BRACKETS ON HOCHSCHILD COHOMOLOGY OF NONCOMMUTATIVE ALGEBRAS

A Dissertation

by

LAUREN ELIZABETH GRIMLEY

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Sarah Witherspoon
Committee Members,	Eric Rowell
	Paulo Lima-Filho
	Andreas Klappenecker
Head of Department,	Emil Straube

May 2016

Major Subject: Mathematics

ABSTRACT

The Hochschild cohomology of an associative algebra is a Gerstenhaber algebra, having a graded ring structure given by the cup product and a compatible graded Lie algebra structure given by the Gerstenhaber bracket. The cup product can be defined generally from multiple perspectives and has been studied for many classes of algebras. The Gerstenhaber bracket, however, has not admitted such a general definition, making computations difficult.

In this dissertation, we characterize the Gerstenhaber algebra structure on the Hochschild cohomology of group extensions of quantum complete intersections. We utilize the notion of twisted tensor products, a noncommutative tensor product, and adapt a technique of Wambst's to compute the graded ring structure on Hochschild cohomology. The bracket structure is computed by employing an alternative description given in recent work of Negron and Witherspoon. When the group is trivial, this work extends the previous computations of the graded ring structure of Hochschild cohomology of quantum complete intersections to include the bracket structure. As an example, we compute the Gerstenhaber algebra structure for two generator quantum complete intersections extended by selected groups.

ACKNOWLEDGEMENTS

I would first like to thank my husband, Richard Veras, for his unconditional love, constant support, and encouragement throughout the graduate process. I would like to thank my parents, Gail and Terry Grimley, for teaching me to have a strong work ethic and to never get discouraged. I would also like to thank my brother, Thomas Grimley, for leading the way in life.

This document would not have existed without the support of my advisor, Dr. Sarah Witherspoon. I would like to thank Sarah for her availability and willingness to help during all stages of this process. I am also thankful for the helpful corrections, comments, and suggestions for future directions of this work given by my other committee members, Dr. Eric Rowell, Dr. Paulo Lima-Filho, and Dr. Andreas Klappenecker.

Finally, I would like to acknowledge the support of the Louis Stokes Alliance for Minority Participation Bridge to the Doctorate program.

TABLE OF CONTENTS

1. INTRODUCTION

Hochschild cohomology is used in many diverse areas of mathematics including representation theory, deformation theory, and the study of the structure of algebras. In low homological degrees, Hochschild cohomology of an algebra reveals some of the structure of the algebra such as the center and possible derivations on the algebra. A large gap in cohomology implies smoothness of finitely generated commutative algebras as shown by Avramov and Iyengar in [2]. Beyond the vector space structure, Hochschild cohomology has graded ring structure given by the cup product and compatible graded Lie algebra structure given by the Gerstenhaber bracket, making it a Gerstenhaber algebra. The cup product on cohomology has been widely studied for many algebras but the Lie bracket on Hochschild cohomology is less understood as it can be difficult to compute.

Both products are typically computed on projective resolutions. Motivated by deformations, Gerstenhaber originally defined the Lie bracket structure of Hochschild cohomology in [10] on the bar resolution. While the bar resolution is a projective resolution for any algebra over a field, some algebras have projective resolutions that are more tuned to the structure of the specific algebra. For example, the Koszul complex is a finite length resolution for the polynomial ring whose differentials involve multiplying by the generators while the bar resolution is an infinite length resolution whose differentials involve summing over increasingly long tensor products. In order to compute the brackets using a different resolution, in general, one needs to construct chain maps between the two resolutions, pass elements through a chain map to the bar resolution, compute the bracket, and then pass back through a chain map to the desired resolution. While such chain maps always exist, they can be difficult to explicitly state. For example, see [26] or [27]. Fortunately, Negron and Witherspoon gave a more computationally minded description of the Lie brackets in [23] on any resolution which is compatible with the bar resolution. Unlike more traditional techniques, this method does not require explicit chain maps to and from the bar resolution. In this dissertation, we utilize this bracket description to study the Lie structure of group extensions of quantum complete intersections.

The behavior of Hochschild cohomology of quantum complete intersections has been shown to vary greatly based on the choice of quantum coefficients, exhibiting behaviors unseen with commutative algebras. Quantum complete intersections of interest here are noncommutative generalizations of truncated polynomial rings. Formally, if k is a field, $m_1, m_2, ..., m_n$ are positive integers, and $\mathbf{q} = \{q_{i,j}\}_{i,j\in\{1,2,...,n\}}$ such that $q_{i,j} \in k^*$ for all $i, j \in \{1, 2, ..., n\}$, $q_{j,i} = q_{i,j}^{-1}$, and $q_{i,i} = -1$, then define

$$\Lambda_{\mathbf{q}}^{(m_1,m_2,...,m_n)} = k\langle x_1, x_2, ..., x_n |\ x_i x_j = -q_{i,j} x_j x_i,\ x_i^{m_i} = 0 \text{ for all } i, j \in \{1, 2, ...n\}\rangle,$$

the quantum complete intersection on $\mathbf{q}$ and $(m_1, ...m_n)$. That is, $\Lambda_{\mathbf{q}}^{(m_1,m_2,...,m_n)}$ is a noncommutative truncated polynomial ring generated by $x_1, x_2, .., x_n$ where x_i is truncated at $x_i^{m_i}$ and multiplication on the set of generators is commutative up to multiplication by non-zero scalars. This noncommutative multiplication can be replicated using iterated twisted tensor products given by Bergh and Oppermann in [6] as will be shown in Example 3.1.1.

The quantum complete intersection $\Lambda_{\mathbf{q}}^{(2,2,...,2)}$ was introduced by Manin in [19, Example 3.2] as an example of a quantum Grassmannian (exterior) algebra. Avramov, Gasharov, and Peeva first formally defined quantum complete intersections in [1] by introducing the notion of quantum regular sequences, generalizing complete intersections and regular sequences. A discussion of other possible classes of noncommutative complete intersections can be found in [15].

Buchweitz, Green, Madsen and Solberg showed that the two generator quantum complete intersections, $\Lambda_{\mathbf{q}}^{(2,2)}$, whose quantum coefficient is not a root of unity have infinite global dimension but finite Hochschild cohomology in [7], the first example of this type. When the quantum coefficient is a root of unity, $\Lambda_{\mathbf{q}}^{(2,2)}$ has infinite Hochschild cohomology with large gaps determined by the root of unity. Bergh and Erdmann generalized this result in [5], showing that the general class of two generator quantum complete intersections, $\Lambda_{\mathbf{q}}^{(m_1,m_2)}$, have finite Hochschild cohomology if and only if the quantum coefficient is not a root of unity. In [25], Oppermann furthered this study by showing that the Hochschild cohomology of finite quantum complete intersections, $\Lambda_{\mathbf{q}}^{(m_1,m_2,\ldots,m_n)}$, is intimately tied to the choice of quantum coefficients.

In this dissertation, we allow a finite group to act diagonally on the quantum complete intersections and study the effect on Hochschild cohomology. In particular, if G is a finite group acting on $\Lambda_{\mathbf{q}}^{(m_1,m_2,\ldots,m_n)}$, we denote by $\Lambda_{\mathbf{q}}^{(m_1,m_2,\ldots,m_n)} \rtimes G$ the group extension of $\Lambda_{\mathbf{q}}^{(m_1,m_2,\ldots,m_n)}$ by G. Work on the structure of Hochschild cohomology of $\Lambda_{\mathbf{q}}^{(m_1,m_2,\ldots,m_n)} \rtimes G$ is related to efforts to understand Hochschild cohomology of $S_{\mathbf{q}}(V) \rtimes G$. The quantum symmetric algebra, $S_{\mathbf{q}}(V)$, of the vector space V is

$$S_{\mathbf{q}}(V) = T(V)/(v_iv_j - q_{i,j}v_jv_i | i,j \in \{1,2,\ldots,n\}) \text{ where } T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$$

and $\{v_1, v_2, \ldots, v_n\}$ is a k-basis for V. The structure of Hochschild cohomology of an algebra reveals information about possible deformations the algebra. In the case of $S_{\mathbf{q}}(V) \rtimes G$, deformations of this algebra include many algebras of interest, such as quantum Drinfeld Hecke algebras, rational Cherednik algebras [9], and braided Cherednik algebras [3]. The explicit relation between Hochschild cohomology of $S_{\mathbf{q}}(V) \rtimes G$ and quantum Drinfeld Hecke algebras is given by Naidu and Witherspoon in [21]. Deformations of $\Lambda_{\mathbf{q}}^{(m_1,m_2,\ldots,m_n)} \rtimes G$ are related to quotients of these interesting algebras.

The document is organized as follows. In Section 2, we give the necessary homological framework and define the Gerstenhaber algebra structure on Hochschild cohomology. As in [5] and [25], the work in this dissertation utilizes the notion of twisted tensor products for computations on quantum complete intersections. We define twisted tensor products in Section 3 and give the results of Nguyen, Witherspoon and myself regarding bracket constructions using twisted tensor products contained in [11]. Section 4 contains a description of the Gerstenhaber algebra structure on Hochschild cohomology of group extensions of quantum complete intersections. When the group is trivial, this work extends the computations of [7] and [25] (when $m_i = 2$ for all i) to include the bracket structure. To demonstrate this result, we compute the structure for several two generator cases, including cases with a specified group action and cases with a specified quantum coefficient, in Section 5. Both cases contain new computations which give a sense of the behaviors that can be found in Hochschild cohomology of noncommutative algebras.

2. HOMOLOGICAL ALGEBRA

We start by providing the homological techniques which we will need in Section 4, the main section of this document. In Section 2.1, we give a brief introduction to homological terminology and use this to define the Hochschild cohomology vector space structure. We give a definition of the Gerstenhaber algebra structure on Hochschild cohomology in Section 2.2 on the bar resolution. In Section 2.3, we discuss alternative descriptions of the cup product and bracket on more general resolutions. We will use the descriptions from Section 2.3 in Section 4.

Unless otherwise noted, $\otimes = \otimes_k$ and all modules are left modules for the remainder of the text.

2.1 Hochschild cohomology

We start by defining the homological terminology which will be used in later sections. A more general discussion of homological algebra can be found in [13], [18], and [30].

Definition 2.1.1. *Let R be a ring. A chain complex $\mathbb{D}$ over R is a sequence of R-module homomorphisms*

$$\mathbb{D} : \ldots \xrightarrow{\delta_{i+2}} D_{i+1} \xrightarrow{\delta_{i+1}} D_i \xrightarrow{\delta_i} D_{i-1} \xrightarrow{\delta_{i-1}} \ldots$$

such that $\delta_i\delta_{i+1} = 0$ for all $i \in \mathbb{Z}$. The R-module homomorphisms δ are called differentials or boundary maps.

If $\mathbb{D}$ is a chain complex over R with boundary maps δ, then, because $\delta_i\delta_{i+1} = 0$, $\operatorname{Im}\delta_{i+1} \subset \operatorname{Ker}\delta_i$ for all $i \in \mathbb{Z}$. Therefore the quotient $\operatorname{Ker}\delta_i / \operatorname{Im}\delta_{i+1}$ is well-defined. The R-module $\operatorname{Ker}\delta_i / \operatorname{Im}\delta_{i+1}$ is called the ith *homology module* and is denoted $\mathrm{H}_i(\mathbb{D})$.

A sequence of R-module homomorphisms

$$\mathbb{D} : \ldots \xrightarrow{\delta_{i+2}} D_{i+1} \xrightarrow{\delta_{i+1}} D_i \xrightarrow{\delta_i} D_{i-1} \xrightarrow{\delta_{i-1}} \ldots$$

is *exact* if $\operatorname{Im} \delta_{i+1} = \operatorname{Ker} \delta_i$ for all $i \in \mathbb{Z}$. An exact sequence is, by definition, a complex with $\mathrm{H}_i(\mathbb{D}) = 0$ for all $i \in \mathbb{Z}$.

If $\mathbb{D}$ and $\mathbb{E}$ are chain complexes over R, a *chain map* $\phi : \mathbb{D} \to \mathbb{E}$ is a family of R-module homomorphisms $\{\phi_i : D_i \to E_i | i \in \mathbb{Z}\}$ such that $\phi_{i-1}\delta_i = \tilde{\delta}_i \phi_i$ where δ and $\tilde{\delta}$ are the families of boundary maps on $\mathbb{D}$ and $\mathbb{E}$ respectively. We can combine the complexes $\mathbb{D}$ and $\mathbb{E}$ to create a new complex called the total complex of the tensor product, a useful construction which we will use throughout the rest of the text.

Definition 2.1.2. *Let $\mathbb{D}$ and $\mathbb{E}$ be chain complexes over R with R-module homomorphisms δ and $\tilde{\delta}$ respectively. The total complex* $\mathrm{Tot}(\mathbb{D} \otimes \mathbb{E})$ *is*

$$\mathrm{Tot}(\mathbb{D} \otimes \mathbb{E}) : \ldots \xrightarrow{\delta'_{i+2}} \bigoplus_{p+q=i+1} D_p \otimes E_q \xrightarrow{\delta'_{i+1}} \bigoplus_{p+q=i} D_p \otimes E_q \xrightarrow{\delta'_i} \bigoplus_{p+q=i-1} D_p \otimes E_q \xrightarrow{\delta'_{i-1}} \ldots$$

where $\delta'_i(d \otimes e) = \delta_p(d) \otimes e + (-1)^p d \otimes \tilde{\delta}_q(e)$ for $d \in D_p$ and $e \in E_q$.

Throughout the text, $\mathbb{D} \otimes \mathbb{E}$ should always be interpreted as the total complex, $\mathrm{Tot}(\mathbb{D} \otimes \mathbb{E})$, with boundary map as described in Definition 2.1.2 unless otherwise stated.

Remark 2.1.3. *Terms with a co- prefix are defined under the given definition with all maps reversed. For example, a cochain complex $\mathbb{D}$ over R is a sequence of R-module homomorphisms*

$$\mathbb{D} : \ldots \xrightarrow{\delta_{i-1}} D_{i-1} \xrightarrow{\delta_i} D_i \xrightarrow{\delta_{i+1}} D_{i+1} \xrightarrow{\delta_{i+2}} \ldots$$

such that $\delta_{i+1}\delta_i = 0$ for all $i \in \mathbb{Z}$.

We finish the basic homological terminology with a key definition for Hochschild cohomology.

Definition 2.1.4. *Let R be a ring. A projective resolution of an R-module X is an exact sequence of R-module homomorphisms*

$$\mathbb{P} : \ldots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} X \to 0$$

where for each $i \geq 0$, P_i is a projective R-module.

That is, a projective resolution of X as an R-module, $\mathbb{P}$, is a chain complex such that P_i is projective for all $i \geq 0$, $P_{-1} = X$, $P_i = 0$ for all $i < -1$, and $\mathbb{P}$ is exact.

Remark 2.1.5. *Projective resolutions exist for any R-module as can be shown by a constructive proof. See for example [13, Chapter IV Section 4].*

Let us give an example of a projective resolution for an algebra which we will refer back to frequently.

Example 2.1.6. *Let k be a field and $R_x = k[x]/(x^2)$, the truncated polynomial ring. Let R_x^{op} be the* opposite ring *and $R_x^e = R_x \otimes R_x^{op}$. Then R_x is an R_x^e-module with action given by left and right multiplication, $(r' \otimes r'')r = r'rr''$ for all $r \in R_x$, $r' \otimes r'' \in R_x^e$. Consider*

$$\mathbb{K}_x : \ldots \xrightarrow{\delta_3} R_x^e \xrightarrow{\delta_2} R_x^e \xrightarrow{\delta_1} R_x^e \xrightarrow{\delta_0} R_x \to 0$$

with $\delta_i(a \otimes b) = (x \otimes 1 + (-1)^i 1 \otimes x)(a \otimes b)$ for all $i > 0$ and $\delta_0(a \otimes b) = ab$ for $a, b \in R_x$. Notice δ_i is an R_x^e-module homomorphism for all $i > 0$ and R_x^e is a free (and thus projective) R_x^e-module. It remains to check that $\mathbb{K}_x$ is exact.

For i odd, $\operatorname{Im}\delta_i = span_{R_x^e}\{(x\otimes 1 - 1\otimes x)\}$ *and* $\operatorname{Ker}\delta_i = span_{R_x^e}\{(x\otimes 1 + 1\otimes x)\}$. *For $i>0$ even,* $\operatorname{Im}\delta_i = span_{R_x^e}\{(x\otimes 1 + 1\otimes x)\}$ *and* $\operatorname{Ker}\delta_i = span_{R_x^e}\{(x\otimes 1 - 1\otimes x)\}$ *and* $\operatorname{Im}\delta_0 = R$. *All of this information together implies that $\mathbb{P}_x$ is exact and thus a projective resolution for R_x as an R_x^e-module.*

If R is a graded ring, then a *graded R-module* X is a module that can be written as a direct sum

$$X = \bigoplus_{m\in\mathbb{N}} X_m$$

such that $R_iX_j \subset X_{i+j}$. Let X and Y be graded R-modules, a homomorphism $f : X \to Y$ is *graded of degree m* if $f(X_i) \subset Y_{i+m}$. A *graded projective resolution* of an R-module X is a projective resolution such that P_i is a graded R-module and the degree of δ_i is 0 for all $i \in \mathbb{Z}$.

Continuing our example, we can assign R_x a natural $\mathbb{Z}$-grading by polynomial degree. In this case, $\mathbb{K}_x$ is not a graded projective resolution because each δ_i has degree 1 for $i > 0$. If we shift the degrees of the modules, however, we can change the degrees of δ_i for $i > 0$. Define $(R_x^e\langle a\rangle)_b = (R_x^e)_{b-a}$. That is, in $R_x^e\langle a\rangle$, the degrees of all homogeneous elements are increased by a from their degree in R_x^e. Then

$$\mathbb{K}'_x : \ldots \xrightarrow{\delta_3} R_x^e\langle 2\rangle \xrightarrow{\delta_2} R_x^e\langle 1\rangle \xrightarrow{\delta_1} R_x^e \xrightarrow{\delta_0} R_x \to 0$$

is a graded projective resolution for R_x as an R_x^e-module, where $\delta_i(a\otimes b) = (x\otimes 1 + (-1)^i 1\otimes x)(a\otimes b)$ for all $i > 0$ and $\delta_0(a\otimes b) = ab$ for $a, b \in R_x$ as before.

To get a projective resolution for $R_x \otimes R_y \cong k[x,y]/(x^2,y^2)$, consider the total complex, $\operatorname{Tot}(\mathbb{K}_x \otimes \mathbb{K}_y)$. Denote $k[x,y]/(x^2,y^2)$ by $\Lambda^{(2,2)}$ to compare with our later constructions. Because the tensor product and direct sum of projective modules is a projective module, each of the $(\Lambda^{(2,2)})^e$-modules in $\operatorname{Tot}(\mathbb{K}_x \otimes \mathbb{K}_y)$ is projective.

Moreover, as a result of the Künneth Theorem, $\mathrm{Tot}(\mathbb{K}_x \otimes \mathbb{K}_y)$ is an exact sequence. See, for example, [13, Chapter V, Theorem 2.1] for the statement of the Künneth Theorem. Therefore $\mathrm{Tot}(\mathbb{K}_x \otimes \mathbb{K}_y)$ is a projective resolution of $\Lambda^{(2,2)}$ as a $(\Lambda^{(2,2)})^e$-module. In general, the total complex can be used to construct a projective resolution for any algebra over a field that can be expressed as the tensor product of two algebras for which we know the projective resolutions.

Now that we have the homological framework, we can work towards the definition of Hochschild cohomology. Let k be a field and Λ be an associative k-algebra. As in Example 2.1.6, let Λ^{op} be the *opposite algebra* which is Λ as a vector space with opposite algebra multiplication and $\Lambda^e = \Lambda \otimes \Lambda^{op}$ be the *enveloping algebra* of Λ. Notice Λ is a left Λ^e-module with action $(\lambda_0 \otimes \lambda_1)\lambda = \lambda_0\lambda\lambda_1$ for $\lambda_0, \lambda_1, \lambda \in \Lambda$.

Let M be a Λ^e-module. Define *Hochschild cohomology* of Λ with coefficients in M, $\mathrm{HH}^\bullet(\Lambda, M)$ by the following procedure: Let

$$\mathbb{P} : \ldots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \Lambda \to 0$$

be a projective Λ^e-module resolution of Λ. Apply $\mathrm{Hom}_{\Lambda^e}(-, M)$ to $\mathbb{P}$ and drop the $\mathrm{Hom}_{\Lambda^e}(\Lambda, M)$ term to get the sequence

$$\mathrm{Hom}_{\Lambda^e}(\mathbb{P}, M) : 0 \to \mathrm{Hom}_{\Lambda^e}(P_0, M) \xrightarrow{\delta_1^*} \mathrm{Hom}_{\Lambda^e}(P_1, M) \xrightarrow{\delta_2^*} \mathrm{Hom}_{\Lambda^e}(P_2, M) \xrightarrow{\delta_3^*} \ldots$$

where $\delta_i^*(f) = f\delta_i$ for $i > 0$. While $\mathbb{P}$ was an exact sequence, $\mathrm{Hom}_{\Lambda^e}(\mathbb{P}, M)$ may not be exact. Moreover, for $f \in \mathrm{Hom}_{\Lambda^e}(P_{i-1}, M)$, we have $\delta_{i+1}^*\delta_i^*(f) = \delta_{i+1}^*(f\delta_i) = f\delta_i\delta_{i+1} = f(0) = 0$ because $\mathbb{P}$ is exact and f is a homomorphism. Therefore $\mathrm{Hom}_{\Lambda^e}(\mathbb{P}, M)$ is a cochain complex on which cohomology is well-defined. Define the

ith Hochschild cohomology k-module of Λ with coefficients in M to be the quotient

$$\mathrm{HH}^i(\Lambda, M) = \mathrm{H}^i(\mathrm{Hom}_{\Lambda^e}(\mathbb{P}, M)) = \mathrm{Ker}\,(\delta_{i+1}^*)/\,\mathrm{Im}(\delta_i^*).$$

Remark 2.1.7. *In this definition of Hochschild cohomology, we are implicitly using the* Ext *functor. As a result, this definition can be shown to be independent of choice of projective Λ^e-module resolution. A thorough discussion of the* Ext *functor and resolutions can be found in [13]. For other equivalent definitions of Hochschild cohomology, including the simplicial definition, see [30, Chapter 9].*

We have a natural grading by homological degree such that elements in $\mathrm{HH}^i(\Lambda, M)$ and $\mathrm{Hom}_{\Lambda^e}(P_i, M)$ have *homological degree i*. Define $\mathrm{HH}^\bullet(\Lambda, M)$ to be the graded vector space

$$\mathrm{HH}^\bullet(\Lambda, M) = \bigoplus_{i\geq 0} \mathrm{HH}^i(\Lambda, M).$$

In the special case that $M = \Lambda$, we shorten the notation $\mathrm{HH}^\bullet(\Lambda, \Lambda)$ to $\mathrm{HH}^\bullet(\Lambda)$ and call $\mathrm{HH}^\bullet(\Lambda)$ the Hochschild cohomology of Λ. We now demonstrate the definition of Hochschild cohomology by giving an example using the projective resolution from Example 2.1.6.

Example 2.1.8. *Let k be a field and $R_x = k[x]/(x^2)$. $\mathbb{K}_x$ is a projective resolution for R_x as an R_x^e-module. To compute $\mathrm{HH}^\bullet(R_x)$, apply $\mathrm{Hom}_{R_x^e}(-, R_x)$ to $\mathbb{K}_x$ and drop $\mathrm{Hom}_{R_x^e}(R_x, R_x)$ to get*

$$0 \to \mathrm{Hom}_{R_x^e}(R_x^e, R_x) \xrightarrow{\delta_1^*} \mathrm{Hom}_{R_x^e}(R_x^e, R_x) \xrightarrow{\delta_2^*} \mathrm{Hom}_{R_x^e}(R_x^e, R_x) \xrightarrow{\delta_3^*} \ldots$$

where $\delta_0^* = 0$, $\delta_i^*(f) = f\delta_i$, and $\delta_i(a \otimes b) = (x \otimes 1 + (-1)^i 1 \otimes x)(a \otimes b)$ for all $i > 0$ and $a, b \in R_x$. Homomorphisms in $\mathrm{Hom}_{R_x^e}(R_x^e, R_x)$ are determined by the image of

$1 \otimes 1$. *Let* $f \in \operatorname{Hom}_{R_x^e}(R_x^e, R_x)$ *with* $f(1 \otimes 1) = ax + b$ *for* $a, b \in k$. *Assume* f *has homological degree* $i - 1$.

If i *is odd,* $\delta_i^* f(1 \otimes 1) = f\delta_i(1 \otimes 1) = f(x \otimes 1 - 1 \otimes x) = xb - bx = 0$ *and therefore* $\operatorname{Im} \delta_i^* = 0$ *and* $\operatorname{Ker} \delta_i^* = \operatorname{Hom}_{R_x^e}(R_x^e, R_x)$. *If* $i > 0$ *is even,* $\delta_i^* f(1 \otimes 1) = f\delta_i(1 \otimes 1) = f(x \otimes 1 + 1 \otimes x) = xb + bx = 2bx$ *and therefore* $\operatorname{Im} \delta_i^* = span_k\{x\}$ *and* $\operatorname{Ker} \delta_i^* = span_k\{x\}$. *Therefore*

$$\mathrm{HH}^0(R_x) = \operatorname{Hom}_{R_x^e}(R_x^e, R_x) \cong R_x,$$

$$\mathrm{HH}^{odd}(R_x) = span_k\{x\}/(0) \cong span_k\{x\}, \text{ and}$$

$$\mathrm{HH}^{even}(R_x) = \operatorname{Hom}_{R_x^e}(R_x^e, R_x)/span_k\{x\} \cong k.$$

2.2 Gerstenhaber algebra structure

In addition to the graded vector space structure, Hochschild cohomology of Λ has a graded ring structure given by the cup product and a compatible Lie algebra structure given by the Gerstenhaber bracket, making $\mathrm{HH}^\bullet(\Lambda)$ a *Gerstenhaber algebra*. We will use this section to discuss the general definition of these products, given at the chain level on the bar resolution.

Definition 2.2.1. *Let* Λ *be an algebra over a field* k. *The bar resolution of* Λ, $\mathbb{B}(\Lambda)$, *is a free* Λ^e*-module resolution of* Λ *given by*

$$\mathbb{B} = \mathbb{B}(\Lambda) : \ldots \xrightarrow{d_3} \Lambda^{\otimes 4} \xrightarrow{d_2} \Lambda \otimes \Lambda \otimes \Lambda \xrightarrow{d_1} \Lambda \otimes \Lambda \xrightarrow{d_0} \Lambda \to 0$$

where

$$d_i(\lambda_0 \otimes \lambda_1 \otimes \ldots \otimes \lambda_{i+1}) = \sum_{m=0}^{i} (-1)^m \lambda_0 \otimes \lambda_1 \otimes \ldots \otimes \lambda_m \lambda_{m+1} \otimes \ldots \otimes \lambda_{i+1}$$

for $\lambda_0, ..., \lambda_{i+1} \in \Lambda$.

The ith homological degree term in the bar resolution of Λ is denoted $\mathbb{B}_i$. Notice $\mathbb{B}_i = \Lambda^{\otimes i+2}$ for each $i \geq 0$. It can be shown that the bar resolution is a free (and therefore projective) Λ^e-module resolution of Λ.

Remark 2.2.2. *If we set* $\mathbb{P} = \mathbb{B}$ *in the definition of Hochschild cohomology given in the previous section, we recover the original definition given by Hochschild in [14].*

By our construction in Section 2.1, we can express elements in $\mathrm{HH}^i(\Lambda)$ at the chain level as equivalence classes of elements in $\mathrm{Hom}_{\Lambda^e}(\mathbb{B}_i, \Lambda) = \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes i+2}, \Lambda)$. We now define the products on Hochschild cohomology at the chain level. Let $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes i+2}, \Lambda)$ and $g \in \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes j+2}, \Lambda)$. Define the *cup product* of f and g by

$$f \smile g(\lambda_0 \otimes \lambda_1 \otimes ... \otimes \lambda_{i+j+1}) = f(\lambda_0 \otimes \lambda_1 \otimes ... \otimes \lambda_i \otimes 1)g(1 \otimes \lambda_{i+1} \otimes ... \otimes \lambda_{i+j} \otimes \lambda_{i+j+1})$$

for any $\lambda_0, \lambda_1, ..., \lambda_{i+j+1} \in \Lambda$. With the cup product, $\mathrm{HH}^\bullet(\Lambda)$ becomes a graded ring, graded by homological degree.

Remark 2.2.3. *Denote by* $\overline{f}$ *and* $\overline{g}$ *the representative functions of* $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes i+2}, \Lambda)$ *and* $g \in \mathrm{Hom}_{\Lambda^e}(\Lambda^{\otimes j+2}, \Lambda)$ *in* $\mathrm{HH}^\bullet(\Lambda)$. *If* $\overline{f} \in \mathrm{HH}^i(\Lambda)$ *and* $\overline{g} \in \mathrm{HH}^j(\Lambda)$, *then* $\overline{f} \smile \overline{g} = (-1)^{ij}\overline{g} \smile \overline{f}$, *making* $\mathrm{HH}^\bullet(\Lambda)$ *a graded commutative ring with respect to* $\smile$. *Proof of this relation can be found in [10, Section 7, Corollary 1].*

We will next describe the compatible Lie algebra structure, as defined by Gerstenhaber in [10]. The Gerstenhaber bracket is defined via the *circle product* of f and g, given by the following equation. Let

$$f \circ g(\lambda_0 \otimes \lambda_1 \otimes ... \otimes \lambda_{i+j}) = \sum_{m=1}^{i} (-1)^{(j-1)(m-1)} f(\lambda_0 \otimes \lambda_1 \otimes ... \otimes \lambda_{m-1} \otimes$$

$$g(1 \otimes \lambda_m \otimes ... \otimes \lambda_{m+j-1} \otimes 1) \otimes \lambda_{m+j} \otimes ... \otimes \lambda_{i+j})$$

for any $\lambda_0, \lambda_1, ..., \lambda_{i+j} \in \Lambda$. Then define the *Gerstenhaber bracket*, in terms of $\circ$, to be

$$[f, g] = f \circ g - (-1)^{(i-1)(j-1)} g \circ f.$$

Let $\overline{f} \in \mathrm{HH}^i(\Lambda)$, $\overline{g} \in \mathrm{HH}^j(\Lambda)$, and $\overline{h} \in \mathrm{HH}^l(\Lambda)$. Then the Gerstenhaber bracket satisfies

$$[\overline{f}, \overline{g}] = -(-1)^{(i-1)(j-1)}[\overline{g}, \overline{f}]$$

and

$$(-1)^{(i-1)(l-1)}[[\overline{f}, \overline{g}], \overline{h}] + (-1)^{(j-1)(i-1)}[[\overline{g}, \overline{h}], \overline{f}] + (-1)^{(l-1)(j-1)}[[\overline{h}, \overline{f}], \overline{g}] = 0,$$

making $\mathrm{HH}^\bullet(\Lambda)$ a graded Lie algebra with respect to $[-,-]$. Proof of these bracket properties is given by Gerstenhaber in [10]. Specifically, in [10, Section 7], it is shown that the circle product defines a pre-Lie system which, by [10, Section 2, Theorem 1] makes $(\mathrm{HH}^\bullet(\Lambda), [-,-])$ a graded Lie algebra. While the Gerstenhaber bracket is well-defined on cohomology, the circle product is not.

Moreover, the two products satisfy

$$[\overline{f} \smile \overline{g}, \overline{h}] = [\overline{f}, \overline{h}] \smile \overline{g} + (-1)^{i(l-1)} \overline{f} \smile [\overline{g}, \overline{h}] \tag{2.2.4}$$

for $\overline{f} \in \mathrm{HH}^i(\Lambda)$, $\overline{g} \in \mathrm{HH}^j(\Lambda)$, and $\overline{h} \in \mathrm{HH}^l(\Lambda)$. That is, the Gerstenhaber bracket behaves as a graded derivation with respect to the cup product. Proof of this result is given in [10, Section 8, Corollary 2]. The graded ring structure, graded Lie algebra structure, and compatibility relation between the structures (2.2.4) makes $\mathrm{HH}^\bullet(\Lambda)$ a

Gerstenhaber algebra.

2.3 Alternative descriptions of products

For our computations in Section 4, we will need definitions of the product structures more targeted to our setting. First, let us introduce yet another multiplication. The bar resolution of Λ admits a comultiplication given by the *diagonal map* on $\mathbb{B}$, $\Delta_{\mathbb{B}} : \mathbb{B} \to \mathbb{B} \otimes_{\Lambda} \mathbb{B}$ which is a chain map given by

$$\Delta_{\mathbb{B}}(\lambda_0 \otimes \lambda_1 \otimes ... \otimes \lambda_{n+1}) = \sum_{m=0}^{n} (\lambda_0 \otimes ... \otimes \lambda_m \otimes 1) \otimes_{\Lambda} (1 \otimes \lambda_{m+1} \otimes ... \otimes \lambda_{n+1})$$

for $\lambda_0, ..., \lambda_{n+1} \in \Lambda$. We will define the product structures on $\mathrm{HH}^{\bullet}(\Lambda)$ on the chain level in terms of this diagonal map.

Let $f \in \mathrm{Hom}_{\Lambda^e}(\mathbb{B}_i, \Lambda)$ and $g \in \mathrm{Hom}_{\Lambda^e}(\mathbb{B}_j, \Lambda)$. Using the diagonal map, we can view $f \smile g$, previously defined in Section 2.2, as the composition

$$f \smile g : \mathbb{B} \xrightarrow{\Delta_{\mathbb{B}}} \mathbb{B} \otimes_{\Lambda} \mathbb{B} \xrightarrow{f \otimes_{\Lambda} g} \Lambda \otimes_{\Lambda} \Lambda \xrightarrow{\mu} \Lambda \tag{2.3.1}$$

where μ is the multiplication map giving the isomorphism $\Lambda \otimes_{\Lambda} \Lambda \cong \Lambda$. If $\mathbb{K}$ is a another resolution for Λ with diagonal chain map, $\Delta_{\mathbb{K}}$, lifting the identity map $\mathbb{1}_{\Lambda} : \Lambda \to \Lambda$, then the cup product is given by (2.3.1) after replacing $\mathbb{B}$ with $\mathbb{K}$. That is, if $f, g \in \mathrm{Hom}_{\Lambda^e}(\mathbb{K}, \Lambda)$, then the cup product is given by

$$f \smile g : \mathbb{K} \xrightarrow{\Delta_{\mathbb{K}}} \mathbb{K} \otimes_{\Lambda} \mathbb{K} \xrightarrow{f \otimes_{\Lambda} g} \Lambda \otimes_{\Lambda} \Lambda \xrightarrow{\mu} \Lambda.$$

In particular, if $\mathbb{K}$ is a subresolution of $\mathbb{B}$ for which $\Delta_{\mathbb{B}}$ induces a comultiplication on $\mathbb{K}$, then the cup product on $\mathbb{K}$ is also induced by the composition in (2.3.1). To demonstrate the ring structure on Hochschild cohomology of an algebra, we will continue developing Example 2.1.6. Notice the resolution used in this example was not the bar resolution.

Example 2.3.2. *Let k be a field and $R_x = k[x]/(x^2)$. Recall, $\mathrm{HH}^0(R_x) \cong R_x$, $\mathrm{HH}^{odd}(R_x) \cong span_k\{x\}$, and $\mathrm{HH}^{even}(R_x) \cong k$ for $i > 0$. Before computing the cup product, let's adopt some new notation to better track homological degree. The projective resolution from which we computed cohomology was*

$$\mathbb{P}_x : \ldots \xrightarrow{\delta_3} R_x^e \xrightarrow{\delta_2} R_x^e \xrightarrow{\delta_1} R_x^e \xrightarrow{\delta_0} R_x \to 0.$$

Let ϵ_i be the copy of $1 \otimes 1$ in homological degree i and ϵ_i^* the dual R_x^e-module homomorphism. Then, in this notation, $\mathrm{HH}^\bullet(R_x) \cong span_k\{1, x, x\epsilon_1^*, \epsilon_2^*, x\epsilon_3^*, \epsilon_4^*, \ldots\}$ as a vector space. It can be shown that a diagonal map on $\mathbb{P}_x$ is

$$\Delta_{\mathbb{P}_x}(\epsilon_i) = \sum_{i=p+q} \epsilon_p \otimes_{R_x} \epsilon_q.$$

Then (2.3.1) gives us for $i, j > 0$,

$$\begin{aligned} x \smile \epsilon_{2i}^* &= x\epsilon_{2i}^*, \\ x \smile x\epsilon_{2i+1}^* &= 0, \\ x\epsilon_{2i+1}^* \smile x\epsilon_{2j+1}^* &= 0, \\ x\epsilon_{2i+1}^* \smile \epsilon_{2j}^* &= x\epsilon_{2(i+j)+1}^*, \text{ and} \\ \epsilon_{2i}^* \smile \epsilon_{2j}^* &= \epsilon_{2(i+j)}^*. \end{aligned}$$

The reader can check that $\mathrm{HH}^\bullet(R_x) \cong R_x \times_k \bigwedge^*(x\epsilon_1^*)[\epsilon_2^*]$, the fiber product of rings. That is, $\mathrm{HH}^\bullet(R_x)$ is the subring of $R_x \oplus \bigwedge^*(x\epsilon_1^*)[\epsilon_2^*]$ consisting of pairs (a, b) such that the images of a and b under the augmentation maps of R_x and $\bigwedge^*(x\epsilon_1^*)[\epsilon_2^*]$ respectively *are equal where x, $x\epsilon_1^*$, and ϵ_2^* are in the kernel of their respective augmentation maps.*

While the cup product can be defined using any projective resolution, there is no such general description of the Lie bracket structure. However, the bracket can be defined on certain resolutions. In [23], Negron and Witherspoon defined the Gerstenhaber bracket on any resolution, $\mathbb{K}$, of Λ satisfying the following conditions:

Conditions 2.3.3. *(a) There is a chain map $\iota : \mathbb{K} \to \mathbb{B}$ lifting $\mathbb{1}_\Lambda$.*

(b) There is a chain map $\pi : \mathbb{B} \to \mathbb{K}$ such that $\pi\iota = \mathbb{1}_\mathbb{K}$.

(c) $\mathbb{K}$ admits a diagonal chain map, $\Delta_\mathbb{K} : \mathbb{K} \to \mathbb{K}\otimes_\Lambda\mathbb{K}$, such that $\Delta_\mathbb{B}\iota = (\iota\otimes_\Lambda\iota)\Delta_\mathbb{K}$.

Notice $\mathbb{K} = \mathbb{B}(\Lambda)$ trivially satisfies the conditions above. It is argued in [23] that if Λ is a Koszul algebra and $\mathbb{K}$ is its Koszul resolution, then $\mathbb{K}$ satisfies the conditions above. Explicit diagonal maps for Koszul algebras can be found in [8] and [22].

Assume $\mathbb{K}$ satisfies Conditions 2.3.3. To define the Gerstenhaber bracket on $\mathbb{K}$, we will need the following definition.

Definition 2.3.4. *Let $\mathbb{C}$ and $\mathbb{D}$ be cochain complexes. A chain map $\mathbb{C} \to \mathbb{D}$ is called a quasi-isomorphism if the induced maps $\mathrm{H}^i(\mathbb{C}) \to \mathrm{H}^i(\mathbb{D})$ are isomorphisms for all $i \in \mathbb{Z}$.*

Let $\mathbf{\Lambda}$ be the cochain complex

$$\mathbf{\Lambda} : ... \to 0 \to 0 \to \Lambda \to 0 \to 0 \to ...$$

with Λ in homological degree 0 and 0 in all other degrees. Let $\mathbf{m} : \mathbb{K} \to \mathbf{\Lambda}$ be a quasi-isomorphism. That is, $\mathbf{m}$ induces the isomorphism $\mathrm{H}^0(\mathbb{K}) \cong \Lambda$. Then, as in [23, Section 3.2], define

$$F_\mathbb{K} = (\mathbf{m} \otimes_\Lambda \mathbb{1}_\mathbb{K} - \mathbb{1}_\mathbb{K} \otimes_\Lambda \mathbf{m}) : \mathbb{K} \otimes_\Lambda \mathbb{K} \to \mathbb{K}. \tag{2.3.5}$$

Let $\phi : \mathbb{K} \otimes_\Lambda \mathbb{K} \to \mathbb{K}$ be a Λ^e-module homomorphism which satisfies $d(\phi) = F_\mathbb{K}$ where d is the boundary map on the double complex $\mathrm{Hom}_{\Lambda^e}(\mathbb{K} \otimes_\Lambda \mathbb{K}, \mathbb{K})$. That is, $d(\phi) = d_\mathbb{K}\phi + \phi d_{\mathbb{K}\otimes\mathbb{K}}$ for $d_{\mathbb{K}\otimes\mathbb{K}}$ the differential on $\mathrm{Tot}(\mathbb{K} \otimes \mathbb{K})$ and $d_\mathbb{K}$ the differential on $\mathbb{K}$. An iterative method for computing such a ϕ map is given in [23, Lemma 3.3.1]. We will call such a ϕ map a *contracting homotopy* for $F_\mathbb{K}$. As given in [23, Definition 3.2.2], for $f \in \mathrm{Hom}_{\Lambda^e}((\mathbb{K})_n, \Lambda)$ and $g \in \mathrm{Hom}_{\Lambda^e}((\mathbb{K})_m, \Lambda)$, we can define the $\circ$-product on the chain level as a composition

$$f \circ g : \mathbb{K} \xrightarrow{\Delta_\mathbb{K}} \mathbb{K} \otimes_\Lambda \mathbb{K} \xrightarrow{\Delta_\mathbb{K} \otimes_\Lambda \mathbb{1}_\mathbb{K}} \mathbb{K} \otimes_\Lambda \mathbb{K} \otimes_\Lambda \mathbb{K} \xrightarrow{\mathbb{1}_\mathbb{K} \otimes_\Lambda g \otimes_\Lambda \mathbb{1}_\mathbb{K}} \mathbb{K} \otimes_\Lambda \mathbb{K} \xrightarrow{\phi} \mathbb{K} \xrightarrow{f} \Lambda \quad (2.3.6)$$

where, in $\mathbb{1}_\mathbb{K} \otimes_\Lambda g \otimes_\Lambda \mathbb{1}_\mathbb{K}$, we are including the identification $\mathbb{K} \otimes_\Lambda \Lambda \cong \mathbb{K}$ and the function is given the Koszul sign convention. That is, for $x_1 \otimes_\Lambda x_2 \otimes x_3 \in \mathbb{K} \otimes_\Lambda \mathbb{K} \otimes_\Lambda \mathbb{K}$ with x_1 homogeneous,

$$(\mathbb{1}_\mathbb{K} \otimes_\Lambda g \otimes_\Lambda \mathbb{1}_\mathbb{K})(x_1 \otimes_\Lambda x_2 \otimes_\Lambda x_3) = (-1)^{m||x_1||} x_1 \otimes_\Lambda g(x_2) \otimes_\Lambda x_3$$

where $||x_1||$ is the homological degree of x_1.

Remark 2.3.7. *We recover the original definition of the $\circ$-product given by Gerstenhaber in [10] when we set $\mathbb{K} = \mathbb{B}$ and $\phi((\lambda_0 \otimes ... \otimes \lambda_i) \otimes_\Lambda (\lambda_{i+1} \otimes ... \otimes \lambda_j)) = \lambda_0 \otimes ... \otimes \lambda_i\lambda_{i+1} \otimes ... \otimes \lambda_j$.*

As in Section 2.2, let $[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f$. According to [23, Theorem 3.2.5], this definition of the Gerstenhaber bracket agrees with the definition given in Section 2.2 on cohomology. Using this definition, we are able to compute brackets on certain "nice" resolutions other than the bar resolution without having to pass to the bar resolution definition. We will use this description of brackets for the remainder of the dissertation.

We conclude this section by computing the bracket structure for the ongoing example, Example 2.1.6.

Example 2.3.8. *Let k be a field and $R_x = k[x]/(x^2)$. To use the methods of this section, we must check that $\mathbb{P}_x$ satisfies Conditions 2.3.3. Let $\iota : \mathbb{P}_x \to \mathbb{B}$ be defined by $\iota(\epsilon_i) = 1 \otimes x^{\otimes i} \otimes 1$ for $i \geq 0$. Choose a map $\pi : \mathbb{B} \to \mathbb{P}_x$ such that $\pi(1 \otimes x^{\otimes i} \otimes 1) = \epsilon_i$ for each $i \geq 0$. Such a map exists because we can make a free R_x^e-basis of $\mathbb{B}_i$ which includes $1 \otimes x^{\otimes i} \otimes 1$. As we saw in Example 2.3.2, a diagonal on $\mathbb{P}_x$ is given by*

$$\Delta_{\mathbb{P}_x}(\epsilon_i) = \sum_{i=p+q} \epsilon_p \otimes_{R_x} \epsilon_q.$$

It can be readily checked that $\Delta_{\mathbb{B}}\iota = (\iota \otimes_{R_x} \iota)\Delta_{\mathbb{P}_x}$. Therefore $\mathbb{P}_x$ satisfies Conditions 2.3.3.

Then by [11, Lemma 4.2], the contracting homotopy ϕ is

$$\phi_{i+j}(\epsilon_i \otimes_{R_x} x^m \epsilon_j) = \delta_{m,1}(-1)^i \epsilon_{i+j+1}$$

where $\delta_{m,1}$ is the Kronecker delta. Using (2.3.6) and these maps, the reader will find that the low degree non-zero brackets, for example, are

$$[x\epsilon_1^*, x] = x \text{ and } [\epsilon_2^*, x\epsilon_1^*] = -2\epsilon_2^*.$$

All other brackets can be computed using the equation (2.2.4) and our previous computations.

3. TWISTED TENSOR PRODUCTS

In this section, we discuss twisted tensor products and study their behavior with Hochschild cohomology. Section 3.1 contains the pertinent development of twisted tensor products and Hochschild cohomology given by Bergh and Oppermann in [6]. The reader is directed to [6] for a more detailed discussion of twisted tensor products. In Section 3.2, we discuss the bracket structure on Hochschild cohomology of twisted tensor products as given by Nguyen, Witherspoon and me in [11].

3.1 Preliminaries

Let R and S be associative algebras over a field k, graded by abelian groups A and B respectively. Let

$$t : A \otimes_{\mathbb{Z}} B \to k^*$$

be a homomorphism of abelian groups which we will call the *twisting map*. We will denote $t(a \otimes_{\mathbb{Z}} b) = t^{<a|b>}$ for all $a \in A$ and $b \in B$. Then, as in [6], define $R \otimes^t S$, the *twisted tensor product* of R and S, to be $R \otimes S$ as a vector space with multiplication given by

$$(r_0 \otimes s_0)(r_1 \otimes s_1) = t^{<|r_1|||s_0|>}(r_0 r_1 \otimes s_0 s_1)$$

for all homogeneous elements $r_0, r_1 \in R$ and $s_0, s_1 \in S$, where $|\cdot|$ denotes the grading degree of the element. Twisted tensor products have been used to construct quantum polynomial rings in [20], quantum exterior algebras in [4], and quantum complete intersections in [5], [11], and [25]. We give now an explicit construction of $\Lambda_{\mathbf{q}}^{(2,2,\ldots,2)}$.

Example 3.1.1. *Let* $\mathbf{q} = \{q_{i,j}\}_{i,j \in \{1,2,\ldots,n\}}$ *such that* $q_{i,j} \in k^*$ *for all* $i, j \in \{1, 2, ..., n\}$,

$q_{j,i} = q_{i,j}^{-1}$, and $q_{i,i} = -1$. Define the quantum complete intersection

$$\Lambda_{\mathbf{q}}^{(2,2,...,2)} = k\langle x_1, x_2, ..., x_n |\ x_i x_j = -q_{i,j} x_j x_i,\ x_i^2 = 0 \text{ for all } i, j \in \{1, 2, ...n\}\rangle.$$

We will construct $\Lambda_{\mathbf{q}}^{(2,2,...,2)}$ by taking an iteration of twisted tensor products of the algebra R_x from Example 2.1.6. The algebras R_{x_1} and R_{x_2} are both $\mathbb{Z}$-graded by polynomial degree. Therefore define

$$R_{x_1} \otimes^{t_1} R_{x_2} \text{ by the twist } t_1^{<1|1>} = -q_{1,2}^{-1}.$$

Then notice $\Lambda_{\mathbf{q}}^{(2,2)} \cong R_{x_1} \otimes^{t_1} R_{x_2}$. We continue building towards $\Lambda_{\boldsymbol{q}}^{(2,2,...,2)}$ by defining

$$(R_{x_1} \otimes^{t_1} R_{x_2}) \otimes^{t_2} R_{x_3} \text{ by } t_2^{<(1,0)|1>} = -q_{1,3}^{-1} \text{ and } t_2^{<(0,1)|1>} = -q_{2,3}^{-1}.$$

Again, by construction, we get $\Lambda_{\mathbf{q}}^{(2,2,2)} \cong (R_{x_1} \otimes^{t_1} R_{x_2}) \otimes^{t_2} R_{x_3}$. We can continue this process to get

$$\Lambda_{\boldsymbol{q}}^{(2,2,...,2)} \cong (...((R_{x_1} \otimes^{t_1} R_{x_2}) \otimes^{t_2} R_{x_3}) \otimes^{t_3} ...) \otimes^{t_{n-1}} R_{x_n}$$

where $t_i^{<[j]|1>} = -q_{j,i+1}^{-1}$ for $i \in \{1, 2, ..., n-1\}$ and $j < i$ and $[j] = (0, ..., 0, 1, 0, ..., 0)$, the i-tuple with a 1 in the jth coordinate and 0 otherwise.

The class of quantum complete intersections $\Lambda_{\mathbf{q}}^{(2,2,...,2)}$ is central to this dissertation and will be discussed in more depth in Section 4. In the meantime, we must develop more techniques for computations with twisted tensor products. Our aim is to construct an $(R \otimes S)^e$-module resolution of $R \otimes^t S$ using resolutions of R and S. To do this, we must give a graded $(R \otimes S)^e$-module structure to the twisted tensor product of graded R^e-modules with graded S^e-modules.

Definition 3.1.2. *Let X be a graded R^e-module and Y be a graded S^e-module. Define a graded $(R \otimes^t S)^e$-module structure on $X \otimes^t Y$ by the action*

$$((r \otimes s) \otimes (r' \otimes s'))(x \otimes y) = t^{<|x|||s|>} t^{<|r'|||y|>} t^{<|r'|||s|>} (rxr' \otimes sys')$$

for $r, r' \in R$, $s, s' \in S$, $x \in X$, and $y \in Y$.

Remark 3.1.3. *There is a subtly about the graded module structure of $X \otimes^t Y$ as it compares to the individual grading of X and Y. See [6, Remark 4.2] for more discussion of this point.*

Let X be a graded R^e-module and Y be a graded S^e -module with graded projective bimodule resolutions

$$\mathbb{P} : ... \to P_2 \to P_1 \to P_0 \to X \to 0$$

$$\mathbb{Q} : ... \to Q_2 \to Q_1 \to Q_0 \to Y \to 0$$

of X and Y respectively. Then by [6, Lemma 4.5], the total complex, $\mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q})$, is a graded projective bimodule resolution of $X \otimes^t Y$. That is, we can use a twisted version of the total complex to construct a resolution for algebras given by a twisted tensor product. Just as $\mathrm{Tot}(\mathbb{P}_x \otimes \mathbb{P}_y)$ is a projective resolution for $\Lambda^{(2,2)}$ (recall this discussion in Section 2.1), $\mathrm{Tot}(\mathbb{P}_{x_1} \otimes^t \mathbb{P}_{x_2})$ is a projective resolution for $\Lambda_{\mathbf{q}}^{(2,2)}$. We will use this twisted total complex in our computations in Section 3.2.1.

3.2 Brackets on Hochschild cohomology

The purpose of this section is to construct the Gerstenhaber bracket on Hochschild cohomology of the twisted tensor product of two algebras when we are given the brackets on Hochschild cohomology of the two algebras. We forgo discussion of the vector space and cup product structures because they are defined for arbitrary resolutions (recall Section 2.3). The results in this section are attributed to Nguyen, Witherspoon and me and can be found in more detail in [11].

Let R and S be associative algebras over a field k, graded by abelian groups A and B respectively. Let $t : A \otimes_{\mathbb{Z}} B \to k^*$ be a twisting map. Assume

$$\mathbb{P} : ... \to P_2 \to P_1 \to P_0 \to R \to 0$$

is a graded projective resolution of R as an R^e-module and

$$\mathbb{Q} : ... \to Q_2 \to Q_1 \to Q_0 \to S \to 0$$

is a graded projective resolution of S as an S^e-module. Assume additionally that $\mathbb{P}$ and $\mathbb{Q}$ satisfy Conditions 2.3.3. That is, there are maps $\phi_{\mathbb{Q}}$, $\phi_{\mathbb{P}}$, $\Delta_{\mathbb{P}}$, and $\Delta_{\mathbb{Q}}$ from which we can define the bracket on $\mathrm{HH}^{\bullet}(R)$ and $\mathrm{HH}^{\bullet}(S)$ using (2.3.6).

As we saw in the previous section, $\mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q})$ is a $A \oplus B$-graded projective bimodule resolution of $R \otimes^t S$. Our aim is to construct the bracket on the total complex using the description in Section 2.3. Thus, assume that we are in the setting in which $\mathbb{K} = \mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q})$ satisfies Conditions 2.3.3. In order to use (2.3.6), we must first determine a contracting homotopy $\phi : \mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q}) \otimes_{R \otimes^t S} \mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q}) \to \mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q})$. We would like to construct this ϕ from the contracting homotopies $\phi_{\mathbb{P}}$ and $\phi_{\mathbb{Q}}$. Because $\phi_{\mathbb{P}}$ and $\phi_{\mathbb{Q}}$ are defined on $(\mathbb{P} \otimes_R \mathbb{P})$ and $(\mathbb{Q} \otimes_S \mathbb{Q})$ respectively, we begin by defining a chain map from $\mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q}) \otimes_{R \otimes^t S} \mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q})$ to $(\mathbb{P} \otimes_R \mathbb{P}) \otimes^t (\mathbb{Q} \otimes_S \mathbb{Q})$.

Lemma 3.2.1 ([11, Lemma 3.3]). *The chain map*

$$\sigma : \mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q}) \otimes_{R \otimes^t S} \mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q}) \to (\mathbb{P} \otimes_R \mathbb{P}) \otimes^t (\mathbb{Q} \otimes_S \mathbb{Q})$$

defined by

$$\sigma((x \otimes y) \otimes (x' \otimes y')) = (-1)^{||y||||x'||} t^{<|x'|||y|>} (x \otimes x') \otimes (y \otimes y') \tag{3.2.2}$$

is an isomorphism of $(R \otimes^t S)^e$-modules. Recall $||y|| \in \mathbb{Z}$ is the homological degree of $y \in \mathbb{Q}$ and $|y| \in B$ is the graded degree of y.

Using σ, we can now define the $F_{\mathbb{K}}$ map given in Section 2.3. Let $\mathbf{m}_{\mathbb{Q}} : \mathbb{Q} \to \Lambda$ and $\mathbf{m}_{\mathbb{P}} : \mathbb{P} \to \Lambda$ be quasi-isomorphisms. Then define

$$F = ((\mathbf{m}_{\mathbb{P}} \otimes_R \mathbb{1}_{\mathbb{P}}) \otimes (\mathbf{m}_{\mathbb{Q}} \otimes_S \mathbb{1}_{\mathbb{Q}}) - (\mathbb{1}_{\mathbb{P}} \otimes_R \mathbf{m}_{\mathbb{P}}) \otimes (\mathbb{1}_{\mathbb{Q}} \otimes_S \mathbf{m}_{\mathbb{Q}}))\sigma.$$

F is in fact the chain map $F_{\mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q})}$. We record this in the following lemma.

Lemma 3.2.3 ([11, Lemma 3.4])**.** *$F = F_{\mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q})}$ as in (2.3.5).*

For this F, we determine a corresponding contracting homotopy ϕ. The following lemma is critical to the computations and constructions for remaining work and is shown by tracing back through the definitions of the involved maps.

Lemma 3.2.4 ([11, Lemma 3.5])**.** *Let ϕ_P, ϕ_Q be contracting homotopies for F_P and F_Q respectively as in (2.3.6). Define*

$$\phi = (\phi_P \otimes (\mathbf{m}_{\mathbb{Q}} \otimes_S \mathbb{1}_{\mathbb{Q}}) + (-1)^{i+p} (\mathbb{1}_{\mathbb{P}} \otimes_R \mathbf{m}_{\mathbb{P}}) \otimes \phi_Q)\sigma$$

on $(P_i \otimes^t Q_j) \otimes_{R \otimes^t S} (P_p \otimes Q_q)$. Then ϕ is a contracting homotopy for F.

With this map ϕ, we now have the necessary tools to compute the bracket structure on $\mathrm{HH}^\bullet(R \otimes^t S)$ given the structures on $\mathrm{HH}^\bullet(R)$ and $\mathrm{HH}^\bullet(S)$ by the formula

(2.3.6),

$$f \circ g : \mathbb{K} \xrightarrow{\Delta_{\mathbb{K}}} \mathbb{K} \otimes_\Lambda \mathbb{K} \xrightarrow{\Delta_{\mathbb{K}} \otimes_\Lambda \mathbb{1}_{\mathbb{K}}} \mathbb{K} \otimes_\Lambda \mathbb{K} \otimes_\Lambda \mathbb{K} \xrightarrow{\mathbb{1}_{\mathbb{K}} \otimes_\Lambda g \otimes_\Lambda \mathbb{1}_{\mathbb{K}}} \mathbb{K} \otimes_\Lambda \mathbb{K} \xrightarrow{\phi} \mathbb{K} \xrightarrow{f} \Lambda,$$

setting $\mathbb{K} = \mathrm{Tot}(\mathbb{P} \otimes^t \mathbb{Q})$, $\Lambda = R \otimes^t S$, and ϕ as in Lemma 3.2.4.

Before computing examples in the subsequent section, we apply these techniques to a more general problem. Let R and S be algebras over a field k, at least one of them finite dimensional. Le and Zhou proved that

$$\mathrm{HH}^\bullet(R \otimes S) \cong \mathrm{HH}^\bullet(R) \otimes \mathrm{HH}^\bullet(S) \tag{3.2.5}$$

as Gerstenhaber algebras in [16]. To prove this result, they define the bracket structure and cup product structure on $\mathrm{HH}^\bullet(R) \otimes \mathrm{HH}^\bullet(S)$ using the brackets and cups on the individual cohomologies as we have in this section. Under the more general setting of the twisted tensor product, Bergh and Oppermann proved in [6] that the isomorphism holds as algebras under the cup product for a subalgebra of the Hochschild cohomologies on both sides of (3.2.5). In [11], Nguyen, Witherspoon, and I used the bracket development restated in this section to prove that the isomorphism of [6] extends to an isomorphism of Gerstenhaber algebras. That is, if the brackets on the Hochschild cohomology of two algebras is known, we can recover some of the brackets on Hochschild cohomology of the twisted tensor product of these algebras, without having to make explicit computations on a resolution of the twisted tensor product.

3.2.1 Quantum complete intersections

In this section we use the techniques of the previous sections to compute the brackets on Hochschild cohomology of the quantum complete intersection, $\Lambda_{\mathbf{q}}^{(2,2)}$.

Buchweitz, Green, Madsen, and Solberg computed the vector space and cup product structures of Hochschild cohomology of $\Lambda_{\mathbf{q}}^{(2,2)}$ in [7]. We utilize the vector space description in this section. The brackets in this section were originally computed by Nguyen, Witherspoon, and me in [11, Section 4]. We include these computations for comparison with our results in Section 5.

As shown in Example 3.1.1, quantum complete intersections can be constructed by iterated twisted tensor products as in Example 3.1.1. In particular, $\Lambda_{\mathbf{q}}^{(2,2)}$ is isomorphic to $R_{x_1} \otimes^{t_1} R_{x_2}$ where $R_x = k[x]/(x^2)$ and twist given by $t_1^{<1,1>} = -q_{1,2}^{-1}$. We have already collected quite a bit of information about $\mathrm{HH}^{\bullet}(R_x)$ in the previous examples which we will use in this section. By Example 2.1.6, $\mathbb{P}_x$ is a projective resolution for R_x as an R_x^e-module given by

$$\mathbb{P}_x : ... \xrightarrow{\delta_3} R_x^e \xrightarrow{\delta_2} R_x^e \xrightarrow{\delta_1} R_x^e \xrightarrow{\delta_0} R \to 0$$

with $\delta_i(a \otimes b) = (x \otimes 1 + (-1)^i 1 \otimes x)(a \otimes b)$ for all $i > 0$ and $\delta_0(a \otimes b) = ab$ for $a \otimes b \in R_x^e$. Therefore $\mathbb{K} = \mathrm{Tot}(\mathbb{P}_{x_1} \otimes^{t_1} \mathbb{P}_{x_2})$ is a graded projective resolution for $\Lambda_{\mathbf{q}}^{(2,2)}$ by [6, Lemma 4.5]. See Definition 2.1.2 for the formal description of $\mathbb{K}$. It can be shown (and will be shown in great detail in Section 4) that $\mathbb{K}$ is isomorphic to the resolution

$$\mathbb{K}' : ... \xrightarrow{\delta_3'} \bigoplus_{i+j=2} \Lambda_{\mathbf{q}}^{(2,2)} \epsilon_{i,j} \Lambda_{\mathbf{q}}^{(2,2)} \xrightarrow{\delta_2'} \bigoplus_{i+j=1} \Lambda_{\mathbf{q}}^{(2,2)} \epsilon_{i,j} \Lambda_{\mathbf{q}}^{(2,2)} \xrightarrow{\delta_1'} \Lambda_{\mathbf{q}}^{(2,2)} \epsilon_{0,0} \Lambda_{\mathbf{q}}^{(2,2)} \xrightarrow{\delta_0'} \Lambda_{\mathbf{q}}^{(2,2)} \to 0$$

where $\epsilon_{i,j} = \epsilon_i \otimes \epsilon_j$ and ϵ_i is the copy of $1 \otimes 1$ in homological degree i and

$$\delta_n'(\epsilon_{i,j}) = x\epsilon_{i-1,j} + (-1)^n q^j \epsilon_{i-1,j} x + q^i y \epsilon_{i,j-1} + (-1)^n \epsilon_{i,j-1} y.$$

By a slight abuse of notation, we will use the resolutions $\mathbb{K}$ and $\mathbb{K}'$ interchangeably according to the situation.

Remark 3.2.6. *The resolution $\mathbb{K}'$ as defined above is the resolution found in [7] by making the identification $\tilde{f}_i^n \leftrightarrow \epsilon_{i,n-i}$.*

Now, because $\Lambda_{\mathbf{q}}^{(2,2)}$ is Koszul and $\mathbb{K}$ is a Koszul resolution, Conditions 2.3.3 hold. Thus to compute brackets using (2.3.6), we first need to compute the maps ϕ and $\Delta_{\mathbb{K}}$ for $\mathbb{K}$ as described above. As given by [11, Lemma 3.5] and used in Example 2.3.8, $\phi_{i+j}(\epsilon_i \otimes_{R_x} x^m \epsilon_j) = \delta_{m,1}(-1)i\epsilon_{i+j+1}$ is a contracting homotopy for $\mathbb{P}_x$. We will call this map $\phi_{\mathbb{P}}$ to distinguish this map from our desired map ϕ for $\mathbb{K}$. Then by Lemma 3.2.4,

$$\begin{aligned}
&\phi(\epsilon_{i,j} \otimes_{\Lambda_{\mathbf{q}}^{(2,2)}} x^l y^m \epsilon_{p,r}) \\
&\quad = \phi((\epsilon_i \otimes \epsilon_j) \otimes_{\Lambda_{\mathbf{q}}^{(2,2)}} (-q_{1,2})^{-pm}(x^l \epsilon_p \otimes y^m \epsilon_r)) \\
&\quad = (\phi_{\mathbb{P}} \otimes (\mathbf{m}_{\mathbb{P}} \otimes_{R_{x_2}} \mathbb{1}_{\mathbb{P}}) + (-1)^{i+p}(\mathbb{1}_{\mathbb{P}} \otimes_{R_{x_1}} \mathbf{m}_{\mathbb{P}}) \otimes \phi_{\mathbb{P}})((-q_{1,2})^{-pm-(l+p)j} x^l \epsilon_p \otimes y^m \epsilon_r) \\
&\quad = (-q_{1,2})^{-pm-(l+p)j}(\delta_{l,1}(-1)^i \epsilon_{i+p+1} \otimes \delta_{j,0} y^m \epsilon_r + (-1)^{i+p}\delta_{p,0}\epsilon_i x^l \otimes \delta_{m,1}(-1)^j \epsilon_{j+r+1}).
\end{aligned}$$

Simplifying, we get $\phi(\epsilon_{i,j} \otimes_{\Lambda_{\mathbf{q}}^{(2,2)}} x^l y^m \epsilon_{p,r})$

$$= \begin{cases}
(-q)^{mi+m}\delta_{l,1}(-1)^i y^m \epsilon_{i+p+1,r}, & \text{if } j = 0, p > 0 \\
(-q)^{mi+m}\delta_{l,1}(-1)^i y^m \epsilon_{i+1,r} + (-q)^{lr+l}\delta_{m,1}(-1)^i \epsilon_{i,r+1} x^l, & \text{if } j = 0, p = 0 \\
(-q)^{lr+l}\delta_{m,1}(-1)^{i+j}\epsilon_{i,j+r+1} x^l, & \text{if } j > 0, p = 0 \\
0, & \text{otherwise.}
\end{cases}$$

The last map which needs to be defined before computing examples is $\Delta_{\mathbb{K}}$. However, this map is given in [7] after making the identification $\tilde{f}_i^n \leftrightarrow \epsilon_{i,n-i}$. That is, by

[7, p. 810], the diagonal on $\mathbb{K}$ is given by

$$\Delta_{\mathbb{K}}(\epsilon_{i,j}) = \sum_{w=0}^{i+j} \sum_{l=max\{0,w-i\}}^{min\{w,j\}} q_{1,2}^{l(i+l-w)} \epsilon_{w-l,l} \otimes_{\Lambda_{\mathbf{q}}^{(2,2)}} \epsilon_{i+l-w,j-l}.$$

It is shown in [11] that $\Delta_{\mathbb{K}}$ satisfies $\Delta_{\mathbb{K}}\iota = (\iota \otimes_{\Lambda_{\mathbf{q}}^{(2,2)}} \iota)\Delta_{\mathbb{K}}$ for the inclusion map given by $\iota(\epsilon_{i,j}) = \tilde{f}_j^{i+j}$.

With $\Delta_{\mathbb{K}}$ and $\phi_{\mathbb{K}}$, we now have enough information to compute brackets on $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}^{(2,2)})$ for some choices of $q_{1,2}$. We will compute the example when $q_{1,2}$ is not a root of unity and when $q_{1,2}$ is a dth root of unity for $d > 1$ odd. All other computations for $q_{1,2} \neq 0$ can be found in [11].

Example 3.2.7. *Assume $q_{1,2}$ is not a root of unity. As given by [7, Section 2.1],*

$$\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}^{(2,2)}) \cong k[xy]/((xy)^2) \times_k \bigwedge\nolimits^*(x\epsilon_{1,0}^*, y\epsilon_{0,1}^*).$$

Therefore we need to compute the brackets for pairs of elements from the set of algebra generators $\{xy, x\epsilon_{1,0}^*, y\epsilon_{0,1}^*\}$. All other brackets can be computed using (2.2.4). The reader can check that

$$x\epsilon_{1,0}^* \circ x\epsilon_{1,0}^* = x\epsilon_{1,0}^*,$$
$$x\epsilon_{1,0}^* \circ y\epsilon_{0,1}^* = 0,$$
$$y\epsilon_{0,1}^* \circ x\epsilon_{1,0}^* = 0,$$
$$y\epsilon_{0,1}^* \circ y\epsilon_{0,1}^* = y\epsilon_{0,1}^*.$$

Therefore all of these circle products result in Gerstenhaber brackets that are 0. Non-

zero brackets arising when pairing generators with the degree 0 element xy are:

$$[x\epsilon_{1,0}^*, xy] = xy \quad \textit{and} \quad [y\epsilon_{0,1}^*, xy] = xy.$$

The algebra in this example, $\Lambda_{\mathbf{q}}^{(2,2)}$ for $q_{1,2}$ not a root of unity, provided an answer to Happel's question [12]: does the vanishing of Hochschild cohomology in high degrees of a finite dimensional algebra over a field imply finite global dimension? In [2], Avramov and Iyengar showed that for commutative algebras, the answer is yes. However, for $\Lambda_{\mathbf{q}}^{(2,2)}$ with $q_{1,2}$ not a root of unity, the answer is no, as shown in [7]. When $q_{1,2}$ is a root of unity, Buchweitz, Green, Madsen, and Solberg showed in [7] that $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}^{(2,2)})$ does not vanish but has large gaps in cohomology corresponding to the root of unity. We now look at the brackets on one such example.

Example 3.2.8. *Assume $q_{1,2}$ is a dth root of unity for $d > 1$ odd. By [7, 3.1], translated into our notation,*

$$\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}^{(2,2)}) \cong k[xy]/((xy)^2) \times_k (\textstyle\bigwedge^*(x\epsilon_{1,0}^*, y\epsilon_{0,1}^*)[\epsilon_{2d,0}^*, \epsilon_{d,d}^*, \epsilon_{0,2d}^*]/(\epsilon_{2d,0}^*\epsilon_{0,2d}^* - (\epsilon_{d,d}^*)^2)).$$

Thus we need to calculate the brackets on pairs of elements from the set

$$\{xy, x\epsilon_{1,0}^*, y\epsilon_{0,1}^*, \epsilon_{2d,0}^*, \epsilon_{d,d}^*, \epsilon_{0,2d}^*\}.$$

The rest will follow by applying (2.2.4). We again leave it to the reader to check that, of these pairs, the non-zero circle products are

$$x\epsilon_{1,0}^* \circ x\epsilon_{1,0}^* = x\epsilon_{1,0}^*,$$
$$(xy)^i\epsilon_{2d,0}^* \circ x\epsilon_{1,0}^* = 2d(xy)^i\epsilon_{2d,0}^*,$$

$$(xy)^i\epsilon^*_{d,d} \circ x\epsilon^*_{1,0} = d(xy)^i\epsilon^*_{d,d},$$

$$y\epsilon^*_{0,1} \circ y\epsilon^*_{0,1} = y\epsilon^*_{0,1},$$

$$(xy)^i\epsilon^*_{2d,0} \circ y\epsilon^*_{0,1} = 2d(xy)^i\epsilon^*_{0,2d}, \text{ and}$$

$$(xy)^i\epsilon^*_{d,d} \circ y\epsilon^*_{0,1} = d(xy)^i\epsilon^*_{d,d}.$$

Then the non-zero Gerstenhaber brackets are

$$[x\epsilon^*_{1,0}, xy] = xy,$$

$$[y\epsilon^*_{0,1}, xy] = xy,$$

$$[\epsilon^*_{2d,0}, x\epsilon^*_{1,0}] = 2d\epsilon^*_{2d,0},$$

$$[\epsilon^*_{d,d}, x\epsilon^*_{1,0}] = d\epsilon^*_{d,d},$$

$$[\epsilon^*_{d,d}, y\epsilon^*_{0,1}] = d\epsilon^*_{d,d}, \text{ and}$$

$$[\epsilon^*_{0,2d}, y\epsilon^*_{0,1}] = 2d\epsilon^*_{0,2d}.$$

4. GROUP EXTENSIONS OF QUANTUM COMPLETE INTERSECTIONS

The behavior of Hochschild cohomology of quantum complete intersections is controlled by the choice of quantum coefficients as was shown in [5], [7], and [25]. In this section, we allow a finite group to act diagonally on the algebra and study the behavior of Hochschild cohomology of group extensions of quantum complete intersections. In this case, Hochschild cohomology is controlled by the quantum coefficients as well as the choice of diagonal group action. We restrict to the case of diagonal group actions for two main reasons: (1) in Section 4.2 this action allows for the decomposition of a complex into subcomplexes, a crucial step in the proof of the vector space structure, while other actions would not directly allow for such a decomposition and (2) allowing for a more general group action would require restrictions on our choice of quantum coefficients.

We utilize the notion of twisted tensor products developed in Section 3 and adapt a technique from [29] to compute the vector space structure in Section 4.2. As in [29], we restrict to the characteristic 0 case as it is required by the contracting homotopy used in the proof. We use techniques adapted from [7] to compute the cup product in Section 4.3. Using the alternative bracket description given by Negron and Witherspoon in [23] and [24], summarized in Section 2.3, we compute the bracket structure in Section 4.4. When the group is trivial, the formula in Section 4.4 yields previously unknown bracket computations on Hochschild cohomology of quantum complete intersections, extending the work of [5], [7], and [25].

For the remainder of the paper, we assume k is a field of characteristic 0.

4.1 Projective resolutions

As in Example 3.1.1, let $\mathbf{q} = \{q_{i,j}\}_{i,j\in\{1,2,...,n\}}$ such that $q_{i,j} \in k^*$ for all $i, j \in \{1, 2, ..., n\}$, $q_{j,i} = q_{i,j}^{-1}$, and $q_{i,i} = -1$ and

$$\Lambda_{\mathbf{q}}^{(2,2,...,2)} = k\langle x_1, x_2, ..., x_n | x_i x_j = -q_{i,j} x_j x_i, x_i^2 = 0 \text{ for all } i, j \in \{1, 2, ...n\}\rangle,$$

the class of quantum complete intersections of interest for this paper. Notice that this definition agrees with the definition of $\Lambda_{\mathbf{q}}^{(2,2,...,2)}$ found in [25] with the additional assumption that $\operatorname{char} k = 0$. For brevity of notation, we will denote $\Lambda_{\mathbf{q}}^{(2,2,...,2)}$ by $\Lambda_{\mathbf{q}}$ unless emphasis is needed.

Let G be a finite group which acts diagonally on the generating set $x_1, x_2, ..., x_n$ of $\Lambda_{\mathbf{q}}$. That is, if we denote the action of g on λ by ${}^g\lambda$ for $g \in G$ and $\lambda \in \Lambda_{\mathbf{q}}$, then for each $i \in \{1, 2, ..., n\}$ and $g \in G$, there exists a $\chi_{g,i} \in k$ such that ${}^g x_i = \chi_{g,i} x_i$. By extending linearly to all of $\Lambda_{\mathbf{q}}$, this action induces an action of G on $\Lambda_{\mathbf{q}}$ by automorphisms. Notice, for all $g \in G$ and $i \in \{1, 2, ..., n\}$, $\chi_{g,i}$ is necessarily a root of unity because G is a finite group. With these structures, we define the *group extension* of $\Lambda_{\mathbf{q}}$ by G.

Definition 4.1.1. *Define $\Lambda_{\mathbf{q}} \rtimes G$, the group extension of $\Lambda_{\mathbf{q}}$ by G (also called skew group algebra, crossed product, or smash product in other contexts), to be $\Lambda_{\mathbf{q}} \otimes kG$ as a vector space with multiplication determined by*

$$(\lambda_0 \otimes g_0)(\lambda_1 \otimes g_1) = \lambda_0({}^{g_0}\lambda_1) \otimes g_0 g_1$$

for $\lambda_0, \lambda_1 \in \Lambda_{\mathbf{q}}$ and $g_0, g_1 \in G$.

We are interested in the Hochschild cohomology of $\Lambda_{\mathbf{q}} \rtimes G$. It is well-known that Hochschild cohomology of group extensions has a particular form given by the isomorphism

$$\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}} \rtimes G) \cong (\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}, \Lambda_{\mathbf{q}} \rtimes G))^G, \tag{4.1.2}$$

where $(\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}, \Lambda_{\mathbf{q}} \rtimes G))^G$ is the G-invariant subspace of $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}, \Lambda_{\mathbf{q}} \rtimes G)$. A comparable result on Hochschild homology of group extensions was given by Lorenz in [17]. In [28], Ştefan proved the isomorphism (4.1.2) in the more general setting of Hopf Galois extensions. As a result of this isomorphism, to compute $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}} \rtimes G)$ we can start with a resolution of $\Lambda_{\mathbf{q}}$ and incorporate the group action later in the computation. With this result in mind, we now construct a resolution of $\Lambda_{\mathbf{q}}$ as a $\Lambda_{\mathbf{q}}^e$-module.

In Example 3.1.1, we saw that $\Lambda_{\mathbf{q}} \cong (...((R_{x_1} \otimes^{t_1} R_{x_2}) \otimes^{t_2} R_{x_3}) \otimes^{t_3} ...) \otimes^{t_{n-1}} R_{x_n}$ where $t_i^{<[j]|1>} = -q_{j,i+1}^{-1}$ for $i \in \{1, 2, ..., n-1\}$ and $j < i$ and $[j] = (0, ..., 0, 1, 0, ..., 0)$, the i-tuple with a 1 in the jth coordinate and 0 otherwise. By Example 2.1.6, each factor, R_x, has a projective resolution

$$\mathbb{K}_x : ... \xrightarrow{\delta_3} R_x^e \xrightarrow{\delta_2} R_x^e \xrightarrow{\delta_1} R_x^e \xrightarrow{\delta_0} R_x \to 0$$

with $\delta_i(a \otimes b) = (x \otimes 1 + (-1)^i 1 \otimes x)(a \otimes b)$ for all $i > 0$ and $\delta_0(a \otimes b) = ab$ for $a, b \in R_x$. Then by [6, Lemmas 4.3, 4.4, 4.5],

$$\mathbb{K} = \mathrm{Tot}(\mathbb{K}_{x_1} \otimes^{t_1} \mathbb{K}_{x_2} \otimes^{t_2} ... \otimes^{t_{n-1}} \mathbb{K}_{x_n})$$

is a projective resolution of $\Lambda_{\mathbf{q}}$ as a $(\Lambda_{\mathbf{q}})^e$-module. By definition of the total complex,

$$(\mathbb{K})_m = \bigoplus_{i_1+i_2+...+i_n=m} (\mathbb{K}_{x_1})_{i_1} \otimes^{t_1} (\mathbb{K}_{x_2})_{i_2} \otimes^{t_2} ... \otimes^{t_{n-1}} (\mathbb{K}_{x_n})_{i_n}$$

$$= \bigoplus_{i_1+i_2+...+i_n=m} R^e_{x_1}\langle i_1\rangle \otimes^{t_1} R^e_{x_2}\langle i_2\rangle \otimes^{t_2} ... \otimes^{t_{n-1}} R^e_{x_n}\langle i_n\rangle$$

where the boundary map is the induced map on $\mathbb{K}$,

$$\begin{aligned}
\delta_m((1\otimes 1)\otimes^{t_1} & ... \otimes^{t_{n-1}} (1\otimes 1)) \\
&= \sum_{k=1}^{n} (1\otimes 1)\otimes^{t_1} ... \otimes^{t_{k-1}} (x_k\otimes 1)\otimes^{t_k} ... \otimes^{t_{n-1}} (1\otimes 1) \\
&\quad + (-1)^{\sum_{l\le k} i_k}(1\otimes 1)\otimes^{t_1} ... \otimes^{t_{k-1}} (1\otimes x_k)\otimes^{t_k} ... \otimes^{t_{n-1}} (1\otimes 1).
\end{aligned}$$

By [6, Lemma 4.3], we have an isomorphism of graded $(R_{x_j}\otimes^{t_j} R_{x_{j+1}})^e$-modules, $R^e_{x_j}\langle i_j\rangle \otimes^{t_j} R^e_{x_{j+1}}\langle i_{j+1}\rangle \cong (R_{x_j}\otimes^{t_j} R_{x_{j+1}})^e\langle i_j, i_{j+1}\rangle$ given by

$$(r_1\otimes r_1')\otimes^{t_j}(r_2\otimes r_2') \mapsto t_j^{-<r_1'|r_2>-<i_j|r_2>-<r_1'|i_{j+1}>}(r_1\otimes^{t_j} r_2)\otimes(r_1'\otimes^{t_j} r_2')$$

for $r_1, r_1' \in R_{x_j}$ and $r_2, r_2' \in R_{x_{j+1}}$. By iterating this isomorphism and changing notation, we get an isomorphism of graded $(\Lambda_{\mathbf{q}})^e$-modules,

$$(\mathbb{K})_m \cong \bigoplus_{i_1+i_2+...+i_n=m} \Lambda_{\mathbf{q}}\epsilon_{i_1,i_2,...,i_n}\Lambda_{\mathbf{q}},$$

where $\epsilon_{i_1,i_2,...,i_n}$ is the copy of $1\otimes 1$ in homological degree i_j in x_j, defined by sending

$$x_1^{\alpha_1}\otimes x_1^{\alpha_1'}\otimes^{t_1} x_2^{\alpha_2}\otimes x_2^{\alpha_2'}\otimes^{t_2} ... \otimes^{t_{n-1}} x_n^{\alpha_n}\otimes x_n^{\alpha_n'} \in (\mathbb{K}_{x_1})_{i_1}\otimes^{t_1}(\mathbb{K}_{x_2})_{i_2}\otimes^{t_2} ... \otimes^{t_{n-1}}(\mathbb{K}_{x_n})_{i_n}$$

to

$$\prod_{k<l}(-q_{k,l})^{\alpha_l\alpha_k'+i_l\alpha_k'+i_k\alpha_l} x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}\epsilon_{i_1,i_2,...i_n}x_1^{\alpha_1'}x_2^{\alpha_2'}...x_n^{\alpha_n'} \in \Lambda_{\mathbf{q}}\epsilon_{i_1,i_2,...,i_n}\Lambda_{\mathbf{q}}.$$

Using the $\epsilon_{i_1,i_2,...,i_n}$ notation, we can better track homological degree of elements.

After making this identification, the boundary map becomes

$$\delta_m(\epsilon_{i_1,i_2,\ldots,i_n}) = \sum_{k=1}^{n}(\prod_{k<l} q_{k,l}^{i_k} x_l \epsilon_{i_1,\ldots i_{l-1},i_l-1,i_{l+1},\ldots,i_n} + (-1)^m \prod_{k>l} q_{l,k}^{i_k} \epsilon_{i_1,\ldots i_{l-1},i_l-1,i_{l+1},\ldots,i_n} x_l). \tag{4.1.3}$$

Because $i_1 + i_2 + \ldots + i_n = m$, we could have also written the boundary map

$$\begin{aligned}\delta_m(\epsilon_{i_1,i_2,\ldots,i_n}) = &\sum_{l=1}^{n}(\prod_{k<l} q_{k,l}^{i_k} x_l \epsilon_{i_1,\ldots i_{l-1},i_l-1,i_{l+1},\ldots,i_n} \\ &+ (-1)^{\sum_{k\leq l} i_k} \prod_{k>l} (-q_{l,k})^{i_k} \epsilon_{i_1,\ldots i_{l-1},i_l-1,i_{l+1},\ldots,i_n} x_l).\end{aligned}$$

The latter description will be more helpful in later calculations. As in the previous section, we will refer to this identified complex as $\mathbb{K}$ with boundary maps δ_m for the remainder of the paper.

Homomorphisms $\eta \in \mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}((\mathbb{K})_m, \Lambda_{\mathbf{q}} \rtimes G)$ are entirely determined by the images, $\eta(\epsilon_{i_1,i_2,\ldots,i_n})$, for $i_1, i_2, \ldots, i_n \in \mathbb{N}$ with $i_1 + i_2 + \ldots + i_n = m$. Denote by $\epsilon^*_{i_1,i_2,\ldots,i_n}$ the dual function in $\mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}(\mathbb{K}, \Lambda_{\mathbf{q}} \rtimes G)$ defined by $\epsilon^*_{i_1,i_2,\ldots,i_n}(\epsilon_{j_1,j_2,\ldots,j_n}) = \delta_{i_1,j_1}\delta_{i_2,j_2}\ldots\delta_{i_n,j_n} \otimes 1$ where δ_{i_m,j_m} is the Kronecker delta. In terms of these $\epsilon^*_{i_1,i_2,\ldots,i_n}$, any homomorphism $\eta \in \mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}((\mathbb{K})_m, \Lambda_{\mathbf{q}} \rtimes G)$ can be written

$$\eta = \sum_{g\in G} \sum_{i_1+i_2+\ldots+i_n=m} (\lambda^g_{i_1,i_2,\ldots,i_n} \otimes g)\epsilon^*_{i_1,i_2,\ldots,i_n}$$

where $\lambda^g_{i_1,i_2,\ldots,i_n} \in \Lambda_{\mathbf{q}}$ depends on $i_1, i_2, \ldots, i_n$ and g. Moreover, because

$$\mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}(\mathbb{K}, \Lambda_{\mathbf{q}} \rtimes G) \cong \bigoplus_{g\in G} \mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}(\mathbb{K}, \Lambda_{\mathbf{q}} \otimes g),$$

we can restrict to only looking at homomorphisms in $\mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}(\mathbb{K}, \Lambda_{\mathbf{q}} \otimes g)$ for each

$g \in G$.

Before we move further, we will need some additional notational conveniences, as in [29]. Elements of $\Lambda_{\mathbf{q}}$ are linear combinations of the monomials $cx_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}$ for $c \in k$ and $\alpha_i \in \{0,1\}$. Denote $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}$, using multi-index notation, for $\alpha \in \{0,1\}^n$. Similarly, for $\beta \in \mathbb{N}^n$, define $\epsilon_\beta = \epsilon_{\beta_1,\beta_2,...,\beta_n}$ and $\epsilon_\beta^* = \epsilon^*_{\beta_1,\beta_2,...,\beta_n}$. Denote $|\beta| = \beta_1 + \beta_2 + ... + \beta_n$. Lastly, as in the twisting maps in the construction of $\Lambda_{\mathbf{q}}$, denote $[i] = (0,...,0,1,0,...,0)$, the n-tuple with 1 in the ith coordinate and 0 otherwise.

Using this notation, we then describe the induced boundary map on $\mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}(\mathbb{K}, \Lambda_{\mathbf{q}} \otimes g)$ as

$$\begin{aligned}\delta_m^*((x^\alpha \otimes g)\epsilon_\beta^*)(\epsilon_\gamma) =&(x^\alpha \otimes g)\epsilon_\beta^*\delta_m(\epsilon_\gamma)\\ =&(x^\alpha \otimes g)\epsilon_\beta^*(\sum_{l=1}^n(\prod_{k<l} q_{k,l}^{\beta_k}x_l\epsilon_{\gamma-[l]} - (-1)^{\sum_{k\le l}\beta_k}\prod_{k>l}(-q_{l,k})^{\beta_k}\epsilon_{\gamma-[l]}x_l))\end{aligned}$$

for $\beta, \gamma \in \mathbb{N}^n$ with $|\beta| = m-1$ and $\alpha \in \{0,1\}^n$. Refer to (4.1.3) for the expression of δ_m. In order to get a non-zero value from $\epsilon_\beta^*(\epsilon_{\gamma-[l]})$, we need $\gamma - [l] = \beta$ or, equivalently $\gamma = \beta + [l]$. That is,

$$\begin{aligned}\delta_m^*((x^\alpha \otimes g)\epsilon_\beta^*) =& \sum_{l=1}^n(\prod_{k<l} q_{k,l}^{\beta_k}x_l(x^\alpha \otimes g)\epsilon^*_{\beta+[l]}\\ &- (-1)^{\sum_{k\le l}\beta_k}\prod_{k>l}(-q_{l,k})^{\beta_k}(x^\alpha \otimes g)x_l\epsilon^*_{\beta+[l]})\\ =& \sum_{l=1}^n(\prod_{k<l} q_{k,l}^{\beta_k}(-q_{k,l})^{-\alpha_k}(x^{\alpha+[l]} \otimes g)\epsilon^*_{\beta+[l]}\\ &- (-1)^{\sum_{k\le l}\beta_k}\prod_{k>l}(-q_{l,k})^{\beta_k}(-q_{l,k})^{-\alpha_k}\chi_{g,l}(x^{\alpha+[l]} \otimes g)\epsilon^*_{\beta+[l]})\end{aligned}$$

$$= \sum_{l=1}^{n} (\prod_{k<l} (-1)^{\beta_k} (-q_{k,l})^{\beta_k - \alpha_k} - (-1)^{\sum_{k \le l} \beta_k} \prod_{k>l} (-q_{l,k})^{\beta_k - \alpha_k} \chi_{g,l})$$
$$(x^{\alpha+[l]} \otimes g) \epsilon^*_{\beta+[l]}$$
$$= \sum_{l=1}^{n} (-1)^{\sum_{k<l} \beta_k} (\prod_{k<l} (-q_{k,l})^{\beta_k - \alpha_k} - (-1)^{\beta_l} \prod_{k>l} (-q_{l,k})^{\beta_k - \alpha_k} \chi_{g,l})$$
$$(x^{\alpha+[l]} \otimes g) \epsilon^*_{\beta+[l]}.$$

If $\alpha_l = 1$ or $(-1)^{\beta_l} \prod_{k \neq l} (-q_{k,l})^{\beta_k - \alpha_k} = \chi_{g,l}$, the coefficient of the $\epsilon^*_{\beta+[l]}$ term is 0 in $\delta^m((x^\alpha \otimes g)\epsilon^*_\beta)$. Accordingly, to simplify the differential expression, define

$$\Omega_g(\alpha, \beta, l) = \begin{cases} 0 & \text{if } \alpha_l = 1 \\ 0 & \text{if } (-1)^{\beta_l} \prod_{k \neq l} (-q_{k,l})^{\beta_k - \alpha_k} = \chi_{g,l} \\ (-1)^{\sum_{k<l} \beta_k} (\prod_{k<l} (-q_{k,l})^{\beta_k - \alpha_k} & \\ \quad -(-1)^{\beta_l} \prod_{k>l} (-q_{l,k})^{\alpha_k - \beta_k} \chi_{g,l}) & \text{otherwise} \end{cases}$$

for $l \in \{1, 2, ..., n\}$. Then, using this notation,

$$\delta^*_m((x^\alpha \otimes g)\epsilon^*_\beta) = \sum_{l=1}^{n} \Omega_g(\alpha, \beta, l)(x^{\alpha+[l]} \otimes g)\epsilon^*_{\beta+[l]} \tag{4.1.4}$$

for $\beta \in \mathbb{N}^n$ with $|\beta| = m - 1$ and $\alpha \in \{0, 1\}^n$.

The construction of Ω_g agrees with and should be compared to the Ω_g used in the proof of [29, Theorem 6.1].

4.2 Hochschild cohomology

To compute Hochschild cohomology, we break the complex $\mathrm{Hom}_{\Lambda^e_{\mathbf{q}}}(\mathbb{K}, \Lambda_{\mathbf{q}} \otimes g)$ into subcomplexes. The decomposition of the complex, given in this section, is an adaptation of a method of Wambst originally used in [29, Section 6] to compute the

Hochschild cohomology of quantum symmetric algebras and was adapted by Naidu, Shroff, and Witherspoon in [20] to compute the Hochschild cohomology of group extensions of quantum symmetric algebras. In this section, we adapt the method again to compute Hochschild cohomology of group extensions of quantum complete intersections.

For each $m \in \mathbb{N}, g \in G$, and $\gamma \in (\mathbb{N} \cup \{-1\})^n$, consider the set

$$K_{g,\gamma}^m = span_k\{(x^\alpha \otimes g)\epsilon_\beta^* | \alpha \in \{0,1\}^n, \beta \in \mathbb{N}^n, |\beta| = m, \text{ and } \beta - \alpha = \gamma\}$$

of all elements in homological degree m with fixed $\beta - \alpha$. Notice, for $\alpha \in \{0,1\}^n$, $\beta \in \mathbb{N}^n$ as in (4.1.4), the difference $\beta - \alpha \in (\{0,1\} \cup \mathbb{N})^n$ is unchanged by the boundary map δ_m^*. Therefore for fixed $g \in G$ and $\gamma \in (\mathbb{N} \cup \{-1\})^n$, $K_{g,\gamma} = \bigoplus_{m \in \mathbb{N}} K_{g,\gamma}^m$ is a subcomplex of $\mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}(\mathbb{K}, \Lambda_{\mathbf{q}} \otimes g)$.

We will show that for some $g \in G$ and $\gamma \in (\mathbb{N} \cup \{-1\})^n$, the subcomplexes $K_{g,\gamma}$ are acyclic and, for others, the differentials are 0. This condition on the subcomplexes is determined by the set

$$C_g = \{\gamma \in (\mathbb{N} \cup \{-1\})^n | \forall\, l, \gamma_l = -1 \text{ or } (-1)^{\gamma_l} \prod_{k \neq l} (-q_{k,l})^{\gamma_k} = \chi_{g,l}\},$$

a set of γ which satisfy a relation between the quantum coefficients and diagonal action by g.

Lemma 4.2.1. *If $g \in G$ and $\gamma \in (\mathbb{N} \cup \{-1\})^n - C_g$, then $K_{g,\gamma}$ is acyclic.*

Proof. Let $g \in G$ and $\gamma \in (\mathbb{N} \cup \{-1\})^n - C_g$. To show this result, we adapt the proofs of [29, Theorem 6.1] and [20, Lemma 4.6] to this setting. That is, we will construct a contracting homotopy for these subcomplexes but first we will need a bit more notation. Let

$$||\gamma||_g = \#\{l \in \{1,2,...,n\} | \gamma_l \neq -1 \text{ and } (-1)^{\gamma_l} \prod_{k \neq l} (-q_{k,l})^{\gamma_k} \neq \chi_{g,l}\}$$

count the number of indices not satisfying the conditions of C_g. Because we assumed $\gamma \in (\mathbb{N} \cup \{-1\})^n - C_g$, some index must not satisfy the conditions of C_g and thus $||\gamma||_g \neq 0$.

Now, let $m \in \mathbb{N}$ be fixed, and define $h_m : K^m_{g,\gamma} \to K^{m-1}_{g,\gamma}$ by

$$h_m((x^\alpha \otimes g)\epsilon^*_\beta) = \frac{1}{||\beta - \alpha||_g} \sum_{l=1}^{n} \omega_g(\alpha, \beta, l)(x^{\alpha-[l]} \otimes g)\epsilon_{\beta-[l]}$$

for $\beta \in \mathbb{N}^n$ with $|\beta| = m$, $\alpha \in \{0,1\}^n$ such that $\beta - \alpha = \gamma$ and

$$\omega_g(\alpha, \beta, l) = \begin{cases} 0 & \text{if } \alpha_l = 0 \text{ or } \beta_l = 0 \\ 0 & \text{if } \prod_{k \neq l} (-q_{k,l})^{\beta_k - \alpha_k} (-1)^{\beta_l} = -\chi_{g,l} \cdot \\ \Omega_g(\alpha - [l], \beta - [l], l)^{-1} & \text{otherwise.} \end{cases}$$

Let us check first that $\omega_g(\alpha, \beta, l)$ is well-defined. That is, that $\Omega_g(\alpha - [l], \beta - [l], l) \neq 0$ for any $\alpha \in \{0,1\}^n$, $\beta \in \mathbb{N}^n$, and $l \in \{1,2,...,n\}$ that do not satisfy the first two cases of ω_g. Note that $\Omega_g(\alpha - [l], \beta - [l], l) = 0$ if and only if $(\alpha - [l])_l = 1$ or $(-1)^{(\beta-[l])_l} \prod_{k \neq l} (-q_{l,k})^{\beta_k - \alpha_k} = \chi_{g,l}$ by definition of Ω_g. Therefore we need only check that we do not consider $\Omega_g(\alpha - [l], \beta - [l], l)^{-1}$ when $(\alpha_l - [l])_l = 1$ or $(-1)^{(\beta-[l])_l} \prod_{k \neq l} (-q_{l,k})^{\beta_k - \alpha_k} = \chi_{g,l}$. Because $\alpha_l = 0$ or 1, $(\alpha - [l])_l \neq 1$ so this condition is never satisfied. For the second condition, if α and β satisfy $(-1)^{(\beta-[l])_l} \prod_{k \neq l} (-q_{l,k})^{\beta_k - \alpha_k} = \chi_{g,l}$, then $(-1)^{\beta_l} \prod_{k \neq l} (-q_{l,k})^{\beta_k - \alpha_k} = -\chi_{g,l}$. But this α, β satisfies the second case of ω_g and therefore $\omega_g(\alpha, \beta, l) = 0$ and we do not consider $\Omega_g(\alpha-[l],\beta-[l],l)^{-1}$.

Remark 4.2.2. *Because we divide by* $||\beta-\alpha||_g$ *in* h_m*, we require that* $\operatorname{char} k = 0$.

I claim that h is a contracting homotopy. That is, $h_{m+1}\delta^{m+1}+\delta^m h_m = \mathbb{1}\big|_{K^m_{g,\gamma}}$. Let $\alpha\in\{0,1\}^n$, $\beta\in\mathbb{N}^n$ with $|\beta|=m$ such that $\beta-\alpha=\gamma$, then

$$\begin{aligned}
(h_{m+1}\delta^{m+1}+&\delta^m h_m)((x^\alpha\otimes g)\epsilon^*_\beta)\\
&=\frac{1}{||\beta-\alpha||_g}\sum_{j=1}^n\sum_{l=1}^n(\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],j)\\
&\qquad\qquad+\Omega_g(\alpha-[j],\beta-[j],l)\omega_g(\alpha,\beta,j))(x^{\alpha+[l]-[j]}\otimes g)\epsilon^*_{\beta+[l]-[j]}\\
&=\frac{1}{||\beta-\alpha||_g}\sum_{l=1}^n(\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)\\
&\qquad\qquad+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l))(x^\alpha\otimes g)\epsilon^*_\beta\\
&\quad+\frac{1}{||\beta-\alpha||_g}\sum_{j\neq l}(\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],j)\\
&\qquad\qquad+\Omega_g(\alpha-[j],\beta-[j],l)\omega_g(\alpha,\beta,j))(x^{\alpha+[l]-[j]}\otimes g)\epsilon^*_{\beta+[l]-[j]}.
\end{aligned}$$

We need to show that

$$\begin{aligned}
\frac{1}{||\beta-\alpha||_g}\sum_{l=1}^n\Big(&\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)\\
&+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l)\Big)(x^\alpha\otimes g)\epsilon^*_\beta\\
&=(x^\alpha\otimes g)\epsilon^*_\beta
\end{aligned}$$

and

$$
\begin{aligned}
\frac{1}{||\beta-\alpha||_g}\sum_{j\neq l}\Big(&\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],j)\\
&+\Omega_g(\alpha-[j],\beta-[j],l)\omega_g(\alpha,\beta,j)\Big)(x^{\alpha+[l]-[j]}\otimes g)\epsilon^*_{\beta+[l]-[j]}\\
&=0.
\end{aligned}
$$

Let us start by showing the second condition. Assume $j\neq l$. Notice, because $(\alpha\pm[r])_s=\alpha_s$ and $(\beta\pm[r])_s=\beta_s$ for $r\neq s$, $\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],j)$ and $\Omega_g(\alpha-[j],\beta-[j],l)\omega_g(\alpha,\beta,j)$ are both either simultaneously 0 or nonzero. If they are 0, we have nothing to prove. Therefore assume these terms are nonzero. If $j<l$, then

$$
\begin{aligned}
&\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],j)+\Omega_g(\alpha-[j],\beta-[j],l)\omega_g(\alpha,\beta,j)\\
&=(-1)^{\sum_{k<l}\beta_k}\Big(\prod_{k<l}(-q_{k,l})^{\beta_k-\alpha_k}-(-1)^{\beta_l}\prod_{k>l}(-q_{l,k})^{-\beta_k+\alpha_k}\chi_{g,l}\Big)\\
&\qquad(-1)^{\sum_{k<j}\beta_k}\Big(\prod_{k<j}(-q_{k,j})^{\beta_k-\alpha_k}+(-1)^{\beta_j}\prod_{k>j}(-q_{j,k})^{-\beta_k+\alpha_k}\chi_{g,j}\Big)^{-1}\\
&\quad+(-1)^{\sum_{k<l}\beta_k}\Big(-\prod_{k<l}(-q_{k,l})^{\beta_k-\alpha_k}+(-1)^{\beta_l}\prod_{k>l}(-q_{l,k})^{-\beta_k+\alpha_k}\chi_{g,l}\Big)\\
&\qquad(-1)^{\sum_{k<j}\beta_k}\Big(\prod_{k<j}(-q_{k,j})^{\beta_k-\alpha_k}+(-1)^{\beta_j}\prod_{k>j}(-q_{j,k})^{-\beta_k+\alpha_k}\chi_{g,j}\Big)^{-1}\\
&=0.
\end{aligned}
$$

If $j>l$, then

$$
\begin{aligned}
&\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],j)+\Omega_g(\alpha-[j],\beta-[j],l)\omega_g(\alpha,\beta,j)\\
&=(-1)^{\sum_{k<l}\beta_k}\Big(\prod_{k<l}(-q_{k,l})^{\beta_k-\alpha_k}-(-1)^{\beta_l}\prod_{k>l}(-q_{l,k})^{-\beta_k+\alpha_k}\chi_{g,l}\Big)
\end{aligned}
$$

$$(-1)^{\sum_{k<j}\beta_k}(-\prod_{k<j}(-q_{k,j})^{\beta_k-\alpha_k}-(-1)^{\beta_j}\prod_{k>j}(-q_{j,k})^{-\beta_k+\alpha_k}\chi_{g,j})^{-1}$$

$$+(-1)^{\sum_{k<l}\beta_k}(\prod_{k<l}(-q_{k,l})^{\beta_k-\alpha_k}-(-1)^{\beta_l}\prod_{k>l}(-q_{l,k})^{-\beta_k+\alpha_k}\chi_{g,l})$$

$$(-1)^{\sum_{k<j}\beta_k}(\prod_{k<j}(-q_{k,j})^{\beta_k-\alpha_k}+(-1)^{\beta_j}\prod_{k>j}(-q_{j,k})^{-\beta_k+\alpha_k}\chi_{g,j})^{-1}$$

$$=0.$$

Thus for every $j, l \in \{1, 2, ..., n\}$ with $j \neq l$,

$$\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],j)+\Omega_g(\alpha-[j],\beta-[j],l)\omega_g(\alpha,\beta,j)=0,$$

giving us our desired expression.

Now we will show

$$\frac{1}{||\beta-\alpha||_g}\sum_{l=1}^{n}(\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)$$
$$+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l))(x^\alpha\otimes g)\epsilon^*_\beta$$
$$=(x^\alpha\otimes g)\epsilon^*_\beta$$

by showing that

$$\sum_{l=1}^{n}(\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l))=||\beta-\alpha||_g.$$

Recall, we assumed $\gamma \in (\mathbb{N} \cup \{-1\})^n - C_g$. Thus for some $l \in \{1, 2, ..., n\}$ and any $\beta \in \mathbb{N}^n$ and $\alpha \in \{0, 1\}^n$ with $\beta - \alpha = \gamma$, we have $\beta_l - \alpha_l \neq -1$ and $(-1)^{\beta_l-\alpha_l}\prod_{k\neq l}(-q_{k,l})^{\beta_k-\alpha_k} \neq \chi_{g,l}$. If $\beta_l - \alpha_l = -1$ or $(-1)^{\beta_l-\alpha_l}\prod_{k\neq l}(-q_{k,l})^{\beta_k-\alpha_k} =$

$\chi_{g,l}$, then $\Omega_g(\alpha,\beta,l)=0$ and $\omega_g(\alpha,\beta,l)=0$ by definition, making

$$\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l)=0.$$

Therefore these indices do not contribute to either side of our desired expression. If $\beta_l-\alpha_l\neq -1$ and $(-1)^{\beta_l-\alpha_l}\prod_{k\neq l}(-q_{k,l})^{\beta_k-\alpha_k}\neq \chi_{g,l}$, then we consider two cases.

Case 1: Assume $\alpha_l=1$, then $\Omega_g(\alpha,\beta,l)=0$ by definition of Ω_g and our other assumptions become $(-1)^{\beta_l}\prod_{k\neq l}(-q_{k,l})^{\beta_k-\alpha_k}\neq -\chi_{g,l}$ and $\beta_l\neq 0$. Then, by definition of ω_g, because $(-1)^{\beta_l}\prod_{k\neq l}(-q_{k,l})^{\beta_k-\alpha_k}\neq -\chi_{g,l}$, $\beta_l\neq 0$, and $\alpha_l\neq 0$, $\omega_g(\alpha,\beta,l)=\Omega_g(\alpha-[l],\beta-[l],l)^{-1}$. Therefore

$$\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l)=1.$$

Case 2: Assume $\alpha_l=0$, then $\Omega_g(\alpha-[l],\beta-[l],l)=0$ by definition of Ω_g and our other assumption becomes $(-1)^{\beta_l}\prod_{k\neq l}(-q_{k,l})^{\beta_k-\alpha_k}\neq \chi_{g,l}$. Therefore, by definition of ω_g, because $\alpha_l+1\neq 0$, $\beta_l+1\neq 0$, and $(-1)^{\beta_l+1}\prod_{k\neq l}(-q_{k,l})^{\beta_k-\alpha_k}\neq -\chi_{g,l}$, $\omega_g(\alpha+[l],\beta+[l],l)=\Omega_g(\alpha,\beta,l)^{-1}$. Thus

$$\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l)=1$$

in this case as well.

We have seen that

$$\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l)$$

detects the $l\in\{1,2,...,n\}$ for which $\beta_l-\alpha_l\neq -1$ and $(-1)^{\beta_l-\alpha_l}\prod_{k\neq l}(-q_{k,l})^{\beta_k-\alpha_k}\neq$

$\chi_{g,l}$, giving us our final result,

$$\sum_{l=1}^{n}(\Omega_g(\alpha,\beta,l)\omega_g(\alpha+[l],\beta+[l],l)+\Omega_g(\alpha-[l],\beta-[l],l)\omega_g(\alpha,\beta,l))=||\beta-\alpha||_g.$$

$\square$

Therefore the subcomplexes $K_{g,\gamma}$ for a fixed $g \in G$ and $\gamma \in (\mathbb{N} \cup \{-1\})^n - C_g$ are acyclic and do not contribute to the cohomology. We now are now ready to compute $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}, \Lambda_{\mathbf{q}} \otimes g)$, focusing on those $K_{g,\gamma}$ for which $g \in G$ and $\gamma \in C_g$.

Theorem 4.2.3. *For each g in G,*

$$\mathrm{HH}^m(\Lambda_{\mathbf{q}}, \Lambda_{\mathbf{q}} \otimes g) \cong \bigoplus_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=m}} \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ \beta-\alpha\in C_g}} span_k\{(x^\alpha \otimes g)\epsilon_\beta^*\}.$$

Therefore $\mathrm{HH}^m(\Lambda_{\mathbf{q}} \rtimes G)$ *is the G-invariant subspace of*

$$\bigoplus_{g\in G} \bigoplus_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=m}} \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ \beta-\alpha\in C_g}} span_k\{(x^\alpha \otimes g)\epsilon_\beta^*\}.$$

Proof. We start by showing that for $\gamma \in C_g$, $\delta^m\big|_{K^m_{g,\gamma}} = 0$. If $\gamma \in C_g$, then for all $l \in \{1,2,...,n\}$, $\gamma_l = -1$ or $(-1)^{\gamma_l}\prod_{k\neq l}(-q_{k,l})^{\gamma_k} = \chi_{g,l}$.

Case 1: If $\gamma_l = -1$, then $\alpha_l = 1$ and $\beta_l = 0$, making $\Omega_g(\alpha,\beta,l) = 0$. Therefore $\delta^*((x^\alpha \otimes g)\epsilon_\beta^*) = 0$ by equation (4.1.4).

Case 2: If $(-1)^{\gamma_l}\prod_{k\neq l}(-q_{k,l})^{\gamma_k} = \chi_{g,l}$, we break into two cases: $\alpha_l = 0$ or $\alpha_l = 1$. Assume $\alpha_l = 1$, then $\Omega_g(\alpha,\beta,l) = 0$ and therefore $\delta^*((x^\alpha \otimes g)\epsilon_\beta^*) = 0$. In this case, the condition that $(-1)^{\gamma_l}\prod_{k\neq l}(-q_{k,l})^{\gamma_k} = \chi_{g,l}$ is unnecessary. Assume $\alpha_l = 0$, then $(-1)^{\beta_l}\prod_{k\neq l}(-q_{k,l})^{\gamma_k} = (-1)^{\gamma_l}\prod_{k\neq l}(-q_{k,l})^{\gamma_k} = \chi_{g,l}$ and therefore $\Omega_g(\alpha,\beta,l) = 0$ in this case as well, making $\delta^*((x^\alpha \otimes g)\epsilon_\beta^*) = 0$.

Thus for $\gamma \in C_g$, $\delta^m\big|_{K^m_{g,\gamma}} = 0$. Then the cohomology of subcomplexes $K^m_{g,\gamma}$ for $\gamma \in C_g$ is $span_k\{(x^\alpha \otimes g)\epsilon^*_\beta | \beta - \alpha = \gamma\}$. By Lemma 4.2.1, the subcomplexes $K^m_{g,\gamma}$ for $\gamma \notin C_g$ are acyclic and thus do not contribute to cohomology. Therefore

$$\mathrm{HH}^m(\Lambda_{\mathbf{q}}, \Lambda_{\mathbf{q}} \otimes g) \cong \bigoplus_{\substack{\beta \in \mathbb{N}^n \\ |\beta| = m}} \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ \beta - \alpha \in C_g}} span_k\{(x^\alpha \otimes g)\epsilon^*_\beta\}.$$

$\square$

Remark 4.2.4. *In order to use Theorem 4.2.3, we will need to specify the G-action on $\mathrm{HH}^m(\Lambda_{\mathbf{q}} \rtimes G)$. The group action is the induced action on $\mathrm{Hom}_{\Lambda^e_{\mathbf{q}}}(\mathbb{K}, \Lambda_{\mathbf{q}} \rtimes G)$. That is, for $f \in \mathrm{Hom}_{\Lambda^e_{\mathbf{q}}}(\mathbb{K}, \Lambda_{\mathbf{q}} \rtimes G)$ and $g \in G$, ${}^g f(\lambda) = (1 \otimes g) f({}^{g^{-1}}\lambda)(1 \otimes g^{-1})$ for $\lambda \in \mathbb{K}$.*

4.3 Cup product

We now have a complete description of the vector space structure of $\mathrm{HH}^m(\Lambda_{\mathbf{q}} \rtimes G)$. To understand the algebra structure, we will need a bit more development.

Recall, by Section 2.3, we can view the cup product, $\smile$ on $\mathbb{B}$ as a composition, for $f \in \mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}(\mathbb{B}_l, A)$ and $g \in \mathrm{Hom}_{(\Lambda_{\mathbf{q}})^e}(\mathbb{B}_m, A)$

$$f \smile g : \mathbb{B}(A) \xrightarrow{\Delta_{\mathbb{B}}} \mathbb{B} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{B} \xrightarrow{f \otimes_{\Lambda_{\mathbf{q}}} g} \Lambda_{\mathbf{q}} \otimes_{\Lambda_{\mathbf{q}}} \Lambda_{\mathbf{q}} \xrightarrow{\mu} \Lambda_{\mathbf{q}}$$

where μ is the multiplication map. To make use of this description, we will define yet another resolution of $\Lambda_{\mathbf{q}}$, $\mathbb{P}$, which is a subcomplex of $\mathbb{B}$ such that the diagonal $\Delta_{\mathbb{B}}$ induces a comultiplication on $\mathbb{P}$. That is, $\Delta_{\mathbb{B}}(\mathbb{P}) \subset \mathbb{P} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{P}$.

We begin by defining an n-dimensional analog of the f^m_i defined in [5] and [7]. Let $f_{(0,0,\ldots,0,0)} = 1$, $f_{[l]} = x_l$ for all $l \in \{1, 2, \ldots, n\}$, and $f_\beta = 0$ for any $\beta \in \mathbb{Z}^n$ with

$\beta_l < 0$ for some $l \in \{1, 2, ..., n\}$. Then for $\beta \in \mathbb{N}^n$, define f_β iteratively by the relation

$$f_\beta = \sum_{l=1}^{n} (\prod_{k>l} q_{l,k}^{\beta_k}) f_{\beta-[l]} \otimes x_l.$$

That is, for $\beta \in \mathbb{N}^n$, f_β is a linear combination of all tensor products of length $|\beta|$ with β_i x_i's for each $i \in \{1, 2, ..., n\}$ and coefficient $\mathbf{q}^\alpha$ determined by the commuting coefficients that appear when moving the generators past each other starting from the configuration with generators in increasing index order. For example,

$$f_{(0,2,1,0,...,0)} = x_2 \otimes x_2 \otimes x_3 + q_{2,3} x_2 \otimes x_3 \otimes x_2 + q_{2,3}^2 x_3 \otimes x_2 \otimes x_2.$$

As in [7], let

$$\tilde{f}_\beta = 1 \otimes f_\beta \otimes 1.$$

Consider the $\Lambda_{\mathbf{q}}^e$-module resolution

$$\mathbb{P} : ... \xrightarrow{d_{\mathbb{P}}^{m+1}} \bigoplus_{\substack{\beta \in \mathbb{N}^n \\ |\beta|=m}} \Lambda_{\mathbf{q}} \otimes f_\beta \otimes \Lambda_{\mathbf{q}} \xrightarrow{d_{\mathbb{P}}^{m}} \bigoplus_{\substack{\beta \in \mathbb{N}^n \\ |\beta|=m-1}} \Lambda_{\mathbf{q}} \otimes f_\beta \otimes \Lambda_{\mathbf{q}} \xrightarrow{d_{\mathbb{P}}^{m-1}} ...$$

where

$$d_{\mathbb{P}}^m(\tilde{f}_\beta) = \sum_{j=1}^{n} (\prod_{l<j} q_{l,j}^{i_l} x_j \tilde{f}_{\beta-[j]} + (-1)^m \prod_{l>j} q_{j,l}^{i_l} \tilde{f}_{\beta-[j]} x_j).$$

Remark 4.3.1. *The motivation for developing f_β is that we have, as we saw in Section 3.1.1, an isomorphism*

$$\bigoplus_{\substack{\beta \in \mathbb{N}^n \\ |\beta|=m}} \Lambda_{\mathbf{q}} \otimes f_\beta \otimes \Lambda_{\mathbf{q}} \cong \bigoplus_{\substack{\beta \in \mathbb{N}^n \\ |\beta|=m}} \Lambda_{\mathbf{q}} \epsilon_\beta \Lambda_{\mathbf{q}}$$

defined by sending $\tilde{f}_\beta$ to ϵ_β. This isomorphism preserves the differential on the re-

spective complexes, making $\mathbb{P} \cong \mathbb{K}$. See Section 4.1 for the definition of $\mathbb{K}$. We introduce both resolutions because the resolution $\mathbb{K}$ is computationally useful while the resolution $\mathbb{P}$ is a subcomplex of the bar resolution, inheriting the diagonal on $\mathbb{B}$ (which we will prove next).

For each $m \in \mathbb{N}$, $(\mathbb{P})_m \subset (\mathbb{B}(\Lambda_{\mathbf{q}}))_m$. I claim, as was shown in [7] for $n = 2$, that $\mathbb{P}$ is a subcomplex of $\mathbb{B}$. To see this, we will show that the differential on the bar resolution, d, induces the map $d_{\mathbb{P}}$ defined above.

Fix $\beta \in \mathbb{N}$ with $|\beta| = m$. On the bar resolution, $d(\lambda_1 \otimes ... \otimes \lambda_m)$ is a sum of terms formed by removing the ith tensor symbol, multiplying the $(i-1)$st and the ith term, and then multiplying the result by $(-1)^{i-1}$ for $i \in \{1, 2, ...m-1\}$. See Definition 2.2.1 for the definition of the differential on the bar resolution. Thus $d(\tilde{f}_\beta)$ contains all of the terms in $\sum_{j=1}^{n}(\prod_{l<j} q_{l,j}^{i_l} x_j \tilde{f}_{\beta-[j]} + (-1)^m \prod_{l>j} q_{j,l}^{i_l} \tilde{f}_{\beta-[j]} x_j)$ which are the result of removing the first and last tensor symbol respectively. Thus what needs to be shown is that the terms that come from removing the ith tensor symbol, $i \in \{2, 3, ..., m-2\}$, do not appear in $d(\tilde{f}_\beta)$.

For $\alpha \in \{1, 2, ..., n\}^m$ define $x_\alpha = x_{\alpha_1} \otimes x_{\alpha_2} \otimes ... \otimes x_{\alpha_m}$. Then we can write

$$f_\beta = \sum_{\substack{\alpha \in \{1,2,...,n\}^m \\ \#\{l | \alpha_l = i\} = \beta_i \ \forall i \in \{1,2,...,n\}}} \mathbf{q}^\alpha x_\alpha$$

where $\mathbf{q}^\alpha$ is determined by the commuting coefficients that appear when moving the generators past each other starting from the configuration with generators in increasing order according to index.

Consider a single term $\mathbf{q}^\alpha \otimes x_\alpha \otimes 1$ in $\tilde{f}_\beta$. If $\alpha_i = \alpha_{i+1}$ for any $i \in \{2, 3, ..., m-2\}$ then, because $x_j^2 = 0$ for all $j \in \{1, 2, ..., n\}$, the term coming from removing the ith tensor symbol is 0 as desired. If $\alpha_i \neq \alpha_{i+1}$ for any $i \in \{2, 3, ..., m-2\}$ then, without loss of generality, assume $\alpha_i < \alpha_{i+1}$. By definition of f_β, $\tilde{f}_\beta$ contains the term $\mathbf{q}^\alpha q_{\alpha_i,\alpha_{i+1}} \otimes x_{\alpha_1} \otimes x_{\alpha_2} \otimes ... \otimes x_{\alpha_{i-1}} \otimes x_{\alpha_{i+1}} \otimes x_{\alpha_i} \otimes x_{\alpha_{i+2}} \otimes ... \otimes x_{\alpha_m} \otimes 1$. When we remove the ith tensor symbol, multiply the $(i-1)$st and ith terms, and multiply by $(-1)^{i-1}$ in $\mathbf{q}^\alpha \otimes x_\alpha \otimes 1$, we get

$$\begin{aligned}
&(-1)^{i+1}\mathbf{q}^\alpha \otimes x_{\alpha_1} \otimes ... \otimes x_{\alpha_{i-1}} \otimes x_{\alpha_i}x_{\alpha_{i+1}} \otimes x_{\alpha_{i+2}} \otimes ... \otimes x_{\alpha_m} \otimes 1 \\
&\qquad = (-1)^{i-1}\mathbf{q}^\alpha(-q_{\alpha_i,\alpha_{i+1}}) \otimes x_{\alpha_1} \otimes ... \otimes x_{\alpha_{i-1}} \otimes x_{\alpha_{i+1}}x_{\alpha_i} \otimes x_{\alpha_{i+2}} \otimes ... \otimes x_{\alpha_m} \otimes 1 \\
&\qquad = (-1)^{i}\mathbf{q}^\alpha q_{\alpha_i,\alpha_{i+1}} \otimes x_{\alpha_1} \otimes ... \otimes x_{\alpha_{i-1}} \otimes x_{\alpha_{i+1}}x_{\alpha_i} \otimes x_{\alpha_{i+2}} \otimes ... \otimes x_{\alpha_m} \otimes 1.
\end{aligned}$$

When we remove the ith tensor and multiply by $(-1)^{i-1}$ in $\mathbf{q}^\alpha q_{\alpha_i,\alpha_{i+1}} \otimes x_{\alpha_1} \otimes x_{\alpha_2} \otimes ... \otimes x_{\alpha_{i-1}} \otimes x_{\alpha_{i+1}} \otimes x_{\alpha_i} \otimes x_{\alpha_{i+2}} \otimes ... \otimes x_{\alpha_m} \otimes 1$, we get

$$(-1)^{i-1}q^\alpha \mathbf{q}_{\alpha_i,\alpha_{i+1}} \otimes x_{\alpha_1} \otimes x_{\alpha_2} \otimes ... \otimes x_{\alpha_{i-1}} \otimes x_{\alpha_{i+1}}x_{\alpha_i} \otimes x_{\alpha_{i+2}} \otimes ... \otimes x_{\alpha_m} \otimes 1.$$

Thus in $d(\tilde{f}_\beta)$, these two terms cancel each other as desired. Notice no other terms in $\tilde{f}_\beta$ contribute to the $1 \otimes x_{\alpha_1} \otimes x_{\alpha_2} \otimes ... \otimes x_{\alpha_{i-1}} \otimes x_{\alpha_{i+1}}x_{\alpha_i} \otimes x_{\alpha_{i+2}} \otimes ... \otimes x_{\alpha_m} \otimes 1$ term in $d(\tilde{f}_\beta)$. Therefore $d(\tilde{f}_\beta) = \sum_{j=1}^n (\prod_{l<j} q_{l,j}^{i_l} x_j \tilde{f}_{\beta-[j]} + (-1)^m \prod_{l>j} q_{j,l}^{i_l} \tilde{f}_{\beta-[j]} x_j) = d_{\mathbb{P}}(\tilde{f}_\beta)$ and $\mathbb{P}$ is a subcomplex of $\mathbb{B}$.

Now, to see that the diagonal on $\mathbb{B}(\Lambda_{\mathbf{q}})$ restricts to a diagonal on $\mathbb{P}$, recall

$$\Delta_{\mathbb{B}}(\lambda_0 \otimes ... \otimes \lambda_{m+1}) = \sum_{i=0}^{m} (\lambda_0 \otimes ... \otimes \lambda_i \otimes 1) \otimes_{\Lambda_{\mathbf{q}}} (1 \otimes \lambda_{i+1} \otimes ... \otimes \lambda_{m+1})$$

for $\lambda_0, ..., \lambda_{m+1} \in \Lambda_{\mathbf{q}}$. For $\beta \in \mathbb{N}^m$ and $0 \leq t \leq |\beta|$ fixed, we have

$$f_\beta = \sum_{\substack{\alpha+\gamma=\beta \\ \alpha,\gamma \in \mathbb{N}^m \text{ and } |\alpha|=t}} \Big(\prod_{\substack{1 \leq l \leq n \\ k<l}} q_{k,l}^{\gamma_k \alpha_l} \Big) f_\alpha \otimes f_\gamma.$$

Therefore define,

$$\Delta_{\mathbb{P}}(\tilde{f}_\beta) = \sum_{\alpha+\gamma=\beta} (\prod_{\substack{1\leq l\leq n\\ k<l}} q_{k,l}^{\gamma_k\alpha_l})\tilde{f}_\alpha \otimes_{\Lambda_{\mathbf{q}}} \tilde{f}_\gamma.$$

Then the diagonal map, $\Delta_{\mathbb{P}}$, is induced by $\Delta_{\mathbb{B}}$. Moreover, by Remark 4.3.1, $\mathbb{K} \cong \mathbb{P}$. Therefore, using this isomorphism, we have our desired diagonal map

$$\Delta_{\mathbb{K}}(\epsilon_\beta) = \sum_{\alpha+\gamma=\beta} \prod_{\substack{1\leq l\leq n\\ k<l}} q_{k,l}^{\gamma_k\alpha_l} \epsilon_\alpha \otimes_{\Lambda_{\mathbf{q}}} \epsilon_\gamma$$

and we are finally ready to describe the cup product.

By Theorem 4.2.3, the vector space, $\mathrm{HH}^m(\Lambda_{\mathbf{q}}, \Lambda_{\mathbf{q}} \rtimes G)$, has a basis given by all $(x^\alpha \otimes g)\epsilon_\beta^*$ such that $\beta - \alpha \in C_g$. Then, the cup product, defined on these basis elements is given by the following formula.

Theorem 4.3.2. *If $\alpha, \gamma \in \{0,1\}^n$, $\beta, \kappa \in \mathbb{N}^n$, and $g, h \in G$, then*

$$(x^\alpha \otimes g)\epsilon_\beta^* \smile (x^\gamma \otimes h)\epsilon_\kappa^* = \prod_{l=1}^{n} \chi_{g,l}^{\gamma_l} \prod_{k<l} q_{k,l}^{\kappa_k\beta_l - \gamma_k\alpha_l}(-1)^{-\gamma_k\alpha_l}(x^{\alpha+\gamma} \otimes gh)\epsilon_{\beta+\kappa}^*.$$

Proof. Let $\alpha, \gamma \in \{0,1\}^n$, $\beta, \kappa, \rho \in \mathbb{N}^n$, and $g, h \in G$. Because we can identify $\mathbb{K}$ as a subcomplex of $\mathbb{B}(\Lambda_{\mathbf{q}})$, we can compute the cup product, $(x^\alpha \otimes g)\epsilon_\beta^* \smile (x^\gamma \otimes h)\epsilon_\kappa^*$ as the composition

$$\mathbb{K} \xrightarrow{\Delta_{\mathbb{K}}} \mathbb{K} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{K} \xrightarrow{(x^\alpha\otimes g)\epsilon_\beta^* \otimes (x^\gamma \otimes h)\epsilon_\kappa^*} (\Lambda_{\mathbf{q}} \rtimes G) \otimes_{\Lambda_{\mathbf{q}}} (\Lambda_{\mathbf{q}} \rtimes G) \xrightarrow{\mu} \Lambda_{\mathbf{q}} \rtimes G.$$

Then

$$\begin{aligned}
&(x^\alpha \otimes g)\epsilon_\beta^* \smile (x^\gamma \otimes h)\epsilon_\kappa^*(\epsilon_\rho)\\
&\qquad = \mu((x^\alpha \otimes g)\epsilon_\beta^* \otimes (x^\gamma \otimes h)\epsilon_\kappa^*(\Delta_{\mathbb{K}}(\epsilon_\rho)))
\end{aligned}$$

$$= \mu((x^\alpha \otimes g)\epsilon_\beta^* \otimes (x^\gamma \otimes h)\epsilon_\kappa^*(\sum_{\rho'+\rho''=\rho} \prod_{\substack{1\le l\le n\\ k<l}} q_{k,l}^{\rho''_k\rho'_l} \epsilon_{\rho'} \otimes_{\Lambda_{\mathbf{q}}} \epsilon_{\rho''})).$$

Notice for the last equality, we need $\rho' = \beta, \rho'' = \kappa$, and $\rho = \kappa + \beta$.

$$\begin{aligned}
(x^\alpha \otimes g)\epsilon_\beta^* \smile (x^\gamma \otimes h)\epsilon_\kappa^*(\epsilon_\rho) &= \mu(\prod_{\substack{1\le l\le n\\ k<l}} q_{k,l}^{\kappa_k\beta_l} (x^\alpha \otimes g) \otimes_{\Lambda_{\mathbf{q}}} (x^\gamma \otimes h)) \\
&= \prod_{\substack{1\le l\le n\\ k<l}} q_{k,l}^{\kappa_k\beta_l} (x^\alpha \otimes g)(x^\gamma \otimes h) \\
&= \prod_{l=1}^{n} \chi_{g,l}^{\gamma_l} \prod_{k<l} q_{k,l}^{\kappa_k\beta_l} (x^\alpha x^\gamma \otimes gh) \\
&= \prod_{l=1}^{n} \chi_{g,l}^{\gamma_l} \prod_{k<l} q_{k,l}^{\kappa_k\beta_l} (-q_{k,l})^{-\gamma_k\alpha_l} (x^{\alpha+\gamma} \otimes gh).
\end{aligned}$$

Thus $(x^\alpha \otimes g)\epsilon_\beta^* \smile (x^\gamma \otimes h)\epsilon_\kappa^* = \prod_{l=1}^{n} \chi_{g,l}^{\gamma_l} \prod_{k<l} q_{k,l}^{\kappa_k\beta_l - \gamma_k\alpha_l} (-1)^{-\gamma_k\alpha_l} (x^{\alpha+\gamma} \otimes gh)\epsilon_{\beta+\kappa}^*$. $\square$

4.4 Gerstenhaber bracket

With a bit more structure, we can compute the brackets on $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}} \rtimes G)$ using the techniques of [23] (discussed in Section 2.3) adapted to this setting. We must first show that $\mathbb{K}$ satisfies Conditions 2.3.3 (a)–(c):

(a) Let $\iota : \mathbb{K} \to \mathbb{B}$ be defined by sending $\epsilon_\beta \mapsto \tilde{f}_\beta$, with ϵ_β, $\tilde{f}_\beta$ defined as in the previous section.

(b) By the Comparison Theorem, there exists a chain map $\pi : \mathbb{B} \to \mathbb{K}$. Choose a map π such that $\tilde{f}_\beta \mapsto \epsilon_\beta$. Such a map exists because $\{\tilde{f}_\beta\}_{|\beta|=m}$ may be extended to a free $\Lambda_{\mathbf{q}}^e$-basis of $\mathbb{B}_{|\beta|}$. By construction, we have $\pi\iota = \mathbb{1}_{\mathbb{K}}$.

(c) Let the diagonal map $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{K}$ be defined by

$$\Delta_{\mathbb{K}}(\epsilon_\beta) = \sum_{\alpha+\gamma=\beta} \prod_{\substack{1\le l\le n\\ k<l}} q_{k,l}^{\gamma_k\alpha_l} \epsilon_\alpha \otimes_{\Lambda_{\mathbf{q}}} \epsilon_\gamma$$

for $\beta \in \mathbb{N}^n$, as in Section 4.3. We saw in Section 4.3,

$$\Delta_{\mathbb{B}}\iota(\epsilon_\beta) = \Delta_{\mathbb{B}}(\tilde{f}_\beta) = \sum_{\alpha+\gamma=\beta} \prod_{\substack{1\le l\le n \\ k<l}} q_{k,l}^{\gamma_k \alpha_l} \tilde{f}_\alpha \otimes_{\Lambda_{\mathbf{q}}} \tilde{f}_\gamma.$$

On the other hand,

$$(\iota \otimes_{\Lambda_{\mathbf{q}}} \iota)\Delta_{\mathbb{K}}(\epsilon_\beta) = (\iota \otimes_{\Lambda_{\mathbf{q}}} \iota)(\sum_{\alpha+\gamma=\beta} \prod_{\substack{1\le l\le n \\ k<l}} q_{k,l}^{\gamma_k \alpha_l} \epsilon_\alpha \otimes_{\Lambda_{\mathbf{q}}} \epsilon_\gamma) = \sum_{\alpha+\gamma=\beta} \prod_{\substack{1\le l\le n \\ k<l}} q_{k,l}^{\gamma_k \alpha_l} \tilde{f}_\alpha \otimes_{\Lambda_{\mathbf{q}}} \tilde{f}_\gamma.$$

Therefore $\Delta_{\mathbb{B}}\iota = (\iota \otimes_{\Lambda_{\mathbf{q}}} \iota)\Delta_{\mathbb{K}}$ as required.

Thus $\mathbb{K}$ satisfies Conditions 2.3.3 and, if $\phi : \mathbb{K} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{K} \to \mathbb{K}$ is a contracting homotopy, we can define the $\circ$-product on $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}, \Lambda_{\mathbf{q}})$ on the chain level as a composition

$$f \circ g : \mathbb{K} \xrightarrow{\Delta_{\mathbb{K}}} \mathbb{K} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{K} \xrightarrow{\Delta_{\mathbb{K}} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{1}_{\mathbb{K}}} \mathbb{K} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{K} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{K} \xrightarrow{\mathbb{1}_{\mathbb{K}} \otimes_{\Lambda_{\mathbf{q}}} g \otimes_{\Lambda_{\mathbf{q}}} \mathbb{1}_{\mathbb{K}}} \mathbb{K} \otimes_{\Lambda_{\mathbf{q}}} \mathbb{K} \xrightarrow{\phi} \mathbb{K} \xrightarrow{f} \Lambda_{\mathbf{q}}$$

for $f \in \mathrm{Hom}_{\Lambda_{\mathbf{q}}^e}((\mathbb{K})_l, \Lambda_{\mathbf{q}})$ and $g \in \mathrm{Hom}_{\Lambda_{\mathbf{q}}^e}((\mathbb{K})_m, \Lambda_{\mathbf{q}})$. However, we would like to define the $\circ$-product on $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}} \rtimes G, \Lambda_{\mathbf{q}} \rtimes G)$. By [24], we can define such a $\circ$-product using a similar technique as above, extended trivially to the group.

Define $\tilde{\mathbb{K}}$ to be the resolution

$$\ldots \xrightarrow{\tilde{\delta}_3} \mathbb{K}_2 \otimes kG \xrightarrow{\tilde{\delta}_2} \mathbb{K}_1 \otimes kG \xrightarrow{\tilde{\delta}_1} \mathbb{K}_0 \otimes kG \xrightarrow{\tilde{\delta}_0} \Lambda_{\mathbf{q}} \rtimes G \to 0$$

where $\tilde{\delta}_m = \delta_m \otimes \mathbb{1}_{kG}$. Let $\Delta_{\tilde{\mathbb{K}}} = \Delta_{\mathbb{K}} \otimes \mathbb{1}_{kG}$ be the induced diagonal map on $\tilde{\mathbb{K}}$. Let $\tilde{\phi} = \phi \otimes \mathbb{1}_{kG}$. Then, by [24, Section 2.2], for $f \in \mathrm{Hom}_{(\Lambda_{\mathbf{q}} \rtimes G)^e}(\tilde{\mathbb{K}}_m, \Lambda_{\mathbf{q}} \rtimes G)$ and $g \in \mathrm{Hom}_{(\Lambda_{\mathbf{q}} \rtimes G)^e}(\tilde{\mathbb{K}}_l, \Lambda_{\mathbf{q}} \rtimes G)$ we can view the ϕ-circle product, $f \circ_\phi g$, as a composition

$$\tilde{\mathbb{K}} \xrightarrow{\Delta_{\tilde{\mathbb{K}}}} \tilde{\mathbb{K}} \otimes_{\Lambda_{\mathbf{q}} \rtimes G} \tilde{\mathbb{K}} \xrightarrow{\mathbb{1}_{\tilde{\mathbb{K}}}} \tilde{\mathbb{K}} \otimes_{\Lambda_{\mathbf{q}} \rtimes G} \tilde{\mathbb{K}} \otimes_{\Lambda_{\mathbf{q}} \rtimes G} \tilde{\mathbb{K}} \xrightarrow{\mathbb{1}_{\tilde{\mathbb{K}}} \otimes g \otimes \mathbb{1}_{\tilde{\mathbb{K}}}} \tilde{\mathbb{K}} \otimes_{\Lambda_{\mathbf{q}} \rtimes G} \tilde{\mathbb{K}} \xrightarrow{\tilde{\phi}} \tilde{\mathbb{K}} \xrightarrow{f} \Lambda_{\mathbf{q}} \rtimes G$$

where the tensor products in $\mathbb{1}_{\tilde{\mathbb{K}}} \otimes g \otimes \mathbb{1}_{\tilde{\mathbb{K}}}$ are over $\Lambda_{\mathbf{q}} \rtimes G$, $\mathbb{1}_{\tilde{\mathbb{K}}} \otimes g \otimes \mathbb{1}_{\tilde{\mathbb{K}}}$ includes the indentification $\tilde{\mathbb{K}} \otimes_{\Lambda_{\mathbf{q}} \rtimes G} \Lambda_{\mathbf{q}} \rtimes G \cong \tilde{\mathbb{K}}$, and this function has Koszul signs as in (2.3.6). By [23, Theorem 3.2.5], the Gerstenhaber bracket on $\mathrm{HH}^{\bullet}(\Lambda_{\mathbf{q}} \rtimes G)$ is given by

$$[f, g] = f \circ_{\tilde{\phi}} g - (-1)^{(m-1)(l-1)} g \circ_{\tilde{\phi}} f$$

at the chain level. Thus the only remaining work is to determine ϕ.

By Lemma 3.2.4, ϕ can be defined iteratively as the complex $\mathbb{K} = \mathrm{Tot}(...((\mathbb{K}_{x_1} \otimes^{t_1} \mathbb{K}_{x_2}) \otimes^{t_2} ...) \otimes^{t_{n-1}} \mathbb{K}_{x_n})$ is defined iteratively. To get a closed form description of ϕ, we need to introduce some additional notation. Define $\mathbb{K}^{(l)} = \mathrm{Tot}(...((\mathbb{K}_{x_1} \otimes^{t_1} \mathbb{K}_{x_2}) \otimes^{t_2} ...) \otimes^{t_{l-1}} \mathbb{K}_{x_l})$ and for $\alpha \in \{0,1\}^l$, let $\alpha_{(l-1)} = (\alpha_1, \alpha_2, ..., \alpha_{l-1})$ the $(l-1)$-tuple consisting of the first $l-1$ entries of α.

As we saw in Example 2.3.8, for $\beta, \gamma \in \mathbb{N}$ and $\alpha \in \{0,1\}$

$$\phi_{\mathbb{K}^{(1)}}(\epsilon_\beta \otimes_{\Lambda_{\mathbf{q}}} x_1^\alpha \epsilon_\gamma) = \delta_{\alpha,1}(-1)^\beta \epsilon_{\alpha+\gamma+1}.$$

Therefore, by Lemma 3.2.4, for $\beta, \gamma \in \mathbb{N}^2$ and $\alpha \in \{0,1\}^2$,

$$\begin{aligned}
\phi_{\mathbb{K}^{(2)}}(\epsilon_\beta \otimes_{\Lambda_{\mathbf{q}}} x^\alpha \epsilon_\gamma) =& (-q_{1,2})^{\alpha_2(\beta_1+1)} \delta_{\beta_2,0} \delta_{\alpha_1,1} (-1)^{\beta_1} x_2^{\alpha_2} \epsilon_{\alpha+\gamma+[1]} \\
&+ (-q_{1,2})^{\alpha_1(\gamma_2+1)} \delta_{\gamma_1,0} \delta_{\alpha_2,1} (-1)^{|\beta|} \epsilon_{\alpha+\gamma+[2]} \\
=& \sum_{l=1}^{2} (-1)^{|\beta|} \delta_{\beta_{l+1}+...+\beta_2,0} \delta_{\gamma_1+...+\gamma_{l-1},0} \delta_{\alpha_l,1} \prod_{l<k\leq 2} (-q_{l,k})^{\alpha_k(\gamma_l+1)} \\
& \prod_{1<k\leq l} (-q_{k,l})^{\alpha_k(\beta_l+1)} \prod_{\substack{1\leq r<s\leq 2 \\ r\neq l\neq s}} (-q_{r,s})^{\alpha_r(\alpha_s+\gamma_s)+\alpha_s\beta_r} \\
& x_{l+1}^{\alpha_{l+1}} ... x_2^{\alpha_2} \epsilon_{\beta+\gamma+[l]} x_1^{\alpha_1} ... x_{l-1}^{\alpha_{l-1}}.
\end{aligned}$$

Compare this $\phi_{\mathbb{K}^{(2)}}$ to the ϕ from Section 3.2.1. As suggested by the formula for

$\phi_{\mathbb{K}^{(2)}}$, we claim that $\phi = \phi_{\mathbb{K}^{(n)}}$ is defined, for $\beta, \gamma \in \mathbb{N}^n$ and $\alpha \in \{0,1\}^n$, by

$$\phi(\epsilon_\beta \otimes_{\Lambda_{\mathbf{q}}} x^\alpha \epsilon_\gamma) = \sum_{l=1}^{n} (-1)^{|\beta|} \delta_{\beta_{l+1}+\ldots+\beta_n,0} \delta_{\gamma_1+\ldots+\gamma_{l-1},0} \delta_{\alpha_l,1} \prod_{l<k\leq n} (-q_{l,k})^{\alpha_k(\gamma_l+1)}$$
$$\prod_{1<k\leq l} (-q_{k,l})^{\alpha_k(\beta_l+1)} \prod_{\substack{1\leq r<s\leq n \\ r\neq l\neq s}} (-q_{r,s})^{\alpha_r(\alpha_s+\gamma_s)+\alpha_s\beta_r}$$
$$x_{l+1}^{\alpha_{l+1}} \ldots x_n^{\alpha_n} \epsilon_{\beta+\gamma+[l]} x_1^{\alpha_1} \ldots x_{l-1}^{\alpha_{l-1}}.$$

We will show this by induction. We have established the base case with $\phi_{\mathbb{K}^{(1)}}$ and $\phi_{\mathbb{K}^{(2)}}$. Now assume the formula holds for $\phi_{\mathbb{K}^{(n-1)}}$. Then for $\beta, \gamma \in \mathbb{N}^n$ and $\alpha \in \{0,1\}^n$,

$$\begin{aligned} \phi_{\mathbb{K}^{(n)}}(\epsilon_\beta \otimes_{\Lambda_{\mathbf{q}}} x^\alpha \epsilon_\gamma) =& \phi(\prod_{k<n} (-q_{k,n})^{\alpha_n\gamma_k} \epsilon_\beta \otimes_{\Lambda_{\mathbf{q}}} x^{\alpha_{(n-1)}} \epsilon_{\gamma_{(n-1)}} \otimes x^{\alpha_n} \epsilon_{\gamma_n}) \\ =& (\phi_{\mathbb{K}^{(n-1)}} \otimes F^l_{\mathbb{K}_{x_n}} + (-1)^{i+p} F^r_{\mathbb{K}^{(n-1)}} \otimes \phi_{\mathbb{K}_{x_n}})\sigma \end{aligned}$$

by Lemma 3.2.4. Recall σ is the chain map isomorphism defined in Lemma 3.2.1 that allows for rearranging terms in the tensor product of complexes. Simplifying, we get

$$\begin{aligned} &\phi_{\mathbb{K}^{(n)}}(\epsilon_\beta \otimes_{\Lambda_{\mathbf{q}}} x^\alpha \epsilon_\gamma) \\ &\quad = (\phi_{\mathbb{K}^{(n-1)}} \otimes F^l_{\mathbb{K}_{x_n}} + (-1)^{i+p} F^r_{\mathbb{K}^{(n-1)}} \otimes \phi_{\mathbb{K}_{x_n}}) \\ &\qquad \Big(\prod_{k<n} (-q_{k,n})^{\alpha_n\gamma_k - \beta_n(\alpha_k+\gamma_k)} (-1)^{\beta_n|\gamma_{(n-1)}|} \epsilon_{\beta_{(n-1)}} \otimes_{\Lambda_{\mathbf{q}}} x^{\alpha_{(n-1)}} \epsilon_{\gamma_{(n-1)}} \otimes \epsilon_{\beta_n} \otimes x^{\alpha_n} \epsilon_{\gamma_n}\Big) \\ &\quad = \Big(\prod_{k<n} (-q_{k,n})^{\alpha_n\gamma_k - \beta_n(\alpha_k+\gamma_k)} (-1)^{\beta_n|\gamma_{(n-1)}|}\Big) \\ &\qquad \Big(\sum_{l=1}^{n-1} (-1)^{|\beta_{(n-1)}|} \delta_{\beta_{l+1}+\ldots+\beta_{n-1},0} \delta_{\gamma_1+\ldots+\gamma_{l-1},0} \delta_{\alpha_l,1} \Big(\prod_{l<k<n-1} (-q_{l,k})^{\alpha_k(\gamma_l+1)} \\ &\qquad\quad \prod_{1\leq k<l} (-q_{k,l})^{\alpha_k(\beta_l+1)} \prod_{\substack{1\leq r<s\leq n-1 \\ r\neq l\neq s}} (-q_{r,s})^{\alpha_r(\alpha_s+\gamma_s)+\alpha_s\beta_r}\Big) \end{aligned}$$

$$x_{l+1}^{\alpha_{l+1}}...x_{n-1}^{\alpha_{n-1}}\epsilon_{\beta_{(n-1)}+\gamma_{(n-1)}+[l]}x_1^{\alpha_1}...x_{l-1}^{\alpha_{l-1}} \otimes \delta_{\beta_n,0}x_n^{\alpha_n}\epsilon_{\gamma_n}$$
$$+ (-1)^{|\beta_{(n-1)}|+|\gamma_{(n-1)}|}\delta_{|\gamma_{(n-1)}|,0}\epsilon_{\beta_{(n-1)}}x^{\alpha_{(n-1)}} \otimes \delta_{\alpha_n,1}(-1)^{\beta_n}\epsilon_{\beta_n+\gamma_n+1}\Big)$$

by the inductive hypothesis. Now, we can rearrange our terms to get

$$\begin{aligned}
\phi_{\mathbb{K}^{(n)}}(\epsilon_\beta \otimes_{\Lambda_{\mathbf{q}}} x^\alpha \epsilon_\gamma) = &\Big(\prod_{k<n}(-q_{k,n})^{\alpha_n\gamma_k-\beta_n(\alpha_k+\gamma_k)}(-1)^{\beta_n|\gamma_{(n-1)}|}\Big) \\
&\Big(\sum_{l=1}^{n-1}(-1)^{|\beta_{(n-1)}|}\delta_{\beta_{l+1}+...+\beta_n,0}\delta_{\gamma_1+...+\gamma_{l-1},0}\delta_{\alpha_l,1}\Big(\prod_{l<k<n-1}(-q_{l,k})^{\alpha_k(\gamma_l+1)} \\
&\quad\prod_{1\le k<l}(-q_{k,l})^{\alpha_k(\beta_l+1)}\prod_{\substack{1\le r<s\le n-1\\ r\neq l\neq s}}(-q_{r,s})^{\alpha_r(\alpha_s+\gamma_s)+\alpha_s\beta_r}\prod_{k<l}(-q_{k,n})^{\alpha_k(\alpha_n+\gamma_n)} \\
&\quad\prod_{k<n}(-q_{k,n})^{\alpha_n(\beta_k+\gamma_k+[l])}\Big)x_{l+1}^{\alpha_{l+1}}...x_n^{\alpha_n}\epsilon_{\beta+\gamma+[l]}x_1^{\alpha_1}...x_{l-1}^{\alpha_{l-1}} \\
&\quad+(-1)^{|\beta|}\delta_{|\gamma_{(n-1)}|,0}\Big(\prod_{k<n}(-q_{k,n})^{\alpha_k(\beta_n+\gamma_n+1)}\Big)\epsilon_{\beta+\gamma+[n]}x^{\alpha_{(n-1)}}\Big) \\
=&\sum_{l=1}^{n-1}(-1)^{|\beta|}\delta_{\beta_{l+1}+...\beta_n,0}\delta_{\gamma_1+...+\gamma_{l-1},0}\delta_{\alpha_l,1}\Big(\prod_{l<k\le n}(-q_{l,k})^{\alpha_k(\gamma_l+1)}\prod_{1<k\le l}(-q_{k,l})^{\alpha_k(\beta_l+1)} \\
&\quad\prod_{\substack{1\le r<s\le n\\ r\neq l\neq s}}(-q_{r,s})^{\alpha_r(\alpha_s+\gamma_s)+\alpha_s\beta_r}\Big)x_{l+1}^{\alpha_{l+1}}...x_n^{\alpha_n}\epsilon_{\beta+\gamma+[l]}x_1^{\alpha_1}...x_{l-1}^{\alpha_{l-1}} \\
&\quad+(-1)^{|\beta|}\delta_{|\gamma_{(n-1)}|,0}\delta_{\alpha_n,1}\Big(\prod_{k<n}(-q_{k,n})^{\alpha_k(\gamma_n+1)}\Big)\epsilon_{\beta+\gamma+[n]}x^{\alpha_{(n-1)}} \\
=&\sum_{l=1}^{n}(-1)^{|\beta|}\delta_{\beta_{l+1}+...\beta_n,0}\delta_{\gamma_1+...+\gamma_{l-1},0}\delta_{\alpha_l,1}\Big(\prod_{l<k\le n}(-q_{l,k})^{\alpha_k(\gamma_l+1)}\prod_{1<k\le l}(-q_{k,l})^{\alpha_k(\beta_l+1)} \\
&\quad\prod_{\substack{1\le r<s\le n\\ r\neq l\neq s}}(-q_{r,s})^{\alpha_r(\alpha_s+\gamma_s)+\alpha_s\beta_r}\Big)x_{l+1}^{\alpha_{l+1}}...x_n^{\alpha_n}\epsilon_{\beta+\gamma+[l]}x_1^{\alpha_1}...x_{l-1}^{\alpha_{l-1}}
\end{aligned}$$

as desired. Therefore

$$\phi(\epsilon_\beta \otimes_{\Lambda_{\mathbf{q}}} x^\alpha \epsilon_\gamma) = \sum_{l=1}^{n} (-1)^{|\beta|} \delta_{\beta_{l+1}+\ldots+\beta_n,0} \delta_{\gamma_1+\ldots+\gamma_{l-1},0} \delta_{\alpha_l,1} \Big(\prod_{l<k\leq n} (-q_{l,k})^{\alpha_k(\gamma_l+1)} \prod_{1<k\leq l} (-q_{k,l})^{\alpha_k(\beta_l+1)} \prod_{\substack{1\leq r<s\leq n \\ r\neq l\neq s}} (-q_{r,s})^{\alpha_r(\alpha_s+\gamma_s)+\alpha_s\beta_r} \Big) x_{l+1}^{\alpha_{l+1}} \ldots x_n^{\alpha_n} \epsilon_{\beta+\gamma+[l]} x_1^{\alpha_1} \ldots x_{l-1}^{\alpha_{l-1}}$$

for $\beta, \gamma \in \mathbb{N}^n$ and $\alpha \in \{0,1\}^n$.

We now have all of the necessary pieces to compute the Gerstenhaber bracket,

$$[f, g] = f \circ_{\tilde{\phi}} g - (-1)^{(m-1)(l-1)} g \circ_{\tilde{\phi}} f \tag{4.4.1}$$

for $f \in \mathrm{Hom}_{(\Lambda_{\mathbf{q}} \rtimes G)^e}((\tilde{\mathbb{K}})_m, \Lambda_{\mathbf{q}} \rtimes G)$ and $g \in \mathrm{Hom}_{(\Lambda_{\mathbf{q}} \rtimes G)^e}((\tilde{\mathbb{K}})_l, \Lambda_{\mathbf{q}} \rtimes G)$. Notice $\{(x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^*\}_{\alpha\in\{0,1\}^n, \beta\in\mathbb{N}^n, g\in G}$ forms a basis of $\mathrm{Hom}_{(\Lambda_{\mathbf{q}} \rtimes G)^e}(\tilde{\mathbb{K}}, \Lambda_{\mathbf{q}} \rtimes G)$. In the following theorem, we give the circle product on elements of this form. While these elements are not necessarily non-zero elements of cohomology, the given formula can be extended linearly to give a well-defined bracket on cohomology by restricting to the elements of the form as in Theorem 4.2.3.

Theorem 4.4.2. *For $\alpha, \gamma \in \{0,1\}^n$, $\beta, \kappa \in \mathbb{N}^n$, and $g, h \in G$,*

$$\begin{aligned} &(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \circ_{\tilde{\phi}} (x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^* \\ &= \sum_{r=1}^{n} \sum_{\substack{\rho'+\rho''=\kappa+\beta-[r] \\ (\rho'-\beta)_l \geq 0 \ \forall l\in\{1,2,\ldots,n\}}} (-1)^{|\rho'-\beta|(|\beta|+1)} \delta_{\rho'_{r+1},\beta_{r+1}} \ldots \delta_{\rho'_n,\beta_n} \delta_{\rho''_1+..+\rho''_{r-1},0} \delta_{\alpha_r,1} \\ &\qquad \mathbf{Q}(x^{\alpha+\gamma-[r]} \otimes hg)(\epsilon_{\kappa+\beta-[r]} \otimes 1)^* \end{aligned}$$

where

$$\mathbf{Q} = \prod_{1\le s<r} \chi_{h,s}^{\alpha_s}(-q_{s,r})^{\alpha_s(\rho'_r-\beta_r+1)} \prod_{1\le k<l<r\le n} q_{k,l}^{\beta_k(\rho'_l-\beta_l)} \prod_{1\le r<k<l\le n} q_{k,l}^{\rho''_k\beta_l} \prod_{r<s\le n} (-q_{r,s})^{\alpha_s(\rho''_r+1)}$$
$$\prod_{\substack{1\le t<u\le n\\ t\neq r\neq u}} (-q_{t,u})^{\alpha_t(\alpha_u+\rho''_u)+\alpha_t(\rho'_u-\beta_u)} \prod_{\substack{1\le s<r\\ s<v\le n}} (-q_{s,v})^{-\alpha_s\gamma_v} \prod_{1\le v<r<s\le n} (-q_{v,s})^{-(\gamma_v+\alpha_v)\alpha_s}$$
$$\prod_{\substack{r<s\le n\\ r\le v<s}} (-q_{v,s})^{-\gamma_v\alpha_s}.$$

Proof.

$$\begin{aligned}
&(x^\gamma\otimes h)(\epsilon_\kappa\otimes 1)^* \circ_{\tilde{\phi}} (x^\alpha\otimes g)(\epsilon_\beta\otimes 1)^*(\epsilon_\rho\otimes 1)\\
&\quad=(x^\gamma\otimes h)(\epsilon_\kappa\otimes 1)^*\tilde{\phi}(\mathbb{1}_{\tilde{\mathbb{K}}}\otimes(x^\alpha\otimes g)(\epsilon_\beta\otimes 1)^*\otimes\mathbb{1}_{\tilde{\mathbb{K}}})\tilde{\Delta}^{(2)}(\epsilon_\rho\otimes 1)\\
&\quad=(x^\gamma\otimes h)(\epsilon_\kappa\otimes 1)^*\tilde{\phi}(\mathbb{1}_{\tilde{\mathbb{K}}}\otimes(x^\alpha\otimes g)(\epsilon_\beta\otimes 1)^*\otimes\mathbb{1}_{\tilde{\mathbb{K}}})(\tilde{\Delta}\otimes\mathbb{1}_{\tilde{\mathbb{K}}})\\
&\qquad\Big(\sum_{\rho'+\rho''=\rho}\prod_{\substack{1\le l\le n\\ k<l}} q_{k,l}^{\rho''_k\rho'_l}\epsilon_{\rho'}\otimes\epsilon_{\rho''}\otimes 1\Big)\\
&\quad=(x^\gamma\otimes h)(\epsilon_\kappa\otimes 1)^*\tilde{\phi}(\mathbb{1}_{\tilde{\mathbb{K}}}\otimes(x^\alpha\otimes g)(\epsilon_\beta\otimes 1)^*\otimes\mathbb{1}_{\tilde{\mathbb{K}}})\\
&\qquad\Big(\sum_{\nu'+\nu''=\rho'}\sum_{\rho'+\rho''=\rho}\prod_{\substack{1\le l\le n\\ k<l}} q_{k,l}^{\rho''_k\rho'_l+\nu''_k\nu'_l}\epsilon_{\nu'}\otimes\epsilon_{\nu''}\otimes\epsilon_{\rho''}\otimes 1\Big).
\end{aligned}$$

In order to get a non-zero output from the function $\mathbb{1}_{\tilde{\mathbb{K}}}\otimes(x^\alpha\otimes g)(\epsilon_\beta\otimes 1)^*\otimes\mathbb{1}_{\tilde{\mathbb{K}}}$, we need $\nu''=\beta$. Set $\nu''=\beta$, then $\nu'=\rho'-\beta$. Applying this map thus gives us the Koszul sign $(-1)^{|\rho'-\beta||\beta|}$, making

$$\begin{aligned}
&(x^\gamma\otimes h)(\epsilon_\kappa\otimes 1)^* \circ_{\tilde{\phi}} (x^\alpha\otimes g)(\epsilon_\beta\otimes 1)^*(\epsilon_\rho\otimes 1)\\
&\quad=(x^\gamma\otimes h)(\epsilon_\kappa\otimes 1)^*\tilde{\phi}\\
&\qquad\Big(\sum_{\substack{\rho'+\rho''=\rho\\ (\rho'-\beta)_l\ge 0\ \forall l\in\{1,2,\dots,n\}}}(-1)^{|\rho'-\beta||\beta|}\prod_{\substack{1\le l\le n\\ k<l}} q_{k,l}^{\rho''_k\rho'_l+\beta_k(\rho'-\beta)_l}
\end{aligned}$$

$$\epsilon_{\rho'-\beta} \otimes (x^\alpha \otimes g) \otimes \epsilon_{\rho''} \otimes 1\Big).$$

We need $(\rho' - \beta)_l \geq 0$ for all $l \in \{1, 2, ..., n\}$ because $\epsilon_{\rho'-\beta}$ is tracking homological degree which is positive in each coordinate. Therefore $\delta_{(\rho'-\beta)_{r+1}+...+(\rho'-\beta)_n,0} = \delta_{\rho'_{r+1},\beta_{r+1}}...\delta_{\rho'_n,\beta_n}$. We will use this in the next expression.

$$\begin{aligned}
&(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \circ_{\tilde{\phi}} (x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^*(\epsilon_\rho \otimes 1) \\
&= (x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \tilde{\phi} \\
&\quad \Bigg(\sum_{\substack{\rho'+\rho''=\rho \\ (\rho'-\beta)_l \geq 0 \ \forall l \in \{1,2,...,n\}}} (-1)^{|\rho'-\beta||\beta|} \prod_{\substack{1 \leq l \leq n \\ k<l}} q_{k,l}^{\rho''_k \rho'_l + \beta_k (\rho'-\beta)_l} \\
&\qquad \epsilon_{\rho'-\beta} \otimes (x^\alpha \otimes g) \otimes \epsilon_{\rho''} \otimes 1\Bigg) \\
&= (x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \\
&\quad \Bigg(\sum_{\substack{\rho'+\rho''=\rho \\ (\rho'-\beta)_l \geq 0 \ \forall l \in \{1,2,...,n\}}} (-1)^{|\rho'-\beta||\beta|} \prod_{\substack{1 \leq l \leq n \\ k<l}} q_{k,l}^{\rho''_k \rho'_l + \beta_k (\rho'-\beta)_l} \\
&\quad \sum_{r=1}^{n} (-1)^{|\rho'-\beta|} \delta_{\rho'_{r+1},\beta_{r+1}} ... \delta_{\rho'_n,\beta_n} \delta_{\rho''_1+..+\rho''_{r-1},0} \delta_{\alpha_r,1} \prod_{r<s\leq n} (-q_{r,s})^{\alpha_s(\rho''_r+1)} \\
&\quad \prod_{1\leq s<r} (-q_{s,r})^{\alpha_s(\rho'_r-\beta_r+1)} \prod_{\substack{1\leq t<u\leq n \\ t\neq r\neq u}} (-q_{t,u})^{\alpha_t(\alpha_u+\rho''_u)+\alpha_t(\rho'_u-\beta_u)} \\
&\quad x_{r+1}^{\alpha_{r+1}}...x_n^{\alpha_n} \epsilon_{\rho'-\beta+\rho''+[r]} x_1^{\alpha_1}...x_{r-1}^{\alpha_{r-1}} \otimes g\Bigg).
\end{aligned}$$

In order to get a non-zero output from the function $(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^*$, we need $\rho' - \beta + \rho'' + [r] = \kappa$. That is, $\kappa + \beta - [r] = \rho' + \rho'' = \rho$. Notice

$$x_{r+1}^{\alpha_{r+1}}...x_n^{\alpha_n} \epsilon_{\rho'-\beta+\rho''+[r]} x_1^{\alpha_1}...x_{r-1}^{\alpha_{r-1}} \otimes g = (x_{r+1}^{\alpha_{r+1}}...x_n^{\alpha_n} \otimes 1)(\epsilon_{\rho'-\beta+\rho''+[r]} \otimes 1)(x_1^{\alpha_1}...x_{r-1}^{\alpha_{r-1}} \otimes g)$$

by the definition of the multiplication on $\Lambda_{\mathbf{q}} \rtimes G$. The second expression makes it clearer how to apply $(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^*$. Then

$$
\begin{aligned}
&(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \circ_{\tilde{\phi}} (x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^*(\epsilon_\rho \otimes 1) \\
&\quad = \sum_{\substack{\rho'+\rho''=\kappa+\beta-[r] \\ (\rho'-\beta)_l \geq 0 \ \forall l \in \{1,2,\ldots,n\}}} (-1)^{|\rho'-\beta||\beta|} \prod_{\substack{1 \leq l \leq n \\ k<l}} q_{k,l}^{\rho''_k \rho'_l + \beta_k(\rho'-\beta)_l} \\
&\qquad \sum_{r=1}^{n} (-1)^{|\rho'-\beta|} \delta_{\rho'_{r+1},\beta_{r+1}} \ldots \delta_{\rho'_n,\beta_n} \delta_{\rho''_1+..+\rho''_{r-1},0} \delta_{\alpha_r,1} \prod_{r<s\leq n} (-q_{r,s})^{\alpha_s(\rho''_r+1)} \\
&\qquad \prod_{1\leq s<r} (-q_{s,r})^{\alpha_s(\rho'_r-\beta_r+1)} \prod_{\substack{1\leq t<u\leq n \\ t\neq r\neq u}} (-q_{t,u})^{\alpha_t(\alpha_u+\rho''_u)+\alpha_t(\rho'_u-\beta_u)} \\
&\qquad (x_{r+1}^{\alpha_{r+1}} \ldots x_n^{\alpha_n} \otimes 1)(x^\gamma \otimes h)(x_1^{\alpha_1} \ldots x_{r-1}^{\alpha_{r-1}} \otimes g) \\
&\quad = \sum_{r=1}^{n} \sum_{\substack{\rho'+\rho''=\kappa+\beta-[r] \\ (\rho'-\beta)_l \geq 0 \ \forall l \in \{1,2,\ldots,n\}}} (-1)^{|\rho'-\beta|(|\beta|+1)} \prod_{\substack{1 \leq l \leq n \\ k<l}} q_{k,l}^{\rho''_k \rho'_l + \beta_k(\rho'-\beta)_l} \\
&\qquad \delta_{\rho'_{r+1},\beta_{r+1}} \ldots \delta_{\rho'_n,\beta_n} \delta_{\rho''_1+..+\rho''_{r-1},0} \delta_{\alpha_r,1} \prod_{r<s\leq n} (-q_{r,s})^{\alpha_s(\rho''_r+1)} \\
&\qquad \prod_{1\leq s<r} (-q_{s,r})^{\alpha_s(\rho'_r-\beta_r+1)} \prod_{\substack{1\leq t<u\leq n \\ t\neq r\neq u}} (-q_{t,u})^{\alpha_t(\alpha_u+\rho''_u)+\alpha_t(\rho'_u-\beta_u)} \prod_{1\leq s<r} \chi_{h,s}^{\alpha_s} \\
&\qquad \prod_{\substack{1\leq s<r \\ s<v\leq n}} (-q_{s,v})^{-\alpha_s\gamma_v} \prod_{\substack{r<s\leq n \\ 1\leq v<r}} (-q_{v,s})^{-(\gamma_v+\alpha_v)\alpha_s} \prod_{\substack{r<s\leq n \\ r\leq v<s}} (-q_{v,s})^{-\gamma_v\alpha_s} \\
&\qquad\qquad x^{\alpha+\gamma-[r]} \otimes hg.
\end{aligned}
$$

This expression can be simplified slightly to eliminate trivial terms. That is,

$$
\begin{aligned}
&(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \circ_{\tilde{\phi}} (x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^* \\
&\quad = \sum_{r=1}^{n} \sum_{\substack{\rho'+\rho''=\kappa+\beta-[r] \\ (\rho'-\beta)_l \geq 0 \ \forall l \in \{1,2,\ldots,n\}}} (-1)^{|\rho'-\beta|(|\beta|+1)} \delta_{\rho'_{r+1},\beta_{r+1}} \ldots \delta_{\rho'_n,\beta_n} \delta_{\rho''_1+..+\rho''_{r-1},0} \delta_{\alpha_r,1} \\
&\qquad \prod_{1\leq s<r} \chi_{h,s}^{\alpha_s} (-q_{s,r})^{\alpha_s(\rho'_r-\beta_r+1)} \prod_{1\leq k<l<r\leq n} q_{k,l}^{\beta_k(\rho'-\beta)_l} \prod_{1\leq r<k<l\leq n} q_{k,l}^{\rho''_k\beta_l}
\end{aligned}
$$

$$\prod_{r<s\leq n}(-q_{r,s})^{\alpha_s(\rho''_r+1)}\prod_{\substack{1\leq t<u\leq n\\ t\neq r\neq u}}(-q_{t,u})^{\alpha_t(\alpha_u+\rho''_u)+\alpha_t(\rho'_u-\beta_u)}\prod_{\substack{1\leq s<r\\ s<v\leq n}}(-q_{s,v})^{-\alpha_s\gamma_v}$$

$$\prod_{1\leq v<r<s\leq n}(-q_{v,s})^{-(\gamma_v+\alpha_v)\alpha_s}\prod_{\substack{r<s\leq n\\ r\leq v<s}}(-q_{v,s})^{-\gamma_v\alpha_s}$$

$$(x^{\alpha+\gamma-[r]}\otimes hg)(\epsilon_{\kappa+\beta-[r]}\otimes 1)^*.$$

$\square$

We will denote $\circ_{\tilde{\phi}}$ as $\circ$ for the remainder of the text to ease notation. By the work in this section, the Gerstenhaber algebra structure of the Hochschild cohomology of group extensions of $\Lambda_{\mathbf{q}}$ is completely formulated. When the group is trivial, the work of this section recovers the Gerstenhaber brackets for Hochschild cohomology of quantum complete intersections, extending previous computations. The Gerstenhaber algebra structure can also be used to collect information about the deformations of these algebras.

5. BRACKETS FOR SOME GROUP EXTENSIONS OF QUANTUM COMPLETE INTERSECTIONS

In the previous section, we gave a general description of the Gerstenhaber structure on group extensions of quantum complete intersections. We use this section to compute these structures for several cases when $n = 2$. In Section 5.1, we allow the diagonal group action to vary and restrict the choice of quantum coefficient. In Section 5.2, we allow the quantum coefficient to be a dth root of unity and fix the diagonal group action to agree with the quantum coefficient. The vector space structure is computed for all d and the bracket structure is computed for $d > 1$ odd. When the characteristic of the field is 0, these example cases are generalizations of, and can be compared to, the Gerstenhaber brackets computed in [11, Section 5] (when $n = 2$) and the vector space computations in [5, Section 3] and [7, Section 3] for quantum complete intersections.

5.1 Two generator quantum complete intersections

Now that we have the general formulas for the Gerstenhaber algebra structure on $\mathrm{HH}^{\bullet}(\Lambda_{\mathbf{q}}^n \rtimes G)$, we can apply them to simple examples very similar to the examples computed in [11, Section 5.1]. Let $n = 2$ and assume $q_{1,2}$ is not a root of unity and G is an abelian group. For simplicity, let $q = q_{1,2}$.

Then, by Theorem 4.2.3,

$$\mathrm{HH}^m(\Lambda_{\mathbf{q}}^2 \rtimes G) \cong \Big(\bigoplus_{g\in G} \bigoplus_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=m}} \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ \beta-\alpha\in C_g}} span_k\{(x^\alpha \otimes g)\epsilon_\beta^*\}\Big)^G$$

where $C_g = \{\gamma \in (\mathbb{N}\cup\{-1\})^2 | \forall i, \gamma_i = -1 \text{ or } (-1)^{\gamma_i} \prod_{k\neq i}(-q_{k,l})^{\gamma_k} = \chi_{g,i}\}$.

Therefore we have two conditions on the $\gamma = \beta - \alpha$ for which $(x^\alpha \otimes g)\epsilon_\beta^*$ is non-trivial in $\mathrm{HH}^m(\Lambda^2_{\mathbf{q}}, \Lambda^2_{\mathbf{q}} \rtimes G)$,

$$\gamma_1 = -1 \text{ or } (-1)^{\gamma_1}(-q^{-1})^{\gamma_2} = \chi_{g,1}$$
$$\text{and } \gamma_2 = -1 \text{ or } (-1)^{\gamma_2}(-q)^{\gamma_1} = \chi_{g,2}.$$

If $\gamma_1 = -1$, then, as q is not a root of unity and $\chi_{g,2}$ must be a root of unity, we cannot have $(-q)^{-1} = (-1)^{\gamma_2}\chi_{g,2}$ and thus, for $\gamma \in C_g$, we need $\gamma_2 = -1$.

Alternatively, if $\gamma_1 \neq -1$, then for $\gamma \in C_g$, we need $(-1)^{\gamma_1}(-q^{-1})^{\gamma_2} = \chi_{g,1}$. But, again because q is not a root of unity, we must have $\gamma_2 = 0$ and $\chi_{g,1} = 1$. Thus $\gamma_1 \neq -1$ forces $\gamma_2 \neq -1$ also. Therefore, for $\gamma \in C_g$, γ_1 must satisfy $(-1)^{\gamma_2}(-q)^{\gamma_1} = \chi_{g,2}$, forcing $\gamma_1 = 0$ and $\chi_{g,2} = 1$.

That is, we have two options for non-trivial elements, either $\gamma = (-1,-1)$ or $\gamma = (0,0)$ and $\chi_{g,1} = \chi_{g,2} = 1$, making

$$\begin{aligned}
\mathrm{HH}^{\bullet}(\Lambda^2_{\mathbf{q}} \rtimes G) &\cong (span_k\{\epsilon^*_{0,0}, \{(x_1x_2 \otimes g)\epsilon^*_{0,0}\}_{g\in G}, \\
&\qquad \{(x_2 \otimes g)\epsilon^*_{0,1}, (x_1 \otimes g)\epsilon^*_{1,0}, (x_1x_2 \otimes g)\epsilon^*_{1,1}\}_{\substack{g\in G \\ \chi_{g,1}=\chi_{g,2}=1}}\})^G \\
&\cong \bigoplus_{g\in G} span_k\{\epsilon^*_{0,0}, \{(x_1x_2 \otimes g)\epsilon^*_{0,0}\}_{\chi_{g,1}\chi_{g,2}=1}, \\
&\qquad \{(x_2 \otimes g)\epsilon^*_{0,1}, (x_1 \otimes g)\epsilon^*_{1,0}, (x_1x_2 \otimes g)\epsilon^*_{1,1}\}_{\chi_{g,1}=\chi_{g,2}=1}\}.
\end{aligned}$$

We can now use our formula from Theorem 4.3.2,

$$(x^\alpha \otimes g)\epsilon^*_\beta \smile (x^\gamma \otimes h)\epsilon^*_\kappa = \prod_{l=1}^{2} \chi_{g,l}^{\gamma_l} \prod_{k<l} q_{k,l}^{\kappa_k\beta_l - \gamma_k\alpha_l}(-1)^{-\gamma_k\alpha_l}(x^{\alpha+\gamma} \otimes gh)\epsilon^*_{\beta+\kappa}$$

to compute cup products. Because of the factor $x^{\alpha+\gamma}$ in the product, the only possible non-zero cup products in homological degree greater than 0 are

$$\begin{aligned}(x_2 \otimes g)\epsilon^*_{0,1} \smile (x_1 \otimes h)\epsilon^*_{1,0} &= \chi^1_{g,1} q^{1-1}(-1)^1 (x_1x_2 \otimes gh)\epsilon^*_{1,1} \\ &= -\chi_{g,1}(x_1x_2 \otimes gh)\epsilon^*_{1,1} \\ &= -(x_1x_2 \otimes gh)\epsilon^*_{1,1}\end{aligned}$$

and, using either the formula given in Theorem 4.3.2 or the graded commutativity of $\smile$ on $\mathrm{HH}^m(\Lambda^2_{\mathbf{q}} \rtimes G)$ given in Remark 2.2.3,

$$(x_1 \otimes h)\epsilon^*_{1,0} \smile (x_2 \otimes g)\epsilon^*_{0,1} = (x_1x_2 \otimes hg)\epsilon^*_{1,1}.$$

The element $\epsilon^*_{0,0}$ is the multiplicative identity with respect to $\smile$ and $(x_1x_2 \otimes g)\epsilon^*_{0,0}$ has cup product 0 with any of the generators except $\epsilon^*_{0,0}$.

Finally, we can use our formula from Theorem 4.4.2, restated for the case $n = 2$,

$$\begin{aligned}&(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \circ (x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^* \\ &\quad= \sum_{r=1}^{2} \sum_{\substack{\rho'+\rho''=\kappa+\beta-[r] \\ (\rho'-\beta)_\ell \geq 0 \ \forall \ell \in \{1,2\}}} (-1)^{|\rho'-\beta|(|\beta|+1)} \delta_{\rho'_{r+1},\beta_{r+1}} \ldots \delta_{\rho'_n,\beta_n} \delta_{\rho''_1+..+\rho''_{r-1},0} \delta_{\alpha_r,1} \\ &\qquad \prod_{1 \leq s < r} \chi^{\alpha_s}_{h,s} (-q_{s,r})^{\alpha_s(\rho'_r-\beta_r+1)} \prod_{r<s\leq 2} (-q_{r,s})^{\alpha_s(\rho''_r+1)} (x^{\alpha+\gamma-[r]} \otimes hg) \\ &\qquad (\epsilon_{\kappa+\beta-[r]} \otimes 1)^*\end{aligned}$$

to compute brackets. Then the non-zero $\circ$-products are

$$\begin{aligned}&(x_2 \otimes h)(\epsilon_{0,1} \otimes 1)^* \circ (x_1x_2 \otimes g)(\epsilon_{0,0} \otimes 1)^* \\ &\quad= \sum_{\rho'+\rho''=(0,0)} \chi_{h,1}(-q)^{1(0-0+1)}(-q)^{-1}(x_1x_2 \otimes hg)(\epsilon_{0,0} \otimes 1)^*\end{aligned}$$

$$= \chi_{h,1}(x_1x_2 \otimes hg)(\epsilon_{0,0} \otimes 1)^*$$

$$= (x_1x_2 \otimes hg)(\epsilon_{0,0} \otimes 1)^*$$

and

$$(x_2 \otimes h)(\epsilon_{0,1} \otimes 1)^* \circ (x_1x_2 \otimes g)(\epsilon_{0,0} \otimes 1)^* = (x_1x_2 \otimes hg)(\epsilon_{0,0} \otimes 1)^*,$$

$$(x_1 \otimes h)(\epsilon_{1,0} \otimes 1)^* \circ (x_1 \otimes g)(\epsilon_{1,0} \otimes 1)^* = (x_1 \otimes hg)(\epsilon_{1,0} \otimes 1)^*,$$

$$(x_1 \otimes h)(\epsilon_{1,0} \otimes 1)^* \circ (x_1x_2 \otimes g)(\epsilon_{0,0} \otimes 1)^* = (x_1x_2 \otimes hg)(\epsilon_{0,0} \otimes 1)^*,$$

$$(x_1x_2 \otimes h)(\epsilon_{1,1} \otimes 1)^* \circ (x_1 \otimes g)(\epsilon_{1,0} \otimes 1)^* = (x_1x_2 \otimes hg)(\epsilon_{1,1} \otimes 1)^*,$$

$$(x_1 \otimes g)(\epsilon_{1,0} \otimes 1)^* \circ (x_1x_2 \otimes h)(\epsilon_{1,1} \otimes 1)^* = (x_1x_2 \otimes gh)(\epsilon_{1,1} \otimes 1)^*,$$

$$(x_2 \otimes h)(\epsilon_{0,1} \otimes 1)^* \circ (x_2 \otimes g)(\epsilon_{0,1} \otimes 1)^* = (x_2 \otimes hg)(\epsilon_{0,1} \otimes 1)^*,$$

$$(x_1x_2 \otimes h)(\epsilon_{1,1} \otimes 1)^* \circ (x_2 \otimes g)(\epsilon_{0,1} \otimes 1)^* = (x_1x_2 \otimes hg)(\epsilon_{1,1} \otimes 1)^*, \text{ and}$$

$$(x_2 \otimes g)(\epsilon_{0,1} \otimes 1)^* \circ (x_1x_2 \otimes h)(\epsilon_{1,1} \otimes 1)^* = (x_1x_2 \otimes gh)(\epsilon_{1,1} \otimes 1)^*.$$

Using our formula (4.4.1), modifying for this notation,

$$\begin{aligned}[(x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^*,&(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^*] \\ &= (x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^* \circ (x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \\ &\quad - (-1)^{(|\kappa|-1)(|\beta|-1)}(x^\gamma \otimes h)(\epsilon_\kappa \otimes 1)^* \circ (x^\alpha \otimes g)(\epsilon_\beta \otimes 1)^*,\end{aligned}$$

we can complete the bracket computations. The non-zero brackets among pairs of the generators of $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}^2 \rtimes G)$ are

$$[(x_2 \otimes h)(\epsilon_{0,1} \otimes 1)^*,(x_1x_2 \otimes g)(\epsilon_{0,0} \otimes 1)^*] = (x_1x_2 \otimes hg)(\epsilon_{0,0} \otimes 1)^* \text{ and}$$

$$[(x_1 \otimes h)(\epsilon_{1,0} \otimes 1)^*,(x_1x_2 \otimes g)(\epsilon_{0,0} \otimes 1)^*] = (x_1x_2 \otimes hg)(\epsilon_{0,0} \otimes 1)^*.$$

Notice that the bracket and cup product computations agree with the results in [11, Section 5.1] and [7, Section 2.1] respectively when $G = 1$, given in Example 3.2.7.

5.2 A specific group action

We will now consider a specific example, on $\Lambda^2_{\mathbf{q}}$, to study the structure in more depth. Let $q_{1,2} = q$ be a primitive dth root of unity. Let $G = \langle g \rangle$ be the cyclic group generated by an element g with $|G| = d$. Assume that G acts on $\Lambda^2_{\mathbf{q}}$ by

$$ {}^{g}x_1 = qx_1 \text{ and } {}^{g}x_2 = q^{-1}x_2. $$

That is, G acts on the generators of $\Lambda^2_{\mathbf{q}}$ by multiplication by powers of the quantum coefficient.

By Theorem 4.2.3, we know that $\mathrm{HH}^m(\Lambda^2_{\mathbf{q}} \rtimes G)$ is isomorphic to the G-invariant subspace of

$$ \bigoplus_{i=1}^{d} \bigoplus_{\substack{\beta \in \mathbb{N}^2 \\ |\beta| = m}} \bigoplus_{\substack{\alpha \in \{0,1\}^2 \\ \beta - \alpha \in C_{g^i}}} span_k\{(x^\alpha \otimes g^i)\epsilon^*_\beta\} $$

where

$$ \begin{aligned} C_{g^i} =& \{\gamma \in (\mathbb{N} \cup \{-1\})^2 |\ \forall\ l, \gamma_l = -1 \text{ or, for } k \neq l, (-1)^{\gamma_l}(-q_{k,l})^{\gamma_k} = q^i_{l,k}\} \\ =& \{\gamma \in (\mathbb{N} \cup \{-1\})^2 |\ \forall\ l, \gamma_l = -1 \text{ or, for } k \neq l, q_{k,l}^{\gamma_k + i} = (-1)^{|\gamma|}\} \\ =& \{\gamma \in (\mathbb{N} \cup \{-1\})^2 |\ \gamma_1 = -1 \text{ or } q^{\gamma_2 + i} = (-1)^{|\gamma|} \text{ and} \\ & \qquad \gamma_2 = -1 \text{ or } q^{\gamma_1 + i} = (-1)^{|\gamma|}\}. \end{aligned} $$

Therefore to find a non-trivial element of cohomology, we need to first check the conditions for which $\gamma \in (\mathbb{N} \cup \{-1\})^2$ satisfies $\gamma_1 = -1$ or $q^{\gamma_2 + i} = (-1)^{|\gamma|}$ and $\gamma_2 = -1$ or $q^{\gamma_1 + i} = (-1)^{|\gamma|}$ and then check for terms that are G-invariant.

To compute the cup product, by Theorem 4.3.2, we know that for $\alpha, \gamma \in \{0,1\}^2$, $\beta, \kappa \in \mathbb{N}^2$, and $g^i, g^j \in G$,

$$\begin{aligned}(x^\alpha \otimes g^i)\epsilon^*_\beta \smile (x^\gamma \otimes g^j)\epsilon^*_\kappa =& \prod_{l=1}^{2}\prod_{l\neq k} q_{l,k}^{i\gamma_l} \prod_{k<l} q_{k,l}^{\kappa_k\beta_l - \gamma_k\alpha_l}(-1)^{-\gamma_k\alpha_l}(x^{\alpha+\gamma}\otimes g^{i+j})\epsilon^*_{\beta+\kappa} \\ =& q^{i(\gamma_1-\gamma_2)+\kappa_1\beta_2-\gamma_1\alpha_2}(-1)^{-\gamma_1\alpha_2}(x^{\alpha+\gamma}\otimes g^{i+j})\epsilon^*_{\beta+\kappa}.\end{aligned}$$

Under these assumptions, Theorem 4.4.2 becomes, for $\alpha, \gamma \in \{0,1\}^2$, $\beta, \kappa \in \mathbb{N}^2$, and $g^i, g^j \in G$,

$$\begin{aligned}&(x^\gamma \otimes g^j)(\epsilon_\kappa \otimes 1)^* \circ (x^\alpha \otimes g^i)(\epsilon_\beta \otimes 1)^* \\ &= \sum_{r=1}^{2} \sum_{\substack{\rho'+\rho''=\kappa+\beta-[r] \\ (\rho'-\beta)_\ell \geq 0 \ \forall \ell\in\{1,2\}}} (-1)^{|\rho'-\beta|(|\beta|+1)}\delta_{\rho'_{r+1},\beta_{r+1}}\ldots\delta_{\rho'_n,\beta_n}\delta_{\rho''_1+\ldots+\rho''_{r-1},0}\delta_{\alpha_r,1} \\ &\qquad \prod_{1\leq s<r} q_{s,r}^{j\alpha_s}(-q_{s,r})^{\alpha_s(\rho'_r-\beta_r+1)} \prod_{r<s\leq 2}(-q_{r,s})^{\alpha_s(\rho''_r+1)} \\ &\qquad \prod_{\substack{1\leq s<r \\ s<v\leq 2}} (-q_{s,v})^{-\alpha_s\gamma_v}(x^{\alpha+\gamma-[r]}\otimes g^{i+j})(\epsilon_{\kappa+\beta-[r]}\otimes 1)^* \\ &= \sum_{\substack{\rho'+\rho''=\kappa+\beta-[1] \\ (\rho'-\beta)_\ell\geq 0 \ \forall\ell\in\{1,2\}}} (-1)^{|\rho'-\beta|(|\beta|+1)}\delta_{\rho'_2,\beta_2}\delta_{\alpha_1,1}(-q_{1,2})^{\alpha_2(\rho''_1+1)} \\ &\qquad (x^{\alpha+\gamma-[1]}\otimes g^{i+j})(\epsilon_{\kappa+\beta-[1]}\otimes 1)^* \\ &\quad + \sum_{\substack{\rho'+\rho''=\kappa+\beta-[2] \\ (\rho'-\beta)_\ell\geq 0 \ \forall\ell\in\{1,2\}}} (-1)^{|\rho'-\beta|(|\beta|+1)}\delta_{\rho''_1,0}\delta_{\alpha_2,1}q^{j\alpha_2}(-q)^{\alpha_1(\rho'_2-\beta_2+1)-\alpha_1\gamma_2} \\ &\qquad (x^{\alpha+\gamma-[2]}\otimes g^{i+j})(\epsilon_{\kappa+\beta-[2]}\otimes 1)^*.\end{aligned}$$

As in [7] and [11], we study the specific structure of Hochschild cohomology for each choice of d. We compute the graded vector space structure for all possible choices of $d \neq 0$ and the Gerstenhaber brackets for the case $d > 1$ odd.

5.2.1 *q a dth root of unity for d > 1 odd*

If $d > 1$ is odd, then

$$\begin{aligned}
C_{g^i} =&\{\gamma \in (\mathbb{N} \cup \{-1\})^2 |\ \gamma_1 = -1 \text{ or } (\exists\ t, k \in \mathbb{N}, \gamma_2 + i = td \text{ and } |\gamma| = 2k) \text{ and} \\
&\qquad \gamma_2 = -1 \text{ or } (\exists\ t, k \in \mathbb{N}, \gamma_1 + i = td \text{ and } |\gamma| = 2k)\} \\
=&\{\gamma \in (\mathbb{N} \cup \{-1\})^2 |\ \gamma_1 = -1 \text{ and } \gamma_2 = -1 \text{ or} \\
&\qquad \gamma_1 = -1 \text{ and } \exists\ t \in \mathbb{N}, \gamma_2 = 2t + 1 \text{ and } i = 1 \text{ or} \\
&\qquad \gamma_2 = -1 \text{ and } \exists\ t \in \mathbb{N}, \gamma_1 = 2t + 1 \text{ and } i = 1 \text{ or} \\
&\qquad \exists\ t, t' \in \mathbb{N}, \gamma_1 = td - i \text{ and } \gamma_2 = t'd - i \text{ and } t + t' \text{ even}\}.
\end{aligned}$$

From this description, we immediately get

$$\begin{aligned}
\mathrm{HH}^\bullet(\Lambda^2_{\mathbf{q}}, \Lambda^2_{\mathbf{q}} \rtimes G) \cong & \bigoplus_{t \in \mathbb{N}} span_k\{(x_1 \otimes g)\epsilon^*_{0,2t+1}, (x_2 \otimes g)\epsilon^*_{2t+1,0}, (x_1x_2 \otimes g)\epsilon^*_{2t,0}, \\
&\qquad (x_1x_2 \otimes g)\epsilon^*_{0,2t}\} \\
& \bigoplus_{i=1}^{d} \bigoplus_{\substack{t,t' \in \mathbb{N} \\ t+t' \text{ even}}} span_k\{(1 \otimes g^i)\epsilon^*_{td-i,t'd-i}, (x_1 \otimes g^i)\epsilon^*_{td-i+1,t'd-i}, \\
&\qquad (x_2 \otimes g^i)\epsilon^*_{td-i,t'd-i+1}, (x_1x_2 \otimes g^i)\epsilon^*_{td-i+1,t'd-i+1}\} \\
& \bigoplus_{i=1}^{d} span_k\{(x_1x_2 \otimes g^i)\epsilon^*_{0,0}, (1 \otimes g^i)\epsilon^*_{0,0}\}
\end{aligned}$$

as a vector space. In this expression, the terms $\epsilon^*_{a,b}$ should be interpreted with $a, b \geq 0$ and thus we only consider t large enough to make the homological degrees nonnegative.

Then, after taking the G-invariant subspace, we get

$$\begin{aligned}\mathrm{HH}^{\bullet}(\Lambda^2_{\mathbf{q}} \rtimes G) \cong & \bigoplus_{t\in\mathbb{N}} span_k\{(x_1\otimes g)\epsilon^*_{0,2td-1}, (x_2\otimes g)\epsilon^*_{2td-1,0}, (x_1x_2\otimes g)\epsilon^*_{2td,0}, \\ & \qquad (x_1x_2\otimes g)\epsilon^*_{0,2td}\} \\ & \bigoplus_{i=1}^{d} \bigoplus_{\substack{t,t'\in\mathbb{N} \\ t+t' \text{ even}}} span_k\{(1\otimes g^i)\epsilon^*_{td-i,t'd-i}, (x_1\otimes g^i)\epsilon^*_{td-i+1,t'd-i}, \\ & \qquad\qquad (x_2\otimes g^i)\epsilon^*_{td-i,t'd-i+1}, (x_1x_2\otimes g^i)\epsilon^*_{td-i+1,t'd-i+1}\} \\ & \bigoplus_{i=1}^{d} span_k\{(x_1x_2\otimes g^i)\epsilon^*_{0,0}, (1\otimes g^i)\epsilon^*_{0,0}\}\end{aligned}$$

as a vector space.

We will forgo the cup product structure in favor of showing the bracket structure. The non-zero brackets are, for $t, t', t'', t''' \in \mathbb{N}$ and $i, j \in \{1, 2, ..., d\}$,

$$\begin{aligned}
[(x_2\otimes g)\epsilon^*_{2t'd-1,0}, (x_1\otimes g)\epsilon^*_{0,2td-1}] &= (2t'd-1)(x_2\otimes g^2)\epsilon^*_{2t'd-2,2td-1}, \\
&\quad - (2td-1)q(x_1\otimes g^2)\epsilon^*_{2t'd-1,2td-2}, \\
[(x_1x_2\otimes g)\epsilon^*_{2t'd,0}, (x_1\otimes g)\epsilon^*_{0,2td-1}] &= (2t'd)(x_1x_2\otimes g^2)\epsilon^*_{2t'd-1,2td-1}, \\
[(1\otimes g^i)\epsilon^*_{t'd-i,t''d-i}, (x_1\otimes g)\epsilon^*_{0,2td-1}] &= (t'd-i)(1\otimes g^{i+1})\epsilon^*_{t'd-i-1,2td+t''d-i-1}, \\
[(x_1\otimes g^i)\epsilon^*_{t'd-i+1,t''d-i}, (x_1\otimes g)\epsilon^*_{0,2td-1}] &= (t'd-i+1)(x_1\otimes g^{i+1})\epsilon^*_{t'd-i,2td+t''d-i-1}, \\
[(x_2\otimes g^i)\epsilon^*_{t'd-i,t''d-i+1}, (x_1\otimes g)\epsilon^*_{0,2td-1}] &= (t'd-i)(x_2\otimes g^{i+1})\epsilon^*_{t'd-i-1,2td+t''d-i} \\
&\quad - (2td-1)q(x_1\otimes g^{i+1})\epsilon^*_{t'd-i,2td+t''d-i}, \\
[(x_1x_2\otimes g^i)\epsilon^*_{t'd-i+1,t''d-i+1}, (x_1\otimes g)\epsilon^*_{0,2td-1}] &= (t'd-i+1)(x_1x_2\otimes g^{i+1})\epsilon^*_{t'd-i,2td+t''d-i}, \\
[(x_1x_2\otimes g^i)\epsilon^*_{0,2t'd}, (x_2\otimes g)\epsilon^*_{0,2td-1}] &= (2t'd)q(x_1x_2\otimes g^2)\epsilon^*_{2td-1,2t'd-1}, \\
[(1\otimes g^i)\epsilon^*_{t'd-i,t''d-i}, (x_2\otimes g)\epsilon^*_{0,2td-1}] &= (t''d-i)q^i(1\otimes g^{i+1})\epsilon^*_{2td+t'd-i-1,t''d-i-1}, \\
[(x_1\otimes g^i)\epsilon^*_{t'd-i+1,t''d-i}, (x_2\otimes g)\epsilon^*_{0,2td-1}] &= (t''d-i)q^i(x_1\otimes g^{i+1})\epsilon^*_{2td+t'd-i,t''d-i-1}
\end{aligned}$$

$$- (2td-1)(x_2 \otimes g^{i+1})\epsilon^*_{2td+t'd-i-1,t''d-i},$$

$$[(x_2 \otimes g^i)\epsilon^*_{t'd-i,t''d-i+1}, (x_2 \otimes g)\epsilon^*_{0,2td-1}] = (t''d-i+1)q^i(x_2 \otimes g^{i+1})\epsilon^*_{2td+t'd-i-1,t''d-i},$$

$$[(x_1x_2 \otimes g^i)\epsilon^*_{t'd-i+1,t''d-i+1}, (x_2 \otimes g)\epsilon^*_{0,2td-1}]$$
$$= (t''d-i+1)q^i(x_1x_2 \otimes g^{i+1})\epsilon^*_{2td+t'd-i,t''d-i},$$

$$[(1 \otimes g^i)\epsilon^*_{t'd-i,t''d-i}, (x_1x_2 \otimes g)\epsilon^*_{2td,0}]$$
$$= (-1)^{t'd-i+1}\sum_{\rho'_2=0}^{t''d-i-1} q^{\rho'_2+i+1}(x_1 \otimes g^{i+1})\epsilon^*_{2td+t'd-i,t''d-i-1},$$

$$[(x_1 \otimes g^i)\epsilon^*_{t'd-i+1,t''d-i}, (x_1x_2 \otimes g)\epsilon^*_{2td,0}]$$
$$= -(2td)(x_1x_2 \otimes g^{i+1})\epsilon^*_{2td+t'd-i,t''d-i},$$

$$[(x_2 \otimes g^i)\epsilon^*_{t'd-i,t''d-i+1}, (x_1x_2 \otimes g)\epsilon^*_{2td,0}]$$
$$= (-1)^{t'd-i}\sum_{\rho'_2=0}^{t''d-i} q^{\rho'_2+i}(x_1x_2 \otimes g^{i+1})\epsilon^*_{2td+t'd-i,t''d-i},$$

$$[(1 \otimes g^i)\epsilon^*_{t'd-i,t''d-i}, (x_1x_2 \otimes g)\epsilon^*_{0,2td}]$$
$$= (-1)^{t'd-i}\sum_{\rho''_1=0}^{t'd-i-1} q^{\rho''_1+1}(x_2 \otimes g^{i+1})\epsilon^*_{t'd-i-1,2td+t''d-i}$$
$$- (-1)^{t'd-i}\sum_{\rho'_2=2td}^{t''d-i-1} q^{\rho'_2-2td+i+1}(x_1 \otimes g^{i+1})\epsilon^*_{t'd-i,2td+t''d-i-1},$$

$$[(x_1 \otimes g^i)\epsilon^*_{t'd-i+1,t''d-i}, (x_1x_2 \otimes g)\epsilon^*_{0,2td}]$$
$$= (-1)^{t'd-i+1}\sum_{\rho''_1=0}^{t'd-i} q^{\rho''_1+1}(x_1x_2 \otimes g^{i+1})\epsilon^*_{t'd-i-1,2td+t''d-i},$$

$$[(x_2 \otimes g^i)\epsilon^*_{t'd-i,t''d-i+1}, (x_1x_2 \otimes g)\epsilon^*_{0,2td}]$$
$$= (-1)^{t'd-i}\sum_{\rho'_2=2td}^{2td+t''d-i} q^{\rho'_2-2td+i}(x_1x_2 \otimes g^{i+1})\epsilon^*_{t'd-i,2td+t''d-i}$$
$$- (2td)q(x_1x_2 \otimes g^{i+1})\epsilon^*_{t'd-i,2td+t''d-i},$$

$$[(x_1\otimes g^j)\epsilon^*_{t''d-j+1,t'''d-j},(1\otimes g^i)\epsilon^*_{td-i,t'd-i}]$$

$$= -(td-i)(1\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j,(t'+t''')d-i-j},$$

$$[(x_2\otimes g^j)\epsilon^*_{t'd-j,t''d-j+1},(1\otimes g^i)\epsilon^*_{td-i,t'd-i}]$$

$$= -(t'd-i)q^i(1\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j,(t'+t''')d-i-j},$$

$$[(x_1x_2\otimes g^j)\epsilon^*_{t'd-j+1,t''d-j+1},(1\otimes g^i)\epsilon^*_{td-i,t'd-i}]$$

$$= (-1)^{t''d-j+1}\sum_{\rho_1''=0}^{td-j-1} q^{\rho_1''+1}(x_2\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j,(t'+t''')d-i-j+1}$$

$$+ (-1)^{td-i}\sum_{\rho_2'=t'''d-i+1}^{(t'+t''')d-i-j} q^{\rho_2'-t'''d+i+j}(x_1\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j+1,(t'+t''')d-i-j},$$

$$[(x_1x_2\otimes g^j)\epsilon^*_{0,0},(1\otimes g^i)\epsilon^*_{td-i,t'd-i}]$$

$$= (-1)^{td-i}\sum_{\rho_1''=0}^{td-i-1} q^{\rho_1''+1}(x_2\otimes g^{i+j})\epsilon^*_{td-i-1,t'd-i}$$

$$+ (-1)^{td-i}\sum_{\rho_2'=0}^{t'd-i-1} q^{\rho_2'+i+1}(x_1\otimes g^{i+j})\epsilon^*_{td-i,t'd-i-1},$$

$$[(x_1\otimes g^j)\epsilon^*_{t''d-j+1,t'''d-j},(x_1\otimes g^i)\epsilon^*_{td-i+1,t'd-i}]$$

$$= ((t+t'')d-i-j)(x_1\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j+1,(t'+t''')d-i-j},$$

$$[(x_2\otimes g^j)\epsilon^*_{t'd-j,t''d-j+1},(x_1\otimes g^i)\epsilon^*_{td-i+1,t'd-i}]$$

$$= (t''d-j)(x_2\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j,(t'+t''')d-i-j+1}$$

$$- (t'd-i)q^i(x_1\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j+1,(t'+t''')d-i-j},$$

$$[(x_1x_2\otimes g^j)\epsilon^*_{t'd-j+1,t''d-j+1},(x_1\otimes g^i)\epsilon^*_{td-i+1,t'd-i}]$$

$$= (t''d-j+1)(x_1x_2\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j+1,(t'+t''')d-i-j+1}$$

$$+ (-1)^{td-i}\sum_{\rho_1''=0}^{td-i} q^{\rho_1''+1}(x_1x_2\otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j+1,(t'+t''')d-i-j+1},$$

$$[(x_1x_2\otimes g^j)\epsilon^*_{0,0},(x_1\otimes g^i)\epsilon^*_{td-i+1,t'd-i}]$$

$$= (-1)^{td-i} \sum_{\rho_1''=0}^{td-i} q^{\rho_1''+1}(x_1x_2 \otimes g^{i+j})\epsilon^*_{td-i,t'd-i},$$

$$[(x_2 \otimes g^j)\epsilon^*_{t'd-j,t''d-j+1}, (x_2 \otimes g^i)\epsilon^*_{td-i,t'd-i+1}]$$

$$[(t'''d-j+1)q^j - (t'd-i+1)q^i](x_2 \otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j,(t'+t''')d-i-j+1},$$

$$\begin{aligned}
&[(x_1x_2 \otimes g^j)\epsilon^*_{t'd-j+1,t''d-j+1}, (x_2 \otimes g^i)\epsilon^*_{td-i,t'd-i+1}] \\
&\quad = (t'''d-j+1)q^j(x_1x_2 \otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j+1,(t'+t''')d-i-j+1} \\
&\qquad - (-1)^{td-i} \sum_{\rho_2''=0}^{t'd-j} q^{\rho_2''-t''d+j+i-1}(x_1x_2 \otimes g^{i+j})\epsilon^*_{(t+t'')d-i-j+1,(t'+t''')d-i-j+1},
\end{aligned}$$

and

$$\begin{aligned}
&[(x_1x_2 \otimes g^j)\epsilon^*_{0,0}, (x_2 \otimes g^i)\epsilon^*_{td-i,t'd-i+1}] \\
&\qquad = (-1)^{td-i+1} \sum_{\rho_2'=0}^{t'd-i} q^{\rho_2'+i}(x_1x_2 \otimes g^{i+j})\epsilon^*_{td-i,t'd-i}.
\end{aligned}$$

We can compare these results to the computations in [11, Section 5.3] and [7, Section 3.1] when $i = 0$, given in Example 3.2.8.

5.2.2 q a dth root of unity for $d > 2$ even

If $d > 2$ is even, then

$$\begin{aligned}
C_{g^i} =&\{\gamma \in (\mathbb{N} \cup \{-1\})^2 | \ \gamma_1 = -1 \text{ or } (\exists\ t, t' \in \mathbb{N}, \gamma_2 + i = t\frac{d}{2} \text{ and } |\gamma| = t' \\
&\qquad \text{and } t + t' \text{ even}) \text{ and} \\
&\qquad \gamma_2 = -1 \text{ or } (\exists\ t, t' \in \mathbb{N}, \gamma_1 + i = t\frac{d}{2} \text{ and } |\gamma| = t' \\
&\qquad \text{and } t + t' \text{ even})\} \\
=&\{\gamma \in (\mathbb{N} \cup \{-1\})^2 | \ \gamma_1 = -1 \text{ and } \gamma_2 = -1 \text{ or} \\
&\qquad \gamma_1 = -1 \text{ and } \exists\ t \in \mathbb{N}, \gamma_2 = t' + 1 \text{ and}
\end{aligned}$$

$$i = t\frac{d}{2} + 1 \text{ and } t + t' \text{ even} \quad \text{or}$$

$$\gamma_2 = -1 \text{ and } \exists\, t \in \mathbb{N}, \gamma_1 = t' + 1 \text{ and}$$

$$i = t\frac{d}{2} + 1 \text{ and } t + t' \text{ even} \quad \text{or}$$

$$\exists\, t, t' \in \mathbb{N}, \gamma_1 = t\frac{d}{2} - i \text{ and } \gamma_2 = t'\frac{d}{2} - i \text{ and } t + t' \text{ even}\}.$$

From this description, we immediately get

$$\begin{aligned}
\mathrm{HH}^\bullet(\Lambda^2_{\mathbf{q}}, \Lambda^2_{\mathbf{q}} \rtimes G) \cong \bigoplus_{t\in\mathbb{N}} & span_k\{(x_1 \otimes g)\epsilon^*_{0,2t+1}, (x_1 \otimes g^{\frac{d}{2}+1})\epsilon^*_{0,2t}, (x_2 \otimes g)\epsilon^*_{2t+1,0}, \\
& (x_2 \otimes g^{\frac{d}{2}+1})\epsilon^*_{2t,0}, (x_1x_2 \otimes g)\epsilon^*_{2t,0}, (x_1x_2 \otimes g^{\frac{d}{2}+1})\epsilon^*_{2t+1,0}, \\
& (x_1x_2 \otimes g)\epsilon^*_{0,2t}, (x_1x_2 \otimes g^{\frac{d}{2}+1})\epsilon^*_{0,2t+1}\} \\
\bigoplus_{i=1}^{d} \bigoplus_{\substack{t,t'\in\mathbb{N} \\ t+t' \text{ even}}} & span_k\{(1 \otimes g^i)\epsilon^*_{t\frac{d}{2}-i,t'\frac{d}{2}-i}, (x_1 \otimes g^i)\epsilon^*_{t\frac{d}{2}-i+1,t'\frac{d}{2}-i}, \\
& (x_2 \otimes g^i)\epsilon^*_{t\frac{d}{2}-i,t'\frac{d}{2}-i+1}, (x_1x_2 \otimes g^i)\epsilon^*_{t\frac{d}{2}-i+1,t'\frac{d}{2}-i+1}\} \\
\bigoplus_{i=1}^{d} & span_k\{(x_1x_2 \otimes g^i)\epsilon^*_{0,0}, (1 \otimes g^i)\epsilon^*_{0,0}\}
\end{aligned}$$

as a vector space.

Then the G-invariant subspace is

$$\begin{aligned}
\mathrm{HH}^\bullet(\Lambda^2_{\mathbf{q}} \rtimes G) \cong \bigoplus_{t\in\mathbb{N}} & span_k\{(x_1 \otimes g)\epsilon^*_{0,td-1}, (x_1 \otimes g^{\frac{d}{2}-1})\epsilon^*_{0,td}, (x_2 \otimes g)\epsilon^*_{td-1,0}, \\
& (x_2 \otimes g^{\frac{d}{2}+1})\epsilon^*_{td,0}, (x_1x_2 \otimes g)\epsilon^*_{td,0}, (x_1x_2 \otimes g^{\frac{d}{2}+1})\epsilon^*_{td-1,0}, \\
& (x_1x_2 \otimes g)\epsilon^*_{0,td}, (x_1x_2 \otimes g^{\frac{d}{2}+1})\epsilon^*_{0,td-1}\} \\
\bigoplus_{i=1}^{d} \bigoplus_{\substack{t,t'\in\mathbb{N} \\ t+t' \text{ even}}} & span_k\{(1 \otimes g^i)\epsilon^*_{t\frac{d}{2}-i,t'\frac{d}{2}-i}, (x_1 \otimes g^i)\epsilon^*_{t\frac{d}{2}-i+1,t'\frac{d}{2}-i},
\end{aligned}$$

$$(x_2 \otimes g^i)\epsilon^*_{t\frac{d}{2}-i,t'\frac{d}{2}-i+1}, (x_1x_2 \otimes g^i)\epsilon^*_{t\frac{d}{2}-i+1,t'\frac{d}{2}-i+1}\}$$

$$\bigoplus_{i=1}^{d} span_k\{(x_1x_2 \otimes g^i)\epsilon^*_{0,0}, (1 \otimes g^i)\epsilon^*_{0,0}\}$$

as a vector space.

5.2.3 $q = -1$

Now assume $q = -1$. That is, $d = 2$ and we are in the commutative truncated polynomial case. Then

$$\begin{aligned}
C_{g^i} =& \{\gamma \in (\mathbb{N} \cup \{-1\})^2 | \ \gamma_1 = -1 \text{ or } \exists\ t \in \mathbb{N}, \gamma_1 - i = 2t \text{ and} \\
& \gamma_2 = -1 \text{ or } \exists\ t \in \mathbb{N}, \gamma_2 - i = 2t\} \\
=& \{\gamma \in (\mathbb{N} \cup \{-1\})^2 | \ \gamma_1 = -1 \text{ and } \gamma_2 = -1 \text{ or} \\
& \gamma_1 = -1 \text{ and } \exists\ t \in \mathbb{N}, \gamma_2 = 2t + i \text{ or} \\
& \gamma_2 = -1 \text{ and } \exists\ t \in \mathbb{N}, \gamma_1 = 2t + i \text{ or} \\
& \exists\ t, t' \in \mathbb{N}, \gamma_1 = 2t + i \text{ and } \gamma_2 = 2t' + i\}.
\end{aligned}$$

From this description, we immediately get

$$\begin{aligned}
\mathrm{HH}^\bullet(\Lambda^2_{\mathbf{q}}, \Lambda^2_{\mathbf{q}} \rtimes G) \cong & \bigoplus_{i=1}^{2} \bigoplus_{t \in \mathbb{N}} span_k\{(x_1 \otimes g^i)\epsilon^*_{0,2t+i}, (x_2 \otimes g^i)\epsilon^*_{2t+i,0}, (x_1x_2 \otimes g^i)\epsilon^*_{2t+i+1,0}, \\
& (x_1x_2 \otimes g^i)\epsilon^*_{0,2t+i+1}\} \\
& \bigoplus_{i=1}^{2} \bigoplus_{t,t' \in \mathbb{N}} span_k\{(1 \otimes g^i)\epsilon^*_{2t+i,2t'+i}, (x_1 \otimes g^i)\epsilon^*_{2t+i+1,2t'+i}, \\
& (x_2 \otimes g^i)\epsilon^*_{2t+i,2t'+i+1}, (x_1x_2 \otimes g^i)\epsilon^*_{2t+i+1,2t'+i+1}\} \\
& \bigoplus_{i=1}^{d} span_k\{(x_1x_2 \otimes g^i)\epsilon^*_{0,0}, (1 \otimes g^i)\epsilon^*_{0,0}\}
\end{aligned}$$

as a vector space.

Then, after taking the G-invariant subspace, we get

$$\begin{aligned}\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}^2 \rtimes G) \cong & \bigoplus_{t\in\mathbb{N}} span_k\{(x_1\otimes g)\epsilon^*_{0,2t+1}, (x_2\otimes g)\epsilon^*_{2t+1,0}, (x_1x_2\otimes g)\epsilon^*_{2t,0}, \\ & \quad (x_1x_2\otimes g)\epsilon^*_{0,2t}\} \\ & \bigoplus_{i=1}^{2}\bigoplus_{t,t'\in\mathbb{N}} span_k\{(1\otimes g^i)\epsilon^*_{2t+i,2t'+i}, (x_1\otimes g^i)\epsilon^*_{2t+i+1,2t'+i}, \\ & \qquad (x_2\otimes g^i)\epsilon^*_{2t+i,2t'+i+1}, (x_1x_2\otimes g^i)\epsilon^*_{2t+i+1,2t'+i+1}\} \\ & \bigoplus_{i=1}^{d} span_k\{(x_1x_2\otimes g^i)\epsilon^*_{0,0}, (1\otimes g^i)\epsilon^*_{0,0}\}\end{aligned}$$

as a vector space.

5.2.4 $q = 1$

Finally, if $q = 1$, we are considering the truncated skew polynomial ring. In this case,

$$\begin{aligned}C_1 =& \{\gamma \in (\mathbb{N}\cup\{-1\})^2 \mid \gamma_1 = -1 \text{ or } \exists\ t\in\mathbb{N}, \gamma_1+\gamma_2 = 2t \text{ and} \\ & \qquad \gamma_2 = -1 \text{ or } \exists\ t\in\mathbb{N}, \gamma_1+\gamma_2 = 2t\} \\ =& \{\gamma \in (\mathbb{N}\cup\{-1\})^2 \mid \gamma_1 = -1 \text{ and } \gamma_2 = -1 \text{ or} \\ & \qquad \gamma_1 = -1 \text{ and } \exists\ t\in\mathbb{N}, \gamma_2 = 2t+1 \text{ or} \\ & \qquad \gamma_2 = -1 \text{ and } \exists\ t\in\mathbb{N}, \gamma_1 = 2t+1 \text{ or} \\ & \qquad \exists\ t\in\mathbb{N}, \gamma_1+\gamma_2 = 2t\}.\end{aligned}$$

From this description, we immediately get

$$\begin{aligned}
\mathrm{HH}^{\bullet}(\Lambda_{\mathbf{q}}^2 \rtimes G) \cong & \bigoplus_{t\in\mathbb{N}} span_k\{(x_1 \otimes 1)\epsilon^*_{0,2t+1}, (x_2 \otimes 1)\epsilon^*_{2t+1,0}, (x_1x_2 \otimes 1)\epsilon^*_{2t,0}, \\
& \quad (x_1x_2 \otimes 1)\epsilon^*_{0,2t}\} \\
& \bigoplus_{\substack{t,t'\in\mathbb{N} \\ t+t' \text{ even}}} span_k\{(1 \otimes 1)\epsilon^*_{t,t'}, (x_1 \otimes 1)\epsilon^*_{t+1,t'}, (x_2 \otimes 1)\epsilon^*_{t,t'+1}, \\
& \quad (x_1x_2 \otimes 1)\epsilon^*_{t+1,t'+1}\} \\
& \bigoplus span_k\{(x_1x_2 \otimes 1)\epsilon^*_{0,0}, (1 \otimes 1)\epsilon^*_{0,0}\}
\end{aligned}$$

as a vector space.

Because the group action is trivial in this case, we can directly compare this result to [7, Section 3.5] and [11, Section 5.6].

6. CONCLUSION

In this dissertation, we discussed the Gerstenhaber algebra structure of the Hochschild cohomology of noncommutative associative algebras over a field. We utilized the notion of twisted tensor products given by Bergh and Oppermann in [6] to compute the vector space and cup product structures of Hochschild cohomology of $\Lambda_{\mathbf{q}}^{(2,2,\ldots,2)}$. Using an alternative bracket definition given by Negron and Witherspoon in [23], we were able to formulate the Gerstenhaber bracket on $\mathrm{HH}^\bullet(\Lambda_{\mathbf{q}}^{(2,2,\ldots,2)})$. As well as characterizing the structure of Hochschild cohomology of a large class of algebras, when the group acting on $\Lambda_{\mathbf{q}}^{(2,2,\ldots,2)}$ is trivial, this work provides a complete description of the Gerstenhaber algebra structure on Hochschild cohomology of this class of quantum complete intersections, extending the previous results of [7] and [25] to include the bracket structure. While brackets are useful for determining formal deformations, the bracket structure is less understood than the other structures on Hochschild cohomology.

The work presented in this dissertation has natural extensions for future work. Continuing my work on quantum complete intersections, I want to investigate the Gerstenhaber algebra structure of Hochschild cohomology of group extensions of more general quantum complete intersections, $\Lambda_{\mathbf{q}}^{(m_1,m_2,\ldots,m_n)}$, allowing generators to be truncated at arbitrary powers. The techniques used by Oppermann in [25] can potentially be extended to the group extensions of quantum complete intersections to assist in computing the vector space and cup structure on cohomology. The bracket structure however may prove to be more difficult in this more general setting. In this dissertation, the quantum complete intersections considered were Koszul algebras and thus brackets could be computed using the techniques given by Negron and

Witherspoon in [23]. When $m_i \neq 2$ for any $i \in \{1, 2, ..., n\}$, the quantum complete intersections are not Koszul. There is a natural choice of projective resolution for single generator truncated polynomial rings as given in [29, Exercise 9.1.4]. Taking the total complex of the twisted tensor product of these natural resolutions would result in a resolution for $\Lambda_{\mathbf{q}}^{(m_1,m_2,...,m_n)}$. However, it is not immediately clear that this resolution construction satisfies Conditions 2.3.3, allowing us to use the techniques of [23] for bracket computations.

In addition, I plan to study quantum complete intersections and their group extensions in more detail by studying the deformations of these algebras. In deformation theory, one deforms or perturbs the algebra structure slightly to create a class of related algebras by introducing a new parameter. By studying the deformations of an algebra, one can characterize a large class of closely related algebras. Deformations of group extensions of quantum complete intersections may be related to other algebras of interest, such as restricted rational Cherednik algebras. To begin this research, I am currently working with Christine Uhl, a graduate student at the University of North Texas who has experience with deformations of similar algebras, to study a particular class of deformations of group extensions of quantum complete intersections. The goal is to achieve a result comparable to that of Naidu and Witherspoon in [21], bridging the gap between the homological perspective of deformations and PBW conditions of quantum Drinfeld Hecke algebras.

REFERENCES

[1] L. L. Avramov, V. N.Gasharov, and I. V. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. **86** (1997), 67—114.

[2] L. L. Avramov and S. Iyengar, *Gaps in Hochschild cohomology imply smoothness for commutative algebras*, Math. Res. Letters **12** (2005), 789–804.

[3] Y. Bazlov and A. Berenstein, *Noncommmutative Dunkl operators and braided Cherednik algebras*, Selecta Math. (N.S.) **14**:3–4 (2009), 325–372.

[4] P. A. Bergh, *On the (co)homology of quantum exterior algebras*, Comm. Algebra **35**:11 (2007), 3440–3450.

[5] P. A. Bergh and K. Erdmann, *Homology and cohomology of quantum complete intersections*, Algebra and Number Theory **2**:5 (2008), 501–522.

[6] P. A. Bergh and S. Oppermann, *Cohomology of twisted tensor products*, J. Algebra **320** (2008), 3327–3338.

[7] R.-O. Buchweitz, E. L. Green, D. Madsen, and Ø. Solberg, *Finite Hochschild cohomology without finite global dimension*, Math. Research Letters **12** (2005), 805–816.

[8] R.-O. Buchweitz, E. L. Green, N. Snashall, and Ø. Solberg, *Multiplicative structures for Koszul algebras*, Quar. J. Math. **59**:4 (2008), 441–454.

[9] I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, Ann. Math. (2) **141**:1 (1995), 191–216.

[10] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. Math. **78**:2 (1963), 267–288.

[11] L. Grimley, V. C. Nguyen, and S. Witherspoon, *Gerstenhaber brackets on Hochschild cohomology of twisted tensor products*, arXiv:1503.03531 (2015), submitted.

[12] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Sèminaire d'Algèbre Paul Debreil et Marie-Paul Malliavin (Paris, 1987/1988), Lecture Notes Math. **1404**, Springer, Berlin (1989), 108–126.

[13] P. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer, 1997.

[14] G. Hochschild, *On the Cohomology Groups of an Associative Algebra*, Ann. Math. **46**:1 (1945), 58–67.

[15] E. Kirkman, J. Kuzmanovich, and J.J. Zhang, *Noncommutative complete intersections*, J. Algebra **429** (2015), 253–286.

[16] J. Le and G. Zhou, *On the Hochschild cohomology ring of tensor products of algebras*, J. Pure App. Algebra **218** (2014), 1463–1477.

[17] M. Lorenz, *On the homology of graded algebras*, Comm. Algebra **20** (1992), 489–507.

[18] S. Mac Lane, *Homology*, Springer-Verlag, 1995.

[19] Y. I. Manin, *Some remarks on Koszul algebras and quantum groups*, Ann. Inst. Fourier (Grenoble) **37**:4 (1987), 191–205.

[20] D. Naidu, P. Shroff, and S. Witherspoon, *Hochschild cohomology of group extensions of quantum symmetric algebras*, Proc. Amer. Math. Soc. **139** (2011), 1553–1567.

[21] D. Naidu and S. Witherspoon, *Hochschild cohomology and quantum Drinfeld Hecke algebras*, arXiv:1111.5243 (2014), to appear in Selecta Math.

[22] C. Negron, *The cup product on Hochschild cohomology for localizations of filtered Koszul algebras*, arXiv:1304.0527 (2014).

[23] C. Negron and S. Witherspoon, *An alternate approach to the Lie bracket on Hochschild cohomology*, arXiv:1406.0036 (2015), to appear in Homology, Homotopy, and Appl.

[24] C. Negron and S. Witherspoon, *The Gerstenhaber bracket as a Schouten bracket for polynomial rings extended by finite groups*, arXiv:1511.02533 (2015).

[25] S. Oppermann, *Hochschild cohomology and homology of quantum complete intersections*, Algebra and Number Theory **4**:7 (2010), 821–838.

[26] S. Sánchez-Flores, *The Lie module structure on the Hochschild cohomology groups of monomial algebras with radical square zero*, J. Algebra **320** (2008), 4249–4269.

[27] A. Shepler and S. Witherspoon, *Group actions on algebras and the graded Lie structure of Hochschild cohomology*, J. Algebra **351** (2012), 350–381.

[28] D. Ştefan, *Hochschild cohomology on Hopf Galois extensions,* J. Pure Appl. Algebra **103** (1995), 221–233.

[29] M. Wambst, *Complexes de Koszul quantiques*, Ann. Inst. Fourier, Grenoble **43**:4 (1993), 1089–1156.

[30] C. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.