GEODESIC EQUIVALENCE IN SUB-RIEMANNIAN GEOMETRY

An Undergraduate Research Scholars Thesis

by

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Submitted to Honors and Undergraduate Research
Texas A&M University
in partial fulfillment of the requirements for the designation as

UNDERGRADUATE RESEARCH SCHOLAR

Approved by
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May 2014

Major: Mathematics

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ABSTRACT

Geodesic Equivalence in Sub-Riemannian Geometry. (May 2014)

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Sub-Riemannian geometry is an intensively developing field of Mathematics lying at the intersection of Differential Geometry, Control Theory with application to Robotics, Hamiltonian dynamics and PDEs.

Our research is devoted to the geodesic equivalence of sub-Riemannian metrics, when one wants to study the metrics not up to isometries but up to the group of transformations preserving all their geodesics considered as unparametrized curves. In Riemannian geometry this equivalence problem is well understood thanks to the classical works of Beltrami, Dini, Levi-Civita. The existence of nontrivial pairs of geodesically equivalent metrics is related to the Liouville integrability of the corresponding geodesic flows with integrals of special type and the separability of the corresponding Hamilton-Jacobi equations.

For proper sub-Riemannian metrics only few classification results are known up to now, mainly concerning sub-Riemannian metrics on generic corank 1 distributions. However, there is strong evidence that a general classification theorem on geodesic equivalence of sub-Riemannian metrics defined on a very general class of distributions exists and it includes the classical Levi-Civita theorem as a particular case. The presented research is a step forward to discovering such a theorem.

DEDICATION

This thesis is dedicated to my wonderful and loving family in San Antonio, TX: John, Laura, Joshua, Tristan, and Evan Castillo. For without your love and support these past three years, I would not be where I am today in my undergraduate career and more importantly; where I am as a brother and son.

We are the Castillo Tribe

ACKNOWLEDGMENTS

There are many people I must thank as they all played a major role in my completing this thesis. In particular, I would like to thank five people:

Laura and John Castillo: Thank you for everything that you guys have done for me, I could not be more grateful. I have the impetus to strive to achieve everything that I have set out to accomplish due to you two. You showed me a different light in this world and have given me reason to keep on moving, even through the darkest of times. You both have always been by my side, and for that words couldn't explain how much I appreciate and love you both. It is because of you mom and dad that I have the goals I have, it is because of you that I have reason to keep fighting the fight, and it is because of you two that I know that in order to accomplish anything in life; all you need is *ganas*. Thank you mom and dad for always being there by me, your Frijolito will make you proud, I promise. I love you both very much.

Joshua, Tristan and Evan My three little big brothers, how I cannot explain how much I love each one of you. Although, you are probably to young to understand everything I say in this paragraph, know that you guys mean so much to me. I look up your pictures every time I study and realize that I couldn't have been blessed with a better family. Thank you guys for everything you all do for me, from helping me move in, coming to visit, moving groceries up four flights of stairs, or helping me release tons of stress by annihilating zombies on Black Ops I and II.

Joshua: Thank you for being the brother you are, you are growing up so fast and you took over the role I played while I was at the house, and I must say you are doing a much better job than I ever have. You are going to be a wonderful young man soon and I only hope that

I can help in anyway possible to get you there. You are my BLB and I wouldn't have it any other way.

Tristan Oh little Tristan, how I love to come home and here you always explain the next big game to me. You are very intelligent little brother, and you remind me a lot of myself when I was your age. Thank you BLLB for always watching the latest TV shows with me when I needed to relax and for always making me laugh. I love you Tristan Matthew, don't you ever forget it.

Evan: Little Evan, the youngest of them all and the only one I have a chance to beat in height. Thank you little brother everything that you have done; from throwing the football with us back home and always letting me know when my favorite TV show is on. I can't wait to see you grow up BLLLB, and see everything that you will accomplish.

Know this guys; I have set the bar high, but I want each of you to pass me in everything I do and want you to accomplish everything that you were *destined* to do. I love you guys very much and words are not enough to explain that.

I would also like to thank Texas A&M University and in particular the Honors and Undergraduate Research Program for giving me the opportunity to tackle such a task. Thank you to everybody that attended at my presentation during Student Research Week, even though most of you did not understand what I was talking about, it means so much to me to have your support. Furthermore, a special thank you Dr. Oksana Shatalov, the pictures for my presentation during Student Research Week are incredible and made the material more lucid. Lastly (but certainly not least) I would like to thank my mentor, supervisor, professor and colleague: Dr. Igor Zelenko. Without your aid sir, this thesis would be long from complete and I would certainly not know as much as I do now. From MATH 439 to MATH 622, I hope I have made great strides in your eyes towards becoming a mathematician.

CHAPTER I

INTRODUCTION

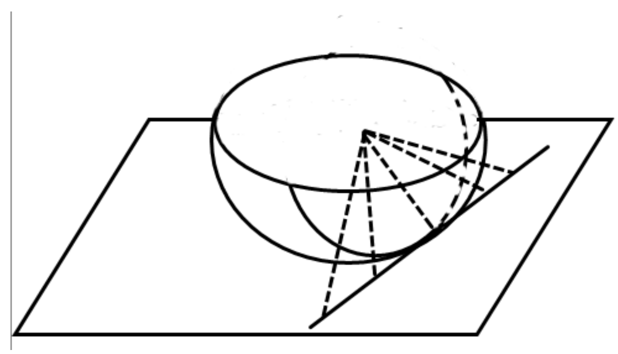
In the last two decades, Sub-Riemannian metrics have attracted substantial attention of researchers in Differential Geometry, Control Theory, Hamiltonian dynamics and PDE's.

In contrast to the Riemannian case, in the proper sub-Riemannian case the inner product is defined not at the whole tangent space at a point but on a distinguished subspace of the tangent space at a point. A field of such distinguished subspaces is called a distribution. Sub-Riemannian metrics appear naturally in Robotics as systems describing car-like robots (cars with trailers) and, more generally, nonholonomic robots [9], when the motion in the configuration space are allowed only in certain direction at any point (belonging to a subspace of a tangent space at this point), and also models of visual perception [16]. Another natural appearance of sub-Riemannian structures are isoperimetric problems [1, 6] and gauge fields in Physics, where they are naturally defined by principal connections on principal bundles over Riemannian manifolds. [15, 11].

Two Riemannian or sub-Riemannian metrics are called (locally) geodesically equivalent if there is a (local) diffeomorphism between the ambient manifolds sending geodesics of one metric, considered as unparametrized curves, to geodesics of another metric, considered as unparametrized curves. We are interested in a local version of this equivalence. For example, the local geodesic equivalence of a metric to the flat one will mean that there is a local change of coordinates such that all geodesics became straight lines after this change.Informally speaking, by studying geodesic equivalence one wants to understand the property of the whole "web" of geodesics forgetting about the metric itself.

The simplest way to produce a metric, which is geodesically equivalent to another given metric, is to multiply it by a constant. We then say that these metrics are geodesically equivalent in a trivial way and are called ,unsurprisingly, trivial. It was the goal of our research to study not these types of metrics but the nontrivial pairs, which are much less understood then their trivial counterparts. The simplest example of nontrivial pairs of locally geodesically equivalent Riemannian metrics are the flat metric and the standard Riemannian metric on a hemisphere via the stereographic projection from the center of this hemisphere.

Fig. I.1. Stereographic Projection



These are both classical problems that one can encounter while enjoying a standard Differential or Riemannian Geometry course. There, one can begin to understand the pertinence of these metrics in standard applications and begin relating it to our special case. Riemannian metrics and their equivalent geodesics have been well understood by the classical works of Beltrami, Dini, and finally of Levi-Civita.[3, 7, 10, 8]. Levi-Civita [10]In fact, it was Levi-Civita who gave an explicit description of all nontrivial pairs of locally geodesically equivalent metrics and discovered that if a given metric g admits a geodesically equivalent metric in a nontrivial way then the geodesic flow of g is Liouville integrable with integrals of some special type depending polynomially on the impulses. In reference to our current project, the existing results suggests that there is a general classification theorem on the existence of nontrivial pairs of geodesically equivalent sub-Riemannian metrics, which includes the classical Riemannian ones as a particular case. In retrospect, the current research in the subject suggest that there exists two "worlds" for these different types of metrics, i.e. one for the Riemannian Case, which was done by Levi-Civita [10], and one for the sub-Riemmanian case, which as just mentioned is the goal of the current paper. Also, recently V.Matveev and his collaborators [5, 13, 14, 12] shed a new light to this relation with integrability and used it for the study of global geodesic equivalence. Other recent related development is the study of separability of Hamilton-Jacobi equations [4].

The notion of geodesic equivalence of sub-Riemannian metrics can be defined completely in the same way as for the Riemannian metrics replacing Riemannian geodesics by sub-Riemannian ones, again recalling that the sub-Riemmanian metrics are defined on a subspace of the tangent space of the Riemannian manifold, and studying such equivalence is important in the same way for understanding the geometric properties of the "web" of sub-Riemannian geodesics. As of today, the only known classification result about the geodesic equivalence of proper sub-Riemannian metrics was obtained by I. Zelenko [17] for the first nontrivial case of sub-Riemannian metric on distributions of corank 1 (i.e., fields of hyperplanes),satisfying a certain generic assumption, namely, of being contact or quasi-contact for the ambient manifold of odd and even dimensions, respectively. These two cases are essentially different, for the case of contact distributions, all pairs of geodesically equivalent metrics are trivial, i.e. constant multiples of one another. However, in the case of quasi-contact distributions there are nontrivial pairs of geodesically equivalent metrics which are described in terms of Cauchy Characteristics. Which are the infinitesimal symmetries of the underlying distributions being also horizontal vector fields. On the other hand, when we consider contact distributions we can see that there are no Cauchy Characteristics.

In the thesis we have sought out to prove the conjecture of Zelenko: the existence of nontrivial pairs of geodesically equivalent sub-Riemannian metrics on general distributions depends on whether or not this distribution possesses Cauchy characteristics and such pairs can be explicitly described using some differential equations along this Cauchy characteristic. The first result of the thesis required the validation of the conjecture for uniformly non-contact sub-distributions of Cauchy Characteristics of rank equal to two and then the classification all pairs of geodesically equivalent distributions in this case, generalizing Zelenko's theorem on the quasi-contact case. The method of the proof is a careful examination of conditions for geodesic equivalence given in terms of the Hamiltonian formalism of the Pontrygin Maximum Principle. The second directions that was partially investigated is the case of rank two distributions on any dimension, as before it was only known for dimension three.

Another objective of the current research thesis is to investigate the relationship between non-trivial pairs of geodesically equivalent metrics and the Liouville integrability of their geodesic flows, by analogy with the Riemannian case. This was done primarily in the Riemannian case by the works Topalov and Mateev [13], where they acquired information regarding the classification of the Liouville Integrability for non-trivial geodesic equivalent metrics. They did this through the formulation of the Levi-Civita integrals acquired through the coordinate version of the Levi-Civita theorem for Riemanninan metrics and the functional independence of certain integrals for the geodesic flow of the Riemannian metrics. It was another goal of this research project to understand these situations in the sub-Riamannian case as it is not yet understood even in the simplest quasi-contact case.

CHAPTER II

METHODS

The main tool in our project is the Pontryagin Maximal Principal of Optimal Control which provides a convenient uniform framework to describe the Hamiltonian flows of sub-Riemannian extremals. The conditions for geodesic equivalence in this language reduce to the orbital equivalence problem of these Hamiltonian flows, which can be rewritten in term of some overdetermined system of PDEs. Solvability of this system implies certain very restrictive algebraic conditions on a pair of metric under consideration (for example, divisibility of certain polynomials associated with a pair of sub-Riemannian metrics), which often significantly restrict the search for a pair of non-trivial geodesically equivalent pairs. This tool proved to be quite efficient. For example, as was shown in [17] it gives an elementary new proof of the classical Levi-Civita theorem.

As preliminary results of the research we examined the case of corank one distributions D with rank 2 subdistributions C of Cauchy characteristic vector fields. We showed that the subdistribution $C^{\perp}$ of D, which is the orthogonal complement to C (with respect to both sub-Riemannian metrics) is not bracket generating, more precisely $(C^{\perp})^2 = [C^{\perp}, C^{\perp}]$ is integrable and has a rank one more than the distribution $C^{\perp}$. Further, we were able to describe all nontrivial pairs of sub-Riemannian metrics in terms of the foliation of integral submanifolds of $(C^{\perp})^2$ as two distinguished vector fields spanning C.

Our method is a generalization of the methods of [17]. First, we compare the coefficients of polynomials in the algebraic part of the mentioned overdetermined system of equations describing our equivalence problem and use the classical Frobenius Theorem on involutive distributions. Second, we analyzed the differential equations along distinguished characteristic vector fields in order to understand the evolution along the flows of these vector fields of the restriction of the metrics to the above mentioned foliation and to these vector fields themselves.This methodology confirmed Zelenko's conjecture in this particular case.

CHAPTER III

RESULTS

We begin by stating and proving a proposition that generalizes a result seen in [17]. In particular, we will show that in the case of a corank one distributions with the subdistribution C of Cauchy characteristic having rank 2 subdistributions , the orthogonal complement of this subdistribution is not bracket generating. Before we do this, lets state a few definitions that will be important for the remainder of the paper.

Definition III.0.1 (Transition Operator) *For a given ordered pair of sub-Riemannian metrics G_1, G_2 and a point p lets can define the following linear operator $T_p : D(p) \mapsto D(p)$: $G_{2p}(v_1, v_2) = G_{1p}(T_p v_1, v_2), \forall v_1, v_2 \in D(p)$. T_p is called the transition operator from the sub-Riemannian metric G_1 to the sub-Riemannian metric G_2 at the point p.*

Definition III.0.2 (Frame for the pair (G_1, G_2)) *Let G_1 and G_2 be sub-Riemannian metrics on an n-dimensional manifold M. Let p_0 be a regular point w.r.t these metrics. We will say that $(X_0, X_1, \ldots, X_{m-1}, X_m)$ is a frame adapted to the ordered pair (G_1, G_2) in some neighborhood of p_0 and the transition operator T_p from the metric G_1 to G_2 has a diagonal matrix representation. In other words, the local frame $(X_1, X_2, \ldots, X_{m-1}, X_m)$ consists of the eigenvectors of the transition operator.*

Proposition III.0.1 *The fiber $C(p)$ of the Cauchy characteristic distribution C is an invariant subspace of the transition operator T_p for any p in a neighborhood of p_0.*

The proof of this proposition follows the lines of in [17, Proposition 9].

Assume that $(X_0, X_1, \ldots, X_{m-1}, X_m)$ is the frame adapted to the pair of sub-Riemannian metrics (G_1, G_2). By the previous Proposition without loss of generality we can assume that

$C = \text{span}(X_{m-1}, X_m)$. Let $C^{\perp}$ be the orthogonal complement of C with respect to the given metrics spanned by the vector fields in the frame. Then

$$(C^{\perp}) = \text{span}\,(X_0, X_1, \ldots, X_{m-3}, X_{m-2}).$$

Proposition III.0.2 *The distribution* $(C^{\perp})^2 = [C^{\perp}, C^{\perp}]$ *is integrable. Moreover, the eigenvalues of the transition operator are constant on each integral submanifold of the distribution* $(C^{\perp})^2$ *. Equivalently, the flows of vector fields* X_{m-1} *and* X_m *preserve the foliation of these integral submanifolds.*

Proof Using identities (2.60) and (4.1) of [17] we can compare coefficients of $u_i u_m$ for $1 \leq i \leq m-2$ to both sides of identity (4.6) of [17] to yield the following equations:

$$(\alpha^2 - \alpha_m^2) c_{ji}^m = r_m c_{ji}^{m+1} + r_i c_{jm}^{m+1}$$

$$(\alpha^2 - \alpha_{m-1}^2) c_{ji}^{m-1} = r_{m-1} c_{ji}^{m+1} + r_i c_{j(m-1)}^{m+1}$$

$\forall 1 \leq i \neq j \leq m-2$.

By hypothesis X_m and X_{m-1} are Cauchy Characteristic, so that $c_{jm}^{m+1} = 0$ and $c_{j(m-1)}^{m+1} = 0$, resulting in the following identities:

$$c_{ji}^m = \frac{r_m c_{ji}^{m+1}}{(\alpha^2 - \alpha_m^2)} \qquad \text{(III.1)}$$

$$c_{ji}^{m-1} = \frac{r_{m-1} c_{ji}^{m+1}}{(\alpha^2 - \alpha_{m-1}^2)}. \qquad \text{(III.2)}$$

This then results in the following observation: $\forall 1 \leq i \leq m-2$

$$[X_i, X_j] \in \mathrm{span}(\{X_k\}_{k=1}^{m-2}, \frac{r_{m-1}}{(\alpha^2 - \alpha_{m-1}^2)} X_{m-1}, \frac{r_m}{(\alpha^2 - \alpha_m^2)} X_m, X_{m+1}).$$

To prove the proposition it is enough to show the involutivity of the vector fields i.e $\forall 1 \leq i, j \leq m-2$

$$[X_i, \frac{r_{m-1}}{(\alpha^2 - \alpha_{m-1}^2)} X_{m-1}, \frac{r_m}{(\alpha^2 - \alpha_m^2)} X_m, X_{m+1}] \in \mathrm{span}(\mathrm{D}, \frac{\mathrm{r_{m-1}}}{(\alpha^2 - \alpha_{\mathrm{m-1}}^2)} \mathrm{X_{m-1}}, \frac{\mathrm{r_m}}{(\alpha^2 - \alpha_{\mathrm{m}}^2)} \mathrm{X_m}, \mathrm{X_{m+1}}). \tag{III.3}$$

By basic calculations one can show that the previous equation is in fact equivalent to the following three conditions:

$$\frac{X_i(r_{m-1})}{(\alpha^2 - \alpha_{m-1}^2)} + \frac{r_{m-1} c_{(m-1)i}^{m-1}}{(\alpha^2 - \alpha_{m-1}^2)} + \frac{r_m c_{mi}^{m-1}}{(\alpha^2 - \alpha_m^2)} + c_{(m+1)i}^{m-1} - r_{m-1} c_{(m-1)i}^{m-1} \tag{III.4}$$

$$\frac{(r_{m-1}) c_{(m-1)i}^{m}}{(\alpha^2 - \alpha_{m-1}^2)} + \frac{X_i(r_{m-1})}{(\alpha^2 - \alpha_m^2)} + \frac{r_m c_{mi}^{m}}{(\alpha^2 - \alpha_m^2)} + c_{(m+1)i}^{m} - r_m c_{mi}^{m} \tag{III.5}$$

$$c_{(m+1)i}^{m+1} = \gamma(t) \tag{III.6}$$

where gamma is a function of one variable.

The duration of the proof will be devoted to showing that the previous identities (III.4 & III.5) indeed do hold.

Using identities (4.11) from [17] we can rewrite the identities for X_m and X_{m-1} in the following way:

$$\vec{h}(\sum_{i=m-1}^{m+1} r_i u_i) - \frac{1}{2}(\sum_{j=m-1}^{m+1} \frac{X_j(\alpha_j^2)u_j}{\alpha_j^2}) \sum_{i=m-1}^{m+1} r_i u_i - Q_{m+1,m+1} \sum_{i=m-1}^{m+1} r_i u_i = \sum_{k=1}^{m} Q_{m+1,k}\alpha_u u_k. \tag{III.7}$$

Note that we made use of the fact that $r_i = 0 \quad \forall 1 \leq i \leq m-2$.

Expanding the previous equation results and comparing the coefficients of identity (2.19) of [17] results in:

$$\vec{h}(\sum_{i=m-1}^{m+1} r_i u_i) - \{\frac{1}{2}(\frac{X_{m-1}(\alpha_{m-1}^2)u_{m-1}}{\alpha_{m-1}^2} + \frac{X_m(\alpha_m^2)u_m}{\alpha_m^2}) \sum_{i=m-1}^{m+1} r_i u_i\} - (r_{m-1}c_{(m-1)i}^{-(m-1)}\alpha_{m-1} + r_m c_{mi}^{-m}\alpha_m)$$

$$= \sum_{k=1}^{m} Q_{m+1,k}\alpha_u u_k.$$

We can then define

$$\beta_m := -\frac{1}{2}\frac{X_m(\alpha_m^2)}{\alpha_m^2} \quad \beta_{m-1} := -\frac{1}{2}\frac{X_{m-1}(\alpha_{m-1}^2)}{\alpha_{m-1}^2}, \tag{III.8}$$

so that the previous expression can be rewritten as:

$$\vec{h}(\sum_{i=m-1}^{m+1} r_i u_i) - \{\beta_{m-1}u_{m-1} + \beta_m u_m \sum_{i=m-1}^{m+1} r_i u_i\} - (r_{m-1}c_{(m-1)i}^{-(m-1)}\alpha_{m-1} + r_m c_{mi}^{-m}\alpha_m) = \sum_{k=1}^{m} Q_{m+1,k}\alpha_u u_k. \tag{III.9}$$

We can now expand the term (III.9) involving the $\beta_i u_i$ and $r_j u_j$ and then compare with the coefficients from (2.24) of [17] which will result in:

$$r_m\{c_{mi}^m + c_{mi}^{m-1}\} + r_{m-1}\{c_{(m-1)i}^m + c_{(m-1)i}^{m-1}\} + r_{m-1}\{c_{(m+1)(m+1)^i} + c_{(m+1)m}^i\}. \tag{III.10}$$

We now now take care of the $\vec{h}(\sum_{i=m-1}^{m+1} r_i u_i)$ term by comparing these coefficients to that of (2.24) of [17] resulting in:

$$\{X_i(r_m) + X_i(r_{m-1})\}. \tag{III.11}$$

Finally, after comparing coefficients of our $\sum_{k=1}^{m} Q_{m+1,k}\alpha_u u_k$ term, we get:

$$(c_{(m+1)(m-1)}^{-i} + c_{(m+1)i}^{-(m-1)})\alpha_{m-1}\alpha + (c_{(m+1)m}^{-i} + c_{(m+1)i}^{-m})\alpha_m\alpha. \tag{III.12}$$

From the calculations getting from (III.8) to (III.9) and using the fact that $c_{(m-1)i}^{-(m-1)} = \frac{c_{(m-1)i}^{m-1}}{\alpha_{m-1}}$ we have that

$$-(r_{m-1}c_{(m-1)i}^{-(m-1)}\alpha_{m-1} + r_m c_{mi}^{-m}\alpha_m) = -r_{m-1}c_{(m-1)i}^{(m-1)} - r_m c_{mi}^m \tag{III.13}$$

After combining (III.10), (III.11), (III.13) and gathering all coefficients results in:

$$X_i(r_{m-1}) + r_m c_{mi}^{m-1} + r_{m-1}c_{mi}^{m-1} + r_{m+1}c_{(m+1)(m-1)}^i - r_{m-1}c_{(m-1)i}^{m-1} \tag{III.14}$$

$$X_i(r_m) + r_m c_{mi}^m + r_{m-1}c_{(m-1)i}^m + r_{m+1}c_{(m+1)(m)}^i - r_m c_{mi}^m. \tag{III.15}$$

Moreover, from (III.12) we have the following identities:

$$c_{(m+1)(m-1)}^{-i} = \frac{\alpha}{\alpha_{m-1}} c_{(m+1)(m-1)}^i, \quad c_{(m+1)i}^{-(m-1)} = \frac{\alpha_{m-1}}{\alpha} c_{(m+1)i}^{m-1} \tag{III.16}$$

$$c^{-i}_{(m+1)m} = \frac{\alpha}{\alpha_m} c^i_{(m+1)(m)}, \quad c^{-m}_{(m+1)i} = \frac{\alpha_m}{\alpha} c^m_{(m+1)i}, \quad r_{m+1} = \alpha^2. \tag{III.17}$$

Using (III.16) and (III.17), (III.12) becomes:

$$\frac{\alpha}{\alpha_{m-1}} c^i_{(m+1)(m-1)} + c^{m-1}_{(m+1)i} \frac{\alpha_{m-1}}{\alpha} \alpha_{m-1}\alpha, \quad (\text{“}m-1\text{”} terms) \tag{III.18}$$

$$\frac{\alpha}{\alpha_m} c^i_{(m+1)(m)} + c^m_{(m+1)i} \frac{\alpha_m}{\alpha} \alpha_m\alpha, \quad (\text{“}m\text{”} terms). \tag{III.19}$$

Lastly, "placing" (III.18) and (III.19) on the right hand sides of (III.14) and (III.15) respectively, we get:

$$X_i(r_{m-1}) + r_{m-1}c^{m-1}_{(m-1)i} + r_m c^{m-1}_{mi} + c^i_{(m-1)i}(\alpha^2 - \alpha^2_{m-1}) - r_{m-1}c^{m-1}_{(m-1)i} \tag{III.20}$$

and

$$X_i(r_m) + r_m c^m_{(m-1)i} + r_m c^m_{mi} + c^m_{(m-1)i}(\alpha^2 - \alpha^2_m) - r_m c^m_{mi} \tag{III.21}$$

which is indeed equivalent to (III.4) and (III.5). Thus it follows that (III.3) holds so that it now follows from Frobenius Theorem that $(C^\perp)^2 = [C^\perp, C^\perp]$ is integrable. ∎

Let $k_0 = dim(M) - 2$. Also let $k_1 = k_2 = 1$, which are the multiplicities of the eigenvalues of S_p different from the eigenvalue corresponding to $C^\perp$.

Theorem III.0.1 *Let (M, D, g_1) and (M, D, g_2) be two sub-Riemannian structures on a corank 1 distribution with $dim(M)$ being odd and $rank(C) = 2$. Then the sub-Riemannian structures are geodesically equivalent if and only if there exists a local coordinate system*

$\bar{x} = (\bar{x}_0, \ldots, \bar{x}_m)$, *where* $\bar{x}_i = (x_i^1, \ldots, x_i^{k_i})$ *such that the quadratic forms of the inner products* g_1 *and* g_2 *have the form*

$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=0}^{2} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

$$g_2(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=0}^{2} \lambda_s(\bar{x}) \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s)$$

where the velocities $\dot{\bar{x}}$ *belong to* D,

$$\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=0}^{2} \beta_l(\bar{x}_l),$$

$$\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|,$$

$\{\beta_s(p_0) \neq \beta_l(p_0)$ *for all* $s \neq l$ *and* β_0 *is constant.*

The proof of this theorem follows the general lines of the Levi-Civita Theorem as seen in [17]. The details and generalization of this theorem will be left for a future paper.

CHAPTER IV

CONCLUSION

The results of this project were interesting, indeed we were not able to generalize our main theorem to general ranks of Cauchy Characteristic subdistributions, but we were able to generalize a result found ten years ago to the case of rank 2 subdistributions. In other words we can say that our work agrees nicely with the work found in [17], and that the ideas were very similar. In addition, our main result does indeed confirm our conjecture that a generalized theorem for geodesic equivalence on general subdistributions does exist, but due to time constraints and other unprecedented issues, were not able to pursue this formulation in full generality. In the near future we will continue working on this problem and it is our hope that we can find this theorem in that the classical Levi CIvita theorem seen in [17] is a particular case. Recall that the Levi-Civita theorem described all pairs of geodesically equivalent *Riemannian* metrics. Moreover we will like for our theorem to hold for contact distributions i.e the case when: $\text{rank}D = \dim\ M - 1$, $\dim M$ is odd, and $\text{rank}C = 0$. Lastly, the theorem will also hold for the case of quasi-contact (even-contact) distributions, i.e. when $\text{rank}D = \dim\ M - 1$, $\dim M$ is even , and $\text{rank}C = 1$ is again a particular case.

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