K-THEORETIC DYNAMICS AND FINITENESS IN C*-CROSSED PRODUCTS

A Dissertation

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2015

Major Subject: Mathematics

ABSTRACT

This work explores the interplay of C*-dynamics and K-theory. More precisely, we study the extent to which various forms of finite-dimensional approximation properties of a topological nature, witnessed in reduced C*-crossed products, are reflections of approximation conditions at the level of the dynamics. Such conditions admit purely algebraic K-theoretical interpretations that we describe and utilize to prove deep structural results.

We introduce the notions of Matricial Field (MF) and Residually Finite Dimensional (RFD) actions of a discrete group Γ on an arbitrary C*-algebra A. These actions have spatial interpretations in the case where the algebra $A = C(X)$ is commutative; these are described. We show that a reduced crossed product $A \rtimes_\lambda \Gamma$ is MF (RFD) if and only if the reduced group C*-algebra $C^*_\lambda(\Gamma)$ is MF (RFD), and the action is MF (RFD). Examples include the limit periodic actions defined by Voiculescu and, in the classical case, the chain recurrent $\mathbb{Z}$-systems of Pimsner. In the presence of sufficiently many projections MF and RFD actions can be expressed by elegant, simple, K-theoretic conditions.

We then focus on actions of free groups on AF-algebras, in which case we prove that a K-theoretic coboundary condition determines whether or not the reduced crossed product is a Matricial Field (MF) algebra. One upshot is the equivalence of stable finiteness and being MF for these reduced crossed product algebras. We also exhibit crossed product algebras for which the *Ext* semigroup is not a group; indeed any action of a free group on a UHF algebra gives rise to an MF crossed product whose *Ext* semigroup is not a group.

Minimal C*-systems (A, Γ) are described by certain filling conditions witnessed

at the level of the induced actions of Γ on $K_0(A)$ and on the Cuntz semigroup $W(A)$. A notion of topological transitivity is defined for noncommutative systems again in terms of the induced action on K-theory. We prove that prime reduced crossed products come from topological transitive actions and, conversely, topologically transitive and properly outer systems yield prime reduced crossed products.

In the presence of sufficiently many projections we associate to each noncommutative C*-system (A, Γ, α) a type semigroup $S(A, \Gamma, \alpha)$ which reflects much of the spirit of the underlying action. We characterize purely infinite, as well as stably finite, crossed products by means of the infinite or rather finite nature of this semigroup. Using ideas of paradoxical decompositions we obtain, for a certain class of simple crossed products, a dichotomy between the stably finite and purely infinite.

ACKNOWLEDGEMENTS

Nearing the end of this significant chapter of my life, it seems good to pause, reflect, and acknowledge those individuals who have made this achievement possible.

Earning a doctoral degree at Texas A&M University has been an unforgettable privilege and a tremendous honor. It is fitting that I first thank the mathematics department at Texas A&M for giving me this opportunity to pursue a PhD. With every original lemma, theorem, and seminar talk I hope to make her proud and faithfully represent the excellence of the department. The synergy and unity of the math community at A&M has left me with a positive impression of academia and is, in my opinion, exemplary to all institutions of higher learning.

None of my accomplishments would have been possible if it were not for the dedicated professors that were true to their calling. I wish to single out those men who were successful in teaching me the beauty of mathematics and molding an academic mind ready to impart the knowledge that was entrusted to me. Thank you to Professors Dykema and Smith who always took time to answer questions and offer their help selflessly. I want to especially thank Professor Smith who was instrumental in recruiting me and guiding me with wisdom throughout my degree. Thank you to Professor Sivakumar who taught me many things, supported me, encouraged me, and offered invaluable advice. I would also like to thank Professor Yu for his insight and persistent encouragement. It is, however, my adviser, Professor David Kerr, that has had the most profound impact on my academic formation and is most deserving of my gratitude. He has imparted to me a deep love for mathematics, in particular operator algebras, and sowed a seed of mathematical inquisitiveness that will follow me well into my future. I thank him for his teaching, his support (financial as well

as emotional), and his selfless devotion to my success. As I prepare to leave College Station I am comforted by the fact that the relationships I formed with the above-mentioned leaders in academia will continue, and as such, my sense of gratitude will persist.

Finally, I am most grateful to my dear family back home in Montreal. I thank them for their love, support, and patience throughout this lengthy journey. One cannot ask for better parents than those I have been blessed with. My mom and dad's prayers are unending and their encouragement is truly refreshing. A very special feeling of gratitude is reserved for my father who has been a dependable rock and anchor throughout my life. He led his children to new heights of achievement far beyond the scope of our imagination.

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1. INTRODUCTION AND PRELIMINARIES

Operator algebras is a rich and diverse field that bridges several disciplines in mathematics. It distinctively blends ideas from functional and harmonic analysis, noncommutative geometry, topology, group theory and dynamics. Initiated in the 1930's and 1940's in a series of papers by Murray and Von Neumann, the theory of operator algebras is broad, interacting with almost every area of mathematics and admitting applications in areas such as mathematical physics, quantum mechanics and the development of quantum field theory. While the treatise of von Neumann algebras progressed and presented with substantial achievements, most notably the great classification theorems of Alain Connes, George Elliott initiated a far reaching program of classifying separable nuclear C*-algebras by their K-theoretic data. As the study of C*-algebras is appropriately termed noncommutative topology in light of Gelfand's theory, this work lies in the framework of noncommutative topological dynamics. More precisely, using classification and K-theoretic techniques, we establish several structure theorems for C*-algebras arising from crossed products.

Dynamical systems and the theory of operator algebras are inextricably related [7], [27], [36]. Topological dynamics has long played a significant role in the study and classification of amenable C*-algebras by providing a wealth of examples that fall under the umbrella of Elliott's classification program as well as examples that lack certain regularity properties [51], [52], [20], [18]. In his recent survey article on operator algebra structure theory [7], Blackadar rightly speaks of the "algebraization of dynamics". The crossed product construction permits the exploitation of symmetry through the acting group and is generous enough to produce a variety of C*-algebraic phenomena [36],[54]. Indeed, the dynamics provide a tool for the coordinatization of algebraic structure. One would like to decipher information about the the crossed product algebra by studying the dynamics and, conversely, describe the nature of the dynamics by looking at the operator algebra's structure and invariants. The transition from classical dynamics to noncommutative topological dynamics presents several challenges and subtleties. One way to approach these issues (which has been very fruitful in my work) is to interpret dynamical conditions K-theoretically and use classification techniques to uncover pertinent information.

1.1 Finite-Dimensional Approximation Properties

Unless otherwise specified, we make the blanket assumption that all C*-algebras A will be considered separable and with unit 1_A, and all groups Γ will be discrete. We will frequently encounter the free group $\mathbb{F}_r$ on r generators. If A and B are C*-algebras we will write $A \odot B$ for the $*$-algebraic tensor product, and $A \otimes B$ for the minimal (spatial) tensor product. Let's recall a few definitions.

A C*-algebra C is said to be *exact* provided that the functor $- \otimes C$ is exact. More precisely, C is exact provided that for every exact sequence of C*-algebras

$$0 \longrightarrow J \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

the sequence

$$0 \longrightarrow J \otimes C \xrightarrow{\varphi \otimes \mathrm{id}_C} A \otimes C \xrightarrow{\psi \otimes \mathrm{id}_C} B \otimes C \longrightarrow 0$$

is also exact.

In 1997 Blackadar and Kirchberg introduced in [8] the so called *matricial field (MF)* algebras. A separable C^*-algebra A is said to be MF if it can be expressed as a generalized inductive system of finite-dimensional algebras, or equivalently, if there is a natural sequence $\mathbf{n} = (n_k)_{k\geq 1}$ and a $*$-monomorphism

$$\iota : A \hookrightarrow Q_{\mathbf{n}} := \prod_{k=1}^{\infty} \mathbb{M}_{n_k} \Big/ \bigoplus_{k=1}^{\infty} \mathbb{M}_{n_k}.$$

Denote by $\pi : \prod_{k=1}^{\infty} \mathbb{M}_{n_k} \to Q_{\mathbf{n}}$ the canonical quotient mapping. If such an embedding ι exists along with a u.c.p. lift, that is, a unital completely positive map $\Phi : A \to \prod_{k=1}^{\infty} \mathbb{M}_{n_k}$ such that $\pi \circ \Phi = \iota$, A is said to be *quasidiagonal*. A good treatise on QD algebras can be found in [11]. It is readily seen that an algebra A is MF (QD) if it satisfies the following local property: for every $\varepsilon > 0$ and finite set $\Omega \subset A$, there is a k and $*$-linear (u.c.p.) map $\psi : A \to \mathbb{M}_k$ such that

$$\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon \qquad \forall a, b \in \Omega,$$
$$\big|\|\psi(a)\| - \|a\|\big| < \varepsilon \qquad \forall a \in \Omega.$$

Recall that a separable algebra A is said to be *residually finite dimensional* (RFD) if there is a sequence of $*$-homomorphisms $\psi_n : A \to \mathbb{M}_{k_n}$ with $\|\psi_n(a)\| \nearrow \|a\|$ for all $a \in A$. Clearly being MF, QD or RFD passes to C*-subalgebras, RFD algebras are QD, and QD algebras are MF. Moreover, MF algebras are stably finite. To see this, suppose $(a_k)_{k\geq 1} \in \prod_{k=1}^{\infty} \mathbb{M}_{n_k}$ is a sequence with

$$1_{Q_{\mathbf{n}}} = \pi((a_k)_k)^* \pi((a_k)_k) = \pi((a_k^* a_k)_k),$$

it follows then that $\|a_k^* a_k - 1_{\mathbb{M}_{n_k}}\| \to 0$. A little spectral theory shows that $\|a_k a_k^* - 1_{\mathbb{M}_{n_k}}\| \to 0$, so that $\pi((a_k))\pi((a_k))^* = 1_{Q_{\mathbf{n}}}$ thus $Q_{\mathbf{n}}$ is finite and hence A, being isomorphic to a unital subalgebra of $Q_{\mathbf{n}}$, is also finite. Since $M_n(A)$ is also MF, A is stably finite. It is still unknown whether or not stably finite algebras are MF, or if there is a countable discrete group Γ for which $C^*_\lambda(\Gamma)$ fails to be MF.

There are several notions of 'rank' in the C*-literature which are meant to reflect notions of 'dimension' in topology. In this piece we shall only need *real rank* and *stable rank*. Write $\mathrm{GL}(A)$ for the set of invertibles in A, and A_{sa} for the set of self-adjoint elements. The C*-algebra A is of stable rank one, written $\mathrm{sr}(A) = 1$, if $\mathrm{GL}(A) \subset A$ is norm-dense, and A is of real rank zero, written $\mathrm{RR}(A) = 0$, if $\mathrm{GL}(A) \cap A_{\mathrm{sa}} \subset A_{\mathrm{sa}}$ is norm-dense. Algebras of real rank zero have an abundance of projections, indeed, one proves that A has real rank zero if and only if every element of the algebra can be approximated by a linear span of finitely many projections. In this case we say that the projections are *total* in A.

1.2 Dynamical Systems and Discrete C*-Crossed Products

Group actions pervade mathematics and much of this piece. Recall that a group action is simply a group homomorphism $h : \Gamma \to \mathrm{Perm}(E)$ from a group Γ to the group of permutations on an arbitrary set E. At times, for economy, we write $\Gamma \curvearrowright E$ to denote the action and $h_s(x) = s.x$ for $s \in \Gamma$ and $x \in X$. When E has additional structure, e.g. when $E = X$ is a topological space, $E = A$ a C*-algebra or $E = (G, G^+, u)$ an ordered abelian group, one imposes extra conditions on the action so that it respects the prescribed category. More precisely, by a *continuous action* $\Gamma \curvearrowright X$, or equivalently a *transformation group* (X, Γ), we mean a group homomorphism $h : \Gamma \to \mathrm{Homeo}(X)$ where $\mathrm{Homeo}(X)$ denotes the group of homeomorphisms of a locally compact Hausdorff space X. In an operator algebraic framework one speaks of a *C*-dynamical system* (A, Γ, α), where A is a C*-algebra, Γ a topological group and $\alpha : \Gamma \to \mathrm{Aut}(A)$ a continuous group homomorphism into $\mathrm{Aut}(A)$; the topological group of automorphisms of A with the point-norm topology. Again we emphasize that since Γ is discrete, we need not worry about the continuity of α. In the case where A

is a commutative algebra, say $A = C_0(X)$ for some locally compact Hausdorff space X, C*-systems $(C_0(X), \Gamma, \alpha)$ are in one-to-one correspondence with transformation groups (X, Γ) via the formula $\alpha_s(f)(x) = f(s^{-1}.x)$ where $s \in \Gamma, f \in C_0(X), x \in X$.

Given a C*-dynamical system (A, Γ, α), we write $A \rtimes_\alpha \Gamma$ to denote the full crossed product C*-algebra whereas $A \rtimes_{\lambda,\alpha} \Gamma$ will stand for the reduced algebra (at times we will omit the α). We briefly recall their construction and refer the reader to [12], [54] and [36] for more details. First consider the algebraic crossed product $A \rtimes_{\mathrm{alg},\alpha} \Gamma$ which is the complex linear space of all finitely supported functions

$$C_c(\Gamma, A) = \{\sum_{s \in F} a_s u_s : F \subset \Gamma, a_s \in A\},$$

equipped with a twisted multiplication and involution: for $s, t \in \Gamma, a, b \in A$

$$\begin{aligned} (au_s)(bu_t) &= a\alpha_s(b)u_{st}, \\ (au_s)^* &= \alpha_{s^{-1}}(a^*)u_{s^{-1}}. \end{aligned}$$

If $A \subset \mathbb{B}(\mathcal{H})$ is faithfully represented (the choice of representation is immaterial), the $*$-algebra $A \rtimes_{\mathrm{alg},\alpha} \Gamma$ can then be faithfully represented as operators on $\mathcal{H} \otimes \ell^2(\Gamma)$ via

$$au_s(\xi \otimes \delta_t) = \alpha_{st}^{-1}(a)\xi \otimes \delta_{st} \qquad \xi \in \mathcal{H}, \quad s, t \in \Gamma.$$

Completing with respect to the operator norm on $\mathbb{B}(\mathcal{H} \otimes \ell^2(\Gamma))$ gives the reduced crossed product $A \rtimes_{\lambda,\alpha} \Gamma$. To realize the full crossed product, for each $x \in A \rtimes_{\mathrm{alg},\alpha} \Gamma$, consider

$$\|x\|_u = \sup \|\pi(x)\|_{\mathbb{B}(\mathcal{H})}$$

where the supremum runs through all $*$-representations $\pi : A \rtimes_{\mathrm{alg},\alpha} \Gamma \to \mathbb{B}(\mathcal{H})$. Then

$$A \rtimes_\alpha \Gamma := A \rtimes_{\mathrm{alg},\alpha} \Gamma^{-\|\cdot\|_u}.$$

If Γ is an amenable group then we have $A \rtimes_\alpha \Gamma = A \rtimes_{\lambda,\alpha} \Gamma$. Furthermore, if Γ is amenable and A is nuclear then $A \rtimes_{\lambda,\alpha} \Gamma$ is nuclear as well.

We will at times make use of the conditional expectation $\mathbb{E} : A \rtimes_{\lambda,\alpha} \Gamma \to A$, which is a unital, contractive, completely positive map satisfying $\mathbb{E}(\sum_{s\in\Gamma} a_s u_s) = a_e$.

1.3 K-Theretical Dynamics and Finiteness

This work is K-theoretic in flavor; the reader may want to consult [6] for a suitable treatment thereof, as well as [2] for the necessary results concerning the Cuntz semigroup. We briefly outline the story-line of $K_0(A)$ and $W(A)$ here.

1.3.1 K-theory and the Cuntz Semigroup

If A is a C*-algebra, $M_{m,n}(A)$ will denote the linear space of all $m \times n$ matrices with entries from A. The square $n \times n$ matrices $M_n(A)$ is a C*-algebra with positive cone $M_n(A)^+$. If $a \in M_n(A)^+$ and $b \in M_m(A)^+$, write $a \oplus b$ for the matrix $\mathrm{diag}(a, b) \in M_{n+m}(A)^+$. Set $M_\infty(A)^+ = \bigsqcup_{n\geq 1} M_n(A)^+$; the set-theoretic direct limit of the $M_n(A)^+$ with connecting maps

$$M_n(A) \to M_{n+1}(A), \quad \text{given by} \quad a \mapsto a \oplus 0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Write $\mathcal{P}(A)$ for the set of projections in A and set $\mathcal{P}_\infty(A) = \bigsqcup_{n\geq 1} \mathcal{P}(M_n(A))$.

Elements a and b in $M_\infty(A)^+$ are said to be *Pedersen-equivalent*, written $a \sim b$, if there is a matrix $v \in M_{m,n}(A)$ with $v^*v = a$ and $vv^* = b$. We say that a is *Cuntz-subequivalent* to (or *Cuntz-smaller* than) b, written $a \precsim b$, if there is a sequence

$(v_k)_{k\geq 1} \subset M_{m,n}(A)$ with $\|v_k^* b v_k - a\| \to 0$ as $k \to \infty$. If $a \precsim b$ and $b \precsim a$ we say that a and b are *Cuntz-equivalent* and write $a \approx b$. Restricting to projections, one can work out that for $p, q \in \mathcal{P}_\infty(A)$, $p \precsim q$ if and only if $p \oplus p' \sim q$ for some $p' \in \mathcal{P}_\infty(A)$ if and only if there is a subprojection $r \leq q$ with $p \sim r \leq q$. With a little work one shows that $\sim$ and $\approx$ are equivalence relations on $M_\infty(A)^+$ and that $a \sim b$ implies $a \approx b$. It is customary to write $V(A) = \mathcal{P}_\infty(A)/\sim$, and $[p]$ for the equivalence class of $p \in \mathcal{P}_\infty(A)$. Also set $W(A) := M_\infty(A)^+/\approx$ and write $\langle a \rangle$ for the class of $a \in M_\infty(A)^+$. $W(A)$ has the structure of a preordered abelian monoid with addition given by $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$ and preorder $\langle a \rangle \leq \langle b \rangle$ if $a \precsim b$. The monoid $W(A)$ embeds into $\mathrm{Cu}(A) := (A \otimes \mathcal{K})_+/\approx$, the *Cuntz semigroup* of A. For this work, the monoid $W(A)$ will be suitable for our purposes and we will refer to it as the Cuntz semigroup as in [2]. With addition and ordering identical to that of $W(A)$, $V(A)$ is also a preordered abelian monoid. There is a cardinal difference between the orderings on $V(A)$ and $W(A)$; the ordering on $W(A)$ extends the algebraic ordering ($x, y, z \in W(A)$ with $x + y = z$ implies $x \leq z$) but only in rare cases agrees with it. With $V(A)$, the ordering agrees with the algebraic one. Indeed, $[p] \leq [q]$ iff $p \precsim q$ iff $p \oplus p' \sim q$ for some p' which gives $[p] + [p'] = [q]$. As a brief reminder, $K_0(A) = \mathcal{G}(V(A))$ the Grothendieck enveloping group of $V(A)$ and $[p]_0 = \gamma([p])$ where $\gamma : V(A) \to K_0(A)$ is the canonical Grothendieck map.

1.3.2 Finiteness, Cancellation, and Refinement

Notions of 'finite' and 'infinite' are widespread throughout all disciplines of mathematics, including the theory of C*-algebras. We learn that a C*-algebra A is finite-dimensional (as a linear space) if and only if $A \cong \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_k}$. But C*-algebraists are more interested in the notion of finiteness that mirrors Dedekind finiteness for sets; recall that a set E is (Dedekind) infinite if and only if it admits a non-surjective injection $\sigma : E \to E$, and finite otherwise. In the same spirit, a concrete algebra $A \subset \mathbb{B}(\mathcal{H})$ should be 'infinite' if it admits a non-unitary isometry $v \in A$. We make this more precise.

Projections $p, q \in \mathcal{P}(A)$ are Murray-vonNeumann equivalent, written $p \sim q$, if there is a $v \in A$ with $v^*v = p$ and $vv^* = q$. A projection p in A is infinite if $p \sim q$ for some subprojection $q \lneq p$. It was shown in [30] that p infinite if and only if $p \oplus b \precsim p$ for some non-zero $b \in M_\infty(A)^+$. A unital C*-algebra A is said to be infinite if 1_A is infinite. Otherwise, A is called finite. If $M_n(A)$ is finite for every $n \in \mathbb{N}$ then A is called stably finite.

A unital, stably finite C*-algebra A yields an ordered abelian group $K_0(A)$ with positive cone $K_0(A)^+ := \gamma(V(A))$ and order unit $[1_A]_0$. In this case, a *state* on the ordered abelian group $(K_0(A), K_0(A)^+, [1]_0)$ is a group homomorphism $\beta : K_0(A) \to \mathbb{R}$ with $\beta(K_0(A)^+) \subset \mathbb{R}^+$ and $\beta([1]_0) = 1$. The collection of all such states is denoted by $S(K_0(A), K_0(A)^+, [1]_0)$. Every tracial state τ on A gives rise to a state $K_0(\tau) : K_0(A) \to \mathbb{R}$ via the formula

$$K_0(\tau)([p]_0) = \tau(p).$$

It is important to note that when A is exact, every state β on $(K_0(A), K_0(A)^+, [1]_0)$ arises in this way, that is, $\beta = K_0(\tau)$ for a tracial state τ on A.

Occasionally we shall require our algebras to have *cancellation*, which simply means that the Grothendieck map γ is injective. It is routine to check that algebras with stable rank one are stably finite and have cancellation. Moreover, when A is stably finite the map $V(A) \to W(A)$, $[p] \mapsto \langle p \rangle$ is injective. To see this, suppose $\langle p \rangle = \langle q \rangle$ for $p, q \in \mathcal{P}_\infty(A)$, then $p \precsim q$ and $q \precsim p$. Therefore, we can find $p', q' \in \mathcal{P}_\infty$

with

$$p \oplus p' \sim q, \quad \text{and} \quad q \oplus q' \sim p.$$

This gives

$$p \sim q \oplus q' \sim p \oplus p' \oplus q'.$$

Since A is stably finite, this is only possible if $p' = q' = 0$. Thus $p \sim q$ so $[p] = [q]$.

Recall that a semigroup K has the Riesz refinement property if, whenever $\sum_{j=1}^{n} x_j = \sum_{i=1}^{m} y_i$, for members $x_1, \ldots, x_n, y_1, \ldots, y_m \in K$, there exist $\{z_{ij}\}_{i,j} \subset K$ satisfying $\sum_i z_{ij} = x_j$ and $\sum_j z_{ij} = y_i$ for each i and j. If A is a stably finite algebra with $\mathrm{RR}(A) = 0$ then S. Zhang showed that $K_0(A)^+$ has the Riesz refinement property [56].

1.3.3 Induced K-Theoretic Dynamics

A C*-dynamical system induces a natural action at the K-theoretical level, and the order theoretical dynamics will reflect information about the nature of the action and will often describe the structure of the crossed product. If (G, G^+, u) and (H, H^+, v) are ordered abelian groups each with their distinguished order units, a morphism in this category is a group homomorphism $\beta : G \to H$ which is positive and order unit preserving, i.e. $\beta(G^+) \subset H^+$, and $\beta(u) = v$ respectively. We also write

$$\mathrm{OAut}(G) := \{\tau \in \mathrm{Aut}(G) : \tau(G^+) = G^+, \tau(u) = u\}$$

for the set of ordered abelian group automorphisms. When the group is $\mathbb{Z}^d$, we employ the standard ordering defined by the positive cone $(\mathbb{Z}^d)^+ := (\mathbb{Z}_{\geq 0})^d$, and whose order unit is $(1, 1, \ldots, 1)$. Recall that $(K_0(\mathbb{M}_d), K_0(\mathbb{M}_d)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$, and if X is a zero-dimensional compact metric space, $K_0(C(X)) \cong C(X; \mathbb{Z})$ with natural point-wise ordering. The K_0-functor is covariant, namely, if $\phi : A \to B$ is a $*$-homomorphism ($*$-automorphism), one obtains a positive group homomorphism (ordered group automorphism) $K_0(\phi) : K_0(A) \to K_0(B)$ defined by $K_0(\phi)([p]) = [\phi(p)]$ where p is a projection living in $\mathbb{M}_n(A)$ for some n. For economy we sometimes write $\hat{\phi} = K_0(\phi)$. Note that for every action $\alpha : \Gamma \to \mathrm{Aut}(A)$, there is an associated action $\hat{\alpha} : \Gamma \to \mathrm{OAut}(K_0(A))$ where $\hat{\alpha}(s) = \hat{\alpha}_s : K_0(A) \to K_0(A)$ is the induced automorphism. Again, in the case of stable finiteness, the positive cone $K_0(A)^+$ is a partially ordered monoid, whose ordering is inherited from $K_0(A)^+$ and coincides with the algebraic ordering. Restricting $\hat{\alpha}$ to $K_0(A)^+$ also gives an action of order isomorphisms. In the same manner a C*-system (A, Γ, α) induces an action $\hat{\alpha} : \Gamma \to \mathrm{OAut}(W(A))$ via $\hat{\alpha}_s(\langle a \rangle) = \langle \alpha_s(a) \rangle$, where $s \in \Gamma$, and $a \in M_\infty(A)^+$. Here $\mathrm{OAut}(W(A))$ will denote the set of monoid isomorphisms of $W(A)$ which respect the ordering.

2. MF CROSSED PRODUCTS[1]

In this chapter we are particularly interested in finite-dimensional approximation properties of C*-algebras. While nuclearity and exactness are measure-theoretic concepts, residual finite dimensionality, quasidiagonality and admitting norm microstates are properties more topological in nature as they concern matricial approximation of both the linear and multiplicative structure of the algebra. In this work we flesh out the appropriate dynamical conditions that give rise to such topological approximations in resulting reduced crossed products, and give K-theoretic expression to these conditions when the underlying algebras have sufficiently many projections. One purpose of this section is to provide a K-theoretic interpretation of dynamical approximation properties such as residual finiteness and quasidiagonal actions as introduced by Kerr and Nowak in [28], and by doing so, extend results found in [10] and [38].

In the classical setting, Pimsner described a purely topological dynamical property for a $\mathbb{Z}$-system $(X, \mathbb{Z})$ that renders the resulting crossed product $C(X) \rtimes_\lambda \mathbb{Z}$ AF-embeddable. He showed in [38] that for a self-homeomorphism T of a compact metrizable space X the following are equivalent: (1) the crossed product embeds into an AF algebra, (2) the crossed product is quasidiagonal, (3) the crossed product is stably finite, (4) "T compresses no open sets", that is, there does not exist a nonempty open set $U \subset X$ for which $T(\overline{U}) \subsetneq U$, which is equivalent to the action being *chain recurrent*, that is, for every $x \in X$ and every $\varepsilon > 0$, there are finitely many points $x = x_1, \ldots, x_n = x$ such that $d(T(x_j), x_{j+1}) < \varepsilon$ for $1 \leq j \leq n$.

It was N. Brown who saw condition (4) as being essentially K-theoretical, at

[1] Part of this chapter is reprinted with permission from "MF Actions and K-theoretic Dynamics" by Timothy Rainone, 2014. *J. Funct. Anal.*, 267, 542-578, Copyright 2014 by Elsevier B.V.

least in the presence of many projections [10]. When X is zero-dimensional we have $K_0(C(X)) = C(X;\mathbb{Z})$, and the chain recurrence condition is expressed as $\hat{T}(f) < f$ for no non-zero $f \in C(X;\mathbb{Z})$, where $\hat{T} : C(X;\mathbb{Z}) \to C(X;\mathbb{Z})$ is the induced order automorphism given by $\hat{T}(f) = f \circ T^{-1}$. Brown was then able to generalize Pimsner's result to the non-commutative setting as follows.

Theorem 2.0.1 (Brown)**.** *Let A be an AF algebra and $\alpha \in \mathrm{Aut}(A)$ an automorphism. Then the following are equivalent:*

1. *$A \rtimes_\alpha \mathbb{Z}$ is AF-embeddable.*

2. *$A \rtimes_\alpha \mathbb{Z}$ is quasidiagonal.*

3. *$A \rtimes_\alpha \mathbb{Z}$ stably finite.*

4. *The induced map $\hat{\alpha} : K_0(A) \to K_0(A)$ "compresses no element", that is, there is no $x \in K_0(A)$ for which $\hat{\alpha}(x) < x$; equivalently, $H_\alpha \cap K_0(A)^+ = \{0\}$, where H_α is the coboundary subgroup $\{x - \hat{\alpha}(x) \ : \ x \in K_0(A)\}$.*

One of the main results of this chapter, Theorem 2.2.14, extends Brown's result to the case of a free group on r generators acting on a unital AF algebra. In this case the coboundary subgroup is given by $H_\sigma = \mathrm{im}(\sigma)$ where $\sigma : \oplus_{j=1}^{r} K_0(A) \to K_0(A)$ is the coboundary morphism in the Pimsner-Voiculescu six-term exact sequence. In abbreviated form our theorem says the following.

Theorem 2.0.2. *Let A be a unital AF algebra and $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ an action of the free group on r generators. Then the following are equivalent:*

1. *$H_\sigma \cap K_0(A)^+ = \{0\}$.*

2. *The reduced crossed product $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is MF.*

3. The reduced crossed product $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is stably finite.

In order to extend the results of Pimsner and Brown to actions of discrete countable groups, one needs the right notion of chain recurrence for arbitrary transformation groups and a corresponding approximation property for C*-dynamical systems. D. Kerr and P. Nowak then introduced residually finite actions and quasidiagonal actions in [28] where it was shown that these systems give rise to MF crossed products provided that the reduced group C*-algebra of the acting group is itself MF. This is a necessary condition as being MF passes to subalgebras and the reduced group C*-algebra sits canonically inside the reduced crossed product. Thus one cannot hope for quasidiagonality, or much less AF-embeddability, when the acting group is non-amenable, for Rosenberg's result ([22]) asserts that a discrete group whose reduced C*-algebra is quasidiagonal must be amenable.

Matricial field (MF) algebras were introduced by Blackadar and Kirchberg in [8]. These are stably finite C*- algebras which arise from generalized inductive limits of finite-dimensional algebras, or, equivalently, which admit norm microstates [12]. The MF property is the C*-analogue of admitting tracial microstates, i.e., embeddability into the ultrapower R^ω of the hyperfinite II_1 factor. Blackadar and Kirchberg remarked that there is no example of a stably finite separable C*-algebra which is known not to be MF, but that a good candidate is $C^*_\lambda(\mathbb{F}_r)$. Then came the deep result of U. Haagerup and S. Thorbjørnsen in [23] that showed that $C^*_\lambda(\mathbb{F}_r)$ is in fact MF. It therefore seems natural to focus our attention on actions of free groups on AF algebras. By studying the induced K-theoretic dynamics of such systems, purely algebraic conditions emerge which are necessary and sufficient for $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ to be MF, one in the form of locally invariant states on $K_0(A)$ and the other in the spirit of a coboundary subgroup as in Brown's work (Theorem 0.2 of [10]). These are summed up in Theorem 2.2.14 below. These K-theoretic conditions enable us to prove that being MF and being stably finite are equivalent for this class of crossed products.

MF algebras are interesting in their own right but are also important in Voiculescu's seminal study of topological free entropy dimension for a family of self-adjoint elements $a_1, \ldots, a_n$ in a unital C*-algebra A [50]. Indeed the latter is well-defined only when $C^*(\{a_1, \ldots, a_n\})$ is MF. There is also a connection between MF algebras and the Brown-Douglas-Fillmore *Ext* semigroup introduced in [13]. We will exhibit several examples of MF algebras whose *Ext* semigroup is not a group.

2.1 Residually Finite, Residually Finite Dimensional, and MF Actions

Definition 2.1.1. Let (X, d) be a compact metric space and Γ a discrete group. A continuous action $h : \Gamma \to \mathrm{Homeo}(X)$ is said to be *residually finite* (RF) if for every $\varepsilon > 0$ and finite set $F \subset \Gamma$, there exists a finite set E which admits an action $k : \Gamma \to \mathrm{Perm}(E)$ and a map $\zeta : E \to X$ such that

1. $d(\zeta(k_s(z)), h_s(\zeta(z))) < \varepsilon$ for each $s \in F$ and $z \in E$,

2. $X \subset_\varepsilon \zeta(E)$, that is, ζ has ε-dense range in X.

This notion of a residually finite action was introduced in [28], from which we mention a few results. It is easily verified that if a group admits a *free* residually finite action on some compact space then the group itself must be residually finite, hence the name. Moreover, a residually finite action $\Gamma \curvearrowright X$ will yield a Γ-invariant probability measure on X which extends in a canonical way (by composition with the conditional expectation) to a trace on $C(X) \rtimes_\lambda \Gamma$. Thus residually finite transformation groups (X, Γ) produce stably finite reduced crossed products. Theorem 2.2.14 below and Lemma 3.9 in [28] together show that the converse holds true when $\dim(X) = 0$ and $\Gamma = \mathbb{F}_r$. It is also shown that when dealing with $\mathbb{Z}$-systems $(X, \mathbb{Z})$, residual finiteness is equivalent to chain recurrence [15].

We introduce here a stronger notion than residual finiteness; one that demands exact and global equivariance.

Definition 2.1.2. Let (X, d) be a compact metric space and Γ a discrete group. A continuous action $h : \Gamma \to \mathrm{Homeo}(X)$ is said to be *residually finite dimensional* (RFD) if for every $\varepsilon > 0$ there exists a finite set E which admits an action $k : \Gamma \to \mathrm{Perm}(E)$ and a map $\zeta : E \to X$ such that

1. $\zeta(k_s(z)) = h_s(\zeta(z))$ for every $z \in E$ and $s \in \Gamma$.

2. $X \subset_\varepsilon \zeta(E)$, that is, ζ has ε-dense range in X.

In other words, a transformation group (X, Γ) is RFD if for every $\varepsilon > 0$ there is finite Γ-invariant subsystem which is ε-dense. Clearly every RFD action is RF, but the converse is false in general; minimal Cantor systems $\mathbb{Z} \curvearrowright X$ yield infinite-dimensional, simple, stably-finite crossed products $C(X) \rtimes \mathbb{Z}$ ([40]). As remarked above such systems are residually finite but cannot be residually finite dimensional by Theorem 2.1.5 below. The nomenclature is justified by Theorem 2.1.5 and Proposition 2.1.7.

As observed by the authors of [28], residually finite actions $\Gamma \curvearrowright X$ have C*-dynamical expressions when looking at the induced action on the algebra $C(X)$ (see Proposition 2.1.6 below). Indeed, what is witnessed at the algebraic level is a finite dimensional approximating property familiar to C*-enthusiasts along with an approximate equivariance. We make similar observations when studying RFD actions (see Proposition 2.1.7). Here are the appropriate definitions at the C*-level.

Definition 2.1.3. Let Γ be a discrete group and A a C*-algebra.

1. An action $\alpha : \Gamma \to \mathrm{Aut}(A)$ is said to be *matricial field* (MF) provided that: given $\varepsilon > 0$, and finite subsets $\Omega \subset A$ and $F \subset \Gamma$, there exist $d \in \mathbb{N}$, a map $v : \Gamma \to \mathcal{U}(\mathbb{M}_d)$ $(s \mapsto v_s)$, and a unital $*$-linear map $\varphi : A \to \mathbb{M}_d$, such that for every $a, b \in \Omega$ and $s, t \in F$

   (a) $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \varepsilon$,

   (b) $\big|\|\varphi(a)\| - \|a\|\big| < \varepsilon$,

   (c) $\|\varphi(\alpha_s(a)) - \mathrm{Ad}_{v_s}(\varphi(a))\| < \varepsilon$,

   (d) $\|v_{st} - v_s v_t\| < \varepsilon$.

   If the unital map φ can be further chosen to be completely positive, α is said to be *quasidiagonal* (QD).

2. The action $\alpha : \Gamma \to \mathrm{Aut}(A)$ is said to be *residually finite dimensional* (RFD) if for every $\varepsilon > 0$ and finite subset $\Omega \subset A$, there is a $d \in \mathbb{N}$, a $*$-homomorphism $\pi : A \to \mathbb{M}_d$ and a unitary representation $v : \Gamma \to \mathcal{U}(\mathbb{M}_d)$ such that

   (a) $\|\pi(b)\| > \|b\| - \varepsilon$ for every $b \in \Omega$,

   (b) $\pi(\alpha_s(a)) = \mathrm{Ad}_{v(s)}(\pi(a))$ for every $a \in A$ and $s \in \Gamma$.

A few remarks and key observations concerning Definition 2.1.3 are in order. Every RFD system (A, Γ) is clearly QD, and every QD system is MF. We show that MF actions are in fact QD when the underlying algebra is amenable (see Proposition 2.1.17). It is obvious that if $\alpha : \Gamma \curvearrowright A$ is RFD (QD, MF), then A is itself RFD (QD, MF). Note that any finite dimensional algebra can be embedded into a full matrix algebra, so we may replace $\mathbb{M}_d$ by any finite dimensional algebra B without changing the notion. Also, when verifying that an action is MF or QD, it suffices to consider finite subsets of a generating set of the acting group Γ.

The properties of being MF, QD, or RFD pass to subalgebras, so if a C*-system (A, Γ, α) yields an MF (QD, RFD) crossed product, one expects the underlying algebra A as well as the group algebra $C^*_\lambda(\Gamma)$ to be MF (QD, RFD). In these cases one can also decipher information about the action α. Indeed, the structure of the reduced crossed product algebra determines the nature of the action.

Proposition 2.1.4. *Let A be a unital C^*-algebra, and $\alpha : \Gamma \to \mathrm{Aut}(A)$ a homomorphism. The following hold.*

1. *If $A \rtimes_{\lambda,\alpha} \Gamma$ is RFD, then $C^*_\lambda(\Gamma)$ is RFD and the action α is residually finite dimensional.*

2. *If $A \rtimes_{\lambda,\alpha} \Gamma$ is QD, then $C^*_\lambda(\Gamma)$ is QD and the action α is quasidiagonal.*

3. *If $A \rtimes_{\lambda,\alpha} \Gamma$ is MF, then $C^*_\lambda(\Gamma)$ is MF and the action α is matricial field. Moreover, if A is nuclear, α is quasidiagonal.*

Proof. Suppose $A \rtimes_{\lambda,\alpha} \Gamma$ is residually finite dimensional. Again, being RFD passes to subalgebras, so $C^*_\lambda(\Gamma)$ is RFD as it sits canonically inside the reduced crossed product. If $\varepsilon > 0$ and if $\Omega \subset A$ is a finite set, then there is a d and a $*$-homomorphism $\phi : A \rtimes_{\lambda,\alpha} \Gamma \to \mathbb{M}_d$ such that $\|\phi(\iota(b))\| > \|\iota(b)\| - \varepsilon = \|b\| - \varepsilon$ for every $b \in \Omega$, where $\iota : A \hookrightarrow A \rtimes_{\lambda,\alpha} \Gamma$ denotes the natural inclusion. Set $\pi = \phi \circ \iota : A \to \mathbb{M}_d$. Now define a unitary representation $v : \Gamma \to \mathcal{U}(d)$ as $v(s) = \phi(1_A u_s)$, where u_s denote the canonical unitaries in the crossed product implementing the action. Set $\gamma_s = \mathrm{Ad}_{v(s)}$ so that $\gamma : \Gamma \curvearrowright \mathbb{M}_d$ is an action. We verify

$$\begin{aligned}\pi(\alpha_s(a)) = \phi(\iota(\alpha_s(a))) = \phi(\alpha_s(a)u_e) = \phi(u_s a u_e u_s^*) = \phi(u_s)\phi(\iota(a))\phi(u_s)^* \\ = v(s)\pi(a)v(s)^* = \gamma_s(\pi(a))\end{aligned}$$

and this completes the proof of (1).

We prove (3) next. Let $\varepsilon > 0$, and let $F \subset \Gamma$, $\mathcal{F} \subset A$ be finite subsets. For $s \in \Gamma$ denote by u_s the canonical unitaries in $A \rtimes_{\lambda,\alpha} \Gamma$ and for economy write $B = A \rtimes_{\lambda,\alpha} \Gamma$. Now set

$$\Omega = \{u_s, u_s^*, u_{st}, u_s a, a \ : \ s, t \in F, \ a \in \mathcal{F}\} \subset B.$$

Next, set $K = \max_{x \in \Omega} \|x\| + 1$. By an elementary perturbation result, there is a $0 < \delta < \min\{1, \varepsilon/4\}$ with the following property: if D is *any* C*-algebra with $d \in D$ satisfying $\|d^*d - 1\| < \delta$ and $\|dd^* - 1\| < \delta$, there is a $v \in \mathcal{U}(D)$ with $\|d - v\| < \varepsilon/4K$. Assuming B is MF, there is a unital $*$-linear map $\psi : B \to \mathbb{M}_d$ such that

$$\|\psi(xy) - \psi(x)\psi(y)\| < \delta \qquad \forall x, y, \in \Omega, \tag{2.1}$$

$$|\|\psi(x)\| - \|x\|| < \delta \qquad \forall x \in \Omega. \tag{2.2}$$

Since $\delta < \varepsilon$, $\mathcal{F} \subset \Omega$ and in light of the inequalities (2.1) and (2.2), all what is needed to show now is approximate equivariance with an appropriate map $v : \Gamma \to \mathcal{U}(\mathbb{M}_d)$. Note that for $r \in F$ or $r = st$ with $s, t \in F$ we have

$$\|\psi(u_r)^*\psi(u_r) - 1\| = \|\psi(u_r^*)\psi(u_r) - \psi(u_r^* u_r)\| < \delta,$$

$$\|\psi(u_r)\psi(u_r)^* - 1\| = \|\psi(u_r)\psi(u_r^*) - \psi(u_r u_r^*)\| < \delta,$$

therefore, by our choice of δ, there are unitaries v_r, in $\mathbb{M}_d$ for each $r \in F$ or $r = st$ with $s, t \in F$ that satisfy $\|v_r - \psi(u_r)\| < \varepsilon/4K$. Extend $v : \Gamma \to \mathcal{U}(\mathbb{M}_d)$ arbitrarily.

We need to show that for every a in $\mathcal{F}$ and $s, t \in F$

$$\|\psi(\alpha_s(a)) - \mathrm{Ad}_{v_s}(\psi(a))\| = \|\psi(u_j a u_j^*) - v_j \psi(a) v_j^*\| < \varepsilon, \tag{2.3}$$

$$\|v_{st} - v_s v_t\| < \varepsilon. \tag{2.4}$$

To this end, first note that (2.2) implies that for each $x \in \Omega$

$$\|\psi(x)\| = \|\psi(x)\| - \|x\| + \|x\| \leq \big|\|\psi(x)\| - \|x\|\big| + \|x\| < \delta + (K-1) < K.$$

Using (2.1) along with the definition of the v_s yields

$$\begin{aligned}
\|\psi(u_s a u_s^*) - v_s \psi(a) v_s^*\| &\leq \|\psi(u_s a u_s^*) - \psi(u_s a)\psi(u_s^*)\| + \|\psi(u_s a)\psi(u_s^*) - \psi(u_s a) v_s^*\| \\
&\quad + \|\psi(u_s a) v_s^* - \psi(u_s)\psi(a) v_s^*\| + \|\psi(u_s)\psi(a) v_s^* - v_s \psi(a) v_s^*\| \\
&< \delta + \|\psi(u_s a)\|\|\psi(u_s)^* - v_s^*\| \\
&\quad + \|\psi(u_s a) - \psi(u_s)\psi(a)\|\|v_s^*\| + \|\psi(u_s) - v_s\|\|\psi(a) v_s^*\| \\
&< \delta + K \cdot \frac{\varepsilon}{4K} + \delta + \frac{\varepsilon}{4K} \cdot K < 4 \cdot \frac{\varepsilon}{4} = \varepsilon,
\end{aligned}$$

which establishes (2.3). To see (2.4),

$$\begin{aligned}
\|v_{st} - v_s v_t\| &\leq \|v_{st} - \psi(u_{st})\| + \|\psi(u_s u_t) - \psi(u_s)\psi(u_t)\| \\
&\quad + \|\psi(u_s)\psi(u_t) - v_s \psi(u_t)\| + \|v_s \psi(u_t) - v_s v_t\| \\
&\leq \frac{\varepsilon}{4K} + \delta + \|\psi(u_t)\| \frac{\varepsilon}{4K} + \frac{\varepsilon}{4K} < \varepsilon.
\end{aligned}$$

The action is thus MF. If A is amenable, Proposition 2.1.17 ensures that α is QD and the proof of (3) is complete. The proof of (2) is identical except for the fact that we may choose ψ to be completely positive provided that B is quasidiagonal. $\square$

We now embark on establishing partial converses to Proposition 2.1.4. Reduced crossed products emerging from a QD C*-system exhibit a finite-dimensional approximating property, they admit norm microstates. Indeed, it was shown in [28], in the separable case, that if α is quasidiagonal and $C^*_\lambda(\Gamma)$ is MF, the reduced crossed product algebra $A \rtimes_{\lambda,\alpha} \Gamma$ is also MF. However, we must point out that the definition of a QD action in [28] is somewhat stronger than Definition 2.1.3. In lieu of a local approximately multiplicative map $v : \Gamma \to \mathcal{U}(\mathbb{M}_d)$, the authors require a legitimate action $\gamma : \Gamma \curvearrowright \mathbb{M}_d$ with

(c′) $\|\varphi(\alpha_s(a)) - \gamma_s(\varphi(a))\| < \varepsilon$, for every $a \in \Omega$ and $s \in F$.

Assuming that such an action γ exists, apply the GNS construction to $(\mathbb{M}_d, \tau)$, where τ is the unique faithful tracial state on $\mathbb{M}_d$, to obtain the faithful representation $\pi_\tau : \mathbb{M}_d \to \mathbb{B}(L^2(\mathbb{M}_d, \tau)) \cong \mathbb{M}_{d^2}$. Then define unitaries in $\mathbb{M}_{d^2}$ by $v_s(\hat{x}) = \widehat{\gamma_s(x)}$ for $s \in \Gamma$ and $x \in \mathbb{M}_d$. It is easily verified that $v : \Gamma \to \mathcal{U}(\mathbb{M}_{d^2})$ is in fact a unitary representation satisfying $v_s \pi_\tau(x) v_s^* = \pi_\tau(\gamma_s(x))$ for $x \in \mathbb{M}_d$. Replacing φ by $\pi_\tau \circ \varphi$ and $\mathbb{M}_d$ by $\mathbb{M}_{d^2}$ we then have an MF (or QD) action in the sense of 2.1.3. Therefore Definition 2.1.3 is a weakening of that given in [28]. With some extra work one can still prove Theorem 3.4 in [28] with our weakened definition of a QD action. We include this result for completeness along with other partial converses to Proposition 2.1.4.

Theorem 2.1.5. *Let Γ be a discrete group and $\alpha : \Gamma \to \mathrm{Aut}(A)$ an action on a separable unital C^*-algebra A.*

1. *The reduced crossed product $A \rtimes_{\lambda,\alpha} \Gamma$ is RFD if and only if $C^*_\lambda(\Gamma)$ is RFD and α is RFD.*

2. *If $C^*_\lambda(\Gamma)$ is MF and α is QD, then the reduced crossed product $A \rtimes_{\lambda,\alpha} \Gamma$ is MF.*

3. *If $C^*_\lambda(\Gamma)$ is QD and α is MF, then the reduced crossed product $A \rtimes_{\lambda,\alpha} \Gamma$ is MF.*

Proof. (1): Consider an RFD action $\alpha : \Gamma \curvearrowright A$. We then have a sequence $*$-homomorphisms $\pi_n : A \to \mathbb{M}_{k_n}$ and unitary representations $v_n : \Gamma \to \mathcal{U}(\mathbb{M}_{k_n})$ such that for every $s \in \Gamma$ and $a \in A$:

(i) $\|\pi_n(a)\| \nearrow \|a\|$ as $n \to \infty$.

(ii) $\pi_n(\alpha_s(a)) = \mathrm{Ad}_{v_n(s)}(\pi_n(a))$.

For economy write $M = \prod_{n=1}^\infty \mathbb{M}_{k_n}$, and $\mathcal{U}(k_n) = \mathcal{U}(\mathbb{M}_{k_n})$. Consider the unitary representation $v : \Gamma \to \mathcal{U}(M) \cong \prod_{n\geq 1} \mathcal{U}(k_n)$ given by $v(s) := (v_n(s))_{n\geq 1}$, and the $*$-homomorphism $\pi : A \to M$ defined by $\pi(a) := (\pi_n(a))_{n\geq 1}$. Also set $\beta_s = \mathrm{Ad}_{v(s)}$, so that $\beta : \Gamma \to \mathrm{Aut}(M)$ is an action. Condition (i) ensures that π is injective, and condition (ii) implies equivariance of π, that is $\pi(\alpha_s(a)) = \beta_s(\pi(a))$ for every $s \in \Gamma$ and $a \in A$. We thus have a monomorphism of C*-dynamical systems $\pi : (A, \Gamma, \alpha) \to (M, \Gamma, \beta)$. Therefore, $A \rtimes_{\lambda,\alpha} \Gamma \hookrightarrow M \rtimes_{\lambda,\beta} \Gamma$. Since β is an inner action, we know that $M \rtimes_{\lambda,\beta} \Gamma \cong M \otimes_{\min} C^*_\lambda(\Gamma)$, whence

$$A \rtimes_{\lambda,\alpha} \Gamma \hookrightarrow M \otimes_{\min} C^*_\lambda(\Gamma).$$

Now simply note that since both M and $C^*_\lambda(\Gamma)$ are RFD algebras, so is their minimal tensor product $M \otimes_{\min} C^*_\lambda(\Gamma)$. Being RFD passes to subalgebras, so we conclude $A \rtimes_{\lambda,\alpha} \Gamma$ is RFD.

(2): (The proof of this is essentially the same proof as Theorem 3.4 in [28].)

(3): We have a sequence of $*$-linear maps $\varphi_n : A \to \mathbb{M}_{k_n}$ and maps $v_n : \Gamma \to \mathcal{U}(\mathbb{M}_{k_n})$ such that for every $s, t \in \Gamma$ and $a, b \in A$:

(i) $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$, as $n \to \infty$.

(ii) $\|\varphi_n(a)\| \to \|a\|$ as $n \to \infty$.

(iii) $\|\varphi_n(\alpha_s(a)) - \mathrm{Ad}_{v_n(s)}(\varphi_n(a))\| \to 0$, as $n \to \infty$.

(iv) $\|v_n(st) - v_n(s)v_n(t)\| \to 0$, as $n \to \infty$.

Write $M = \prod_{n=1}^{\infty} \mathbb{M}_{k_n} / \bigoplus_{n=1}^{\infty} \mathbb{M}_{k_n}$ and $\pi : \prod_{n=1}^{\infty} \mathbb{M}_{k_n} \to M$ the canonical quotient map. Now consider the maps $\Phi : A \to M$ and $v : \Gamma \to \mathcal{U}(M)$ given by

$$\Phi(a) := \pi[(\phi_n(a))_n] \qquad v(s) := \pi[(v_n(s))_n].$$

By properties (i) and (ii) Φ is an $*$-monomorphism, and by property (iv) v is a unitary representation. Therefore, we have an inner action $\beta : \Gamma \curvearrowright M$ defined by $\beta_s := \mathrm{Ad}_{v_s}$ for $s \in \Gamma$. Note that property (iii) implies that $\beta_s(\Phi(a)) = \Phi(\alpha_s(a))$ for every $a \in A$ and $s \in \Gamma$. Indeed

$$\begin{aligned}\beta_s(\Phi(a)) = \mathrm{Ad}_{v_s}(\Phi(a)) = \pi[(v_n(s))_n]\pi[(\phi_n(a))_n]\pi[(v_n(s))_n]^* = \pi[(v_n(s)\phi_n(a)v_n(s)^*)_n] \\ = \pi[(\phi_n(\alpha_s(a)))_n] = \Phi(\alpha_s(a)).\end{aligned}$$

We thus have a monomorphism of C*-dynamical systems $\Phi : (A, \Gamma, \alpha) \to (M, \Gamma, \beta)$. Therefore, $A \rtimes_{\lambda,\alpha} \Gamma \hookrightarrow M \rtimes_{\lambda,\beta} \Gamma$. Since β is an inner action, by Example 9.11 of [36] we know that $M \rtimes_{\lambda,\beta} \Gamma \cong M \otimes_{\min} C^*_\lambda(\Gamma)$, whence

$$A \rtimes_{\lambda,\alpha} \Gamma \hookrightarrow M \otimes_{\min} C^*_\lambda(\Gamma).$$

Since $C^*_\lambda(\Gamma)$ is QD, Γ is amenable, and so $C^*_\lambda(\Gamma)$ is nuclear. It follows from Proposition 3.3.6 of [8] that $M \otimes_{\min} C^*_\lambda(\Gamma)$ is MF, which implies that $A \rtimes_{\lambda,\alpha} \Gamma$ is MF. $\square$

The following is Proposition 3.3 in [28] and justifies the comments made prior to defining MF actions.

Proposition 2.1.6. *Let (X, d) be a compact metric space and $h : \Gamma \to \mathrm{Homeo}(X)$ a continuous action with induced action α on $C(X)$. If h is residually finite, then α is quasidiagonal.*

The question emerges of whether the converse to the previous result holds. The authors of [28] showed that in the case of an action $h : \mathbb{F}_r \curvearrowright X$ on a compact zero-dimensional metric space, h is residually finite if and only if the induced action on $C(X)$ is quasidiagonal.

In the same vein we relate RFD actions on compact metric spaces with RFD actions at the algebraic level.

Proposition 2.1.7. *Let $h : \Gamma \to \mathrm{Homeo}(X)$ be a continuous action on a compact metric space (X, d), and $\alpha : \Gamma \to \mathrm{Aut}(C(X))$ the induced action. Then h is RFD if and only if α is RFD.*

Proof. Consider first a RFD transformation group (X, Γ). Let $g \in C(X)$ and $\varepsilon > 0$ be given. By compactness there is a $\delta > 0$ such that

$$x, y \in X, \quad d(x, y) < \delta \Longrightarrow |g(x) - g(y)| < \varepsilon.$$

We then obtain a finite set E, an action $\Gamma \curvearrowright E$ and a map $\zeta : E \to X$ with $\zeta(E) \subset_\delta X$ and $\zeta(s.z) = s.\zeta(z)$ for every $z \in E$ and $s \in \Gamma$. Dualize by defining $\pi : C(X) \to C(E)$ as $\pi(f) = f \circ \zeta$ for $f \in C(X)$ and $\gamma_s(k)(z) := k(s^{-1}.z)$ for $k \in C(E)$, $s \in \Gamma$ and $z \in E$. The equivariance is straightforward, indeed for $f \in C(X)$, $s \in \Gamma$, $z \in E$ we have

$$\pi(\alpha_s(f))(z) = \alpha_s(f)(\zeta(z)) = f(s^{-1}.\zeta(z)) = f(\zeta(s^{-1}.z)) = \gamma_s(f \circ \zeta)(z) = \gamma_s(\pi(f))(z)$$

which implies $\pi(\alpha_s(f)) = \gamma_s(\pi(f))$ for every $f \in C(X)$ and $s \in \Gamma$.

For the approximate isometry condition, fix x in X, and pick up a $z_x \in E$ with $d(x, \zeta(z_x)) < \delta$. Then observe

$$|g(x)| \leq |g(x) - g(\zeta(z_x))| + |g(\zeta(z_x))| < \varepsilon + \sup_{z \in E} |g \circ \zeta(z)| = \varepsilon + \|\pi(g)\|.$$

Taking a supremum over all $x \in X$ gives $\|g\| \leq \varepsilon + \|\pi(g)\|$.

Conversely, suppose that α is an RFD action. We then have a sequence of finite dimensional representations $\pi_n : C(X) \to \mathbb{M}_{k_n}$ and actions $\gamma_n : \Gamma \curvearrowright \mathbb{M}_{k_n}$ such that

(i) $\|\pi_n(f)\| \nearrow \|f\|$ for each $f \in C(X)$

(ii) $\pi_n(\alpha_s(f)) = \gamma_{n,s}(\pi_n(f))$ for every $s \in \Gamma$, $f \in C(X)$ and $n \geq 1$.

Fix an $n \geq 1$ and note that $\pi_n(C(X))$ is a finite dimensional commutative algebra, therefore isomorphic to $C(E_n)$ for some finite set E_n. Also note that $\pi_n(C(X))$ is invariant under the action γ_n by condition (ii), so by restricting, we may suppose Γ acts on $C(E_n)$ via γ_n. We therefore have maps $\zeta_n : E_n \to X$ with $\pi_n(f) = f \circ \zeta_n$ for each $f \in C(X)$, and actions $\Gamma \curvearrowright E_n$ implemented by the homomorphisms γ_n. We verify the promised equivariance. Indeed, for $f \in C(X)$, $z \in E_n$ and $s \in \Gamma$ we have

$$\begin{aligned} f(s^{-1}.\zeta_n(z)) &= \alpha_s(f)(\zeta_n(z)) = \pi_n(\alpha_s(f))(z) = \gamma_{n,s}(\pi_n(f))(z) = \pi_n(f)(s^{-1}.z) \\ &= f(\zeta_n(s^{-1}.z)). \end{aligned}$$

Recall that $C(X)$ separates points of X so that $s^{-1}.\zeta_n(z) = \zeta_n(s^{-1}.z)$ for all $s \in \Gamma$

and $z \in E$, and equivariance follows.

Consider now an arbitrary $\varepsilon > 0$. We claim that for some n large we have $X \subset_\varepsilon \zeta_n(E_n)$. Suppose not. Then for every n there is an $x_n \in X$ with $d(\zeta_n(z), x_n) \geq \varepsilon$ for every $z \in E_n$. Passing to a subsequence we may assume that $(x_n)_n$ converges to some $x_0 \in X$. Note that for some N large we have that for every $n \geq N$ and $z \in E_n$

$$\varepsilon \leq d(\zeta_n(z), x_n) \leq d(\zeta_n(z), x_0) + d(x_0, x_n) \leq d(\zeta_n(z), x_0) + \varepsilon/2,$$

thus $d(\zeta_n(z), x_0) \geq \varepsilon/2$ holds for every such n and $z \in E_n$. Now choose a continuous $f : X \to [0,1]$ with $f(x_0) = 1$ and $\mathrm{supp}(f) \subset \overline{B(x_0, \varepsilon/3)}$. Thus $\|f\| = 1$, but $\pi_n(f) = f \circ \zeta_n = 0$ since $\zeta_z(E_n) \subset \mathrm{supp}(f)^c$. This contradicts condition (i), so the claim holds and the proof is complete. $\square$

We now want to look at some examples of MF actions. The first class of C*-systems seems tailored to admit finite-dimensional approximating dynamics.

Example 2.1.8. For a fixed discrete group Γ, let $(A_n, \Gamma, \alpha^{(n)})_{n\geq 1}$ be a sequence of C*-dynamical systems with each A_n finite dimensional. By standard inductive limit techniques one constructs the C*-system (A, Γ, α) where $A := \bigotimes_{n\geq 1} A_n$ and $\alpha := \otimes_{n\geq 1}\alpha^{(n)}$ acts via

$$\alpha_s(a_{n_1} \otimes \cdots \otimes a_{n_k}) = \alpha_s^{(n_1)}(a_{n_1}) \otimes \cdots \otimes \alpha_s^{(n_k)}(a_{n_k}) \qquad s \in \Gamma.$$

Given a finite subset $\Omega \subset A$, one can find a large enough m and approximate each member of Ω by elements from the subalgebra $B_m := \bigotimes_{n=1}^m A_n$. The identity map on B_m lifts to a u.c.p map $\varphi : A \to B_m$, and Γ acts on B_m as $\beta^{(m)} = \alpha^{(1)} \otimes \cdots \otimes \alpha^{(m)}$. The conditions for a QD action are now easily verified.

More instances of QD actions will surface as we uncover their theory, but we can immediately provide a wide class of examples. Recall that for a unital C*-algebra A,

$$\mathrm{Inn}(A) = \{\mathrm{Ad}_u : u \in \mathcal{U}(A)\} \leq \mathrm{Aut}(A)$$

denotes the normal subgroup of inner automorphisms, while $\overline{\mathrm{Inn}}(A) \leq \mathrm{Aut}(A)$ is the normal subgroup of all approximately inner automorphisms. Given $\varepsilon > 0$, a finite set $\mathcal{F} \subset A$ and $\alpha \in \overline{\mathrm{Inn}}(A)$, there is an inner automorphism Ad_u with $\| \mathrm{Ad}_u(x) - \alpha(x)\| \leq \varepsilon$ for every $x \in \mathcal{F}$.

Proposition 2.1.9. *Let A be a unital AF algebra with $\overline{\mathrm{Inn}}(A) = \mathrm{Aut}(A)$. Then any action $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ of a free group on A is quasidiagonal. In particular, if A is AF with $K_0(A)$ totally ordered and Archimedean, or if A is UHF, then any action of $\mathbb{F}_r$ on A is quasidiagonal.*

Proof. Let $0 < \varepsilon < 1$ and let $\Omega \subset A$ be a finite set. Also, denote the generators of $\mathbb{F}_r$ by $s_1, \ldots, s_r$. Set $K = \max_{a \in \Omega} \|a\| + 1$ and put $\delta = \min\{\varepsilon/(3 + 2K), \varepsilon/4K, \varepsilon/2\}$. Since A is AF and unital, locate a unital finite-dimensional subalgebra $1_A \in B \subset A$ and a finite subset $\Omega' \subset B$ with $\|a - b\| < \delta$ for a in Ω and b in Ω'.

Since every automorphism on A is approximately inner, there are unitaries $u_1, \ldots, u_r$ in $\mathcal{U}(A)$ such that

$$\|u_j b u_j^* - \alpha_{s_j}(b)\| < \delta \qquad \forall b \in \Omega',\ \forall j \in \{1, \ldots, r\}. \tag{2.5}$$

By standard perturbation results, we can find a unital finite-dimensional algebra $1_A \in D \subset A$ along with unitaries $v_1, \ldots, v_r$ in D such that $B \subset D$ and $\|u_j - v_j\| < \delta$ for every j. We then have automorphisms of D for each $j = 1, \ldots, r$, namely $\mathrm{Ad}_{v_j} : D \to D$ given by $\mathrm{Ad}_{v_j}(x) = v_j x v_j^*$ for $x \in D$. By the universal property of the free group, the map $\{s_1, \ldots, s_r\} \to \mathrm{Aut}(D)$ where $s_j \mapsto \mathrm{Ad}_{v_j}$ extends to a group homomorphism

$$\gamma : \mathbb{F}_r \to \mathrm{Aut}(D) \qquad \gamma_{s_j} = \mathrm{Ad}_{v_j} \quad \forall j \in \{1, \ldots, r\}.$$

Appealing to Arveson's extension theorem, let $\varphi : A \to D$ be the u.c.p extension of $\mathrm{id}_D : D \to D$. We work out the necessary estimates. First, since each $a \in \Omega$ is δ-close to a $b \in \Omega'$, we have

$$\begin{aligned}
\|\varphi(\alpha_{s_j}(a)) - \gamma_{s_j}(\varphi(a))\| &\le \|\varphi(\alpha_{s_j}(a)) - \varphi(\alpha_{s_j}(b))\| \\
&\quad + \|\varphi(\alpha_{s_j}(b)) - \gamma_{s_j}(\varphi(b))\| + \|\gamma_{s_j}(\varphi(b)) - \gamma_{s_j}(\varphi(a))\| \\
&\le \|a - b\| + \|\varphi(\alpha_{s_j}(b)) - \gamma_{s_j}(\varphi(b))\| + \|a - b\| \\
&\le 2\delta + \|\varphi(\alpha_{s_j}(b)) - \gamma_{s_j}(\varphi(b))\|.
\end{aligned}$$

Next, we use the fact that $\gamma_{s_j}(\varphi(b)) = \gamma_{s_j}(b) = v_j b v_j^* = \varphi(v_j b v_j^*)$ since $\varphi|_D = \mathrm{id}_D$ and the elements b and $v_j b v_j^*$ all belong to D. This together with (1) gives

$$\begin{aligned}
\|\varphi(\alpha_{s_j}(b)) - \gamma_{s_j}(\varphi(b))\| &\le \|\varphi(\alpha_{s_j}(b)) - \varphi(u_j b u_j^*)\| + \|\varphi(u_j b u_j^*) - \varphi(v_j b v_j^*)\| \\
&\le \|\alpha_{s_j}(b) - u_j b u_j^*\| + \|u_j b u_j^* - v_j b v_j^*\| \\
&< \delta + \|u_j b u_j^* - v_j b v_j^*\|.
\end{aligned}$$

The unitaries u_j and v_j are δ-close so we get

$$\begin{aligned}
\|u_j b u_j^* - v_j b v_j^*\| &\le \|u_j b u_j^* - u_j b v_j^*\| + \|u_j b v_j^* - v_j b v_j^*\| \\
&= \|u_j b (u_j^* - v_j^*)\| + \|(u_j - v_j) b v_j^*\| \\
&\le \|b\| \|u_j^* - v_j^*\| + \|u_j - v_j\| \|b\| \le 2K\delta
\end{aligned}$$

All of the above estimates yield $\|\varphi(\alpha_{s_j}(a)) - \gamma_{s_j}(\varphi(a))\| < (2K+3)\delta \leq \varepsilon$ for every generator s_j and every $a \in \Omega$. This gives the desired approximate equivariance. We still have yet to show that φ is approximately isometric and approximately multiplicative on Ω. To that end, let $x, y \in \Omega$ and let $x', y' \in \Omega'$ be their δ-approximations. Note that since $\delta < 1$, $\|x\| \leq K-1$ and $\|x-x'\| < \delta$, it easily follows that $\|x'\| \leq K$. A simple triangle inequality gives

$$\|xy - x'y'\| \leq \|xy - x'y\| + \|x'y - x'y'\| \leq \|x - x'\|\|y\| + \|x'\|\|y - y'\| \leq 2\delta K.$$

Similarly $\|\varphi(x)\varphi(y) - \varphi(x')\varphi(y')\| \leq 2\delta K$, since φ is contractive. Recalling that φ restricted to D is the identity, our above estimates yield

$$\begin{aligned} \|\varphi(xy) - \varphi(x)\varphi(y)\| &\leq \|\varphi(xy) - \varphi(x'y')\| + \|\varphi(x'y') - \varphi(x')\varphi(y')\| + \|\varphi(x')\varphi(y') - \varphi(x)\varphi(y)\| \\ &\leq \|xy - x'y'\| + 0 + \|\varphi(x)\varphi(y) - \varphi(x')\varphi(y')\| \leq 4\delta K < \varepsilon. \end{aligned}$$

This gives the approximate multiplicativity. Finally, φ is easily seen to be approximately isometric:

$$\begin{aligned} \big|\|\varphi(x)\| - \|x\|\big| &\leq \big|\|\varphi(x)\| - \|\varphi(x')\|\big| + \big|\|\varphi(x')\| - \|x\|\big| \leq \|\varphi(x) - \varphi(x')\| + \big|\|x'\| - \|x\|\big| \\ &\leq \|\varphi(x - x')\| + \|x - x'\| \leq 2\|x - x'\| \leq 2\delta \leq \varepsilon, \end{aligned}$$

which confirms that α is indeed quasidiagonal.

If A is a unital AF algebra such that $K_0(A)$ is totally ordered and Archimedean then $\overline{\mathrm{Inn}(A)} = \mathrm{Aut}(A)$, which is indeed the case for UHF algebras (see Corollary IV.5.8 in [16]). $\square$

The next example of a QD action is a generalization of Voiculescu's notion of an action of $\mathbb{Z}$ admitting *pseudo-orbits* described in [49]. For an algebra A, write $\mathcal{F}(A)$ for the collection all of finite-dimensional subalgebras of A. Also, if B, C are C*-subalgebras of A and $\varepsilon > 0$, we shall write $B \subset_\varepsilon C$ if $\sup_{b \in \mathrm{Ball}(B)} d(b, \mathrm{Ball}(C)) < \varepsilon$, and $d(B, C)$ is defined by

$$d(B, C) = \inf\{\varepsilon > 0 : B \subset_\varepsilon C \text{ and } C \subset_\varepsilon B\}.$$

Definition 2.1.10. Let (A, Γ, α) be a C*dynamical system. The action α is said to have the *pseudo-orbit property* if for every $\varepsilon > 0$, $F \subset \Gamma$ finite subset and $D \in \mathcal{F}(A)$, there is a finite quotient $\pi : \Gamma \to \Lambda$ along with a map $\zeta : \Lambda \to \mathcal{F}(A)$ such that

1. $D \subset B_t := \zeta(t)$, for every $t \in \Lambda$,

2. $d(\alpha_s(B_t), B_{\pi(s)t}) < \varepsilon$ for every $t \in \Lambda$ and $s \in F$.

Before stating the proposition, we remind the reader of a perturbation result due to E. Christensen (see [14]) which reads as follows.

Lemma 2.1.11. *For every $\delta > 0$, there is a $\delta_1 > 0$ such that whenever B and C are C*-subalgebras of a unital C*-algebra A with B finite dimensional and $C \subset_{\delta_1} B$, then there is a unitary $u \in A$ with $\|u - 1\| < \delta$ and $\mathrm{Ad}_u(B) \subset C$.*

Proposition 2.1.12. *Let A be an AF-algebra, $r \in \mathbb{N}$ and $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ an action with the pseudo-orbit property. Then α is quasidiagonal.*

Proof. Let $\varepsilon > 0$, $\Omega \subset A$ a finite subset and $F = \{e = s_0, s_1, \ldots, s_r\}$, where $s_1, \ldots, s_r$ are the standard generators for $\mathbb{F}_r$. Let $C := \max_{x \in \Omega} \|x\| + 1$ and let δ be so small that $2\delta < \varepsilon$, $4C\delta < \varepsilon$ and $2\delta(2 + C) < \varepsilon$. Since A is an AF algebra, we may choose a finite dimensional subalgebra $D \subset A$ with $\alpha_s(\Omega) \subset_\delta D$ for every $s \in F$. Let $\delta_1 = \delta_1(\delta) > 0$ be a perturbation constant as in Christensen's result above. By our hypothesis, there is a finite quotient $\pi : \mathbb{F}_r \to \Lambda$ and a map $\zeta : \Lambda \to \mathcal{F}(A)$ with $D \subset B_t$ and $d(\alpha_s(B), B_{\pi(s)t}) < \delta_1$ for each $t \in \Lambda$ and $s \in F$. Thus for each pair $(s,t) \in F \times \Lambda$, find unitaries $u_{s,t} \in \mathcal{U}(A)$ with $\|u_{s,t} - 1\| < \delta$ and $\mathrm{Ad}_{u_{s,t}}(\alpha_s(B_t)) \subset B_{\pi(s)t}$.

Now set $B = \bigoplus_{t\in\Lambda} B_t$ and for each $s \in F$ consider the automorphism of B given by

$$\sigma_s((b_t)_{t\in\Lambda}) = (u_{s,t}\alpha_s(b_t)u_{s,t}^*)_{t\in\Lambda}.$$

We thus have an action $\sigma : \mathbb{F}_r \to \mathrm{Aut}(B)$. Let $\varphi : A \to B$ be the u.c.p. extension of the inclusion $D \hookrightarrow B$ given by $a \mapsto (a)_{t\in\Lambda}$. Fixing an $s \in F$ and $x, y \in \Omega$, we know that there are elements $a, b, c \in D$ with $\|a-x\| < \delta$, $\|c-y\| < \delta$ and $\|d-\alpha_s(x)\| < \delta$. Note that $\|\alpha_s(a) - d\| \leq \|\alpha_s(a) - \alpha_s(x)\| + \|\alpha_s(x) - d\| < 2\delta$. We may now verify the approximate equivariance:

$$\begin{aligned}
\|\sigma_s(\varphi(x)) - \varphi(\alpha_s(x))\| &\leq \|\sigma_s(\varphi(x)) - \sigma_s(\varphi(a))\| + \|\sigma_s(\varphi(a)) - \varphi(d)\| + \|\varphi(d) - \varphi(\alpha_s(x))\| \\
&< 2\delta + \|\sigma_s(\varphi(a)) - \varphi(d)\| = 2\delta + \|(u_{s,t}\alpha_s(a)u_{s,t}^*)_{t\in\Lambda} - (d)_{t\in\Lambda}\| \\
&\leq 2\delta + \max_{t\in\Lambda} \|u_{s,t}\alpha_s(a)u_{s,t}^* - d\| \\
&\leq 2\delta + \max_{t\in\Lambda} \left\{\|u_{s,t}\alpha_s(a)u_{s,t}^* - \alpha_s(a)\| + \|\alpha_s(a) - d\|\right\} \\
&\leq 2\delta + 2\|a\|\delta + 2\delta \leq 2\delta(2 + C) < \varepsilon.
\end{aligned}$$

As for approximate multiplicativity, a simple estimate gives $\|xy - ab\| < 2C\delta$ as well as $\|\varphi(a)\varphi(b) - \varphi(x)\varphi(y)\| < \delta$. Also note that φ is multiplicative on D, so

$$\begin{aligned}
\|\varphi(xy) - \varphi(x)\varphi(y)\| &\leq \|\varphi(xy) - \varphi(ab)\| + \|\varphi(a)\varphi(b) - \varphi(x)\varphi(y)\| \\
&\leq \|xy - ab\| + \|\varphi(a)\varphi(b) - \varphi(x)\varphi(y)\| < 4C\delta < \varepsilon.
\end{aligned}$$

Finally, since $\|\varphi(a)\| = \|a\|$, we have

$$\big|\|\varphi(x)\| - \|x\|\big| \leq \big|\|\varphi(x)\| - \|\varphi(a)\|\big| + \big|\|a\| - \|x\|\big| \leq \|\varphi(x) - \varphi(a)\| + \|a - x\| \leq 2\delta < \varepsilon,$$

which concludes the proof. $\square$

As mentioned above, we pay attention to residually finite actions, and quasidiagonal actions in the non-commutative case, for such actions will determine the structure of the resulting reduced crossed product algebras. To employ Theorem 2.1.5, we need a quasidiagonal C*-system (A, Γ) where $C^*_\lambda(\Gamma)$ is MF. A remarkable result of Haagerup and Thorbjørnsen in [23] states that the reduced group C*-algebra $C^*_\lambda(\mathbb{F}_r)$ is MF. However, by Rosenberg's result, $C^*_\lambda(\mathbb{F}_r)$ is not quasidiagonal when $r \geq 2$. Therefore a reduced crossed product where the acting group is a non-abelian free group can never be quasidiagonal. A wonderful result connecting the Brown-Douglas-Fillmore theory of extensions [13] to quasidiagonal C*-algebras and MF algebras reads as follows. A proof of this result can be found in [11].

Theorem 2.1.13. *Let B be a unital separable MF algebra which fails to be quasidiagonal. Then* $\mathrm{Ext}(B)$ *is not a group.*

We have the following corollaries.

Corollary 2.1.14. *Let A be a unital AF algebra satisfying* $\overline{\mathrm{Inn}(A)} = \mathrm{Aut}(A)$*, then $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is an MF algebra. In particular, if A is AF with $K_0(A)$ totally ordered and Archimedean, or if A is UHF, then $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is always MF. For such algebras A and $r \geq 2$ we have that* $\mathrm{Ext}(A \rtimes_{\lambda,\alpha} \mathbb{F}_r)$ *is not a group.*

Corollary 2.1.15. *Let A be an AF-algebra, $r \in \mathbb{N}$ and $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ an action with the pseudo-orbit property. Then $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is an MF algebra. If $r \geq 2$ then* $\mathrm{Ext}(A \rtimes_{\lambda,\alpha} \mathbb{F}_r)$ *is not a group.*

For actions on nuclear algebras, we aim to show that QD and MF actions coincide. Before embarking on the details, a bit of notation is in order. Given a (separable) MF algebra B, by an *MF approximating sequence for B* we mean a sequence of $*$-linear, unital maps $(\psi_n : B \to \mathbb{M}_{k_n})_{n\geq 1}$ which are asymptotically multiplicative and asymptotically isometric. If the ψ_n are completely positive, then B is a quasidiagonal algebra and the sequence $(\psi_n)_{n\geq 1}$ will be referred to as a *QD approximating sequence.*

Lemma 2.1.16. *Let A be a unital, separable, nuclear MF algebra. Suppose $(\psi_n : A \to \mathbb{M}_{k_n})_{n\geq 1}$ is an MF approximating sequence for A. Then there exists a QD approximating sequence $(\varphi_n : A \to \mathbb{M}_{k_n})_{n\geq 1}$ for A satisfying*

$$\|\varphi_n(a) - \psi_n(a)\| \longrightarrow 0 \qquad \forall a \in A.$$

Proof. If $\pi : \prod_{n=1}^\infty \mathbb{M}_{k_n} \to \frac{\prod_{n=1}^\infty \mathbb{M}_{k_n}}{\bigoplus_{n=1}^\infty \mathbb{M}_{k_n}}$ denotes the canonical quotient mapping, the MF approximating sequence $(\psi_n)_{n\geq 1}$ provides an embedding

$$\Psi : A \hookrightarrow \frac{\prod_{n=1}^\infty \mathbb{M}_{k_n}}{\bigoplus_{n=1}^\infty \mathbb{M}_{k_n}},$$

namely $\Psi(a) := \pi((\psi_n(a))_{n\geq 1})$. Now nuclearity of A ensures a u.c.p. lifting

$$\Phi : A \to \prod_{n=1}^\infty \mathbb{M}_{k_n}$$

with $\pi\circ\Phi = \Psi$. Set for each n, $\varphi_n := \pi_n\circ\Phi : A \to \mathbb{M}_{k_n}$, where $\pi_n : \prod_{m=1}^\infty \mathbb{M}_{k_m} \to \mathbb{M}_{k_n}$ is the natural projection mapping. The maps φ_n are clearly u.c.p, and note that for each a in A,

$$\pi((\psi_n(a))_n) = \Psi(a) = \pi \circ \Phi(a) = \pi((\varphi_n(a))_n),$$

which means that $(\varphi_n(a)-\psi_n(a))_{n\geq 1}\in\bigoplus_{n=1}^{\infty}\mathbb{M}_{k_n}$ for each a, that is

$$\|\varphi_n(a)-\psi_n(a)\|\longrightarrow 0 \qquad \forall a\in A.$$

From this, the approximating properties of the sequence $(\varphi_n)_{n\geq 1}$ follow from those of $(\psi_n)_{n\geq 1}$. Indeed, for each $a,b\in A$

$$\begin{aligned}\|\varphi_n(ab)-\varphi_n(a)\varphi_n(b)\| &\leq \|\varphi_n(ab)-\psi_n(ab)\|+\|\psi_n(ab)-\psi_n(a)\psi_n(b)\|\\ &\quad+\|\psi_n(a)\psi_n(b)-\varphi_n(a)\varphi_n(b)\|\end{aligned}$$

with each term tending to zero as $n\to\infty$. Note that one needs a standard $\varepsilon/3$ argument to show that the last term tends to zero. Also

$$\big|\|\varphi_n(a)\|-\|a\|\big|\leq\|\varphi_n(a)-\psi_n(a)\|+\big|\|\psi_n(a)\|-\|a\|\big|\overset{n\to\infty}{\longrightarrow}0$$

for every $a\in A$. $\square$

Proposition 2.1.17. *Let (A,Γ,α) be a C^*-dynamical system with A nuclear and separable. Then α is a quasidiagonal action if and only if it is an MF action.*

Proof. That QD implies MF is obvious. Assume that α is MF. We then have an MF approximating sequence $(\psi_n : A\to\mathbb{M}_{k_n})_{n\geq 1}$ as well as a sequence of actions $\gamma_n\curvearrowright\mathbb{M}_{k_n}$ with

$$\|\gamma_{n,s}(\psi_n(a))-\psi_n(\alpha_s(a))\|\overset{n\to\infty}{\longrightarrow}0 \qquad \forall a\in A,\forall s\in\Gamma.$$

Use the above Lemma 2.1.16 to generate a QD approximating sequence $(\phi_n : A\to\mathbb{M}_{k_n})_{n\geq 1}$ with $\|\varphi_n(a)-\psi_n(a)\|\to 0$ for every $a\in A$. For a fixed $a\in A$ and

$s \in \Gamma$, a simple estimate now gives

$$\|\gamma_{n,s}(\phi_n(a)) - \phi_n(\alpha_s(a))\| \leq \|\gamma_{n,s}(\phi_n(a)) - \gamma_{n,s}(\psi_n(a))\| + \|\gamma_{n,s}(\psi_n(a)) - \psi_n(\alpha_s(a))\| \\ + \|\psi_n(\alpha_s(a)) - \phi_n(\alpha_s(a))\|$$

which tends to zero as $n \to \infty$ since $\|\gamma_{n,s}(\phi_n(a)) - \gamma_{n,s}(\psi_n(a))\| \leq \|\varphi_n(a) - \psi_n(a)\|$ which goes to zero. The action is thus QD. $\square$

We mention one more example taken from [21] which stems from [39].

Example 2.1.18. Consider an action of the integers $\alpha : \mathbb{Z} \to \mathrm{Aut}(A)$ which admits an *almost periodic* condition: there is a natural sequence $(n_k)_{k\geq 1}$ for which $(\alpha_{n_k})_k \to \mathrm{id}_A$ in $\mathrm{Aut}(A)$ as $k \to \infty$. Call such an action (AP). Pimsner and Voiculescu showed (see [39]) that if A is separable, unital, and quasidiagonal and $\mathbb{Z} \curvearrowright A$ satisfies (AP), then $A \rtimes_\lambda \mathbb{Z}$ is also quasidiagonal. Hadwin and Shen proved an analogous result in the context of MF algebras. In Theorem 4.2 of [21], they prove that if A is MF, unital and finitely generated $\mathbb{Z} \curvearrowright A$ satisfies condition (AP), then $A \rtimes_\lambda \mathbb{Z}$ is again MF. From their work and applying Proposition 2.1.4, we conclude that (AP) actions of the integers on unital QD algebras are QD, and (AP) actions of the integers on unital finitely generated MF algebras are MF. We mention that this notion of an almost periodic action was generalized to actions of amenable countable residually finite discrete groups by Orfanos [34] where he extended Pimsner and Voiculescu's result.

The results obtained thus far have a concise formulation when the underlying algebra is nuclear.

Theorem 2.1.19. *Let A be a unital separable nuclear C^*-algebra, Γ a countable discrete group and $\alpha : \Gamma \to \mathrm{Aut}(A)$ an action. The following hold.*

1. $A \rtimes_{\lambda,\alpha} \Gamma$ is MF if and only if $C^*_\lambda(\Gamma)$ is MF and α is MF.

2. $A \rtimes_{\lambda,\alpha} \Gamma$ is QD if and only if $C^*_\lambda(\Gamma)$ is QD and α is QD.

3. $A \rtimes_{\lambda,\alpha} \Gamma$ is RFD if and only if $C^*_\lambda(\Gamma)$ is RFD and α is RFD.

When $A = C(X)$ is abelian, α is RFD if and only if the induced action $\Gamma \curvearrowright X$ is RFD. Moreover, if $\Gamma = \mathbb{F}_r$ and X is a zero-dimensional metrizable space, then α is QD if and only if the induced action $\Gamma \curvearrowright X$ is residually finite.

Proof. (1): This follows from Theorem 2.1.5, Proposition 2.1.17 and Proposition 2.1.4. Recall that being MF passes to subalgebras.

(2): If α is QD and $C^*_\lambda(\Gamma)$ is QD then Γ is amenable by Rosenberg's result and $A \rtimes_{\lambda,\alpha} \Gamma$ is MF by Theorem 2.1.5. Since A is nuclear and Γ is amenable, then $A \rtimes_{\lambda,\alpha} \Gamma$ is nuclear. Now recall that nuclear and MF implies QD. The converse is again Proposition 2.1.4.

(3): This is Theorem 2.1.5.

A residually finite action $\Gamma \curvearrowright X$ by any discrete group on any compact metric space always induces a quasidiagonal action on $C(X)$ as shown in Proposition 2.1.6. Moreover, it is shown in [28] that if the reduced crossed product $C(X) \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is MF, the induced action $\mathbb{F}_r \curvearrowright X$ is residually finite, provided that X is a zero-dimensional compact space. □

2.2 K-Theoretical Dynamics

In this section our aim is to model classical and noncommutative C*-dynamics K-theoretically. In the presence of sufficiently many projections, the properties of residually finite, RFD, and MF actions admit simple K-theoretic characterizations that will aid us to prove structure theorems for the resulting reduced crossed products. Proposition 2.2.6 below shows how RFD systems (A, Γ) admit Γ-invariant, integer-valued states on $K_0(A)$. Quasidiagonal actions are likewise described via local, Γ-invariant faithful states. For an action of a free group on an AF algebra this characterization leads to the coboundary condition $H_\sigma \cap K_0(A)^+ = \{0\}$ from which the main result Theorem 2.2.14 ensues.

2.2.1 Commutative Case

We first restrict our attention to transformation groups $h : \Gamma \curvearrowright X$ where X is zero-dimensional. As usual we shall denote by α the corresponding action on $A = C(X)$ and by $\hat{\alpha}$ the induced action on $K_0(A)$. Introducing some further notation for this result, if A is any C*-algebra, write

$$\Sigma(A) = \{[p]_0 : p \in \mathcal{P}(A)\}$$

for the scale of A, and given subsets $F \subset \Gamma$, and $S \subset K_0(A)$ write

$$S_F = \{\hat{\alpha}_t(x) : \ x \in S, \ t \in F \cup \{e\} \ \}$$

for the subset of $K_0(A)$ containing S and all F-iterates of S.

Proposition 2.2.1. *Let X be a zero-dimensional compact metrizable space, and Γ a discrete group. Suppose $h : \Gamma \curvearrowright X$ is a continuous action with induced action $\alpha : \Gamma \to \mathrm{Aut}(C(X))$. Then the following statements are equivalent:*

1. *$\Gamma \curvearrowright X$ is residually finite.*
2. *Given finite subsets $S \subset K_0(C(X))^+$ and $F \subset \Gamma$ there exist d in $\mathbb{N}$, an action $\sigma : \Gamma \to \mathrm{OAut}(\mathbb{Z}^d)$ of ordered abelian groups, and a morphism of ordered abelian groups $\beta : K_0(A) \to \mathbb{Z}^d$ such that*

   (a) $\beta \circ \hat{\alpha}_t(g) = \sigma_t \circ \beta(g)$ for each $g \in S$ and $t \in F$,

(b) $\beta(g) \neq 0$ *for every* $0 \neq g \in S$.

3. *Given finite subsets* $S \subset K_0(C(X))^+$ *and* $F \subset \Gamma$ *there exist* d *in* $\mathbb{N}$*, a subgroup* $H \leq K_0(C(X))$*, along with an action* $\sigma : \Gamma \to \mathrm{OAut}(\mathbb{Z}^d)$ *by ordered abelian group automorphisms, and a positive group homomorphism* $\beta : H \to \mathbb{Z}^d$ *such that*

   (a) $[1] \in H$*,* $S_F \subset H$*, and* $\beta([1]) = (1, 1, \ldots, 1)$*,*

   (b) $\beta \circ \hat{\alpha}_t(g) = \sigma_t \circ \beta(g)$ *for each* $g \in S$ *and* $t \in F$*,*

   (c) $\beta(g) \neq 0$ *for every* $0 \neq g \in S$.

Proof. (1) $\Rightarrow$ (2): First consider $S' = \{[p_j] : p_j \in \mathcal{P}(A), j = 1, \ldots n\} \subset \Sigma(C(X))$, a finite subset of the scale of $C(X)$, and $F \subset \Gamma$ a finite subset. Let $0 < \varepsilon < 1$. Since h is residually finite, by the proof of Proposition 2.1.6 in [28] there is a unital $*$-homomorphism $\varphi : A \to \mathbb{C}^d$ for some $d \in \mathbb{N}$, and an action $\gamma : \Gamma \to \mathrm{Aut}(\mathbb{C}^d)$ such that for each $j \in \{1, \ldots, n\}$ and $t \in F$

$$\|\varphi(p_j)\| > \|p_j\| - \varepsilon,$$
$$\|\varphi(\alpha_t(p_j)) - \gamma_t(\varphi(p_j))\| < \varepsilon.$$

Applying the K_0 functor yields a positive group homomorphism $\beta := K_0(\varphi) : K_0(A) \to K_0(\mathbb{C}^d) \cong \mathbb{Z}^d$, with $\beta([1_A]) = [\varphi(1_A)] = [1_{\mathbb{C}^d}] \cong (1, \ldots, 1)$. As in the above discussion we also have an induced action $K_0(\gamma) : \Gamma \to \mathrm{OAut}(K_0(\mathbb{C}^d))$. Write $\sigma_t = K_0(\gamma)(t) = K_0(\gamma_t)$. After composing by a suitable isomorphism of ordered abelian groups, we may assume β takes values in $\mathbb{Z}^d$, and $\sigma_t \in \mathrm{OAut}(\mathbb{Z}^d)$. We may now verify equivariance: for $t \in F$ and every j we have

$$\begin{aligned}\beta \circ \hat{\alpha}_t([p_j]) &= \hat{\varphi} \circ \hat{\alpha}_t([p_j]) = [\varphi \circ \alpha_t(p_j)] \\ &= [\gamma_t \circ \varphi(p_j)] = \hat{\gamma}_t \circ \hat{\varphi}([p_j]) = \sigma_t \circ \beta([p_j]).\end{aligned}$$

Suppose $\beta([p_j]) = 0$ for some j. Then by definition of β, $[\varphi(p_j)] = 0$ in $K_0(\mathbb{C}^d)$, which gives $\varphi(p_j) \sim_0 0$ and so $\varphi(p_j) = 0$. However, we read above that $\|\varphi(p_j)\| > \|p_j\| - \varepsilon$, which is absurd when $p_j \neq 0$.

Since $C(X)$ is AF, the positive cone $K_0(C(X))^+$ is generated by its scale. Therefore, if $S = \{[q_i]\}_{i=1}^m \subset K_0(C(X))^+$, for each i there are elements of the scale $\{[p_{ij}]\}_{j=1}^{n_i}$ and positive integers k_{ij} with $[q_i] = \sum_{j=1}^{n_i} k_{ij}[p_{ij}]$. Set $S' = \{[p_{ij}] : i = 1, \ldots, m, \quad j = 1, \ldots, n_i\}$ and find d, β, and σ as above. Clearly β remains equivariant on S. Since β is faithful on S', it remains faithful on S.

(2) $\Rightarrow$ (3): This is obvious; simply take $H = K_0(C(X))$.

(3) $\Rightarrow$ (1): Fix a finite set $F \subset \Gamma$ and let $\varepsilon > 0$. By compactness and the zero-dimensionality of X, we can choose a clopen partition $X = \bigsqcup_{j=1}^n Y_j$ with $\mathrm{diam}(Y_j) < \varepsilon/2$. Set $p_j = \mathbf{1}_{Y_j}$ and note that these are orthogonal projections with $\sum_j^n p_j = \mathbf{1}_X$. Consider now

$$B = C^*(\{\alpha_s(p_j) : \ s \in F, j = 1, \ldots, n\}) \quad \text{and} \quad S = \{[p_1], \ldots, [p_n]\} \subset K_0(C(X))^+.$$

Apply (3) and obtain suitable d, H, β, and σ. If $\iota : B \hookrightarrow C(X)$ denotes inclusion, $\hat{\iota} = K_0(\iota) : K_0(B) \to K_0(C(X))$ is a positive group homomorphism. By hypothesis, the subgroup $H \leq K_0(C(X))$ contains all the classes of iterates $\{[\alpha_s(p_j)] : s \in$

$F \cup \{e\}, j = 1, \ldots, n\}$ and $[1_A]$. This guarantees H will contain the image of $\hat{\imath}$, and so we can therefore compose and define the positive group homomorphism

$$\tau := \beta \circ \hat{\imath} : K_0(B) \to \mathbb{Z}^d.$$

After composing with a suitable isomorphism of ordered abelian groups, we may assume

$$\tau : K_0(B) \to K_0(\mathbb{C}^d), \qquad \sigma : \Gamma \to \mathrm{OAut}(K_0(\mathbb{C}^d)),$$

and these satisfy $\tau([1_A]) = [1_{\mathbb{C}^d}]$, and $\sigma_t([1_A]) = [1_{\mathbb{C}^d}]$ for each t in Γ. By Lemma 1.3.4 of [44] there is a unital $*$-morphism $\varphi : B \to \mathbb{C}^d$ with $K_0(\varphi) = \tau$, and an action $\gamma : \Gamma \to \mathrm{Aut}(\mathbb{C}^d)$ with $K_0(\gamma_t) = \sigma_t$. We then extend φ to all of $C(X)$. The conditions then read as follows: for each $t \in F$ and $j = 1, \ldots, n$

$$\begin{aligned}
[\varphi(\alpha_t(p_j))] &= \hat{\varphi}([\alpha_t(p_j)]) = \tau([\alpha_t(p_j)]) = \beta \circ \hat{\imath}([\alpha_t(p_j)]) \\
&= \beta \circ \hat{\alpha}_t([p_j]) = \sigma_t \circ \beta([p_j]) = \sigma_t \circ \beta \circ \hat{\imath}([p_j]) \\
&= \sigma_t \circ \tau([p_j]) = \hat{\gamma}_t \circ \hat{\varphi}([p_j]) = [\gamma_t(\varphi(p_j))].
\end{aligned}$$

This equality holds true in $K_0(\mathbb{C}^d)$ where there is cancellation, whence $\varphi(\alpha_t(p_j)) \sim_0 \gamma_t(\varphi(p_j))$, and commutativity then yields the equality $\varphi(\alpha_t(p_j)) = \gamma_t(\varphi(p_j))$. Moreover, if $\varphi(p_j) = 0$, it follows that

$$\beta([p_j]) = \beta \circ \hat{\imath}([p_j]) = \tau([p_j]) = \hat{\varphi}([p_j]) = [\varphi(p_j)] = 0,$$

which entails, by the condition on β, that $[p_j] = 0$ and thus $p_j = 0$. Thus $\|\varphi(p_j)\| = 1$ whenever p_j is a non-zero projection.

Let $\zeta : \{1, \ldots, d\} \to X$ be the map for which $\varphi(f) = f \circ \zeta$. Moreover, there is an action $\Gamma \curvearrowright \{1,\ldots,d\}$ such that $\gamma_t(g)(z) = g(t^{-1}.z)$ for every $z \in \{1,\ldots,d\}$ and $t \in \Gamma$. The above equivariance of φ implies that for each $j = 1,\ldots,n$, $t \in F$ and $z \in \{1,\ldots,d\}$

$$\begin{aligned} p_j(\zeta(t^{-1}.z)) = \varphi(p_j)(t^{-1}.z) = \gamma_t(\varphi(p_j))(z) = \varphi(\alpha_t(p_j))(z) \\ = \alpha_t(p_j)(\zeta(z)) = p_j(t^{-1}.\zeta(z)). \end{aligned}$$

This shows that for such t and z, $d(t^{-1}.\zeta(z), \zeta(t^{-1}.z)) < \varepsilon$, for otherwise $t^{-1}.\zeta(z)$ and $\zeta(t^{-1}.z)$ would be separated by some p_j and the above equality would fail. Next, the faithfulness of φ means that for each fixed j

$$\max_{z\in\{1,\ldots,d\}} |p_j(\zeta(z))| = \|\varphi(p_j)\| = 1.$$

This proves that $X \subset_\varepsilon \zeta(\{1,\ldots,d\})$, for if $x \in X$, x belongs to some Y_{j_0} and the above equality applied to p_{j_0} ensures that there is a z_0 with $\zeta(z_0) \in Y_{j_0}$ which gives $d(\zeta(z_0), x) < \varepsilon$. The action is thus residually finite, completing the proof.

□

2.2.2 Perturbation Lemmas

Modeling non-commutative C*-dynamics at the K-theoretical level will involve some perturbation results. Recall that two projections determine the same class in K_0 provided that they are sufficiently close. This allows us some much needed flexibility when applying the K_0 functor. The next few results are sufficient for our purposes. The first perturbation lemma is quite standard, and may be found in Davidson's book [16], Lemma III.3.2. We therefore state it without proof.

Lemma 2.2.2. *Given $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists a $\delta = \delta(\varepsilon, n)$ with the following property: Given a unital C^*-algebra A and C^*-subalgebras $C, D \subset A$ with $\dim(C) = n$ and system of matrix units $\mathcal{E}$ for C satisfying $\mathcal{E} \subset_\delta D$, there is a unitary $u \in A$ with $\|1 - u\| < \varepsilon$ such that $uCu^* \subset D$.*

The next result is crucial to the main theorem of this paper, and so we offer a full detailed proof.

Lemma 2.2.3. *Let $B \cong \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_s}$ be a finite dimensional C^*-algebra with $\dim(B) = d$ and system of canonical matrix units $\mathcal{E} = \{e^r_{i,j}\}$. Then for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, d) > 0$ with the following property:*

Given a u.c.p. map $\varphi : B \to \mathbb{M}_k$ which is approximately multiplicative on $\mathcal{E}$ within δ, there is a unital $*$-homomorphism $\sigma : B \to \mathbb{M}_k$ with

$$\|\sigma(x) - \varphi(x)\| < \varepsilon\|x\| \qquad \text{for every } x \in B.$$

Proof. Let B and $\varepsilon > 0$ be given, we will later choose an appropriate δ depending only on ε and on $d = \dim(B)$. By Stinespring's dilation theorem, the u.c.p. map $\varphi : B \to \mathbb{M}_k$ is the compression of a unital $*$-homomorphism. More precisely, there is an isometry $V : \ell_2^k \to \ell_2^l$ and a unital $*$-homomorphism $\pi : B \to \mathbb{M}_l$ such that $\varphi(x) = V^*\pi(x)V$ for every $x \in B$. Let $P = VV^*$ denote the Stinespring projection in $\mathbb{M}_l$.

Claim 1. $\|[P, \pi(e)]\| < \sqrt{\delta}$ for every $e \in \mathcal{E}$, provided φ is approximately multiplicative on $\mathcal{E}$ within δ.

Using the identity $Pa - aP = Pa(1 - P) - (1 - P)aP$ for $a \in \mathbb{M}_l$, we get

$$\|Pa - aP\| = \max\{\|Paa^*P - PaPa^*P\|^{1/2}, \|Pa^*aP - Pa^*PaP\|^{1/2}\}$$

If $e \in \mathcal{E}$, so is e^*, so setting $a = \pi(e)$, we get

$$\begin{aligned}
\|P\pi(e)\pi(e)^*P - P\pi(e)P\pi(e)^*P\| &= \|VV^*\pi(ee^*)VV^* - VV^*\pi(e)VV^*\pi(e^*)VV^*\| \\
&= \|V\varphi(ee^*)V^* - V\varphi(e)\varphi(e^*)V^*\| \\
&= \|V(\varphi(ee^*) - \varphi(e)\varphi(e^*))V^*\| \\
&\le \|\varphi(ee^*) - \varphi(e)\varphi(e^*)\| < \delta.
\end{aligned}$$

Similarly, $\|P\pi(e)^*\pi(e)P - P\pi(e)^*P\pi(e)P\| < \delta$, and together with the above estimate we get $\|P\pi(e) - \pi(e)P\| < \sqrt{\delta}$ as claimed.

Claim 2. Let $C = \pi(B) \subset \mathbb{M}_l$, then $\dim(C) \le d$ and $\|[P, u]\| < \sqrt{\delta}d$ for every $u \in \mathrm{Ball}(C)$, in particular for every unitary $u \in \mathcal{U}(C)$.

If $u \in \mathrm{Ball}(C)$, we can lift u to an $x \in \mathrm{Ball}(B)$ with $\pi(x) = u$. Write

$$x = \sum_{i,j,r} \alpha_{i,j}^{(r)} e_{i,j}^{(r)}, \qquad |\alpha_{i,j}^{(r)}| \le 1.$$

Straightforward estimates yield

$$\begin{aligned}
\|Pu - uP\| = \|P\pi(x) - \pi(x)P\| &= \|\sum_{i,j,r} \alpha_{i,j}^{(r)}(P\pi(e_{i,j}^{(r)}) - \pi(e_{i,j}^{(r)})P)\| \\
&\le \sum_{i,j,r} |\alpha_{i,j}^{(r)}| \|P\pi(e_{i,j}^{(r)}) - \pi(e_{i,j}^{(r)})P\| \le d\sqrt{\delta}
\end{aligned}$$

where we've used Claim 1 and the fact that $|\alpha_{i,j}^{(r)}| \le 1$. This proves Claim 2.

Now $C \subset \mathbb{M}_l$ is a finite dimensional subalgebra, so we have a conditional expectation $\mathbb{E} : \mathbb{M}_l \to C^{'} \cap \mathbb{M}_l$ given by

$$\mathbb{E}(a) = \int_{\mathcal{U}(C)} uau^* du.$$

where du is the normalized Haar measure on $\mathcal{U}(C)$. Using the estimate from Claim 2 we have

$$\begin{aligned}\|\mathbb{E}(P) - P\| &= \left\| \int_{\mathcal{U}(C)} uPu^* du - \int_{\mathcal{U}(C)} P du \right\| = \left\| \int_{\mathcal{U}(C)} (uPu^* - P) du \right\| \\ &\leq \int_{\mathcal{U}(C)} \|uPu^* - P\| du = \int_{\mathcal{U}(C)} \|uP - Pu\| du \leq d\sqrt{\delta}.\end{aligned}$$

Now let $0 < \eta = \eta(\varepsilon) < 1$, to be determined later. We know from standard perturbation results that there is a $\delta^{'} > 0$ with the following property: if A is any unital C*-algebra, $B \subset A$ is a unital subalgebra and $p \in \mathcal{P}(A)$ a projection with $\|p - b\| < \delta^{'}$, then there is a projection $q \in \mathcal{P}(B)$ with $\|p - q\| < \eta$. Making sure that $d\sqrt{\delta} < \delta^{'}$, there is a projection $q \in \mathcal{P}(C^{'} \cap \mathbb{M}_l)$ with $\|P - q\| < \eta$. We may then find a unitary u in $\mathbb{M}_l$ with $u^* P u = q$ and $\|1 - u\| \leq \sqrt{2}\eta$. Now we define

$$\sigma : B \to \mathbb{M}_k \qquad \sigma(b) = V^* u \pi(b) u^* V.$$

We claim that σ is a unital $*$-homomorphism. Indeed, $\sigma(1) = V^* u \pi(1) u^* V = V^* u u^* V = V^* V = 1$, and for a and b in B,

$$\begin{aligned}\sigma(a)\sigma(b) &= V^* u \pi(a) u^* V V^* u \pi(b) u^* V = V^* u \pi(a) u^* P u \pi(b) u^* V \\ &= V^* u \pi(a) q \pi(b) u^* V = V^* u \pi(a) \pi(b) q u^* V = V^* u \pi(ab) q u^* V \\ &= V^* u \pi(ab) u^* P V = V^* u \pi(ab) u^* V V^* V = V^* u \pi(ab) u^* V = \sigma(ab).\end{aligned}$$

We now compute the desired perturbation

$$\|\sigma(x) - \varphi(x)\| = \|V^* u\pi(x)u^* V - V^*\pi(x)V\| = \|V^*(u\pi(x)u^* - \pi(x))V\|$$
$$\leq \|u\pi(x)u^* - \pi(x)\| \leq 2\|u - 1\|\|\pi(x)\| \leq 2\sqrt{2}\eta\|x\|.$$

Now simply choose η so small that $2\sqrt{2}\eta < \varepsilon$.

□

The next lemma is a straightforward application of spectral theory.

Lemma 2.2.4. *Let A be an AF algebra, $\mathcal{F} \subset A_{sa}$ a finite subset, and $\varepsilon > 0$ be given. Then there is a finite dimensional subalgebra $B \subset A$ such that for every $a \in \mathcal{F}$ there are orthogonal projections $p_1, \ldots, p_n$ in B and scalars $\lambda_1, \ldots, \lambda_n$ with*

$$\|a - \sum_{j=1}^{n} \lambda_j p_j\| < \varepsilon.$$

Proof. Fixing a self-adjoint matrix C in $\mathbb{M}_k$, find a $k \times k$ unitary U with $C = UDU^*$ where $D = \operatorname{diag}(t_1, \ldots t_k) = \sum_{j=1}^{k} t_j e_{jj}$, the t_j being real scalars. Then

$$C = U \sum_{j=1}^{k} t_j e_{j,j} U^* = \sum_{j=1}^{k} t_j U e_{j,j} U^* = \sum_{j=1}^{k} t_j P_j,$$

where the projections $P_j := U e_{j,j} U^*$ remain orthogonal.

Now given a self-adjoint $C = (C_1, \ldots, C_s)$ in $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_s}$, by above, write each C_i as

$$C_i = \sum_{j=1}^{k_i} t_{ij} P_{ij},$$

where for each fixed $i \in \{1, \ldots, s\}$ the family of projections $P_{ij} \in \mathbb{M}_{k_i}$ are orthogonal. We regard the P_{ij} as members of the larger algebra $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_s}$, and as such, they are all orthogonal therein. Then,

$$C = \sum_{i=1}^{s} \sum_{j=1}^{k_i} t_{ij} P_{ij},$$

and so every self-adjoint element in a finite dimensional algebra can be written as a linear combination of orthogonal projections.

Given that A is AF, locate a finite dimensional algebra $B \subset A$, and elements $b_i \in B$ for $i = 1, \ldots, n$ with $\|a_i - b_i\| < \varepsilon$. Set $h_i = (b_i + b_i^*)/2$. Clearly $\|a_i - h_i\| < \varepsilon$. From our work above, each h_i is the linear combination of orthogonal projections in B, say $h_i = \sum_{j=1}^{J_i} t_{ij} p_{ij}$, where $p_{ij} \perp p_{il}$ for $j \neq l$ which is what was needed. □

2.2.3 Noncommutative Case

We now wish to explore the K-theoretic expressions that describe RFD, QD and MF C*-systems which in turn shed light on the structure of reduced crossed product. We begin with the more restrictive case; RFD actions.

Definition 2.2.5. Let A be a unital stably finite algebra. An action $\alpha : \Gamma \to \mathrm{Aut}(A)$ is said to be K_0-*RFD* if the following holds: Given any non-zero $g \in K_0(A)^+$, there is a positive group homomorphism $\mu : K_0(A) \to \mathbb{Z}$ with

1. $\mu([1_A]) > 0$, and $\mu(g) > 0$.

2. $\mu(\hat{\alpha}_s(x)) = \mu(x)$ for every $x \in K_0(A)$.

Proposition 2.2.6. *Let A be a unital stably finite algebra and $\alpha : \Gamma \to \mathrm{Aut}(A)$ an action. Consider the following properties:*

1. $A \rtimes_{\lambda,\alpha} \Gamma$ is RFD.

2. The action α is RFD.

3. The action α is K_0-RFD.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Moreover, if A is AF and $\Gamma = \mathbb{F}_r$, then* (3) $\Rightarrow$ (2), *whence all three properties are equivalent.*

Proof. (1) $\Leftrightarrow$ (2) was shown in Theorem 2.1.5.

(2) $\Rightarrow$ (3): Let α be an RFD action and let $g = [p]$ be a non-zero element in $K_0(A)$, in which case $p \neq 0$. By amplifying the action we may assume that p is a non-zero projection in A. Setting $\varepsilon = 1/2$, there is a $*$-homomorphism $\pi : A \to \mathbb{M}_d$ and an inner action $\gamma \curvearrowright \mathbb{M}_d$ such that

(i) $\|\pi(q)\| \geq \|q\| - 1/2 = 1/2$,

(ii) $\pi(\alpha_s(a)) = \gamma_s(\pi(a))$ for every $s \in \Gamma$ and $a \in A$.

Applying the K_0 functor we get a positive group homomorphism $\hat{\pi} : K_0(A) \to K_0(\mathbb{M}_d)$ with $\hat{\pi}([1_A]) = [1_{\mathbb{M}_d}]$. The action γ induces the trivial action at the K_0-level so that condition (ii) implies $\hat{\pi}(\hat{\alpha}_s([q])) = \hat{\pi}([q])$ for every $q \in \mathcal{P}_\infty(A)$. Recall that $K_0(A) = K_0(A)^+ - K_0(A)^+$, so that $\hat{\pi}(\hat{\alpha}_s(x)) = \hat{\pi}(x)$ for every $x \in K_0(A)$. Now let

$$\mu = \beta \circ \hat{\pi} : K_0(A) \to \mathbb{Z}$$

where $\beta : (K_0(A), K_0(A)^+, [1_A]) \to (\mathbb{Z}, \mathbb{Z}^+, d)$ is an isomorphism of ordered abelian groups. Clearly μ is a positive group homomorphism that satisfies the required equivariance condition as well as $\mu([1]) = d > 0$ and $\mu(g) \geq 0$. Also, by stable finiteness

$$\mu(g) = 0 \Rightarrow \beta([\pi(q)]) = 0 \Rightarrow [\pi(q)] = 0 \Rightarrow \pi(q) = 0,$$

a contradiction.

We establish the implication (3) $\Rightarrow$ (2) in the case where A is an AF algebra and $\Gamma = \mathbb{F}_r$ is a free group. Suppose now that α satisfies the K_0-RFD condition. Let $\varepsilon > 0$ and $b = b^* \in A$. Since A has real rank zero, there are orthogonal non-zero projections $p_1, \ldots, p_n$ in A and scalars $t_1, \ldots, t_n$ such that

$$\|b - \sum_{j=1}^{n} t_j p_j\| < \varepsilon/2.$$

We may as well assume that $\|\sum_{j=1}^{n} t_j p_j\| = \max_{1 \leq j \leq n} |t_j| = |t_1|$. Set $g = [p_1]$ and apply the K_0-RFD condition. We obtain a positive group homomorphism μ and set $\mu([1_A]) = d$. By composing with an ordered group isomorphism we may suppose that μ takes values in $K_0(\mathbb{M}_d)$ and $\mu([1]) = [1_{\mathbb{M}_d}]$. By Lemma 1.3.4 of [44] there is a unital $*$-homomorphism $\pi : A \to \mathbb{M}_d$ such that $\hat{\pi} = \mu$. Fix a generator $s_j \in \mathbb{F}_r$ where $1 \leq j \leq r$. The two $*$-homomorphisms π and $\pi \circ \alpha_{s_j}$ from A to $\mathbb{M}_d$ agree at the K-theoretic level by condition (ii) of K_0-RFD. Utilizing once more Lemma 1.3.4 of [44] there is a sequence of unitaries $(u_n)_{n \geq 1}$ in $\mathcal{U}(\mathbb{M}_d)$ with

$$\mathrm{Ad}_{u_n} \circ \pi(a) \overset{n \to \infty}{\longrightarrow} \pi \circ \alpha_{s_j}(a) \quad \forall a \in A.$$

By the compactness of $\mathcal{U}(d)$, we may assume $\|u_n - u_j\| \to 0$ for some unitary $u_j \in \mathbb{M}_d$. Thus $\pi \circ \alpha_{s_j}(a) = \mathrm{Ad}_{u_j} \circ \pi(a)$ for every a in A. By the universal property of the free group we may now define an inner action $\gamma : \mathbb{F}_r \curvearrowright \mathbb{M}_d$ by $\gamma_{s_j} = \mathrm{Ad}_{u_j}$. Thus $\gamma_s(\pi(a)) = \pi(\alpha_s(a))$ holds for every $s \in \mathbb{F}_r$ and $a \in A$.

By condition (i) of K_0-RFD $\pi(p_1)$ is a non-zero projection in $\mathbb{M}_d$. Write $c = \sum_{j=1}^{n} t_j p_j$, so $\|c\| = \|\pi(c)\| = |t_1|$. Then note that

$$\|b\| \leq \|b - c\| + \|c\| \leq \varepsilon/2 + \|\pi(c)\| \leq \varepsilon/2 + \|\pi(c) - \pi(b)\| + \|\pi(b)\| \leq \varepsilon + \|\pi(b)\|,$$

so that α is an RFD action.

$\square$

Recall that for a stably finite unital C*-algebra A, a *state* on $(K_0(A), K_0(A)^+, [1])$ is a group homomorphism $\beta : K_0(A) \to \mathbb{R}$ with $\beta(K_0(A)^+) \subset \mathbb{R}^+$ and $\beta([1]) = 1$. Given an action $\Gamma \curvearrowright A$, a state β is Γ-*invariant* if $\beta(\hat{\alpha}_s(x)) = \beta(x)$ for every $x \in K_0(A)$ and $s \in \Gamma$. Therefore, in a sense, an RFD system (A, Γ, α) is one that admits an *integer*-valued invariant state on K-theory that is locally faithful. A word of caution is in order. The fact that the invariant state emerging from an RFD action is integer valued is much more restrictive. We may consider minimal Cantor systems $(X, \mathbb{Z})$ for example. These always admit an invariant tracial state on $C(X)$, but the induced invariant state on $K_0(C(X))$ can never be integer valued by virtue of the previous proposition and the fact that $C(X) \rtimes \mathbb{Z}$ is simple.

We proceed to look at QD actions K-theoretically. As in the commutative case we focus our attention on AF algebras, in which case the notions of QD and MF actions coincide by Proposition 2.1.17.

Definition 2.2.7. Let (A, Γ, α) be a C*-dynamical system with A unital. We say that α is K_0-QD if the induced action $\hat{\alpha} : \Gamma \to \mathrm{OAut}(K_0(A))$ satisfies the following condition:

Given finite subsets $S \subset K_0(A)^+$ and $F \subset \Gamma$ there is a subgroup $H \leq K_0(A)$, along with a group homomorphism $\beta : H \to \mathbb{Z}$ satisfying

1. $[1_A] \in H$ and $S_F \subset H$,
2. $\beta([1_A]) > 0$ and $\beta(g) > 0$ for each $0 \neq g \in S$,
3. $\beta(\hat{\alpha}_s(g)) = \beta(g)$ for all $g \in S$ and $s \in F$.

One may thus paraphrase condition K_0-QD by saying that the action admits faithful Γ-invariant states in a local sense.

Proposition 2.2.8. *Let A be a unital AF algebra, Γ a discrete group and $\alpha : \Gamma \to \mathrm{Aut}(A)$ an MF-action. Then the induced action α is K_0-QD.*

Proof. Approximately finite dimensional algebras are nuclear, so by Proposition 2.1.17 we may assume that $\alpha : \Gamma \to \mathrm{Aut}(A)$ is a quasidiagonal action. Fix a finite subset $F = \{e = s_1, \ldots, s_m\} \subset \Gamma$. We shall first consider a finite subset $S = \{[p_1], \ldots, [p_n]\} \subset \Sigma(A)$ of the scale of A. Since A is AF we can locate a unital finite dimensional subalgebra $B \subset A$ with $\{\alpha_{s_i}(p_j)\}_{i,j} \subset_{1/4} B$. By perturbing, there are projections $q_{i,j} \in B$ with $\|\alpha_{s_i}(p_j) - q_{i,j}\| < 1/4$, whence $[\alpha_{s_i}(p_j)] = [q_{i,j}]$ in $K_0(A)$. Consider the natural inclusion $\iota : B \hookrightarrow A$ which induces a positive group homomorphism

$$\hat{\iota} : K_0(B) \to K_0(A), \quad \text{where} \quad \hat{\iota}([q]) = [\iota(q)] = [q].$$

Set $K = \ker(\hat{\iota}) \leq K_0(B)$. Since $K_0(B)$ is a finitely generated abelian group, so is K, say $K = \langle t_1, \ldots, t_l \rangle = \mathbb{Z}t_1 + \cdots + \mathbb{Z}t_l$. By the continuity of the functor K_0, there is a finite dimensional subalgebra D of A containing B with the following property: if $j : B \hookrightarrow D$ denotes inclusion, $\hat{j}(t_i) = 0$ in $K_0(D)$ for $i = 1, \ldots, l$.

Let $\varepsilon > 0$ (to be determined later), and choose $\delta = \delta(\varepsilon, \dim(D)) < \varepsilon$ according to the above perturbation Lemma 2.2.2. Also, set $G = \{e^r_{i,j}\} \cup \{q_{i,j}\}$ where $\{e^r_{i,j}\}$ is a system of matrix units for D. Since α is quasidiagonal, there are a positive integer d, a u.c.p. map $\varphi : A \to \mathbb{M}_d$, and an action $\gamma : \Gamma \to \mathrm{Aut}(\mathbb{M}_d)$ such that for every

$a, b \in G$ and $s \in F$

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \delta,$$

$$\|\varphi(a)\| > \|a\| - \delta,$$

$$\|\varphi(\alpha_s(a)) - \gamma_s(\varphi(a))\| < \delta.$$

Utilizing the perturbation Lemma 2.2.3, there is a unital $*$-homomorphism $\pi : D \to \mathbb{M}_d$ such that

$$\|\pi(a) - \varphi(a)\| < \varepsilon \qquad \text{for every} \quad a \in \mathrm{Ball}(D).$$

With the positive group homomorphism $\hat{\pi} : K_0(D) \to K_0(\mathbb{M}_d)$ at hand, we define the subgroup $H = \hat{\imath}(K_0(B)) \leq K_0(A)$ and the map

$$\beta : H \to K_0(\mathbb{M}_d) \cong \mathbb{Z} \qquad \beta(\hat{\imath}(g)) := \hat{\pi}(\hat{\jmath}(g)).$$

Claim 1. β is a well defined group homomorphism satisfying condition (2) of K_0-QD.

Given $g, g' \in K_0(B)$, with $\hat{\imath}(g) = \hat{\imath}(g')$, we have $0 = \hat{\imath}(g) - \hat{\imath}(g') = \hat{\imath}(g - g')$, so that $g - g' \in K$. By construction, $\hat{\jmath}(g - g') = 0$, so $\hat{\jmath}(g) = \hat{\jmath}(g')$ and thus

$$\beta(\hat{\imath}(g)) = \hat{\pi}(\hat{\jmath}(g)) = \hat{\pi}(\hat{\jmath}(g')) = \beta(\hat{\imath}(g')),$$

showing that β is well defined. Clearly β is additive on H, and observe that

$$\beta([1_A]) = \beta(\hat{\imath}([1_A])) = \hat{\pi}(\hat{\jmath}([1_A])) = [\pi(1)] = [1_{\mathbb{M}_d}] \cong d.$$

Now let $0 \neq g = [p_j] = [q_{1j}]$ be in S, which implies by cancellation that q_{1j} is a non-zero projection. Then $\beta(g) = \hat{\pi}(\hat{j}([q_{1j}])) = [\pi(q_{1j})]$ which is clearly positive in $K_0(\mathbb{M}_d)$. Finally, if $[\pi(q_{1j})] = 0$, then by cancellation, $\pi(q_{1j}) = 0$. However,

$$|\|\pi(q_{1j})\| - \|q_{1j}\|| \leq |\|\pi(q_{1j})\| - \|\varphi(q_{1j})\|| + |\|\varphi(q_{1j})\| - \|q_{1j}\|| < \varepsilon + \delta < 2\varepsilon,$$

and since $\|q_{1j}\| = 1$, by choosing $\varepsilon < 1/3$, $\pi(q_{1j})$ must be a non-zero projection as well, an absurdity. The claim is thus proved.

We now verify the promised equivariance with the induced action $\sigma := K_0(\gamma) : \Gamma \to \mathrm{OAut}(K_0(\mathbb{M}_d))$. If $g = [p_j] \in S$ and $s_i \in F$, note that

$$\hat{\alpha}_{s_i}(g) = \hat{\alpha}_{s_i}([p_j]) = [\alpha_{s_i}(p_j)] = [q_{i,j}] = [\iota(q_{i,j})] = \hat{\iota}([q_{i,j}])$$

belongs to $\hat{\iota}(K_0(B)) = H$, so we may apply β to this element and obtain

$$\beta(\hat{\alpha}_{s_i}(g)) = \hat{\pi}(\hat{j}([q_{i,j}])) = [\pi(q_{i,j})].$$

On the other hand, first applying β followed by the action σ gives

$$\begin{aligned}\sigma_{s_i} \circ \beta(g) = \hat{\gamma}_{s_i} \circ \beta([p_j]) = \hat{\gamma}_{s_i} \circ \beta([q_{1,j}]) = \hat{\gamma}_{s_i} \circ \beta(\hat{\iota}[q_{1,j}]) \\ = \hat{\gamma}_{s_i} \circ \hat{\pi} \circ \hat{j}([q_{1,j}]) = [\gamma_{s_i}(\pi(q_{1,j}))].\end{aligned}$$

Claim 2. For each i, j, $[\pi(q_{i,j})] = [\gamma_{s_i}(\pi(q_{1,j}))]$ in $K_0(\mathbb{M}_d)$, which will give us the desired equivariance.

Note that

$$\begin{aligned}\|\gamma_{s_i}(\pi(q_{1,j})) - \pi(\alpha_{s_i}(q_{1,j}))\| &\leq \|\gamma_{s_i}(\pi(q_{1,j})) - \gamma_{s_i}(\varphi(q_{1,j}))\| \\ &\quad + \|\gamma_{s_i}(\varphi(q_{1,j})) - \varphi(\alpha_{s_i}(q_{1,j}))\| + \|\varphi(\alpha_{s_i}(q_{1,j})) - \pi(\alpha_{s_i}(q_{1,j}))\| \\ &< \varepsilon + \delta + \varepsilon < 3\varepsilon.\end{aligned}$$

Choosing $\varepsilon < 1/6$ we guarantee that $[\gamma_{s_i}(\pi(q_{1,j}))] = [\pi(\alpha_{s_i}(q_{1,j}))]$. Now $s_1 = e$ so we have

$$\begin{aligned}\|\alpha_{s_i}(q_{1,j}) - q_{i,j}\| &\leq \|\alpha_{s_i}(q_{1,j}) - \alpha_{s_i} \circ \alpha_{s_1}(p_j)\| + \|\alpha_{s_i} \circ \alpha_{s_1}(p_j) - q_{i,j}\| \\ &\leq \|q_{1,j} - \alpha_{s_1}(p_j)\| + \|\alpha_{s_i}(p_j) - q_{i,j}\| < 1/4 + 1/4 = 1/2.\end{aligned}$$

Therefore $\|\pi(\alpha_{s_i}(q_{1,j})) - \pi(q_{i,j})\| < 1/2$ and so $[\pi(\alpha_{s_i}(q_{1,j}))] = [\pi(q_{i,j})]$ in $K_0(\mathbb{M}_d)$ and our claim holds.

For a positive integer d, any order automorphism σ of $(\mathbb{Z}, \mathbb{Z}^+, d)$ must be trivial. Indeed, since σ is a positive isomorphism $\sigma(1) > 0$, and since $d = \sigma(d) = d \cdot \sigma(1)$, we must have $\sigma(1) = 1$. Therefore, any action $\sigma : \Gamma \to \mathrm{OAut}(\mathbb{Z}, \mathbb{Z}^+, d)$ of ordered abelian groups must be trivial, and the above equivariance now reads:

$$\beta(\hat{\alpha}_{s_i}(g)) = \beta(g) \qquad \forall g \in S, \quad \forall s_i \in F.$$

Next, we must consider an arbitrary finite subset of the positive cone $K_0(A)^+$ and not restrict ourselves to its scale $\Sigma(A)$. This is not problematic, for as A is AF, its scale $\Sigma(A)$ generates the positive cone $K_0(A)^+$. So, given a finite set $S = \{[p_1], \ldots, [p_m]\} \subset K_0(A)^+$, where $p_j \in \mathcal{P}_\infty(A)$, write each $[p_j] = \sum_{i=1}^{I_j} t_{ji}[q_{ji}]$ where the t_{ji} are positive integers and the q_{ji} are non-zero projections in A. Set $S' = \{[q_{ji}]; j = 1, \dots, m, \ i = 1, \dots, I_j\} \subset \Sigma(A)$, and by our above work, obtain a suitable H, and β satisfying the required conditions for the given finite set $F \subset \Gamma$. Using the fact that $\hat{\alpha}_t$ is additive for each $t \in F$, and β is faithful on non-zero elements of S' the properties $(1), (2)$, and (3) of K_0-QD will hold. Since the t_{ji} are positive integers, and $\beta([q_{ji}]) \geq 0$, observe that for every j

$$0 = \beta([p_j]) = \sum_i t_{ji}\beta([q_{ji}]) \quad \Longrightarrow \quad \beta([q_{ji}]) = 0, \, \forall i \quad \Longrightarrow \quad [q_{ji}] = 0 \, \forall i \quad \Longrightarrow \quad [p_j] = 0,$$

so that β is indeed faithful on non-zero elements of S, completing the proof. □

When a free group is acting on a unital AF-algebra, K_0-QD actions coincide with QD actions.

Theorem 2.2.9. *Let A be a unital AF-algebra and $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ an action, where $r \in \{1, 2, \dots, \infty\}$. Then α is quasidiagonal if and only if α is K_0-QD.*

Proof. Having shown the 'only if' above, we embark on the proof of sufficiency. Denote the generators of $\mathbb{F}_r$ by $s_1, \dots, s_r$ and set $e_{\mathbb{F}_r} = s_0$. We abbreviate $\alpha_{s_i} = \alpha_i$ for $i = 0, \dots, r$, and to ease notation write $K_0(\alpha_i) = \hat{\alpha}_i$ to denote the induced order automorphism at the K_0-level. Let $\delta > 0$, to be determined later, and let us first consider the case where we are given a finite set of non-zero projections $p_1, \dots, p_n$ belonging to a finite dimensional subalgebra $B \subset A$. Find $\delta' = \delta'(\delta, \dim(B))$ as in Lemma 2.2.2. The algebras $B_i = \alpha_i(B)$ are finite dimensional and admit systems of matrix units $\mathcal{E}_i$ for each i. Since A is AF, there is a finite dimensional $D \subset A$ containing B with $\mathcal{E}_i \subset_{\delta'} D$ for every i. Lemma 2.2.2 then provides us with unitaries u_i in A satisfying $\|u_i - 1\| < \delta$ and $u_i B_i u_i^* \subset D$.

Choose $F = \{s_1, \dots, s_r\}$ and $S \subset K_0(A)^+$ as $S = \{[p_1], \dots, [p_n], e_1, \dots, e_k, f_1, \dots, f_l\}$, where the e_j generate $K_0(B)$ and the f_j generate $K_0(D)$. More precisely, $e_j = [e_{11}^{(j)}]$ and $f_j = [f_{11}^{(j)}]$ where $\{e_{s,t}^{(j)}\}$ and $\{f_{s,t}^{(j)}\}$ are appropriate systems of matrix units for B and D respectively. Since α is K_0-QD, we obtain the subgroup $H \leq K_0(A)$ and the group morphism $\beta : H \to \mathbb{Z}$ satisfying all the desired properties. Suppose $\beta([1_A]) = d > 0$. By composing with an isomorphism of ordered abelian groups

$$(\mathbb{Z}, \mathbb{Z}^+, d) \cong (K_0(\mathbb{M}_d), K_0(\mathbb{M}_d)^+, [1_{\mathbb{M}_d}]),$$

we may assume β takes values in $K_0(\mathbb{M}_d)$ and $\beta([1_A]) = [1_{\mathbb{M}_d}]$. Denote by ι the inclusion $\iota : D \hookrightarrow A$, and note that for any generator f_j of $K_0(D)$ we have $\hat{\iota}(f_j) = f_j \in S \subset H$ whence the map

$$\beta \circ \hat{\iota} : K_0(D) \to K_0(\mathbb{M}_d)$$

is a well defined group homomorphism. Since $K_0(D) = \mathbb{Z}^+ f_1 + \cdots + \mathbb{Z}^+ f_l$, and β takes positive values on S, $\beta \circ \hat{\iota}$ is certainly a positive map. Also, $\beta \circ \hat{\iota}([1_A]) = \beta([1_A]) = [1_{\mathbb{M}_d}]$, so there is a unital $*$-homomorphism $\varphi : D \to \mathbb{M}_d$ with $\hat{\varphi} = \beta \circ \hat{\iota}$. Appealing to the invariance of β, we obtain

$$\begin{aligned}
\hat{\varphi}(e_j) &= \beta \circ \hat{\iota}(e_j) = \beta \circ \hat{\iota}([e_{11}^{(j)}]) = \beta([e_{11}^{(j)}]) \\
&= \beta \circ \hat{\alpha}_i([e_{11}^{(j)}]) = \beta([\alpha_i(e_{11}^{(j)})]) = \beta([u_i \alpha_i(e_{11}^{(j)}) u_i^*]) = \beta \circ \hat{\iota}([u_i \alpha_i(e_{11}^{(j)}) u_i^*]) \\
&= \hat{\varphi}([u_i \alpha_i(e_{11}^{(j)}) u_i^*]) = [\varphi \circ \mathrm{Ad}_{u_i} \circ \alpha_i(e_{11}^{(j)})] = K_0(\varphi \circ \mathrm{Ad}_{u_i} \circ \alpha_i)(e_j).
\end{aligned}$$

Therefore the homomorphisms $\varphi|_B$ and $\varphi \circ \mathrm{Ad}_{u_i} \circ \alpha_i|_B$ agree at the K_0 level, as morphisms from $K_0(B)$ to $K_0(\mathbb{M}_d)$, and by the finite-dimensionality of B we know that there are unitaries v_i in $\mathbb{M}_d$ with

$$\mathrm{Ad}_{v_i} \circ \varphi|_B = \varphi \circ \mathrm{Ad}_{u_i} \circ \alpha_i|_B.$$

By the universal property of the free group, we may define an action $\gamma : \mathbb{F}_r \to \mathrm{Aut}(\mathbb{M}_d)$ by $\gamma_i := \gamma_{s_i} := \mathrm{Ad}_{v_i}$, which gives us

$$\gamma_i \circ \varphi|_B = \varphi \circ \mathrm{Ad}_{u_i} \circ \alpha_i|_B.$$

By Arveson's extension theorem, we may extend φ to a unital completely positive map $\varphi : A \to \mathbb{M}_d$. For each p_j and each s_i a simple estimate using the fact that $\|1 - u_i\| < \delta$ gives

$$\begin{aligned} \|\gamma_i \circ \varphi(p_j) - \varphi \circ \alpha_i(p_j)\| &= \|\varphi \circ \mathrm{Ad}_{u_i} \circ \alpha_i(p_j) - \varphi \circ \alpha_i(p_j)\| \\ &\leq \|\mathrm{Ad}_{u_i} \circ \alpha_i(p_j) - \alpha_i(p_j)\| = \|u_i \alpha_i(p_j) u_i^* - \alpha_i(p_j)\| \leq 2\delta. \end{aligned}$$

Now φ is multiplicative on D and hence on the p_j and is clearly injective on $\{p_1, \ldots, p_n\}$. Indeed, by the condition on β and cancellation,

$$\varphi(p_j) = 0 \Rightarrow \hat{\varphi}([p_j]) = 0 \Rightarrow \beta([p_j]) = 0 \Rightarrow [p_j] = 0 \Rightarrow p_j = 0,$$

a contradiction.

We now can proceed to the general case. To verify quasidiagonality of the action, it suffices to consider a finite set of self-adjoint elements $a_1, \ldots, a_m \in A$ with $\|a_j\| \leq 1$, the finite set of standard generators $\{s_1, \ldots, s_r\}$ of $\mathbb{F}_r$ and an arbitrary $\varepsilon > 0$. By

Lemma 2.2.4, we find a finite-dimensional subalgebra $B \subset A$ such that for each a_j

$$\left\| a_j - \sum_{l=1}^{L_j} t_{jl} p_{jl} \right\| < \eta \qquad 0 \neq p_{jl} \in \mathcal{P}(B), \quad p_{jl} p_{jk} = 0, \ l \neq k$$

where $\eta = \eta(\varepsilon) > 0$ will be determined later. Set $b_j = \sum_{l=1}^{L_j} t_{jl} p_{jl}$. Note that fixing j, the projections p_{jl} are orthogonal, whence

$$\max_l |t_{jl}| = \|b_j\| \leq \|b_j - a_j\| + \|a_j\| \leq \eta + 1.$$

Apply all our above work to the set of projections $\{p_{jl}\}_{j,l} \subset B$ in order to obtain φ, d, γ, as above for an arbitrary $\delta > 0$. We estimate

$$\begin{aligned}
&\|\gamma_i \circ \varphi(a_j) - \varphi \circ \alpha_i(a_j)\| \\
&\qquad \leq \|\gamma_i \circ \varphi(a_j) - \gamma_i \circ \varphi(b_j)\| + \|\gamma_i \circ \varphi(b_j) - \varphi \circ \alpha_i(b_j)\| + \|\varphi \circ \alpha_i(b_j) - \varphi \circ \alpha_i(a_j)\| \\
&\qquad \leq 2\|a_j - b_j\| + \left\| \sum_{l=1}^{L_j} t_{jl} (\gamma_i \circ \varphi(p_{jl}) - \varphi \circ \alpha_i(p_{jl})) \right\| \\
&\qquad \leq 2\eta + \sum_{l=1}^{L_j} |t_{jl}| \|(\gamma_i \circ \varphi(p_{jl}) - \varphi \circ \alpha_i(p_{jl}))\| \\
&\qquad \leq 2\eta + \sum_{l=1}^{L_j} (1+\eta) 2\delta = 2\eta + L_j(1+\eta)2\delta \leq 2\eta + L(1+\eta)2\delta
\end{aligned}$$

where $L = \max_j L_j$. To verify approximate multiplicativity, observe

$$\|a_i a_j - b_i b_j\| \leq \|a_i a_j - a_i b_j\| + \|a_i b_j - b_i b_j\| \leq \|a_i\| \|a_j - b_j\| + \|a_j - b_j\| \|b_j\| \leq \eta + \eta(1+\eta).$$

A similar estimate yields $\|\varphi(a_i)\varphi(a_j) - \varphi(b_i)\varphi(b_j)\| \leq \eta + \eta(1+\eta)$. Note that φ, being multiplicative on all the projections p_{jl}, will also be multiplicative on the b_j, therefore

$$\|\varphi(a_ia_j)-\varphi(a_i)\varphi(a_j)\| \leq \|\varphi(a_ia_j)-\varphi(b_ib_j)\|+\|\varphi(a_i)\varphi(a_j)-\varphi(b_i)\varphi(b_j)\| \leq 2(\eta+\eta(1+\eta)).$$

Now since φ is faithful on the p_{jl}, φ will be isometric on the b_j. Indeed, using the fact that $\varphi(p_{jl})\varphi(p_{jk}) = \varphi(p_{jl}p_{jk}) = 0$ for $k \neq l$, we have

$$\|\varphi(b_j)\| = \left\| \sum_{l=1}^{L_j} t_{jl}\varphi(p_{jl}) \right\| = \max_l |t_{jl}| = \|b_j\|.$$

Finally we estimate

$$|\|\varphi(a_j)\| - \|a_j\|| \leq |\|\varphi(a_j)\| - \|\varphi(b_j)\||+|\|b_j\| - \|a_j\|| \leq \|\varphi(a_j)-\varphi(b_j)\|+\|a_j-b_j\| \leq 2\eta.$$

We need only choose the right η and δ. Given $\varepsilon > 0$, choose η so that $\eta < \varepsilon/4$, and $2(\eta + \eta(1 + \eta)) < \varepsilon$. Then simply choose $\delta < \varepsilon/(4L(1 + \eta))$. By our above estimates this choice will ensure the approximate equivariance $\|\gamma_i \circ \varphi(a_j) - \varphi \circ \alpha_i(a_j)\| < \varepsilon$, the approximate multiplicativity $\|\varphi(a_ia_j) - \varphi(a_i)\varphi(a_j)\| < \varepsilon$, and the approximate isometricity $|\|\varphi(a_j)\| - \|a_j\|| < \varepsilon$, so that $\mathbb{F}_r \curvearrowright A$ is quasidiagonal. □

Combining the last few results with Theorem 2.1.5 we obtain:

Corollary 2.2.10. *Let $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ be an action on a unital AF algebra. The following are equivalent:*

1. *α is MF.*
2. *α is QD.*
3. *α satisfies K_0-QD.*
4. *The reduced crossed product $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is an MF algebra.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) is Theorem 2.1.19.

(2) $\Leftrightarrow$ (3): This is Theorem 2.2.9.

$\square$

We seek yet another equivalent K-theoretic condition, this time in the spirit of a coboundary subgroup analogous to N. Brown's main result in [10]. Now we insist that our discrete group be a free group $\Gamma = \mathbb{F}_r = \langle s_1, \ldots, s_r \rangle$ of finitely many generators, which acts on a unital AF algebra A. Denote this action by α and $\alpha_i = \alpha_{s_i}$. By the Pimsner-Voiculescu six term exact sequence (consult [6] p.78) and the fact that $K_1(A) = \{0\}$ for an AF algebra, the sequence

$$\bigoplus_{j=1}^{r} K_0(A) \xrightarrow{\sigma} K_0(A) \xrightarrow{\hat{\iota}} K_0(A \rtimes_{\lambda,\alpha} \mathbb{F}_r) \longrightarrow 0$$

is exact, where $\iota : A \hookrightarrow A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is the canonical inclusion, and

$$\sigma(g_1, \ldots, g_r) = \sum_{j=1}^{r} (g_j - \hat{\alpha}_j(g_j)).$$

Write $H_\sigma = \mathrm{im}(\sigma) \leq K_0(A)$, so that $K_0(A)/H_\sigma \cong K_0(A \rtimes_{\lambda,\alpha} \mathbb{F}_r)$. First, a preliminary result about the subgroup H_σ.

Lemma 2.2.11. *In the above context, the subgroup $H_\sigma \leq K_0(A)$ is generated by the set*

$$\{g - \hat{\alpha}_w(g) : g \in K_0(A), w \in \mathbb{F}_r\}.$$

Proof. One direction being clear from the definition, we claim that every element of the form $g - \hat{\alpha}_w(g)$ will belong to H_σ. To that end, write the alphabet for $\mathbb{F}_r$ as

$$\mathcal{A} = \{e, s_1, \ldots, s_r, s_1^{-1}, \ldots, s_r^{-1}\}.$$

First note that for every letter $a \in \mathcal{A}$, and for all $g \in K_0(A)$, $g - \hat{\alpha}_a(g) \in H_\sigma$. For $a = e$ it's clear. Suppose $a = s_i$, for some $1 \leq i \leq r$, then $g - \hat{\alpha}_{s_i}(g) = \sigma(0, \ldots, 0, g, 0, \ldots, 0) \in H_\sigma$, where g is in the ith spot. Next, say $a = s_i^{-1}$ for some $1 \leq i \leq r$, then

$$\begin{aligned} g - \hat{\alpha}_{s_i^{-1}}(g) &= \widehat{\alpha_{s_i} \circ \alpha_{s_i^{-1}}}(g) - \hat{\alpha}_{s_i^{-1}}(g) = \hat{\alpha}_{s_i} \circ \hat{\alpha}_{s_i^{-1}}(g) - \hat{\alpha}_{s_i^{-1}}(g) \\ &= \hat{\alpha}_{s_i}(\hat{\alpha}_{s_i^{-1}}(g)) - \hat{\alpha}_{s_i^{-1}}(g) = -(\hat{\alpha}_{s_i^{-1}}(g) - \hat{\alpha}_{s_i}(\hat{\alpha}_{s_i^{-1}}(g))) = -(f - \hat{\alpha}_{s_i}(f)) \in H_\sigma, \end{aligned}$$

where $f = \hat{\alpha}_{s_i^{-1}}(g)$. Now let $w \in \mathbb{F}_r$ be a (reduced) word in symbols from $\mathcal{A}$. We have shown that if $|w| = 1$ the claim holds, so proceed by strong induction on $|w|$. If $|w| = l$, write $w = aw'$ where $a \in \mathcal{A} \setminus \{e\}$ so $|w'| < l$. For $g \in K_0(A)$:

$$\begin{aligned} g - \hat{\alpha}_w(g) &= g - \hat{\alpha}_{aw'}(g) = g - \hat{\alpha}_{w'}(g) + \hat{\alpha}_{w'}(g) - \hat{\alpha}_a \circ \hat{\alpha}_{w'}(g) \\ &= g - \hat{\alpha}_{w'}(g) + \hat{\alpha}_{w'}(g) - \hat{\alpha}_a(\hat{\alpha}_{w'}(g)) = g - \hat{\alpha}_{w'}(g) + f - \hat{\alpha}_a(f) \in H_\sigma \end{aligned}$$

by the inductive hypothesis, where $f = \hat{\alpha}_{w'}(g)$. This completes the proof. □

We shall make use of the following key lemma which is due to Spielberg. Consult [48] for a clear argument. Note that this result relies on the theorem of Effros, Handelman and Shen [17] on dimension groups.

Lemma 2.2.12 (Spielberg)**.** *If K is a dimension group and H is a subgroup of K with $H \cap K^+ = \{0\}$, then there is a dimension group G and a positive group homomorphism $\theta : K \to G$ such that*

1. $H \subset \ker(\theta)$,

2. $\ker(\theta) \cap K^+ = \{0\}$.

Proposition 2.2.13. *Let $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ be an action of a free group on a unital AF algebra. Then the following are equivalent:*

1. *α is K_0-QD.*

2. *$H_\sigma \cap K_0(A)^+ = \{0\}$.*

Proof. (1) $\Rightarrow$ (2): Suppose $x = \sum_{j=1}^{r}(g_j - \hat{\alpha}_j(g_j)) > 0$ in $K_0(A)^+$. For each j write $g_j = x_j - y_j$ with $x_j, y_j \in K_0(A)^+$. By setting $S = \{x_1, \ldots, x_r, y_1, \ldots, y_r, x\}$ and $F = \{s_1, \ldots, s_r\}$ as our finite sets, we obtain a suitable H and $\beta : H \to \mathbb{Z}$ with the desired conditions in the definition of K_0-QD. Observe then that

$$0 < \beta(x) = \beta\left(\sum_{j=1}^{r}(g_j - \hat{\alpha}_j(g_j))\right) = \sum_{j=1}^{r}(\beta(g_j) - \beta(\hat{\alpha}_j(g_j))) = \sum_{j=1}^{r}(\beta(g_j) - \beta(g_j)) = 0,$$

a contradiction. Therefore, $x = 0$, and (2) holds.

(2) $\Rightarrow$ (1): Since $H_\sigma \cap K_0(A)^+ = \{0\}$ using Lemma 2.2.12 we get a dimension group (G, G^+) and positive group homomorphism $\theta : K_0(A) \to G$ satisfying $H_\sigma \subset \ker(\theta)$ and $\ker(\theta) \cap K_0(A)^+ = \{0\}$. Given finite subsets $F \subset \mathbb{F}_r$ and $S \subset K_0(A)^+$, consider the finitely generated subgroup H of $K_0(A)$ given by

$$H = \langle \hat{\alpha}_s(x) : x \in S \cup \{[1]\}, s \in F \cup \{e\} \rangle.$$

This H will be the desired subgroup for verifying that α is K_0-QD, and so what is needed is the correct $\beta : H \to \mathbb{Z}$. Restricting θ to H we note that the subgroup $\theta(H) \leq G$ is also generated by the finitely many positive elements $\theta(\hat{\alpha}_t(x))$ for $x \in S \cup \{[1]\}$ and $t \in F \cup \{e\}$. To ease notation, label these as $k_1, \ldots, k_n \in G^+$. Since G is a dimension group, write $(G, (\beta_i))$ for the limit of an inductive sequence of ordered abelian groups (G_i, G_i^+)

$$G_1 \xrightarrow{h_1} G_2 \xrightarrow{h_2} G_3 \xrightarrow{h_3} \dots$$

where the h_i are positive group homomorphisms, the $\beta_i : G_i \to G$ are the connecting positive group homomorphisms and each (G_i, G_i^+) is order isomorphic to $(\mathbb{Z}^{p_i}, \mathbb{Z}^{p_i}_{\geq 0})$ for some positive integers p_i. There is an m large enough so that $k_1, \dots, k_n \in \beta_m(G_m^+)$. Set $k_i = \beta_m(y_i)$ for some $y_i \in G_m^+$. The group G_m is abelian and finitely generated, and so is its subgroup $K = \ker(\beta_m)$, say $K = \langle g_1, \dots, g_l \rangle$. Now choose k large enough so that $k \geq m$ and such that $h_{k,m}(g_j) = 0$ for all $j = 1, \dots, l$. Identify $(G_k, G_k^+) = (\mathbb{Z}^p, \mathbb{Z}^p_{\geq 0})$ for some $p \in \mathbb{N}$ and define $\psi : G_k \to \mathbb{Z}$ by

$$\psi((z_1, \dots, z_p)) = \sum_{i=1}^{p} z_i.$$

Clearly, ψ is a positive group homomorphism which is faithful on the positive cone G_k^+. We now may define $\phi : \beta_m(G_m) \to \mathbb{Z}$ by $\phi(\beta_m(g)) := \psi(h_{k,m}(g))$. Observe that, by our choice of k,

$$\begin{aligned}\beta_m(g) = \beta_m(g') \Leftrightarrow g - g' \in K \Rightarrow h_{k,m}(g - g') = 0 \Leftrightarrow h_{k,m}(g) = h_{k,m}(g') \\ \Leftrightarrow \psi(h_{k,m}(g)) = \psi(h_{k,m}(g')) \Leftrightarrow \phi(\beta_m(g)) = \phi(\beta_m(g')),\end{aligned}$$

verifying that ϕ is well defined. It is routine to check that ϕ is additive on $\beta_m(G_m)$. Naturally, we now compose and define

$$\beta := \phi \circ \theta|_H : H \longrightarrow \mathbb{Z}.$$

Since the t_i lie in $\beta_m(G_m)$, we have $\theta(H) = \langle t_1, \dots, t_n \rangle \leq \beta_m(G_m)$ and thus β is a well defined group homomorphism. Let $x \in S$. Then from our notation $\theta(x) = t_i$ for some i. So

$$\beta(x) = \phi(\theta(x)) = \phi(t_i) = \phi(\beta_m(y_i)) = \psi(h_{k,m}(y_i)) \geq 0,$$

since ψ and $h_{k,m}$ are positive maps and $y_i \in G_m^+$. To see that β is faithful on S, we use the fact that ψ is faithful on G_k^+: if $x \in S$ then

$$\beta(x) = 0 \Rightarrow \psi(h_{k,m}(y_i)) = 0 \Rightarrow h_{k,m}(y_i) = 0 \Rightarrow \beta_k(h_{k,m}(y_i)) = 0 \Rightarrow \beta_m(y_i) = 0$$
$$\Rightarrow t_i = 0 \Rightarrow \theta(x) = 0 \Rightarrow x \in \ker(\theta) \cap K_0(A)^+ \Rightarrow x = 0.$$

Finally, we verify the invariance of β. For $x \in S$ and $s \in F$, Lemma 2.2.11 ensures that $x - \hat{\alpha}_s(x)$ belongs to H_σ, which in turn lives inside $\ker(\theta)$, so that $\theta(x - \hat{\alpha}_s(x)) = 0$. Therefore,

$$\beta(x - \hat{\alpha}_s(x)) = \phi \circ \theta(x - \hat{\alpha}_s(x)) = 0 \Longrightarrow \beta(x) = \beta(\hat{\alpha}_s(x)),$$

completing the proof.

□

While MF algebras are always stably finite, the authors of [8] remarked that there are no known examples of stably finite C*-algebras which are not MF. With the right K_0 condition at our disposal we can give an answer to this inquiry for a special class of crossed product algebras. Here is the crucial result, reminiscent of N. Brown's main result in [10].

Theorem 2.2.14. *Let A be a unital AF algebra and $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ an action of the free group on r generators. Then the following are equivalent:*

1. *α is MF.*

2. *α is quasidiagonal.*

3. *α is K_0-QD.*

4. *$H_\sigma \cap K_0(A)^+ = \{0\}$.*

5. *The reduced crossed product $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is MF.*

6. *The reduced crossed product $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ is stably finite.*

Proof. For such an action, the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are contained in Theorem 2.1.19, Propositions 2.2.8 and 2.2.13. Now every MF algebra is stably finite, so it suffices to show (6) $\Rightarrow$ (4). To that end, suppose $x \in H_\sigma \cap K_0(A)^+$. Then $x = [p]$ where $p \in \mathcal{P}_\infty(A)$. From the Pimsner-Voiculescu exact sequence, $\ker(\hat{\iota}) = H_\sigma$, so $0 = \hat{\iota}(x) = \hat{\iota}([p]) = [\iota(p)]$ in $K_0(A \rtimes_{\lambda,\alpha} \mathbb{F}_r)$. However, the stable finiteness of $A \rtimes_{\lambda,\alpha} \mathbb{F}_r$ ensures $\iota(p) = 0$, which implies that $p = 0$ since ι is inclusion. Thus $x = 0$ and (3) holds.

$\square$

Example 2.2.15. If A is an AF-algebra and $(A, \mathbb{F}_r, \alpha)$, $(A, \mathbb{F}_r, \beta)$ are C*-dynamical systems which agree on K-theory, that is $\hat{\alpha} = \hat{\beta}$, then Theorem 2.2.14 ensures that α is MF if and only if β is MF. In particular, recall that actions α and β are said to be *exterior equivalent* provided there is a map $u : \Gamma \to \mathcal{U}(A)$ which satisfies the cocycle condition $u_{st} = u_s \alpha_s(u_t)$ and $\beta_s = \mathrm{Ad}_s \circ \alpha_s$ for each $s, t \in \Gamma$. In this case α and β clearly agree on K-theory and the above discussion applies.

3. C*-FINITENESS AND PARADOXICAL DECOMPOSITIONS

This chapter explores the deep theme common to groups, dynamical systems and operator algebras; that of finiteness, infiniteness, and proper infiniteness, the latter expressed in terms of paradoxical decompositions. The remarkable alternative theorem of Tarski establishes, for discrete groups, the dichotomy between amenability and paradoxical decomposability. This carries over into the realm of operator algebras. Indeed, if a discrete group Γ acts on itself by left-translation, the Roe algebra $C(\beta\Gamma) \rtimes_\lambda \Gamma$ is properly infinite if and only if Γ is Γ-paradoxical and this happens if and only if Γ is non-amenable [45]. This is mirrored in the von Neumann algebra setting as well; all projections in a II_1 factor are finite and the ordering of Murray-von-Neumann subequivalence is determined by a unique faithful normal tracial state. Alternatively type III factors admit no traces since all non-zero projections therein are properly infinite. As for unital, simple, separable and nuclear algebras, the C*-enthusiast of old hoped that the trace/traceless divide determined a similar dichotomy between stable finiteness and pure infiniteness (the C*-algebraic analog of type III). This hope was laid to rest with Rørdam's example of a unital, simple, separable, nuclear C*-algebra containing both an infinite and a non-zero finite projection [43]. The conjecture for such a dichotomy remains open for those algebras whose projections are total. Theorem 3.0.18 below is a result in this direction.

Despite the failure of the above dichotomy, the classification program of Elliott in its original K-theoretic formulation has witnessed much success for stably finite algebras [44], [19], as well as in the purely infinite case with the spectacular complete classification results of Kirchberg and Phillips [37], [29] modulo the UCT. One motivation for studying purely infinite algebras stems from the fact that Kirchberg algebras (unital, simple, separable, nuclear, and purely infinite) are classified by their K- or KK-theory. The C*-literature has produced examples of purely infinite C*-algebras arising from dynamical systems [4], [31], [32], [45]. In many cases the underlying algebra is abelian with spectrum the Cantor set. For example, Archbold, Spielberg, and Kumjian (independently) proved that there is an action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on the Cantor set so that the corresponding crossed product C*-algebra is isomorphic to $\mathcal{O}_2$ [47]. Laca and Spielberg [32] construct purely infinite and simple crossed products that emerge from strong boundary actions. Jolissaint and Robertson [26] generalized the idea of strong boundary action to noncommutative systems with the concept of an n-filling action. They showed that $A \rtimes_\lambda \Gamma$ is simple and purely infinite provided that the action is properly outer and n-filling and every corner pAp of A is infinite dimensional. When the algebra A has a well behaved $K_0(A)$ group we will in fact give a K-theoretic proof of their result (see Proposition 3.2.20).

The transition from classical topological dynamics to noncommutative C*-dynamics presents several challenges and subtleties. One way to approach these issues is to interpret dynamical conditions K-theoretically via the induced actions on $K_0(A)$ and on the Cuntz semigroup $W(A)$ and use tools from the classification literature as well as developed techniques of Cuntz comparison to uncover pertinent algebraic information. Such an approach is seen in Brown's work [10] as well as that of the author in the previous chapter (see [41]. We continue this philosophy here. For instance, the classical version of topological transitivity has a natural extension to noncommutative systems (Definition 3.1.9), and, as in the commutative case, is tied to the primitivity of the algebra (see Theorem 3.1.12). The idea of a group acting paradoxically on a set and the construction of the type semigroup goes back to the work of Tarski (the reader is encouraged to read Wagon's book [53] for a good treatment). Rørdam and Sierakowski [45] looked at the type semigroup $S(X, \Gamma)$ built from an action of a discrete group on the Cantor set and tied pure infiniteness of the resulting reduced crossed product to the absence of traces on this semigroup. In effect, they prove that if a countable, discrete, and exact group Γ acts continuously and freely on the Cantor set X, and the preordered semigroup $S(X, \Gamma)$ is almost unperforated, then the following are equivalent: (i) The reduced crossed product $C(X) \rtimes_\lambda \Gamma$ is purely infinite, (ii) $C(X) \rtimes_\lambda \Gamma$ is traceless, (iii) $S(X, \Gamma)$ is purely infinite (that is $2x \leq x$ for every $x \in S(X, \Gamma)$), and (iv) $S(X, \Gamma)$ is traceless. Inspired by their work, we construct a type semigroup $S(A, \Gamma, \alpha)$ for noncommutative systems (A, Γ, α) and establish a generalized result. This is Theorem 3.2.21 below which, in particular, implies the following.

Theorem 3.0.16. *Let A be a unital, separable, and exact C^*-algebra with stable rank one and real rank zero. Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal and properly outer action with $S(A, \Gamma, \alpha)$ almost unperforated. Then the following are equivalent:*

1. *The semigroup $S(A, \Gamma, \alpha)$ is purely infinite.*
2. *The C^*-algebra $A \rtimes_\lambda \Gamma$ is purely infinite.*
3. *The C^*-algebra $A \rtimes_\lambda \Gamma$ is traceless.*
4. *The semigroup $S(A, \Gamma, \alpha)$ admits no non-trivial state.*

As a suitable quotient of $K_0(A)^+$, this type semigroup $S(A, \Gamma, \alpha)$ is purely infinite if and only if every positive element of $K_0(A)^+$ is paradoxical under the induced action with covering multiplicity at least two. Taking covering multiplicities into account, Kerr and Nowak [28] consider completely non-paradoxical actions of a discrete group on the Cantor set. We do the same for noncommutative systems using ordered K-theory, and inevitably resort to Tarski's deep result (Theorem 3.2.11) to prove Theorem 3.2.13; of which the following is a special case.

Theorem 3.0.17. *Let A be a unital, separable and exact C*-algebra with stable rank one and real rank zero . Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal action. Then the following are equivalent:*

1. *$A \rtimes_\lambda \Gamma$ admits a faithful tracial state.*
2. *$A \rtimes_\lambda \Gamma$ is stably finite.*
3. *α is completely non-paradoxical.*

Moreover, if A is AF and Γ is a free group, then (1) *through* (3) *are all equivalent to $A \rtimes_\lambda \Gamma$ being MF in the sense of Blackadar and Kirchberg [8].*

Combining these two results we obtain the desired dichotomy, albeit for a certain class of crossed products.

Theorem 3.0.18. *Let A be a unital, separable, and exact C*-algebra with stable rank one and real rank zero. Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal and properly outer action with $S(A, \Gamma, \alpha)$ almost unperforated. Then the reduced crossed product $A \rtimes_\lambda \Gamma$ is simple and is either stably finite or purely infinite.*

3.1 Minimality and Topological Transitivity

In this section we develop K-theoretic descriptions of minimality and topological transitivity for C*-systems, primarily in the noncommutative setting. These formulations will be useful when describing the structure of the resulting reduced crossed product algebra.

For a general C*-dynamical system $\alpha : \Gamma \to \mathrm{Aut}(A)$, we say that α is *minimal* (or equivalently we call A Γ-*simple*) if A admits no non-trivial invariant ideals, that is, there does not exist an ideal $(0) \neq I \subsetneq A$ with $\alpha_s(I) = I$ for every $s \in \Gamma$. Note that *ideals* in the category of C*-algebras will always be assumed to be closed, and the term *algebraic ideal* will be reserved for ideals in the algebraic sense, that is, not necessarily closed. If A has a unit, it is routine to check that A admits a non-trivial invariant (closed) ideal if and only if A contains a non-trivial invariant algebraic ideal. Since every ideal in $M_n(A)$ is of the form $M_n(I)$ for an ideal $I \subset A$, it follows easily that if A is Γ-simple, then $M_n(A)$ is Γ-simple as well, where the action $\Gamma \curvearrowright M_n(A)$ is given by amplification $s \mapsto \alpha_s^{(n)} \in \mathrm{Aut}(M_n(A))$.

The notion of a minimal action $\alpha : \Gamma \curvearrowright A$ is tied to the simplicity of the corresponding reduced crossed product $A \rtimes_{\lambda,\alpha} \Gamma$. Recall that a C*-algebra is simple if it contains no non-trivial (closed) ideals. Indeed, given a action $\alpha : \Gamma \curvearrowright A$, with a non-trivial Γ-invariant ideal $I \subset A$, one readily sees that $I \rtimes_{\lambda,\alpha} \Gamma$ is a non-trivial ideal in $A \rtimes_{\lambda,\alpha} \Gamma$, since $(I \rtimes_{\lambda,\alpha} \Gamma) \cap A = I \neq A = (A \rtimes_{\lambda,\alpha} \Gamma) \cap A$. Therefore, a necessary condition for the reduced crossed product to be simple is minimality of the action. However, the absence of invariant ideals does not always ensure simplicity of the crossed product algebra. In some cases, however, minimality is enough to ensure a simple reduced crossed product. We record here some of the these examples.

A discrete group Γ is said to be *exact* provided that its reduced group C*-algebra $C^*_\lambda(\Gamma)$ is exact, or equivalently, if it admits an amenable action on some compact space. Exact groups include all amenable groups and all free groups $\mathbb{F}_r$ for $r \in \{1, 2, \ldots, \infty\}$. An action $\Gamma \curvearrowright X$ is said to be *free* if for each $x \in X$, the isotropy group $\{s \in \Gamma : s.x = x\}$ is trivial. It is shown in [46] that if $\Gamma \curvearrowright X$ is a free action of an exact group on a locally compact Hausdorff space, the reduced crossed product $C_0(X) \rtimes_\lambda \Gamma$ is simple if and only if the action is minimal.

A group Γ is called a *Powers group* if the following holds: For every finite set $F \subset \Gamma$ and integer $n \in \mathbb{N}$ there is a partition $\Gamma = E \sqcup D$ and elements $t_1, \ldots, t_n \in \Gamma$ such that

1. $sD \cap rD = \emptyset$ for every $s, r \in F$ with $s \neq r$,

2. $t_j E \cap t_k E = \emptyset$ for every $j, k \in \{1, \dots, n\}$ with $j \neq k$.

It was shown in [25] that Powers' groups are non-amenable and have infinite conjugacy classes. Also, Powers showed that non-abelian free groups are Powers groups. In [24] P. de la Harpe and G. Skandalis showed that an action $\alpha : \Gamma \to \mathrm{Aut}(A)$ of a Powers' group on a unital algebra A is minimal if and only if $A \rtimes_{\lambda,\alpha} \Gamma$ is simple.

Recently, the authors of [9] have shown that a minimal C*-system (A, Γ, α) yields a simple crossed product $A \rtimes_\lambda \Gamma$ provided that the group Γ is C*-simple, that is, $C^*_\lambda(\Gamma)$ is simple. Examples of such groups can be found in [5].

For general C*-systems (A, Γ, α), an extra condition is needed over and above minimality to ensure a simple reduced crossed product. Recall that an automorphism α in $\mathrm{Aut}(A)$ is said to be *properly outer* if and only if for every invariant ideal $I \subset A$ and inner automorphism β in $\mathrm{Inn}(I)$ we have $\|\alpha|_I - \beta\| = 2$. An action $\alpha : \Gamma \to \mathrm{Aut}(A)$ is said to be *properly outer* if for every $e \neq t \in \Gamma$, α_t is properly outer. The following result is Theorem 7.2 in [35].

Theorem 3.1.1. *Let (A, Γ, α) be a C*-dynamical system with Γ discrete and A separable. If α is minimal and properly outer, then $A \rtimes_{\lambda,\alpha} \Gamma$ is simple.*

3.1.1 K-theoretic Minimality

In the classical setting, a continuous action $\Gamma \curvearrowright X$ of a discrete group on a compact Hausdorff space is said to be *minimal* if the action admits no non-trivial closed invariant sets, that is, there is no closed subset $\emptyset \neq Y \subsetneq X$ with $s.Y = Y$ for every $s \in \Gamma$. A well known example of a minimal action is that of an irrational rotation $\mathbb{Z} \curvearrowright \mathbb{T}$, given by $n.z = \omega^n z$, where $\omega = \exp(2\pi i\theta)$ for an irrational θ. This, of course, agrees with the notion of a minimal action above. The equivalence of (1), (2), and (4) in the following proposition is well known and standard in dynamics, whereas statement (3) is tailored here to serve as motivation for our work below.

Proposition 3.1.2. *Let $\Gamma \curvearrowright X$ be a continuous action on a compact Hausdorff space, and let $\alpha : \Gamma \curvearrowright C(X)$ denote the induced action. The following are equivalent:*

1. *The action is minimal.*

2. *For every x in X, the orbit $\operatorname{Orb}(x) = \{s.x \mid s \in \Gamma\}$ is dense in X.*

3. *For any non-empty open set $E \subset X$, there are elements $t_1, \ldots, t_n$ in Γ such that*
$$\bigcup_{j=1}^{n} t_j.E = X.$$

4. *The Γ-algebra $C(X)$ is Γ-simple under the associated action.*

Proof. (1) $\Rightarrow$ (2): Fix $x \in X$, and set $Y = \overline{\operatorname{Orb}(x)}$. For $s \in \Gamma$, note that $s.\operatorname{Orb}(x) = \operatorname{Orb}(x)$, so taking closures we get

$$s.Y = s.\overline{\operatorname{Orb}(x)} = \overline{s.\operatorname{Orb}(x)} = \overline{\operatorname{Orb}(x)} = Y.$$

Since the action is minimal and $\emptyset \neq Y$, we have that $\overline{\operatorname{Orb}(x)} = Y = X$.

(2) $\Rightarrow$ (3): Let $\emptyset \neq E \subset X$ be open. For each finite subset $F = \{t_1, \ldots, t_k\} \subset \Gamma$, put $E_F = \cup_{j=1}^{k} t_j.E$. Denoting by $\mathfrak{F}$ the collection of all finite sets of Γ, we claim that $\cup_{F \in \mathfrak{F}} E_F = X$. Given the claim, compactness allows for a finite subcover $\cup_{j=1}^{J} E_{F_j} = X$, and thus $E_F = X$ where $F = \cup_{j=1}^{J} F_j$ which proves (2) $\Rightarrow$ (3).

To prove the claim, assume there is an $x \in X \setminus \cup_{F \in \mathfrak{F}} E_F$. By hypothesis, $\operatorname{Orb}(x)$ is dense in X, and since $\cup_{F \in \mathfrak{F}} E_F$ is open, there is a $z \in \cup_{F \in \mathfrak{F}} E_F \cap \operatorname{Orb}(x)$. We can then write $z = s.x \in E_F$ for some finite set F and some $s \in \Gamma$, so that $z = s.x \in t.E$ for a certain t, yielding $x \in (s^{-1}t).E$, a contradiction.

(3) $\Rightarrow$ (4): This direction is even easier. Suppose there is a non-trivial closed invariant set Y. Then $\emptyset \neq X \setminus Y =: E$ By assumption there are group elements $t_1, \ldots, t_n$ with $\bigcup_{j=1}^n t_j.E = X$. Thus for a point $y \in Y$, we have that $y \in t_j.E$ for some j whence $t_j^{-1}.y$ belongs to $E \cap Y = \emptyset$ by invariance, which is absurd.

(4) $\Leftrightarrow$ (1): Every ideal in $C(X)$ is of the form $J_Y = \{f \in C(X) | \ f|_Y = 0\}$ for some closed set $Y \subset X$. Note that J_Y is a non-trivial and invariant if and only if Y is non-trivial and invariant. $\square$

An important remark on statement (3) is in order. Jolissaint and Robertson ([26]) introduced the notion of an *n-filling action* for general C*-systems (A, Γ, α), which in the commutative case is equivalent to a generalized global version of hyperbolicity [32]. More precisely, for a given integer $n \geq 2$, an action $\Gamma \curvearrowright X$ of a discrete group on a compact Hausdorff space is n-filling if and only if for any non-empty open subsets of X, $E_1, \ldots, E_n$, there are group elements $t_1, \ldots, t_n$ with $t_1.E_1 \cup \cdots \cup t_n.E_n = X$. Thus, by Proposition 3.1.2, an n-filling action is minimal. We shall see in Proposition 3.1.13 below that the n-filling property is equivalent to the apparently weaker condition: given any non-empty open subset E, there are group elements $t_1, \ldots, t_n$ with $t_1.E \cup \cdots \cup t_n.E = X$. The subtle difference is that the given integer n is fixed in the n-filling property whereas it is not necessarily bounded in Proposition 3.1.2.

When the space X is zero-dimensional, other characterizations of minimality will be useful, indeed, they will motivate a suitable notion of K-theoretic minimality in the noncommutative case. Here we write $C(X; \mathbb{Z})$ for the dimension group of all continuous integer-valued functions on X, and $\mathcal{C}_X$ for the collection of all clopen subsets of a topological space X. The action on the underlying space induces a natural action of order automorphisms $\beta : \Gamma \to \mathrm{OAut}(C(X; \mathbb{Z}))$, given by $\beta_s(f)(x) =$

$f(s^{-1}.x)$ for $s \in \Gamma$ and $f \in C(X;\mathbb{Z})$.

Proposition 3.1.3. *Let $\Gamma \curvearrowright X$ be a continuous action on a compact, zero-dimensional metrizable space. Then the following are equivalent:*

1. *The action is minimal.*

2. *For any non-empty clopen set $E \subset X$, there are elements $t_1, \ldots, t_n$ in Γ such that*
$$\bigcup_{j=1}^{n} t_j.E = X.$$

3. *For every non-zero positive function $f \in C(X;\mathbb{Z})^+$, there are elements $t_1, \ldots, t_n$ in Γ such that*
$$\sum_{j=1}^{n} \beta_{t_j}(f) \geq \mathbf{1}_X.$$

Proof. (1) $\Leftrightarrow$ (2): Identical to the proof in Proposition 3.1.2, use the fact that since our space is now zero-dimensional and therefore every open set (more precisely Y^c in the proof above) contains a clopen set E.

(2) $\Rightarrow$ (3): Let $0 \neq f \in C(X;\mathbb{Z})^+$. Such an f has the form $f = \sum_{j=1}^{m} n_j \mathbf{1}_{E_j}$ where the n_j are non-negative integers, not all zero, and the E_j are clopen sets. Pick a non-empty $E_j := E$ with $n_j \neq 0$, there is one by our assumption on f. Assuming (2), find elements $t_1, \ldots, t_n$ such that $\bigcup_{j=1}^{n} t_j.E = X$. Now since the β_{t_j} are order preserving and $\mathbf{1}_E \leq f$,

$$\mathbf{1}_X \leq \sum_{j=1}^{n} \mathbf{1}_{t_j.E} = \sum_{j=1}^{n} \beta_{t_j}(\mathbf{1}_E) \leq \sum_{j=1}^{n} \beta_{t_j}(f).$$

(3) $\Rightarrow$ (2): Given a non-empty clopen set E, $f := \mathbf{1}_E$ is a non-negative, non-zero, integer-valued continuous function. We then are granted group elements $t_1, \ldots, t_n$ in

Γ such that $\sum_{j=1}^{n} \beta_{t_j}(f) \geq \mathbf{1}_X$. Then

$$\mathbf{1}_X \leq \sum_{j=1}^{n} \beta_{t_j}(f) = \sum_{j=1}^{n} \beta_{t_j}(\mathbf{1}_E) = \sum_{j=1}^{n} \mathbf{1}_{t_j.E},$$

which shows $\bigcup_{j=1}^{n} t_j.E = X$.

$\square$

Recall that when X is the Cantor set, $K_0(C(X))$ is order isomorphic to $C(X;\mathbb{Z})$ via the dimension map $\dim : K_0(C(X)) \to C(X;\mathbb{Z})$ given by $\dim([p]_0)(x) = \mathrm{Tr}(p(x))$. Here p represents a projection over the matrices of $C(X)$; $M_n(C(X)) \cong C(X;\mathbb{M}_n)$, and Tr denotes the standard (non-normalized) trace on $\mathbb{M}_n$. Now given a continuous action $\Gamma \curvearrowright X$, let $\alpha : \Gamma \to \mathrm{Aut}(C(X))$ denote the associated action on the algebra $C(X)$, and write $\hat{\alpha} : \Gamma \to \mathrm{OAut}(K_0(C(X)))$ for the induced action on the ordered group $K_0(C(X))$. Moreover, as above, we have a natural action $\beta : \Gamma \to \mathrm{OAut}(C(X;\mathbb{Z}))$, given by $\beta_s(f)(x) = f(s^{-1}.x)$ for $s \in \Gamma$ and $f \in C(X;\mathbb{Z})$. One may inquire about the possible equivariance of $\hat{\alpha}$ and β through the isomorphism dim. Indeed, these actions are the same; we show that for each $s \in \Gamma$, the following diagram is commutative.

$$\begin{array}{ccc} K_0(C(X)) & \xrightarrow{\hat{\alpha}_s} & K_0(C(X)) \\ \downarrow{\scriptstyle \dim} & & \downarrow{\scriptstyle \dim} \\ C(X;\mathbb{Z}) & \xrightarrow{\beta_s} & C(X;\mathbb{Z}) \end{array}$$

To see this, consider any $n \in \mathbb{N}$, a projection $p \in \mathcal{P}_n(C(X))$, an $s \in \Gamma$ and any $x \in X$. We compute:

$$\beta_s \circ \dim([p]_0)(x) = \dim([p]_0)(s.^{-1}x) = \mathrm{Tr}(p(s.^{-1}x)) = \mathrm{Tr}(\alpha_s^{(n)}(p)(x)) = \dim([\alpha_s^{(n)}(p)]_0)(x)$$
$$= \dim \circ \hat{\alpha}_s([p]_0)(x)$$

which shows that $\beta_s \circ \dim([p]_0) = \dim \circ \hat{\alpha}_s([p]_0)$ as functions on X, and consequently that $\beta_s \circ \dim = \dim \circ \hat{\alpha}_s$ by uniqueness of the Grothendieck extension.

Condition (3) in the above Propostion and this discussion motivate a suitable definition for *minimal* actions at the K-theoretic level in the noncommutative case, at least for stably finite algebras where the K_0 group is ordered.

Definition 3.1.4. Let Γ be a discrete group, A a unital, stably finite C*-algebra, and $\alpha : \Gamma \curvearrowright A$ an action with induced action $\hat{\alpha}$ on $K_0(A)$.

1. We say that α is K_0-*minimal* provided that for every $0 \neq g \in K_0(A)^+$, there are $t_1, \ldots, t_n$ in Γ such that $\sum_{j=1}^n \hat{\alpha}_{t_j}(g) \geq [1]_0$.

2. Fix an integer $n \in \mathbb{N}$. We say that α is K_0-n-*minimal* provided that for every $0 \neq g \in K_0(A)^+$, there are $t_1, \ldots, t_n$ in Γ such that $\sum_{j=1}^n \hat{\alpha}_{t_j}(g) \geq [1]_0$.

3. Fix an integer $n \in \mathbb{N}$. We say that α is K_0-n-*filling* provided that for all non-zero $g_1, \ldots, g_n \in K_0(A)^+$, there are $t_1, \ldots, t_n$ in Γ such that $\sum_{j=1}^n \hat{\alpha}_{t_j}(g_j) \geq [1]_0$.

There is a significant difference between K_0-minimal actions and K_0-n-minimal actions. Of course every K_0-n-minimal action is K_0-minimal, but the converse is far from true. We shall see that when $K_0(A)$ has suitable properties K_0-n-minimal actions along with proper outerness guarantee that the reduced crossed product is simple and purely infinite, whereas K_0-minimal actions along with proper outerness may generate simple stably finite crossed product algebras.

Proposition 3.1.3 and the remarks proceeding it imply that a Cantor system (X, Γ) is minimal if and only if the algebra $C(X)$ is Γ-simple if and only if α is K_0-minimal, where $\alpha : \Gamma \curvearrowright C(X)$ is, of course, the induced action. With some work, we will show that for a stably finite algebra that admits sufficiently many projections, K_0-minimality and Γ-simplicity are equivalent notions. Due to the rigid structure of K_0, it turns out to be easier to work with the Cuntz semigroup $W(A)$. Also, when dealing with Cuntz comparability we need not make any restrictions on the underlying algebra. Here are the parallel definitions.

Definition 3.1.5. Let Γ be a discrete group, A a unital C*-algebra, and $\alpha : \Gamma \curvearrowright A$ an action with induced action $\hat{\alpha}$ on the Cuntz semigroup $W(A)$.

1. We say that α is W-*minimal* provided that for every $0 \neq g \in W(A)$, there are $t_1, \dots, t_n$ in Γ such that $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(g) \geq \langle 1 \rangle$.

2. Fix an integer $n \in \mathbb{N}$. We say that α is W-n-*minimal* provided that for every $0 \neq g \in W(A)$, there are $t_1, \dots, t_n$ in Γ such that $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(g) \geq \langle 1 \rangle$.

3. Fix an integer $n \in \mathbb{N}$. We say that α is W-n-*filling* provided that for all nonzero $g_1, \dots, g_n \in W(A)$, there are $t_1, \dots, t_n$ in Γ such that $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(g_j) \geq \langle 1 \rangle$.

Using topological transitivity we show below (Proposition 3.1.13) that W-n-minimal and W-n-filling actions coincide. But first, we justify our choice of nomenclature.

Proposition 3.1.6. *Let (A, Γ, α) be a C*-dynamical system with induced action $\hat{\alpha} : \Gamma \curvearrowright W(A)$ on the Cuntz semigroup of A. Then A is Γ-simple if and only if α is W-minimal.*

Proof. Suppose the action is W-minimal and let $(0) \neq I \subset A$ be a Γ-invariant ideal. Take a nonzero x in I^+ and find group elements $t_1, \dots, t_n$ with $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(\langle x \rangle) \geq \langle 1 \rangle$.

This means

$$\langle \alpha_{t_1}(x) \oplus \cdots \oplus \alpha_{t_n}(x) \rangle = \sum_{j=1}^{n} \langle \alpha_{t_j}(x) \rangle = \sum_{j=1}^{n} \hat{\alpha}_{t_j}(\langle x \rangle) \geq \langle 1 \rangle = \langle 1 \oplus 0_{n-1} \rangle.$$

This implies that $1 \oplus 0_{n-1}$ is Cuntz smaller than $\alpha_{t_1}(x) \oplus \cdots \oplus \alpha_{t_n}(x)$ and there is a sequence $(y_k)_{k \geq 1}$ in $M_n(A)$ with $y_k^*(\alpha_{t_1}(x) \oplus \cdots \oplus \alpha_{t_n}(x))y_k \to 1 \oplus 0_{n-1}$. Now each $\alpha_{t_j}(x)$ belongs to I so that $\alpha_{t_1}(x) \oplus \cdots \oplus \alpha_{t_n}(x)$ belongs to $M_n(I)$, a (closed) ideal in $M_n(A)$. Furthermore, each $y_k^*(\alpha_{t_1}(x) \oplus \cdots \oplus \alpha_{t_n}(x))y_k \in M_n(I)$ so that $1 \oplus 0_{n-1}$ lives in $M_n(I)$ ($M_n(I)$ is closed) which implies that $1 \in I$ and $I = A$. The action is thus Γ-simple.

Conversely, assume α admits no non-trivial invariant ideals, and let $g = \langle a \rangle \in W(A)$, for some $a \in M_n(A)^+$. Since the algebraic ideal generated by $\{\alpha_s^{(n)}(a) : s \in \Gamma\}$ is all of $M_n(A)$, there are lists of elements $t_1, \ldots, t_m \in \Gamma$, and $x_1, \ldots, x_m; y_1, \ldots, y_m$ in $M_n(A)$ such that

$$\sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(a) y_j^* = \frac{1}{2} \mathbf{1}_{M_n(A)}.$$

Now set $z_j := x_j + y_j$ and observe that

$$\begin{aligned}\sum_{j=1}^{m} z_j \alpha_{t_j}^{(n)}(a) z_j^* &= \sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(a) y_j^* + \sum_{j=1}^{m} y_j \alpha_{t_j}^{(n)}(a) x_j^* + \sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(a) x_j^* + \sum_{j=1}^{m} y_j \alpha_{t_j}^{(n)}(a) y_j^* \\ &\geq \sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(a) y_j^* + \big(\sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(a) y_j^*\big)^* = \mathbf{1}_{M_n(A)} \geq 1_A \oplus 0_{m-1},\end{aligned}$$

the first inequality following from the fact that the last two sums on the first line are positive. A simple Cuntz comparison now gives

$$\begin{aligned} 1_A \approx 1_A \oplus 0_{m-1} &\precsim \sum_{j=1}^{m} z_j \alpha_{t_j}^{(n)}(a) z_j^* = (z_1, \ldots, z_m)(\alpha_{t_1}^{(n)}(a) \oplus \cdots \oplus \alpha_{t_m}^{(n)}(a))(z_1, \ldots, z_m)^* \\ &\precsim \alpha_{t_1}^{(n)}(a) \oplus \cdots \oplus \alpha_{t_m}^{(n)}(a). \end{aligned}$$

Therefore, in the ordering on $W(A)$,

$$\langle 1 \rangle \leq \langle \alpha_{t_1}^{(n)}(a) \oplus \cdots \oplus \alpha_{t_m}^{(n)}(a) \rangle = \sum_{j=1}^{m} \langle \alpha_{t_j}^{(n)}(a) \rangle = \sum_{j=1}^{m} \hat{\alpha}_{t_j}(\langle a \rangle)$$

which gives the W-minimality of the action. □

It is well known that if a C*algebra A is unital and stably finite, $(K_0(A), K_0(A)^+, [1]_0)$ is a ordered abelian group with order unit $u = [1]_0$, and so the above definition of K_0-minimality applies. With the added assumption of sufficiently many projections, all the notions of minimality mentioned above will coincide as the next result shows. Recall that a subgroup H of an abelian ordered group (G, G^+) is said to be an *order ideal* provided that its positive cone is spanning and hereditary, that is, $H = H^+ - H^+$ and $0 \leq g \leq h \in H^+$ implies $g \in H$, where by definition $H^+ = H \cap G^+$. In the context of an action $\beta : \Gamma \to \mathrm{OAut}(G)$, a subset $H \subset G$ is called Γ-invariant if for every $t \in \Gamma$, $\beta_t(H) \subset H$.

Theorem 3.1.7. *Let A be a unital, stably finite C*-algebra with the property that every ideal in A admits a non-trivial projection. Consider an action $\alpha : \Gamma \to \mathrm{Aut}(A)$ with induced action $\hat{\alpha} : \Gamma \to \mathrm{OAut}(K_0(A))$. The following are equivalent:*

1. *A is Γ-simple.*

2. *α is W-minimal.*

3. α is K_0-minimal.

4. There are no non-trivial Γ-invariant order ideals $H \subset K_0(A)$.

Proof. The equivalence of (1) and (2) was shown in Propostion 3.1.6.

(2) $\Rightarrow$ (3): Let $0 \neq x \in K_0(A)^+$, then $x = [p]_0$ for some non-zero $p \in \mathcal{P}_m(A)$. By hypothesis there are group elements $t_1, \ldots, t_n$ such that

$$\langle \alpha_{t_1}(p) \oplus \cdots \oplus \alpha_{t_n}(p) \rangle = \sum_{j=1}^{n} \langle \alpha_{t_j}(p) \rangle = \sum_{j=1}^{n} \hat{\alpha}_{t_j}(\langle p \rangle) \geq \langle 1 \rangle.$$

By definition $1 \precsim r := \alpha_{t_1}(p) \oplus \cdots \oplus \alpha_{t_n}(p)$ and so $1 \sim q \leq r$ where q is a subprojection of r in $M_{mn}(A)$. Since $r - q \perp q$, a small computation will give the desired inequality, indeed:

$$\begin{aligned}[1]_0 \leq [1]_0 + [r-q]_0 = [q]_0 + [r-q]_0 = [r-q+q]_0 = [r]_0 = [\alpha_{t_1}(p) \oplus \cdots \oplus \alpha_{t_n}(p)]_0 \\ = \sum_{j=1}^{n} \hat{\alpha}_{t_j}([p]_0) = \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x).\end{aligned}$$

(3) $\Rightarrow$ (1): Suppose $(0) \neq I \subset A$ is a Γ-invariant ideal. By our assumption on A, we can find a nonzero projection $p \in I$. Now find group elements $t_1, \ldots, t_n$ such that

$$[\alpha_{t_1}(p) \oplus \cdots \oplus \alpha_{t_n}(p)]_0 = \sum_{j=1}^{n} [\alpha_{t_j}(p)]_0 = \sum_{j=1}^{n} \hat{\alpha}_{t_j}([p]_0) \geq [1]_0.$$

Apply the order embedding $V(A) \hookrightarrow W(A)$ which gives $\langle \alpha_{t_1}(p) \oplus \cdots \oplus \alpha_{t_n}(p) \rangle \geq \langle 1 \rangle$ so that

$$1 \oplus 0_{n-1} \approx 1 \precsim \alpha_{t_1}(p) \oplus \cdots \oplus \alpha_{t_n}(p).$$

Now follow the exact reasoning as Propostion 3.1.6 to deduce that $1 \in I$ and $I = A$.

(3) $\Rightarrow$ (4): Suppose $(0) \neq H \subset K_0(A)$ is a Γ-invariant order ideal. Since H^+

is spanning, we can locate a non-zero x in $H^+ := H \cap K_0(A)^+$. By (3) there are $t_1, \dots, t_n \in \Gamma$ with $\sum_{j=1}^n \hat{\alpha}_{t_j}(x) \geq [1]_0 \geq 0$. Each $\hat{\alpha}_{t_j}(x)$ is in H so $\sum_{j=1}^n \hat{\alpha}_{t_j}(x) \in H$, and H being hereditary implies that $[1]_0$ is in H. Now given any $z \in K_0(A)^+$, there is an $n \in \mathbb{Z}^+$ such that $0 \leq z \leq n[1]_0$. Using again the fact that H is hereditary we have $z \in H$, thus $K_0(A)^+ \subset H$ whence $K_0(A) = H$.

(4) $\Rightarrow$ (3): Let $0 \neq x \in K_0(A)^+$. Consider the set

$$L := \left\{ y \in K_0(A) : \exists n \in \mathbb{Z}^+, \exists t_1, \dots, t_n \in \Gamma \quad \text{such that} \quad 0 \leq y \leq \sum_{j=1}^n \hat{\alpha}_{t_j}(x) \right\}.$$

Two facts are fairly clear about $L \subset K_0(A)^+$: $L + L \subset L$ and L is hereditary, that is, if $z \in K_0(A)$ and $y \in L$ with $0 \leq z \leq y$ then $z \in L$. It is natural to then define the subgroup $H = L - L$. We show that H is in fact a non-zero Γ-invariant order ideal. To that end set $H^+ = H \cap K_0(A)^+$ and note that $L \subset H^+$. Then

$$H = L - L \subset H^+ - H^+ \subset H,$$

so $H = H^+ - H^+$. Also, if $z \in K_0(A)$ with $0 \leq z \leq y - y' \in H$, with $y, y' \in L$, then since $y - y' \leq y$ and L is hereditary, we have $z \in L \subset H$ so H is hereditary as well. $H \neq (0)$ since $x \in H$. Finally, if $y \in L$ and $t \in \Gamma$, then $0 \leq y \leq \sum_{j=1}^n \hat{\alpha}_{t_j}(x)$ for certain group elements $t_1, \dots, t_n$. Applying the order isomorphism $\hat{\alpha}_t$ we get

$$0 \leq \hat{\alpha}_t(y) \leq \hat{\alpha}_t\big(\sum_{j=1}^n \hat{\alpha}_{t_j}(x)\big) = \sum_{j=1}^n \hat{\alpha}_{tt_j}(x),$$

which implies that $\hat{\alpha}_t(y) \in L$ and $\hat{\alpha}_t(L) \subset L$. So $\hat{\alpha}_t(H) = \hat{\alpha}_t(L) - \hat{\alpha}_t(L) \subset L - L = H$ which is what we wanted. By our hypothesis, $H = K_0(A)$, so that $[1]_0 \in H$. Writing $[1]_0 = y - y' \leq y$ for some y, y' in L and recalling that L is hereditary ensures $[1]_0 \in L$, which means that there are group elements $t_1, \ldots, t_n$ with $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x) \geq [1]_0$, and α is thus K_0-minimal.

$\square$

3.1.2 K-theoretic Topological Transitivity

We now aim to develop a notion of topological transitivity in the noncommutative setting, and we do this using K-theory. An action $\Gamma \curvearrowright X$ of a group on a locally compact Hausdorff space is termed *topologically transitive* if for every pair U, V of non-empty open subsets of X, there is a group element $s \in \Gamma$ with $s.U \cap V \neq \emptyset$. When X is compact, it is routine to check that every minimal action is topologically transitive (see Proposition 3.1.2 above) but the converse is false in general as witnessed by the translation action $\mathbb{Z} \curvearrowright \mathbb{Z}_\infty$ on the one-point compactification of the integers with the point ∞ being fixed. An action $\Gamma \curvearrowright X$ is said to have the *intersection property* if each non-zero ideal of $C_0(X) \rtimes_\lambda \Gamma$ has non-zero intersection with $C_0(X)$. As minimality of an action is linked with simplicity of the crossed product, topological transitivity is associated with primitivity. The following is an abbreviated form of Proposition 2.8 of [33].

Proposition 3.1.8. *Consider a continuous action of a discrete group on a locally compact Hausdorff space X. If $C_0(X) \rtimes_\lambda \Gamma$ is prime, then the action is topologically transitive. Conversely, if the action is topologically transitive and has the intersection property, then $C_0(X) \rtimes_\lambda \Gamma$ is prime.*

After we develop a notion of topological transitivity in the noncommutative setting we will establish a more general result (see Theorem 3.1.12).

Definition 3.1.9. Let (A, Γ, α) be a C*-system. Call an action α *topologically transitive* if for every pair of non-zero $x, y \in W(A)$, there is group element $t \in \Gamma$ and a non-zero $z \in W(A)$ with $z \leq x$ and $\hat{\alpha}_t(z) \leq y$.

The following Proposition shows that this definition is consistent with the established notion of topological transitivity in the commutative setting. Recall that for $f, g \in M_\infty(C(X))^+$, we have $f \precsim g$ if and only if $\operatorname{supp}(f) \subset \operatorname{supp}(g)$, where $\operatorname{supp}(\cdot)$ denotes the support.

Proposition 3.1.10. *Let X be a locally compact space, and let $k : \Gamma \curvearrowright X$ be a continuous action with induced action $\alpha : \Gamma \to \operatorname{Aut}(C_0(X))$. Then k is topologically transitive if and only if α is topologically transitive.*

Proof. Assume that that k is topologically transitive, and let $x = \langle g \rangle$, $y = \langle f \rangle$ be non-zero elements in $W(C_0(X))$. Since f and g are continuous matrix valued functions on X, $U = \{x | f(x) \neq 0\}$ and $V = \{x | g(x) \neq 0\}$ are open and non-empty. Therefore, there is a $s \in \Gamma$ such that $s.U \cap V \neq \emptyset$. Consider any non-empty open subset $Y \subset s.U \cap V$ and find a non-zero continuous function $h : X \to [0, 1]$ with $\operatorname{supp}(h) \subset Y$. Since $\operatorname{supp}(h) \subset V \subset \operatorname{supp}(g)$ we have that $h \precsim g$ whence $0 \neq z := \langle h \rangle \leq \langle g \rangle = x$. Also,

$$\operatorname{supp}(s^{-1}.h) = s^{-1}.\operatorname{supp}(h) \subset s^{-1}.Y \subset s^{-1}.(s.U) = U \subset \operatorname{supp}(f),$$

thus $s^{-1}.h \precsim f$ which gives $\hat{\alpha}_{s^{-1}}(z) = \langle s^{-1}.h \rangle \leq \langle f \rangle = y$.

Conversely, now suppose $\alpha : \Gamma \curvearrowright C_0(X)$ is topologically transitive and consider a pair U, V of non-empty open subsets of X. Find continuous non-zero mappings $f, g : X \to [0, 1]$ with $\operatorname{supp}(f) \subset U$ and $\operatorname{supp}(g) \subset V$. There is then a non-zero $z \in W(C_0(X))$ and $t \in \Gamma$ with $z \leq \langle f \rangle$ and $\hat{\alpha}_t(z) \leq \langle g \rangle$. Say $z = \langle h \rangle$ for some continuous $h \in C_0(X, \mathbb{M}_n^+)$. Then $\operatorname{supp}(h) \subset \operatorname{supp}(f) \subset U$ and $t.\operatorname{supp}(h) = \operatorname{supp}(t.h) \subset \operatorname{supp}(g) \subset V$. Now set $Y := \{x | h(x) \neq 0\}$, a non-empty open set and observe that $\emptyset \neq Y \subset \operatorname{supp}(h) \subset U \cap t^{-1}.V$. $\square$

The next result shows that, as in the commutative case, every minimal action is topologically transitive. A standard piece of notation will be used in the proof: if $a \in A^+$, and $\varepsilon > 0$, set $(a-\varepsilon)_+ := f(a)$ where $f : [0,\infty) \to [0,\infty)$ is the continuous function $f(t) = \max\{0, t-\varepsilon\}$.

Proposition 3.1.11. *Let A be a unital C^*-algebra. If $\alpha : \Gamma \to \mathrm{Aut}(A)$ is a minimal action, then it is topologically transitive.*

Proof. Let $x, y \in W(A)$ be non-zero, without loss of generality we may assume $x = \langle a \rangle$ and $y = \langle b \rangle$ with $a, b \in A^+$. By minimality there are group elements $t_1, \dots, t_n \in \Gamma$ with

$$\langle \alpha_{t_1}(a) \oplus \cdots \oplus \alpha_{t_n}(a) \rangle = \sum_{j=1}^{n} \langle \alpha_{t_j}(a) \rangle = \sum_{j=1}^{n} \hat{\alpha}_{t_j}(\langle a \rangle) = \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x) \geq \langle 1 \rangle.$$

There is a sequence $(v_k)_{k \geq 1}$ in $M_{n\times 1}(A)$ with $v_k^*(\alpha_{t_1}(a) \oplus \cdots \oplus \alpha_{t_n}(a))v_k \to 1$ in A as $k \to \infty$. For each k write $v_k = (v_{k,1}, \dots, v_{k,n})^T$ so that

$$\left(\sum_{j=1}^{n} v_{k,j}^* \alpha_{t_j}(a) v_{k,j} \right)_{k \geq 1} \longrightarrow 1, \quad \text{as} \quad k \to \infty.$$

With k large enough we have $\left\| 1 - \sum_{j=1}^{n} v_{k,j}^* \alpha_{t_j}(a) v_{k,j} \right\| < 1/2$. There is a $y \in A$ with

$$(1_A - 1/2)_+ = y^* \left(\sum_{j=1}^{n} v_{k,j}^* \alpha_{t_j}(a) v_{k,j} \right) y,$$

which gives $1_A = \sum_{j=1}^{n} u_j^* \alpha_{t_j}(a) u_j$ where $u_j = 2^{1/2} v_{k,j} y$. It follows that for every $j = 1, \dots, n$

$$b = \sum_{j=1}^{n} b^{1/2} u_j^* \alpha_{t_j}(a) u_j b^{1/2} \geq b^{1/2} u_j^* \alpha_{t_j}(a) u_j b^{1/2} \geq 0.$$

Choose an i such that $b^{1/2}u_i^*\alpha_{t_i}(a)u_ib^{1/2} \neq 0$ (there is one since $b \neq 0$), and set

$$c = \alpha_{t_i^{-1}}(b^{1/2}u_i^*\alpha_{t_i}(a)u_ib^{1/2}) = (\alpha_{t_i^{-1}}(u_ib^{1/2}))^*a\alpha_{t_i^{-1}}(u_ib^{1/2}).$$

Then $c \neq 0$, $c \precsim a$ and $\alpha_{t_i}(c) \precsim b$. With $z := \langle c\rangle$, we have $z \leq x$ and $\hat{\alpha}_{t_i}(z) \leq y$ so α is topologically transitive. $\square$

Recall that a C*-algebra B is *prime* if for every pair of non-trivial ideals $I, J \subset B$, $IJ = I \cap J \neq (0)$. It natural to ask what dynamical conditions give rise to prime reduced crossed products. We briefly study this issue.

A C*-system (A, Γ, α) is said to have the *intersection property* if every ideal $I \subset A \rtimes_{\lambda,\alpha} \Gamma$ has non-trivial intersection with A. If the action α is properly outer, then the intersection property follows (see lemma 3.2.15). When $A = C_0(X)$, proper outerness is equivalent to topological freeness, and it well known that if the action is topologically free, the reduced crossed product $C_0(X) \rtimes_\lambda \Gamma$ is prime if and only if the action $\Gamma \curvearrowright X$ is topologically transitive. We now can generalize this to the noncommutative setting.

Theorem 3.1.12. *Let A be a C*-algebra, Γ a countable discrete group and $\alpha : \Gamma \to \operatorname{Aut}(A)$ an action. If $A \rtimes_{\lambda,\alpha} \Gamma$ is prime then α is topologically transitive. Conversely, if (A, Γ, α) has the intersection property and α is topologically transitive then $A \rtimes_{\lambda,\alpha} \Gamma$ is prime.*

Proof. Assume α is topologically transitive and that (A, Γ, α) has the intersection property. Let I and J be non-zero ideals in $A \rtimes_{\lambda,\alpha} \Gamma$. By the intersection property there are $0 \neq x \in I \cap A$ and $0 \neq y \in J \cap A$. Set $a = x^*x \in I \cap A^+$ and $b = y^*y \in J \cap A^+$. By topological transitivity there is a $0 \neq z \in W(A)$ and $t \in \Gamma$ with $z \leq \langle a\rangle$ and $\hat{\alpha}_t(z) \leq \langle b\rangle$. Writing $z = \langle c\rangle$ for some $c \in M_n(A)^+$, we have $c \precsim a$ and $\alpha_t(c) \precsim b$.

There is a sequence $v_k \in M_{1\times n}(A)$ with $v_k^* a v_k \to c$ as $k \to \infty$. If $v_k = (v_{k,1}, \ldots, v_{k,n})$, and $c = (c_{i,j})_{i,j}$ then for every $1 \le i, j \le n$ we get $v_{k,i}^* a v_{k,j} \to c_{i,j}$ as $k \to \infty$. Note that $v_{k,i}^* a v_{k,j} \in A \cap I$ for each i, j, k, so $c_{i,j} \in A \cap I$ for every i, j. Since $A \cap I$ is a Γ-invariant ideal in A we know that $\alpha_t(c_{i,j}) \in A \cap I$ for every i, j.

Similarly there is a sequence $u_k \in M_{1\times n}(A)$ with $u_k^* b u_k \to \alpha_t(c)$ as $k \to \infty$ giving $u_{k,i}^* b u_{k,j} \to \alpha_t(c_{i,j})$ for every i, j where $u_k = (u_{k,1}, \ldots, u_{k,n})$. Since $u_{k,i}^* b u_{k,j}$ belongs to $A \cap J$ for every i, j so do the $\alpha_t(c_{i,j})$. With c non-zero, there is a $c_{i,j} \neq 0$ so that $\alpha_t(c_{i,j}) \in (A \cap I) \cap (A \cap J) \subset I \cap J$. Thus $A \rtimes_\lambda \Gamma$ is prime.

Conversely, now suppose $A \rtimes_\lambda \Gamma$ is prime. Let $x, y \in W(A)$ be nonzero. We can write $x = \langle a \rangle$ and $y = \langle b \rangle$ with $a, b \in M_n(A)^+$. Since $A \rtimes_\lambda \Gamma$ is prime, $M_n(A \rtimes_\lambda \Gamma) \cong M_n(A) \rtimes_{\lambda, \alpha^{(n)}} \Gamma$ is prime, so we can find a non-zero $c \in M_n(A)$ and $s \in \Gamma$ with

$$0 \neq b^{1/2} c u_{s^{-1}} a^{1/2} = b^{1/2} c u_{s^{-1}} a^{1/2} u_s u_{s^{-1}} = b^{1/2} c \alpha_{s^{-1}}(a^{1/2}) u_{s^{-1}}.$$

Multiplying on the right by the unitary u_s, we get $v := b^{1/2} c \alpha_{s^{-1}}(a^{1/2})$ is non-zero in $M_n(A)$. Setting $w = \alpha_s(v)$ we get $z := \langle w w^* \rangle \le \langle a \rangle = x$ since

$$w w^* = \alpha_s(v) \alpha_s(v)^* = \alpha_s(b^{1/2} c) a^{1/2} (\alpha_s(b^{1/2} c) a^{1/2})^* = \alpha_s(b^{1/2} c) a \alpha_s(b^{1/2} c)^* \precsim a.$$

On the other hand,

$$\begin{aligned} w^* w &= \alpha_s(v)^* \alpha_s(v) = \alpha_s(v^* v) = \alpha_s(\alpha_{s^{-1}}(a^{1/2}) c^* b c \alpha_{s^{-1}}(a^{1/2})) \\ &= a^{1/2} \alpha_s(c)^* \alpha_s(b) \alpha_s(c) a^{1/2} = (\alpha_s(c) a^{1/2})^* \alpha_s(b) \alpha_s(c) a^{1/2} \precsim \alpha_s(b), \end{aligned}$$

which says that $z = \langle w w^* \rangle = \langle w^* w \rangle \le \langle \alpha_s(b) \rangle = \hat{\alpha}_s(\langle b \rangle) = \hat{\alpha}_s(y)$. Therefore we have found $0 \neq z \in W(A)$, and $t := s^{-1} \in \Gamma$ with $z \le x$ and $\hat{\alpha}_t(z) \le y$ as was required. $\square$

We end with a cute result that will be needed later on.

Proposition 3.1.13. *Let (A, Γ, α) be a C^*-system. Then α is W-n-minimal if and only if α is W-n-filling.*

Proof. The n-minimal property easily follows from the n-filling property. For the converse, let $x_1, \ldots, x_n$ be non-zero in $W(A)$. Let $t_1 = e$. Since α is n-minimal, α is topologically transitive, so we can find $0 \neq z_1 \leq x_1$ and $t_2 \in \Gamma$ with $z_1 \leq t_2.x_2$. Next, again by transitivity find $0 \neq z_2 \leq z_1$ and $t_3 \in \Gamma$ with $z_2 \leq t_3.x_3$. We continue in this fashion until we find $0 \neq z_{n-1} \leq z_{n-2}$ and $t_n \in \Gamma$ with $z_{n-1} \leq t_n.x_n$. Now apply the n-minimal property to locate $s_1, \ldots, s_n$ in Γ with $\sum_{j=1}^{n} s_j.z_{n-1} \geq \langle 1 \rangle$. From these orderings we get

$$s_1 t_n.x_n \geq s_1.z_{n-1}, \quad s_2 t_{n-1}.x_{n-1} \geq s_2.z_{n-2} \geq s_2.z_{n-1}, \ldots$$
$$\ldots, s_{n-1} t_2.x_2 \geq s_{n-1}.z_1 \geq s_{n-1}.z_{n-1}, \quad s_n.x_1 \geq s_n.z_1 \geq s_n.z_{n-1}.$$

We thus obtain
$$\sum_{j=1}^{n} s_{n-j+1} t_j.x_j \geq \sum_{j=1}^{n} s_j.z_{n-1} \geq \langle 1 \rangle$$
so that α is indeed W-n-filling. □

3.2 Finiteness, Paradoxical Decompositions, and The Type Semigroup

In this section we study K-theoretic conditions, in the form of paradoxical phenomena, that characterize finite and infinite crossed products. As a brief reminder, a projection $p \in A$ is properly infinite if there are two subprojections $q, r \leq p$ with $qr = 0$ and $q \sim p \sim r$. The algebra A is properly infinite if 1_A is properly infinite. A simple algebra A is termed purely infinite if every hereditary C*-subalgebra of A contains a properly infinite projection. In the simple case, S. Zhang showed that A

is purely infinite if and only in $\mathrm{RR}(A) = 0$ and every projection in A is properly infinite [55]. It was a longstanding open question whether there existed a unital, simple, separable, and nuclear C*-algebra which was neither stably finite or purely infinite. M. Rørdam settled the issue in [43] by exhibiting a unital, simple, nuclear, and separable C*-algebra D containing a finite and infinite projection p, q. It follows that $A = qDq$ is unital, separable, nuclear, simple, and properly infinite, but not purely infinite. It is natural to ask if there is a smaller class of algebras for which such a dichotomy exists. Theorem 3.2.22 below is a result in this direction.

3.2.1 Paradoxical Decompositions

We first construct infinite algebras arising from crossed products by generalizing the notion of a local boundary action to the noncommutative setting. A continuous action $\Gamma \curvearrowright X$ of a discrete group on a locally compact space is called a *local boundary action* if for every non-empty open set $U \subset X$ there is an open set $V \subset U$ and $t \in \Gamma$ with $t.\overline{V} \subsetneq V$. Laca and Spielberg showed in [32] that such actions yield infinite projections in the reduced crossed product $C_0(X) \rtimes_\lambda \Gamma$. Sierakowski remarked that the condition $t.\overline{V} \subsetneq V$ for *some* non-empty open set V and group element $t \in \Gamma$ is equivalent to the existence of open sets $U_1, U_2 \subset X$ and elements $t_1, t_2 \in \Gamma$ such that $U_1 \cup U_2 = X$, $t_1.U_1 \cap t_2.U_2 = \emptyset$, and $t_1.U_1 \cup t_2.U_2 \neq X$. He generalized this by defining *paradoxical* actions. A transformation group (X, Γ) is n-paradoxical if there exist open subsets $U_1, \dots, U_n \subset X$ and elements $t_1, \dots, t_n \in \Gamma$ such that

$$\bigcup_{j=1}^{n} U_j = X, \qquad \bigsqcup_{j=1}^{n} t_j.U_j \subsetneq X.$$

He then showed that the algebra $C(X) \rtimes_\lambda \Gamma$ is infinite provided that X is compact and the action $\Gamma \curvearrowright X$ is n-paradoxical for some n. We do the same here in the noncommutative setting.

Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a C*-dynamical system where Γ is a discrete group. Once again, we look at the induced actions $\hat{\alpha} : \Gamma \curvearrowright K_0(A)^+$ and $\hat{\alpha} : \Gamma \curvearrowright W(A)$ given by $t.x = \hat{\alpha}_t(x)$ for $t \in \Gamma$ and $x \in K_0(A)^+$ or $W(A)$.

Proposition 3.2.1. *Let A be a stably finite C*-algebra with cancellation and such that $K_0(A)^+$ has Riesz refinement. Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a K_0-paradoxical action in the sense that there exist $x_1, \dots, x_n \in K_0(A)^+$ and group elements $t_1, \dots, t_n \in \Gamma$ with*

$$\sum_{j=1}^{n} x_j \geq [1_A]_0, \quad \text{and} \quad \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) < [1_A]_0.$$

Then $A \rtimes_\lambda \Gamma$ is infinite.

Proof. Denote by $\iota : A \to A \rtimes_\lambda \Gamma$ the canonical embedding. Given that $\sum_{j=1}^{n} x_j \geq [1_A]_0$, there is an $r \in \mathcal{P}_\infty(A)$ with $\sum_{j=1}^{n} x_j = [1_A]_0 + [r]_0$. With the refinement property one can find elements $\{y_j\}_j^n, \{z_j\}_j^n \subset K_0(A)^+$ with

$$x_j = y_j + z_j, \quad \sum_{j=1}^{n} y_j = [1_A]_0, \quad \sum_{j=1}^{n} z_j = [r]_0.$$

Then $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(y_j) \leq \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) < [1_A]_0$. Cancellation implies there are mutually orthogonal projections $p_1, \dots, p_n$ in A with $[p_j]_0 = y_j$, as well as mutually orthogonal projections $q_1, \dots, q_n$ in A with $[q_j]_0 = \hat{\alpha}_{t_j}(x_j) = \hat{\alpha}_{t_j}([p_j]_0) = [\alpha_{t_j}(p_j)]_0$. It also implies that $q_j \sim \alpha_{t_j}(p_j)$ as projections in A for each j, whence $\iota(q_j) \sim \iota(\alpha_{t_j}(p_j)) \sim \iota(p_j)$ as projections in $A \rtimes_\lambda \Gamma$. Setting $p = \sum_j p_j$, and $q = \sum_j q_j$ we obtain

$$\iota(q) = \sum_{j=1}^{n} \iota(q_j) \sim \sum_{j=1}^{n} \iota(p_j) = \iota(p).$$

On the other hand $[p]_0 = \left[\sum_j p_j\right]_0 = \sum_j [p_j]_0 = \sum_j y_j = [1_A]_0$. Cancellation once more implies $p \sim 1_A$ and therefore $\iota(p) \sim \iota(1_A) = 1_{A \rtimes_\lambda \Gamma}$. Thus we have $\iota(q) \sim 1_{A \rtimes_\lambda \Gamma}$.

All is needed to show is that $\iota(q) \neq 1_{A \rtimes_\lambda \Gamma}$. To this end we observe that

$$[q]_0 = \Big[\sum_{j=1}^{n} q_j\Big]_0 = \sum_{j=1}^{n} [q_j]_0 = \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) < [1]_0,$$

so that $[1_A]_0 - [q]_0 = [1_A - q]_0 \neq 0$, which implies $1 - q \neq 0$ by stable finiteness. Therefore $\iota(q) \neq \iota(1_A) = 1_{A \rtimes_\lambda \Gamma}$ and $A \rtimes_\lambda \Gamma$ is infinite as claimed. $\square$

A similar result holds with less restrictions on the underlying algebra A but with a slight strengthening on the dynamics. For this result we will make the following convention: for $x, y \in W(A)$ we shall write $x < y$ to mean $x + z \leq y$ for some non-zero $z \in W(A)$.

Proposition 3.2.2. *Let A be a unital C^*-algebra and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action which is W-paradoxical in the sense that there exist $x_1, \ldots, x_n \in W(A)$ and group elements $t_1, \ldots, t_n \in \Gamma$ with $\sum_{j=1}^{n} x_j \geq \langle 1_A \rangle$ and $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) < \langle 1_A \rangle$. Then $A \rtimes_\lambda \Gamma$ is infinite.*

Proof. Again let $\iota : A \to A \rtimes_\lambda \Gamma$ denote the canonical embedding and for $t \in \Gamma$ write u_t for the canonical unitary in $A \rtimes_\lambda \Gamma$ that implements the action $\alpha_t : A \to A$, so that $\iota(\alpha_t(a)) = u_t \iota(a) u_t^* \approx \iota(a)$ for every $a \in A$ and $t \in \Gamma$. If $a \in M_n(A)^+$ then by amplification we have $\iota^{(n)}(\alpha_t^{(n)}(a)) = (u_t \otimes 1_A)\iota^{(n)}(a)(u_t \otimes 1_n)^* \approx \iota^{(n)}(a)$ for every $t \in \Gamma$. For economy we will omit denoting the amplification when the context is understood.

For each $j = 1, \ldots, n$ set $x_j = \langle a_j \rangle$ for $a_j \in M_\infty(A)^+$. Then we have

$$\langle 1_A \rangle \leq \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \langle a_j \rangle = \langle a_1 \oplus \ldots \oplus a_n \rangle,$$

which implies $1_A \precsim \oplus_{j=1}^n a_j$ in $M_\infty(A)^+$. Applying ι we get $1_{A\rtimes_\lambda\Gamma} \precsim \oplus_{j=1}^n \iota(a_j) \approx \oplus_{j=1}^n \iota(\alpha_{t_j}(a_j))$ in $M_\infty(A \rtimes_\lambda \Gamma)^+$.

By our convention we have

$$\langle \alpha_{t_1}(a_1) \oplus \ldots \oplus \alpha_{t_n}(a_n) \oplus b\rangle = \sum_{j=1}^n \hat{\alpha}_{t_j}(x_j) + \langle b\rangle \leq \langle 1_A\rangle$$

for some non-zero $b \in M_\infty(A)^+$. Thus $\alpha_{t_1}(a_1)\oplus\ldots\oplus\alpha_{t_n}(a_n)\oplus b \precsim 1_A$ and $\iota(\alpha_{t_1}(a_1))\oplus \ldots \oplus \iota(\alpha_{t_n}(a_n)) \oplus \iota(b) \precsim 1_{A\rtimes_\lambda\Gamma}$. Together we get

$$1_{A\rtimes_\lambda\Gamma} \oplus \iota(b) \precsim \iota(\alpha_{t_1}(a_1)) \oplus \ldots \oplus \iota(\alpha_{t_n}(a_n)) \oplus \iota(b) \precsim 1_{A\rtimes_\lambda\Gamma}.$$

Since $1_{A\rtimes_\lambda\Gamma} \oplus \iota(b) \precsim 1_{A\rtimes_\lambda\Gamma}$ and$\iota(b) \neq 0$, work in [30] implies that $A \rtimes_\lambda \Gamma$ is infinite as claimed. □

We make the brief remark that an action $\Gamma \curvearrowright A$ is K_0-paradoxical in the above sense with $n = 2$ if and only if there is a non-zero $x \in \Sigma(A)$ (the scale of A) and $t \in \Gamma$ with $\hat{\alpha}_t(x) < x$.

Perhaps what has been called *paradoxical* is misleading because, in a sense, paradoxicality implies the idea of duplication of sets. Gleaning from the ideas explored in [28], we define a notion of paradoxical decomposition with covering multiplicity in the noncommutative setting.

Definition 3.2.3. Let A be a C*-algebra, Γ a discrete group and $\alpha : \Gamma \to \mathrm{Aut}(A)$ an action with its induced action $\hat{\alpha}$. Let $0 \neq x \in K_0(A)^+$ and $k > l > 0$ be positive integers. We say x is *(Γ, k, l)-paradoxical* if there are $x_1, \ldots, x_n$ in $K_0(A)^+$

and $t_1, \dots, t_n$ in Γ such that

$$\sum_{j=1}^{n} x_j \geq kx, \qquad \text{and} \qquad \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \leq lx.$$

If an element $x \in K_0(A)^+$ fails to be (Γ, k, l)-paradoxical for all integers $k > l > 0$ we call x *completely non-paradoxical*. The action α will be called *completely non-paradoxical* if every member of $K_0(A)^+$ is completely non-paradoxical.

The notion of a quasidiagonal action was first introduced in [28] and further studied in the previous chapter from a K-theoretic viewpoint. We observed that MF (or equivalently QD) actions of discrete groups Γ on AF algebras admit, in a local sense, Γ-invariant traces on $K_0(A)$, so it should come to no surprise that these actions do not allow paradoxical decompositions at the K-theoretic level. The next proposition illustrates this principle and provides us with our first class of examples of completely non-paradoxical actions.

Proposition 3.2.4. *If* $\alpha : \Gamma \to \mathrm{Aut}(A)$ *is an MF action of a discrete group* Γ *on a unital AF algebra, then* α *is completely non-paradoxical.*

Proof. Suppose $0 \neq x \in K_0(A)^+$ is (Γ, k, l)-paradoxical for some positive integers $k > l > 0$, so that there are $x_1, \dots, x_n$ in $K_0(A)^+$ and $t_1, \dots, t_n$ in Γ such that

$$y := \sum_{j=1}^{n} x_j \geq kx \qquad \text{and} \qquad z := \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \leq lx.$$

Consider the finite sets $F = \{t_1, \dots, t_n\} \subset \Gamma$, and $S = \{y - kx, lx - z, x_1, \dots, x_n, x\} \subset K_0(A)^+$. Since α is quasidiagonal, Proposition 2.2.8 guarantees existence of a subgroup $H \leq K_0(A)$ which contains all the F-iterates of S, and a group homomorphism $\beta : H \to \mathbb{Z}$ with $\beta(\hat{\alpha}_t(g)) = \beta(g)$ for each $t \in F$ and $g \in S$. Also, $\beta(g) > 0$ for

$0 < g \in S$. Clearly y, z, kx, lx all belong to the subgroup H, and since $\beta(y - kx) \geq 0$, we have $k\beta(x) = \beta(kx) \leq \beta(y)$. Similarly, $\beta(z) \leq l\beta(x)$. Now using the Γ-invariance of β,

$$\begin{aligned} k\beta(x) \leq \beta(y) = \beta\left(\sum_{j=1}^{n} x_j\right) = \sum_{j=1}^{n} \beta(x_j) &= \sum_{j=1}^{n} \beta(\hat{\alpha}_{t_j}(x_j)) \\ &= \beta\left(\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j)\right) = \beta(z) \leq l\beta(x). \end{aligned}$$

This is absurd since $\beta(x) > 0$ and $l < k$. Thus no such non-zero x exists. $\square$

It was shown by Kerr and Nowak [28] that quasidiagonal actions by groups whose reduced group algebras are MF give rise to MF crossed products, which are always stably finite. Indeed, it is the finiteness of the crossed product that is an obstruction to a positive element being paradoxical.

Proposition 3.2.5. *Consider a C^*-dynamical system (A, Γ, α) with stably finite reduced crossed product $A \rtimes_\lambda \Gamma$. Then the induced $\hat{\alpha} : \Gamma \curvearrowright K_0(A)^+$ is completely non-paradoxical.*

Proof. Suppose on the contrary that $0 \neq [p]_0 := x \in K_0(A)^+$ is (Γ, k, l) paradoxical for some integers $k > l > 0$ where $p \in \mathcal{P}_m(A)$. We then have elements $x_1, \ldots, x_n$ in $K_0(A)^+$ and $t_1, \ldots, t_n \in \Gamma$ with

$$\sum_{j=1}^{n} x_j \geq kx \qquad \text{and} \qquad \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \leq lx.$$

If $\iota : A \hookrightarrow A \rtimes_\lambda \Gamma$, $\iota : a \mapsto au_e$, denotes the canonical embedding, apply $\hat{\iota} : K_0(A)^+ \to K_0(A \rtimes_\lambda \Gamma)^+$ which is order preserving to obtain

$$k\hat{\iota}(x) = \hat{\iota}(kx) \leq \hat{\iota}\big(\sum_{j=1}^{n} x_j\big) = \sum_{j=1}^{n} \hat{\iota}(x_j) = \sum_{j=1}^{n} \hat{\iota}\hat{\alpha}_{t_j}(x_j) = \hat{\iota}\big(\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j)\big) \leq \hat{\iota}(lx) = l\hat{\iota}(x).$$

Here we used the fact that for a projection q in A and $s \in \Gamma$ we have

$$\begin{aligned}\hat{\iota}([q]_{K_0(A)}) &= [\iota(q)]_{K_0(A\rtimes_\lambda\Gamma)} = [u_s q u_s^*]_{K_0(A\rtimes_\lambda\Gamma)} = [\alpha_s(q)]_{K_0(A\rtimes_\lambda\Gamma)} \\ &= \hat{\iota}[\alpha_s(q)]_{K_0(A)} = \hat{\iota}\hat{\alpha}_s([q]_{K_0(A)}),\end{aligned}$$

so that $\hat{\iota} = \hat{\iota}\hat{\alpha}_s$ agree as maps $K_0(A)^+ \to K_0(A \rtimes_\lambda \Gamma)^+$.

The fact that $A \rtimes_\lambda \Gamma$ is stably finite now implies that $\hat{\iota}(x) = 0$,which means that $\iota(p) = 0$, so $p = 0$, a contradiction. □

3.2.2 A Noncommutative Type Semigroup

We wish to establish a converse to Proposition 3.2.5. For this we shall need more machinery. Analogous to the type semigroup of a general group action (see [53]), we associate to each suitable C*-system (A, Γ, α) a preordered abelian monoid $S(A, \Gamma, \alpha)$ which correctly reflects the above notion of paradoxicality in $K_0(A)$, and then resort to an extension result (Theorem 3.2.11 below) in the spirit of Tarski's theorem tying the existence of states on $S(A, \Gamma, \alpha)$ to non-paradoxicality. We embark on the details.

Let us first recall the notion of equidecomposability for group actions and the construction of the type semigroup. Suppose a group Γ acts on a set X, and let $\mathcal{C}$ be a Γ-invariant subalgebra of the power set $\mathcal{P}(X)$. Orthogonality is then built in as we enlarge the action as follows. Let $Y = X \times \mathbb{N}_0$, and $G = \Gamma \times \mathrm{Perm}(\mathbb{N}_0)$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We then have a canonical action $G \curvearrowright Y$ given by

$$(t, \sigma).(x, n) = (t.x, \sigma(n)).$$

For a set $E \subset Y$, and $j \in \mathbb{N}_0$ the *jth level* of E is the set $E_j = \{x \in X : (x, j) \in E\}$. We say that E is *bounded* if only finitely many levels E_j are non-empty. Now consider the algebra of G-invariant subsets

$$S(X,\mathcal{C})=\{E\subset Y : E \quad \text{is bounded and} \quad E_j\in\mathcal{C},\ \forall j\in\mathbb{N}_0\}.$$

Subsets $E,F\in S(X,\mathcal{C})$ are said to be *G-equidecomposable*, and we write $E\sim_G F$, if there are $E_1,\dots,E_n\in S(X,\mathcal{C})$, and $g_1,\dots,g_n\in G$ such that:

$$E=\bigsqcup_{j=1}^{n}E_j,\quad \text{and} \quad F=\bigsqcup_{j=1}^{n}g_j.E_j.$$

The notation $\sqcup$ is used to emphasize the fact that the partitioning sets are disjoint. Reflexivity and symmetry of the relation $\sim_G$ are straightforward, and transitivity follows from taking refined partitions. We quotient out by the equivalence relation $\sim_G$, setting

$$S(X,\Gamma,\mathcal{C}):=S(X,\mathcal{C})/\sim_G,$$

and write $[E]$ for the equivalence class of $E\in S(X,\mathcal{C})$. Addition is then defined on classes via

$$\left[\bigcup_{j=1}^{n}E_j\times\{j\}\right]+\left[\bigcup_{i=1}^{m}F_i\times\{i\}\right]=\left[\bigcup_{j=1}^{n}E_j\times\{j\}\cup\bigcup_{i=1}^{m}F_i\times\{n+j\}\right].$$

A little work shows that addition is well defined and $[\emptyset]$ is a neutral element. Endowed with the algebraic ordering, $S(X,\Gamma,\mathcal{C})$ has the structure of a preordered abelian monoid, often referred to as the type semigroup [53].

We aim to construct a similar monoid for noncommutative C*-systems (A,Γ,α), at least in the presence of sufficiently many projections. The philosophy is that elements of the positive cone $K_0(A)^+$ would represent our "subsets" as it were, and the idea of refined partitions is reflected by suitable refinement properties displayed in the additive structure of $K_0(A)^+$. If we are to translate the notion of equidecomposability to the K_0-setting, we shall require that A be an algebra for which the monoid $K_0(A)^+$ has the the Riesz refinement property. This discussion thus motivates the following definition.

Definition 3.2.6. Let A be a C^*-algebra, Γ a discrete group, and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ an action. We define a relation on $K_0(A)^+$ as follows:

$$x \sim_\alpha y \quad (x, y \in K_0(A)^+)$$

$$\Longleftrightarrow$$

$$\exists \, \{u_j\}_{j=1}^k \subset K_0(A)^+, \; \{t_j\}_{j=1}^k \subset \Gamma, \quad \text{such that} \quad \sum_{j=1}^k u_j = x \quad \text{and} \quad \sum_{j=1}^k \hat{\alpha}_{t_j}(u_j) = y.$$

Lemma 3.2.7. *If A is a stably finite C^*-algebra such that $K_0(A)^+$ has the Riesz refinement property, then $\sim_\alpha$ as defined above is an equivalence relation.*

Proof. Let $x, y \in K_0(A)^+$. Clearly $x \sim_\alpha x$, simply take $u_1 = x$ and $t_1 = e$. If $x \sim_\alpha y$, via the decomposition $x = \sum_{j=1}^k u_j$ and $y = \sum_{j=1}^k \hat{\alpha}_{t_j}(u_j)$, set $v_j = \hat{\alpha}_{t_j}(u_j)$ and $s_j = t_j^{-1}$ for $j = 1, \ldots k$. It clearly follows that

$$\sum_{j=1}^k v_j = y \quad \text{and} \quad \sum_{j=1}^k \hat{\alpha}_{s_j}(v_j) = \sum_{j=1}^k \hat{\alpha}_{t_j^{-1}}(\hat{\alpha}_{t_j}(u_j)) = \sum_{j=1}^k u_j = x$$

whence $y \sim_\alpha x$. Transitivity is a little harder, and here is where the fact that $K_0(A)^+$ has the Riesz refinement property will surface. To that end, suppose $x \sim_\alpha y \sim_\alpha z$ via

$$x = \sum_{j=1}^k u_j, \quad y = \sum_{j=1}^k \hat{\alpha}_{t_j}(u_j) \quad \text{and} \quad y = \sum_{j=1}^l v_j, \quad z = \sum_{j=1}^l \hat{\alpha}_{s_j}(v_j).$$

Since $\sum_{j=1}^k \hat{\alpha}_{t_j}(u_j) = \sum_{j=1}^l v_j$ and $K_0(A)$ has the interpolation properties, there are elements $\{w_{ij} : 1 \leq j \leq l, 1 \leq i \leq k\} \subset K_0(A)^+$ such that

$$\sum_{j=1}^{l} w_{ij} = \hat{\alpha}_{t_i}(u_i) \quad \text{and} \quad \sum_{i=1}^{k} w_{ij} = v_j.$$

We then compute

$$\sum_{i,j} \hat{\alpha}_{s_j t_i}(\hat{\alpha}_{t_i^{-1}}(w_{ij})) = \sum_{i,j} \hat{\alpha}_{s_j}(w_{ij}) = \sum_{j} \hat{\alpha}_{s_j}\big(\sum_{i} w_{ij}\big) = \sum_{j} \hat{\alpha}_{s_j}(v_j) = z,$$

while

$$\sum_{i,j} \hat{\alpha}_{t_i^{-1}}(w_{ij}) = \sum_{i} \hat{\alpha}_{t_i^{-1}}\big(\sum_{j} w_{ij}\big) = \sum_{i} \hat{\alpha}_{t_i^{-1}}(\hat{\alpha}_{t_i}(u_i)) = \sum_{i} u_i = x.$$

which gives the desired decomposition for $x \sim_\alpha z$. $\square$

We can now make the following definition.

Definition 3.2.8. Let A be a C*-algebra such that $K_0(A)^+$ has the Riesz refinement property. Let $\Gamma \to \mathrm{Aut}(A)$ be an action. We set $S(A, \Gamma, \alpha) := K_0(A)^+/ \sim_\alpha$, and write $[x]_\alpha$ for the equivalence class with representative $x \in K_0(A)^+$.

For a general group action $G \curvearrowright X$ on an arbitrary set, it is not difficult to see that we may define addition on equidecomposability classes. Indeed if $E, F, H, K \subset X$ with $E \cap H = \emptyset$, $F \cap K = \emptyset$, $E \sim F$ and $H \sim K$ then it is routine to verify that $(E \sqcup H) \sim (F \sqcup K)$. This gives an idea for a well defined additive structure on $S(A, \Gamma, \alpha)$. Define addition on classes simply by $[x]_\alpha + [y]_\alpha := [x + y]_\alpha$ for x, y in $K_0(A)^+$. It is routine to check that this operation is well defined; indeed if $z \sim_\alpha x$

via $x = \sum_{j=1}^{k} u_j$ and $z = \sum_{j=1}^{k} \hat{\alpha}_{t_j}(u_j)$,then

$$[z]_\alpha + [y]_\alpha = [z+y]_\alpha = \left[\sum_{j=1}^{k} \hat{\alpha}_{t_j}(u_j) + y \right]_\alpha = \left[\sum_{j=1}^{k} u_j + y \right]_\alpha = [x+y]_\alpha = [x]_\alpha + [y]_\alpha.$$

We make a few elementary observations concerning $S(A,\Gamma,\alpha)$ when A is stably finite. Firstly, $S(A,\Gamma,\alpha)$ is not just a semigroup but an abelian monoid as $[0]_\alpha$ is clearly the neutral additive element. Impose the algebraic ordering on $S(A,\Gamma,\alpha)$, that is, set $[x]_\alpha \leq [y]_\alpha$ if there is a $z \in K_0(A)^+$ with $[x]_\alpha + [z]_\alpha = [y]_\alpha$. This gives $S(A,\Gamma,\alpha)$ the structure of an abelian preordered monoid. Notice at once that if $x, y \in K_0(A)^+$ with $x \leq y$ (in the ordering of $K_0(A)$) then $[x]_\alpha \leq [y]_\alpha$ in $S(A,\Gamma,\alpha)$. To see this, $x \leq y$ implies $y - x := z \in K_0(A)^+$, so $[y]_\alpha = [x+z]_\alpha = [x]_\alpha + [z]_\alpha$ which gives $[x]_\alpha \leq [y]_\alpha$. Next, we observe that if $[x]_\alpha = [0]_\alpha$, for some x in $K_0(A)^+$, then in fact $x = 0$. Indeed, say $x = \sum_i u_i$, and $\sum_i \hat{\alpha}_{t_i}(u_i) = 0$ for some elements $t_i \in \Gamma$ and $u_i \in K_0(A)^+$, then for each i, $\hat{\alpha}_{t_i}(u_i) = 0$ and so $u_i = 0$ which gives $x = 0$. Here we used the important fact that for stably finite algebras A, $K_0(A)^+ \cap (-K_0(A)^+) = (0)$. All together, there is an order preserving, faithful, monoid homomorphism

$$\rho : K_0(A)^+ \to S(A,\Gamma,\alpha) \quad \text{given by} \quad \rho(g) = [g]_\alpha.$$

This next fact shows that we have in fact constructed a noncommutative analogue of the type semigroup construction studied in [53].

Proposition 3.2.9. *Let X be the Cantor set, Γ a discrete group, and $\Gamma \curvearrowright X$ a continuous action with corresponding action $\alpha : \Gamma \to \mathrm{Aut}(C(X))$. Write $\mathcal{C}$ for the Γ-invariant algebra of all clopen subsets of X. Then the type semigroup $S(X,\Gamma,\mathcal{C})$ is isomorphic to $S(C(X),\Gamma,\alpha)$ constructed above.*

Proof. Let $f \in K_0(C(X))^+ = C(X;\mathbb{Z})^+$, then we can write $f = \sum_{j=1}^{n} \mathbb{1}_{E_j}$ where the

E_j are clopen subsets of X. Note that such a representation is not unique.

Claim. Suppose $f = \sum_{j=1}^{n} \mathbb{1}_{E_j} = \sum_{j=1}^{m} \mathbb{1}_{F_j}$, then

$$\bigsqcup_{j=1}^{n} E_j \times \{j\} := E \sim_\Gamma F := \bigsqcup_{j=1}^{m} F_j \times \{j\},$$

so that $[E] = [F]$ in the type semigroup $S(X, \Gamma, \mathcal{C})$.

It is clear that $\cup_{j=1}^{n} E_j = \cup_{j=1}^{m} F_j$. By choosing a common clopen refinement, we may assume that there are disjoint clopen sets $H_1, \ldots, H_r$, where $r \geq n, m$, such that each E_j and each F_j is a union of distinct H_i. For each $i = 1, \ldots, r$ set the multiplicities of the H_i as

$$n_i := \big|\{j : H_i \subset E_j\}\big| = \big|\{j : H_i \subset F_j\}\big|.$$

In this case we have $f = \sum_{i=1}^{r} n_i \mathbb{1}_{H_i}$. For each pair (i, j) set

$$\Delta_{i,j} = \begin{cases} H_i, & \text{if } H_i \subset E_j \\ \emptyset & \text{if } H_i \cap E_j = \emptyset \end{cases}$$

With a j fixed we run through all the H_i and get $\bigsqcup_{i=1}^{r} \Delta_{i,j} \times \{j\} = E_j \times \{j\}$. Then

$$E = \bigsqcup_{j=1}^{n} E_j \times \{j\} = \bigsqcup_{j=1}^{n} \bigsqcup_{i=1}^{r} \Delta_{i,j} \times \{j\} = \bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{n} \Delta_{i,j} \times \{j\} \sim \bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{n_i} H_i \times \{j\} := H.$$

By a similar argument $F \sim H$, and transitivity gives $E \sim F$ and the Claim is thus proved.

We now define a map $\psi : K_0(C(X))^+ \to S(X, \Gamma, \mathcal{C})$ by

$$\psi(f) = \left[\bigsqcup_{j=1}^{n} E_j \times \{j\} \right]$$

where f has representation $f = \sum_{j=1}^{n} \mathbb{1}_{E_j}$ with $E_j \subset X$ clopen. Thanks to the Claim, this map is well defined as any representation of f will do. Also, it is routine to check that ψ is additive and onto. Moreover, ψ is invariant under the equivalence $\sim_\alpha$. To see this, suppose $f, g \in K_0(C(X))^+$ and $f \sim_\alpha g$. By definition and by writing members of $K_0(C(X))^+$ as sums of indicator functions on clopen sets we can find clopen sets $E_1, \ldots, E_n \in \mathcal{C}$ and group elements $t_1, \ldots, t_n \in \Gamma$ with

$$f = \sum_{j=1}^{n} \mathbb{1}_{E_j}, \quad \text{and} \quad g = \sum_{j=1}^{n} \mathbb{1}_{t_j.E_j}.$$

Since $\bigsqcup_{j=1}^{n} E_j \times \{j\} \sim \bigsqcup_{j=1}^{n} t_j.E_j \times \{j\}$ we get that $\psi(f) = \psi(g)$. The map ψ thus descends to a surjective monoid homomorphism $\overline{\psi} : S(C(X), \Gamma, \alpha) \to S(X, \Gamma, \mathcal{C})$ with $\overline{\psi}([f]_\alpha) = \psi(f)$. To establish injectivity we construct a left inverse $\varphi : S(X, \Gamma, \mathcal{C}) \to S(C(X), \Gamma, \alpha)$ as follows. Set

$$\varphi\left(\left[\bigsqcup_{j=1}^{n} E_j \times \{j\} \right] \right) = \left[\sum_{j=1}^{n} \mathbb{1}_{E_j} \right]_\alpha.$$

To show that φ is well defined, suppose $E = \bigsqcup_{j=1}^{n} E_j \times \{j\} \sim F = \bigsqcup_{j=1}^{m} F_j \times \{j\}$, then there exist $l \in \mathbb{N}$, $C_k \in \mathcal{C}$, $t_k \in \Gamma$ and natural numbers n_k, m_k for $k = 1, \ldots, l$, such that

$$E = \bigsqcup_{k=1}^{l} C_k \times \{n_k\}, \qquad F = \bigsqcup_{k=1}^{l} t_k.C_k \times \{m_k\}.$$

For each fixed j, we see that $\bigsqcup_{\{k:\ n_k=j\}} C_k = E_j$, so $\sum_{\{k:\ n_k=j\}} \mathbb{1}_{C_k} = \mathbb{1}_{E_j}$. Therefore

$$\sum_{j=1}^{n} \mathbb{1}_{E_j} = \sum_{j=1}^{n} \sum_{\{k:\ n_k=j\}} \mathbb{1}_{C_k} = \sum_{k=1}^{l} \mathbb{1}_{C_k} \sim_\alpha \sum_{k=1}^{l} \mathbb{1}_{t_k.C_k} = \sum_{j=1}^{n} \mathbb{1}_{F_j}.$$

where the last equality follows from same reasoning. It follows that $\varphi([E]) = \varphi([F])$. Also φ is clearly additive and onto. For an element $[f]_\alpha \in S(C(X), \Gamma, \alpha)$, where f has representation $f = \sum_{j=1}^{n} \mathbb{1}_{E_j}$, we see that

$$\varphi \circ \overline{\psi}([f]_\alpha) = \varphi \circ \psi(f) = \varphi\left(\left[\bigsqcup_{j=1}^{n} E_j \times \{j\}\right]\right) = \left[\sum_{j=1}^{n} \mathbb{1}_{E_j}\right]_\alpha = [f]_\alpha.$$

We conclude that $\overline{\psi}$ is a monoid isomorphism. Since both monoids are preordered with the algebraic ordering $\overline{\psi}$ is actually an isomorphism of preordered monoids. $\square$

Next we look at how (Γ, k, l)-paradoxically is reflected in our monoid $S(A, \Gamma, \alpha)$.

Lemma 3.2.10. *Let A be a stably finite C^*-algebra such that $K_0(A)^+$ has Riesz refinement, and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action. Then an element $0 \neq x \in K_0(A)^+$ is (Γ, k, l)-paradoxical if and only if $k[x] \leq l[x]$ in $S(A, \Gamma, \alpha)$.*

Proof. Suppose $0 \neq x \in K_0(A)^+$ is (Γ, k, l)-paradoxical. Then $kx \leq \sum_{j=1}^{n} x_j$ and $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \leq lx$ for some x_j in $K_0(A)^+$ and t_j in Γ. Then from our above remarks:

$$k[x]_\alpha = [kx]_\alpha \leq \left[\sum_{j=1}^{n} x_j\right]_\alpha = \left[\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j)\right]_\alpha \leq [lx]_\alpha = l[x]_\alpha.$$

Now assume $k[x]_\alpha \leq l[x]_\alpha$ for integers $k > l > 0$. Then for some z in $K_0(A)^+$ we have

$$[kx + z]_\alpha = [kx]_\alpha + [z]_= k[x]_\alpha + [z]_\alpha = l[x]_\alpha = [lx]_\alpha$$

. By definition there are elements $x_1, \dots, x_n$ in $K_0(A)^+$ and $t_1, \dots, t_n \in \Gamma$ with

$$kx \leq kx + z = \sum_{j=1}^{l} x_j \quad \text{and} \quad \sum_{j=1}^{l} \hat{\alpha}_{t_j}(x_j) = lx,$$

which witnesses the (Γ, k, l)-paradoxicality of x. The proof is complete. $\square$

Before going any further let us recall some terminology. Let $(W, \leq)$ be a preordered abelian monoid. For positive integers $k > l > 0$, we say that an element $\theta \in W$ is (k,l)-*paradoxical* provided that $k\theta \leq l\theta$. If θ fails to be paradoxical for all pairs of integers $k > l > 0$, call θ *completely non-paradoxical*. Note that θ is completely non-paradoxical if and only if $(n+1)\theta \not\leq n\theta$ for all $n \in \mathbb{N}$. The above lemma basically states that in its setting, an element $x \in K_0(A)^+$ is completely non-paradoxical with respect to the action $\hat{\alpha}$ exactly when $[x]_\alpha$ is completely non-paradoxical in the preordered abelian monoid $S(A, \Gamma, \alpha)$. An element θ in W is said to *properly infinite* if $2\theta \leq \theta$, that is, if it is $(2,1)$-paradoxical. If every member of W is properly infinite then W is said to be *purely infinite*. A *state* on W is a map $\nu : W \to [0, \infty]$ which is additive, respects the preordering $\leq$, and satisfies $\nu(0) = 0$. If a state β assumes a value other than 0 or ∞, β it said to be *non-trivial*. The monoid W is said to be *almost unperforated* if, whenever $\theta, \eta \in W$, and $n, m \in \mathbb{N}$ are such that $n\theta \leq m\eta$ and $n > m$, then $\theta \leq \eta$.

The following result is a main ingredient in the proof of what is known as Tarski's theorem. It is a Hahn-Banach type extension result and is essential in establishing a converse to Proposition 3.2.4. A proof can be found in [53].

Theorem 3.2.11. *Let $(W, +)$ be an abelian monoid equipped with the algebraic ordering, and let θ be an element of W. Then the following are equivalent:*

1. *$(n+1)\theta \not\leq n\theta$ for all $n \in \mathbb{N}$, that is θ is completely non-paradoxical.*

2. *There is a non-trivial state $\nu : W \to [0, \infty]$ with $\nu(\theta) = 1$.*

We mean to apply Theorem 3.2.11 to our preordered monoid $S(A, \Gamma, \alpha)$. Note that such a ν, which arises in the landscape of complete non-paradoxicality will not in general be finite on all of $S(A, \Gamma, \alpha)$. One needs the right condition on the action α, or more precisely, $\hat{\alpha}$, to guarantee finiteness everywhere. Suppose we considered $\theta = [u]_\alpha$ as in Theorem 3.2.11, where $u = [1]_0$ is the order unit in $K_0(A)$. If we compose the state ν with the the above $\rho : K_0(A)^+ \to S(A, \Gamma, \alpha)$, this would give us, in a sense, an invariant 'state' at the K-theoretic level, but perhaps not finitely valued everywhere, but with a finite value at $[1]_0$. To ensure finiteness at every $x \in K_0(A)^+$ we would require that finitely many Γ-iterates of x lie above $[1]_0$. This is exactly the notion of K-theoretic minimality we looked at in Section 3.1.

Proposition 3.2.12. *Let A be a stably finite unital C^*-algebra for which $K_0(A)^+$ has Riesz refinement ($\mathrm{sr}(A) = 1$ and $\mathrm{RR}(A) = 0$ for example). Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action on A. Consider the following properties.*

1. *For every $0 \neq g \in K_0(A)^+$, there is a faithful Γ-invariant positive group homomorphism $\beta : K_0(A) \to \mathbb{R}$ with $\beta(g) = 1$, (Γ-invariant in the sense that $\beta \circ \hat{\alpha} = \beta$ on $K_0(A)$).*

2. *There is a faithful Γ-invariant state β on $(K_0(A), K_0(A)^+, [1]_0)$.*

3. *α is completely non-paradoxical.*

Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If the action α is minimal, then (3) $\Rightarrow$ (1) whence all the conditions are equivalent.

Proof. (1) $\Rightarrow$ (2): Simply take $g = [1]_0$.

(2) $\Rightarrow$ (3): Assume that $x \in K_0(A)^+$ is (Γ, k, l)-paradoxical for some integers $k > l > 0$ with paradoxical decomposition $\sum_j^n x_j \geq kx$ and $\sum_j^n \hat{\alpha}_{t_j}(x_j) \leq lx$ for certain $x_j \in K_0(A)^+$ and $t_j \in \Gamma$. Apply the $\hat{\alpha}$-invariant state β and get

$$\begin{aligned} k\beta(x) = \beta(kx) \leq \beta\big(\sum_j^n x_j\big) = \sum_j^n \beta(x_j) = \sum_j^n \beta(\hat{\alpha}_{t_j}(x_j)) = \beta\big(\sum_j^n \hat{\alpha}_{t_j}(x_j)\big) \\ \leq \beta(lx) = l\beta(x). \end{aligned}$$

Now since β is faithful, we may divide by $\beta(x) > 0$ and get $k \leq l$ which is absurd.

Assuming the action α is minimal we prove (3) $\Rightarrow$ (1). Fix a non-zero $g \in K_0(A)^+$. Since the action is completely non-paradoxical, it follows from Lemma 3.2.10 that for every positive integer n, $(n+1)[g]_\alpha \not\leq n[g]_\alpha$. Theorem 3.2.11 then states that $S(A, \Gamma, \alpha)$ admits a non-trivial state $\nu : S(A, \Gamma, \alpha) \to [0, \infty]$ with $\nu([g]_\alpha) = 1$.

Claim: ν is finite.

To see this, employ K-minimality of the action to obtain group elements $t_1, \dots, t_n$ such that $\sum_{j=1}^n \hat{\alpha}_{t_j}(g) \geq [1]_0$. Now for an arbitrary $[x]_\alpha$ in $S(A, \Gamma, \alpha)$ with x belonging to $K_0(A)^+$, there is a positive integer m with $x \leq m[1]_0 \leq m \sum_{j=1}^n \hat{\alpha}_{t_j}(g)$. Therefore

$$[x]_\alpha \leq \left[m \sum_{j=1}^n \hat{\alpha}_{t_j}(g) \right]_\alpha = m \left[\sum_{j=1}^n \hat{\alpha}_{t_j}(g) \right]_\alpha = m[ng]_\alpha = mn[g]_\alpha.$$

Applying ν yields $\nu([x]_\alpha) \leq \nu(mn[g]_\alpha) = mn\nu([g]_\alpha) = mn$. The Claim is therefore proved.

We now compose ν with our above $\rho : K_0(A)^+ \to S(A, \Gamma, \alpha)$ to yield $\beta' : K_0(A)^+ \to ([0, \infty), +)$ a finite order preserving monoid homomorphism given by $\beta'(x) = \nu([x]_\alpha)$. Note how β' is invariant under the action $\hat{\alpha} : \Gamma \curvearrowright K_0(A)^+$. Indeed, for t in Γ, and x in $K_0(A)^+$,

$$\beta'(\hat{\alpha}_t(x)) = \nu([\hat{\alpha}_t(x)]_\alpha) = \nu([x]_\alpha) = \beta'(x).$$

By universality of the Grothendieck enveloping group construction, there is a unique extension of β' to a group homomorphism on all of $K_0(A)$, which we will denote as β, given simply by $\beta(x-y) = \beta'(x) - \beta'(y)$ for x, y in $K_0(A)^+$. Clearly β is still Γ-invariant. The final product is a bona fide Γ-invariant positive group homomorphism $\beta : K_0(A) \to \mathbb{R}$, with $\beta(g) = 1$. We now show how β is faithful which will complete this direction. Assume $0 \neq x \in K_0(A)^+$. Minimality ensures the existence of group elements $t_1, \dots t_n$ with $\sum_{j=1}^n \hat{\alpha}_{t_j}(x) \geq [1]_0$. Now we find a positive integer m for which $m[1]_0 \geq g$, so that $m\big(\sum_{j=1}^n \hat{\alpha}_{t_j}(x)\big) \geq g$. Applying β gives

$$1 = \beta(g) \leq \beta\big(m\big(\sum_{j=1}^n \hat{\alpha}_{t_j}(x)\big)\big) = m\big(\sum_{j=1}^n \beta(\hat{\alpha}_{t_j}(x))\big) = m\big(\sum_{j=1}^n \beta(x)\big) = mn\beta(x)$$

thus $\beta(x) \neq 0$ and β is indeed faithful. $\square$

We now are ready to establish the long desired converse.

Theorem 3.2.13. *Let A be a stably finite unital C^*-algebra for which $K_0(A)^+$ has Riesz refinement* ($\mathrm{sr}(A) = 1$ *and* $\mathrm{RR}(A) = 0$ *for example). Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal action on A. Consider the following properties.*

1. *There is an Γ-invariant faithful tracial state $\tau : A \to \mathbb{C}$.*

2. *$A \rtimes_\lambda \Gamma$ admits a faithful tracial state.*

3. *$A \rtimes_\lambda \Gamma$ is stably finite.*

4. *α is completely non-paradoxical.*

5. There is a faithful Γ*-invariant state* β *on* $(K_0(A), K_0(A)^+, [1]_0)$.

Then we have the following implications:

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5).$$

If A *is exact and projections are total in* A *(e.g.* $RR(A) = 0$*) then* $(5) \Leftrightarrow (1)$*. Furthermore, if* A *is AF and* Γ *is a free group, then* (1) *through* (5) *are all equivalent to* $A \rtimes_\lambda \Gamma$ *being MF.*

Proof. It is well known that $(1) \Leftrightarrow (2) \Rightarrow (3)$. Also, $(3) \Rightarrow (4)$ is Proposition 3.2.5 and $(4) \Rightarrow (5)$ is Proposition 3.2.12.

$(5) \Rightarrow (1)$: Since A exact, such a β arises from a tracial state $\tau : A \to \mathbb{C}$, via $\tau(p) = \beta([p])$ for any projection $p \in A$ ([44]). We need only to show the Γ-invariance of τ. For any $s \in \Gamma$ and projection p in A,

$$\tau(\alpha_s(p)) = \beta([\alpha_s(p)]) = \beta \circ \hat{\alpha}_s([p]) = \beta([p]) = \tau(p).$$

Using linearity, continuity, and the fact that the projections are total in A, it follows that $\tau(\alpha_s(a)) = \tau(a)$ for every $a \in A$ and $s \in \Gamma$ which yields the invariance.

Now we let $\Gamma = \mathbb{F}_r$ and A an AF algebra. In [41] the author shows that $A \rtimes_\lambda \mathbb{F}_r$ is MF if and only if it is stably finite.

□

Recall that a continuous affine action of an amenable group Γ on a compact convex subset K of a locally convex space admits a fixed point.

Corollary 3.2.14. *Let* A *be a simple, unital, AF algebra and* Γ *a discrete amenable group. Then any action* $\alpha : \Gamma \to \mathrm{Aut}(A)$ *is completely non-paradoxical.*

Proof. Let $T(A)$ denote the compact convex set of all tracial states on A viewed as a subset of the locally convex space A^* with the weak*-topology. The group Γ acts continuously and affinely on $T(A)$ by $t.\tau(a) = \tau(\alpha_{t^{-1}}(a))$ for $t \in \Gamma$ and $a \in A$. Since Γ is amenable, $T(A)$ has a fixed point. Now apply Theorem 3.2.13. $\square$

3.2.3 Purely Infinite Crossed Products

A continuous action $\Gamma \curvearrowright X$ of a discrete group on a compact Hausdorff space is called a *strong boundary action* if X has at least three points and for every pair U, V of non-empty open subsets of X there exists $t \in \Gamma$ with $t.U^c \subset V$. Laca and Spielberg showed in [32] that if $\Gamma \curvearrowright X$ is a strong boundary action and the induced action $\Gamma \curvearrowright C(X)$ is properly outer then $C(X) \rtimes_\lambda \Gamma$ is purely infinite and simple.

Jolissaint and Robertson [26] made a generalization valid in the noncommutative setting. They termed an action $\alpha : \Gamma \to \mathrm{Aut}(A)$ as *n-filling* if, for all $a_1, \dots, a_n \in A^+$, with $\|a_j\| = 1$, $1 \le j \le n$, and for all $\varepsilon > 0$, there exist $t_1, \dots, t_n \in \Gamma$ such that $\sum_{j=1}^{n} \alpha_{t_j}(a_j) \ge (1-\varepsilon)1_A$. They showed that $A \rtimes_\lambda \Gamma$ is purely infinite and simple provided that the action is properly outer and n-filling and every corner pAp of A is infinite dimensional. Using ordered K-theoretic dynamics we shall provide an alternate simpler proof of this result below, albeit for a smaller class of algebras.

The following lemma contains ideas from Lemma 3.2 of [45].

Lemma 3.2.15. *Let (A, Γ, α) be a C*-dynamical system with A separable and Γ countable and discrete. Assume that α is properly outer. Then for every non-zero $b \in (A \rtimes_\lambda \Gamma)^+$ there is a non-zero $a \in A^+$ with $a \precsim b$.*

Proof. We know that $\mathbb{E}(b) \neq 0$ since b is non-zero and $\mathbb{E}$ is faithful. Set $b_1 = b/\|\mathbb{E}(b)\|$ so that $\|\mathbb{E}(b_1)\| = 1$. Let $0 < \varepsilon < 1/16$. Find a $\delta > 0$ with $\frac{\delta(1+\|b_1\|)}{1-\delta} < \varepsilon$. Next find a non-zero positive $c \in C_c(\Gamma, A)^+$ with $\|c - b_1\| < \delta$. Write $c = \sum_{s \in F} c_s u_s$ where F

is a finite subset of Γ. Note that $\mathbb{E}(c) = c_e \neq 0$, and also $\big|1 - \|c_e\|\big| \leq \delta$. Setting $d = c/\|c_e\|$ we estimate

$$\begin{aligned}\|b_1 - d\| = \frac{1}{\|c_e\|}\big\|\|c_e\|b_1 - c\big\| &= \frac{1}{\|c_e\|}\big\|\|c_e\|b_1 - b_1 + b_1 - c\big\| \\ &\leq \frac{1}{\|c_e\|}\big(|\|c_e\| - 1|\|b_1\| + \|b_1 - c\|\big) \\ &\leq \frac{1}{1-\delta}(\delta\|b_1\| + \delta) = \frac{\delta}{1-\delta}(1 + \|b_1\|) < \varepsilon.\end{aligned}$$

Now let $\eta > 0$ be so small that $|F|\eta < 1/8$. Since A is separable and α is properly outer, we apply Lemma 7.1 of [35] and obtain an element $x \in A^+$ with $\|x\| = 1$ satisfying

$$\|x\mathbb{E}(d)x\| = \|xd_ex\| > \|d_e\| - \eta = 1 - \eta, \qquad \|xd_s\alpha_s(x)\| < \eta \quad \forall s \in F \setminus \{e\}.$$

Therefore we have

$$\begin{aligned}\|x\mathbb{E}(d)x - xdx\| &\leq \Big\|\sum_{s\in F\setminus\{e\}} xd_su_sx\Big\| \leq \sum_{s\in F\setminus\{e\}} \|xd_su_sx\| \\ &= \sum_{s\in F\setminus\{e\}} \|xd_su_sxu_s^*\| = \sum_{s\in F\setminus\{e\}} \|xd_s\alpha_s(x)\| \leq |F|\eta < 1/8.\end{aligned}$$

A straightforward use of the triangle inequality now gives

$$\|x\mathbb{E}(b_1)x - xb_1x\| \leq 2\varepsilon + 1/8 < 1/4, \qquad \|x\mathbb{E}(b_1)x\| \geq 3/4.$$

Let $a := (x\mathbb{E}(b_1)x - 1/2)_+$. Then $a \in A$ and $a \neq 0$ since $\|x\mathbb{E}(b_1)x\| > 1/2$. Also by Proposition 2.2 of [42] we know $a \precsim xb_1x \precsim b_1 \precsim b$. $\square$

Theorem 4.1 in [45] concentrates on the commutative case. We, however, make the observation that the same proof holds true for noncommutative algebras. Recall that a C*-algebra A has property (SP) if every non-zero hereditary subalgebra admits a non-zero projection.

Theorem 3.2.16. *Let (A, Γ, α) be a C*-dynamical system with A separable with property (SP) and Γ countable and discrete. Assume that α is minimal and properly outer (so that $A \rtimes_\lambda \Gamma$ is simple). Then the following are equivalent:*

1. *$A \rtimes_\lambda \Gamma$ is purely infinite.*

2. *Every non-zero projection p in A is properly infinite in $A \rtimes_\lambda \Gamma$.*

Proof. (1) $\Rightarrow$ (2): Every non-zero projection in any purely infinite algebra is properly infinite.

(2) $\Rightarrow$ (1): By Theorem 7.2 in [35] we know that the reduced crossed product $A \rtimes_\lambda \Gamma$ is simple. Therefore, it suffices to show that every hereditary subalgebra admits an infinite projection. To this end, let $B \subset A \rtimes_\lambda \Gamma$ be a hereditary C*-subalgebra and let $0 \neq b \in B$. By lemma 3.2.15 there is a non-zero a in A with $a \precsim b$. Since A has property (SP), the hereditary subalgebra of A generated by a, $H_a = \overline{aAa}$, contains a non-zero projection $q \in H_a$. By our assumption q is properly infinite relative to $A \rtimes_\lambda \Gamma$, and $q \precsim a \precsim b$. Since q is a projection, there is a $z \in A \rtimes_\lambda \Gamma$ with $q = z^* b z$. Now consider $v := b^{1/2} z$. Then $q = v^* v \sim v v^* = b^{1/2} z z^* b^{1/2} \in B$. Thus $p := vv^*$ is the desired properly infinite projection in B. $\square$

We now embark on studying to what extent paradoxical systems (A, Γ, α) characterize purely infinite reduced crossed product algebras $A \rtimes_\lambda \Gamma$.

Proposition 3.2.17. *Let (A, Γ, α) be a C*-system for which A has cancellation and $K_0(A)^+$ has the Riesz refinement property. Let $0 \neq r \in \mathcal{P}(A)$ and set $g = [r]_0 \in K_0(A)^+$. The following properties are equivalent:*

1. *There exist $x, y \in C_c(\Gamma, A)$ that satisfy $x^*x = r = y^*y$, $xx^* \perp yy^*$, $xx^* \leq r$, $yy^* \leq r$, and whose coefficients are partial isometries.*

2. *g is $(k, 1)$-paradoxical for some $k \geq 2$.*

3. *$\theta = [g]_\alpha$ is properly infinite in $S(A, \Gamma, \alpha)$.*

Proof. (1) $\Rightarrow$ (2): Write $x = \sum_{s\in F} u_s v_s$ and $y = \sum_{s\in L} u_s w_s$ where $F, L \subset \Gamma$ are finite subsets, and $v_s, w_s \in A$ are partial isometries. For each s in F set $p_s := v_s^* v_s$ and $p'_s := v_s v_s^*$. Similarly for every $s \in L$ set $q_s := w_s^* w_s$ and $q'_s := w_s w_s^*$. If we apply the conditional expectation $\mathbb{E} : A \rtimes_\lambda \Gamma \to A$ to the equality $r = x^*x$ we get

$$r = \mathbb{E}(r) = \mathbb{E}\bigg(\sum_{s,t\in F} v_s^* u_s^* u_t v_t\bigg) = \sum_{s,t\in F} \mathbb{E}(v_s^* u_s^* u_t v_t) = \sum_{s\in F} v_s^* v_s = \sum_{s\in F} p_s.$$

The second to last equality follows from the fact that for $s, t \in F$ we have

$$\mathbb{E}(v_s^* u_s^* u_t v_t) = \mathbb{E}(v_s^* u_{s^{-1}t} v_t (u_{s^{-1}t})^* u_{s^{-1}t}) = \mathbb{E}(v_s^* \alpha_{s^{-1}t}(v_t) u_{s^{-1}t}) = \delta_{s,t} v_s^* v_s.$$

Therefore, the projections p_s are mutually orthogonal subprojections of r that sum to r. Similarly all the q_s, for $s \in L$, are mutually orthogonal subprojections of r with $r = \sum_{s\in L} q_s$. Thus, in $K_0(A)^+$ we have

$$\sum_{s\in F} [p_s]_0 + \sum_{s\in L} [q_s]_0 = \bigg[\sum_{s\in F} p_s\bigg]_0 + \bigg[\sum_{s\in F} q_s\bigg]_0 = 2[r]_0.$$

Now we note that for s, t in F with $s \neq t$ we have $v_s v_t^* = v_s v_s^* v_s v_t^* v_t v_t^* = v_s p_s p_t v_t^* = 0$. Computing xx^* we get

$$xx^* = \sum_{s,t\in F} u_s v_s v_t^* u_t^* = \sum_{s\in F} u_s v_s v_s^* u_s^* = \sum_{s\in F} \alpha_s(p'_s).$$

Similarly $yy^* = \sum_{s\in L} \alpha_s(q'_s)$. From

$$\sum_{s\in F} \alpha_s(p'_s) + \sum_{s\in L} \alpha_s(q'_s) = xx^* + yy^* \leq r$$

we conclude that the projections $\alpha_s(p'_s), \alpha_s(q'_s)$ are mutually orthogonal subprojections of r whence in $K_0(A)$ we have

$$\begin{aligned}[r]_0 \geq \left[\sum_{s\in F} \alpha_s(p'_s) + \sum_{s\in L} \alpha_s(q'_s)\right]_0 &= \sum_{s\in F} [\alpha_s(p'_s)]_0 + \sum_{s\in L} [\alpha_s(q'_s)]_0 \\ &= \sum_{s\in F} \hat{\alpha}_s([p'_s]_0) + \sum_{s\in L} \hat{\alpha}_s([q'_s]_0) = \sum_{s\in F} \hat{\alpha}_s([p_s]_0) + \sum_{s\in L} \hat{\alpha}_s([q_s]_0).\end{aligned}$$

Therefore $g = [r]_0$ is (2,1)-paradoxical.

(2) $\Rightarrow$ (1): Suppose $\sum_{j=1}^n x_j \geq k[r]_0$ and $\sum_{j=1}^n \hat{\alpha}_{t_j}(x_j) \leq [r]_0$ for some $k \geq 2$, group elements $t_1, \ldots, t_n \in \Gamma$, and $x_j \in K_0(A)^+$. Since $k[r]_0 \geq 2[r]_0$ we may assume $k = 2$. For some $u \in K_0(A)^+$ we then have $\sum_{j=1}^n x_j = [r]_0 + [r]_0 + u$. Refinement implies that there are subsets $\{y_j\}_{j=1}^n$, $\{z_j\}_{j=1}^n$ and $\{u_j\}_{j=1}^n$ of $K_0(A)^+$ with

$$\sum_{j=1}^n y_j = [r], \quad \sum_{j=1}^n z_j = [r], \quad \sum_{j=1}^n u_j \geq 0, \quad \text{and} \quad x_j = y_j + z_j + u_j, \quad \forall j.$$

Using the fact that A has cancellation we know that there are mutually orthogonal projections $p_j \in \mathcal{P}(A)$ with $[p_j]_0 = y_j$ for $j = 1, \ldots, n$. Similarly there are mutually orthogonal projections $q_j \in \mathcal{P}(A)$ with $[q_j]_0 = z_j$ for $j = 1, \ldots, n$. Therefore,

$$\begin{aligned}\sum_j [\alpha_{t_j}(p_j)]_0 + \sum_j [\alpha_{t_j}(q_j)]_0 &= \sum_j \hat{\alpha}_{t_j}(y_j) + \sum_j \hat{\alpha}_{t_j}(z_j) \\ &\leq \sum_j \hat{\alpha}_{t_j}(y_j) + \sum_j \hat{\alpha}_{t_j}(z_j) + \sum_j \hat{\alpha}_{t_j}(u_j) = \sum_j \hat{\alpha}_{t_j}(x_j) \leq [r]_0.\end{aligned}$$

We again use the fact that A has cancellation and find mutually orthogonal subprojections of r $e_1, \dots, e_n; f_1, \dots, f_n \in \mathcal{P}(A)$ with $[e_j]_0 = [\alpha_{t_j}(p_j)]_0$ and $[f_j]_0 = [\alpha_{t_j}(q_j)]_0$ for every j. Cancellation also implies that there are partial isometries v_j and w_j in A with

$$v_j^* v_j = \alpha_{t_j}(p_j), \quad v_j v_j^* = e_j, \quad w_j^* w_j = \alpha_{t_j}(q_j), \quad w_j w_j^* = f_j.$$

Now set $a := \sum_{j=1}^n v_j u_j$ and $b := \sum_{j=1}^n w_j u_j$ where $u_j = u_{t_j}$. Note that for $i \neq j$ we compute $v_j^* v_i = v_j^* v_j v_j^* v_i v_i^* v_i = v_j^* e_j e_i v_i = 0$, so

$$a^* a = \sum_{i,j} u_j^* v_j^* v_i u_i = \sum_j u_j^* v_j^* v_j u_j = \sum_j u_j^* \alpha_{t_j}(p_j) u_{t_j} = \sum_j \alpha_{t_j^{-1}}(\alpha_{t_j}(p_j)) = \sum_j p_j := p.$$

In order to compute aa^* we note that for $i \neq j$ we have

$$\begin{aligned} v_j u_j u_i^* v_i^* &= v_j v_j^* v_j u_j u_i^* v_i^* v_i v_i^* = v_j \alpha_{t_j}(p_j) u_j u_i^* \alpha_{t_i}(p_i) v_i^* = v_j u_j p_j u_j^* u_j u_i^* u_i p_i u_i^* v_i^* \\ &= v_j u_j p_j p_i u_i^* v_i^* = 0, \end{aligned}$$

whence

$$aa^* = \sum_{i,j} v_j u_j u_i^* v_i^* = \sum_j v_j u_j u_j^* v_j^* = \sum_j v_j v_j^* = \sum_j e_j := e.$$

Similarly $b^* b = \sum_j q_j := q$, and $bb^* = \sum_j f_j = f$.

Now define $x := av$ where v is the partial isometry in A with $v^* v = r$ and $vv^* = p$. Such a v exists because $[p]_0 = \left[\sum_j p_j \right]_0 = \sum_j [p_j]_0 = \sum_j y_j = [r]_0$ and A has cancellation. Similarly define $y := bw$ where $w \in A$ satisfies $w^* w = r$ and $ww^* = q$. We compute

$$x^* x = v^* a^* a v = v^* p v = v^* v v^* v = r^2 = r,$$

and

$$y^*y = w^*b^*bw = w^*qw = w^*ww^*w = r^2 = r.$$

Moreover, since a and b are partial isometries, and $e \perp f$ we have

$$xx^*yy^* = avv^*a^*bww^*b^* = avv^*a^*aa^*bb^*bww^*b^* = avv^*a^*efbww^*b^* = 0.$$

Next we observe that xx^* is a subprojection of r; indeed, since $e \leq r$,

$$rxx^* = ravv^*a^* = raa^*avv^*a^* = reavv^*a^* = eavv^*a^* = aa^*avv^*a^* = avv^*a^* = xx^*$$

Similarly yy^* is a subprojection of r.

Finally we verify that the coefficients of x and y are partial isometries. Write

$$x = av = \sum_{j=1}^{n} v_j u_j v = \sum_{j=1}^{n} v_j \alpha_{t_j}(v) u_j,$$

and compute

$$(v_j\alpha_{t_j}(v))^* v_j \alpha_{t_j}(v) = \alpha_{t_j}(v^*) v_j^* v_j \alpha_{t_j}(v) = \alpha_{t_j}(v^*)\alpha_{t_j}(p_j)\alpha_{t_j}(v) = \alpha_{t_j}(v^* p_j v),$$

but since $p_j \leq p$ for every j, v^*p_jv is a projection: $(v^*p_jv)^2 = v^*p_jvv^*p_jv = v^*p_jpp_jv = v^*p_jv$. Therefore $\alpha_{t_j}(v^*p_jv)$ is a projection for each j and so the coefficients of x, $v_j\alpha_{t_j}(v)$, are partial isometries. An identical argument works for the coefficients of y. This completes the implication (2) $\Rightarrow$ (1).

(2) $\Leftrightarrow$ (3): By definition $[g]_\alpha$ is infinite in $S(A, \Gamma, \alpha)$ if and only if $2[g]_\alpha \leq [g]_\alpha$, and by Proposition 3.2.10, we know this occurs if and only if g is $(2, 1)$-paradoxical. Clearly g is $(2, 1)$-paradoxical if and only if g is $(k, 1)$-paradoxical for some $k \geq 2$. $\square$

At this point we can supply an alternate proof of Jolissaint and Robertson's result using ordered K-theory, but first, two basic lemmas. Recall that a partially ordered group (G, G^+) is said to be *non-atomic* if, for every non-zero $g > 0$, there is an $h \in G$ with $0 < h < g$.

Lemma 3.2.18. *If A is a unital stably finite C^*-algebra with property (SP) such that pAp is infinite dimensional for every projection $p \in A$, then $(K_0(A), K_0(A)^+)$ is non-atomic.*

Proof. Let $0 < g = [q]_0$ belong to $K_0(A)^+$ for some non-zero $q \in \mathcal{P}_n(A)$. Then clearly there is a non-zero $b \in A^+$ with $b \precsim q$. By property (SP) there is a non-zero projection $p \in \overline{bAb}$. A little work gives $p \precsim b$. By hypothesis the corner pAp is infinite dimensional and thus every masa of pAp is infinite dimensional. Inside such an infinite dimensional masa we can find positive elements a_1, a_2 of norm one with $a_1 a_2 = 0$. Now find non-zero projections $p_i \in \overline{a_i A a_i}$ for $i = 1, 2$. Then p_1, p_2 are non-zero orthogonal subprojections of p. It follows that $g > [p_1]_0 > 0$. $\square$

Lemma 3.2.19. *Let A be a unital C^*-algebra and $\alpha : \Gamma \to \mathrm{Aut}(A)$ an action. Consider the following properties:*

1. *The action α is n-filling.*

2. *The action α is W-n-minimal.*

3. *The action α is W-n-filling.*

4. *The action α is K_0-n-filling..*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

Proof. (1) $\Rightarrow$ (2): Let $x \in W(A)$. We can find a positive norm-one element $b \in A^+$ with $b \precsim x$. By hypothesis there are group elements $t_1, \dots, t_n$ with

$$\sum_{j=1}^{n} \alpha_{t_j}(b) \geq (1/2)1_A.$$

The result follows since

$$\langle 1_A \rangle = \langle (1/2)1_A \rangle \leq \Big\langle \sum_{j=1}^{n} \alpha_{t_j}(b) \Big\rangle \leq \langle \oplus_j \alpha_{t_j}(b) \rangle = \sum_{j=1}^{n} \langle \alpha_{t_j}(b) \rangle = \sum_{j=1}^{n} \hat{\alpha}_{t_j}(\langle b \rangle) \leq \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x).$$

(2) $\Leftrightarrow$ (3): This was shown in Proposition 3.1.13 above.

(3) $\Rightarrow$ (4): This follows from the fact that if $p, q \in \mathcal{P}_\infty(A)$ and $\langle p \rangle \leq \langle q \rangle$ in $W(A)$, then $[p]_0 \leq [q]_0$ in $K_0(A)$. $\square$

Proposition 3.2.20. *Let A be a separable C^*-algebra with cancellation, property (SP), and for which $(K_0(A), K_0(A)^+)$ is non-atomic and $K_0(A)^+$ has Riesz refinement (an algebra of real rank zero and stable rank one will do). Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a properly outer action which is K_0-n-filling for some $n \in \mathbb{N}$. Then $A \rtimes_\lambda \Gamma$ is simple and purely infinite.*

Proof. By theorem 3.2.16 it suffices to prove that every projection p in A is properly infinite in $A \rtimes_\lambda \Gamma$. Now by Proposition 3.2.17 we need only show that $g = [p]_0$ in $K_0(A)^+$ is $(2,1)$-paradoxical. Since $K_0(A)^+$ is non-atomic we may find non-zero elements $x_1, \dots, x_{2n} \in K_0(A)^+$ with $\sum_{j=1}^{2n} x_j \leq g$. By the n-filling property there are group elements $t_1, \dots, t_{2n}$ with

$$\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \geq [1]_0, \quad \text{and} \quad \sum_{j=n+1}^{2n} \hat{\alpha}_{t_j}(x_j) \geq [1]_0.$$

Together $\sum_{j=1}^{2n} x_j \leq g$ and $\sum_{j=1}^{2n} \hat{\alpha}_{t_j}(x_j) \geq 2[1]_0 \geq 2g$ and thus g is $(2,1)$-paradoxical.

□

The following result generalizes Theorem 5.4 of [45] to the noncommutative case.

Theorem 3.2.21. *Let A be a unital, separable, exact C*-algebra whose projections are total. Moreover, suppose A has cancellation and $K_0(A)^+$ has the Riesz refinement property. Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal and properly outer action. Consider the following properties:*

1. *The semigroup $S(A, \Gamma, \alpha)$ is purely infinite.*

2. *Every non-zero element in $K_0(A)^+$ is $(k, 1)$-paradoxical for some $k \geq 2$.*

3. *The C*-algebra $A \rtimes_\lambda \Gamma$ is purely infinite.*

4. *The C*-algebra $A \rtimes_\lambda \Gamma$ is traceless.*

5. *The semigroup $S(A, \Gamma, \alpha)$ admits no non-trivial state.*

Then the following implications always hold: $(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$*. If the semigroup $S(A, \Gamma, \alpha)$ is almost unperforated then* $(5) \Rightarrow (1)$ *and all properties are equivalent.*

Proof. $(1) \Leftrightarrow (2)$: We have already seen that $x \in K_0(A)^+$ is $(k, 1)$-paradoxical for some $k \geq 2$ if and only if $\theta = [x]_\alpha$ is properly infinite in $S(A, \Gamma, \alpha)$.

$(2) \Rightarrow (3)$: Let r be a non-zero projection in A. By assumption $[r]_0$ is $(2, 1)$-paradoxical, so by lemma 3.2.17 r is properly infinite in $A \rtimes_\lambda \Gamma$. Then $A \rtimes_\lambda \Gamma$ is purely infinite by Theorem 3.2.16.

$(3) \Rightarrow (4)$: Purely infinite C*-algebras are always traceless.

$(4) \Rightarrow (5)$: Suppose $\nu : S(A, \Gamma, \alpha) \to [0, \infty]$ is a non-trivial state. Suppose $0 < \nu([x]_\alpha) < \infty$ where $x \in K_0(A)^+$ is non-zero. Composing with the quotient map

$\rho : K_0(A)^+ \to S(A, \Gamma, \alpha)$ we get an order preserving monoid homomorphism $\beta' = \nu \circ \rho : K_0(A)^+ \to [0, \infty]$ with $0 < \beta'(x) < \infty$. As in the proof of Proposition 3.2.12, minimality of the action ensures that β' is finite on all of $K_0(A)^+$. Extending β' to $K_0(A)$ gives a Γ-invariant positive group homomorphism, β, on $K_0(A)$. Since A is exact and projections are total, β comes from a Γ-invariant tracial state on A (see Theorem 1.1.11 in [44]), so that $A \rtimes_\lambda \Gamma$ admits a trace, a contradiction.

Now we assume that $S(A, \Gamma, \alpha)$ is almost unperforated and prove (5) $\Rightarrow$ (1). Let $\theta = [x]_\alpha$ be a non-zero element in $S(A, \Gamma, \alpha)$. If θ is completely non-paradoxical then by Tarski's Theorem $S(A, \Gamma, \alpha)$ admits a non-trivial state. So, assuming (5), we must have $(k+1)\theta \leq k\theta$ for some $k \in \mathbb{N}$. So

$$(k+2)\theta = (k+1)\theta + \theta \leq k\theta + \theta = (k+1)\theta \leq k\theta.$$

Repeating this trick we get $(k+1)2\theta \leq k\theta$. Since $S(A, \Gamma, \alpha)$ is almost unperforated we conclude $2\theta \leq \theta$ and θ is properly infinite. $\square$

We conclude this chapter with a result that combines Theorem 3.2.13 and Theorem 3.2.21. In this way we obtain the desired dichotomy, albeit under suitable conditions.

Theorem 3.2.22. *Let A be a unital, separable, exact C^*-algebra whose projections are total. Moreover suppose A has cancellation and $K_0(A)^+$ has the Riesz refinement property. Let Γ be a countable discrete group and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal and properly outer action such that $S(A, \Gamma, \alpha)$ is almost unperforated. Then the reduced crossed product $A \rtimes_\lambda \Gamma$ is a simple C^*-algebra which is either stably finite or purely infinite.*

4. SUMMARY AND CONCLUDING REMARKS

The overarching theme in chapter two is that of finite-dimensional approximation properties of a topological nature witnessed in reduced crossed products. These emerge as consequences of approximation properties at the level of the dynamics. For example, we studied MF actions and noticed the presence of norm microstates in the reduced crossed product. We showed that an obstruction to this property is also an obstruction to stable finiteness. Indeed, the main achievement there was that we described MF crossed products using a K-theoretic coboundary condition. Consequently, in the case of a free group $\mathbb{F}_r$ acting on an AF algebra A, we saw that $A \rtimes_\lambda \mathbb{F}_r$ is MF if and only if $A \rtimes_\lambda \mathbb{F}_r$ is stably finite (Theorem 2.2.14). Then in chapter 3 we looked at the grand theme of finiteness in C*-crossed products. Under suitable conditions on the underlying algebra−conditions that ensure that the K_0 group is well-behaved, we learned that stable finiteness is characterized by a complete non-paradoxicality property at the level of the induced dynamics on K-theory (see Theorems 3.2.13 and 3.2.21). Combining Theorem 3.2.22 and Theorem 2.2.14 we obtain the following dichotomous result.

Corollary 4.0.23. *Let A be an AF algebra and let $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ be a minimal, properly outer action with almost unperforated type-semigroup $S(A, \mathbb{F}_r, \alpha)$. Then $A \rtimes_\lambda \Gamma$ is either MF or purely infinite.*

We end our discussion by mentioning a few interesting questions and avenues for future research.

It is unknown to the author if there are examples of minimal and properly outer actions on C*-algebras satisfying the conditions in Theorem 3.2.22 for which the type semigroup is *not* almost unperforated. In particular, is there a free and action of the free group $\mathbb{F}_2$ on the Cantor set X for which $S(X, \mathbb{F}_2, \mathcal{C})$ is not almost unperforated? Although Ara and Exel construct actions of a finitely generated free group on the Cantor set for which the type semigroup is not almost unperforated, these actions are not minimal [1]. Moreover, almost unperforation may be too strong a condition to establish (5) $\Rightarrow$ (1) in Theorem 3.2.21. What is required is that every 'infinite element' (in the sense that $(k+1)x \leq kx$ for some k) is properly infinite. This is a priori a weaker condition than almost unperforation.

The author is convinced that Theorem 2.2.14 can be extended to actions of free groups on $A\mathbb{T}$-algebras, or even a larger class of separable C*-algebras that are classifiable. If such an extension holds, then a similar result as 4.0.23 would hold for these algebras.

We observed in Proposition 2.1.9 that any action of a free group on a UHF algebra A is quasidiagonal; consequently $A \rtimes_\lambda \mathbb{F}_r$ is always MF. It is unknown if the same permanence holds for countable discrete groups Γ whose group C*-algebra $C^*_\lambda(\Gamma)$ is MF. That is, if $C^*_\lambda(\Gamma)$ is MF, and Γ acts on a UHF algebra A, is $A \rtimes_\lambda \Gamma$ also MF?

The Pimsner-Voiculescu sequence leaves much to be desired when one is interested in the order structure of $K_0(A \rtimes_\lambda \mathbb{F}_r)$ in the case where it is known that $A \rtimes_\lambda \mathbb{F}_r$ is stably finite. More precisely, in the notation of Theorem 2.2.14, if A is an AF-algebra, is the group isomorphism

$$K_0(A \rtimes_\lambda \mathbb{F}_r) \cong K_0(A)/H_\sigma$$

an isomorphism of *ordered* abelian groups? The question boils down to whether or not the K-theory map $\hat{\iota} : K_0(A)^+ \to K_0(A \rtimes_\lambda \mathbb{F}_r)^+$ is onto, where $\iota A \hookrightarrow A \rtimes_\lambda \mathbb{F}_r$ is the canonical inclusion. In the same spirit we can also ask the following: is the well-defined map

$$S(A,\Gamma,\alpha) = K_0(A)^+/\sim_\alpha \longrightarrow K_0(A \rtimes_\lambda \mathbb{F}_r)^+ \quad ([x]_\alpha \mapsto [\iota(x)]_{K_0(A\rtimes_\lambda \mathbb{F}_r)})$$

injective? A positive answer to both these questions would give us a complete description of $K_0(A \rtimes_\lambda \mathbb{F}_r)$ in terms of the dynamics and relate almost unperforation of the type semigroup to that of the K_0-group of the crossed product. These queries seem to be elusive both to the author and experts in the field.

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