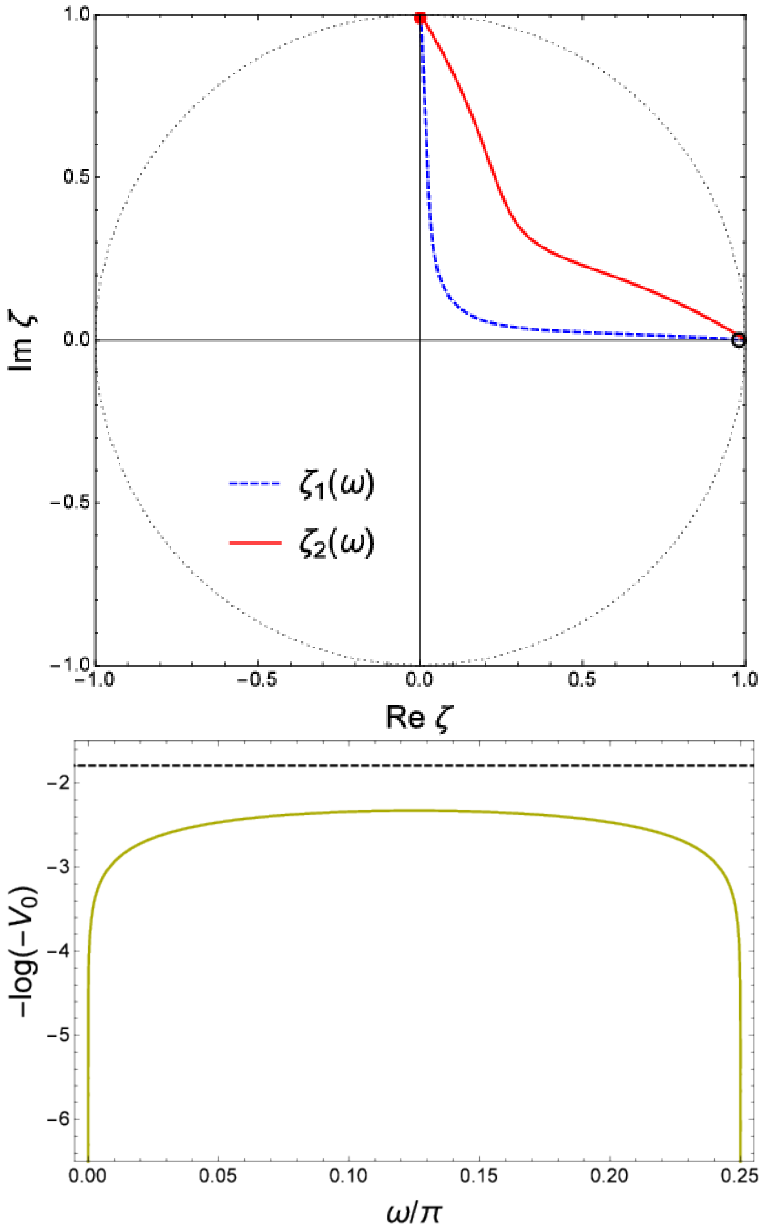


Figure 2.3: The ω dependence of the $\mathcal{N} = 0$ $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ critical point. (Red dot: $\omega = 0$. Black dot: $\omega = \pi/4$. Dashed line of the bottom plot corresponds to $V_0 = -6$.) Figure reprinted with permission from [20].

3. REFINED TEST OF ADS/CFT CORRESPONDENCE*

3.1 Romans term and Chern-Simons conformal field theory

We have explained how to get (non)supersymmetric (warpped) AdS solution of 10 dimensional massive type IIA supergravity, by uplifting critical points of four dimensional ISO(7) gauged $\mathcal{N} = 8$ critical points. Now we will explain what the dual field theory is. We know the Wess-Zumino coupling [46] on D2 brane is give by

$$S_{WZ} = (2\pi l_s^2)^2 \mu_{D2} \int_{\mathbf{R}^{1,2}} F_0 \cdot \frac{1}{2} \mathrm{tr}[\epsilon^{\mu\nu\rho} A_\mu^a \partial_\nu A_\rho^a + \frac{2}{3} A_\mu^a A_\nu^a A_\rho^a] \tag{3.1}$$

where μ_{D2} is the D2 brane charge. This is exactly the Chern-Simons kinetic term. Another way to get Chern-Simons terms on branes is through the Wess-Zumino coupling on D4 branes, with the D4 branes wrap a two-cycle of the internal space, the two form $F_{(2)}$ induce Chern-Simons terms on the three transverse dimension, which is exactly the mechanism in ABJM theory [3]. For our $\mathcal{N} = 3$ solution, the internal space is S^6, which has no two-cycle, it is then making sense to assume the dual theory has only one gauge group.

The second important fact is that our solution preserves three supersymmetry, the dual theory then need to be an $\mathcal{N} = 3$ superconformal Chern-Simons field theory. As we mentioned in the introduction, an $\mathcal{N} = 3$ supersymmetric Chern-Simons theory was constructed in [8], and was studied in [6], if one take the gauge generator in

*Part of the result reported in this chapter is reprinted with permission from *$\mathcal{N}$=3 solution in dyonic ISO(7) gauged maximal supergravity and its uplift to massive type IIA supergravity* by Yi Pang and Junchen Rong, published in Phys. Rev. D **92**, no. 8, 085037 (2015), Copyright [2015] by American Physical Society, [DOI:10.1103/PhysRevD.92.085037]; *Evidence for the Holographic dual of $\mathcal{N}$ = 3 Solution in Massive Type IIA* by Yi Pang and Junchen Rong, published in Phys. Rev. D **93**, no. 6, 065038 (2016), Copyright [2016] by American Physical Society, [DOI:10.1103/PhysRevD.93.065038].

(1.16) to be the generators in the adjoint representation of the gauge group, (1.15) could be written in components as [6]:

$$\begin{aligned}\mathcal{L} = &\frac{k}{4\pi}\big[CS(A) + \mathrm{Tr}(D^{ab}s_{ab} - \frac{1}{2}\chi^{ab}\chi_{ab} + \chi\chi + \frac{1}{6}s^{ab}[s_{bc}, s^{c}{}_{a}])\big] \\ &+\frac{1}{2}|\nabla_\mu q_{Ia}|^2 + \frac{1}{2}q_{Ia}D^{ab}q^{I}{}_{b} - \frac{1}{4}|s_{ab}q^{Ic}|^2 \\ &+\frac{\mathrm{i}}{2}\psi_{Ia}\gamma^\mu\nabla_\mu\psi^{Ia} - \frac{1}{2}\psi_{I}{}^{a}s_{ab}\psi^{Ib} + \mathrm{i}q_{I}{}^{a}\chi_{ab}\psi^{Ib} + \mathrm{i}q_{Ia}\chi\psi^{Ia}.\end{aligned} \tag{3.2}$$

This would be the dual theory of our $\mathcal{N} = 3$ solution.

Before talking about the detailed tests of this AdS/CFT correspondence, let say a few words about the relation between the known critical point in dyonic ISO(7) gauged $\mathcal{N} = 8$ supergravity, which are listed in Figure 3.1, according to the hight of cosmological constant. The $\mathcal{N} = 2$ point found in [18] is also included. The figure

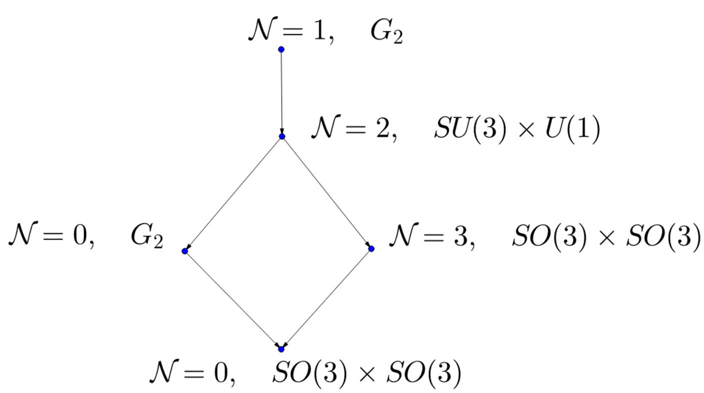


Figure 3.1: List of critical points according to the heights of their cosmological constants. Figure reprinted with permission from [21].

indicates one could construct domain wall solution interpolating between these critical point, correspondingly, there should be RG flow running between these conformal field theories.

3.2 Short Kaluza-Klein multiplets and BPS protected operators

Even though the original proposed AdS/CFT correspondence (1.6) was between maximal supersymmetric solution and maximal supersymmetric conformal field theories, it is widely believed that AdS/CFT correspondence is also valid in backgrounds with reduced supersymmetry. As an example, we are going to study the warped $AdS_4 \times \mathcal{M}_6$ solution of type IIA supergravity (2.61), which has $\mathcal{N} = 3$ reduced supersymmetry (six supercharges). To figure out the dual superconformal field theory, let's use a principle of AdS/CFT correspondence:

Principle 1: *Isometries of the background solution becomes global symmetries of the dual field theory.*

A round six sphere has SO(7) isometry, the solution (2.61) however has only $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_D$ isometry. Pay attention to the term $(d\mu^A \eta^i_{AB} \mu^B \nu^i)^2$, $d\nu^i$ is a three vector of $\mathrm{SO}(3)_\nu$, while $d\mu^A$ and μ^A are both four vectors of $\mathrm{SO}(4) \sim \mathrm{SO}(3)_L \times \mathrm{SO}(3)_R$. With the help of t'Hooft matrix, $d\mu^A \eta^i_{AB} \mu^B$ is now a vector of $\mathrm{SO}(3)_R$. It is then clear that $(d\mu^A \eta^i_{AB} \mu^B \nu^i)^2$ breaks SO(7) into $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_D = \mathrm{SO}(3)_L \times \{\mathrm{SO}(3)_R \times \mathrm{SO}(3)_V\}_D$.

The dual $\mathcal{N} = 3$ superconformal field theory should preserve two SO(3) $\sim$SU(2) global symmetry group. One type of $\mathcal{N} = 3$ superconformal field theories were proposed in [6]. Depending on the number of chiral multiplets, the theory had $\mathrm{USp}(2N_f) \times \mathrm{SU}(2)_\mathcal{R}$ global symmetry, where $\mathrm{SU}(2)_\mathcal{R}$ is contained in the superconformal group $OSp(3|4)$. As $\mathrm{USp}(2) \sim \mathrm{SU}(2)$, the $N_f = 1$ theory is then a candidate of the dual superconformal field theory, the symmetry of the two side of the correspondence match as:

$$\mathrm{SO}(3)_L \leftrightarrow \mathrm{USp}(2) \tag{3.3}$$

$$\mathrm{SO}(3)_D \leftrightarrow \mathrm{SU}(2)_{\mathcal{R}} \tag{3.4}$$

Notice $\mathrm{SO}(3)_D$ becomes the R symmetry, which is in accord with the branching rules (2.14), where the eight supersymmetry generators transforms as triplets of $\mathrm{SO}(3)_D$.

To test this proposal, we need to use another principle of AdS/CFT correspondence:

Principle 2: *Short Kaluza-Klein multiplets are dual to single trace gauge invariant BPS operators of the dual superconformal field theory.*

On the gauge theory side, one need to know the spectrum of the boundary CFT. The BPS operators of this superconformal field theory was studied in [47], where the following three BPS multiplets was shown to exist:

- *short graviton multiplet* $\mathrm{DS}(2, J_{\mathcal{R}}+3/2, J_{\mathcal{R}}|3)$, with $j_F = 0$ and $j_{\mathcal{R}} \in \mathbb{Z}^+ \cup \{0\}$,
- *short gravitino multiplet* $\mathrm{DS}(3/2, J_{\mathcal{R}}+1, J_{\mathcal{R}}|3)$, with $j_F = j_{\mathcal{R}} \in \frac{1}{2}\mathbb{Z}^+$,
- *short vector multiplet* $\mathrm{DS}(1, J_{\mathcal{R}}, J_{\mathcal{R}}|3)$, with $j_F = j_{\mathcal{R}} \in \frac{1}{2} + \frac{1}{2}\mathbb{Z}^+$.

If the proposed SCFT is indeed dual to the $\mathcal{N} = 3$ solution (2.61), we'd better find the same multiplets in Kaluza-Klein modes. The internal space of (2.61) is clearly not homogeneous, this makes the study of Kaluza-Klein reduction quite complicated. The spin-2 modes, on the other hand, is calculable. We consider fluctuations of the metric around the $\mathcal{N} = 3$ background

$$\hat{g}_{MN} \to \bar{g}_{MN} + \hat{h}_{MN}. \tag{3.5}$$

Similar to the cases studied in [48–52], applying the separation of variables to the transverse and traceless (with respect to the AdS$_4$ metric $g_{4\mu\nu}$ without the warp factor) part of $\hat{h}_{\mu\nu}$,

$$\hat{h}_{\mu\nu} = h_{\mu\nu}(x)Y(y), \quad \nabla_4^\mu h_{\mu\nu} = 0, \quad g_4^{\mu\nu} h_{\mu\nu} = 0, \tag{3.6}$$

we find that the spin-2 modes solving the homogenous linearized Einstein equation satisfy

$$Y(y)L_0^{-2}(\Box_4 + 2)h_{\mu\nu} + h_{\mu\nu}\mathcal{O}Y(y) = 0, \quad L_0^2 = \frac{3\sqrt{3}}{16}, \tag{3.7}$$

where $\Box_4$ is the Laplacian on the unit AdS$_4$, and the operator $\mathcal{O}$ is given by

$$\begin{aligned}\mathcal{O}Y(y) &= \frac{\Delta^{-1}}{\sqrt{-\bar{g}_{10}}}\partial_M(\sqrt{-\bar{g}_{10}}\bar{g}^{MN}\partial_N)Y(y) \\ &= \frac{1}{\sqrt{\mathring{g}_6}}\partial_m(\Delta^{-1}\sqrt{\mathring{g}_6}\bar{g}^{mn}\partial_n)Y(y),\end{aligned} \tag{3.8}$$

where $\mathring{g}_6$ is the metric on the round S^6. The operator $\mathcal{O}$ can be written explicitly as

$$L_0^2\mathcal{O} \equiv \widetilde{\mathcal{O}} = \frac{1}{2}\partial_\xi^2 + \frac{1}{2}(3\cot\xi - 2\tan\xi)\partial_\xi + \frac{1}{2}\sec^2\xi C_V + (2\csc^2\xi - 1)C_F + \frac{C_{\mathcal{R}} - C_R}{2}, \tag{3.9}$$

where C_V, C_F, $C_{\mathcal{R}}$ and C_L are the quadratic Casimirs associated with the subgroups of $SO(7)$:

$$\mathrm{SO}(3)_V, \quad \mathrm{SO}(3)_F \equiv SO(3)_L, \quad \mathrm{SO}(3)_{\mathcal{R}} \equiv \mathrm{SO}(3)_D, \quad \text{and } \mathrm{SO}(3)_R \tag{3.10}$$

Notice we have changed the notation of these subgroup, so as to make clear the field theory correspondence, where F stands for "Flavor", "$\mathcal{R}$" stands for $\mathcal{R}$-symmetry, and "R" stands "right-handed". When acting on scalars, these Casimirs can be expressed as bilinears of Lie derivatives associated with Killing vectors generating the corresponding SO(3). Killing vectors associated with $SO(3)_V$ are given by

$$\xi_V^i = \epsilon^{ijk}\nu^j \frac{\partial}{\partial \nu^k}, \tag{3.11}$$

whilst Killing vectors associated with $SO(3)_F$ and $SO(3)_R$ take the form

$$\xi_F^i = -\mu^A (T_F^i)_{AB} \frac{\partial}{\partial \mu^B}, \quad \xi_R^i = -\mu^A (T_L^i)_{AB} \frac{\partial}{\partial \mu^B}. \tag{3.12}$$

In the expressions above ,

$$T_F^1 = -\frac{1}{2}(R^{12} - R^{34}), \quad T_F^2 = -\frac{1}{2}(R^{13} - R^{42}), \quad T_F^3 = -\frac{1}{2}(R^{14} - R^{23}), \tag{3.13}$$

$$T_R^1 = \frac{1}{2}(R^{12} + R^{34}), \quad T_R^2 = \frac{1}{2}(R^{13} + R^{42}), \quad T_R^3 = \frac{1}{2}(R^{14} + R^{23}), \tag{3.14}$$

where the R^{ij} are the SO(4) generators, with $(R^{ij})_{ij} = -(R^{ij})_{ji} = 1$, and all other elements equal to zero. Then the quadratic Casimirs are given by

$$C_F = \mathcal{L}_{\xi_F^i}\mathcal{L}_{\xi_F^i}, \quad C_R = \mathcal{L}_{\xi_R^i}\mathcal{L}_{\xi_R^i}, \quad C_V = \mathcal{L}_{\xi_V^i}\mathcal{L}_{\xi_V^i}, \quad C_{\mathcal{R}} = (\mathcal{L}_{\xi_R^i} + \mathcal{L}_{\xi_V^i})(\mathcal{L}_{\xi_R^i} + \mathcal{L}_{\xi_V^i}). \tag{3.15}$$

The harmonic function $Y(y)$ satisfies $\widetilde{\mathcal{O}}Y(y) = -m^2 Y(y)$ leading to

$$(\Box_4 + 2)h_{\mu\nu} - m^2 h_{\mu\nu} = 0. \tag{3.16}$$

From the equation above, one can solve for the AdS energies carried by the spin-2 modes. For each m^2, we have

$$E_0 = \frac{1}{2}(3 + \sqrt{9 + 4m^2}). \tag{3.17}$$

To find eigenmodes for the operator $\widetilde{\mathcal{O}}$, it is useful to know the eigenfunctions of various Casimirs. We recall that spin-0 harmonics on a round 6-sphere are characterized by $(n, 0, 0)$, $n = 1, 2 \cdots$ representations of SO(7) and also form a complete basis for smooth scalar functions on manifold with S^6 topology. Thus, the decomposition of the SO(7) harmonics under the $\mathrm{SO}(3)_F \times \mathrm{SO}(3)_{\mathcal{R}}$ subgroup should give rise to a complete functional basis on the internal space of the $\mathcal{N} = 3$ solution (2.61) which is a smooth deformation of S^6. Since the $\mathrm{SO}(3)_F \times \mathrm{SO}(3)_{\mathcal{R}}$ subgroup is embedded in SO(7) via the chain

$$\mathrm{SO}(7) \supset \mathrm{SO}(4) \times \mathrm{SO}(3)_V \simeq \mathrm{SO}(3)_F \times \mathrm{SO}(3)_R \times \mathrm{SO}(3)_V \supset \mathrm{SO}(3)_F \times \mathrm{SO}(3)_{\mathcal{R}}, \tag{3.18}$$

we first branch the $(n, 0, 0)$ irrep under the $\mathrm{SO}(4) \times \mathrm{SO}(3)_V$ subgroup. This yields a sequence of irreps of $\mathrm{SO}(4) \times \mathrm{SO}(3)_V$ of the form $(\ell, 0)_{j_V}$, where $(\ell, 0)$ correspond to the highest weights of the SO(4) irrep. Here ℓ, j_V are non-negative integers. Under the isomorphism $\mathrm{SO}(4) \simeq \mathrm{SO}(3)_F \times \mathrm{SO}(3)_R$, the highest weights (ℓ_1, ℓ_2) of SO(4) are related to the isospins (j_F, j_R) of $\mathrm{SO}(3)_F \times \mathrm{SO}(3)_R$ by

$$j_F = \tfrac{1}{2}(\ell_1 + 2\ell_2), \qquad j_R = \tfrac{1}{2}\ell_1. \tag{3.19}$$

This means further branching of $(\ell, 0)$ under $\mathrm{SO}(3)_F \times \mathrm{SO}(3)_R$ leads to a sequence of irreps with $j_F = j_R$. The analysis above suggests that the eigenfunctions of the Casimirs C_F, C_R, C_V should take the form

$$f(\xi)(\alpha_{A_1 A_2 \cdots A_p} \prod_{k=1}^{p=2j_F} \widetilde{\mu}^{A_k})(\beta_{i_1 i_2 \cdots i_q} \prod_{m=1}^{q=j_V} \widetilde{\nu}^{i_m}), \quad j_F \in \frac{1}{2}\mathbb{Z}^+ \cup \{0\}, \quad j_V \in \mathbb{Z}^+ \cup \{0\}, \tag{3.20}$$

where

$$\widetilde{\mu}^A = \frac{\mu^A}{\sin\xi}, \quad A = 1\cdots 4, \qquad \widetilde{\nu}^i = \frac{\nu^i}{\cos\xi}, \quad i = 1\cdots 3, \tag{3.21}$$

and $f(\xi)$ is a function of ξ which cannot be determined by group theoretical analysis. Coefficients $\alpha_{A_1A_2\ldots A_p}$ and $\beta_{i_1i_2\ldots i_q}$ are totally symmetric, traceless with respect to their indices and transform according to the (j_F, j_F, j_V) irrep of $\mathrm{SO}(3)_F \times \mathrm{SO}(3)_R \times \mathrm{SO}(3)_V$. Since $\mathrm{SO}(3)_{\mathcal{R}}$ is the diagonal of $\mathrm{SO}(3)_R \times \mathrm{SO}(3)_V$, the eigenfunctions of its Casimir can be obtained by decomposing the product of $\alpha_{A_1A_2\cdots A_p}$ and $\beta_{i_1i_2\cdots i_q}$ in terms of irreps of $\mathrm{SO}(3)_{\mathcal{R}}$ using Clebsch-Gordan coefficients. In the end, we achieve the mutual eigenfunctions for C_F, C_R, C_V, $C_{\mathcal{R}}$ labeled by the quantum numbers

$$(j_F, j_F, j_V, j_{\mathcal{R}}), \quad j_{\mathcal{R}} = |j_V - j_F|, \cdots, j_V + j_F, \quad j_F \in \frac{1}{2}\mathbb{Z}^+ \cup \{0\}, \quad j_V \in \mathbb{Z}^+ \cup \{0\}. \tag{3.22}$$

For simplicity, we denote the eigenfunction obtained through the above procedure by the abstract symbol

$$|\psi\rangle = |j_F, j_F, j_V, j_{\mathcal{R}}\rangle. \tag{3.23}$$

It satisfies

$$\begin{aligned}
C_F|\psi\rangle &= c_F|\psi\rangle, \qquad & c_F &= -j_F(j_F+1),\\
C_R|\psi\rangle &= c_R|\psi\rangle, \qquad & c_R &= -j_F(j_F+1),\\
C_V|\psi\rangle &= c_V|\psi\rangle, \qquad & c_V &= -j_V(j_V+1),\\
C_{\mathcal{R}}|\psi\rangle &= c_{\mathcal{R}}|\psi\rangle, \qquad & c_{\mathcal{R}} &= -j_{\mathcal{R}}(j_{\mathcal{R}}+1),
\end{aligned} \tag{3.24}$$

which also illustrates the normalization of the Casimirs. Substituting the ansatz

$$Y(y) = f(\xi)|j_F, j_F, j_V, j_{\mathcal{R}}\rangle, \tag{3.25}$$

into (3.9) and making the change of variable

$$u = \cos^2 \xi, \quad \widetilde{f}(u) \equiv f(\xi), \tag{3.26}$$

we arrive at an equation for $\widetilde{f}(u)$

$$\begin{aligned}(1-u)^2 u^2 \widetilde{f}''(u) &+ \frac{1}{2}(7u^3 - 10u^2 + 3u)\widetilde{f}'(u) \\ &+ \left(\frac{1}{4}u(3u+1)c_F + \frac{1}{4}(1-u)c_V + \frac{1}{4}u(1-u)(c_{\mathcal{R}} + 2m^2)\right)\widetilde{f}(u) = 0.\end{aligned} \tag{3.27}$$

By a further change of variable

$$\widetilde{f}(u) \equiv u^{j_V/2}(1-u)^{j_F} H(u), \tag{3.28}$$

the equation above is brought to the form of a standard hypergeometric differential equation

$$u(1-u)\frac{d^2 H}{du^2} + \big(c - (a+b+1)u\big)\frac{dH}{du} - abH(u) = 0, \tag{3.29}$$

where the constants are given by

$$\begin{aligned} a &= \frac{1}{4}(-\sqrt{12j_F^2 + 12j_F - 4j_{\mathcal{R}}^2 - 4j_{\mathcal{R}} + 8m^2 + 25} + 4j_F + 2j_V + 5), \\ b &= \frac{1}{4}(\sqrt{12j_F^2 + 12j_F - 4j_{\mathcal{R}}^2 - 4j_{\mathcal{R}} + 8m^2 + 25} + 4j_F + 2j_V + 5), \\ c &= j_V + \frac{3}{2}, \end{aligned} \tag{3.30}$$

There are two independent solutions to the hypergeometric differential equation above

$${}_2F_1(a,b,c,u), \quad \text{and} \quad u^{1-c}{}_2F_1(1+a-c, 1+b-c, 2-c, u). \tag{3.31}$$

The second solution should be discarded, since the corresponding $f(u)$ is singular at $u = 0$. The first solution converges for $|u| < 1$. It can be proved that for $(1-u)^{j_F} 2F_1(a, b, c, u)$ to be regular at $u = 1$, the coefficient a must be a non-positive integer. Regularity of the solution thus dictates the mass squared m^2 to depend on the quantum numbers quadratically

$$m^2 = \frac{1}{2}\left(2n(4j_F + 2j_V + 5) + 4j_F j_V + j_F^2 + 7j_F + 4n^2 + j_{\mathcal{R}}^2 + j_{\mathcal{R}} + j_V^2 + 5j_V\right), \quad (3.32)$$

where $n \in \mathbb{Z}^+ \cup \{0\}$. A typical spin-2 excitation with AdS energy being an integer is given by $n = 0$, $j_F = 0$ and $j_V = j_{\mathcal{R}}$, which leads to

$$m^2 = j_{\mathcal{R}}(j_{\mathcal{R}} + 3), \quad E_0 = j_{\mathcal{R}} + 3, \quad j_{\mathcal{R}} \in \mathbb{Z}^+ \cup \{0\}. \quad (3.33)$$

It should be noted that gravitons with the same $\mathrm{SO}(3)_F \times \mathrm{SO}(3)_{\mathcal{R}}$ quantum numbers and AdS energies appear in the short graviton multiplet $\mathrm{DS}(2, j_{\mathcal{R}} + 3/2, j_{\mathcal{R}}|3)$ of $\mathrm{OSP}(3|4)$ [53], which we quote in Table 3.1.

Since the supergravity background preserves $\mathcal{N} = 3$ superconformal symmetry, the spin-2 states (3.33) must form complete $\mathrm{DS}(2, j_{\mathcal{R}}+3/2, j_{\mathcal{R}}|3)$ multiplets together with other lower spin states with proper quantum numbers and AdS energies. The spin-2 states (3.33) are singlets with respect to $\mathrm{SO}(3)_F$, which means all the states belonging to the short graviton multiplets are singlets of the flavor symmetry. On the CFT side, the spectrum of BPS operators in the $\mathcal{N} = 3$ superconformal $\mathrm{SU}(N)$ Chern-Simons-matter theory with 2 adjoint chirals has been studied by [47]. It was shown that the short multiplets $\mathrm{DS}(2, j_{\mathcal{R}} + 3/2, j_{\mathcal{R}}|3)$ composed by gauge invariant operators are singlets of the flavor symmetry. Therefore, our results demonstrate a perfect matching between the short graviton multiplets in the KK spectrum of

$\Delta\backslash s$	2	3/2	1	1/2	0
$j_{\mathcal{R}}+4$			$[J_{\mathcal{R}}-1]$	$[J_{\mathcal{R}}]$ $[J_{\mathcal{R}}-1]$ $[J_{\mathcal{R}}-2]$	
$j_{\mathcal{R}}+7/2$		$[J_{\mathcal{R}}]$,$[J_{\mathcal{R}}-1]$			
$j_{\mathcal{R}}+3$	$[J_{\mathcal{R}}]$		$[J_{\mathcal{R}}+1]$, $[J_{\mathcal{R}}]$ $[J_{\mathcal{R}}]$, $[J_{\mathcal{R}}-1]$ $[J_{\mathcal{R}}-1]$, $[J_{\mathcal{R}}-2]$		$[J_{\mathcal{R}}]$ $[J_{\mathcal{R}}-1]$ $[J_{\mathcal{R}}-2]$
$j_{\mathcal{R}}+5/2$		$[J_{\mathcal{R}}+1]$ $[J_{\mathcal{R}}]$ $[J_{\mathcal{R}}-1]$		$[J_{\mathcal{R}}+1]$, $[J_{\mathcal{R}}]$ $[J_{\mathcal{R}}]$, $[J_{\mathcal{R}}-1]$ $[J_{\mathcal{R}}-1]$, $[J_{\mathcal{R}}-2]$	
$j_{\mathcal{R}}+2$			$[J_{\mathcal{R}}+1]$ $[J_{\mathcal{R}}]$ $[J_{\mathcal{R}}-1]$		$[J_{\mathcal{R}}]$ $[J_{\mathcal{R}}-1]$
$j_{\mathcal{R}}+3/2$				$[J_{\mathcal{R}}]$	

Table 3.1: Short graviton multiplet DS$(2, j_{\mathcal{R}}+3/2, j_{\mathcal{R}}|3)$of OSP$(3|4)$. The red one denote the spin-2 mode. Table reprinted with permission from [22].

fluctuations around the $\mathcal{N}=3$ vacuum in massive IIA and the short multiplets involving spin-2 operators in the $\mathcal{N}=3$ superconformal SU(N) Chern-Simons matter theory with two adjoint chirals.

A list of the bulk spin-2 states labeled by their quantum numbers is given in Table 3.2, from which one can see that the spectrum includes long graviton multiplets with rational dimensions. This feature has been observed for other M-theory and string theory backgrounds [48, 54–57]. A class of long multiplets with rational dimensions was termed as the "shadow" multiplets [57]. From the bulk point of view, shadowing mechanism is related to the fact that the same harmonics also appear in other fields belonging to short multiplets. In the spectrum obtained here, the long graviton labeled by $(j_F, j_V, j_{\mathcal{R}}, n) = (1, r, r, 0)$ carries $E_0 = r+4$. The corresponding long graviton multiplets are shadows of vector multiplets.

3.3 Euclidean Action and Free Energy

Another test of AdS/CFT correspondence involves the third principle:

Principle 3: *Euclidean Action of the bulk solution is equal to $F = -\log(Z)$, where Z is the partition function of the dual superconformal field theory on a Euclidean S^3.*

3.3.1 Supersymmetric Localization and Free Energy

Supersymmetric Localization is a powerful non-perturbative technique that allows us to calculate exactly free energy and BPS Wilson line in certain supersymmetric conformal field theories. It have been applied to various superconformal field theories such as $\mathcal{N} = 4$ super Yang-Mills theory [58] and ABJM theory [59]. Let's follow [60], and consider an $\mathcal{N} = 2$ theory on S_3, whose quiver diagram are show in Figure 3.2. Every node denotes a gauge group, while every line denotes chiral multiplets. Lines connects different nodes are bi-fudamental in $(\mathbf{N}, \bar{\mathbf{N}})$ of $\mathrm{U(N)}_a \times \mathrm{U(N)}_b$, while lines connecting each nodes itself transform as adjoint representation of the gauge group. We consider the case where the kinetic term of each group are Chern-Simons terms with level the same Chern-Simons level k [7]:

$$\mathcal{L}_{CS} = \sum_{a=1}^{G} \frac{k_a}{4\pi} \mathrm{tr}[\epsilon^{\mu\nu\rho} A^a_\mu \partial_\nu A^a_\rho + \frac{2\mathrm{i}}{3} A^a_\mu A^a_\nu A^a_\rho - \bar{\chi}\chi + 2D\sigma] \tag{3.34}$$

the partition function are given by

$$Z_{S^3} = \frac{1}{(N!)^G} \int \prod_{a=1}^{G} \left[\prod_{i=1}^{N} \frac{d\lambda_i^a}{2\pi} \right] \exp\left[\frac{\mathrm{i}k}{4\pi} \sum_{N}^{i=1} (\lambda_i^a)^2 \right] \prod_{i\neq j}^{N} \sinh^2\left(\frac{\lambda_i^a - \lambda_j^a}{2} \right) e^{-F_{\mathrm{matter}}} \tag{3.35}$$

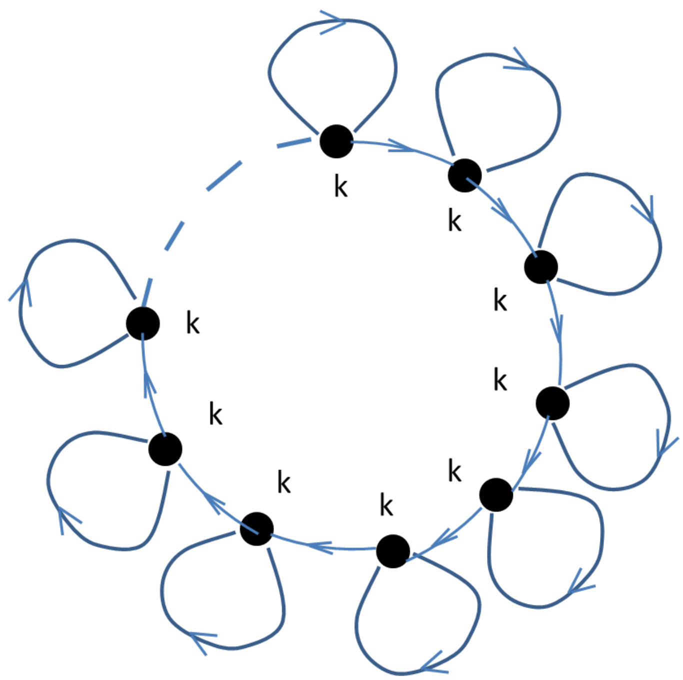


Figure 3.2: A quiver diagram for the $\mathcal{N}=2$ Chern-Simons matter theory. Notice all the gauge sectors have equal Chern-Simons level.

where F_{matter} depends on the representation of chiral matter field. If the chiral multiplets live in the bi-fudamental $(\mathbf{N}, \bar{\mathbf{N}})$ of $U(N)_a \times U(N)_b$:

$$F^{ab}_{\text{matter}} = -\sum_{i,j=1}^{N} l\left[1 - \Delta^{ab} + \frac{\mathrm{i}}{2\pi}(\lambda^a_i - \lambda^b_j)\right] \tag{3.36}$$

If the chiral multiplets live in the adjoint representation of $U(N)_c$:

$$F^{\text{adj},c}_{\text{matter}} = -\sum_{i,j=1}^{N} l\left[1 - \Delta^{c} + \frac{\mathrm{i}}{2\pi}(\lambda^c_i - \lambda^c_j)\right] \tag{3.37}$$

where we have defined

$$l(z) = -z\log(1 - e^{2\pi \mathrm{i} z}) + \frac{\mathrm{i}}{2}\left[\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2(e^{2\pi \mathrm{i} z}) - \frac{\mathrm{i}\pi}{12}\right]. \tag{3.38}$$

In general when the rank of the gauge group is not too small, the integral is difficult to perform. A saddle point method was however proposed in [61] based on numerical estimation in various examples. At leading order of $O(1/N)$, the eigenvalues λ_i^a could be approximated by

$$\lambda_i^a = N^\nu(x_i + \mathrm{i}y_i) \tag{3.39}$$

Using such an approximation, we could get a general formula for such theories [60]:

$$\begin{aligned} F_{SCFT} = & \frac{3\sqrt{3}\pi}{20\cdot 2^{1/3}}\left\{G + \sum_{I\in \text{matter fields}} (1-\Delta^I)\big(1-2(1-\Delta^I)^2\big)\right\}^{2/3} k^{1/3}N^{5/3}. \\ & +O(N^{5/3}) \end{aligned} \tag{3.40}$$

Δ_I is the scaling dimension of chiral multiplets living in the adjoint or bifudamental representation of $U(N)$. For $\mathcal{N}=2$ theories, chiral multiplets are BPS protected and alway have scaling dimension:

$$\Delta_I = \mathcal{R}_I, \tag{3.41}$$

where $\mathcal{R}_I$ is the $\mathrm{U}(1)_\mathcal{R} \in \mathrm{OSP}(4|2)$ charge of the chiral multiplets. Even though BPS protected, since $\mathrm{U}(1)_\mathcal{R}$ group could mix with other flavor U(1) group, the $\mathcal{R}$-charge and Δ_I of chiral multiplets could be renormalized. $\mathcal{N}=3$ theories could be viewed as special cases of $\mathcal{N}=2$ theories where OSP(4|2) is enhanced to OSP(4|3). Since $\mathrm{U}(1)_\mathcal{R} \in \mathrm{SU}(2)_\mathcal{R}$, $\mathrm{U}(1)_\mathcal{R}$ charge is fixed, and therefore not renormalized. For the theory we are interested in, chiral multiplets are doublets of $\mathrm{SU}(2)_\mathcal{R}$, therefore:

$$\Delta_I = \mathcal{R}_I = \frac{1}{2}. \tag{3.42}$$

Plug it in (3.40), we get

$$F_{SCFT} = \frac{9\pi}{40} 3^{1/6} k^{1/3} N^{5/3}. \tag{3.43}$$

The bulk calculation of the gravitational free energy is also easy to preform. The number of massive D2-branes which is equal to the rank of the gauge group is determined by the quantized Page charge [62, 63].

$$\int_{S^6} \tilde{F}_{(6)} = \int_{S^6} e^{\frac{1}{2}\phi} \hat{*} F_{(4)} + A_{(2)} \wedge dA_{(3)} + \frac{1}{6} m A_{(2)} \wedge A_{(2)} \wedge A_{(2)} = -(2\pi\ell_s)^5 N. \tag{3.44}$$

Plugging the $\mathcal{N}=3$ solution, we get

$$\frac{1}{(2\pi\ell_s)^5 g^5} \frac{16\pi^3}{3} = N. \tag{3.45}$$

On the other hand, the gravitational free energy is inversely proportional to the effective $D=4$ Newton's constant

$$F_{\text{gravity}} = \frac{\pi\ell^2}{2G_4}, \quad \ell^2 = \frac{3\sqrt{3}}{16} g^{-7/3} (m/2)^{1/3}, \tag{3.46}$$

where ℓ is the radius of AdS$_4$ and the effective $D=4$ Newton's constant is related to the string length by

$$\frac{1}{16\pi G_4} = \frac{2\pi}{(2\pi\ell_s)^8 g^6} \text{Vol}(S^6). \tag{3.47}$$

In the equation above , $\text{Vol}(S^6) = \frac{16}{15}\pi^3$ is the area of a unit S^6. Finally, using the relation between the Romans mass parameter and the induced Chern-Simons level [9]

$$m = F_{(0)} = \frac{k}{2\pi\ell_s}, \tag{3.48}$$

we can express the free energy of the $\mathcal{N}=3$ supergravity solution in terms of k and N:

$$F_{\text{gravity}} = \frac{9\pi}{40} 3^{1/6} k^{1/3} N^{5/3}. \tag{3.49}$$

This obvious agrees with the CFT localization result, which is another strong evidence for the proposed AdS/CFT correspondence.

$(j_F, j_V, j_{\mathcal{R}})$	n		m^2		E_0	S(hort)/L(ong)
(0, 0, 0)	0		0		3	S
(0, 0, 0)	1		7		$\frac{1}{2}(3+\sqrt{37})$	L
(0, 0, 0)	2		18		6	L
(0, 1, 1)	0		4		4	S
(0, 1, 1)	1		13		$\frac{1}{2}(3+\sqrt{61})$	L
(0, 1, 1)	2		26		$\frac{1}{2}(3+\sqrt{113})$	L
$(\frac{1}{2},\ 0,\ \frac{1}{2})$	0		$\frac{9}{4}$		$\frac{1}{2}(3+3\sqrt{2})$	L
$(\frac{1}{2},\ 0,\ \frac{1}{2})$	1		$\frac{45}{4}$		$\frac{1}{2}(3+3\sqrt{6})$	L
$(\frac{1}{2},\ 0,\ \frac{1}{2})$	2		$\frac{97}{4}$		$\frac{1}{2}(3+\sqrt{106})$	L
$(\frac{1}{2},\ 1,\ \frac{1}{2})$	0		$\frac{25}{4}$		$\frac{1}{2}(3+\sqrt{34})$	L
$(\frac{1}{2},\ 1,\ \frac{1}{2})$	1		$\frac{69}{4}$		$\frac{1}{2}(3+\sqrt{78})$	L
$(\frac{1}{2},\ 1,\ \frac{1}{2})$	2		$\frac{129}{4}$		$\frac{1}{2}(3+\sqrt{138})$	L
$(\frac{1}{2},\ 1,\ \frac{3}{2})$	0		$\frac{31}{4}$		$\frac{1}{2}(3+2\sqrt{10})$	L
$(\frac{1}{2},\ 1,\ \frac{3}{2})$	1		$\frac{75}{4}$		$\frac{1}{2}(3+2\sqrt{21})$	L
$(\frac{1}{2},\ 1,\ \frac{3}{2})$	2		$\frac{135}{4}$		$\frac{15}{2}$	L
(1, 0, 1)	0		5		$\frac{1}{2}(3+\sqrt{29})$	L
(1, 0, 1)	1		16		$\frac{1}{2}(3+\sqrt{73})$	L
(1, 0, 1)	2		31		$\frac{1}{2}(3+\sqrt{133})$	L
(1, 1, 0)	0		9		$\frac{1}{2}(3+3\sqrt{5})$	L
(1, 1, 0)	1		22		$\frac{1}{2}(3+\sqrt{97})$	L
(1, 1, 0)	2		39		$\frac{1}{2}(3+\sqrt{165})$	L
(1, 1, 1)	0		10		5	L
(1, 1, 1)	1		23		$\frac{1}{2}(3+\sqrt{101})$	L
(1, 1, 1)	2		40		8	L

Table 3.2: An incomplete list of the KK spectrum of spin-2 states. The "Short" and "Long" refer to the short and long multiplets which the spin-2 states belong to. Here we remind the reader that $j_{\mathcal{R}} = |j_V - j_F|, \cdots, j_V + j_F$, $j_F \in \frac{1}{2}\mathbb{Z}^+ \cup \{0\}$, $\quad j_V \in \mathbb{Z}^+ \cup \{0\}$. Table adapted with permission from [22].

4. CONFORMAL BOOTSTRAP

4.1 Introduction to Conformal Bootstrap

Scalar four point function for any CFT could be written as

$$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) >= \sum_{\mathcal{O}\in\phi\times\phi} \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \lambda_{\mathcal{O}}^2 g_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v)$$

where $g_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v)$ is the so called “conformal block”, which is a general function that does not depend on the specific CFT. In four dimensions the conformal block is given by

$$\begin{aligned}
&g_{\Delta,l}(u,v) \equiv \frac{z\bar{z}}{z-\bar{z}}(k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z}))\\
&k_\beta \equiv x^{\beta/2}{}_2F_1(\beta/2,\beta/2,\beta,x),\\
&u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = (1-z)(1-\bar{z}) = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}
\end{aligned} \tag{4.1}$$

as computed by Dolan and Osborn [64, 65]. The crucial information here is that conformal block $g_{\Delta,l}(u,v)$ is fully fixed by conformal algebra, while the information of any given CFT is contained solely in OPE coefficient $\lambda_{\mathcal{O}}$ and operator spectrum. Equating s-channel and t-channel, we get:

$$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) > -(x_1 \leftrightarrow x_3) = \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 F_{\Delta,l}^{\Delta_\phi}(z,\bar{z}) = 0 \tag{4.2}$$

where

$$F_{\Delta,l}^{\Delta_\phi}(z,\bar{z}) \equiv \frac{v^{\Delta_\phi} g_{\Delta,l}(u,v) - u^{\Delta_\phi} g_{\Delta,l}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}. \tag{4.3}$$

Let's make the following assumption:

Assumption: *In $\phi \times \phi$ OPE, all spin-0 operators has scaling dimension $\Delta \geq \Delta_0$.* (4.4)

If one finds that $F_{d,\Delta,l}(z,\bar{z}) > 0$ for all the operator satisfying [66]:

$$\begin{aligned} \Delta &\geq \Delta_0, && \text{for} \quad l = 0 \\ \Delta &\geq l + 2 \text{ (Unitary bound)}, && \text{for} \quad l > 0 \end{aligned} \tag{4.5}$$

Then (4.2) tells us that the assumption is invalid for "unitary" CFT's, as such CFT's has real OPE coefficients, hence $\lambda_{\mathcal{O}}^2 > 0$.

Instead of working with $F_{\Delta,l}^{\Delta_\phi}(z,\bar{z})$ themselves, we could instead search for a linear functional α such that

$$\begin{aligned} &\alpha(F_{\Delta,l}^{\Delta_\phi}(z,\bar{z})) \geq 0 \\ &\text{for} \begin{cases} \Delta \geq \Delta_0, & \text{when} \quad l = 0 \\ \Delta \geq \Delta_{unitary}, & \text{when} \quad l > 0 \end{cases} \end{aligned} \tag{4.6}$$

A commonly used basis for such a functional is that

$$\alpha = \sum_{m+n<\Lambda} a_{mn} \partial_z^m \partial_{\bar{z}}^n \tag{4.7}$$

The problem of searching for the coefficients a_{mn} that satisfies (4.6) could be converted to a numerical semi-definite programming problem and solved using a semidefinite program solver call "SDPB", which are specially designed for conformal bootstrap problems [67]. We will come back to this, but for now, let's warm up with a two dimensional example—solving the two dimension Ising model. A throughout discussion of this example is could be found in [68].

4.1.1 2D Example

In two dimensions, the conformal block is given by

$$g_{\Delta,l}(u,v) \equiv k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) - (z \leftrightarrow \bar{z}) \tag{4.8}$$

Defined a vector

$$\vec{v}^{\Delta_\phi}_{\Delta,l} = \begin{pmatrix} F^{\Delta_\phi}_{\Delta,l}(0.5, 0.55) - F^{\Delta_\phi}_{\Delta,l}(0.5, 0.4) \\ F^{\Delta_\phi}_{\Delta,l}(0.5, 0.6) - F^{\Delta_\phi}_{\Delta,l}(0.43, 0.35) \end{pmatrix}. \tag{4.9}$$

The crossing equation (4.2) then tells us that

$$\sum_{\mathcal{O}\in\phi\times\phi} \lambda^2_{\mathcal{O}} \vec{v}^{\Delta_\phi}_{\Delta,l} = 0 \tag{4.10}$$

Notice the crossing equation is supposed to be satisfied by any value of z and $\bar{z}$, we have just taken some specific value for them here. Also, since we are dealing with Euclidean conformal field theory, z and $\bar{z}$ should be taken to be independent.

In Figure 4.1 and 4.2, we plot such vectors on a two dimensional surface [1]. We have taken $\Delta_\phi = 1/8$. Compare the two figures, we see that after making the assumption that the first scalar has scaling dimension $\Delta > 1.03$, there is no points above the dashed line, which simply means that there is no chance that (4.10) could be solved by any set of real OPE coefficients. Hence we know that the first scalar

[1] The vectors are rescaled by the following function

$$f(l) = \begin{cases} 1/(l/20), & l \neq 0 \\ 1 - (\Delta - l)/10, & l = 0. \end{cases} \tag{4.11}$$

to make the plot look nicer.

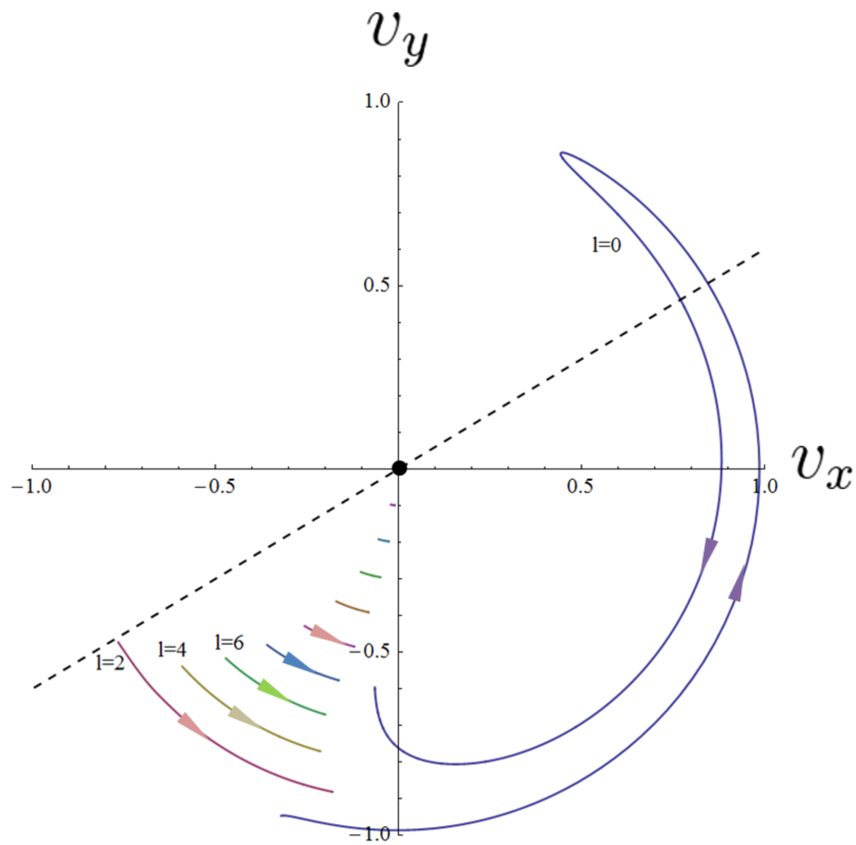


Figure 4.1: Plot of the vector defined in (4.9) with $\Delta_\phi = 1/8$. Each curve stands for vectors with different spin. The arrow shows the direction of increasing Δ, all curves start with the vector with $\Delta = \Delta_{uintary}$. The black dot in the center stands for the identity operator.

operator (except for the identity operator) appearing in the OPE has to have the scaling dimension less that 1.03. Let's remind ourself that the two dimensional Ising model has the spectrum:

$$\Delta_\sigma = 1/8, \quad \Delta_\epsilon = 1, \tag{4.12}$$

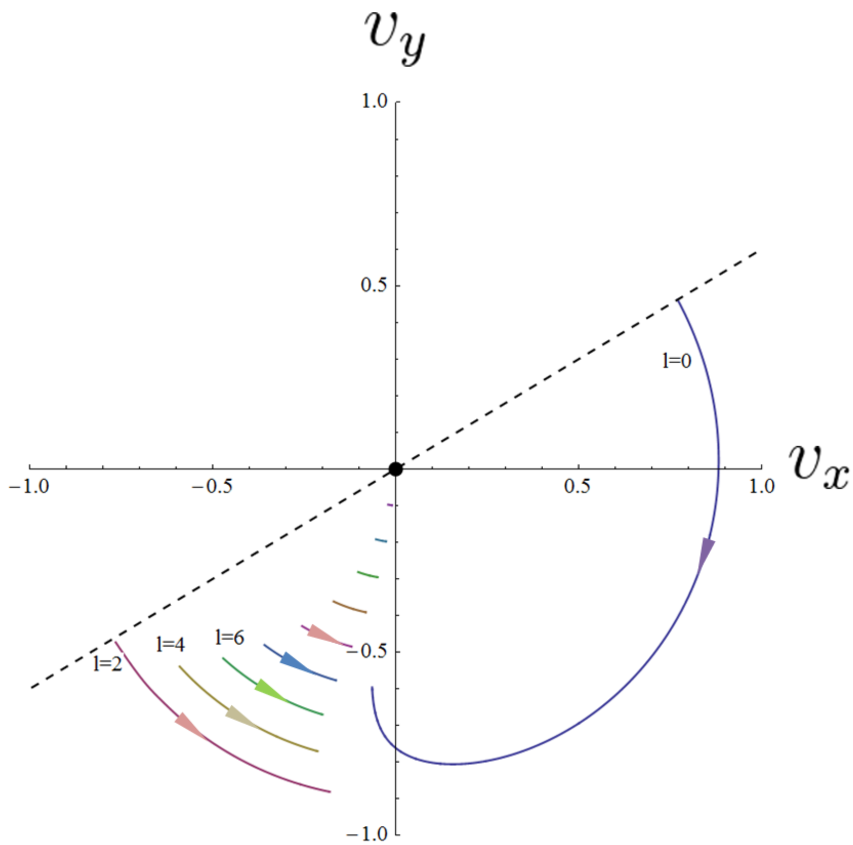


Figure 4.2: Plot of the vector defined in (4.9) with $\Delta_\phi = 1/8$. Each curve stands for vectors with different spin. The arrow shows the direction of increasing Δ. All curve start with the vector with $\Delta = \Delta_{uintary}$, except for the spin-0 curve which starts with $\Delta = 1.03$. The black dot in the center stands for the identity operator.

and the following fusion rules:

$$
\begin{aligned}
\sigma \times \sigma &\sim \mathbf{1} + \epsilon \\
\epsilon \times \epsilon &\sim \mathbf{1} + \epsilon \\
\epsilon \times \sigma &\sim \sigma.
\end{aligned}
\tag{4.13}
$$

Taking $\Delta_\phi = 1/8$ is equivalent to saying we are studying the first channel. Notice the bound we got is with in 3% of the actual value $\Delta_\epsilon = 1$. Considering we are doing the bootstrap by simply plotting dots, this is pretty remarkable.

4.1.2 CFT's with Global Symmetry

It is possible to consider conformal field with global symmetry. Now the decomposition of four point functions becomes:

$$
< \phi_I(x_1)\phi_J(x_2)\phi_K(x_3)\phi_L(x_4) >= \frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\sum_i \mathbf{P}^{(i)}_{IJKL}\left(\sum_{\mathcal{O}\in i}\lambda^2_{\mathcal{O}}g_{\Delta,l}(u,v)\right)
$$

$$
\text{where} \quad i \in \{\text{all irreps appearing in } \phi\times\phi\}. \tag{4.14}
$$

take O(N) vector model as an example, the projector $\mathbf{P}^{(i)}_{IJKL}$ are now

$$
\begin{aligned}
\mathbf{P}^{(S)}_{IJKL} &= \frac{1}{N}\delta_{IJ}\delta_{KL} \\
\mathbf{P}^{(T)}_{IJKL} &= \frac{1}{2}\delta_{IL}\delta_{JK} + \frac{1}{2}\delta_{IK}\delta_{JL} - \frac{1}{N}\delta_{IJ}\delta_{KL} \\
\mathbf{P}^{(A)}_{IJKL} &= \frac{1}{2}\delta_{IL}\delta_{JK} - \frac{1}{2}\delta_{IK}\delta_{JL}
\end{aligned} \tag{4.15}
$$

equating the s-channel and t-channel :

$$
< \phi_I(x_1)\phi_J(x_2)\phi_K(x_3)\phi_L(x_4) > -\{x_1 \leftrightarrow x_3, I \leftrightarrow K\} = 0 \tag{4.16}
$$

and collect the coefficient of independent tensor structure $\delta_{IJ}\delta_{KL}, \delta_{IL}\delta_{JK}$ and $\delta_{IK}\delta_{JL}$. We got the following three crossing equation:

$$
\sum_{\mathbf{S}+} p_S \begin{pmatrix} 0 \\ F \\ H \end{pmatrix} + \sum_{\mathbf{T}+} p_7 \begin{pmatrix} F \\ (1-\frac{2}{N})F \\ -(1+\frac{2}{N})H \end{pmatrix} + \sum_{\mathbf{A}-} p_{14} \begin{pmatrix} -F \\ F \\ -H \end{pmatrix} = 0 \tag{4.17}
$$

These crossing equations were worked in [69] in a slightly different notation. They were later applied to three dimensional O(N) vector model in [28].

Instead of repeating their work, we are going to consider the exceptional group G_2. The product of two fundamental irreps of G_2 gives

$$\mathbf{7}\times\mathbf{7}=\mathbf{1}+\mathbf{7}+\mathbf{14}+\mathbf{27} \tag{4.18}$$

in terms of OPE, it is given by:

$$\phi_I\times\phi_J\sim\delta_{IJ}O^{(\mathbf{1})}+d_{IJK}O^{(\mathbf{7})}_K+O^{(\mathbf{14})}_{[IJ]}+O^{(\mathbf{27})}_{(IJ)} \tag{4.19}$$

$$\text{satisfying}\quad O^{(\mathbf{14})}_{[IJ]}d_{IJK}=0,\quad O^{(\mathbf{27})}_{(IJ)},\delta_{IJ}=0 \tag{4.20}$$

where d_{IJK} are the famous invariant three form of G_2. The constrains (4.20) guarantee that $O^{(\mathbf{14})}_{[IJ]}$ and $O^{(\mathbf{27})}_{(IJ)}$ indeed lives in irreducible representation of G_2. The projector are now give by

$$\begin{aligned}
\mathbf{P}^{(1)}_{IJKL}&=\frac{1}{7}\delta_{IJ}\delta_{KL}\\
\mathbf{P}^{(7)}_{IJKL}&=-d_{IJM}d_{KLM}\\
\mathbf{P}^{(14)}_{IJKL}&=\frac{1}{2}\delta_{IL}\delta_{JK}-\frac{1}{2}\delta_{IK}\delta_{JL}+d_{IJM}d_{KLM}\\
\mathbf{P}^{(27)}_{IJKL}&=\frac{1}{2}\delta_{IL}\delta_{JK}+\frac{1}{2}\delta_{IK}\delta_{JL}-\frac{1}{7}\delta_{IJ}\delta_{KL}
\end{aligned} \tag{4.21}$$

we have assumed the normalization $d_{IMN}d_{NMJ}=-\delta_{IJ}$, and the signs are fixed by reflection-positivity. One could contract the projectors with $\delta_{IL}\delta_{JK}$ to get the correct dimension of each representation. When we go from s-channel to t-channel, hence make the replacement $\{1, I, u\} \to \{3, K, v\}$, one need to use the relation

$$d_{IJM}d_{KLM} = -\frac{1}{3}(\frac{1}{2}\delta_{IL}\delta_{JK} - \frac{1}{2}\delta_{IK}\delta_{JL}) - \frac{1}{\sqrt{6}}f_{IJKL} \tag{4.22}$$

where $f_{IJKL} = \frac{1}{3!}\epsilon_{IJKLMNP}d_{MNP}$, corresponding to the Hodge dual of G_2 invariant three form. The coefficients has to do with our normalization. Collecting the coefficients of $\{\delta_{IJ}\delta_{KL}, \delta_{IL}\delta_{JK} - \delta_{IK}\delta_{JL}, \delta_{IL}\delta_{JK} + \delta_{IK}\delta_{JL}, f_{IJKL}\}$, we get the crossing equations:

$$\sum_{\mathbf{1}+} p_1 \begin{pmatrix} 0 \\ 0 \\ \frac{F}{7} \\ -\frac{H}{7} \end{pmatrix} + \sum_{\mathbf{7}-} p_7 \begin{pmatrix} -\frac{F}{2} \\ H \\ \frac{F}{6} \\ \frac{H}{6} \end{pmatrix} + \sum_{\mathbf{14}-} p_{14} \begin{pmatrix} -F \\ -H \\ \frac{F}{3} \\ \frac{H}{3} \end{pmatrix} + \sum_{\mathbf{27}+} p_{27} \begin{pmatrix} \frac{3F}{2} \\ 0 \\ \frac{5F}{14} \\ \frac{9H}{14} \end{pmatrix} = 0 \tag{4.23}$$

One could try to bound the scalar operator in the singlet representation of G_2, such a plot is given in Figure 4.3, which appears to be equal to the bound for the first scalar in **1** representation O(7). This is a very general feature for conformal bootstrap, as observed in [70]. The bound for singlets of the subgroup turns to be equal to be equal the bound for singlets of the parent group. Non-trivial results start to appear when we introduce and bound the gap $\Delta \geq \Delta_{\mathbf{7}}^{l=1}$ for the first non-conserved current, hence spin-1 operator in $\mathbf{7}^-$ channel of G_2. The branching rule from O(7) to G_2:

$$\mathbf{21} \to \mathbf{21} + \mathbf{7} \tag{4.24}$$

tells us that one need conserved current in **7** of G_2 for the global symmetry to be enhanced to O(7). Introducing a gap in this channel therefore allows us to study G_2

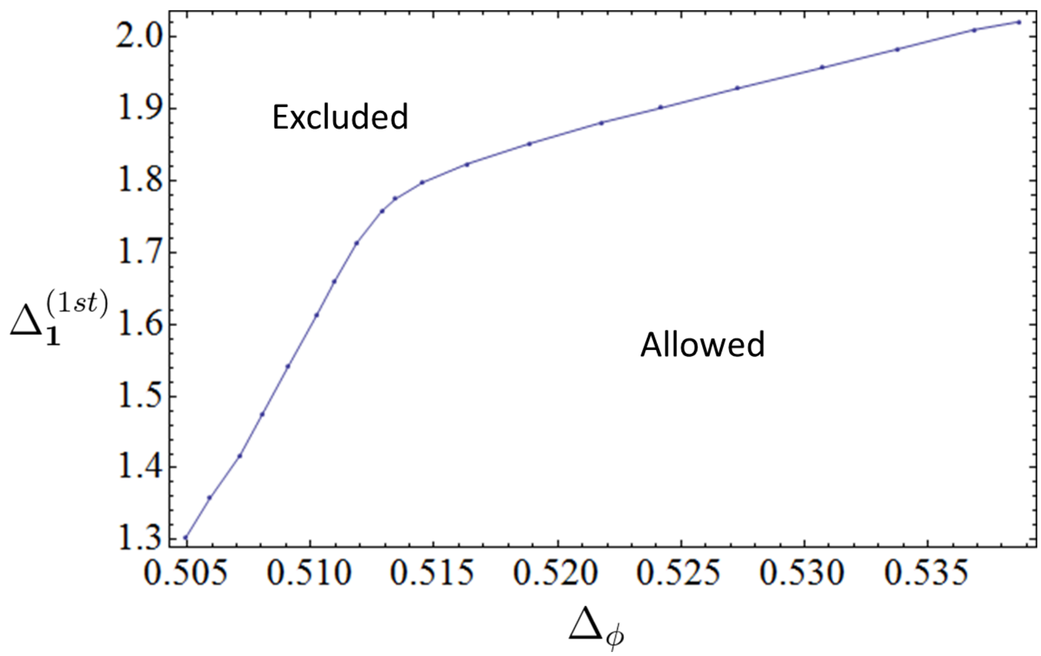


Figure 4.3: Bootstrap study of G_2 invariant theory in $D = 3$. Bounds on the first scalar in **1** representation of G_2 are plotted, which appears to be equal to the bound for the first scalar in **1** representation O(7).

invariant theory which are not O(7) invariant. Bounds for the massive spin-1 current in four dimension are presented in Figure 4.4. Notice there seems to be a "kink" resembling the two dimensional Ising model [71]. We are not sure whether such a kink corresponds to an actual conformal field theory or not. A confirmation using other methods such as $d = 4 - \epsilon$ loop calculation would be extremely interesting. Our ultimately objective is, however, to use the G_2 conformal bootstrap to study the dual theory of the solution we found, this however requires the knowledge of the full Kaluza-Klein spectrum, which has not been worked out yet. We hope to come back to this problem in the future.

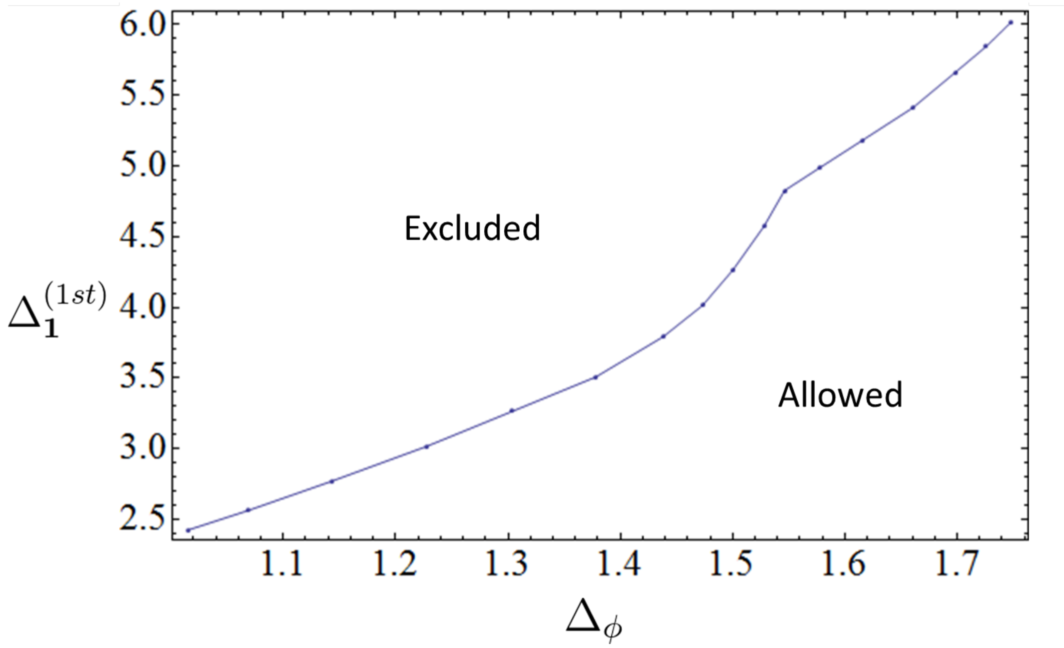


Figure 4.4: Bootstrap study of G_2 invariant theory in $D = 4$. Bounds on the non-conserved current operator (spin-1) in **7** representation of G_2.

4.2 CFT with $\mathrm{F}_4/\mathrm{SU}(3)$ flavor symmetry in $6 - \epsilon$ dimension

4.2.1 One loop calculation in $6 - \epsilon$ dimension

As an interesting application of conformal bootstrap, let's consider a theory with $\mathrm{F}_4/\mathrm{SU}(3)$ flavor symmetry in $6 - \epsilon$ dimensions:

$$\mathcal{L}_{int}^{6-\epsilon} = \frac{g}{6}\mu^{\epsilon/2} d_{IJK}\phi^I\phi^J\phi^K, \tag{4.25}$$

where d_{IJK} is the totally symmetric invariant tensor of $\mathrm{F}_4/\mathrm{SU}(3)$. For the case of F_4, the flavor symmetry group is taken to be the compact real form of the lie algebra f_4, which is also known as the isometry group of octonionic projective plane $\mathbf{OP}^2$ [72].

We will first use ϵ-expansion to show that these theories have stable IR fixed points. The idea of ϵ-expansion in $6 - \epsilon$ dimensions could be dated back to [73].

Where the interaction

$$\mathcal{L}_{int}^{6-\epsilon} = g\mu^{\epsilon/2}\phi^3. \tag{4.26}$$

was studied, and the 1-loop beta function was found to be

$$\beta(g) = -\frac{\epsilon}{2}g - \frac{3}{256\pi^3}g^3 \tag{4.27}$$

Notice the beta function has a zero at imaginary vale of g, which is related to the the so called "Yang-Lee Edge Singularity" [74, 75]. The model with O(N) global symmetry and the interaction

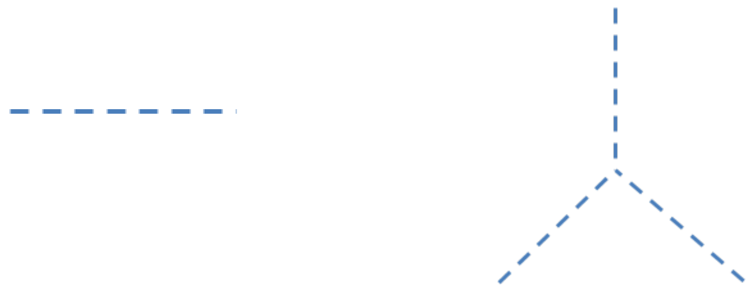
$$\mathcal{L}_{int}^{6-\epsilon} = \mu^{\epsilon/2}\frac{1}{6}g_1\tau^3 + \frac{1}{2}g_2\tau(\phi^i\phi^i)^2. \tag{4.28}$$

was considered more recently [76, 77], where τ is an O(N) singlet and ϕ^i is a O(N) vector, in the pursuit of studying the critical O(N) vector model in five dimension (by taking $\epsilon = 1$). An IR fixed point was found for large enough $N > N_{critical}$. In 1-loop calculation [76], $N_{critical}$ was found to be 1038, while in three loop calculation $N_{critical} = 64$ [77]. It is therefore an interesting problem to figure out the actual $N_{critical}$ in five dimensions. In [78], the method of conformal bootstrap was implemented and a kink was found for N as low as 35, which seems to indicated a much lower $N_{critical}$ than the three loop result.

Throughout this subsection, we will follow the convention in [76]. The Feynman rules could be summarized as in Figure 4.5.

Before calculation the beta function, we first need to consider the wave function renormalization, as indicated by Figure 4.6. The graph is given by

$$D_1 = \frac{1}{2}(-g^2)d_{Iab}d_{Jab}I_1 = \frac{1}{2}g^2\delta_{IJ}I_1 = \frac{1}{2}g^2\delta_{IJ}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(p+q)^2}\frac{1}{q^2}$$



$\frac{1}{p^2}\delta_{IJ}$ $\qquad$ $-gd_{IJK}$

Figure 4.5: Feynman rules.

Figure 4.6: Wave function renormalization.

where we have used the relation

$$d_{Iab}d_{Jab} = \delta_{IJ} \tag{4.29}$$

which is in fact a choice of the normalization of the invariant tensor d_{IJK}. The integral I_1 was preformed in the appendix A of [76],

$$\begin{aligned} I_1 &= \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)\Gamma(D/2-1)^2}{(2-D/2)\Gamma(D-2)} (\frac{1}{p^2})^{2-D/2} \\ &= -\frac{p^2}{6(4\pi)^2} \frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}} + \mathcal{O}(1) \end{aligned} \tag{4.30}$$

We have used the renormalization condition $p^2 = M^2$. The $1/\epsilon$ pole in D_1 must be canceled by the counter term $-p^2\delta_\phi$, therefore

$$\delta_\phi = -\frac{g^2}{12(4\pi)^3}\frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}}. \tag{4.31}$$

The Feynman diagram related to vertex renormalzation is Figure 4.7. which is

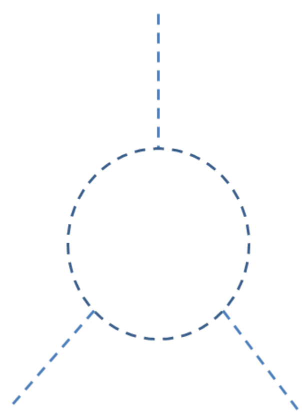


Figure 4.7: Vertex renormalization.

$$D_2 = (-g)^3 d_{Iab}d_{Jbc}d_{K_ca}I_2 = -g^3\beta I_2 = -g^3\beta' \int \frac{d^Dk}{(2\pi)^D}\frac{1}{(p-k)^2}\frac{1}{(k+q)^2}\frac{1}{k^2}. \tag{4.32}$$

We have defined the constant β' by

$$d_{Iab}d_{Jbc}d_{K_ca} = -\beta' d_{IJK}, \tag{4.33}$$

which we will calculate later. The diagram has a $1/\epsilon$ pole

$$D_2 = \frac{\beta' g^3}{2(4\pi)^3}\frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}}. \tag{4.34}$$

canceled by the contour term $-\delta g$, so we have

$$\delta_g = \frac{\beta' g^3}{2(4\pi)^3} \frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}}. \tag{4.35}$$

For the calculation of the integral, we again refer the readers to the appendix A of [76].

The Callan-Symanzik equation [79–81] for Green's function with m external legs is:

$$(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + m\gamma_\phi)G^{(m)} = 0. \tag{4.36}$$

Apply it to $G^{(2)}$, we get

$$-\frac{1}{p^2}M\frac{\partial}{\partial M}\delta_\phi + 2\gamma_\phi\frac{1}{p^2} = 0. \tag{4.37}$$

Then the anamolous dimension is

$$\gamma_\phi = \frac{1}{2}M\frac{\partial}{\partial M}\delta_\phi = \frac{1}{(4\pi)^3}\frac{g^2}{12}. \tag{4.38}$$

Apply the Callan-Symanzik equation to $G^{(3)}$, the beta function becomes

$$\beta(g) = -\frac{\epsilon}{2}g + M\frac{\partial}{\partial M}(-\delta_g + \frac{1}{2}g(3\delta_\phi)) = -\frac{\epsilon}{2}g + \frac{1+4\beta'}{256\pi^3}g^3 \tag{4.39}$$

Notice $\beta' = -1$ give us back the single scalar result (4.26). We could now see that for

$$\beta' > -\frac{1}{4}, \tag{4.40}$$

the beta function has a real zero at

$$g_*^2 = \frac{128\pi^3\epsilon}{1+4\beta}. \tag{4.41}$$

Plug in (4.38), we get

$$\gamma_\phi = \frac{\epsilon}{6}\frac{1}{1+4\beta}. \tag{4.42}$$

Of course, for the critical point to exist, the flavor symmetry group we pick needs to have at least a rank-3 totally symmetric tensor. F_4 group is know to have such a tensor carrying three fundamental (**26** or $(0,0,0,1)$) index, while SU(N) group have such tensors carrying three adjoint index. For F_4, β' was calculated in [82, 83] to be

$$\beta'_{F_4} = 3/7. \tag{4.43}$$

While for SU(N), one could use various Mathematica packages such as [84] to get

$$\beta'_{SU(N)} = -\frac{N\left(N-\frac{12}{N}\right)}{2\left(N^2-4\right)}. \tag{4.44}$$

Therefore, one loop result suggest that there exist a IR fixed point at $6-\epsilon$ dimensions

	F_4	$SU(3)$	$SU(4)$
β'	3/7	3/10	-1/6

Table 4.1: The constant β' defined in (4.33).

for F_4 group and SU(N) with $N \leq 4$, see Table 4.1.

Next we will calculated the anomalous dimension of scalar operators $O^{IJ} \sim \phi^I\phi^J$, we need to consider the three point function $<\phi_I(p)\phi^J(q)O^{IJ}(p+q)>$, as indicated by the Feynman diagrams in Figure 4.8. The operator $O^{IJ} \sim \phi^I\phi^J$ forms a reducible representation of the flavor group, we therefore need to decompose them into

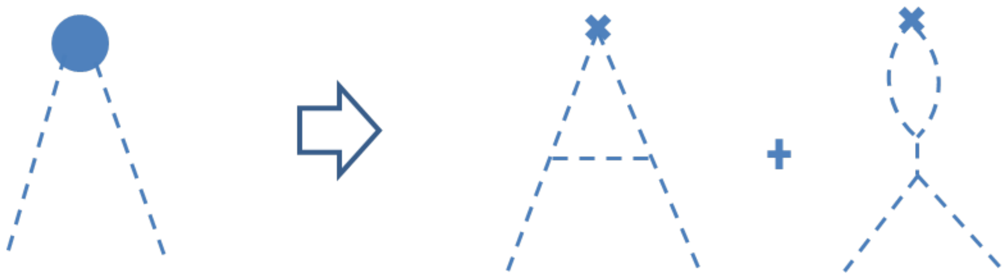


Figure 4.8: Renormalization of operator $O^{IJ} \sim \phi^I \phi^J$.

irreducible representation. For F_4, this could be achieved using the projectors:

$$
\begin{aligned}
\mathbf{P}^{(\mathbf{1})}_{IJKL} &= \frac{1}{26}\delta_{IJ}\delta_{KL} \\
\mathbf{P}^{(\mathbf{26})}_{IJKL} &= d_{IJM}d_{KLM} \\
\mathbf{P}^{(\mathbf{324})}_{IJKL} &= \frac{1}{2}\delta_{IL}\delta_{JK} + \frac{1}{2}\delta_{IK}\delta_{JL} - d_{IJM}d_{KLM} - \frac{1}{26}\delta_{IJ}\delta_{KL} \\
\mathbf{P}^{(\mathbf{52})}_{IJKL} &= (\frac{2}{9})\left(\frac{1}{2}\delta_{IL}\delta_{JK} - \frac{1}{2}\delta_{IK}\delta_{JL} + 7(\frac{1}{2}d_{ILM}d_{JKM} - \frac{1}{2}d_{JLM}d_{IKM})\right) \\
\mathbf{P}^{(\mathbf{273})}_{IJKL} &= (\frac{7}{9})\left(\frac{1}{2}\delta_{IL}\delta_{JK} - \frac{1}{2}\delta_{IK}\delta_{JL} - 2(\frac{1}{2}d_{ILM}d_{JKM} - \frac{1}{2}d_{JLM}d_{IKM})\right)
\end{aligned}
\tag{4.45}
$$

One could contract the generators with $\delta_{IL}\delta_{JK}$ and check that the projectors give the correct dimension for each representation. The scalar operators form a irreducible representation could be written as

$$
O^{(i)IJ} \sim P^{(i)}_{IJKL}\phi^K\phi^L. \tag{4.46}
$$

Use (4.29) and (4.33), one could check the following relations:

$$
\mathbf{P}^{(i)}_{KLPQ}d_{PMI}d_{QMJ} = A_i \cdot \mathbf{P}^{(i)}_{IJKL}, \quad \mathbf{P}^{(i)}_{IJPQ}d_{PQM}d_{MKL} = B_i \cdot \mathbf{P}^{(i)}_{IJKL}. \tag{4.47}
$$

with a little bit calculation, we get

$$
\begin{aligned}
A_{\mathbf{1}} &= 1, \quad A_{\mathbf{26}} = -\beta, \quad A_{\mathbf{324}} = \frac{1}{14} \\
B_{\mathbf{1}} &= 0, \quad B_{\mathbf{26}} = 1, \qquad B_{\mathbf{324}} = 0
\end{aligned} \tag{4.48}
$$

In Figure 4.8, the first diagram gives

$$
D_3 = (-g)^2 \mathbf{P}^{(i)}_{KLPQ} d_{PMI} d_{QMJ} I_2 = A_i g^2 \mathbf{P}^{(i)}_{IJKL} I_2, \tag{4.49}
$$

while the second diagram gives

$$
D_4 = (-g)^2 \mathbf{P}^{(i)}_{IJPQ} d_{PQM} d_{MKL} I_2 = -B_i g^2 \mathbf{P}^{(i)}_{IJKL} I_2. \tag{4.50}
$$

The $1/\epsilon$ pole is canceled by

$$
\delta_{\phi^2 \in i} = -A_i \frac{g^2}{2(4\pi)^3} \frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}} + B_i \frac{g^2}{12(4\pi)^3} \frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}} \tag{4.51}
$$

then

$$
\begin{aligned}
\delta_{\phi^2 \in \mathbf{1}} &= -\frac{g^2}{2(4\pi)^3} \frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}} + 0 = 6\delta_\phi \\
\delta_{\phi^2 \in \mathbf{26}} &= \beta \frac{g^2}{2(4\pi)^3} \frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}} + \frac{g^2}{12(4\pi)^3} \frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}} = -(6\beta+1)\delta_\phi \\
\delta_{\phi^2 \in \mathbf{324}} &= -\frac{1}{14} \frac{g^2}{2(4\pi)^3} \frac{\Gamma(\epsilon/2)}{(M^2)^{\epsilon/2}} + 0 = \frac{3}{7}\delta_\phi
\end{aligned} \tag{4.52}
$$

The anomalous dimension is given by

$$\begin{aligned}
\gamma_{\phi^2\in\mathbf{26}} &= M\frac{\partial}{\partial M}(-\delta_{\phi^2\in\mathbf{26}}+\delta_\phi) = (12\beta+4)\gamma_\phi \\
\gamma_{\phi^2\in\mathbf{1}} &= M\frac{\partial}{\partial M}(-\delta_{\phi^2\in\mathbf{1}}+\delta_\phi) = -10\gamma_\phi \\
\gamma_{\phi^2\in\mathbf{324}} &= M\frac{\partial}{\partial M}(-\delta_{\phi^2\in\mathbf{324}}+\delta_\phi) = \frac{8}{7}\gamma_\phi.
\end{aligned} \tag{4.53}$$

The anomalous dimension of $d_{IJK}\phi^I\phi^J\phi^K$, however, could simply be calculated by the second derivative of beta function

$$\Delta_{\phi^3\in\mathbf{1}} = d + \left.\frac{\partial\beta(g)}{\partial g}\right|_{g=g^*} = d+\epsilon = 6. \tag{4.54}$$

These are all the operators related to our later bootstrap study. Notice for the case of SU(3) and SU(4), (4.52) is also valid, simply because the projectors take the same form for these three irreps, as long as one make the replacement

$$\mathbf{26}\to\mathbf{Adj}, \quad \mathbf{324}\to(\mathbf{S},\bar{\mathbf{S}}) \tag{4.55}$$

For later reference, we summarize the scaling dimension of operators in $D=5.95$ and $D=5$ in Table 4.2 and Table 4.3. Notice for flavor group SU(4), in $D=5$, the scaling dimension of $\Delta_{\phi^2\in\mathbf{1}}$ violates the unitary bound. It is possible that such a IR fixed point would not survive in when one takes $\epsilon\to 1$. We will skip this theory when we do conformal bootstrap.

Another remark need to be mentioned, for SU(N) group, the interaction $\mathcal{L}_I = gd_{IJK}\phi^I\phi^J\phi^K$ has a hidden assumption that there is no fields transforms in the

	F_4	SU(3)	SU(4)
ϕ^I	1.97807	1.97879	2
$\phi^2 \in \mathbf{1}$	3.91930	3.91212	3.7
$\phi^3 \in \mathbf{1}$	6	6	6
$\phi^2 \in \mathbf{26}/\mathbf{Adj}$	3.97807	3.97879	4

Table 4.2: Scaling dimension of operators in $D = 5.95$ from one loop calculation.

	F_4	$SU(3)$	$SU(4)$
ϕ^I	1.56140	1.57576	2
$\phi^2 \in \mathbf{1}$	2.38596	2.24242	-2
$\phi^3 \in \mathbf{1}$	6	6	6
$\phi^2 \in \mathbf{26}/\mathbf{Adj}$	3.56140	3.57576	4

Table 4.3: Scaling dimension of operators in $D = 5$ from one loop calculation.

fundamental representation of SU(N). A more general type of interaction is

$$\mathcal{L}_I = \frac{g_1}{6}\mu^{\epsilon/2} d_{ABC}\phi^A\phi^B\phi^C + g_2\mu^{\epsilon/2}\eta^i(\phi^A T^A)_i^j\bar{\eta}_j \tag{4.56}$$

where η^i are SU(N) fundamentals. We leave this theory for future research.

4.2.2 Conformal Bootstrap study of CFT with $\mathrm{F}_4/\mathrm{SU}(3)$ *in* $6-\epsilon$ *dimensions*

The crossing equations for four point function of scalar operators carrying adjoint index of SU(N) has been worked out in [85],

$$\sum_{\mathbf{1}+}\lambda_1^2\begin{pmatrix}F\\0\\0\\-H\\-H\end{pmatrix}+\sum_{(\mathbf{S},\bar{\mathbf{S}})+}\lambda_{(S,\bar{S})}^2\begin{pmatrix}\frac{3F}{2}\\\frac{9F}{5}\\F\\\frac{9H}{10}\\\frac{5H}{2}\end{pmatrix}+\sum_{(\mathbf{S},\bar{\mathbf{A}})-}\lambda_{(S,\bar{A})}^2\begin{pmatrix}F\\-\frac{3}{2}F\\-\frac{F}{6}\\H\\\frac{5H}{9}\end{pmatrix}$$

$$+\sum_{\mathbf{Adj}+}\lambda_{Adj+}^2\begin{pmatrix}0\\F\\0\\\frac{4H}{3}\\0\end{pmatrix}+\sum_{\mathbf{Adj}-}\lambda_{Adj-}^2\begin{pmatrix}0\\0\\-F\\0\\\frac{4H}{3}\end{pmatrix}=0. \quad (4.57)$$

Here we will simply work out the F_4 crossing equations. Remember the four point could be decomposed as

$$<\phi_I(\overline{x_1)\phi_J(}x_2)\phi_K(\overline{x_3)\phi_L(}x_4)>=\frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\sum_i\mathbf{P}_{IJKL}^{(i)}\left(\sum_{\mathcal{O}\in i}\lambda_{\mathcal{O}}^2g_{\Delta,l}(u,v)\right)$$
$$\text{where}\quad i\in\{\mathbf{1}^+,\mathbf{26}^+,\mathbf{324}^+,\mathbf{52}^-,\mathbf{273}^-\}. \quad (4.58)$$

where the projectors are given in (4.45). To go from s-channel to t-channel, hence make the replacement $\{I,x_1\to J,x_3\}$, one need to use the following relation:

$$d_{IJM}d_{KLM}+d_{KJM}d_{ILM}+d_{IKM}d_{JLM}=\frac{1}{14}(\delta_{IJ}\delta_{KL}+\delta_{IL}\delta_{JK}+\delta_{IK}\delta_{JL}) \quad (4.59)$$

This equation could also be viewed as the defining relation of f_4 algebra. The tensor $d_{IJM}d_{KLM}$ has originally three independent choice of index structure, because of the above relation, we are left with two independent tensors $d_{JLM}d_{IKM}$and $d_{IJM}d_{KLM} - d_{KJM}d_{ILM}$, which have definite parity (ignoring the δ's) under $I \leftrightarrow K$. Make the replacement

$$\begin{aligned} d_{IJM}d_{KLM} &\to \frac{1}{2}(d_{IJM}d_{KLM} - d_{KJM}d_{ILM}) - \frac{1}{2}d_{IKM}d_{JLM} \\ &+\frac{1}{28}(\delta_{IJ}\delta_{KL} + \delta_{IL}\delta_{JK} + \delta_{IK}\delta_{JL}) \\ \frac{1}{2}d_{ILM}d_{JKM} - \frac{1}{2}d_{JLM}d_{IKM} &\to -\frac{1}{4}(d_{IJM}d_{KLM} - d_{KJM}d_{ILM}) - \frac{3}{4}d_{JLM}d_{IKM} \\ &+\frac{1}{56}(\delta_{IJ}\delta_{KL} + \delta_{IL}\delta_{JK} + \delta_{IK}\delta_{JL}), \end{aligned} \tag{4.60}$$

in

$$< \phi_I(x_1)\phi_J(x_2)\phi_K(x_3)\phi_L(x_4) > -\{I, x_1 \leftrightarrow J, x_3\} = 0 \tag{4.61}$$

and collect independent tensor structures, we get

$$\begin{aligned} &\sum_{\mathbf{1}+}\lambda_1^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{F}{26} \\ -\frac{H}{26} \end{pmatrix} + \sum_{\mathbf{26}+}\lambda_{26}^2 \begin{pmatrix} \frac{H}{2} \\ -F \\ \frac{F}{7} \\ \frac{F}{14} \\ 0 \end{pmatrix} + \sum_{\mathbf{273}-}\lambda_{273}^2 \begin{pmatrix} \frac{7H}{18} \\ \frac{7F}{3} \\ -\frac{5}{3}F \\ \frac{F}{3} \\ \frac{7H}{18} \end{pmatrix} \\ &+ \sum_{\mathbf{52}-}\lambda_{52}^2 \begin{pmatrix} -\frac{7}{18}H \\ -\frac{7}{3}F \\ -\frac{F}{3} \\ \frac{F}{6} \\ \frac{H}{9} \end{pmatrix} + \sum_{\mathbf{324}+}\lambda_{324}^2 \begin{pmatrix} -\frac{H}{2} \\ F \\ \frac{13F}{7} \\ \frac{71F}{182} \\ \frac{7H}{13} \end{pmatrix} = 0. \end{aligned} \tag{4.62}$$

We will now use these bootstrap equation to study conformal field theory in $D = 5.95$ with and $D = 5$ respectively.

Make the assumptions that:

- The first scalar operator in the $\mathbf{1}^+$ channel has scaling dimension $\Delta = \Delta_{\mathbf{1}}^{1st}$,
- the second scalar operator in the $\mathbf{1}^+$ channel has scaling dimension $\Delta \geq \Delta_{\mathbf{1}}^{(2nd)}$,
- the second scalar operator in the $\mathbf{26}^+/\mathbf{Adj}^+$ channel has scaling dimension $\Delta \geq \Delta_{\mathbf{26}/\mathbf{Adj}}^{(2nd)}$. (The first scalar in this channel is simply Δ_ϕ.)

We will test whether a conformal with the specific choice of $\{\Delta_\phi, \Delta_{\mathbf{1}}^{(1st)}, \Delta_{\mathbf{1}}^{(2nd)}, \Delta_{\mathbf{26}/\mathbf{Adj}}^{(2nd)}\}$ is allowed by conformal bootstrap.

For F_4 invariant theory in $D = 5.95$, the result is given in Figure 4.9. The blue curve assumes that there is only one relevant scalar which is F_4 singlet in the spectrum, while the red curve assume the second scalar which is F_4 singlet has scaling dimension bigger than $\Delta_{\mathbf{1}}^{(2nd)} = 6.05$. For both curves, the second scalar operator which lives in $\mathbf{26}$ representation is assumed to have scaling dimension bigger than mean field theory value, hence $\Delta_{\mathbf{26}}^{(2nd)} = 2\Delta_\phi$. Notice both curves has a sudden change of slope around $(\Delta_\phi, \Delta^{1st}) = (1.97807, 3.91930)$, which are the value calculated from 1-loop ϵ-expansion.

For SU(3) invariant theory in $D = 5.95$, the result is given in Figure 4.10. The blue curve assumes that there is only one relevant scalar which is SU(3) singlet in the spectrum, while the red curve assumes the second scalar which is SU(3) singlet has scaling dimension bigger than $\Delta_{\mathbf{1}}^{(2nd)} = 6.05$. For both curves, the second scalar operator which lives in $\mathbf{Adj}$ representation is assumed to have scaling dimension bigger than free theory value, hence $\Delta_{\mathbf{26}}^{(2nd)} = 2\Delta_\phi$. Notice the curves have a sudden change of slope around $(\Delta_\phi, \Delta^{1st}) = (1.97879, 3.91212)$, which are the value calculated from 1-loop ϵ-expansion. This is a nontrivial check of our previous calculation.

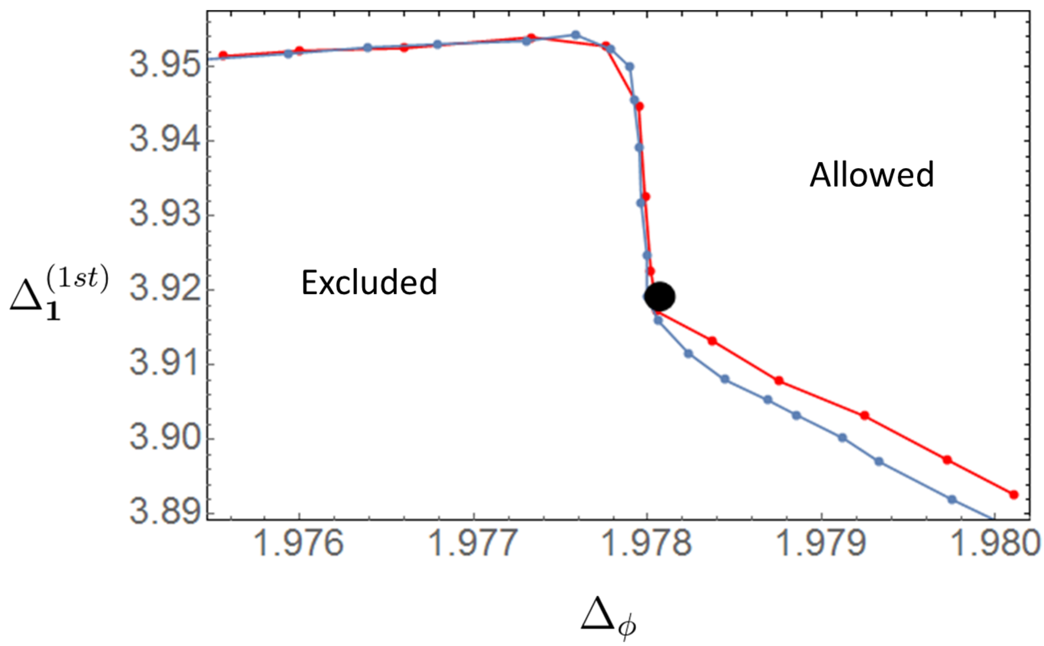


Figure 4.9: Bootstrap study of F_4 invariant theory in $D = 5.95$. Allowed choice of $\{\Delta_\phi, \Delta^{1st}\}$ is indicated. The second scalar operator in $\mathbf{26}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{26}}^{(2nd)} = 2\Delta_\phi$. For the red curve, the second scalar in $\mathbf{1}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{1}}^{(2nd)} = 5.95$. For the blue curve, the second scalar in $\mathbf{1}^+$ channel is assumed to have scaling bigger than $\Delta \geq \Delta_{\mathbf{1}}^{(2nd)} = 6.05$. The black dot indicates the result form 1-loop calculation.

Now let's focus to theories in five dimensions. We study F_4 and SU(3) invariant theory in $D = 5$ in Figure 4.11 and Figure 4.12, we make the assumption that there is only one relevant scalar operator which are F_4/SU(3) singlet. The three curves correspond to three different choices of $\Delta_{\mathbf{26/Adj}}^{(2nd)}$—the second scalar operator lives $\mathbf{26/Adj}$ of F_4/SU(3). Notice that as we increase the value of $\Delta_{\mathbf{26}}^{(2nd)}$, the curve starts to surround the black circle. Our guess is that the actual value of $(\Delta_\phi, \Delta_{\mathbf{1}}^{1st})$ is somewhere around the circle. Notice the location of the circle does not match the 1-loop calculation (black dot). This is however not to be worried. Since one has to take the limit $\epsilon \to 1$ to extrapolate the ϵ-expansion result to five dimensions,

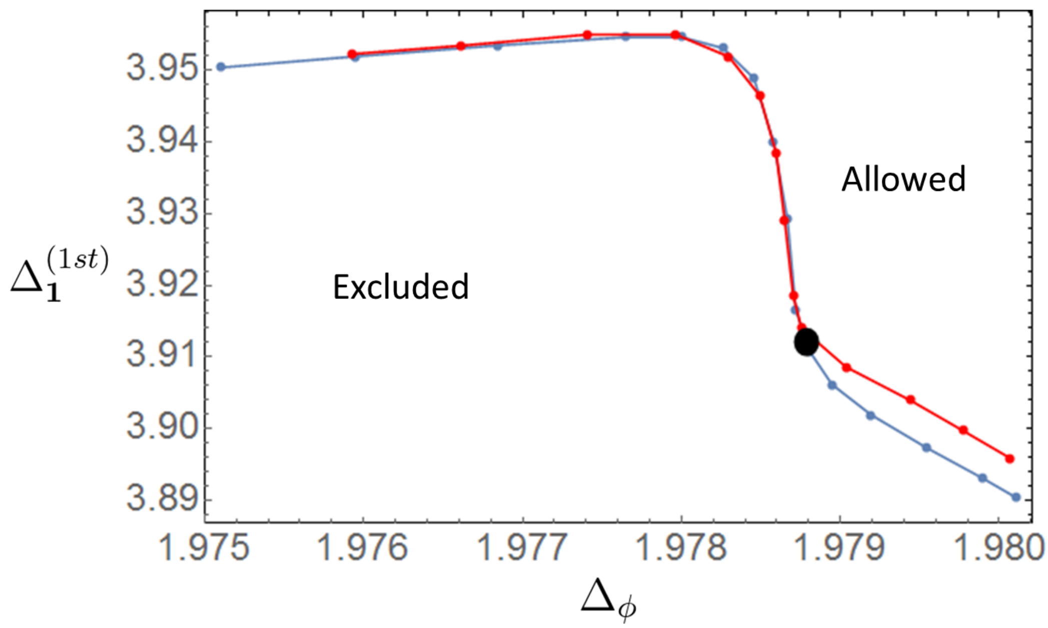


Figure 4.10: Bootstrap study of SU(3) invariant theory in $D = 5.95$. Allowed choice of $\{\Delta_\phi, \Delta^{1st}\}$ is indicated. The second scalar operator in $\mathbf{Adj}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{Adj}}^{(2nd)} = 2\Delta_\phi$. For the red curve, the second scalar in $\mathbf{1}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{1}}^{(2nd)} = 5.95$. For the blue curves, the second scalar in $\mathbf{1}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{1}}^{(2nd)} = 6.05$. The black dot indicates the result from 1-loop calculation.

there is no guarantee that one loop calculation should be correct. Remember that for three dimensional Ising model, the $4-\epsilon$ 1-loop expansion tells us that the critical exponent [86]:

$$\nu = \frac{1}{2} + \frac{1}{12}\epsilon \approx 0.583333 \tag{4.63}$$

corresponds to the scaling dimension

$$\Delta_\epsilon = 1.28571. \tag{4.64}$$

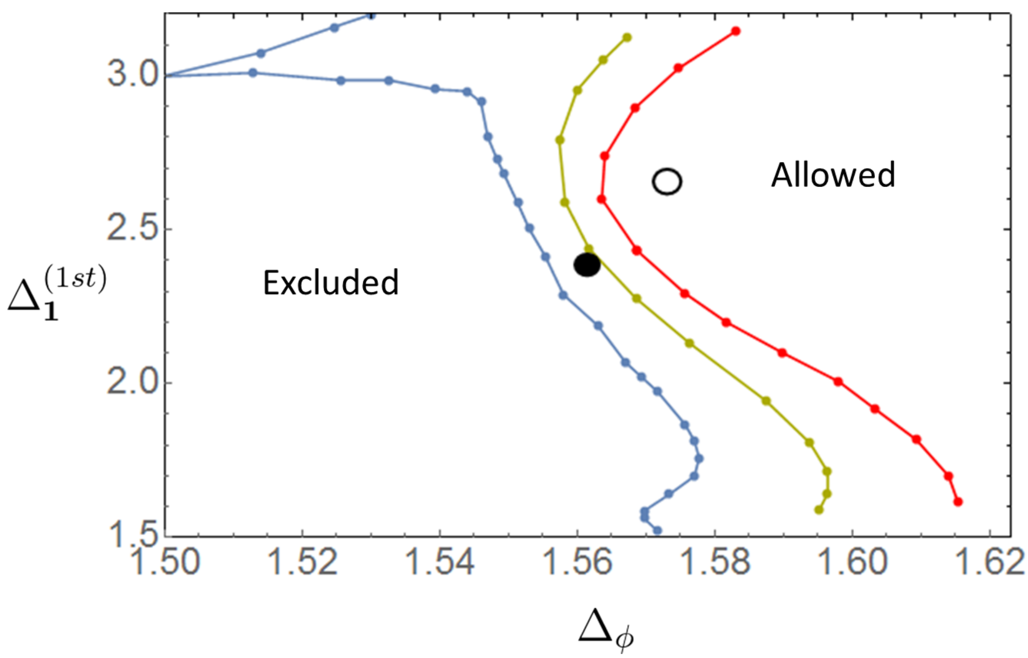


Figure 4.11: Bootstrap study of F_4 invariant theory in $D = 5$. Allowed choices of $\{\Delta_\phi, \Delta^{1st}\}$ is indicated. The second scalar operator in $\mathbf{1}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{1}}^{(2nd)} = 5$. For the blue, yellow, and red curve, the second scalar in $\mathbf{26}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{26}}^{(2nd)} = 2\Delta_\phi, 3.3$ and 3.5 respectively. The black dot indicates the result from 1-loop calculation.

The more precise result form conformal bootstrap is however [29]:

$$\nu = 0.629971(4), \quad \Delta_\epsilon = 1.412625(10). \tag{4.65}$$

It is of course of great interest to study the precise spectrum using either higher loop ϵ-expansion [87], or mixed operator conformal bootstrap [27].

4.3 Superconformal Bootstrap and AdS/CFT

As we mentioned in the introduction, AdS/CFT correspondence provides us a new method to study conformal field theory. Especially, the supergravity limit of

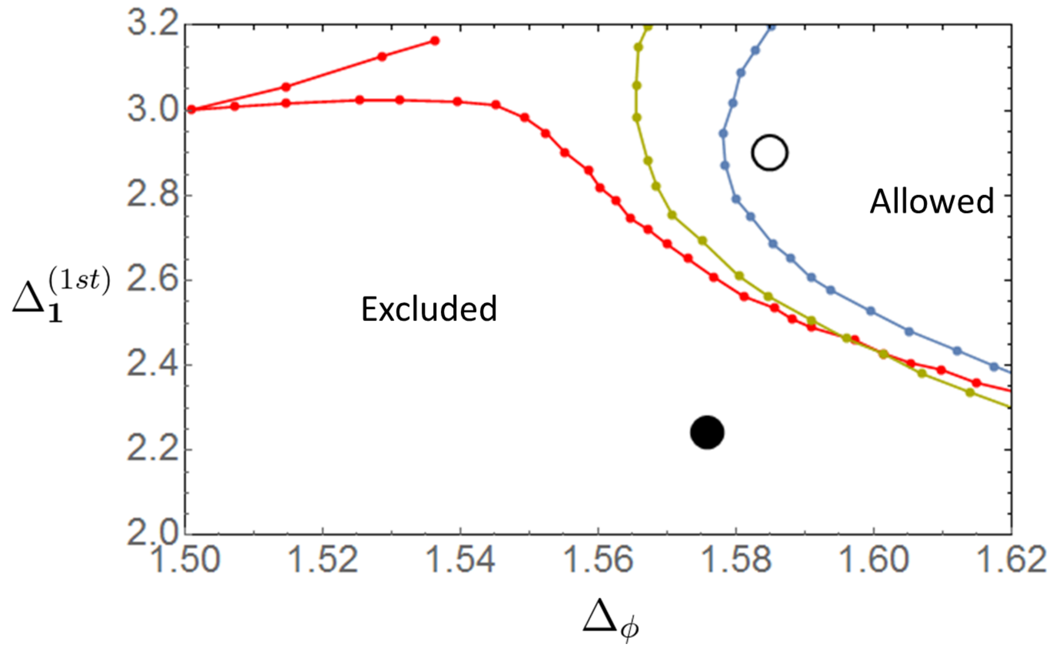


Figure 4.12: Bootstrap study of SU(3) invariant theory in $D = 5$. Allowed choices of $\{\Delta_\phi, \Delta^{1st}\}$ is indicated. The second scalar operator in $\mathbf{1}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{1}}^{(2nd)} = 5$. For the blue, yellow, and red curve, the second scalar in $\mathbf{26}^+$ channel is assumed to have scaling bigger than $\Delta_{\mathbf{1}}^{(2nd)} = 2\Delta_\phi, 3.2$ and 3.3 respectively. The black dot indicates the result from 1-loop calculation.

string theory is dual to

$$N \gg 1, \lambda \gg 1 \tag{4.66}$$

limit of the corresponding conformal field theory. There has been a lot nontrivial tests of the correspondence. For example, the spectrum of BPS protected operators match precisely with the Kaluza-Klein spectrum of $AdS_5 \times S^5$ [88]. In general, some kind of BPS protection mechanism was necessary to carry out the calculation form both the conformal field theory side and the supergravity side. When this is not the case, the field theory side of calculation becomes extremely difficult.

In [89,90], the authors calculated the anomalous dimension of the double trace operator $O^I O^J$, by first calculating the four point function $< O^I(x_1) O^J(x_2) O^K(x_3) O^L(x_4) >$

using Witten diagrams [91] in AdS_5, and then take the limit $x_3 \to x_4$. The operator $O^I \sim \mathrm{Tr}[\phi^i \phi^j]$ is the chiral primary operators of $\mathcal{N} = 4$ Super Yang-Mills theory. The result turns out to be

$$\gamma_{O^I O^I} = \frac{1}{16N^2} + \mathcal{O}(1/N^4) \tag{4.67}$$

where N is the rank of the gauge group SU(N). Notice the double trace operator is not BPS-protected, and therefore one in general do not know how to calculate its anomalous dimension in CFT side. It was until very recently, such a prediction form AdS/CFT was tested using $\mathcal{N} = 4$ super conformal bootstrap [32]. Interestingly, the result matches the AdS calculation beautifully.

Another interesting phenomenon happens when the supergravity solution is not maximally supersymmetric. Take the $\mathcal{N}$=3 solution discussed in Section 2.3.1 as an example. We have shown in Table 3.2 that the corresponding Kaluza-Klein spectrum contains long multiplets. Unlike string state, whose dual operator has large anomalous dimension of the order $\lambda^{1/4}$ [1], these supergravity states has finite anomalous dimension as $\lambda \to \infty$. In terms of standard AdS/CFT dictionary, the dual operators should be "single trace" operators which are not BPS-protected. In this section, we prepare the basic information for implementing superconformal bootstrap to study the spectrum of a superconformal field theory with known AdS dual [92], we will leave the numerical bootstrap as a future project.

4.3.1 $\mathcal{N} = 1$ Superconformal bootstrap

Before discussing applying conformal bootstrap to ADS/CFT, we need to prepare some basics about superconformal bootstrap. The basic ingredients are worked out in [31, 93]. We will give a short review here. Consider the four point function $< \phi\phi\phi^\dagger\phi^\dagger >$. For simplicity, we take ϕ (and $\phi^\dagger$) to be chiral (and anti-chiral) operators, or simply the lowest (power in θ) component of the chiral superfield Φ (and $\Phi^\dagger$). Its scaling dimension and $\mathrm{U}(1)_{\mathcal{R}}$ charge of this operator are related by

$$\Delta_\phi = \frac{3}{2}r. \tag{4.68}$$

As fixed by chiral condition.

Before performing conformal bootstrap, the first task one encounter is to work out the "selection rules", or in other words, which operator could appear in the operator product expansion of $\phi \times \phi$ or $\phi \times \phi^\dagger$. Equivalently, one could ask the question, for the three point function

$$< \Phi\Phi O_I^\dagger >, \quad \text{or} \ < \Phi\Phi^\dagger O_I > \tag{4.69}$$

to be non-vanishing, what condition does O_I need to satisfy. In [94], the general structure of three point in four dimensional superconformal field theory has been worked out. Applying the condition that Φ is chiral [31], one could work out the selection rules. Instead of using superspace method, one could requires that the operators appearing in $\phi\times\phi$ OPE to be annihilated by the superconformal transformation generator $\bar{Q}$ (contained in superconformal algebra). The final conclusion are equivalent. There are three type of superfields O_I that with non vanishing $< \Phi\Phi O_I^\dagger >$:

- Chiral multiplet Φ^2, with $r = (2r_\phi)$ and $l = 0$. This operator lives in the $\mathcal{N}$=1 chiral ring, and are clearly BPS-protected:

  $$\Delta = 2\Delta_\phi.$$

  Only one operator in this multiplet appears in $\phi \times \phi$ OPE, which is the superconformal primary, with

$$\Delta = \Delta_\phi, \quad l = 0. \tag{4.70}$$

- Non-chiral short multiplet $O^{\alpha_1 \ldots \alpha_{s1}}{}_{\dot{\alpha}_1 \ldots \dot{\alpha}_{s2}}$ with $s_1 - s_2 = 1/2$, spin $l = 2s_2$ odd, $r = 2r_\phi - 1$. This operator is also BPS-protected, and therefore have the scaling dimension

$$\Delta = 2\Delta_\phi + l + 1.$$

Only one operator in this multiplet appears in $\phi \times \phi$ OPE, which is the level-1 SUSY descendant, with

$$\Delta = 2\Delta_\phi + l, \quad l \text{ even}. \tag{4.71}$$

- Non-chiral long multiplet $O^{\alpha_1 \ldots \alpha_{s1}}{}_{\dot{\alpha}_1 \ldots \dot{\alpha}_{s2}}$ with $l = 2s_1 = 2s_2$ even, $r_\phi = 2r - 2$. These operators are not BPS-protected, their scaling dimension just need to satisfy unitarity bound

$$\Delta \geq 2|\Delta_\phi - 3| + l + 2.$$

Only one operator in this multiplet appears in $\phi \times \phi$ OPE, which is the level-2 SUSY descendant, with

$$\Delta \geq |2\Delta_\phi - 3| + l + 3, \quad l \text{ even}. \tag{4.72}$$

Notice the first two type have the same expression for their scaling dimension and are both BPS-protected, we will call them "BPS" when we write down bootstrap equations. The third family will be referred as "non-BPS".

For $\Phi \times \Phi^\dagger$ OPE, the condition is greatly relaxed, the only condition for $<$

$\Phi\Phi^{\dagger}O_I >$ to be non vanishing is that the superconformal multiplet O_I needs to be neutral

$$r = 0, \quad \Delta \geq l + 2. \tag{4.73}$$

where l could be either even or odd. We will call these operators "'neutral' operator when we write bootstrap equations. More than one operator in this multiplet appears in $\phi \times \phi$ OPE, which will be clear shortly.

The four point function of chiral primaries are:

$$< \phi(x_1)\phi^{\dagger}(x_2)\phi(x_3)\phi^{\dagger}(x_4) >= \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\Delta \geq l+2} \lambda_{\mathcal{O}}^2 \mathcal{G}_{\Delta,l}(u,v) \tag{4.74}$$

where the superconformal blocks $\mathcal{G}_{\Delta,l}(u,v)$ is simply a linear combination of conformal blocks:

$$\begin{aligned} \mathcal{G}_{\Delta,l}(u,v) = g_{\Delta,l}(u,v) &+ \frac{1}{4}\frac{\Delta+l}{\Delta+l+1} g_{\Delta+1,l+1}(u,v) + \frac{1}{4}\frac{\Delta-l-2}{\Delta-l-1} g_{\Delta+1,l-1}(u,v) \\ &+ \left(\frac{1}{4}\frac{\Delta+l}{\Delta+l+1}\right)\left(\frac{1}{4}\frac{\Delta-l-2}{\Delta-l-1}\right) g_{\Delta+2,l}(u,v), \end{aligned} \tag{4.75}$$

as computed in [93]. Notice the superconformal block are determined by the SO(4,2) and $\mathrm{U}(1)_{\mathcal{R}}$ quantum numbers of the "superconformal" primaries. Conformal primaries in the same supermultiplet have OPE coefficients proportional to each other, as they are related by supersymmetry. The conformal primaries in a superconformal multiplets with spin $s_1 = s_2 = l/2$ and scaling dimension Δ_0 is listed in Table 4.4, the special case $l = 1$ "long graviton multiplet" could be found in [95]. When the unitary bound $\Delta_0 \geq l + 2$ is saturated, some conformal primaries decouples, which are generally referred as the phenomenon of "multiplet shorten" [96]. The same phenomenon could be observed in the expression of superconformal block, when you

$(s_1+l/2, s_2+l/2)$	Δ	$\mathcal{R}$-charge	
$(1/2,1/2)$	Δ_0+1	0	*
$(1/2,0)$	$\Delta_0+1/2$	-1	
$(0,1/2)$	$\Delta_0+1/2$	$+1$	
$(0,1/2)$	$\Delta_0+3/2$	-1	
$(1/2,0)$	$\Delta_0+3/2$	$+1$	
$(0,0)$	Δ_0	0	*
$(0,0)$	Δ_0+1	$+2$	
$(0,0)$	Δ_0+1	-2	
$(0,0)$	Δ_0+2	0	*
$(1/2,-1/2)$	Δ_0+1	0	
$(-1/2,1/2)$	Δ_0+1	0	
$(0,-1/2)$	$\Delta_0+1/2$	$+1$	
$(-1/2,0)$	$\Delta_0+1/2$	-1	
$(0,-1/2)$	$\Delta_0+3/2$	-1	
$(-1/2,0)$	$\Delta_0+3/2$	1	
$(-1/2,-1/2)$	Δ_0+1	0	*

Table 4.4: $\mathcal{N}$=1 long multiplets with spin-l, satisfies the unitary bound $\Delta_0 \geq l+2$, in terms of quantum numbers of the conformal group SO(4,2) and the U$(1)_{\mathcal{R}}$ group. The last column denotes the conformal primaries appear in (4.75).

take $\Delta = l+2$ in (4.75), the last two terms vanish. "Multiplet shorten" for most general multiplets (for example, $s_1 \neq s_2$) are summeried in [97].

The crossing equation:

$$< \phi(\overbracket{x_1)\phi^\dagger(}x_2)\phi(\overbracket{x_3)\phi^\dagger(}x_4) > -(x_1 \leftrightarrow x_3) = 0 \tag{4.76}$$

tells us that

$$\sum_{\Delta \geq l+2} \lambda_{\mathcal{O}}^2 \mathcal{F}_{\Delta,l} = 0. \tag{4.77}$$

This equation itself could give us bounds for the scaling dimension of operators appear in $\phi \times \phi^\dagger$, which is the approach take in [93]. However, to get more stringent bound, one need to consider another four point function [31]

$$< \phi(\overbrace{x_1)\phi^\dagger(}x_2)\phi(\overbrace{x_4)\phi^\dagger(}x_3) > = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \mathcal{G}_{\Delta,l}(u/v, 1/v)$$

$$= \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 (-1)^l \tilde{\mathcal{G}}_{\Delta,l}(u,v) \qquad (4.78)$$

where

$$\tilde{\mathcal{G}}_{\Delta,l}(u,v) = g_{\Delta,l}(u,v) - \frac{1}{4}\frac{\Delta+l}{\Delta+l+1} g_{\Delta+1,l+1}(u,v) - \frac{1}{4}\frac{\Delta-l-2}{\Delta-l-1} g_{\Delta+1,l-1}(u,v)$$
$$+ (\frac{1}{4}\frac{\Delta+l}{\Delta+l+1})(\frac{1}{4}\frac{\Delta-l-2}{\Delta-l-1}) g_{\Delta+2,l}(u,v), \qquad (4.79)$$

we have used the general relation for conformal blocks

$$g(u/v, 1/v) = (-1)^l g_{\Delta,l}(u,v). \qquad (4.80)$$

Consider a different order of contraction of (4.78), and using the selection rules, we get:

$$< \phi(\overbrace{x_1)\phi(}x_4)\phi^\dagger(\overbrace{x_2)\phi^\dagger(}x_3) >= \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \Bigg(\sum_{\text{BPS, } l \text{ even}} \lambda_{\mathcal{O}}^2 g_{2\Delta_\phi+l,l}(v,u)$$
$$+ \sum_{\text{non-BPS, } l \text{ even}} \lambda_{\mathcal{O}}^2 g_{\Delta,l}(v,u) \Bigg). \qquad (4.81)$$

Collecting (4.77), (4.78) and (4.81), we get

$$\sum_{\text{netural}\pm} \lambda_{\mathcal{O}}^2 \begin{pmatrix} \mathcal{F}_{\Delta,l} \\ \tilde{\mathcal{F}}_{\Delta,l} \\ \tilde{\mathcal{H}}_{\Delta,l} \end{pmatrix} + \sum_{\text{BPS}+} \lambda_{\mathcal{O}}^2 \begin{pmatrix} 0 \\ F_{2\Delta_\phi+l,l} \\ -H_{2\Delta_\phi+l,l} \end{pmatrix} + \sum_{\text{non-BPS}+} \lambda_{\mathcal{O}}^2 \begin{pmatrix} 0 \\ F_{\Delta,l} \\ -H_{\Delta,l} \end{pmatrix} = 0 \qquad (4.82)$$

The crossing equation was derived in [31, 93], and could be easily generalized to superconformal field theories with extra flavor symmetries. Especially in [98], the O(N) SCFT's were considered, of which the O(4) case it of special interest for our case, as will be explicit in the next section.

4.3.2 The CFT dual of $\mathrm{AdS}_5 \times \mathrm{T}^{11}$

The dual theory of $AdS_5 \times T^{11}$ was identified in [92] by Igor R. Klebanov and Edward Witten, hence sometimes referred as Klebanov-Witten theory. T^{11} is a five dimensional space which could be written in coset form

$$M_5 = \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathrm{U}(1)} = \frac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)}{\mathrm{U}(1) \times \mathrm{U}(1)} \tag{4.83}$$

The dual theory is a four dimensional $\mathcal{N} = 1$ superconformal field theory with SU(2)×SU(2) global symmetry, and the gauge group SU(N)×SU(N). The field content could be represented by the same quiver diagram as the ABJM theory [3], see Figure 1.1. See [99] for the relation between Klebanov-Witten theory and ABJM theory. The theory contains two set of bi-fundamental of SU(N)×SU(N), A_i and B_i, transforms as doublet of either of the SU(2)'s. A superpotential of the form

$$W \sim \epsilon^{ij}\epsilon^{kl}Tr[A_i B_k A_j B_l]. \tag{4.84}$$

is present and the SU(2)×SU(2)∼O(4) flavor could be seen explicitly.

The superpotential (4.83) alway has $\mathrm{U}(1)_{\mathcal{R}}$-charge $\mathcal{R} = 2$, which means the fields A_i and B_i has

$$\mathcal{R} = 1/2. \tag{4.85}$$

When the theory flow to IR fixed point, superconformal algebra fixed the scaling dimension of chiral primaries to be

$$\Delta_I = \frac{3}{2}\mathcal{R} = 3/4. \tag{4.86}$$

To do conformal bootstrap, one need to consider four point function of gauge invariant operators. The chiral operator

$$\Phi_I \sim \mathrm{Tr}[A_i B_j]. \tag{4.87}$$

is in the chiral-ring and therefore has the scaling dimension $\Delta_\Phi = \frac{3}{2}$, they live in the $(\mathbf{1/2}, \mathbf{1/2})$ representation of SU(2)×SU(2). Knowing the scaling dimension Δ_Φ is in fact one of the advantage of picking $AdS_5 \times T^{11}$ as the solution to implement bootstrap test of AdS/CFT correspondence. This save us a lot of CPU time as we do not need to change Δ_Φ when doing numerics. We could now use superconformal bootstrap method introduced in previous subsection to study the four point function

$$< \phi_I(x_1)\phi_J(x_2)\phi^\dagger_K(x_3)\phi^\dagger_L(x_4) > \tag{4.88}$$

where ϕ_I are the lowest component of the superfield Φ_I. According to the spin production rule

$$(\mathbf{1/2}, \mathbf{1/2})_1 \times (\mathbf{1/2}, \mathbf{1/2})_{-1} \to (\mathbf{0}, \mathbf{0})_0 + (\mathbf{0}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{0})_0 + (\mathbf{1}, \mathbf{1})_0, \tag{4.89}$$

$U(1)_\mathcal{R}$ neutral operators which are SU(2)×SU(2) singlets could appear in $\phi_I \times \phi^\dagger_J$ OPE. Another advantage of picking $AdS_5 \times T^{11}$ as to implement bootstrap test of AdS/CFT correspondence is that the full Kaluza-Klein spectrum is known for this solution, which was calculated in [95]. For readers who are not familiar with the conventions of Kaluza-Klein reduction, we recommend the reference [100]. Kaluza-Klein spectrum tells us that the first $U(1)_{\mathcal{R}}$ neutral operators which are also SU(2)×SU(2) singlets such has the scaling dimension

$$\Delta = 2\sqrt{7} - 2 \approx 3.2915. \tag{4.90}$$

We hope to get this number from numerical bootstrap. As a generalization to this direction of research, it would be interesting to test AdS/CFT correspondence for solutions/dual theories which are not supersymmetric. It should be mentioned that AdS/CFT has been used to study condense matter system [101], and the whole idea relies on the assumption that AdS/CFT works even without supersymmetry. We hope to come back to this in the future.

5. CONCLUSION AND DISCUSSION

In this dissertation, we first classified all the critical points of both SO(8) gauged and dyonic ISO(7) gauged $\mathcal{N} = 8$ supergravity in four dimensions with residue symmetry bigger that $\mathrm{SO}(3)_D \times \mathrm{SO}(3)_L$ subgroup of the gauge group. The first $\mathcal{N} = 3$ solution of ten dimensional Romans deformed type IIA supergravity in ten dimensions was constructed by uplifting the corresponding $\mathcal{N} = 3$ critical point in four dimensions.

The CFT dual of this solution was proposed to be the three dimensional $\mathcal{N} = 3$ Chern-Simons matter theory [6, 8]. In order to test the correspondence, we studied the spin-2 Kaluza-Klein reduction of the solution, and found that it match with the spectrum of short BPS protected operators in the dual field theory. We also calculated the Euclidean gravitational action, and found that it match precisely with the free energy of the dual CFT at leading order in $1/N$. Based on the two non-trivial tests, we can conclude that the proposed AdS/CFT correspondence is indeed correct.

We also reviewed the non-perturbative method "Conformal Bootstrap", and applied it to CFT's with $\mathrm{F}_4/\mathrm{SU}(3)$ flavor symmetry in $6 - \epsilon$ dimension. The scaling dimension of the first few operators was first calculated using traditional Feynman diagrams method up to 1-loop. To test the Feynman diagrams calculation, numerical conformal bootstrap was preformed. In $D = 5.95$, the curve of bounds for operator dimensions shows a sudden change of slope at the valued calculated at 1-loop, therefore provide strong evidence for the correctness of our Feynman diagram result. In $D = 5$, as we chage our assumptions for the operator spectrum, the curve of bounds start to form a peninsula surrounding certain regions in the $(\Delta_\phi, \Delta_{\mathbf{1}}^{(1st)})$

plane, we therefore conjecture that the CFT's survive in five dimensions as one take $\epsilon \to 1$. The possibility of using this method to test AdS/CFT correspondence was also discussed, where we argued that the CFT dual of $AdS_5 \times T^{11}$ solution in type IIB supergravity is a good candidate to implement numerical bootstrap.

As a newly born technology for strongly coupled system, conformal bootstrap has been extremely successful in getting critical exponents for various models. For three dimensional Ising model, the critical exponents provided by conformal bootstrap is up to two orders of magnitude more precise than the result coming form lattice Monte Carlo simulation [29]. Also, we expect conformal bootstrap to be applicable to systems that are not accessible by lattice method. Famously, the Nielsen–Ninomiya no-go theorem [102] says that it is in general very difficult to define chiral fermions on a lattice in even dimensions. Such difficulty is however not present in conformal bootstrap. We hope that within years of development, conformal bootstrap would be able to answer some of the long-standing unsolved problems in conformal field theory, such as what's the lower bound of N_f for Banks-Zaks fixed point [103, 104] to appear in QCD with $N_c = 3$? Is there any conformal field theories in space-time dimension higher that six?

We also want to mention that our discussion of AdS/CFT correspondence has been focused on solutions of string/M theory and their dual superconformal field theories. Starting with [101], there has been various attempts in trying to apply gauge/gravity duality to the down-to-earth problems of condensed matter physics. It would be very interesting to see whether these ideas could give us some more insights on dealing with strong coupled systems.

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