FINITELY CONSTRAINED GROUPS

A Dissertation

by

ANDREW DANIEL PENLAND

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Zoran Šunik
Committee Members,	Rostislav Grigorchuk
	Volodymyr Nekrashevych
	Renata Ivanek-Miojevic
Head of Department,	Emil Straube

August 2015

Major Subject: Mathematics

ABSTRACT

This work investigates three aspects of the theory of finitely constrained groups, motivated by questions first asked by Rostislav Grigorchuk when he introduced the subject in 2005. The first topic is Hausdorff dimension of finitely constrained groups of p-adic tree automorphisms. The set of possible values of Hausdorff dimension for such a group is known, and we are able to show that every value in this set actually occurs. The second topic, related to the first, is topological finite generation of finitely constrained groups of p-adic tree automorphisms. Relatively little is known about which values of Hausdorff dimension occur for topologically finitely generated, finitely constrained groups of p-adic tree automorphisms. We are able to show that certain values can not occur as the Hausdorff dimension a topologically finitely generated, finitely constrained group of p-adic automorphisms defined by patterns of size d. We discuss finitely constrained groups of binary tree automorphisms with pattern size $d \geq 5$ and Hausdorff dimension $1 - \frac{2}{2^{d-1}}$; the issue of topological finite generation for these groups is more challenging. We provide explicit constructions of new examples of finitely constrained groups and calculate their Hausdorff dimension. Finally, we study the portraits of self-similar groups using well-known ideas from the theory of tree automata, with particular focus on examples which separate certain classes of tree languages. These self-similar groups generalize the usual notion of self-similar groups, and we show that some well-known results extend to this more general case. From the tree language perspective, self-similar groups whose portraits form sofic tree shifts are of particular interest. We conclude by posing many questions for future study.

DEDICATION

For Katie Penland

ACKNOWLEDGEMENTS

I would like to thank first and foremost my advisor Zoran Šunik. In addition to being exceptionally well-versed in many areas of mathematics, he is an outstanding teacher and mentor. He consistently provided patient guidance and gentle correction throughout my time at Texas A&M. I have learned much from him about mathematics, both as a subject and as a profession. I will always feel grateful that I had the opportunity to be his student.

I would also like to thank to the other members of my Ph. D. committee: Rostislav Grigorchuk, Volodymir Nekrashevych, and Renata Ivanek-Miojevic. Both Dr. Grigorchuk and Dr. Nekrashevych are leaders in contemporary group theory, and their feedback has proved very valuable throughout the writing of this dissertation. I am thankful to Dr. Ivanek-Miojevic for pointing out potential applications of this work to other fields. I hope to explore these connections more in the future.

There were many other people at Texas A&M who also influenced me profoundly during my Ph. D. studies. I will not attempt to list them all, since such an attempt, no matter how lengthy, would remain incomplete. I hope that these people will know who they are, and I hope that they will know how much I have appreciated their efforts.

TABLE OF CONTENTS

Page

LIST OF FIGURES

FIGURE Page

1. INTRODUCTION

This work investigates aspects of finitely constrained groups, which are topological groups defined via labeled trees corresponding to finite quotients of the group. They are related to the theory of both finite and infinite groups, as well as symbolic dynamics and the theory of computation. Like several other areas of interest in contemporary group theory, their study is motivated by a property of the first Grigorchuk group.

Finitely constrained groups of tree automorphisms were introduced by Grigorchuk in 2005 [28]. This dissertation examines three topics, corresponding to three questions asked in that work. The first topic is determining the topological finite generation of a finitely constrained group, given its set of defining patterns (see [28, Problem 7.3.i]). The second topic is determining the Hausdorff dimension of the closure of a finitely generated, self-similar group (see [28, Problem 7.1(iii)]); we address this question in the special circumstance when the closure is finitely constrained. The final question is the appropriate analog of sofic systems for self-similar groups (see [28, Problem 7.4]). We should mention here that we are not the first to address these questions, and previous work by others will be discussed and used throughout to obtain new results. We should also mention at this point that some of the results in this dissertation were obtained through collaborative work with Zoran Šunić, and this will be acknowledged whenever it is the case.

An outline of the remainder of this dissertation is as follows. Section 2 gives necessary background, establishing both definitions and notation, as well as giving a review of relevant literature. We will establish connections between the different perspectives on finitely constrained groups. We especially emphasize the connection between groups of tree automorphisms and symbolic dynamics on arbitrary semigroups. The definition of self-similar and finitely constrained groups that we give is more general than that given for tree automorphisms, since we allow arbitrary groups whose action on the tree may not be faithful. We give the details of the construction explicitly.

Section 3 is dedicated to Hausdorff dimension and topological finite generation of finitely constrained groups of p-adic tree automorphisms. We prove an upper bound (as a function of pattern size) on the Hausdorff dimension of a topologically finitely generated, finitely constrained group of p-adic tree automorphisms. While obtaining this upper bound, we also provide an explicit description of the finite pattern groups which can be used to define such a finitely constrained group. Next, we discuss finitely constrained groups of binary tree automorphisms defined by pattern size d and having Hausdorff dimension $1 - \frac{2}{2^{d-1}}$. We are again able to completely characterize patterns which define these finitely constrained groups. For pattern size $d \geq 5$, the question of topological finite generation for finitely constrained groups defined by patterns of size d is more subtle. We discuss certain cases where topological finite generation can not occur. We provide examples, both known and new, of topologically finitely generated, finitely constrained groups. We also define two special classes of patterns which together contain all examples of finitely constrained groups of which we are aware. We conclude by exhibiting new examples, verified using the computer program `GAP`.

Section 4 discusses the computational aspects of finitely constrained groups. We discuss some work done both classically and recently related to languages of tree patterns, which motivates the study of self-similar groups as tree languages. We generalize some known results about self-similar groups to the more general case considered here. We give examples which separate distinct classes in a computational hierarchy of tree languages. Finally, we discuss self-similar groups whose portraits define sofic tree shifts. These groups give an answer to Grigorchuk's question on the analog of sofic systems for self-similar groups. We give a sufficient (though not necessary) condition for which the sofic and finitely constrained class coincide. Whether these classes always coincide is still open in general.

Section 5 discusses potential avenues of future work related to these topics. Our work is still far from giving a complete answer to Grigorchuk's questions, and even in the special cases we investigate, some work remains to be done. The questions we suggest are related to Grigorchuk's questions, and give related questions whose answers might provide insight into the larger ones. We believe that the many remaining open problems surrounding finitely constrained groups are intriguing and deserve further study.

2. BACKGROUND

This section has several purposes. The most important is to introduce the topics which will be the focus of this work. Along the way, we review basic concepts, introduce the notation we will use, and give an overview of known results in the field. There is a vast overlap between between the areas of group theory, symbolic dynamics, and the theory of computation, so we will restrict our attention to the most relevant background. On the other hand, we do want to emphasize that there are many different ways of thinking about finitely constrained groups, and there are many connections between these complementary perspectives. Where appropriate, we will also use the opportunity to generalize known results.

2.1 Trees, semigroups, and groups

2.1.1 Basic background on semigroups and groups

In general, we will assume that the reader is familiar with basic concepts of sets, semigroups and groups (at least to the level of the first 3 chapters of [32]), as well as some topology (at the level of Chapter 1 in [16]).

If X is a set, we write X^n for the set of all words of length n in X. If I is an indexing set, $\{X_i\}_{i\in I}$ is a collection of sets, and $\prod_{i\in I} X_i$ is the direct product of the sets equipped with its standard projection maps π_i, we write $x_{(i)}$ for $\pi_i(x)$, and we call $x_{(i)}$ the *label of x at i.* Generally, if X and Y are sets, we use the notation Y^X to indicate the set of all functions from X to Y. We also define

$$X^{(n)} = \bigcup_{i=0}^{n} X^i \text{ and } X^{[n]} = \bigcup_{i=0}^{n} X^i$$

If T is a semi-group and Q is a set, a *left action of T on Q* is a map $T \times Q \to$

$Q : (t, q) \mapsto t.q$ such that $s.(t.q) = (st).q$ for all $s, t \in T$ and $q \in Q$. A *right action of T on Q* is a map $Q \times T \to T.(q, t) \to q^t$ such that $q^{(st)} = (q^s)^t$. We call a set Q a *left (respectively, right) T-set* if T has a left (respectively, right) action on Q. In particular, if G is a group, a *left* (respectively, *right*) G-set is a set on which G has a left (respectively, *right*) action. A group H is called a G-group if G acts on H by automorphisms. A frequently used group action is the action of a group G on itself by conjugation, with $h^g = g^{-1}hg$ for $g, h \in G$.

We now review the iterated wreath product, which is crucial for the construction of finite patterns and finitely constrained groups. Let G and H be groups and let X be a left G-set. The direct product $H^X = \prod_{x \in X} H$ is a group under componentwise multiplication. The left action of G on X extends to a right action by automorphisms on H^X given by $((h_x)_{x \in X})^g = \left(h_{g(x)}\right)_{x \in X}$.

Definition 2.1.1 (Semi-Direct Product)**.** If a group G has a right action on a group K by automorphisms, we can define the *semi-direct product of K and G* as the group $G \ltimes K$ with underlying set $G \times K$ and binary operation given by $(g_1, k_1)(g_2, k_2) = (g_1 g_2, k_1^{g_2} k_2)$.

Definition 2.1.2 (Permutational Wreath Product)**.** Let G and H be a groups and X be a left G-set. The group $G \ltimes H^X$ is called the *permutational wreath product* of G and H, and is denoted $G \wr_X H$.

The elements of the permutational wreath product $G \ltimes H^X$ are ordered pairs $(g, (h_x)_{x \in X})$, but it is often convenient to omit the outside parentheses and the reference to X, writing an element as $g(h_x)$. Then multiplication of two elements $g_1(h_x)$ and $g_2(h'_x)$ in $G \wr_X H$ is then given as

$$g_1(h_x)g_2(h'_x) = g_1(g_2 g_2^{-1})(h_x)g_2(h'_x) = g_1 g_2(h_x)^{g_2}(h'_x) = g_1 g_2(h_{g_2(x)} h'_x)$$

This construction can be used repeatedly to obtain the *iterated wreath product*. If X is a finite set and A is a finite group with some left action ϕ on X, the elements of the permutational wreath product $A \wr_X A$ belong to the set $A^{X^{(2)}}$, and the group $A \wr_X A$ acts on X^2 . The action of A on X naturally induces an action of A on $(A \wr_X A)^X$, leading to the permutational wreath product

$$A \wr_X (A \wr_X A),$$

which acts on X^3 and whose elements correspond to those of the set $A^{X^{(3)}}$. The groups $(A \wr_X A) \wr_X A$ and $A \wr_X (A \wr_X A)$ are canonically isomorphic, so we omit parentheses.

If A is a finite group with a left action ϕ on a set X, we define the *n-fold iterated wreath product of A over X with action ϕ* inductively, as follows. We set the group $W_{(\phi,X)}(A,1)$ to be A, and for $n > 1$ we define $W_{(\phi,X)}(A,n) = A \wr_{(\phi,X)} W_{(\phi,X)A(n-1)}$. If X and ϕ are understood, we write $W_{(\phi,X)}(A,n)$ as $W_A(n)$.

In the special case when p is a prime number and A is the cyclic group C_p with its standard action on the set $X = \{0, 1, \ldots, p-1\}$, we will write the $W_{(\phi,X)}(A,n)$ as $W_p(n)$. These groups were originally studied by Kaloujnine [34] and are famous as the Sylow p-subgroups of the symmetric group on p^n letters. They are also important subgroups of the automorphism groups of finite trees, which will be a major topic in this work.

2.1.2 Trees

Nearly everything discussed in this work is related to trees. Trees are fundamental to both theoretical and practical aspects of computation, and they are important in many areas of mathematics. Trees are ubiquitious in graph theory, and they serve as natural discrete models of negatively curved metric spaces. Groups acting on rooted trees have also spurred many important developments in group theory, as we will discuss in this section.

We begin with necessary definitions. Let X be a non-empty finite set. For $n \geq 1$, we define X^n as the set of all words of length n in X. We write $|w| = n$ to indicate $w \in X^n$. We let

$$X^* = \bigcup_{i=0}^{\infty} X^i$$

be the set of all words in X of any length, including the empty word ϵ of length zero. With the product of two words w and v defined to be their concatenation wv, the set X^* forms a semigroup, called *the free semigroup on X.*

For two words w, w' in X^*, we say w' is a *descendant* of w if $w' = wv$ for some $v \in X^*$. In this case, we say w is a prefix of w' and write $w \leq w'$. We write $w < w'$ if $w \leq w'$ and $w \neq w'$. If $w' = wx$ for some $x \in X$, we say that w' is a *child of* w.

The elements of the set X^* can be identified with the vertices of a regular $|X|$-ary tree with the empty word ϵ as the root, where each vertex $w \in X^*$ is connected to its children $\{wx \mid x \in X\}$. We typically identify X^* with the tree representing it, and call X^* the *infinite* $|X|$*-ary tree.* When $|X| = 2$, X^* is an *infinite binary tree.* The set X^n is called *level* n of the tree X^*. For a word w, the *infinite subtree rooted at* w is the set $wX^* = \{wu \ \mid \ u \in X^*\}$ consisting of w and all of its descendants. The tree X^* is *self-similar* in the sense that the graphs X^* and wX^* are isomorphic for any word w.

We are also interested in finite trees. Recall that

$$X^{(n)} = \bigcup_{i=0}^{n-1} X^i, \qquad X^{[n]} = \bigcup_{i=0}^{n} X^i.$$

The members of the finite set $X^{[(d)]}$ correspond to the vertices of a regular $|X|$-ary rooted tree having d levels, which we say *has size* d.

2.1.3 Groups of tree automorphisms

For our purposes, a *tree automorphism* is a graph automorphism of a rooted tree. All the rooted trees we consider will be of the type defined in the previous subsection, i.e. $|X|$-regular rooted trees representing the words of X^* or $X^{[d]}$ for some finite set X. These automorphisms give permutations of the elements of either X^* or $X^{[d]}$ which preserve word length, prefixes, and the empty word ϵ.

Certain groups of finite tree automorphisms are important in the the theory of finite p-groups. When $|X|$ is a prime number, we may assume that $X = \{0, 1, \ldots, p-1\}$ and let γ denote the cyclic permutation $(0\ 1\ \ldots\ p-1)$. The groups $W_p(n)$ were studied by Kaloujnine [34] as the Sylow p-subgroups of the Symmetric group on p^n letters. These groups correspond to *locally cyclic* groups of tree automorphisms, denoted $\mathrm{Aut}_p(X^{[d]})$ i.e. groups where the action of the group element on the children of each vertex is as a cyclic permutation. The structure of these groups plays an important role in classification problems related to finite p-groups (see [39, Chapters 3 and 4]).

Infinite groups of tree automorphisms are also an important topic in contemporary group theory. Interest in groups of tree automorphisms has been driven by the discovery of intriguing examples, leading to the development of the general theory of self-similar groups of tree automorphisms. One such example is the first Grigorchuk group, introduced by Grigorchuk as a solution to the Burnside Problem on infinite torsion groups. The first Grigorchuk group has a fascinating structure and has been used as a solution to many open problems in group theory. The interested reader may consult [21, Chapter 8] and [28] for a more thorough overview of the first Grigorchuk group. The *Gupta-Sidki p-groups*, also introduced as a solution to the

Burnside problem [31], provided other interesting examples and additional impetus for the study of groups of rooted tree automorphisms. The monograph [44] provides an overview of groups of automorphisms of infinite rooted trees.

Several classes of groups of tree automorphisms have been introduced and studied, many of them based on or motivated by the first Grigorchuk group and the Gupta-Sidki p-group. One particularly interesting class is that of *self-similar* groups, which we now define. By the previously-discussed similarity of the infinite tree X^*, an automorphism g of X^* induces an automorphism of the tree $g(w)X^*$. For $g \in \mathrm{Aut}(X^*)$ and $w \in X^*$, we define the *section* g_w as the unique element of $\mathrm{Aut}(X^*)$ such that $g(wv) = g(w)g_w(v)$ for all $v \in X^*$. A subgroup G of $\mathrm{Aut}(X^*)$ is *self-similar* if for any $g \in G$ and $w \in X^*$, the section $g_w \in G$ as well. The group $\mathrm{Aut}(X^*)$ can be viewed as the wreath product $\mathrm{Sym}(X) \wr_X \mathrm{Aut}(X^)$, so an element $g \in \mathrm{Aut}(X^*)$ decomposes as $\sigma(g_{x_1}, g_{x_2}, \ldots, g_{x_n})$, where $\sigma \in \mathrm{Sym}(X)$, $n = |X|$, and each $g_{x_i} \in \mathrm{Aut}(X^*)$.

There are many interesting classes of self-similar groups which have been studied. We will discuss some of them in more detail when we survey known results on Hausdorff dimension and finitely constrained groups of tree automorphisms in Section 2.5.3.

2.1.4 Profinite groups

Many of the groups we are interested in are examples of *profinite groups*. The general theory of profinite groups as presented in a standard reference like [50] is more than we need, so we will give more specifialized definitions adapted to the cases we consider. What is important for our purposes is that profinite groups have a natural metric structure which agrees with the metric structure of the full shift in symbolic dynamics. We should note that the Ph.D. thesis of Siegenthaler [53] offers a very thorough and insightful discussion of the profinite theory of groups acting on infinite rooted trees.

Let $\{G(i)\}_{i=1}^{\infty}$ a collection of finite groups such that for each $n \geq 1$, there is a surjective homomorphism $\theta_n : G(n+1) \to G(n)$. We let π_0 be the trivial homomorphism with domain $G(1)$ and call the collection $\{(G(i), \pi_i\}_{i=1}^{\infty}$ a *projective system of finite groups.* Let

$$\mathcal{G} = \prod_{i=1}^{\infty} G(i)$$

be the infinite direct product group. Define the *inverse limit* of the projective system (G_i, π_i) to be the subgroup of $\mathcal{G}$ defined as

$$G = \{g \in \mathcal{G} \mid \theta_n(g_{(n+1)}) = g_{(n)}\} \text{ for all } n \geq 1.$$

Definition 2.1.3. A *profinite group* G is the inverse limit of a projective system $\{G(i), \pi_i\}_{i=1}^{\infty}$ of finite groups.

A profinite group is a metrizable compact Hausdorff space, and each projection map $\pi_i : G \to G(i)$ is a group homomorphism. The standard metric for the inverse limit of the projective system $(G(i), \theta_i)$ is given by $d(g, h) = 0$ if $g = h$ and for $g \neq h$,

$$d(g, h) = \frac{1}{|G(i)|},$$

where i is the least value such that $g_{(i)} \neq h_{(i)}$. Setting $G_i = \ker \pi_i$, we see that $G_1 \supseteq G_2 \supseteq G_3 \ldots$ is a descending sequence of normal subgroups of G.

Definition 2.1.4. Let G be a profinite group. A subgroup $H \leq G$ is *topologically finitely generated* if H is the topological closure of a finitely generated subgroup of G.

When X is a finite set, the infinite tree automorphism group $\mathrm{Aut}(X^*)$ is the inverse limit of the projective system of finite groups $\{\mathrm{Aut}(X^{[n]}), \theta_n\}$ where θ_n is the natural projection map from $\mathrm{Aut}(X^{[n+1]}$ to $\mathrm{Aut}(X^{[n]})$. Again, the case when $|X| = p$ for some prime number p, we write $\mathrm{Aut}_p(X^*)$ for the inverse limit of the groups $W_p(n)$, and we call $\mathrm{Aut}_p(X^*)$ *the group of p-adic automorphisms.* Our purpose in introducing profinite groups is to eventually generalize certain ideas of self-similar groups of tree automorphisms to that of more general structures arising in the theory of symbolic dynamics and computation on trees.

2.2 Computation and symbolic dynamics

In this section we review the necessary background for computation and symbolic dynamics for general semigroups, and for free semigroups in particular. All of the material we present is known, and in general our presentation gives a synthesis of material which closely follows that of [20] and [18]. However, we will note a few places where our conventions differ from the usual ones.

Traditionally, symbolic dynamics has dealt with shifts over the abelian semigroups $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Z}^t$ for $t > 1$. Such shifts are used both in applications to coding theory and as discrete models for studying dynamical systems in general. The textbooks by Kitchens [37] and Marcus and Lind [40] offer very readable introductions to this classical viewpoint.

2.2.1 Symbolic dynamics on semigroups

A *dynamical system* is a pair (X, T), where X is a compact space and T is a semi-group acting on X by continuous transformations.

Let T be a finitely generated semigroup and A be a finite alphabet. The *full shift (over T with alphabet A)* is A^T, the set of all maps from T to A. The full shift is a compact Hausdorff space which is homeomorphic to a Cantor set if $|A| > 1$.

Elements of the full shift are called *configurations*. The value of a configuration x at a point t is called the *label of x at t* and is denoted $x_{(t)}$.

The *shift action* of T on A^T is a continuous right semigroup action given by $[\rho_s(x)]_{(t)} = x_{(st)}$ for all $s, t \in T, x \in X$. With this action, the pair (A^T, T) is a dynamical system.

Remark 2.2.1. We have defined the shift action of T on A^T as a right action. It is common to define a *left shift action* λ of T on A^T given by $[\lambda_t(x)]_s = x_{(st)}$. This distinction is meaningless in the classical case, when the underlying semigroup is abelian, but it is very important here. Consider the following example. Let X be a finite set and A be a finite alphabet. Let $w, v \in X^*$ and f be a configuration of A^{X^*}. The value of $[\lambda_w(f)]_{(v)}$ is the value of f at vw, while the value of $[\rho_w(f)]_{(v)}$ is $f_{(wv)}$. Thus all of the labels in the right-shifted configuration come from the subtree wX^*.

Henceforth, when we refer to the shift action, we will always mean the right shift action. We also use the notation x_s for $\rho_s(x)$. Note that with our notation, $(x_s)_{(t)} = x_{(st)}$. A subset $X \subseteq A^T$ is *shift-invariant* if $\rho_t(X) \subseteq X$ for all $t \in T$. A *subshift* is a closed, shift-invariant subset of A^T.

A *pattern* is a map $p : \Omega \to A$ for some finite $\Omega \subseteq T$. We say a pattern p *appears in* a configuration x if there exists $t \in T$ such that the restriction of x_t to Ω is equal to p. If F is a set of patterns, we define

$$\mathcal{X}_F = \{x \in A^T \ \mid \ \text{if } p \text{ appears in } x, \text{ then } p \in F\},$$

and say that F is the *set of allowed patterns* for $\mathcal{X}_F$. It is also possible - and in fact more common traditionally - to define shifts in terms of *forbidden* patterns. Allowed patterns are more natural for our purposes, but it is not difficult to translate between the two equivalent notions. It is a well-known fact that *any* subshift can be defined in terms of allowed (or forbidden) patterns.

Proposition 2.2.2. *Let T be a semigroup and A be a finite alphabet. A set $\mathcal{Y} \subseteq A^T$ is a subshift if and only if there exists a set F of patterns such that $\mathcal{X}_F = \mathcal{Y}$*

Definition 2.2.3. Let A be a finite alphabet, let T be a semi-group, and let $\mathcal{Y}$ be a subshift of A^T. If $\mathcal{Y} = \mathcal{X}_F$ for some *finite* set of patterns, we say that $\mathcal{Y}$ is a *shift of finite type.*

Now that we know that we have defined shift spaces as the objects of study, it is natural to consider the maps between them. These are given by *cellular automata*.

Definition 2.2.4. Given a semigroup T and two finite alphabets A and B, a map $\tau : A^T \to B^T$ is called a *cellular automaton* if there exists a finite subset $M \subset T$ and a map $\mu : A^M \to B$ such that

$$\tau(f)_{(w)} = \mu((f_w)_{|M}).$$

The set M is called the memory set for τ, and μ is called the *local defining map* for τ.

Definition 2.2.5. Let T be a semi-group and let A and B be finite alphabets. A subset $\mathcal{Y} \subset B^T$ is *sofic* if there exists a cellular automaton τ and a shift of finite type $\mathcal{X} \subseteq A^T$ such that $\mathcal{Y} = \tau(\mathcal{X})$.

It is clear that any shift of finite type $\mathcal{X} \subseteq A^T$ is sofic, since we can simply take the cellular automaton τ in the definition to be the identity map on $\mathcal{X}$.

Most of our attention in this work will be paid to the case when the semigroup T is equal to the free semigroup X^* for some finite set X, We consider subshifts of the full tree shift A^{X^*} for some finite set X and finite alphabet A. In this case, configurations correspond to infinite trees whose vertices are labeled with elements of A. The shift A^{X^*} is a metric space with distance d defined as follows. If $f, g \in A^{X^*}$ and $f = g$, define $d(f, g) = 0$. If $f \neq g$, then there exists some $w \in X^*$ such that $f_{w)} \neq g_{(w)}$, and we define

$$d(f, g) = \frac{1}{|X^{[n]}|},$$

where n is taken to be the length of the shortest word w such that $f_{(w)} \neq g_{(w)}$.

2.2.2 Symbolic dynamics and profinite groups

In the case that the alphabet A is a finite group, it is natural to consider group structures on the full shift A^T. Although there are different ways to do this, the most obvious group structure is the direct product, with group operation given componentwise, i.e. $(gh)_{(t)} = g_{(t)}h_{(t)}$ for all $g, h \in A^T$. Kitchens [36] studied *group shifts* defined over $\mathbb{Z}$, i.e. subshifts which are also subgroups of $A^{\mathbb{Z}}$, where A is a finite group. Here we review his results and give a straightforward generalization from $\mathbb{Z}$ to the case of an arbitrary semigroup.

Some additional background is necessary. Let (X, T) and (Y, T) be dynamical systems. A *conjugacy between* X *and* Y is a homeomorphism $\pi : X \to Y$ such that $\pi(x_t) = [\pi(x)]_t$ for all $x \in X$.

If G is a topological group and (G, T) is a dynamical system, we say that $U \subseteq G$ *is expansively open* if U is an open set and for any $x, y \in G$, there exists $t \in T$ such that $t(x) \notin t(y)U$. We say (G, T) is *expansive* if there exists an expansively open set $U \subseteq G$.

We will also utilize the following standard fact about topological groups (just as Kitchens' original proof does).

Theorem 2.2.6. *Let G be a zero-dimensional topological group. If U is an open subset of G which contains e_G, then U contains an open normal subgroup.*

Kitchens proves the following theorem in the case that T is an infinite cyclic subgroup. Our proof follows his and relies on Theorem 2.2.6. It is also related to [20, Theorem 2.6].

Theorem 2.2.7 (Kitchens,1987)**.** *Let G be a profinite group and let T be a semigroup with a right action by expansive endomorphisms of G. The dynamical system (G, T) is conjugate to a subshift of A^T, where A is a finite group.*

Proof. Since T is expansive, there exists an open subset U with $e_G \in U$ such that for any $x, y \in G$, there exists $t \in T$ with $t(x) \notin t(y)U$. By Theorem 2.2.6, G has an open normal subgroup N such that $N \subseteq U$, and obviously N is expansively open as well. Note that N has finite index, since the cosets of N form an open cover of G and G is compact. Let $A = G/N$. Define $\pi : G \to A^T$ by $[\pi(g)]_{(t)} = (g^t)N$. Composition of π with each of the projection maps $A^T \to A$ is equal to the composition of translation by t and projection onto G/N, each of which is continuous. Thus π is continuous. It follows from the expansiveness of (G, T) that π is injective, and thus π is a homeomorphism onto its image. Also, π is T-equivariant since for $t, s \in T$ and $g \in G$,

$$\begin{aligned}
[\pi(g)_s]_{(t)} &= [\pi(g)]_{(st)} \\
&= (g^{st})N \\
&= ((g^s)^t)N \\
&= [\pi(g^s)]_{(t)}
\end{aligned}$$

$\square$

In the same work, Kitchens also proved the following theorem characterizing group shifts over $\mathbb{Z}$.

Theorem 2.2.8 (Kitchens,1987)**.** *Let A be a finite group and let G be a subgroup of the direct product group $A^{\mathbb{Z}}$ such that G is also a shift over $\mathbb{Z}$. Then G is a shift of finite type.*

This result is not true for groups which are shifts over arbtirary semigroups. However, it does extend to $\mathbb{N}$, which corresponds to the rooted tree X^* when $|X| = 1$. We will provide proofs of these facts in Section 4.

2.2.3 Symbolic dynamics and groups of tree automorphisms

As observed by Grigorchuk in [28, Section 7], each automorphism of the infinite rooted tree X^* corresponds to a *portrait*, i.e. a labeling of the vertices of X^* by elements of the finite alphabet $\mathrm{Sym}(X)$. Using the homomorphism $\pi_1 : \mathrm{Aut}(X^*) \to \mathrm{Sym}(X)$ and the section map in $\mathrm{Aut}(X^*)$, we define the *portrait map* $\alpha : \mathrm{Aut}(X^*) \to (\mathrm{Sym}(X))^{X^*}$ to be $[\alpha(g)]_{(w)} = \pi_1(g_w)$.

The portrait map gives a correspondence between the elements in the group $\mathrm{Aut}(X^*)$ and the portraits of the full tree shift A^{X^*}. Under this correspondence, the portraits of a topologically closed, self-similar subgroup of $\mathrm{Aut}(X^*)$ form a tree subshifts, and elements of $\mathrm{Aut}(X^{[d]})$ correspond to patterns of size d.

The portrait makes it possible to easily visualize the action of a tree automorphism. Moreover, Grigorchuk noted in [28, Section 7] that the closure of the first Grigorchuk group can be defined by a finite set of allowed patterns of size 4 corresponding to a subgroup of $\mathrm{Aut}(X^{[4]})$ (for $|X| = 2$). We call such groups *finitely constrained groups of tree automorphisms* (Grigorchuk used the term *groups of finite type*). A more precise definition and deeper discussion about these groups will be given later in this section.

Finite state automata are an important construct in computer science and in the theory of self-similar groups. Many of the interesting constructions of self-similar groups are given by groups generated by finite-state automata. Groups generated by a finite state automata have been studied for at least five decades, with much of the early development occurring in the former Soviet Union. The interested reader should consult the Introduction to [14] for more information. The definition we give is specialized to our purposes.

Definition 2.2.9. Let X be a finite set. A *finite state automaton* is a finite, self-similar subset of $\mathrm{Aut}(X^*)$.

2.3 Hausdorff dimension

Hausdorff dimension and box-counting dimension are often associated to fractal geometry, but they can be defined for any metric space. Our presentation of these concepts is standard (though minimal). Additional background can be found in the textbooks by Falconer [23] and Edgar [22].

Let (X, d) be a metric space and Y be a subset of X. An *δ-cover of Y* is a countable collection of subsets $\{U_i\}_{i=1}^{\infty}$ such that $\mathrm{diam}(U_i) \leq \delta$ for all i and $Y \subseteq \bigcup_{i=1}^{\infty} U_i$. For $r \geq 0$, we define $\mathcal{H}_{\epsilon}^{r}(X)$ to be the infimum over all ϵ-covers of the quantity $\sum_{i=1}^{\infty}(\mathrm{diam}(U_i))^r$, and we define *the r-dimensional Hausdorff measure* of Y as

$$\mathcal{H}^r(Y) = \lim_{\delta \to 0} \mathcal{H}_{\delta}^{r}(Y).$$

Finally, the *Hausdorff dimension* of Y, written $\dim_{\mathrm{H}}(Y)$, is given as

$$\dim_{\mathrm{H}}(Y) = \sup\{r \geq 0 : \mathcal{H}^r(Y) = 0\}.$$

Another dimension used in fractal geometry is the *lower box-counting dimension.*

(There is also an upper box-counting dimension, which we will not need.) Let $N_\epsilon(Y)$ be the minimum of the cardinalities of all ϵ-covers of Y ($N_\epsilon(Y)$ is finite if and only if Y is bounded). The *lower box-counting dimension* of Y is

$$\underline{\dim_{\mathrm{B}}}(Y) = \liminf_{\epsilon \to 0} \frac{\log N_\epsilon(Y)}{-\log \delta}.$$

Proposition 2.3.1 (see [23], p.46)**.** *Let (X, d) be a metric space. For any $Y \subseteq X$, $\dim_{\mathrm{H}}(Y) \leq \underline{\dim_{\mathrm{B}}}(Y)$.*

The values of these dimensions take for a metric space depend strongly on the metric, and topologically equivalent metrics on a space may lead to different Hausdorff or box-counting dimension functions.

2.4 Hausdorff dimension and entropy in symbolic dynamics

Entropy is an important notion for dynamical systems in general and shift spaces in particular. If A is a finite alphabet and $\mathcal{X}$ is a subshift of the full shift $A^{\mathbb{N}}$, the *entropy* of $\mathcal{X}$ is given by

$$\mathrm{ent}(\mathcal{X}) = \limsup_{n \to \infty} \frac{\log |\mathcal{B}_n(\mathcal{X})|}{n}.$$

Furstenburg [25] showed that if A is a finite alphabet, then for a subshift $\mathcal{X} \subseteq A^{\mathbb{N}}$,

$$\dim_{\mathrm{H}}(\mathcal{X}) = \underline{\dim_{\mathrm{B}}}(\mathcal{X}) = \frac{1}{\log |A|} \, \mathrm{ent}(\mathcal{X}).$$

A similar result was proven for shifts over any finitely generated, free abelian semigroup or group by Stephens [55].

2.4.1 Hausdorff dimension for profinite groups

Abercrombie [1] initiated the general study of Hausdorff dimension in profinite groups (with respect to the profinite metric discussed in Section 2.1.4). He showed that if G is a profinite group and H is a closed subgroup of G, then

$$\dim_{\mathrm{H}}(H) \geq \liminf_{n\to\infty} \frac{\log[H : (H \cap G_n)]}{\log[G : G_n]}.$$

Barnea and Shalev [6] noted that in this situation,

$$\underline{\dim_{\mathrm{B}}}(H) = \liminf_{n\to\infty} \frac{\log[H : H \cap G_n]}{\log[G : G_n]},$$

and so applying Proposition 2.3.1, the following theorem holds.

Theorem 2.4.1 (Abercrombie, 1994; Barnea and Shalev, 1997)**.** *Let G be a profinite group with a filtration $\{G_n\}_{n=0}^{\infty}$ and let H be a closed subgroup of G. Then*

$$\dim_{\mathrm{H}}(H) = \underline{\dim_{\mathrm{B}}}(H) = \liminf_{n\to\infty} \frac{\log|H : H \cap G_n|}{\log|G : G_n|}$$

Barnea and Shalev [6] studied several aspects of the Hausdorff dimension of pro-p groups. They defined the *Hausdorff spectrum* of a pro-p group G as the set of all values of Hausdorff dimension for closed subgroups of G. They also showed that the Hausdorff spectrum of a p-adic analytic group consists of a finite set of rational numbers.

Several authors have examined aspects of the Hausdorff spectrum of the profinite group $\mathrm{Aut}_p(X^*)$. Grigorchuk [27] showed that the Hausdorff dimension of the closure of the first Grigorchuk group is $\frac{5}{8}$. Abért and Virág [2] studied the Hausdorff spectrum of the pro-p group $\mathrm{Aut}_p(X^*)$ and showed that for any $\lambda \in [0, 1]$, there exists a 3-

generated subgroup of $\mathrm{Aut}_p(X^*)$ whose closure has Hausdorff dimension equal to λ. Abért and Virág used probabilistic methods on trees to prove this result, so it does not give any explicit examples of Hausdorff dimension for subgroups of $\mathrm{Aut}_p(X^*)$.

The Hausdorff dimension has been explicitly calculated for several classes of groups of p-adic tree automorphisms. Many of these examples are regular branch groups, and the calculations often make use of the group's branching structure. We will discuss Hausdorff dimension and finitely constrained groups of p-adic tree automorphisms more later in this Section.

2.5 Self-similar and finitely constrained groups

The portrait map discussed in Subsection 2.2.3 associates to each tree automorphism g a labeled tree which encodes the action of g. These labeled trees naturally correspond to elements of a full shift. In particular, the portraits of the group $\mathrm{Aut}(X^*)$ correspond to the full shift $(\mathrm{Sym}(X))^{X^*}$, and the group of p-adic tree automorphisms corresponds to the full shift $(C_p)^{X^*}$. In the same way, an automorphism of the finite tree $X^{[d]}$ corresponds to a pattern with domain $X^{(d)}$. These observations were the key to Grigorchuk's definition of *groups of finite type* (which we call *finitely constrained groups*).

From the perspective of symbolic dynamics, the requirement that the finite alphabet A in the full tree shift A^{X^*} be a subgroup of $\mathrm{Sym}(X)$ is needlessly restrictive. Thus, in this section, we give a definition of finitely constrained group which generalizes the definition in the case of tree automorphisms. The main difference here is that we do not require the action of our group on the tree to be faithful. For any finite set X and finite group A which acts on X, we explain how to give a natural group structure to the full shift A^{X^*}. This construction is related to both the study of group shifts over $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Z}^2$, and to that of closed, self-similar subgroups of tree automorphisms.

2.5.1 The general definition of self-similar groups

Here we give a general definition of *self-similar group* as a profinite group whose elements will be identified with those of a full tree shift. We will write out the details of the construction carefully, although it it will be natural to those familiar with self-similar groups of tree automorphisms.

Recall that if A is a finite group and X is a left A-set with left action ϕ, we write $W_{(X,\phi)}(A,n)$ for the n-fold iterated wreath product of A over X. We use the recursion inherent in the construction to define *finite sections* and *labels* for elements of $W_{X,\phi}(A,n)$ and words of length less than n. We take the base case to be $n = 1$, in which case $g_\epsilon = g$.

Now assume that g_w is defined whenever $g \in W_A(n-1)$ and $w \in X^{(n-1)}$. An element $g \in W_{(\phi,X)}(A,n)$ can be written as $(g_{(\epsilon)}, (g_x)_{x\in X})$, where $g_{(\epsilon)} \in A$ and each $g_x \in W(A, n-1)$. For a word $v \in X^{(n)}$, we write $v = xv'$ for $v' \in X^{(n-1)}$, and we define $g_v = (g_x)_{v'}$.

There is an obvious homomorphism $\alpha : W_{(\phi,X)}(A,n) \to A$ given by $\alpha(g) = g_{(\epsilon)}$. We call $\alpha(g)$ the *root portrait of* g. For $w \in A^{(n)}$, we define the *label of* g *at* w to be $g_{(w)} = \alpha(g_w)$.

Now, the use of the term *pattern group* is justified, since the map from $W_A(n)$ to $A^{X^{(n)}}$ which takes g to the pattern $(g_{(w)})_{w\in X^{(n)}}$ with domain $X^{(n)}$. In general, we exploit this bijection and freely identify the elements of $W_{(X,\phi)}(A,n)$ with their corresponding patterns in $A^{X^{(n)}}$, though there are times where some care is needed since this construction depends on the initial choice of the left action ϕ. For the moment, we continue to assume that X and ϕ are understood and will continue with writing $W_A(n)$ for the pattern group over A with pattern size n.

For $n \geq 2$, there is a surjective homomorphism

$$\theta_n : W_A(n+1) \to W_A(n),$$

given by restriction of the pattern, i.e. $[\theta_n(g)]_{(w)} = g_{(w)}$ for all w with $|w| < n$. Thus we have a projective system of finite groups $\{(W_A(n), \theta_n)\}_{n=1}^{\infty}$, and we define the *full tree shift group* $\mathcal{F}(W, A, \phi)$ to be the inverse limit of this system. It is not hard to see that for each $g \in \mathcal{F}(W, A, \phi)$, there is exactly one $f \in A^{X^*}$ such that $g_{(n)}$ corresponds to $f_{|X^{(n)}}$ for all $n \geq 1$. We call this f the *portrait* of g, and henceforth we will make no distinction between g and its portrait, so that the *label of g at w* is defined to be the label of its portrait at w, the section g_w is the image of its portrait under the shift ρ_w, etc.

Note in particular that the profinite metric on $\mathcal{F}(A, X, \phi)$ is the same as the metric on A^{X^*} as a tree shift. Again, if there is no risk of confusion, we may identify the group $\mathcal{F}(A, X, \phi)$ with its set of portraits and refer to this group as A^{X^*}.

We record some basic properties of these self-similar groups, which are well-known for self-similar groups of tree automorphisms and not difficult to show in the more general case.

Lemma 2.5.1. *Let X be a finite set and A be a finite group acting on X via ϕ. Let $g, h \in F(A, X, \phi)$, $x \in X$, and and $u, v, w \in X^*$. Then the following hold.*

1. $(gh)_w = g_{h(w)} h_w$

2. $(gh)_{(w)} = g_{(h(w))} h_{(w)}$

3. $(g_u)_v = g_{uv}$

4. *If g and h are supported on disjoint subtrees wX^* and vX^*, then $gh = hg$.*

For $u \in X^*$ and a subgroup $G \leq A^{X^*}$, we define the stabilizer of u as

$$\mathrm{Stab}_G(u) = \{f \in A^{X^*} \ \mid \ f(u) = u\}.$$

and *the level n stabilizer* (for $n \geq 0$) is the subgroup defined as

$$Stab_G(n) = \{f \in A^{X^*} \ \mid \ f(s) = s \text{ for all } s \in X^n\} = \bigcap_{u \in X^n} \mathrm{Stab}_G(u).$$

The subgroup $\mathrm{Triv}(n)$ of A^{X^*} is given by

$$\mathrm{Triv}(n) = \{f \in A^{X^*} \ \mid \ f_{(u)} = e_A \text{ whenever } |u| < n\}.$$

For a subgroup $G \leq A^{X^*}$, we define

$$\mathrm{Triv}_G(n) = \mathrm{Triv}(n) \cap G.$$

For any $G \leq \mathcal{F}(A, X, \phi)$, $\mathrm{Triv}_G(n)$ is a normal subgroup of G, corresponding to the kernel of $\pi_n : G \to G(n)$ (given by the restriction of the map π_n defined on A^{X^*}). The groups $\mathrm{Triv}_G(n)$ and $\mathrm{Stab}_G(n)$ are the same if and only if the action of A on X is faithful.

We also need the notion of a *regular branch group*. Regular branch groups are a special class of branch groups, which are important in the study of self-similar groups of tree automorphisms – see [10] for an introduction and overview of branch groups.

Definition 2.5.2. A self-similar group H acting on a tree X^* with $|X| = n$ is a *regular branch group over* a group K if K is a finite index, normal subgroup of H such that whenever $k_1, \ldots, k_n \in K$, then $(k_1, k_2, \ldots, k_n) \in K$.

2.5.2 Finitely constrained groups

We can now give a general definition of finitely constrained groups.

Definition 2.5.3. A *finitely constrained group* is a self-similar group whose portraits form a tree shift of finite type.

Definition 2.5.4. Let X be a set and A be a finite group. A *pattern group of size d* is a subgroup of $W_A(d) = A^{X^{(d)}}$, $d \geq 1$. A pattern group P is an *essential pattern group* if for all $g \in P$ and $i = 0, 1$, there exists $h_i \in P$ such that $(h_i)_{(w)} = g_{(iw)}$ for all $w \in X^{(d-1)}$.

Remark 2.5.5. It is not hard to see that P is an essential pattern group with pattern size n if and only if there exists a self-similar group A such that $P = A(n)$. Indeed, the patterns of any size for any self-similar group will have the essential pattern property, while if P is an essential pattern group, the finitely constrained group G_P is a self-similar group such that $G_P(n) = P$.

Let us give a basic example of a finitely constrained group defined by allowed patterns, first given by Grigorchuk in [28, Section 7]. It is also discussed as Example 1 in [57] and Example 2.9 in [17]

Example 2.5.6 ("Monochrome Children")**.** Let P be the subgroup of $W_2(2)$ defined by

$$p \in P \Leftrightarrow h_{(0)} + h_{(1)} = 0.$$

This is an essential pattern group. The patterns of this group are given in Figure 2.5.2, with w labeled by $\circ$ if $g(wx) = g(w)x$, and w labeled by $\bullet$ otherwise. A binary tree automorphism g is in G_P if and only if $g_{(w0)} + g_{(w1)} = 0$ for all $w \in X^*$.

Figure 2.1: An illustration of the size two patterns of the *monochrome children* group, a finitely constrained group of binary tree automorphisms originally defined by Grigorchuk [28, page 174]

Remark 2.5.7. Let us illustrate one difference between classical symbolic dynamics on $\mathbb{N}$ or $\mathbb{Z}$ and that on X^* when $|X| > 2$. It is known that for a shift over $\mathbb{N}$ or $\mathbb{Z}$, Hausdorff dimension can only decrease under cellular automata (see [40, Proposition 4.1.9]. Let $X = \{0,1\}$. There is a two-to-one cellular automaton ϕ from G_P to $\mathrm{Aut}(X^*)$ given by $\phi(g) = g_{(\epsilon)} + g_{(0)}$. We will see in the next section (see Proposition 3.1.8) that the Hausdorff dimension of G_P is $\frac{1}{2}$, and we know that the Hausdorff dimension of $\mathrm{Aut}(X^*)$ is 1. Thus ϕ is a cellular automaton which increases the Hausdorff dimension of its image.

The next example shows that there exist closed, self-similar groups which are not finitely constrained. This example was suggested to us by Zoran Šunik.

Example 2.5.8. Let $X = \{0,1\}$, $A = \mathrm{Sym}(X) = \{\mathrm{id}, \sigma\}$, and let A act faithfully on X by permutations, so that the group $A^{X^*} = \mathrm{Aut}(X^*)$. Let $a = \sigma(1,a) \in \mathrm{Aut}(X^*)$. In terms of labels,

$$a_{(w)} = \begin{cases} \sigma, \text{ if } w = 1^n \text{ for some } n \\ \mathrm{id}, \text{ otherwise }. \end{cases} .$$

Any section of a is either the identity or a, so the group generated by $\{1, a\}$ is self-similar. This group $\mathcal{O}$ is often called the *odometer group*, as it "rolls over" any word consisting of all 1's. We claim that $\overline{\mathcal{O}}$ is not a finitely constrained group. This follows immediately from the fact that $\overline{\mathcal{O}}$ is abelian as the closure of an abelian group, any finitely constrained group is a regular branch group (Theorem 3.6) and it is known that a branch group of tree automorphisms must have trivial center [27, Theorem 2(c)]. However, we will present a different proof which relies only on the structure of the portraits of $\overline{\mathcal{O}}$.

It follows from induction that

$$a^{2n} = \mathrm{id}(a^n, a^n)$$

and

$$a^{2n+1} = \sigma(a^n, a^{n+1}).$$

Also, we note that $a^{2n}_{(\epsilon)} = \mathrm{id}$, while $a^{2n+1}_{(\epsilon)} = \sigma$. Finally, it is not hard to see that $a^{2^n} \in \mathrm{Triv}(n)$ for all $n \in \mathbb{N}$.

Since $\mathcal{O}$ is self-similar, the closure $\overline{\mathcal{O}}$ is a tree shift group. We will show that $\overline{\mathcal{O}}$ is not finitely constrained.

Suppose that $\overline{\mathcal{O}}$ is finitely constrained by some set F of allowed patterns of size $n+1$. Consider the element $g \in A^{X^*}$ with root portrait $g_\epsilon = \mathrm{id}$ and sections given by $g_0 = 1_G$, $g_1 = a^{2^{n+1}}$. Each pattern in g is a pattern which appears in a, so g must be in the shift space $X_{\mathcal{F}}$.

Since we assumed $X_F = \overline{\mathcal{O}}$, there must be a sequence of elements in $\mathcal{O}$ which converge to g. Since $g_\epsilon = \mathrm{id}$, this sequence must eventually consist of even powers of a, so

$$a^{2n_i} \to g.$$

Then $(a^{2n_i})_0 \to g_0$ and $(a^{2n_i})_1 \to g_1$. However, $a_0^{2n_i} = a_1^{2n_i}$, but $g_0 \neq g_1$. Therefore we have a contradiction, and $\overline{\mathcal{O}}$ is not finitely constrained.

Finitely constrained groups of tree automorphisms are characterized in the following theorem. The direction (i.) $\to$ (ii.) was proven by Grigorchuk [28, Proposition 7.5], while the direction (ii.) $\to$ (i.) was proven by Šunić [56, Theorem 3].

Theorem 2.5.9 (Grigorchuk, 2005; Šunić, 2007)**.** *Let G be a group of tree automorphisms of X^* and $s \geq 0$. The following are equivalent.*

(i) The group G is the closure of some self-similar, regular branch group H, branching over its level s stabilizer H_s.

(ii) The group G is a finitely constrained group defined by forbidden patterns of size $s+1$.

The analog of this theorem holds for the more general finitely constrained groups introduced in this section. We will prove this result in Section 4.

We record now some basic and useful facts about essential pattern groups and finitely constrained groups.

Proposition 2.5.10. *Let A be a finite group, X be a finite set, and let ϕ be a left action of A on X. Let $G = \mathcal{F}(A, X, \phi)$ be the full tree shift group of A over X induced by ϕ, and let d be the standard metric on G. Let $d > 1$, let P be an essential pattern subgroup of $G(d)$, and let G_P be the finitely constrained group defined by P. Then the following hold.*

(i.) Let $n \in \mathbb{N}$ and let $g, g' \in G$. Then $d(g, g') < \frac{1}{|G(n)|}$ if and only if $\pi_n(g) = \pi_n(g')$.

(ii.) If H is a subgroup of A^{X^*}, then $g \in \overline{H}$ if and only if $\pi_n(g) \in H(n)$ for all $n \in \mathbb{N}$.

(iii.) For any $g \in G$, $g \in G_P$ if and only if $\pi_n(g_w) \in P$ for all $w \in X^*$.

(iv.) If H is a self-similar subgroup of G, then $H \leq G_{H(n)}$ for all $n \in \mathbb{N}$.

(v.) If H is a self-similar subgroup of G and $m < n$, then $G_{H(m)} \geq G_{H(n)}$.

(vi.) If H is a self-similar subgroup of G, then $\overline{H} = \bigcap_{n\in\mathbb{N}} G_{H(n)}$.

Proof. (i.) This is clear from the definition of the standard metric on G.

(ii.) Let $g \in G$ and suppose $g \in \overline{H}$. For any $n \in \mathbb{N}$, there exists h_n such that $d(g, h_n) < \frac{1}{|G(n)|}$, and thus $\pi_n(g) = \pi_n(h_n) \in H(n)$. Thus $\pi_n(g) \in H(n)$ for all $n \in \mathbb{N}$. Now suppose $\pi_n(g) \in H(n)$ for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists h_n such that $\pi(g) = \pi(h_n)$. These h_n form a sequence which converges to g, and so $g \in \overline{H}$.

(iii.) This follows from the definition of G_P and the observation that $\pi_d(g_w)$ gives the pattern of size d which appears at $w \in g$.

(iv.) Let $h \in H$ and $n \in \mathbb{N}$. Since H is self-similar, $h_w \in H$ for all $w \in X^*$, so $\pi_n(h_w) \in H(n)$ for all $w \in X^*$. Thus $h \in G_{H(n)}$ by (iii.)

(v.) If $g \in G_{H(n)}$, then $\pi_n(g_w) \in H(n)$ for all $w \in X^*$, so $\pi_m(g_w) \in H(m)$ for all $w \in X^*$.

(vi.) First we show that $\overline{H} \subseteq \bigcap_{n\in\mathbb{N}} G_{H(n)}$. Each $G_{H(n)}$ is a closed set which contains H by (iv.), so $\overline{H} \subseteq G_{H(n)}$ for all n, and the result follows. For the other direction, suppose $g \in \bigcap_{n\in\mathbb{N}} G_{H(n)}$. Then $\pi_n(g) \in H(n)$ for all n, so $g \in \overline{H}$ by (ii.)

□

2.5.3 Hausdorff dimension and finitely constrained groups of tree automorphisms

In this section, we survey some known examples of self-similar groups of tree automorphisms. We particularly focus on groups whose closures are finitely constrained, or whose closures have had their Hausdorff dimension calculated. The list is intended to be as comprehensive as possible, without giving all details about the groups being discussed.

The Hausdorff dimension of a finitely constrained group of p-adic automorphisms defined by patterns of size d must be an element of the set

$$\{0, \frac{1}{p^{d-1}}, \ldots, 1 - \frac{1}{p^{d-1}}, 1\},$$

which follows from [7, Proposition 2.7] (it also follows independently from [56, Proposition 6]). The value 0 occurs only for finite groups, and the value 1 occurs only for the group of all p-adic automorphisms (which is finitely constrained by allowing all patterns). Note that these facts will follow also independently from Proposition 3.1.8.

As noted earlier, Grigorchuk [28, Section 7] first discussed the notion of finitely constrained groups of tree automorphisms and drew attention to the several questions concerning the Hausdorff dimension of self-similar groups. He had previously showed, using Theorem 2.4.1, that the closure of the first Grigorchuk group in $\mathrm{Aut}(X^*)$ has Hausdorff dimension $\frac{5}{8}$ [27].

Šunić [56] gave the first explicitly constructed (non-random) examples of finitely generated groups with Hausdorff dimension approaching 1. Each example is defined by a prime p and an integer m. He proved that these examples are finitely constrained (defined by patterns of size $m + 2$). For $p \neq 2$ and $d \geq 4$, the corresponding example of pattern size d has Hausdorff dimension equal to $1 - \frac{p}{p^{d-1}}$. For $p = 2$

and $d \geq 4$, the corresponding example of pattern size d has Hausdorff dimension equal to $1 - \frac{3}{2^{d-1}}$. More generally, Šunić gave a formula for the Hausdorff dimension of self-similar groups of p-adic tree automorphisms(see [57, Proposition 6]) and for finitely constrained groups of p-adic tree automorphisms (see [57, Theorem 4]).

Bartholdi and Nekrashevych studied groups generated by finite state automata whose structure is determined by either a word $v \in X^*$ or a pair of words $v, w \in X^*$. These groups include the iterated monodromy groups of quadratic polynomials. We denote such groups by $\mathcal{R}(v)$ and $\mathcal{R}(w, v)$. The Hausdorff dimension of these groups was calculated by Pink [46]. His work shows that for $d \geq 5$, there exist topologically finitely generated, finitely constrained groups of binary tree automorphisms with pattern size d and Hausdorff dimension $1 - \frac{2}{2^{d-1}}$.

Spinal groups are studied by Bartholdi and Šunić in [12]. Siegenthaler [52] gave a formula for the Hausdorff dimension of the closure of a spinal group and used it to produce specific examples of finitely generated groups having transcendental values of Hausdorff dimension. By embedding spinal groups of increasing Hausdorff dimension into a larger group, Siegenthaler also constucted a concrete example of a topologically finitely generated group with Hausdorff dimension equal to 1.

The *GGS (Grigorchuk-Gupta-Sidki) groups* (a term coined by Baumslag in [13, Chapter 2])) are groups of p-adic tree automorphisms defined by a vector from the vector space $(\mathbb{F}_p)^{p-1}$. Fernández-Alcober and Zugadi-Reizabal [24] calculated the Hausdorff dimension of the closures of all GGS-groups, based on properties of the vector used to define the group. They showed that any GGS-group with non-constant defining vector is a regular branch group over its level two stabilizer. Although they did not explicitly mention it, this result implies by Theorem 3.6 that each such group is a finitely constrained group defined by patterns of size 3. Siegenthaler and Zugadi-Reizabal [54] gave an explicit description of the defining patterns of the GGS-groups. They also made the observation that the portraits of these groups form an abelian group under the operation of componentwise addition. We will consider new examples of groups which satisfy this property in the next section.

Bondarenko and Samoilovych considered topological finite generation of finitely constrained groups, proving two theorems which make it possible to determine this property from some set of finite patterns which appear in the group. We will see these theorems in the next section. Using these theorems and the computer algebra system `GAP` [26],they showed that there are no topologically finitely generated, finitely constrained groups defined by patterns of size $d = 3$, while there are 32 such groups having pattern size $d = 4$ (including the closure of the first Grigorchuk group and the closure of the Iterated Monodromy Group of the polynomial $z^2 + i$). The Hausdorff dimension of these 32 examples is not discussed in that work, but it can be deduced from information which is given therein that each of these 32 groups has Hausdorff dimension $\frac{5}{8}$.

Thus for a given $d \geq 5$, there are only two values of the possible Hausdorff dimension known in the literature to occur for some finitely constrained, topologically finitely generated group of binary tree automorphisms with pattern size d. The connection between Hausdorff dimension and topological finite generation of finitely constrained groups will be the primary focus of the next section.

3. TOPOLOGICAL FINITE GENERATION AND HAUSDORFF DIMENSION OF FINITELY CONSTRAINED GROUPS *

This section investigates the structure of finitely constrained groups of p-adic tree automorphisms. In general, given a positive integer d, we seek information about the finitely constrained subgroups of p-adic tree automorphisms defined by patterns of size d. This will obviously depend on what we can say about the essential pattern groups with pattern size d. One goal is to then use this information to produce examples of topologically finitely generated self-similar groups with known Hausdorff dimension, since we will show that the Hausdorff dimension of a finitely constrained group can be easily calculated.

For the same reason, Hausdorff dimension serves as a natural parameter in the investigation of finitely constrained groups of pattern size d. The challenge in this approach is to understand whether or not a given finitely constrained group is topologically finitely generated, beginning only with a description of its patterns. This problem is inherently combinatorial and seems very difficult in general, but we are able to address certain cases. In particular, we can give some very definite results about the two largest possible values of Hausdorff dimension for finitely constrained groups of binary tree automorphisms with pattern size d.

We also seek specific examples of topologically finitely generated, finitely constrained groups, preferably ones with easily describable patterns. To this end, we define two specific types of essential pattern groups, which we call *full pattern groups* and *linearly constrained groups*. These classes are natural as first objects of study,

*Some of the results in this section are based on material from the article "Finitely Constrained Groups of Maximal Hausdorff Dimension", by Andrew Penland and Zoran Šunić, 2015, To Appear In *Journal of Australian Mathematical Society*, Copyright 2015 by Australian Mathematical Publishing Association Incorporated.

since all essential pattern groups corresponding to the examples of finitely constrained groups in the literature fall into at least one of these two classes.

3.1 Topological finite generation, Hausdorff dimension, and patterns for finitely constrained groups of p-adic tree automorphisms

In this section we introduce some notation and review some known results related to finitely constrained groups of p-adic tree automorphisms.

In the case of p-adic tree automorphisms, it is natural to identify the cyclic group C_p with $\mathbb{F}_p$, the finite field with p elements. Under this identification, the portraits of a tree automorphism give a function $X^* \to \mathbb{F}_p$. Also, a pattern of size d is a function $X^{(d)} \to \mathbb{F}_p$, which can be expressed as a polynomial whose variables are the (evaluation at the labels of) words in $X^{(d)}$. Siegenthaler [53] used this fact to study closed subgroups of $\mathrm{Aut}(X^*)$ and $\mathrm{Aut}_p(X^*)$ via methods from algebraic geometry. In particular, he defined the *branching ideal* of a self-similar subgroup, and used the functions in this branching ideal to give both criteria for topological finite generation (see Sections 1.8 and 2.2, especially Theorem 1.8.6 and Theorem 2.2.9 of [53]) and a formula for Hausdorff dimension (see Theorem 5.3.5 and the following discussion in [53]). Some of what we say in this work could also be said in this language, but we avoid that perspective in order to emphasize the combinatorial properties of the finite patterns, which Siegenthaler does not discuss.

If A is a self-similar group of p-adic tree automorphisms, then there is some first level k such that $A(k) \neq W_p(k)$. This is the first time that the patterns of A are interesting to us, as this is the first time the patterns of A define a finitely constrained group distinct from $\mathrm{Aut}_p(X^*)$. The following class of pattern groups correspond to this situation.

Definition 3.1.1. A subgroup $P \leq W_p(d)$ is a *full pattern group of pattern size d* if

$|P(d-1)| = |W_p(d-1)|$, i.e. if for any pattern $q \in W_p(d-1)$, there exists a $p \in P$ such that $\pi_{d-1}(p) = q$.

If the pattern size d is understood, we will simply use the term *full pattern group* to refer to a full pattern group of pattern size d. Note that a full pattern group is obviously an essential pattern group.

Now we introduce some important background and tools to use in this investigation. Bondarenko and Samoilovych [15] studied finitely constrained groups of tree automorphisms, and our investigation will utilize some of their results, as well as straightforward corollaries.

Bondarenko and Samoilovych provided the following two criteria which are useful in determining topological finite generation of a finitely constrained group. Recall that if G is a group, $[G, G]$ denotes the *commutator subgroup* of G, which is generated by all elements of the form $g^{-1}h^{-1}gh$ for $g, h \in G$.

Theorem 3.1.2 (Theorem 3, [15]). *Let X be a finite set and let G_P be a level-transitive, finitely constrained subgroup of* $\mathrm{Aut}(X^*)$ *defined by an essential pattern group P of pattern size d. Then G_P is topologically finitely generated if and only if there exists an n such that* $[\mathrm{Triv}_{G_P(n)}(d-1), \mathrm{Triv}_{G_P(n)}(d-1)]$ *contains* $\mathrm{Triv}_{G_P(n)}(n-1)$.

Proposition 3.1.3 (Proposition 4, [15]). *Let X be a finite set and let G_P be a finitely constrained subgroup of* $\mathrm{Aut}(X^*)$ *defined by an essential pattern subgroup P of pattern size d. If there exists an $n \geq d$ such that $[G_P(n), G_P(n)]$ does not contain* $\mathrm{Triv}_{G_P(n)}(n-1)$*, then G_P is not topologically finitely generated.*

Recall that the *Frattini subgroup of a group H*, denoted $\Phi(H)$, is the intersection of all maximal subgroups of H. If H is a p-group, then it is well known that $\Phi(H)$ is the group generated by commutators and p'th powers in H, and that $\Phi(H)$ is the smallest normal subgroup such that the quotient $H/\Phi(H)$ is elementary abelian p-group. (See Section 13 in [33]). We note the following corollary of Proposition 3.1.3, which was also essentially given as a result by Siegenthaler [53, Theorem 2.2.9].

Corollary 3.1.4. *Let p be a prime number, let X be a finite set with $|X| = p$, and let G_P be a finitely constrained subgroup of $\mathrm{Aut}_p(X^*)$ defined by an essential pattern subgroup P of pattern size d. If there exists an $n \geq d$ and a homomorphism $\phi : G_P(n) \to C_p$ such that $\mathrm{Triv}_{G_P(n)}(n-1)$ is not contained in the kernel of ϕ, then G_P is not topologically finitely generated.*

Proof. If there exists such an n and such a ϕ, then $\ker\phi$ is a maximal subgroup of $G_P(n)$ which does not contain $\mathrm{Triv}_{G_P(n)}(n-1)$. It follows that $\Phi(G_P(n))$ does not contain $\mathrm{Triv}_{G_P(n)}(n-1)$, and thus $[G_P(n), G_P(n)]$ does not contain $\mathrm{Triv}_{G_P(n)}(n-1)$. Applying Proposition 3.1.3, it follows that G_P is not topologically finitely generated. □

A homomorphism ϕ as described in Corollary 3.1.4 can be recognized by the fact that there are two elements of $\mathrm{Triv}_{G_P(n)}(n-1)$ for which ϕ takes different values. We also observe some other weaker, simpler corollaries of Proposition 3.1.3 which apply to certain situations.

Corollary 3.1.5. *Let p be a prime number and let P be an essential pattern group contained in $W_p(d)$. If the extension*

$$1 \to \mathrm{Triv}_P(d-1) \to P \to P/P_{d-1} \to 1$$

splits, then the finitely constrained group G_P is not topologically finitely generated.

Proof. Since the extension splits, there is a subgroup $K \leq P$ such that $K \cap \mathrm{Triv}_P(d-1)$ is trivial and $K\,\mathrm{Triv}_P(d-1) = P$. Let M be a maximal subgroup of P such that

$K \leq M$. Note that it is not possible for M to contain $\mathrm{Triv}_P(d-1)$, since then we would have that M contains $K \,\mathrm{Triv}_P(d-1) = P$. Then $[P : M] = p$, and the kernel of the homomorphism $\phi : P \to P/M \cong C_p$ does not contain $\mathrm{Triv}_P(d-1)$. Applying Corollary 3.1.4, we conclude that G_P is not topologically finitely generated. $\square$

The following corollary is useful in telling us where *not* to look for topologically finitely generated, finitely constrained groups.

Corollary 3.1.6. *Let P be a full essential pattern subgroup of $W_p(d)$. If P has a maximal subgroup Q which is also a full essential pattern subgroup, then G_P is not topologically finitely generated.*

Proof. Since Q is maximal, $P/Q \cong C_p$, and since Q is a full group, it follows that $\mathrm{Triv}_Q(d-1)$ is a proper subgroup of $\mathrm{Triv}_P(d-1)$. Thus the homomorphism $\phi : P \to C_p$ which has Q as kernel is a map from $P = G_P(d)$ to C_2 which is not constant on cosets of $\mathrm{Triv}_P(d-1)$, so G_P is not topologically finitely generated. $\square$

The following formula for the number of size n patterns of a finitely constrained group defined by patterns of size d is also due to Bondarenko and Samoilovych. Its proof relies on recursively counting the size n patterns, using the fact that a finitely constrained group is a regular branch group.

Proposition 3.1.7 (see Proposition 1, [15])**.** *Let X be a finite set and let G_P be a finitely constrained subgroup of $\mathrm{Aut}(X^*)$ defined by an essential pattern subgroup P with pattern size d. For $n \geq d$, $|G_P(n)| = |P||\,\mathrm{Triv}_P(d-1)|^{|X|+|X|^2+\ldots+|X|^{n-d}}$.*

We now show how Proposition 3.1.7 simplifies the task of calculating the Hausdorff dimension of finitely constrained groups.

Proposition 3.1.8. *Let p be a prime number, let $X = \{0, 1, \ldots, p-1\}$. If P is an essential pattern subgroup of $W_p(n)$, then $\dim_{\mathrm{H}}(G_P) = \frac{\log_p |P_{d-1}|}{p^{d-1}}$.*

Proof. Since G_P is a profinite group, we know that the Hausdorff dimension of G_P is given by

$$\dim_{\mathrm{H}}(G_P) = \liminf_{n\to\infty} \frac{\log_p |G_P(n)|}{\log_p |W_p(n)|}.$$

Since $|W_p(n)| = p^{\frac{p^n-1}{p-1}}$, this becomes

$$\begin{aligned}\dim_{\mathrm{H}}(G_P) &= \liminf_{n\to\infty} \frac{\log_p |G_P(n)|}{\log_p(p^{\frac{p^n-1}{p-1}})} \\ &= \liminf_{n\to\infty} \log_p |G_P(n)| \frac{p-1}{p^n-1}.\end{aligned}$$

We know from Proposition 3.1.7 that $|G_P(n)| = |P||P_{d-1}|^{p+\ldots+p^{n-d}}$. Substituting this in the previous expression, we can calculate that

$$\begin{aligned}\dim_{\mathrm{H}}(G_P) &= \liminf_{n\to\infty} \log_p |P||P_{d-1}|^{p+\ldots+p^{n-d}} \frac{p-1}{p^n-1} \\ &= \liminf_{n\to\infty} \left(\log_p |P| + (p+\ldots+p^{n-d}) \log_p |P_{d-1}|\right) \frac{p-1}{p^n-1} \\ &= \liminf_{n\to\infty} \frac{\log_p |P|}{p^n-1} + \frac{(p-1)(p+\ldots+p^{n-d}) \log_p |P_{d-1}|}{p^n-1} \\ &= \liminf_{n\to\infty} \frac{\log_p |P|}{p^n-1} + \frac{((p^{n-d+1}+\ldots+p) - (p^{n-d}+\ldots+1) \log_p(|P_{d-1}|)}{p^n-1}\end{aligned}$$

We can cancel the telescoping sum in the numerator of the second term to yield $p^{n+d+1}-1$, and the terms whose numerators involve constants $\log_p |P|$ go to zero as $n \to \infty$, so this becomes

$$\begin{aligned}\dim_{\mathrm{H}}(G_P) &= \liminf_{n\to\infty} \frac{p^{n-d+1}\log_p |P_{d-1}|}{p^n - 1} \\ &= \liminf_{n\to\infty} \frac{\log_p |P_{d-1}|}{p^{d-1}} \\ &= \frac{\log_p |P_{d-1}|}{p^{d-1}}\end{aligned}$$

□

Before proving the main results of this section, we need some additional background about certain essential pattern groups of p-adic tree automorphisms. For the remainder of this section we reserve the letter G for the group $\mathrm{Aut}_p(X^*)$ of all p-adic tree automorphisms, and we write $G(d)$ for $\mathrm{Aut}_p(X^{[d]})$, the d-fold iterated wreath product of C_p. As usual, if A is either a group of infinite tree automorphisms or an essential pattern group of finite tree automorphisms, we write $A(n)$ for the patterns of size n which appear in A.

We recall the following well-known facts about $G(d)$. Proofs may be found in [39, Section 3].

Proposition 3.1.9. *Let $G(d)$ be the group of p-adic tree automorphisms of depth d.*

(i.) $G(d)$ is generated by the set $\{a_0, a_1, a_2, \ldots, a_{d-1}\}$, where a_i is the element with $\alpha_{(w)}(a_i) = \gamma$ if $w = 0^i$, and trivial otherwise.

(ii.) With respect to this generating set, $G(d)$ has a presentation $\langle \{a_i\}_{i=0}^{d-1} \mid \{a_i^p\}_{i=0}^{d-1}, \{[a_i^{a_j}, a_k]\}_{0\le j,k<i\le(d-1)} \rangle$.

(iii.) The Frattini subgroup of $G(d)$ is equal to the commutator of $G(d)$, and the abelianization of $G(d)$ is given by the surjective homomorphism $G(d) \to \prod_{i=0}^{d-1} C_p$, $g \mapsto \left[\sum_{w\in X^i} g_{(w)}\right]_{i=0}^{d-1}$.

(iv.) $\mathrm{Triv}_{G(d)}(d-1)$*, the level* $(d-1)$ *stabilizer of* $G(d)$*, is naturally identified with* $\prod_{w \in X^{d-1}} C_p$*, a* p^{d-1}*-dimensional vector space over* $\mathbb{F}_p$*.*

(v.) For any *pattern group* $H \leq G(d)$*, the level* $(d-1)$ *stabilizer* $\mathrm{Triv}_H(d-1)$ *is a normal elementary abelian subgroup.*

Remark 3.1.10. Regarding item (iv.): The notion of *uniseriality* of a group acting on a vector space has been examined in various contexts related to both finite and infinite groups. Plesken [47] showed that the action of $W_p(d)$ on the module $V_p(d) = \prod_{w \in X^d} C_p$ is uniserial. As a consequence, there exists a unique, properly descending filtration of $W(d)$-invariant submodules

$$V_p(d) = V^{(0)} \supseteq V^{(1)} \supseteq V^{(2)} \supseteq \ldots \supseteq V^{(p^d+1)} = \{0\}$$

with $|V_p^i| = p^{p^d - i}$.

Uniseriality of the action of tree automorphism groups has also been discussed by Ceccherini-Silberstein, Leonov, Scaraboti, and Tolli [19] and Bartholdi and Grigorchuk [9], as well as more recently by Grigorchuk, Leonov, Nekrashevych, and Suschansky [29].

The uniserial filtration allows us to construct finitely constrained groups with any desired Hausdorff dimension.

Proposition 3.1.11. *Let* p *be a prime number,* d *be a positive integer, and let* a *be a positive integer such that* $1 \leq a \leq p^{d-1}$*. If* $|X| = p$*, then there exists a finitely constrained subgroup of* $\mathrm{Aut}_p(X^*)$ *with pattern size* $d+1$ *and Hausdorff dimension* $\frac{a}{p^d}$*.*

Proof. Let $V(d) = \prod_{w \in X^d} C_p$, viewed as a vector space over the finite field with p

elements, and,consider the descending chain of $W(d)$-invariant submodules

$$(d) = V^{(0)} \supseteq V^{(1)} \supseteq V^{(2)} \supseteq \ldots \supseteq V^{(p^d+1)} = \{0\}$$

with $|V^i| = p^{p^d - i}$. Note that for each i with $0 \leq i \leq p^d$, the group $G(d) \ltimes V_p^i$ naturally embeds in $W_p(d+1)$ as the subgroup $P_d^{(i)}$, defined as follows. The patterns which appear in $P_d^{(i)}$ consist of all patterns appear up to level $d-2$, and the patterns which appear on level $d-1$ of elements in $P_d^{(i)}$ are precisely those of $V^{(i)}$. The group $P_d^{(i)}$ is a full pattern subgroup of pattern size d with $|\operatorname{Triv}_{(P_d^{(i)}}(d)| = |V_i|$, and thus we have $\dim_{\mathrm{H}}(G_{P_d^{(i)}}) = \frac{p^d - i}{p^d}$. □

Remark 3.1.12. It is clear from the description of the groups in the previous proof that $P_d^{(i)}$ is a split extension of $W_p(d)$ by $V_p^{(i)}$. Thus, by Corollary 3.1.5, none of these groups are topologically finitely generated.

This leads to the subject of the possible values of Hausdorff dimension for *topologically finitely generated*, finitely constrained groups, which we address in the next section.

3.2 Finitely constrained groups of p-adic tree automorphisms having maximal Hausdorff dimension

For the remainder of this section, we will consider only finitely constrained groups of binary tree automorphisms. This corresponds to finitely constrained groups where the tree alphabet is $X = \{0, 1\}$ and the label alphabet is C_2, with C_2 acting faithfully on X. For the remainder of this section we reserve the letter G for the group $\operatorname{Aut}(X^*)$ of all infinite binary tree automorphisms. Throughout the remainder of this section, we write $W(d)$ for $W_2(d) = \operatorname{Aut}(X^{[d]})$.

In this section, we work with finitely constrained subgroups of $\operatorname{Aut}_p(X^*)$, where p

is an arbitrary prime. For a given positive integer d, we consider finitely constrained groups with pattern size d and having Hausdorff dimension $1 - \frac{1}{p^{d-1}}$, which is the largest possible for such a group. We show that these finitely constrained groups can not be topologically finitely generated. Note that this result was shown for $p = 2$ and $d = 2$ by Šunić [57], and for $p = 2$ and $d = 3$ and $d = 4$ by Bondarenko and Samoilovych [15]. This result is also known for arbitrary p in the case of pattern size $d = 2$, due to Bondarenko and Samoilovych, and our proof consists of reducing arbitrary pattern size to that case.

As a key step in obtaining this result, we also characterize the essential pattern subgroups with pattern size d which define such finitely constrained groups. Many of the key results in this section are generalizations of those for the case $p = 2$, which were obtained as a joint work with Zoran Šunić [45]. Thus the outline of this section is very similar to that of [45], though some of the avenues of proof are different.

Using items (iii.) and (iv.) in Proposition 3.1.9 and some basic linear algebra, we describe all maximal subgroups of $G(d)$. Since $G(d)$ is a p-group, the commutator of $G(d)$ is contained in every maximal subgroup of $G(d)$. Thus there is a one-to-one correspondence between maximal subgroups of $G(d)$ and maximal subspaces of the $\mathbb{F}_p$ vector space $\prod_{i=0}^{d-1} C_p$. Recall that a maximal subspace $V \leq \prod_{i=0}^{d-1} C_p$ can be defined by giving a nonzero vector $\mathbf{c} \in \prod_{i=0}^{d-1} C_p$ which is orthogonal to all $\mathbf{v} \in V$ under the usual inner product $\langle \mathbf{c}, \mathbf{v} \rangle = \sum_{i=0}^{d-1} \mathbf{c}_{(i)} \mathbf{v}_{(i)}$. The vector $\mathbf{c}$ is unique only up to scalar multiplication. However, if we require that $\mathbf{c}$ be normalized so that the last nonzero entry of $\mathbf{c}$ is equal to 1, then we can make a one-to-one correspondence between a maximal subspace M and a vector $\mathbf{c}$ which is orthogonal to all $\mathbf{v} \in M$. Accordingly, we call a vector $\mathbf{c} \in \prod_{i=0}^{d-1} C_p$ a *defining vector* if its last nonzero entry is equal to 1.

This correspondence leads to a one-to-one correspondence between defining vectors and maximal subgroups of $G(d)$, as follows. Given a defining vector $\mathbf{c} \in \prod_{i=0}^{d-1} C_p$, let $\alpha_{\mathbf{c}}$ be the homomorphism given by

$$\alpha_{\mathbf{c}}(g) = \sum_{i=0}^{d-1} \sum_{w \in X^i} \mathbf{c}_{(i)} g_{(w)}.$$

We write $P_{\mathbf{c}}$ for $\ker \alpha_{\mathbf{c}}$.

We also need the following result, which summarizes results known in the literature.

Proposition 3.2.1. *Let G_P be a finitely constrained subgroup of p-adic tree automorphisms. The following are equivalent.*

1. *G_P is infinite.*

2. *For each $j \geq 0$, G_P acts transitively on X^j.*

3. *The Hausdorff dimension of G_P is positive.*

Proof. The argument given in [14, Lemma 3] proves the equivalence of (i) and (ii) for any self-similar group of p-adic tree automorphisms (this equivalence does not hold for self-similar groups of tree automorphisms in general) . The equivalence of (i) and (iii) is shown in [56, Theorem 4(a)]. □

Lemma 3.2.2. *Let P be an essential pattern subgroup of $G(d)$ such that $[Triv_{G(d)}(d-1) : \mathrm{Triv}_P(d-1)] = p$. Then*

$$\mathrm{Triv}_P(d-1) = \{p \in G(d) \mid \sum_{w \in X^{d-1}} p_{(w)} = 0 \textit{ and } p_{(w)} = 0 \textit{ whenever } |w| < d-1\}$$

Proof. The fact that $p_{(w)} = 0$ whenever $|w| < d-1$ follows from the definition of $\mathrm{Triv}_P(d-1)$. Since $\mathrm{Triv}_P(d-1)$ is nontrivial, G_P has positive Hausdorff dimension by Proposition 3.2.1, and thus P acts transitively on X^{d-1}. Since $[\mathrm{Triv}_{G(d)}(d-1) : \mathrm{Triv}_P(d-1)] = p$, there exists a nonzero vector $\mathbf{v} \in \mathrm{Triv}_{G(d)}(d-1)$ such that $\sum_{w \in X^{d-1}} v_{(w)} p_{(w)} = 0$. This is true for p^g for any $g \in G(d)$, so it follows that each $\mathbf{v}_i$ must be nonzero, else we could conjugate p by an element which moves an index i with a zero coefficient to an index j with a nonzero coefficient, changing the sum. We must also have $\mathbf{v}$ is a constant vector, since otherwise we could conjugate p by an element which changes the value of the sum. We can normalize $\mathbf{v}$ to have the value 1 in each index. Thus $\sum_{w \in X^{d-1}} p_{(w)} = 0$. □

Theorem 3.2.3. *Let $X = \{0, 1, \ldots, p-1\}$ and let G_P be a finitely constrained subgroup of $\mathrm{Aut}_p(X^*)$ defined by an essential pattern group P of pattern size d, $d \geq 2$. The following conditions are equivalent.*

1. *G_P has Hausdorff dimension equal to $1 - \frac{1}{p^{d-1}}$.*

2. *P is a maximal subgroup of $G(d)$ that does not contain the generator a_{d-1}.*

3. *$P = P_{\mathbf{c}} = \ker \alpha_{\mathbf{c}}$ for some defining vector $\mathbf{c} \in \prod_{i=0}^{d-1} C_p$ such that $\mathbf{c}_{(d-1)} = 1$.*

4. *P is a proper subgroup of the group $G(d)$, the group of p-adic automorphisms of $X^{[d]}$, such that P contains the commutator of $G(d)$.*

5. *P is a maximal subgroup of $G(d)$ that does not contain $\mathrm{Triv}_{G(d)}(d-1)$, the stabilizer of level $d-1$ in $G(d)$.*

Proof. **(i.)** $\to$ **(ii.)** If G_P has Hausdorff dimension equal to $1 - \frac{1}{p^{d-1}} = \frac{p^{d-1}-1}{p^{d-1}}$, then it follows by Proposition 3.1.8 that $\mathrm{Triv}_P(d-1) = p^{p^{d-1}-1}$. Thus $\mathrm{Triv}_P(d-1)$ is a maximal subspace of $\mathrm{Triv}_{G(d)}(d-1)$, and we apply Lemma 3.2.2. We claim now that $|P(d-1)| = |G(d-1)|$. To see this, note that $p \in \mathrm{Triv}_P(d-1)$ if and only if $\sum_{w \in X^{d-1}} p_{(w)} = 0$ and $p_{(w)} = 0$ whenever $|w| < d-1$.We claim that for any

$0 \leq j < d-1$ and any pattern r with support on the level X^j, we can produce an element $q \in P(d-1)$ such that the restriction of q to X^j is precisely r. To do this, we define an element $q' \in \mathrm{Triv}_P(d-1)$ defined as follows.

$$q' = \begin{cases} r_{(w)}, w \in 0^j X^{d-1-j} \\ p - \sum_{w \in X^j} r_{(w)}, w = (p-1)^{d-1} \\ 0, \text{ otherwise} \end{cases}$$

Since P is an essential pattern group, for each i with $0 \leq i \leq d-1$, there exists an element $q'_{0^i} \in P$ such that $(q'_{0^i})_{(w)} = q'_{(0^i w)}$ for each $w \in X^{d-1-i}$. Thus, we can now obtain any pattern on levels 0 through $d-2$ as a product of elements obtained using the process just described. It follows that P is a maximal subgroup of $G(d)$.

To see that P does not contain a_{d-1}, note that if $a_{d-1} \in P$, then $\mathrm{Triv}_P(d-1) = \mathrm{Triv}_{G(d)}(d-1)$, from which it would follow that $|\,\mathrm{Triv}_P(d-1)| = p^{p^{d-1}}$, and this contradicts our assumption that G_P has Hausdorff dimension $1 - \frac{1}{p^{d-1}}$. Thus $a_{d-1} \notin P$.

(ii.) $\Leftrightarrow$ **(iii.)** Let P be a maximal subgroup of $G(d)$ with defining vector $\mathbf{c}$. Note that by the definition of defining vector, the last nonzero entry of $\mathbf{c}$ is 1. If this last nonzero entry is in position $d-1$, we have $\mathbf{c}_{(d-1)} = 1$, and otherwise we have either $\mathbf{c}_{(d-1)} = 0$. Since the only nonzero label of a_{d-1} is the label 1 on 0^{d-1}, it follows that $a_{d-1} \in \ker \alpha_{\mathbf{c}} = P_{\mathbf{c}}$ if and only if $\mathbf{c}_{d-1} = 0$.

(iii.) $\implies$ **(iv.)** This follows from the fact that any maximal subgroup of a p-group contains the commutator subgroup.

(ii.) $\Leftrightarrow$ **(v.)** Since a maximal subgroup of $G(d)$ has index p, each maximal subgroup is a normal. Since $\mathrm{Triv}_{G(d)}(d-1)$ is the normal closure in $G(d)$ of the group generated by a_{d-1}, a normal subgroup of $G(d)$ contains a_{d-1} if and only if it contains $\mathrm{Triv}_{G(d)}(d-1)$.

(iv.) $\implies$ **(i.)** Let P be an essential pattern group which is a proper subgroup of $G(d)$ such that P contains $[G(d), G(d)]$. Then $[a_0, a_{d-1}] \in P_{d-1}$. Since the action of P on X^{d-1} is transitive, it follows that P_{d-1} contains the unique $G(d)$-invariant maximal subspace of $\mathrm{Triv}_{G(d)}(d-1)$ described in Lemma 3.2.2. If $P_{d-1} = G(d)_{d-1}$, then by Proposition 3.1.8, we have $\dim_{\mathrm{H}}(G_P) = 1$, which contradicts the assumption that P was a proper subgroup of $G(d)$. Thus P_{d-1} is a maximal subspace of $G(d)_{d-1}$, from which it follows that P is a maximal subgroup of $G(d)$. $\square$

Remark 3.2.4. Note that for a positive integer $d \geq 2$, there are $p^{\frac{p^d-1}{p-1}}$ maximal subgroups of $G(d)$. Note also that if $\mathbf{c}$ is a defining vector with $\mathbf{c}_{d-1} = 1$, then there are p choices for each of the entries $\mathbf{c}_{(0)}$,$\mathbf{c}_{(1)}$, …, $\mathbf{c}_{(d-2)}$, and thus there are p^{d-1} defining vectors of this type. Thus, according to Theorem 3.2.3, only p^{d-1} of the maximal subgroups of $G(d)$ give essential pattern subgroups which can be used to define a finitely constrained subgroup of $\mathrm{Aut}_p(X^*)$.

Now our goal is to prove the following theorem, which puts a bound on the Hausdorff dimension of a topologically finitely generated, finitely constrained group (as a function of pattern size).

Theorem 3.2.5. *Let p be a prime number, and let G_P be a finitely constrained subgroup of $\mathrm{Aut}_p(X^*)$ defined by an essential pattern group P of pattern size d. If G_P has Hausdorff dimension $1 - \frac{1}{p^{(d-1)}}$ (the largest possible for such a finitely constrained group), then G_P is not topologically finitely generated.*

Now we quote a result of Bondarenko and Samoilovych which applies to pattern size $d = 2$. It clearly applies in our situation, since we are dealing only with p-groups and every p-group is nilpotent.

Proposition 3.2.6 (see [15], Corollary 5). *Let X be a finite set and let C be a cyclic subgroup of* $\mathrm{Sym}(X)$*. Consider the group $C \wr_X C$ as a subgroup of* $\mathrm{Sym}(X) \wr_X \mathrm{Sym}(X)$*. For any nilpotent pattern group $P \leq C \wr_X C$, the commutator $[P, P]$ contains* $\mathrm{Triv}_P(1)$ *if and only if* $\mathrm{Triv}_P(1)$ *is trivial.*

Let $d > 2$ and P be a maximal subgroup of $W_p(d)$. From our previous discussion, P has some defining vector $\mathbf{c}$ with $\mathbf{c}_{d-1} = 1$. We define a map $\alpha_{\mathbf{c}} : W_P(d-1) \to W_P(2)$ given by

$$\alpha_c^{'} : \sum_{k=0}^{d-2} \sum_{w \in X^k} \mathbf{c}_{k+1} g_{(w)}.$$

Recall that for any $g \in W_p(d)$, we write g as $(g_{(\epsilon)}, (g_i)_{i=0}^{p-1})$, where $g_{(\epsilon)} \in C_p$ and each $g_i \in W_p(d-1)$. Then we define $\beta_c : W_p(d) \to W_p(2)$ by

$$[\beta_c(g)]_{(\epsilon)} = g_{(\epsilon)} \qquad [\beta_{\mathbf{c}}(g)]_{(i)} = \alpha_{\mathbf{c}}^{'}(g_i).$$

We remark that $\alpha_{\mathbf{c}}^{'}$ is the defining map of a cellular automaton, though that observation will not be needed again. The following properties of β_c are essential to what follows, so we provide verification even though they are routine.

Proposition 3.2.7. *Let $d > 2$. For a maximal subgroup $P_{\mathbf{c}}$ of $W_p(d)$ with defining vector $\mathbf{c}$, the map $\beta_c : W_p(d) \to W_p(2)$ defined above has the following properties.*

(i.) *β_c is a well-defined homomorphism from $W_p(d)$ to $W_p(2)$.*

(ii.) *The kernel of β_c is contained in $P_{\mathbf{c}}$.*

(iii.) *The image of the level $(d-1)$ stabilizer of $P_{\mathbf{c}}$ under the map β_c is contained in the level 2 stabilizer of the image, i.e. we have $\beta_c((P_{\mathbf{c}})_{d-1}) \leq (\beta_c(P))_2$.*

(iv.) *We have $\beta_c([P_c, P_c]) = [\beta_c(P_c), \beta_c(P_c)]$.*

Proof. (i.) For $g, h \in W_p(d)$, we write $g = \sigma^j(g_i)$ and $h = \sigma^k(h_i)$, where $g_i, h_i \in W_p(d-1)$ for $i = 0$ to $p-1$. Then we calculate

$$\begin{aligned}
\beta_c(gh) &= \beta_c(\sigma^{j+k}(g_{\sigma^j(i)}h_i) \\
&= \sigma^{j+k}(\alpha_{\mathbf{c}}^{'}(g_{\sigma^k(i)}h_i)) \\
&= \sigma^j\sigma^k(\alpha_{\mathbf{c}}^{'})(g_{\sigma^k(i)}\alpha_{\mathbf{c}}^{'}(h_i)) \\
&= \sigma^j\sigma^k(\alpha_{\mathbf{c}}^{'}(g_i)^{\sigma^k}(\alpha^{'}(h_i)) \\
&= \sigma^j(\alpha_{\mathbf{c}}^{'}(g_i))\sigma^k(\alpha_{\mathbf{c}}^{'}(h_i))
\end{aligned}$$

(ii.) Suppose $g \in \ker \beta_{\mathbf{c}}$. Then $g_{(\epsilon)} = 0$ and $\alpha_{\mathbf{c}}^{'}(g_i) = 0$ for all i between 0 and $d-2$. But

$$\alpha_{\mathbf{c}}(g) = \sum_{i=0}^{p-1} \alpha_c^{'}(g_i) + c_0 g_{(\epsilon)}$$

and thus if $g \in \ker \beta_{\mathbf{c}}$, it follows that $g \in \ker \alpha_{\mathbf{c}} = P_{\mathbf{c}}$.

(iii.) Recall that if $g \in (P_{\mathbf{c}})_{d-1}$, then $\sum_{w \in X^{d-1}} g_{(w)} = 0$ and $g_{(w)} = 0$ for $|w| < d-1$. Therefore for $g \in \mathrm{Triv}_{P_{\mathbf{c}}}(d-1)$, we have

$$\begin{aligned}
\sum_{i=0}^{d-1} [\beta_{\mathbf{c}}(g)]_{(i)} &= \sum_{i=0}^{d-1} \alpha_{\mathbf{c}}^{'}(g_i) \\
&= \sum_{w \in X^{d-1}} g_{(w)} \\
&= 0.
\end{aligned}$$

Thus $\beta_{\mathbf{c}}(g) \in [\beta_c(P)]_2$, and the result follows.

(iv.) This is true for any homomorphism.

□

Now we prove Theorem 3.2.5.

Proof of Theorem 3.2.5. By Proposition 3.2.7, the group $P_{\mathbf{c}}$ is above $\ker \beta_c$, so the image $\beta_c(P_{\mathbf{c}})$ is a maximal subgroup of $W_p(2)$. If $\mathrm{Triv}_{(P_c)}(d-1) \subseteq [P_{\mathbf{c}}, P_{\mathbf{c}}]$, then we would have $\mathrm{Triv}_{\beta_c(P_c)}(2) \subseteq [\beta_c(P_c), \beta_c(P_c)]$. However, Proposition 3.2.6 tells us that this can not occur, so $P_c(d-1)$ is not contained in $[P_c, P_c]$, and the desired result follows. □

3.3 Saturation of some finitely constrained groups of binary tree automorphisms

We begin with some background and notation necessary for this section only. We define a *self-isomorphism* of a group G as an isomorphism from G to itself. Self-isomorphisms are usually called *automorphisms* in group theory, but we will avoid that term because of the great potential for confusion between group automorphisms and tree automorphisms. The set of self-isomorphisms of a group form a group under the operation of composition, which we denote by $\mathrm{Iso}(G)$. Recall that if G is a group and H is a subgroup of G, the *normalizer of H in G* is given by

$$N_G(H) = \{g \in G | h^g \in H \text{ for all } h \in H\}.$$

Conjugation by an element of G is obviously a self-isomorphism of H. Thus for any $g \in G$ there is a self-isomorphism $\phi_g : H \to H$ given by $\phi_g(h) = h^g$ (though it is possible for conjugation by distinct elements of G to induce the same self-isomorphism). In this case we say that g *induces* the self-isomorphism ϕ_g.

The objective of this section is to prove that in certain cases, the self-isomorphism group of a finitely constrained group G_P coincides with the normalizer of G_P in $\mathrm{Aut}(X^*)$. Our approach is to use the sufficient condition given by Lavreniuk and Nekrashevych [38].

Stating their result will require some additional definitions. If $v \in t^*$ and $G \leq \mathrm{Aut}(t^*)$, the *rigid vertet stabilizer in v of G* is denoted $\mathrm{RiSt}_v(G)$ and is equal to the set

$$\{g \in G | \text{ if } w \notin vX^*, \text{ then } g(w) = w\}.$$

The *rigid stabilizer of level n* is the subgroup $\mathrm{RiSt}_G(n)$ generated by all elements of $\mathrm{RiSt}_v(G)$ for all $v \in X^n$. A subgroup $G \leq \mathrm{Aut}(X^*)$ is called *weakly branch* if for any $v \in X^*$, $\mathrm{RiSt}_G(n)$ is infinite. Any infinite, finitely constrained group G_P is weakly branch, since in this case G_P is a regular branch group over an infinite subgroup.

We say that a group $G \leq \mathrm{Aut}(X^*)$ is *saturated* if for every $n \geq 0$, there exists a characteristic subgroup $H_n \leq \mathrm{Triv}_G(n)$ such that H_n such that for each $w \in t^n$, H_n acts transitively on wt^*. Note that if $H \leq K$ and H is saturated, then K is saturated, as well, since any characteristic subgroup of H is also a characteristic subgroup of K.

The following theorem, proven in [30], will be crucial for the following discussion.

Theorem 3.3.1 (Theorem 7.5, [38])**.** *Let $G \leq \mathrm{Aut}(X^*)$ be saturated and weakly branch. Then for any $\phi \in \mathrm{Iso}(G)$, there exists an element $h \in N_{\mathrm{Aut}(X^*)}(G)$ such that $\phi_h = \phi$ for all $g \in G$.*

To apply Theorem 3.3.1 to a finitely constrained group G_P of binary tree automorphisms of matimal Hausdorff dimension discussed in the previous section, it suffices to prove that G_P contains a saturated subgroup.

Throughout the remainder of the section, we fix a pattern size d and let J be a subset of $\{0, \ldots, d-1\}$. For each such J, there is a corresponding essential pattern subgroup of P_J and finitely constrained group $G_{P_J} \leq \mathrm{Aut}(X^*)$.

First, we consider the case when $|J|$ is even.

Let $\mathcal{O}$ denote the odometer group defined in Example 2.5.8.

Lemma 3.3.2. *Let P_J be an essential pattern group of pattern size d such that G_{P_J} has Hausdorff dimension $1 - \frac{1}{2^{d-1}}$. If $|J|$ is even, then G_{P_J} contains $\mathcal{O}$.*

Proof. If $|J|$ is even, then we have $\alpha_J(a) = 0$, since $\text{supp}(a) = \{1^n | n \geq 0\}$ and thus a has odd total activity on each level. Thus $\mathcal{O}(d) \leq P_J$ and $\mathcal{O} \leq G_{P_J}$ by *some reference*. □

Proposition 3.3.3. *The group $\mathcal{O}$ is saturated.*

Proof. It is not hard to see that the action of $\mathcal{O}$ on X^* is level-transitive, since for each d the finite group $\mathcal{O}(d)$ acts as a cyclic permutation on all elements of X^d. Moreover, for each $n \geq 0$, the group $\mathcal{O}^{2^n} = \langle g^{2^n} \mid g \in \mathcal{O} \rangle$ is characteristic and contained in the level n stabilizer. Let $g = a^{2^n}$. Applying the calculations done in Example 2.5.8, we see that for any $n \geq 0$ and any $w \in X^n$, $g_w = a$. Since a acts transitively on X^*, it follows that g_w acts transitively on wX^* for all $w \in X^n$, and thus the group is saturated. □

Proposition 3.3.4. *Let $d \geq 1$, let $J \subset \{0, \ldots, d-1\}$ such that $d - 1 \in J$ and $|J|$ is even, and let P_J be the maximal subgroup of $W(d)$ given by $\ker \alpha_J$. For any $\phi \in \text{Iso}(G_{P_J})$, there exists $h \in N_{\text{Aut}(X^*)}(G_{P_J}$ such that $\phi = \phi_h$.*

Proof. This follows from Lemma 3.3.2, Proposition 3.3.3, and Theorem 3.3.1. □

The previous proposition covers half of the possible cases for J and leaves the cases when $|J|$ is odd. We consider one subcase. Suppose that $|J|$ is odd, and $|J \cap \{0, 1\}|$ is even. We will show that in this case, the group P_J is saturated by showing that P_J contains a saturated subgroup. This subgroup is a well-known example called the *Lamplighter.*

Let L be the self-similar group generated by the finite state automaton

$$a = \sigma(a, b), \quad b = (a, b).$$

This is a well-known group in geometric group theory (see Section 8 in [42]) and the automaton we consider here is discussed in Section 4 of [30]. Although its structure is well-understood, we will prove some facts about it from scratch.

Lemma 3.3.5. *Let $d \geq 1$, let $J \subset \{0, \ldots, d-1\}$ such that $d-1 \in J$, $|J|$ is odd and $|J \cap \{0,1\}$ is even. Let P_J be the maximal subgroup of $W(d)$ given by* $\ker \alpha_J$. *Then $L \leq G_{P_J}$.*

Proof. If $w \in X^*$ ends in 0, then $a_w = b_w = a$, and if $w \in X^*$ ends in 1, then $a_w = b_w = b$. Thus $a_{(w)} = b_{(w)}$ for all w of positive length, and $a_{(w)} = b_{(w)}$ is trivial if and only if w ends in 0. From these observations, it is clear that a is active at the root, and both a and b have odd activity on level 1. It is also clear that a and b both have even total activity on level n for $n \geq 2$. $\square$

Proposition 3.3.6. *The group L is saturated.*

In order to prove Proposition 3.3.6, we need to review some known facts and prove some preliminary results. We will write $\langle X \rangle$ for the group generated by X, and write $< X >^G$ for the normal closure of $\langle X \rangle$ in G.

We recall the standard notion of the *lower central series* of a group. Recall that if H and K are subgroups of a group G, the group $[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle$. Note that if H and K are characteristic subgroups of G, then so is $[H, K]$. For a group G, we inductively define $\gamma_1(G = L$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$. The *lower central series* is the decreasing sequence of subgroups $G = \gamma_1(G) \leq \gamma_2(G) \leq \gamma_3(G) \ldots$.

Further, if X is a set, we define a *simple commutator in* X to be an element of the form $[g, h]$ for $g, h \in X$, and for $i > 1$ we define a *simple i-fold commutator in* X to be an element of the form $[g, h]$, where g is a simple $(i-1)$-fold commutator in X and $h \in X$. The following proposition is a well-known fact in group theory (see Section 5 of [41]).

Proposition 3.3.7. *Let G be a finitely generated group and X be a generating set for X. The group $\gamma_{n+1}(G)$ is equal to the normal closure of the group generated by all simiple n-fold commutators in X.*

Our objective now is to describe the lower central series of the Lamplighter defined above. We prove a few results which will simplify this task.

To faciliate our arguments, we introduce a new element $t = ab^{-1} = \sigma(a, b)(a^{-1}, b^{-1}) = \sigma(1, 1)$. Any two elements of the set $\{a, b, t\}$ can serve as a generating set for L, and we will often make arguments using the generating set $\{b, t\}$. Note that t has order 2, so $t = t^{-1}$ and thus $(ab^{-1}) = (ab^{-1})^{-1} = ba^{-1}$.

Lemma 3.3.8. *For any $g \in L$, $(gt)^b = b^{-1}ga$ and $(tg)^b = a^{-1}gb$.*

Proof.

$$(gt)^b = b^{-1}(gt)b = b^{-1}gab^{-1}b = b^{-1}ga$$

and

$$(tg)^b = b^{-1}tgb = b^{-1}ba^{-1}gb = a^{-1}gb$$

$\square$

Lemma 3.3.9. *For any $n \geq 0$, $t^{b^n} = \sigma(t^{b^n}t^{b^{n-1}} \ldots t^{b^2}t^b, t^bt^{b^2} \ldots t^{b^{n-1}}t^{b^n})$.*

Proof. The proof is by induction on n. We compute first that $t^b = (a^-1, b^-1)\sigma(1, 1)(a, b) = \sigma(b^{-1}a, a^{-1}b) = \sigma(t^b, t^b)$. Now assume that the result is true for $n = k \geq 1$, we calculate that

$$
\begin{aligned}
t^{b^{k+1}} &= b^{-1}(t^{b^k})b \\
&= (a^{-1}, b^{-1})t^{b^k}(a, b) \\
&= (a^{-1}, b^{-1})\sigma(t^{b^k}t^{b^{k-1}} \dots t^{b^2}t^b, \\
&t^b t^{b^2} \dots t^{b^{k-1}}t^{b^k})(a, b) \\
&= (b^{-1}(t^{b^k}t^{b^{k-1}} \dots t^{b^2}t^b)a, \\
&a^{-1}(t^b t^{b^2} \dots t^{b^{k-1}}t^{b^k})b)
\end{aligned}
$$

Applying Lemma 3.3.8, we rewrite this as

$$
(t^{b^k}t^{b^{k-1}} \dots t^{b^2}t^b t)^b, (tt^b t^{b^2} \dots t^{b^{k-1}}t^{b^k})^b)
$$

which is equivalent to the desired result. □

Corollary 3.3.10. *The group $\langle t \rangle^G$ is abelian.*

Proof. Follows from the fact that $\mathbb{Z} \wr_{\mathbb{Z}} C_2$ is an abelian group and the proof of [30, Proposition 4.1], which shows that $< t >^G$ is the base group of $\mathbb{Z} \wr_{\mathbb{Z}} C_2$. □

Corollary 3.3.11. *If $g \in G'$, then t commutes with g.*

Proof. By Proposition 3.3.7, $G' = \langle [t, b] \rangle^G$, and we simply note that

$$
\langle [t, b] \rangle^G \leq \langle \{t^{b^k}\}_{k=0}^{\infty} \rangle^G = \langle t \rangle^G.
$$

□

In the next proposition, we describe the structure of the lower central series of this group.

Proposition 3.3.12. *Let $X = \{b^{-1}, t\}$. Define sequences of elements $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ as follows: $y_0 = z_0 = t$, and for any $i \geq 1$, $y_{i+1} = [a^{-1}, t]$ and $z_{i+1} = [b^{-1}, t]$. Then the following properties hold for all $n \geq 1$.*

1. $z_n = y_n$
2. $z_n = (z_{n-1}, z_{n-1})$
3. *z_n is the only nontrivial simple n-fold commutator in X*
4. $\langle z_n \rangle^L = \gamma_n(L)$
5. $(z_n)_w = \begin{cases} e_L, & \text{if } |w| < n \\ t, & \text{if } |w| = n \end{cases}$
6. $\gamma_n(L) \leq \mathrm{Triv}_n(G)$

Proof. We will write t as σ rather than $\sigma(1,1)$.

1. (By induction on n) This is cearly true for $n = 0$. Assume the statement is true for some $k \geq 0$. Then for $k+1$, we note that $z_k, y_k \in G'$ and thus by Corollary 3.3.11 and the induction hypothesis,

$$\begin{aligned} z_{k+1} &= [a^{-1}, z_k] = [tb^{-1}, z_k] = btz_k^{-1} \\ &= tb^{-1}z_k = bty_k^{-1}tb^{-1}y_k = by_k^{-1}t^2b^{-1}y_k \\ &= by_k^{-1}b^{-1}y_k \\ &= y_{k+1} \end{aligned}$$

2. (By induction on n) For $n = 1$, we have

$$[b^{-1}, t] = (a, b)\sigma(a^{-1}, b^{-1})\sigma = \sigma^2(ab^{-1}, ba^{-1})^{\sigma} = (ba^{-1}, ab^{-1} = (t, t).$$

Assume there is some k such that the statement holds for all $k \geq n \geq 1$. Then for $k+1$, we have

$$z_{k+1} = [b^{-1}, z_k] = (a,b)(z_{k-1}, z_{k-1})(a^{-1}, b^{-1})(z_{k-1}^{-1}, z_{k-1}^{-1})$$
$$= ([a, z_{k-1}], [b, z_{k-1}]) = (y_k, z_k) = (z_k, z_k)$$

3. This follows immediately by induction using Corollary 3.3.11

4. This follows from the previous observation and Proposition 3.3.7.

5. (By induction on n) For $n = 1$, we have $z_1 = (t,t)$ by the calculation in in (ii.). Assume there is some k it is true whenever $k \geq |w| \geq 1$. Assume $k+1 \leq |w| \leq 1$ and write $w = xw'$ for $x \in \{0,1\}$ and $w' \in X^{|w|-1}$. Then since $(z_{k+1})_x = z_k$, we have

$$(z_{k+1})_w = ((z_{k+1})_x)_w = (z_k)_w = \begin{cases} e_L, \text{ if } |w| < k \\ t, \text{ if } |w| = k \end{cases} .$$

This completes the proof.

□

Now we are ready to prove Proposition 3.3.6

Proof of Proposition 3.3.6. By Proposition 3.3.12, for each $n \geq 0$ we have that $\gamma_n(L)$ is a characteristic subgroup of G contained in $\mathrm{Triv}_G(n)$. Suppose $w \in X^n$. the element $z_n \in \gamma_n(G)$ and z_n acts transitively on wX since $(z_n)_w = t$ and t acts transitively on X. Thus for each $k \geq n$, $\gamma_k(G)$ contains an element which acts transitively on wX^k, and it follows that $\gamma_n(G)$ acts transitively on wX. Thus L is saturated. □

Theorem 3.3.13. *Let $d \geq 1$, let $J \subset \{0, \ldots, d-1\}$ such that $d-1 \in J$ and both J is odd and $|J \cap \{0,1\}|$ is even. Let P_J be the maximal subgroup of $W(d)$ given by* $\ker \alpha_J$. *For any $\phi \in \mathrm{Iso}(G_{P_J})$, there exists $h \in N_{\mathrm{Aut}(X^*)}(G_{P_J}$ such that $\phi = \phi_h$.*

Proof. Follows immediately from Theorem 3.3.1 since we have shown that G_P is a weakly branch group which contains the saturated subgroup L. □

3.4 Finitely constrained groups of binary tree automorphisms with Hausdorff dimension $1 - \frac{2}{2^{d-1}}$

Our primary objective in this section is to investigate finitely constrained groups of binary tree automorphisms with Hausdorff dimension equal to $1 - \frac{2}{2^{d-1}}$. We note first that for the case $p = 2$, our previous discussion of maximal subgroups of $G(d)$ can be simplified somewhat by associating to a maximal subgroup M a defining subset $J \subseteq \{0, 1, \ldots, d-1\}$ such that $d-1 \in J$, (instead of a defining vector in the arbitrary p case). For each such subset J, we define a homomorphism $\phi_J : G(d) \to C_2$ by

$$\alpha_J(g) = \sum_{j \in J} \sum_{w \in X^j} g_{(w)}.$$

and let $P_J = \ker \alpha_J$.

Remark 3.4.1. Let $J \subseteq \{0, 1, \ldots, d-1\}$ such that $d-1 \in J$. From the definition of P_J we see that the generator $a_j \in P_J$ if and only if $j \notin J$ and $a_j a_{d-1} \in P_J$ if and only if $j \in J$.

Remark 3.4.2. Ceccherini-Silberstein, Leonov, Scaraboti, and Tolli give a very clear description of the patterns of the subspaces in the uniserial filtration. In particular, they show that $V^{(1)}$ is generated by the closure of $[a_0, a_{d-1}]$ under the action of $W_p(d)$, and that for $p = 2$, $V^{(2)}$ is generated by the closure of $[a_1, a_{d-1}]$ under this action. .

Proposition 3.4.3. *Let G_P be a finitely constrained group of binary tree automorphisms defined by an essential pattern subgroup P with pattern size d. If G_P has Hausdorff dimension $1 - \frac{2}{2^{d-1}}$, then P must be full.*

Proof. Assume that there is some P such that G_P has Hausdorff dimension $1 - \frac{2}{2^{d-1}}$, but P is not full, and assume that d is the smallest pattern size such that there is such a P. Observe first that we must have $|\operatorname{Triv}_P(d-1)| = 2^{2^{d-1}-2}$. Let $Q = P(d-1)$, and consider $\operatorname{Triv}_Q(d-2)$. Since P is an essential pattern group, so is the group Q. By assumption, $\operatorname{Triv}_Q(d-2) \neq \operatorname{Triv}_{G(d-1)}(d-2)$, and so we have

$$|\operatorname{Triv}_Q(d-2)| \leq 2^{2^{d-1}-1}.$$

Since $P \leq G_Q(d)$ by Proposition 2.5.10, it follows that

$$|\operatorname{Triv}_P(d-1) \leq |\operatorname{Triv}_{G_Q}(d-1)|.$$

But from the fact that G_Q is regular branch over $\operatorname{Triv}_Q(d-1)$, it follows that $|\operatorname{Triv}_{G_Q}(d-1)|. = |\operatorname{Triv}_{G_Q}(d-2)|^2$, so we have

$$\begin{aligned} |\operatorname{Triv}_{G_{Q(d)}}(d-1)| &= |\operatorname{Triv}_Q(d-2)|^2 \\ &\leq 2^{(2^{d-2}-1)^2} \\ &= 2^{2(2^{d-2}-1)} \\ &= 2^{2^{d-1}-2} \\ &= |\operatorname{Triv}_P(d-1)| \end{aligned}$$

Thus, since $\operatorname{Triv}_P(d-1) \leq \operatorname{Triv}_{G_Q}(d-1)$ and $|\operatorname{Triv}_P(d-1)| \geq |\operatorname{Triv}_{G_Q}(d-1)|$, these two finite groups are actually equal.

But then, for all $n \geq d$, we have

$$\begin{aligned}
|G_P(n)| &= |P||P_{d-1}|^{2+4+\ldots+2^{n-d}} \\
&= |Q||Q_{d-2}|^2|P_{d-1}|^{2+4+\ldots+2^{n-d}} \\
&= |Q||Q_{d-2}|^2|P(d-1)_{d-2}|^{4+8+\ldots+2^{n-d+1}} \\
&= |Q||Q_{d-2}|^{2+4+8+\ldots+2^{n-(d-1)}} \\
&= |G_Q(n)|
\end{aligned}$$

Since $G_P(n) \leq G_{P(d-1)}(n)$ for all n by Proposition 2.5.10 and the finite groups $G_P(n)$ and $G_Q(n)$ have the same size, it follows that $G_P(n) = G_{P(d-1)}(n)$ for all n. Thus $G_P = G_Q$, since both groups are closed. This means that G_P is actually defined by patterns of size $(d-1)$, contradicting our assumption that G_P was defined by patterns of size d. Thus P is full. $\square$

It follows that if G_P is defined by an essential pattern subgroup P of pattern size d such that Hausdorff dimension equal to $1 - \frac{2}{2^{d-1}}$, then P has index 4 in $G(d)$. Our goal is to describe and count these essential pattern subgroups.

Remark 3.4.4. From uniseriality, specifically the description of patterns in the invariant subspaces of $V_2(d)$ under the action of $W_2(d)$ given in [19, Section 2], it follows that $\mathrm{Triv}_P(d-1)$ must have the following form. An element $p \in \mathrm{Triv}_P(d-1)$ and the sum of all labels on the last level of each subtree (i.e. words of the form $0X^{d-2}$ and $1X^{d-2}$) is equal to 0. Equivalently, $\mathrm{Triv}_P(d-1)$ is the normal closure in $G(d)$ of the element $[a_1, a_{d-1}]$.

Lemma 3.4.5. *Let P be an essential pattern subgroup of $G(d)$ such that $[G(d) : P] = 4$. If M is a maximal subgroup of $G(d)$ such that $P \leq M$, then we must have $a_0 a_{d-1} \notin P$, $a_{d-1} a_0 \notin P$, and $a_0 \in M$.*

Proof. Since P has index 2 in M, there is a homomorphism $M \to C_2$ such that $P = \ker\phi$, and since ϕ is a map onto an elementary abelian p-group, it follows that $\Phi(M) \subseteq P$. Note that $M = M_J = \ker\alpha_J$ for some $J \subseteq \{0, \ldots, d-1\}$. Note that if $a_0 a_{d-1} \in P$, then would have $(a_0 a_{d-1})^2 = [a_0, a_{d-1}] \in P$ since the Frattini subgroup contains all squares of elements in M. But P can not contain $[a_0, a_{d-1}]$, since then $\mathrm{Triv}_P(d-1)$ would contain the normal closure of $< [a_0, a_{d-1}] >$, which is a maximal subspace of $\mathrm{Triv}_{G(d)}(d-1)$. If $a_0 \notin M$, then, following Remark 3.4.1, we must have $a_0 a_{d-1} \in M$. $\square$

Proposition 3.4.6. *Let $d > 1$. There are at most 2^{2d-3} essential pattern subgroups of index 4 in $G(d)$.*

Proof. Algebraically, P is an extension of P_{d-1} by P/P_{d-1}, so a generating set for P is given by a generating set for P_{d-1} and a transversal for $P(d-1)$ in P which satisfies certain conditions imposed by the presentation of P/P_{d-1} (see [33], Section 11). Translated into the language of patterns, this corresponds to decomposing each pattern of a generator $p \in P$ into subpatterns $p_{|_{X(d-1)}}$ and $p_{|_{X^{d-1}}}$, and so to each $a_i \in G(d-1)$, there is some coset of $\mathrm{Triv}_P(d-1)$.

There are 4 cosets for $\mathrm{Triv}_P(d-1)$ in $\mathrm{Triv}_{G(d)}(d-1)$. Note that $a_0 a_{d-1} \notin P$ and $a_0 a_{d-1}^{a_0} = a_{d-1} a_0 \notin P$ by the previous Lemma. So there are at most 2 choices of pattern on the last level corresponding to a_0, and at most 4 choices of coset representative for each a_i, $1 \le i \le d-2$. Thus there are at most $(2)4^{d-2} = 2^{2d-3}$ such groups. $\square$

Remark 3.4.7. We may take the following standard representatives for each coset: 1. the identity, 2. a_{d-1} (corresponding to odd total activity on $0X^{d-2}$ and even total activity on $1X^{d-2}$), 3. $a_{d-1}^{a_0}$ (corresponding to even total activity on $0X^{d-2}$ and odd total activity on $1X^{d-2}$), and 4. $[a_0, a_{d-1}]$ (corresponding to odd total activity on both $0X^{d-2}$ and $1X^{d-2}$).

Our goal is now to prove that this upper bound is also a lower bound. In order to do so, we describe combinatorially the patterns of the index 4 essential pattern subgroups. For this purpose, it is important to note that the action of $G(d)$ on $X^{(d)}$ extends to an action by bijections on the power set of $X^{(d)}$. The fixed points of this action are precisely the sets of the form $X^J = \bigcup_{j \in J} X^j$. We let Δ denote the symmetric difference operation on two subsets of $X^{(d)}$. Given a set J which contains $d-1$, suppose there exist sets $S_1, S_2 \subset X^{(d)}$ which satisfy the following properties:

1. $S_1 \Delta S_2 = X^J$
2. for all $p \in P_J$ and $i \in \{1, 2\}$, we have $p(S_i) \in \{S_1, S_2\}$.

The second condition says that P_J acts by permutations on the set $\{S_1, S_2\}$, which forces S_1 and S_2 to have the same cardinality. Given such sets, we define the set

$$H_{(S_1,S_2)} = \{p \in P_J \mid \sum_{w \in S_1} g_{(w)} = \sum_{w \in S_2} g_{(w)} = 0\}.$$

Proposition 3.4.8. *For S_1, S_2 given above, $H_{(S_1,S_2)}$ is an index 4 subgroup of $G(d)$.*

htp. First, we show that $H_{(S_1,S_2)} \subseteq P_J$. Note that

$$\begin{aligned}
0 &= \sum_{v \in S_1} g_{(v)} + \sum_{w \in S_2} g_{(w)} \\
&= \sum_{v \in S_1 \setminus (S_1 \cap S_2)} g_{(v)} + \sum_{v \in S_1 \cap S_2)} g_{(v)} + \sum_{w \in S_2 \setminus (S_1 \cap S_2)} g_{(w)} + \sum_{w \in S_1 \cap S_2} g_{(w)} \\
&= \sum_{v \in S_1 \setminus (S_1 \cap S_2)} g_{(v)} + \sum_{w \in S_2 \setminus (S_1 \cap S_2)} g_{(w)} \\
&= \sum_{w \in S_1 \Delta S_2} g_{(w)} \\
&= \sum_{w \in X^J} g_{(w)}
\end{aligned}$$

Thus $H_{(S_1,S_2)} \subseteq P_J$.

Note that the above implies that

$$\sum_{w \in S_1} g_{(w)} + \sum_{w \in S_2} g_{(w)} = \sum_{w \in X^J} g_{(w)} = 0$$

for all $g \in P_J$. Thus, for any $g \in P_J$, $\sum_{w \in S_1} g_{(w)} = \sum_{v \in S_2} g_{(v)}$. It follows that for any $h \in P_J$,

$$\sum_{w \in S_i} g_{h(w)} = \sum_{v \in h(S_i)} g_v = \sum_{w \in S_i} g_{(w)}.$$

Now we define a map $\phi : P_J \to C_2$ by $\phi(g) = \sum_{w \in S_1} g_{(w)}$. We claim that ϕ is a homomorphism and that $H_{(S_1,S_2)} = \ker \phi$.

To see that ϕ is a homomorphism, we use the above properties to calculate that

$$\begin{aligned}\phi(gh) &= \sum_{w\in S_1} (gh)_{(w)} \\ &= \sum_{w\in S_1} g_{(h(w))} + h_{(w)} \\ &= \sum_{w\in S_1} g_{(h(w))} + \sum_{w\in S_1} h_{(w)} \\ &= \sum_{w\in S_1} g_{(w)} + \sum_{w\in S_1} h_{(w)} \\ &= \phi(g) + \phi(h)\end{aligned}$$

It is immediate from the definition that $H_{(S_1,S_2)} \subseteq \ker\phi$.

On the other hand, since we showed above that $\sum_{v\in S_1} g_{(v)} = \sum_{w\in S_2} g_{(w)}$, it follows that $\ker\phi \subseteq H_{(S_1,S_2)}$. Thus $H_{(S_1,S_2)}$ has index 2 in P_J, and so it has index 4 in $G(d)$. $\square$

Proposition 3.4.9. *Let $d > 1$. There are at least 2^{2d-3} essential pattern subgroups of index 4 in $G(d)$.*

Proof. We count subgroups of the form $H_{(S_1,S_2)}$ discussed in the previous Proposition. To define such a group, we choose a subset J of $\{1,\ldots,d-1\}$ such that $(d-1)\in J$, (this is the same as choosing an arbitrary subset of $\{1,2,\ldots,d-2\}$, There are clearly 2^{d-2} distinct choices for J. In choosing S_1, note that we must choose for each $j\in J$, whether to put $0X^{j-1}$ or $1X^{j-1}$ in S_1, and for each k in the complement of J, choosing whether or not to include X^k in S_1 (so there are $2^{d-1-|J|}$ such choices). This S_1 determines S_2, and thus determines the subgroup $H_{(S_1,S_2)}$ uniquely. Thus, for each subset J of $\{1,\ldots,d-2,d-1\}$ such that $d-1\in J$, there are $2^{|J|}2^{d-1-|J|} = 2^{d-1}$ subgroups of P_J, and in total $2^{d-2}2^{d-1} = 2^{2d-3}$ such subgroups. $\square$

From the results and discussion in this section, particularly Proposition 3.4.6 and Proposition 3.4.9, we have the following theorem.

Theorem 3.4.10. *For a given $d \geq 2$, there are exactly 2^{2d-3} finitely constrained groups of binary tree automorphisms defined by patterns of size d and having Hausdorff dimension $1 - \frac{2}{2^{d-1}}$.*

3.5 Known examples

Now we discuss examples of linearly constrained groups with index 4 which define topologically finitely generated, finitely constrained groups. These examples come from the patterns of certain groups considered by Bartholdi and Nekrashevych in [11].

Fix a pattern size $d \geq 5$. We now define a particular finitely generated subgroup of G. Define the tree automorphism $r_0 = (01)(1_G, 1_G)$, which is active only at the root. For $1 \leq j \leq d-3$, recursively define automorphisms $r_j = (r_{j-1}, 1_G)$. Let q be the tree automorphism satisfying $q = (r_{d-3}, q)$. The subgroup $R \leq G$ generated by the set $\{r_0, r_1, \ldots, r_{d-3}, q\}$ is precisely the group $\mathfrak{R}(0^{d-3}, 1)$ defined by Bartholdi and Nekrashevych in [11, Section 4]. (The generators $r_0, r_1, \ldots, r_{d-3}$ correspond to what they call the pre-periodic generators $b_1, b_2, \ldots, b_{d-2}$, and the element we call q corresponds to the periodic generator a_1 in their notation.) For $0 \leq j \leq d-3$, we have $\operatorname{supp}(r_j) = \{0^j\}$, and one can show that $\operatorname{supp}(q) = \{1^n 0^{d-3} \mid n \geq 0\}$.

The following theorem summarizes facts that are consequences of results proven in [11, Section 4].

Theorem 3.5.1. *The self-similar group R has the following properties.*

1. *The commutator subgroup $[R, R]$ is the kernel of the map $\phi : R \to \prod_{i=0}^{d-2} C_2$ given by $[\phi(g)]_i = \sum_{w \in X^i} g_{(w)}$.*

2. *R is a regular branch group over its commutator $[R, R]$.*

3. *The commutator $[R, R]$ contains the level $d-1$ stabilizer R_{d-1}.*

Proof.

1. See [11, Proposition 4.2].

2. See [11, Theorem 4.10].

3. Clear, since any element in R_{d-1} is annihilated by the map ϕ given in (i).

□

The previous two theorems and Theorem 3.6 imply that the topological closure of the finitely generated group R is equal to the finitely constrained group $G_{R(d)}$. The Hausdorff dimension of $G_{R(d)}$ was calculated by Pink (see Section 2.5.3) To calculate the Hausdorff dimension of $G_{R(d)}$, we will describe the patterns of $R(d)$. First, we define a homomorphism from a maximal subgroup of $G(d)$, and then we show that the kernel of this homomorphism coincides with $R(d)$.

For the remainder of this section, let $A = X^{d-2} \cup X^{d-1}$. We define subsets

$$A_0 = 0X^{d-3} \cup 1X^{d-2} \text{ and } A_1 = 1X^{d-3} \cup 0X^{d-2}$$

which form a partition of A. (This partition splits the last two levels of $X^{[d-1]}$ in a "crossing" manner, giving those of the top left and bottom right to A_0 and those of the top right and bottom left to A_1).

The map $\theta_0 : P_J \to C_2$ given by $\theta_0(g) = \sum_{w \in A_0} g_{(w)}$ is a homomorphism, by Proposition 3.4.8.

Remark 3.5.2. Let $P_{A_0} = \ker\theta$. A generating set for P_{A_0} consists of all elements of one of the following four types.

(**Type 1.**) $g \in P_J$ such that $\mathrm{supp}(g) \subseteq X^{[d-3]}$

(**Type 2.**) $g \in P_J$ having exactly two nontrivial labels, both contained in $0X^{d-3}$.

(**Type 3.**) $g \in P_J$ having exactly two nontrivial labels, both contained in $1X^{d-2}$

(**Type 4.**) $g \in P_J$ for which $\mathrm{supp}(g) \subseteq A_0$ and $\sum_{w\in 0X^{d-3}} g_{(w)} = \sum_{w\in 1X^{d-3}} = 1$.

Proposition 3.5.3. *Let $R(d)$ be the patterns of size d which appear in R, and let P_{A_0} be the subgroup of $G(d)$ defined by $\ker\theta_0$. Then $R(d) = P_{A_0}$.*

Proof. The projection $\pi_d : R \to R/R_d$ gives the patterns of $R(d)$, with the image of the generators of R given in terms of the standard generators of $G(d)$ by $\pi_d(r_i) = a_i$ for $0 \le i \le d-3$ and $\pi_d(q) = a_{d-2}(a_{d-1})^{a_0}$. Setting $b = \pi_d(q)$ for convenience, we observe that $\mathrm{supp}(b) = \{0^{d-2}, 10^{d-1}\}$. It follows that all generators of $R(d)$ are contained in P_J and annihilated by ϕ, so $R(d) \subseteq P_{A_0}$.

To show that $P_{A_0} \subseteq R(d)$, it suffices to prove that $R(d)$ contains each of the four types of patterns listed in Remark 3.5.2.

(**Type 1**.) Since $R(d)$ contains the elements $a_0, a_1, \ldots, a_{d-3}$, it contains all elements of Type 1.

(**Type 2**.) We first obtain elements which have $\mathrm{supp}(p) = \{0^{d-2}, w\}$, where $w \in X^{d-2}$ and $w \neq 0^{d-2}$. Let T be the subgroup of $R(d)$ consisting of Type 1 elements whose support is in $0X^{[d-2]}$ (i.e., the group is generated by $a_1, a_2, \ldots, a_{d-3}$). It is clear that T acts transitively on the set $0X^{d-3}$, so we can take $t \in T$ such that $t^{-1}(w) = 0^{d-2}$. Then, by the wreath product group multiplication, we have

$$(b^t)_{(w)} = b_{t(w)} = b_{0^{d-2}} = 1,$$

so $\mathrm{supp}(b^t b) = \{0^{d-2}, w\}$. If w_1, w_2 are distinct words in $0X^{d-3}$ and $p_1, p_2 \in R(d)$ such that $\mathrm{supp}(p_1) = \{0^{d-2}, w_1\}$ and $\mathrm{supp}(p_2) = \{0^{d-2}, w_2\}$, then $\mathrm{supp}(p_1p_2) = \{w_1, w_2\}$. Thus all Type 2 generators of P_{A_0} are in $R(d)$.

(**Type 3**) Let S_1 be the subgroup of $R(d)$ consisting of Type 1 elements whose support is in $1X^{[d-2]}$ (i.e. the group generated by $a_1^{a_0}, a_2^{a_0}, \dots, a_{d-1}^{a_0}$), and let S_2 consist of elements of the form h^{a_0}, where h is a Type 2 element. Let S be the group generated by $S_1 \cup S_2$. Then S consists of all patterns which are supported on $1X^{[d-3]}$ and have even total activity on $1X^{d-3}$. The patterns of size $(d-2)$ which on the subtree $1X^{[d-2]}$ in the elements of S correspond to those of a certain maximal subgroup M of $G(d-2)$ such that M has total activity even on the last level. By Proposition 3.2.1, it follows that the action of S on $1X^{d-2}$ is transitive. Thus, we can apply the same reasoning as for Type 2 to see that $R(d)$ contains all elements supported on a set $\{10^{d-1}, v\}$, and therefore all Type 3 elements are in $R(d)$.

(**Type 4**) Taking nontrivial elements $t \in T$ and $s \in S$, we have that $\mathrm{supp}(b^{(ts)}) = \{t(0^{d-2}), s(10^{d-2})\}$. Elements of this form can be used to produce all Type 4 elements.

Thus $R(d)$ contains a generating set for P_{A_0}, and the two subgroups are equal. $\square$

Corollary 3.5.4. *The group $G_{R(d)}$ is a topologically finitely generated, finitely constrained group defined by patterns of size d with Hausdorff dimension $1 - \frac{2}{2^{d-1}}$.*

Proof. By Proposition 3.1.8, it suffices to calculate the size of $R(d)_{d-1}$. Since $R(d)$ is a maximal subgroup of P_J, we have that $[G(d) : R(d)] = 4$. It is not hard to see from the patterns that $R(d-1) = G(d-1)$. Thus $[G(d)_{d-1} : R(d)_{d-1}] = 4$ and $|R(d)_{d-1}| = 2^{2^{d-3}}$. The result follows immediately. $\square$

The argument of Bartholdi and Nekrashevych used in Theorem 3.5.1 to show that $\overline{\mathfrak{R}(0^{d-3}, 0)}$ is finitely constrained can be applied to the closure of any group $\mathfrak{R}(w, v)$ when $|w| \geq 2$ and $|v| \geq 2$, or $|w| \geq 3$ and $|v| \geq 1$. Moreover, if $w, v \in X^*$ with $|w| \geq 3$, $|v| = 1$, then similar arguments to those used in Proposition 3.5.3 show that the patterns of size d in $\mathfrak{R}(w,v)$ are the same as those of $\mathfrak{R}(0^{d-3},1)$, so $\mathfrak{R}(w,v)$ and $\mathfrak{R}(0^{d-3},1)$ have the same topological closure in $\mathrm{Aut}(X^*)$.

3.6 New examples

In this section we discuss a family of self-similar groups inspired by the first Grigorchuk group. For each $d \geq 5$, this family contains an example of a topologically finitely generated, finitely constrained group defined by patterns of size d and which has Hausdorff dimension $1 - \frac{2}{2^{d-1}}$. We will also explicitly describe the defining patterns of these groups and show that they are distinct from those of the groups discussed in the previous section.

Fix k a positive integer, and define A to be the self-similar group generated by the finite-state automaton

$$X = \{r_0 = \sigma(1,1), r_i = (r_{i-1},1)(\text{ for } 1 \leq i \leq k), b = (r_k, c), c = (r_k, d), d = (1, b)\}.$$

We will first establish some basic facts about A. Most (but not all) of these facts and their proofs parallel known results for the Grigorchuk group.

Lemma 3.6.1. *Every element in X has order 2.*

Proof. It is clear that r_0^2 is trivial. Thus $r_1^2 = (r_0^2, 1) = (1,1)$, and it follows recursively that $r_i^2 = (r_{i-1}^2, 1) = (1,1)$ for all $0 \leq i \leq k$. Thus $b^2 = (a_k, c^2) = (1, c^2)$, $c^2 = (a_k, d^2) = (1, d^2)$, and $d^2 = (1, b^2)$. Since b^2 is in $\mathrm{Triv}_A(1)$, d^2 is in $\mathrm{Triv}_A(2)$, and c^2 is in $\mathrm{Triv}_A(3)$. Inductively, It follows that b^2, c^2, d^2 are in $\mathrm{Triv}_A(n)$ for all n, and thus they are all equal to the identity. $\square$

Remark 3.6.2. It follows that $x^{-1} = x$ and $[x,y] = (xy)^2$ for any $x, y \in X$.

Remark 3.6.3. The portraits of the generators of A can be described as follows:

$\mathrm{supp}(r_i) = 0^i$ for $0 \leq i \leq k$, $\mathrm{supp}(b) = \{1^t 0^{k+1} \mid t \equiv 0, 1 \mod 3\}$, $\mathrm{supp}(c) = \{1^t 0^{k+1} \mid t \equiv 0, 2 \mod 3\}$, $\mathrm{supp}(d) = \{1^t 0^{k+1} \mid t \equiv 1, 2 \mod 3\}$.

Lemma 3.6.4. *Let $h = (h_0, 1) \in A$. Then, for any $g \in A$, there exists h' such that $h' = (h_0^g, 1)$.*

Proof. First, we show that this holds for the generating set of A. Let $y \in X$. From the definition of X, there exists $x \in X$ and $i \in \{0, 1\}$ such that $x_i = y$.

If $i = 0$, then we take

$$h^{x^{a_0}} = (y, x_0)(h_0, 1)(y, x_0) = (h_0^y, x_0^2) = (h_0^y, 1)$$

If $i = 1$, we take

$$h^x = (y, x_1)(h_0, 1)(y, x_1) = (h_0^y, x_1^2) = (h_0^y, 1).$$

For the case of an arbitrary group element, we simply note that conjugation is an action of A on itself, so we can obtain $(h_0^g, 1)$ through repeated conjugation by generators. □

Proposition 3.6.5. *A is a regular branch group, branching over the commutator.*

Proof. Let K be the commutator subgroup of A.

First, we show that K has finite index. Since each element of X has order two, each element in the generator of the abelianization of A has order at most two. We know that an abelian group generated by a finite set of elements with finite order is finite. Thus A is a regular branch group branching over K.

It is well-known that K is equal to the normal closure of the group generated by commutators $[x, y]$, for $x, y \in X$. Thus, we need to show that for any $x, y \in X$, the element $([x, y], 1) \in K$. We make the following observations.

- $d = bc$, so we do not have to consider commutators involving d.

- b and c commute, so we do not need to consider $[b, c]$.

- For $1 \leq i \leq k$, $[a_i, b] = [a_i, c]$, since

$$[a_i, b] = (a_{i-1}, 1)(a_k, c)((a_{i-1}, 1)(a_k, c) = ([a_{i-1}, a_k), 1)$$

- If $0 \leq i, j \leq k - 1$, then

$$[a_{i+1}, a_{j+1}] = (a_i, 1)(a_j, 1)(a_i, 1)(a_j, 1) = ([a_i, a_j], 1)$$

  and

$$[a_{i+1}, b] = (a_i, 1)(a_k, c)(a_i, 1)(a_k, c) = ([a_i, a_k], c^2) = ([a_i, a_k], 1).$$

- It remains to show that for any $0 \leq i \leq k$, the element $([a_i, b], 1) \in K$. For $0 \leq i \leq k - 1$, we have

$$[a_{i+1}, d^a] = (a_i, 1)(b, 1)(a_i, 1)(b, 1) = ([a_{i-1}, b], 1)$$

  and for $i = k$

$$[b, d^a] = (a_k, c)(b, 1)(a_k, c)(b, 1) = ([a_k, b], 1).$$

It follows from Lemma that $([x, y]^g, 1) \in K$ for any $x, y \in X$ and $g \in A$. This completes the proof.

□

Proposition 3.6.6. *The groups $G(k+3) = W(k+3)$ are the same.*

Proof. To show that $G(k+3) = W(k+3)$, it suffices to show that each of the standard generators $\{a_0, \ldots, a_{k+2}\}$ of $W(k+3)$ is contained in $A(k+3)$. Note that $\mathrm{supp}(\pi_j(g)) = \mathrm{supp}(g) \cap X^{(j)}$ for any $g \in A$ and any $j \geq 0$. It is obvious that $\pi_{k+3}(r_i) = a_i$ for each $i = 0, \ldots, k$, while $\pi_{k+3}(b) = a_{k+1}a_{k+2}^{a_0}$, $\pi_{k+3}(c) = a_{k+1}$. Thus each a_i for $0 \leq i \leq k+1 \in A(k+3)$. Since $\pi_{k+3}(cb) = a_{k+2}^{a_0}$ and $a_0 \in G(k+3)$, we conclude that $a_{k+2} \in A(k+3)$ and that $G(k+3) = W(k+3)$. $\square$

Proposition 3.6.7. *As above, let K denote the commutator subgroup of A. K contains* $\mathrm{Triv}_G(k+3)$.

Proof. Since A projects onto the group $W(k+3)$ and A is generated by elements of order 2, the abelianization of A is isomorphic to $\prod_{j=0}^{k+2} C_2$. Let π_{ab} be the map from A to $\prod_{j=0}^{k+2} C_2$ given by $[\pi_{\mathrm{ab}}(g)]_{(i)} = \sum_{w \in X^i} g_{(w)}$. This map is surjective, and thus K must be equal to the kernel of π_{ab}. It is also clear that elements of the group $\mathrm{Triv}_A(k+3)$ are annihilated by π_{ab}. $\square$

Corollary 3.6.8. *The group $\overline{A}$ is a finitely constrained group defined by patterns of size $k+3$.*

Proof. This follows immediately from Theorem and the previous 3 propositions. $\square$

Proposition 3.6.9. *Let $d = k+4$. The Hausdorff dimension of $G_{A(d)}$ is $1 - \frac{2}{2^{d-1}}$.*

Proof. Again, we take $W(d)$ to be generated by the set $\{a_0, a_1, \ldots, a_{d-1}$ given in Proposition 3.1.9 , and let π_d be the homomorphism $A \to A(d)$. ISince $H_{A(d)}$ is a finitely constrained, topologically finitely generated group, it follows that $A(d)$ can not be a maximal subgroup of $A(d)$, and thus $[W(d) : A(d)] > 2$. We want to show that $[W(d) : A(d)] = 4$. From Remark 3.4.4, it suffices to show that $[a_1, a_{d-1}] \in A(d)$.

Note that $\pi_d(c) = a_{d-2}^{a_0} a_{d-1}^{a_1 a_0}$, and thus $\pi_d([r_1, c^{r_0}]) = [a_1, (a_{d-2})^{a_0} a_{d-1}^{a_1}]$. Since $a_{d-2}^{a_0}$ has support contained in a disjoint subtree from a_1 and $a_{d-1}^{a_1}$, it follows that $a_{d-2}^{a_0}$ commutes with a_1 and $a_{d-1}^{a_1}$, Thus we calculate

$$\begin{aligned}
[a_1, (a_2)^{a_0} a_{d-1}^{a_1}] &= a_1 a_{d-2}^{a_0} a_{d-1}^{a_1} a_1 a_{d-2}^{a_0} a_{d-1}^{a_1} \\
&= (a_{d-2}^{a_0})^2 [a_1, a_{d-1}^{a_1}] \\
&= a_1 a_1 a_{d-1} a_1 a_1 a_1 a_{d-1} a_1 \\
&= a_{d-1} a_1 a_{d-1} a_1 \\
&= [a_{d-1}, a_1] \\
&= [a_1, a_{d-1}]
\end{aligned}$$

It follows that $[W(d) : A(d)] \geq 4$, and hence $[W(d) : A(d)] = 4$. This completes the proof. □

Remark 3.6.10. We now describe he allowed patterns of the finitely constrained group $\overline{A}$, which are given by the finite group $A(d)$. Let $J = \{k+1, k+2, k+3\}$, and let $X^J = \bigcup_{j \in J} X^j$. By inspecting patterns, we see that $A(k+4)$ is contained in the maximal subgroup P_J of $W(d)$ which is equal to the kernel of the homorphism $\phi_J : W(d) \to C_2, g \mapsto \sum_{w \in X^J} g_{(w)}$.

Next, we take $S_0 = 0X^k \cup 1X^{k+1} \cup 1X^{k+2}$ and $S_1 = 1X^k \cup 0X^{k+1} \cup 0X^{k+2}$. By Proposition 3.4.8, the subgroup $H_{(S_0,S_1)}$ is an index 4, essential pattern subgroup of $W(d)$. Again, it is clear that for each $x \in X$, $\operatorname{supp}(x) \cap S_0$ is even, so $A(k+3) \leq H_{(S_0,S_1)}$. However, we have already shown that $[W(k+3) : A(k+3)] = 4$, so we must have that $A(k+3) = H_{(S_0,S_1)}$. Thus a binary tree automorphism g is contained in

$G_{A(k+3)}$ if and only if for all $w \in X^*$,

$$\sum_{v \in w0X^k \cup w1X^{k+1} \cup w1X^{k+2}} g_{(w)} = \sum_{v \in w1X^k \cup w0X^{k+1} \cup w0X^{k+2}} g_{(w)} = 0$$

All of the groups discussed so far in this section have been defined using the sum of labels on certain invariant subsets of the tree. We now turn our attention to such groups.

3.7 Linearly constrained groups of binary tree automorphisms

We say that an essential pattern subgroup $P \leq W_2(d)$ of finite binary tree automorphisms is *linearly constrained* if there exist subsets $S_1, S_2, \ldots, S_n \subseteq X^{(d)}$ such that

$$p \in P \Leftrightarrow \sum_{w \in S_i} p_{(w)} = 0$$

for all $1 \leq i \leq n$.

We call the sets $\{S_i\}_{i=1}^n$ the *linear constraints of P.*

We say that a set of portraits $\mathcal{X}$ contained in A^{X^*} is *additive* if it is closed under the operation of pointwise addition, i.e. if it forms a subgroup of A^{X^*} when we endow the full shift A^{X^*} with the group structure of a direct product. We write $\oplus$ for the pointwise addition of configurations in this group.

We are interested in groups whose portraits are additive because the patterns defining such groups can be described succinctly, and some important examples in the literature have this property. Recall (from Section 2.5.3) that Siegenthaler and Zugadi-Reizabal proved that certain GGS groups have additive portraits. There is also a connection between these shifts and the subject of *linear cellular automata* as discussed in [18, Section 8]

Proposition 3.7.1. *Let G_P be a finitely constrained group defined by an essential* *pattern subgroup P. If P is* linearly constrained, *then G_P is additive.*

Proof. Suppose P is linearly constrained with constraints $S_1, S_2, \ldots, S_n$. Then, by definition, $g \in G_P$ if and only if $\sum_{v \in S_i}(g_w) = 0$ for all $v \in X^*$. Then for $g, h \in G_P$ and all $v \in X^*$, we have

$$(g \oplus h)_{(v)} = g_{(v)} \oplus h_{(v)}$$

so that

$$\sum_{w \in S_i}(g \oplus h)_{vw} = \sum_{w \in S_i} g_{(vw)} \oplus \sum_{w \in S_i} h_{(vw)} = 0,$$

so $g \oplus h \in G_P$. □

The following theorem follows immediately from the results in the previous two sections.

Theorem 3.7.2. *Let G_P be a finitely constrained group of binary tree automorphisms defined by an essential pattern subgroup P with pattern size d. If $\dim_{\mathrm{H}}(G_P) \geq 1 - \frac{2}{2^{d-1}}$, then the portraits of G_P are additive.*

Remark 3.7.3. The portraits of the first Grigorchuk group are not additive, as is shown in the description of the defining patterns given in [3] .

Proposition 3.7.4. *Let P be an essential pattern subgroup of index 4 and pattern size $d \geq 4$ such that G_P has Hausdorff dimension $1 - \frac{2}{2^{d-1}}$. Let S be a constraint of P which is neither X^J for some $J \subseteq \{0, \ldots, d-1\}$ nor empty. If there exists $y \in X$ such that $S \subseteq yX^{(d-3)}$, then G_P is not topologically finitely generated.*

Proof. Let S_0 be the constraint such that $S_0 \subseteq 0X^{(d-3)}$ and S_1 be the constraint such that $S_1 \subsetneq 1X^{(d-3)}$. Define

$$T_1 = S_0 \cap 00X^{(d-2)}$$
$$T_2 = S_0 \cap 01X^{(d-2)}$$
$$T_3 = S_1 \cap 10X^{(d-2)}$$
$$T_4 = S_1 \cap 11X^{(d-2)}$$

Now define a homomorphism $\phi : P \to C_2$ by

$$\phi(g) = \sum_{w \in T_1} g_{(w)} + \sum_{w \in T_3} g_{(w)}$$

We claim that ϕ is a homomorphism which is not constant on the cosets of $\mathrm{Triv}_P(d-1)$. Since

$$\sum_{w \in T_1} g_{(w)} + \sum_{w \in T_2} g_{(w)} = \sum_{w \in S_1} = 0 \text{ and } \sum_{w \in T_3} g_{(w)} + \sum_{w \in T_4} g_{(w)} = \sum_{w \in S_2} = 0,$$

it follows that the value of ϕ is constant under any permutations of $\{T_1, T_2, T_3, T_4\}$, so ϕ is a homomorphism. To see that ϕ is not constant on cosets of $\mathrm{Triv}_{G(d)}(d-1)$, let g to be an element of P_{d-1} with exactly one nontrivial label on $T_1 \cap X^{d-1}$ and exactly one nontrivial label on $T_2 \cap X^{d-2}$, and let h be the element of P_{d-1} with exactly two nontrivial labels in T_1 and all other labels trivial. These two elements take different values on ϕ even though they are the same pattern up to level $(d-2)$ and thus are in the same coset of $\mathrm{Triv}_P(d-1)$. By Corollary 3.1.4, G_P is not topologically finitely generated. $\square$

Remark 3.7.5. For $d \geq 4$, there are 2^{d-3} essential pattern subgroups of index 4 covered by the preceding proposition - one for each of the maximal subgroups P_J

with $J \subseteq \{2, 3, \ldots, d-1\}$. Thus for $d \geq 4$, we can say that at least 2^{d-3} of the 2^{2d-3} finitely constrained groups with pattern size d and Hausdorff dimension $1 - \frac{2}{2^{d-1}}$ are not topologically finitely generated.

3.8 Computation for examples of topologically finitely generated, finitely constrained groups of binary tree automorphisms

It is clear that there are still large gaps in our knowledge about the actual values which occur as the Hausdorff dimension of some topologically finitely generated, finitely constrained group of binary tree automorphisms. As discussed in Section 2, the values which are known to occur come from explicit families of examples. In this section, we discuss a strategy for finding new examples, with the aim towards extending single examples to new families of examples. The groups discussed in the previous sections were discovered using this strategy.

This approach exploits the fact that in some instances, Theorem 3.1.2 and Proposition 3.1.8 can be used to verify topological finite generation and Hausdorff dimension from the finite quotients of a finitely constrained group. These finite quotients can be determined using the computational group theory program `GAP` [26]. There are two `GAP` packages specifically designed for calculating with self-similar groups, the `FR` package of Bartholdi [8] and the `AutomGrp` package developed by Muntyan and Savcychuk [43]. For a given n, these packages can produce the patterns of size n which appear in a self-similar group (up to limitations on computer hardware). For a finite group, `GAP` can give the relevant subgroups such as level stabilizers and commutators.

Thus, the following heuristic computational approach can be used to produce new examples of Hausdorff dimension .

1. Begin with a self-similar group A, such that $\overline{A}$ is conjectured to be finitely

constrained defined by patterns of size d. Define A as a self-similar group in GAP.

2. Calculate $A(d)$ and $|\operatorname{Triv}_{A(d)}(d-1)|$ in GAP, using one of the packages listed above.

3. Choose some value of $n \geq d$, and test if $|A(n)|$ (which is calculated in GAP) is equal to $|G_{A(d)}(n)|$ (calculated using Proposition 3.1.7).

4. If the $|A(n)| < |G_{A(d)}(n)|$, then $\overline{A}$ can not be a finitely constrained group defined by patterns of size d.

5. If $|A(n)| = |G_{A(d)}(n)|$, then note that $A(n) = G_{A(d)}(n)$ as a finite group. Test if $[\operatorname{Triv}_{A(n)}(d-1), \operatorname{Triv}_{A(n)}(d-1)]$ contains $\operatorname{Triv}_{A(n)}(n-1)$. If this containment holds, then $\operatorname{Triv}_{G_{A(d)}}(d-1), \operatorname{Triv}_{G_{A(d)}}(d-1)]$ contains $\operatorname{Triv}_{G_{A(d)}}(n-1)$ and it follows from Theorem 3.1.2 that $G_{A(d)}$ is a topologically finitely generated, finitely constrained group defined by the size d patterns in $A(d)$.

6. If containment holds in Step 5, then note that the Hausdorff dimension of $G_{A(d)}$ is $\frac{\log_2 |\operatorname{Triv}_{A(d)}(d-1)|}{2^{d-1}}$.

7. If $|A(n)| = |G_{A(d)}(n)|$, but $[\operatorname{Triv}_{A(n)}(d-1), \operatorname{Triv}_{A(n)}(d-1)]$ does not contain $\operatorname{Triv}_{A(n)}(n-1)$, try a larger value of n. If n becomes too large for GAP to work with, try the procedure with a different self-similar group A.

Now we discuss some examples found using this heuristic process. Unfortunately, none of the following examples have computer-independent proofs. We let σ be the nontrivial element of $\operatorname{Sym}(\{0,1\})$, and we recall that a tree automorphism g can be written in the form of a wreath recursion $\sigma(g_0, g_1)$ (if $g(0) = 1$) or (g_0, g_1) (if $g(0) = 0$).

Example 3.8.1.

Let H be the group generated by the following finite-state automaton.

$$b = (f, 1), c = \sigma(d, d), d = (b, 1), f = (c, f)$$

Using the procedure just discussed, it can be verified that $G_{H(5)}$ is a finitely constrained group defined by patterns of size 5, and that $\overline{H}$ has Hausdorff dimension $\frac{9}{16}$. This is smaller than the previously known examples of Hausdorff dimension for topologically finitely generated, finitely constrained groups of binary tree automorphisms. We should note also that this example is not a spinal group, since by examining the patterns of size 5, it can be seen that the root permutation $\sigma(1,1)$ is not part of the group $\overline{H}$.

Example 3.8.2. Consider the group A generated by the following finite-state automaton.

$$a = \sigma(1,1), b = (a,a), c = (b,d), d = (1,c)$$

Using the procedure outlined above, it can be shown computationally that $G_{A(5)}$ is a topologically finitely generated, finitely constrained group defined by patterns of size 5 such that A has Hausdorff dimension $5/16$. We believe this is the smallest known Hausdorff dimension for a topologically finitely generated, finitely constrained group of binary tree automorphisms.

We will discuss the search for additional examples more in the final section of this work.

4. SELF-SIMILAR GROUPS AND TREE LANGUAGES

This section examines self-similar groups using languages accepted by tree automata. This hierarchy includes sofic tree shifts and tree shifts of finite type. We give examples of self-similar groups whose portraits can be used to separate some of the classes in this hierarchy. We also provide a criterion under which the classes of finitely constrained groups and sofic tree shift groups coincide.

The results in this section were obtained as a joint work with Zoran Šunić.

4.1 Classes of tree automata

Recall that in Section 2, we defined tree shifts of finite type and sofic tree shifts, and discussed the connection between tree shifts of finite type and profinite groups. We should mention that there is continued interest in tree shifts from the perspective of symbolic dynamics and theoretical computer science, apart from their connections to group theory and Hausdorff dimension. In particular, Aubrun and Béal have studied tree shifts of finite type [4] and sofic tree shifts [5]. Ceccherini-Silberstein, Coornaert, Fiorenzi and Šunić also studied sofic tree shifts [17]. Groups of tree automorphisms whose portraits form sofic tree shifts answer a question of Grigorchuk [28, Problem 7.4].

In Section 2, we defined a *sofic tree shift* as one which was the image of a tree shift of finite type under a cellular automaton. In the case when $|X| = 1$, a shift $\mathcal{X}$ is called *sofic* if the blocks of X form a regular language (accepted by some finite state automaton). Analogously, sofic tree shifts are exactly the class of those which can be accepted by a particular type of tree automaton, as we now explain.

Definition 4.1.1. An *unrestricted Rabin graph* is a 4-tuple $\mathcal{A} = (S, X, A, \mathcal{T})$ with X and A non-empty finite sets, S a non-empty set, and $\mathcal{T}$ a subset of $S \times A \times S^X$.

X is called the *tree alphabet*, S is called the *state set* or *vertex set*, A is called the *label alphabet*, and $\mathcal{T}$ is called the *set of transition bundles*.

Definition 4.1.2. An *unrestricted Rabin automaton* is an unrestricted Rabin graph with a finite state set.

To any configuration $f \in A^{X^*}$, we can associate an unrestricted Rabin graph $\mathcal{A}_f = (X^*, X, A, \mathcal{T}_f)$ with

$$\mathcal{T}_f = \{(w; f(w); (wx)_{x \in X}) \ \mid \ w \in X^*\}.$$

Given two unrestricted Rabin graphs $\mathcal{A}_1 = (S_1, X, A_1, \mathcal{T}_1)$ and $\mathcal{A}_2 = (S_2, X, A_2, \mathcal{T}_2)$, a *homomorphism from* $\mathcal{A}_1$ *to* $\mathcal{A}_2$ is a map $\alpha : S_1 \to S_2$ such that $(\alpha(s); a; (\alpha(s_x))_{x \in X}) \in \mathcal{T}_2$ whenever $(s; a; (s_x)_{x \in X}) \in \mathcal{T}_1$. We may also say, in an overloading of notation, that α is a homomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$.

Definition 4.1.3. Let $\mathcal{A}$ be an unrestricted Rabin automaton. An element f of A^{X^*} is *accepted* by $\mathcal{A}$ if there exists a homomorphism $\alpha_f : \mathcal{A}_f \to \mathcal{A}$. The *language* $\mathcal{A}$ is the set of all configurations accepted by $\mathcal{A}$.

For any unrestricted Rabin automaton $\mathcal{A}$, the language of $\mathcal{A}$ is a tree shift, denoted by $\mathcal{X}_{\mathcal{A}}$. This leads us to a characterization of sofic tree shfits in terms of unrestricted Rabin automata.

Theorem 4.1.4. *([17, Corollary 3.20]) Let A be a finite alphabet, and let X be a finite set. A subset of $\mathcal{X} \subseteq A^{X^*}$ is a sofic tree shift if and only if it is there exists an unrestricted Rabin automaton $\mathcal{A}$ such that $\mathcal{X} = \mathcal{X}_{\mathcal{A}}$.*

We now discuss two additional classes of tree automata, *Büchi automata* and *Rabin automata*, whose acceptance conditions are defined using *rays* in the tree.

Definition 4.1.5. Let X be a finite set. A *ray* π in X^* is a subset of X^* such that $\epsilon \in \pi$ and such that each $w \in \pi$ has exactly one child in π.

The set of all rays forms a compact topological space called *the boundary of* X^*, which we denote ∂X^*.

Definition 4.1.6. Let A be a finite alphabet and X be a non-empty set. A Büchi automaton $\mathcal{B}$ (over X with alphabet set A) is a 6-tuple $\mathcal{B} = (S, X, A, \mathcal{T}, \mathcal{I}, \mathcal{F})$ where $(S, X, A, \mathcal{T})$ is an unrestricted Rabin automaton, $\mathcal{I}$ is a non-empty subset of S (called the *set of initial states*) and $\mathcal{F} \subset S$ (called the set of *accepting states*).

Definition 4.1.7. Let A be a finite alphabet and X be a non-empty set. A Rabin automaton $\mathcal{R}$ (over X with alphabet set A) is a 6-tuple $\mathcal{R} = (S, X, A, \mathcal{T}, \mathcal{I}, \mathcal{F})$, where $(S, X, A, \mathcal{T})$ is an unrestricted Rabin automaton, $\mathcal{I}$ is a non-empty subset of S (called the *set of initial states*) and $\mathcal{F}$ (the *set of accepting sets*) is a collection of subsets of S.

Acceptance in these classes of automata is based on the notion of a *successful run*.

Definition 4.1.8. Let $f \in A^{X^*}$ be a configuration and $\mathcal{A} = (S, X, A, \mathcal{T}, \mathcal{I}, \mathcal{F})$ be either a Büchi or Rabin automaton over X with alphabet A. A *run* of $\mathcal{A}$ on f is a map $r : X^* \to S$ such that

- r is a homomorphism from the unrestricted Rabin automaton $\mathcal{A}_f$ to the unrestricted Rabin automaton $(S, X, A, \mathcal{T})$
- $r(\epsilon) \in \mathcal{I}$

For a configuration $f \in A^{X^*}$, a ray $\pi \in \partial X^*$, and a run r of either a Büchi or Rabin automaton, let $r_\infty(\pi, f) = \{s \in S \mid r^{-1}(s) \cap \pi \text{ is infinite }\}$. A run r of a

Büchi automaton $\mathcal{B}$ on a configuration f is *successful* if $r_\infty(\pi, f) \cap \mathcal{F} \neq \emptyset$, for all $\pi \in \partial X^*$. For a Rabin automaton $\mathcal{R}$, a run r is successful if for all $\pi \in \partial X^*$, there exists $F \in \mathcal{F}$ (which may depend on π) such that $r_\infty(\pi, f) = F$. A configuration f is *accepted* by a Rabin (or Büchi) automaton $\mathcal{A}$ if there exists a successful run r of $\mathcal{A}$ on f.

For a Rabin (Büchi) automata $\mathcal{A}$, the *language of* $\mathcal{A}$ is written as $\mathcal{L}(A)$ and defined as

$$\mathcal{L}(\mathcal{A}) = \{f \in A^{X^*} \mid \text{ there exists a successful run } r \text{ of } \mathcal{A} \text{ on } f\}.$$

A set $W \subset A^{X^*}$ is Rabin (Büchi) recognizable if there exists a Rabin (Büchi) automaton $\mathcal{A}$ such that $W = \mathcal{L}(\mathcal{A})$.

It is routine (see [35, Sections 5.1 and 5.2]) to show that both Rabin and Büchi languages are closed under taking finite unions, finite intersections, and projections. It is well-known, but much more challenging to prove, that Rabin languages are closed under taking complements, while Büchi languages are not (see [35, Section 5.12]).

Remark 4.1.9. We may view any Büchi tree automaton $\mathcal{B} = (S, X, A, \mathcal{T}, \mathcal{I}, \mathcal{F})$ as a Rabin tree automaton $\mathcal{R} = (S, X, A, \mathcal{T}, \mathcal{I}, \mathcal{F}')$ where $\mathcal{F}' = \{F \subseteq S \mid F \cap \mathcal{F} \neq \emptyset\}$, so any Büchi recognizable set is Rabin recognizable. Similarly, any sofic shift is Büchi recognizable, since we may consider an unrestricted Rabin automaton as a Büchi automaton by taking all states to be both initial and final (initial and final states are not needed to accept a closed, shift-invariant set – see [17, Section 9] for more details). Also, as discussed in Section 2, any tree shift of finite type is a sofic tree shift.

Remark 4.1.10. The automata given here for trees are generalizations of automata accepting right-infinite words. Their applications to trees were given by Rabin in [49] and [48], respectively. What we call Büchi automata were called *special automata* when introduced by Rabin. They are the generalization of a class of automaton introduced by Büchi. Some authors use the term *Muller automata* for what we call Rabin automata. The interested reader should consult [35, Section 3] for additional historical background.

Recall that a *finitely constrained group* is a self-similar group whose portraits form a tree shift of finite type, while a self-similar group whose portraits form a sofic tree shift is called a *sofic tree shift group*. The reader should be warned that the term *sofic* is used in group theory, as in [59]. However, we use the word *sofic* to describe the tree shift, as in the older sense of symbolic dynamics.

For arbitrary trees, the different classes of configuration subspaces discussed so far form a hierarchy as follows:

$$\text{SFT} \subsetneq \text{SOFIC} \subsetneq \text{BÜCHI RECOGNIZABLE} \subsetneq \text{RABIN RECOGNIZABLE}.$$

See the graph in Figure 4.1 for more details on these relationships.In addition, it is shown in [17, Theorem 1.7] that the class of sofic tree shifts is exactly equal to the class of topologically closed subsets of A^{X^*} which are accepted by some Rabin automaton.

We will discuss this hierarchy for self-similar groups shortly. First, we discuss finitely constrained groups and the notion of *branching*, which is a concept utilized in computer science (see [35, page 266])

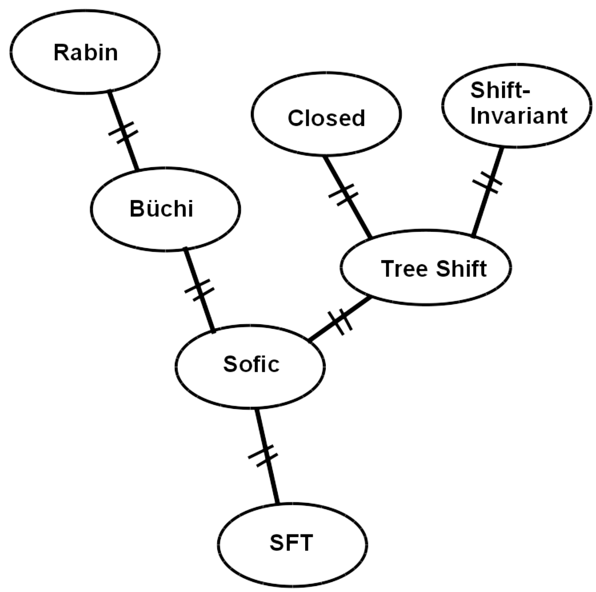

Figure 4.1: Classes of tree configuration spaces. In this graph, Class A is contained in Class B if there is an edge between A and B and if B lies above A in the figure. The dashed lines on the edges indicate strict containment.

4.2 Branching and finitely constrained groups

The notion of *grafting* is well-known in computer science as a way of combining parts of two labeled trees. We use it here to describe the concept of a *regular branch group* and characterize finitely constrained groups.

Definition 4.2.1. Let $a, b \in A^{X^*}$. The *grafting* of b on a at $v \in X^*$ is the element $g_{[a,b,v]}$ of A^{X^*} given by

$$\left(g_{[a,b,v]}\right)_{(w)} = \begin{cases} b_{(u)}, v = wu \in wX^* \\ a_{(w)}, v \notin wX^* \end{cases}$$

Lemma 4.2.2 (Grafting Lemma)**.** *Let $\mathcal{A}$ be an unrestricted Rabin automaton and suppose $a, b \in \mathcal{L}(\mathcal{A})$ and $v \in X^*$ such that*

(i.) $a_{(v)} = b_{(\epsilon)}$

(ii.) there exist homomorphisms $\alpha_a : X^* \to S$ by which $\mathcal{A}$ accepts a and $\alpha_b : X^* \to S$ by which $\mathcal{A}$ accepts b such that $\alpha_a(v) = \alpha_b(\epsilon)$.

Then $\mathcal{A}$ accepts the grafting of b on a at v.

Proof. Define a map $\alpha_{[a,b,v]} : X^* \to S$ by

$$\alpha_{[a,b,v]}(w) = \begin{cases} \alpha_b(u), w \in vX^* \text{ and } w = vu \\ \alpha_a(w), \text{ otherwise} \end{cases} .$$

We claim that this is a homomorphism by which $\mathcal{A}$ accepts $g_{[a,b,v]}$. We must show that for any $w \in X^*$, the transition bundle

$\left(\alpha_{[a,b,v]}(w); g_{[a,b,v]}(w); (\alpha_{[a,b,v]}(wx))_{x\in X}\right) \in \mathcal{T}$.

There are three cases. If $w \in vX^*$ and $w = vu$, then we have

$$\left(\alpha_{[a,b,v]}(w); (g_{[a,b,v]})_{(w)}; (\alpha_{[a,b,v]}(wx))_{x\in X}\right) = \left(\alpha_b(u)); b_{(u)}; (\alpha_b(ux))_{x\in X}\right),$$

and $(\alpha_b(u)); b_{(u)}; (\alpha_b(ux))_{x\in X}) \in \mathcal{T}$ since α_b accepts b. If $w \notin vX^*$ and $wx \neq v$ for any $x \in X$, then $\alpha_{[a,b,v]}(w) = \alpha_a(w)$, and

$$\left(\alpha_{[a,b,v]}(w); g_{[a,b,v]})_{(w)}; (\alpha_{[a,b,v]}(wx))_{x\in X}\right) = \left(\alpha_a(w); a_{(w)}; (\alpha_{[a,b,v]}(wx))_{x\in X}\right)$$

where $\left(\alpha_a(w); a_{(w)}; (\alpha_{[a,b,v]}(wx))_{x\in X})\right) \in \mathcal{T}$ since α_a accepts a. Finally, if $v = wx$ for some $x \in X$, then using the fact that $\alpha_{[a,b,v]}(wx) = \alpha_b(v) = \alpha_a(v)$ gives that

$$\left(\alpha_{[a,b,v]}(w); (g_{[a,b,v]})_{(w)}; (\alpha_{[a,b,v]}(wx))_{x\in X}\right) = \left(\alpha_a(w); a_{(w)}; (\alpha_a(wx))_{x\in X}\right)$$

and, again, $\left(\alpha_a(w); a_{(w)}; (\alpha_a(wx))_{x\in X}\right) \in \mathcal{T}$ since α_a accepts a. This completes the proof. □

Definition 4.2.3. For $v \in X^*$ and $f \in A^{X^*}$, we denote $g_{[e_G,f,v]}$ by $\delta_v(f)$. Note that from the definition, $\delta_v(g)$ is given by

$$(\delta_v(g))_{(w)} = \begin{cases} g_{(z)}, w = vz \text{ for some } z \in X^* \\ e_A, \text{ otherwise} \end{cases}$$

The following useful properties of the δ operator can be easily verified.

Lemma 4.2.4. *For all $g, h \in A^{X^*}$ and $v, w \in X^*$, the following hold.*

(i.) If $v \leq w$ and $w = vu$, then $[\delta_w(g)]_v = \delta_u(g)$

(ii.) $\delta_v(\delta_w(g)) = \delta_{vw}(g)$

(iii.) If $|w| = k$ and $g \in \mathrm{Triv}_G(n)$, then $\delta_w(g) \in \mathrm{Triv}_G(n+k)$.

(iv.) $\delta_v(gh) = \delta_v(g)\delta_v(h)$

Definition 4.2.5. Let G be a subgroup of the full tree shift A^{X^*}, and $k \geq 1$. We say G is a *regular branch group* over the subgroup $\mathrm{Triv}_G(k)$ if $\delta_x(g) \in \mathrm{Triv}_G(k+1)$ for all $g \in \mathrm{Triv}_G(k)$ and $x \in X$.

The following theorem generalizes Theorem 3.6 to the more general case of self-similar groups which are considered here. The proofs given are essentially the same as those found in [28] and [56].

Theorem 4.2.6 (after Theorem 3.6)**.** *Let G be a subgroup of the full tree shift A^{X^*}. G is a finitely constrained group defined by patterns of size $n \geq 2$ if and only if G is the closure in of a group $H \leq A^{X^*}$ which is a regular branch group over* $\mathrm{Triv}_H(n-1)$.

Proof. Let A be a finite group and $X = \{x_1, x_2, \ldots, x_m\}$ be a finite set on which A acts. Assume that $G = G_P$ is a finitely constrained subgroup of A^{X^*} which is defined by an essential pattern group P with pattern size n. Let $g \in \mathrm{Triv}_G(n-1)$ and $x \in X$. It is clear that that $\delta_x(g) \in \mathrm{Triv}(n)$, so we must show that $\delta_x(g)$ is also in G. All size n patterns which appear in $\delta_x(g)$ are either size n patterns which appear in g, or equal to e_P. Since $g \in G_P$ and $e_P \in P$, it follows that $\delta_x(g) \in G$, so G is a regular branch group over its level $(n-1)$ stabilizer.

Suppose now that $G = \overline{H}$, where H is a regular branch group, branching over $\mathrm{Triv}_H(n-1)$. Let $P = H(n)$ be the patterns of size n which appear in the quotient group $H/\operatorname{Triv} H(n)$. Since $G = \overline{H}$, these are exactly the same as the patterns of size

n which appear in $G(n) = G/\operatorname{Triv}_G(n)$. Let G_P be the self-similar group defined by P. We claim that $G = G_P$.

The fact that $G \leq G_P$ follows immediately from Proposition 2.5.10, (iv.)

Now we show that $G_P \subseteq G$. To do this, we will show that for any $g \in G_P$, there exists a sequence $\{h_n\}$ of elements in H which converges to g, i.e. that for each $j \geq 0$, there exists an element $h_j \in H$ with $\pi_{n+j}(h_j) = \pi_{n+j}(g)$. The proof will proceed by induction on j.

Suppose $g \in G_P$. For the base case $j = 0$, note that since $g \in G_P$, $\pi_n(g) \in P = H(n)$, so there exists an element $h \in H$ such that $\pi_n(h) = \pi_n(g)$. Now assume that whenever $k \leq n + j$, there exists an h_k with $h_k^{-1}g \in \operatorname{Triv}(k)$. Note that each $h_k \in G_P$ by Proposition 2.5.10, (iv.). Let $m = n + (j + 1)$. By the previous assumption, there exists $h_{m-1} \in H$ with $h_{m-1}^{-1}g \in \operatorname{Triv}(m - 1)$. We let $f = h_{m-1}^{-1}g$. Note that f is an element of G_P since both h_{m-1} and g are. Thus, for each $x \in X$, the section $f_x \in \operatorname{Triv}_{G_P}(m - 2)$, and we can write $f = \prod_{x \in X} \delta_x(f_x)$. Applying the induction hypothesis, we have that for each $x \in X$, there exists $q_x \in H$ such that $q_x^{-1}f_x \in \operatorname{Triv}(m - 1)$. Note that this implies that each $q_x \in \operatorname{Triv}(m - 1)$, since f_x is. Since H is a regular branch group over $\operatorname{Triv}_H(m - 1)$, it follows that $\delta_x(q_x) \in \operatorname{Triv}_H(m - 1)$ for each $x \in X$. We set $q = \prod_{x \in X} \delta_x(q_x)$, and it follows that $q^{-1}f = \prod_{x \in X} \delta_x((q_x^{-1}f_x)$ is in $\operatorname{Triv}(m)$. Since $q^{-1}h_{n-1}^{-1}g \in \operatorname{Triv}(m)$, and we can define h_n to be $h_{n-1}q$. This Completes the proof. $\square$

4.3 The special case when $|X| = 1$

When $|X| = 1$ and A is a finite alphabet, the full shift A^{X^*} is naturally identified with $A^{\mathbb{N}}$, the set of sequences with entries in A. When A is a finite group, $A^{\mathbb{N}}$ is a group with the direct product structure.

Definition 4.3.1. A *group subshift* is a topologically closed, shift invariant subset of $A^{\mathbb{N}}$.

Here we provide a proof that the result of Kitchens about group shifts over $\mathbb{Z}$ extends to group shifts over $\mathbb{N}$. Our proof uses the same ideas as that of Kitchens, but is somewhat simplified by the fact that we can use Theorem .

Before giving the proof, we need a bit of notation specifically for this case. Let A be a finite group. We write elements of A^n as $[g_0, g_2, g_3, \ldots, g_{n-1}]$, to avoid confusion with products of elements in A, but we write $[\underbrace{e_A, e_A, \ldots, e_A}_{k\text{times}}]$as $[e^n]$. If G is a subgroup of A, we write $G(n)$ for elements of length n which appear in G. Since there is only one $x \in X$, we write $\delta_x(g)$ as $\delta(g)$, and we note that $\delta([g_0, \ldots, g_{n-1}]) = [e_A, g_0, \ldots, g_{n-1}]$. Finally, for $w \in G(n)$, we define *the follower set of w in G* to be

$$\mathrm{fol}_G(w) = \{a \in A \mid [w, a] \in G(n+1)\}.$$

Proposition 4.3.2. *Let A be a finite alphabet. If G is a group subshift of $A^{\mathbb{N}}$, then G is a finitely constrained group.*

Proof. First, we claim that for any $k \geq 0$, the set $\mathrm{fol}([e^k])$ is a subgroup of A. First, note that we must have $[e^m] \in G(m)$ for all $m \geq 0$ since $G(m)$ is a subgroup of A^m. Thus $e \in \mathrm{fol}[e^k]$ and the set is non-empty. Since A is a finite group, it suffices to show closure under the binary operation of A. If $a, b \in \mathrm{fol}([e^k])$, then $[e^k, a]$ and $[e^k, b]$ are in $G(k+1)$, and thus so is their product $[e^k, a][e^k, b] = [e^k, (ab)]$. Thus $gh \in \mathrm{fol}[e^k]$, and $\mathrm{fol}[e^k]$ is a subgroup of A for all $k \geq 0$.

It follows that $\mathrm{fol}([e]) \supseteq \mathrm{fol}([e^2]) \supseteq \mathrm{fol}([e^3]) \ldots$ is a descending chain of subgroups of the finite group A. Thus there exists N such that $\mathrm{fol}([e^n]) = \mathrm{fol}([e^N])$ for all $n \geq N$. Now we claim that G is a regular branch group over $\mathrm{Triv}_G(N)$. Indeed, if $[g] \in \mathrm{Triv}_N$, then $g = [e^N, a]$ for some $a \in \mathrm{fol}(e^N)$. But then $a \in \mathrm{fol}(e^{N+1}$, so

$\delta(g) = [e^{N+1}, a] = [e, e^N, a] = \delta(g) \in \mathrm{Triv}_G(n+1)$. It is obvious that $\mathrm{Triv}_G(N)$ has finite index in G, since it is the kernel of the map $G \to G(N)$, which is a surjective map onto a finite group. Thus G is regular branch over $\mathrm{Triv}_G(n)$, and G is finitely constrained. $\square$

4.4 Language hierarchy for subgroups of full tree shift groups

This section is dedicated to examples of self-similar groups lie in the various classes given in Figure 4.1. Henceforth, whenever A is a finite group which has a left action ϕ on a finite set X, we will write the infinite iterated wreath product $F(A, X, \phi)$ as A^{X^*} (suppressing reference to ϕ).

Our first example shows that there are Büchi-recognizable tree shift groups which are not sofic tree shift groups.

Example 4.4.1 (A Büchi-recognizable self-similar group which is not a sofic tree shift group)**.** Let $G = A^{X^*}$ for some finite set X, some finite group A, and some left action ϕ. Let B be a proper subgroup of A. We define the subset H_{fin} to be

$$H_{\text{fin}} = \{h \in A^{X^*} \mid \text{ there exists an } N_h \text{ such that } |v| > N_h \text{ implies that } h_{(v)} \in B\}.$$

Note that H_{fin} is self-similar. However, H_{fin} is not closed. In fact, H_{fin} is dense in G, since for any $g \in G$, we can build a sequence of elements $\{h_n\}$ in H_{fin} which converge to g by letting h_n and g agree on $X^{[n]}$, and taking h_n to be trivial everywhere else. Since H_{fin} is not topologically closed, H_{fin} is not sofic.

We will show now that H_{fin} is a subgroup of G. If $h_1, h_2 \in H_{\text{fin}}$, then there exists N_1, N_2 such that $h_{1(v)} \in B$ whenever $|v| > N_1$, and there exists N_2 such that ${h_2^{-1}}_{(v)} \in B$ whenever $|v| \in N_2$. Then, taking $N = \max\{N_1, N_2\}$, it follows that whenever $|v| > N$, we have

$$h_1 h_2^{-1}{}_{(v)} = h_{1\left(h_2^{-1}(v)\right)} h_{2(v)} \in B,$$

as $|h_2^{-1}(v)| = |v|$. Also, it is clear that H_{fin} is self-similar and self-replicating. Moreover, if the action of H on X is transitive, then H_{fin} is level-transitive, as well. We will show that H_{fin} is Büchi. Consider the Büchi automaton $\mathcal{B} = (S, X, A, \mathcal{T}, \mathcal{I}, \mathcal{F})$, where

- $S = \{s_1, s_2\}$
- $\mathcal{T}$ consists of transition bundles of the following forms:

  for all $a \in A$, $T_{1,a} = (s_1; a; (s_1)_{x \in X}) \in \mathcal{T}$;

  for all $b \in B$, $T_{1,2,b} = (s_1; b; (s_2)_{x \in X}) \in \mathcal{T}$;

  for all $b \in B$, $T_{2,b} = (s_2; b; (s_2)_{x \in X}) \in \mathcal{T}$
- $\mathcal{I} = \{s_1\}$
- $\mathcal{F} = \{s_2\}$

The computation of the Büchi automaton $\mathcal{B}$ is given as follows. The automaton begins in the initial state s_1, after which $\mathcal{B}$ can remain in s_1 by reading any element of A, or it can transition to s_2 by reading any element of B. Once in s_2, it will remain at s_2, at which point it can only read elements of B. Thus a configuration is accepted by $\mathcal{B}$ if there exists a run which eventually reaches s_2 and never leaves. We show now that the set of elements accepted by $\mathcal{B}$ is the same as the subgroup H_{fin}.

If $h \in H_{\text{fin}}$, with N_h such that $h_{(v)} \in H$ whenever $|v| > N_h$, we can define a successful run $r_h : \mathcal{A}_f \to \mathcal{B}$ by

$$r_h(v) = \begin{cases} s_1, |v| \leq N_h \\ s_2, |v| > N_h \end{cases} .$$

Thus $H_{\text{fin}} \subset \mathcal{L}(\mathcal{B})$.

Now suppose $h \in \mathcal{L}(\mathcal{B})$. Let r_h be a successful run of h on $\mathcal{B}$. By the definition of Büchi acceptance, each ray π must take the value s_2 infinitely often. However, every transition bundle in $\mathcal{B}$ which begins at s_2 also ends at s_2, so $r_h(w) = s_2$ implies that $r_h(wx) = s_2$ for all $x \in X$. It follows from induction that for every ray π, there is a $w \in \pi$ such that r_h only takes the value s_2 on wX^*. Since each ray π is contained in some such wX^*, and the sets wX^* are open in the topology of ∂X^*, these sets form an open cover of ∂X^*. Since ∂X^* is compact, we can take a finite collection $\{w_1, w_2, \ldots, w_n\}$ such that each ray π is contained in at least one open set from the finite collection $\{w_iX^*\}_{i=1}^n$ and such that for each $v \in w_iX^*$, $r_h(v) = s_2$. Taking

$$N = \max\{|w_1|, |w_2|, \ldots, |w_n|\},$$

we have that $r_h(v) = s_2$ whenever $|v| > N$. However, the only transition bundles from s_2 to itself are labeled by elements of B, so we have that $v_{(h)} \in B$ whenever $|v| > N$. Thus $h \in H_{\text{fin}}$.

In the case that A is a cyclic group of prime order with its usual action on $X = \{0, 1, \ldots, p-1\}$, the previous example is exactly that of the *finitary tree automorphisms* studied by Sidki in [51].

The next example utilizes the standard construction of a Rabin recognizable subset which is not Büchi recognizable. The key observation is that this tree language describes the portraits of a self-similar group.

Example 4.4.2 (A Rabin-recognizable self-similar group which is not Büchi-recognizable)**.** Let $X = \{0,1\}$ be a finite set and $A = C_2 = \{\mathrm{id}, \sigma\}$ be the cyclic group of order 2 acting transitively on X. Let

$$H = \{g \in A^{X^*} \mid \text{ every ray in } g \text{ has only finitely many vertices with nontrivial label}\}.$$

The set H is a well-known example of a tree language which is Rabin but not Büchi (see [58]). It is clear that H is self-similar. First we show that H is a subgroup of A^{X^*}. Indeed, an element $h \in A^{X^*}$ is in H if and only if for every ray π, there exists an N such that for all $v \in \pi$ with $|v| > N$, $h_{(v)} = \mathrm{id}$. Let $h_1, h_2 \in H$, and let π be a ray in $h_1 h_2^{-1}$. Since $h_1 \in H$ and $h_2^{-1}(\pi)$ is a ray in X^*, there exist an N_1 such that $(h_1)_{(h_2^{-1}(v))}$ is the identity whenever $|h_2^{-1}(v)| = |v| > N_1$. Since $h_2 \in H$, there exists an N_2 such that $(h_2)_{(v)}$ is the identity whenever $|v| > N_2$. Taking $N = \max\{\, N_1, N_2 \,\}$, it follows that whenever $|v| > N$

$$(h_1 h_2^{-1})_{(v)} = h_{1(h_2^{-1}(v))} h_{2(v)} = \mathrm{id}\,.$$

Thus H is a self-similar, self-replicating, level-transitive subgroup which is Rabin-recognizable but is not Büchi-recognizable. Note that $H_{\mathrm{fin}} \leq H$, so H is dense in A^{X^*} as well.

At present, we do not know if all sofic tree shift groups are finitely constrained. The remainder of this section will be dedicated to describing sufficient conditions to ensure that a sofic tree shift group is finitely constrained.

4.4.1 Branching structure and sofic tree shift groups

In this part, we examine the structure of certain elements and subgroups of sofic tree shift groups.

For the remainder of this subsection, we assume A is a finite group with identity element e_A, and let X be a finite alphabet. We also fix an action ϕ of A on X, and (as usual) identify the full tree shift group $\mathcal{F}(A, X, \phi)$ with the full tree shift A^{X^*}. Also, we let G be a sofic tree shift group, i.e. a subgroup of $\mathcal{F}(A, X, \phi) = A^{X^*}$ such that the portraits of G form a sofic tree subshift of A^{X^*}. The identity of G is denoted by e_G. We let $\mathcal{A} = (S, X, A, \mathcal{T})$ be an unrestricted Rabin automaton so that $G = \mathcal{L}(\mathcal{A})$, and assume that $\mathcal{A}$ has exactly N states.

Lemma 4.4.3. *If $g \in \mathrm{Triv}_G(N)$ and $\alpha_g : X^* \to S$ be a homomorphism by which $\mathcal{A}$ accepts g, then there exists an integer $k = k(g)$ satisfying the following conditions:*

(i.) *for any $w \in X^{[k]}$, the restriction of α to the vertices in the path from ϵ to w is not injective.*

(ii.) $|\alpha_g(X^{[k]})| = |\alpha_g(X^{[k+1]})|$

(iii.) $0 \leq k \leq 2N - 1$

Proof. Condition (i.) is satisfied for any $n \geq N - 1$, by applying the Pigeonhole Principle to the labels of the vertices in the path from ϵ to a vertex $w \in X^{[n]}$. To see (ii.), note that the map $\phi : n \mapsto |\alpha(X^{[n]}|$ is a nondecreasing function, bounded above by N, with $\phi(N - 1) \geq 1$. Thus, there must be a $k \leq 2N - 1$ such that conditions (i.) and (ii.) are satisfied. $\square$

Lemma 4.4.4. *If $g \in \mathrm{Triv}_G(2N - 1)$ via a homomorphism α_g, then there exists a homomorphism α_g' and k with $1 \leq k \leq 2N - 1$ such that α_g' and α_g agree on $X^{[k]}$. In particular, $\alpha_g'(\epsilon) = s$, so s accepts the identity.*

Proof. Assume that s accepts some $g \in \mathrm{Triv}_G(2N-1)$. By the previous lemma, there exists k with $0 \leq k \leq 2N-1$ such that $\alpha_g\left(X^{[k]}\right) = \alpha_g\left(X^{[k+1]}\right)$. Note that for all $w \in X^{[k+1]}$, the transition bundle $(\alpha_g(w); e_A; (\alpha_g(wx))_{x \in X})$ must be in $\mathcal{T}$. Define a function $\beta_g : S \to X^{[k+1]}$ such that $\alpha(\beta_g(s)) = s$. (One possibility is to order the vertices of $X^{[k+1]}$ lexicographically, then let $\beta_g(s)$ be the least element v such that $\alpha(v) = s$.)

We now define a homomorphism $\alpha'_e : X^* \to S$ by which $\mathcal{A}$ accepts e_G. If $w \in X^{[k]}$, set $\alpha_e(w) = \alpha_g(w)$. For $n \geq k+1$, we recursively define α_e on X^n by setting $\alpha_e(wx) = \alpha_g(\beta_g(\alpha(w))x)$. For $w \in X^n$, $n \geq k+1$, let w^* denote $\beta_g(\alpha(w))$.

To see that α_e is a homomorphism by which $\mathcal{A}$ accepts the identity, note that α_g and α_e agree on $X^{[k]}$, and that for all v of length greater than k, we must have the transition bundle $(\alpha_e(v); e_A; (\alpha_e(vx))_{x \in X}) \in \mathcal{T}$, since by construction

$$(\alpha_e(v); e_A; (\alpha_e(vx))_{x \in X})) = (\alpha_g(v^*); e_A; (\alpha_g(v^*x))_{x \in X})$$

for some $v^* \in X^k$. This completes the proof.

$\square$

Remark 4.4.5. Note that for a given $g \in \mathrm{Triv}_g(2N-1)$, the β_g used to construct α_g involves a choice, and thus we can obtain a different α_g' via a different choice of β_g. In particular, for any $v \in X^{[k]}$ and any u with $|u| < |v|$ such that $\alpha_g(u) = \alpha_g(v) = t$, we could define $\beta(t) = u$ and still obtain an α'_g with the desired properties.

Proposition 4.4.6. *Assume $g \in \mathrm{Triv}_G(2N-1)$ and $v \in X^k$ and $u \leq v$. If there exists a homomorphism $\alpha_g : X^* \to S$ by which $\mathcal{A}$ accepts g such that $\alpha_g(u) = \alpha_g(v)$, then $\delta_v(g_u) \in \mathrm{Triv}(2N-1+|v|-|u|)$.*

Proof. Assume $\alpha_g(u) = \alpha_g(v) = t \in S$. By the reasoning above, there exists α_g'

which accepts e_G such that $\alpha_g^{'}$ and α_g agree on $X^{[k]}$. Since G is self-similar, there is a homomorphism α_{g_u} by which $\mathcal{A}$ accepts g_u such that $\alpha_{g_u}(\epsilon) = \alpha_g(u)$. (The map α_{g_u} given by $\alpha_{g_u}(w) = \alpha_g(vw)$ is easily seen to be such a homomorphism.) Now we have that

$$\alpha_{g_u}(\epsilon) = \alpha_g^{'}(u) = \alpha_g^{'}(v)$$

and

$$(g_u)_{(\epsilon)} = g_{(u)} = e_A = (e_G)_{(v)}.$$

Applying the Grafting Lemma yields the desired result. ☐

Corollary 4.4.7. *Assume $g \in \mathrm{Triv}_G(2N-1)$ and $u, v \in X^{[k]}$ with $u \leq v$ and $|u| = j$. If there exists a homomorphism $\alpha_g : X^* \to S$ by which $\mathcal{A}$ accepts g such that $\alpha_g(u) = \alpha_g(v)$, then there exists $u' \in X^{j+1}$ such that $\delta_{u'}(g_u) \in \mathrm{Triv}(2N)$*

Proof. By the previous proposition, we know that $\delta_v(g_u) \in \mathrm{Triv}(2N - 1 + |v| - |u|)$. Since $v > u$, can write $v = v_1u'$ where v_1 is some word (possibly empty) and $|u'| = |u| + 1$. Then we have that $[\delta_v(g_u)]_{v_1} = \delta_{u'}(g_u)$. ☐

Recall that for any group, *conjugation* is a right action of the group on itself given by $g^h = h^{-1}gh$. Given $G \leq A^{X^*}$, we let the *normalizer of G* $N_{A^{X^*}}(G)$ be the elements of A^{X^*} which leave G fixed under conjugation, i.e.

$$N_{A^{X^*}}(G) = \{h \in A^{X^*} \ | \ g^h \in G \text{ for all } g \in G\}.$$

The following lemma is proven in [57] for self-similar groups of tree automorphisms. We will not reproduce the proof here, since it is a lengthy computation which generalizes easily to our current setting.

Lemma 4.4.8. *Let* $g, h \in A^{X^*}$ *and* $u \in X^*$. *Then* $(\delta_u(h))^g = \delta_v(h^{(g_v)})$, *where* $v = g^{-1}(u)$.

Proposition 4.4.9. *Let* G *be a subgroup of* A^{X^*} *such that* $N_{A^{X^*}}(G)$ *contains a self-similar, self-replicating, level-transitive subgroup. If* $\delta_u(g) \in G$ *for some* $g \in G$ *and* $u \in X^n$, *then* $\delta_v(g) \in G$ *for all* $v \in X^n$.

Proof. Suppose $\delta_u(g) \in G$ for some $u \in X^n$ and $g \in G$. Let $v \in X^n$ be arbitrary. Let $N = N_{A^{X^*}}(G)$ be the normalizer of G, and assume that N contains a self-similar, self-replicating, level-transitive subgroup M. Since M is level-transitive, there exists $f \in M$ such that $f(v) = u$. Since M is self-similar, $(f_v)^{-1} \in M$, and since M is self-replicating, there exists $f' \in \mathrm{Stab}_v(M)$ such that $f'_v = (f_v)^{-1}$. Then $(\delta_u(g))^{ff'^{-1}} \in G$ since M normalizes G. Moreover, from these observations and Lemma 4.4.8, it follows that

$$\begin{aligned}
(\delta_u(g))^{f(f')} &= \left((\delta_u(g))^f\right)^{(f')} \\
&= \left(\delta_{f^{-1}(u)}(g^{f_v}))\right)^{f'} \\
&= \left(\delta_v(g^{f_v})\right)^{f'} \\
&= \delta_{f'^{-1}(v)}\left(g^{f_v f'_v}\right) \\
&= \delta_v\left(g^{f_v (f_v)^{-1}}\right) \\
&= \delta_v(g)
\end{aligned}$$

□

4.4.2 Conditions for equivalence of sofic and finitely constrained tree shift groups

With the results of the previous subsection in hand, we can prove the following theorem.

Theorem 4.4.10. *Let G be a subgroup of A^{X^*}. If $N_{A^{X^*}}(G)$ contains a self-similar, self-replicating, level-transitive subgroup, then G is a sofic tree shift group if and only if G is a finitely constrained group.*

Proof. Since every finitely constrained group is a sofic tree shift group, we only need to prove one direction. Since G is a sofic tree shift group, there exists an unrestricted Rabin automaton $\mathcal{A}_G$ such that $\mathcal{L}(\mathcal{A}) = G$. Assume that $\mathcal{A}_G$ has a state set S such that $|S| = N$. We will prove that G is a regular branch group over the subgroup $\mathrm{Triv}_G(2N-1)$. Let $g \in \mathrm{Triv}_G(2N-1)$ and $\alpha_g : X^* \to S$ be a homomorphism by which $\mathcal{A}$ accepts g. By the argument of Lemma 4.4.3, there exists a $k \leq 2N-1$ such that for any $w \in X^k$, some state used by $\mathcal{A}$ on the vertices in the path from ϵ to w is repeated at least once.

For $w \in X^{[k]}$, let $\mu(w)$ be the least element such that $\mu(w) < w$ and the state $\alpha_g(\mu(w))$ is repeated in the path from ϵ to w. Let

$$B_g = \{\mu(w) \mid w \in X^{k(g)}\},$$

and construct a set C_g from B_g as follows: if $b, b' \in B_g$ with $b < b'$, remove b'. It is clear that after the inevitable termination of this procedure, C satisfies the following conditions

(i.) for any $u \in X^{k(g)}$, there is a prefix of u in C

(ii.) if $u, v \in C$ such that $u \leq v$, then $u = v$.

Then, for any $g \in \mathrm{Triv}_G(2N-1)$, we can write $g = \prod_{c \in C(g)} \delta_c(g_c)$. For distinct elements $c, c' \in C$, the element $\delta_c(g_c)$ and $\delta_{c'}(g_{c'})$ commute, as their supports are subsets of disjoint subtrees. Let $x \in X$. By Proposition 4.4.9, the element $\delta_{xc}(g_x) \in$

$\mathrm{Triv}_G(2N)$ for all $c \in C$. Since $\mathrm{Triv}_G(2N)$ is a group, it also contains the product $\prod_{c\in C} \delta_{xc}(g_c)$. Further, we note that

$$\begin{aligned}
\prod_{c\in C} \delta_{xc}(g_c) &= \prod_{c\in C} \delta_x(\delta_c(g_c)) \\
&= \delta_x\left(\prod_{c\in C} \delta_c(g_c))\right) \\
&= \delta_x(g)
\end{aligned}$$

Therefore G is a regular branch group over the subgroup $\mathrm{Triv}_G(2N-1)$, and G is finitely constrained.

□

Theorem 4.4.10 allows us to give our first example of a self-similar group which is not a sofic tree shift group.

Corollary 4.4.11. *The closure of the odometer is not a sofic tree shift group.*

Proof. Let $\mathcal{O}$ represent the odometer group. Since $\mathcal{O}$ is a self-similar, self-replicating, level-transitive subgroup of $\overline{\mathcal{O}}$ and we have shown that $\overline{\mathcal{O}}$ is not finitely constrained, this result follows immediately from Theorem 4.4.10. □

We will discuss other potentially interesting questions related to the computational aspects of finitely constrained groups in the next section.

5. CONCLUSIONS AND FUTURE WORK

In the previous three sections, we have presented various aspects of finitely constrained groups and related topics. In this sections, we present several aspects of this subject which remain to be explored.

In particular, we focus on questions related to the four major topics addressed in this work. The first is the Hausdorff spectrum of topologically finitely generated, finitely constrained groups. The second is an understanding of the essential pattern groups used to define finitely constrained groups. The final topic is the theory of self-similar groups from the computational and symbolic dynamics point of view, especially the relationship between various types of self-similar groups and tree automata.

5.1 The Hausdorff spectrum of topologically finitely generated, finitely constrained groups

In Section 2, we discussed the examples in the litertaure of topologically finitely generated, finitely constrained groups with known Hausdorff dimension. In particular, we saw that the values $1 - \frac{2}{2^{d-1}}$ and $1 - \frac{3}{2^{d-1}}$ occur as the Hausdorff dimension of topologically finitely generated, finitely constrained groups of binary tree automorphisms. Section 3 was dedicated to expanding the knowledge in this area. We also showed that for any prime p and any k with $1 \leq k \leq p^{d-1} - 1$, there does exist a finitely constrained group of p-adic automorphisms with Hausdorff dimension $\frac{k}{p^{d-1}}$. The construction produced groups which are not topologically finitely generated, which naturally leads to the question of which values of Hausdorff dimension can occur for a topologically finitely generated, finitely constrained group.

We addressed this question in Section 3. In particular, we showed that for an arbitrary prime p and pattern size d, there does not exist any topologically finitely generated, finitely constrained group with Hausdorff dimension $1 - \frac{1}{p^{d-1}}$. We then turned our attention to the case $p = 2$ and the next largest Hausdorff dimension for groups defined by pattern size d, $1 - \frac{2}{2^{d-1}}$. We showed that there are 2^{2d-3} such groups, and that their portraits form an abelian group under pointwise addition. We showed that 2^{d-3} of these groups are not topologically finitely generated. For $d \geq 5$, we discussed examples (due to Bartholdi and Nekrashevych) of topologically finitely generated, finitely constrained binary tree automorphisms defined by patterns of size d and having Hausdorff dimension $1 - \frac{2}{2^{d-1}}$. We introduced another family of examples satisfying the same properties. We concluded the section by giving other new examples, verified computationally, finitely generated group whose closures are finitely constrained and have Hausdorff dimension $\frac{9}{16}$ and $\frac{5}{16}$, respectively.

The possible values of Hausdorff dimension which are actually known to occur for a topologically finitely generated, finitely constrained group of binary tree automorphisms is still very limited. This leads to the following questions.

Question 5.1.1. Which values can occur as the Hausdorff dimension of a topologically finitely generated, finitely constrained group of binary tree automorphisms?

Question 5.1.2. Are there other values of k besides $2^{d-1} - 1$ such that $\frac{k}{2^{d-1}}$ can not occur as the Hausdorff dimension of a topologically finitely generated, finitely constrained group with Hausdorff dimension d?

One initial approach would be to use the procedure discussed at the end of Section 3 for finding examples computationally. Using `GAP`, one can produce examples of finitely generated self-similar groups, and in certain circumstances these examples can be shown computationally to have finitely constrained closures. The finitely constrained groups of binary tree automorphisms defined by pattern size $d \leq 4$ are completely understood through the work of Bondarenko and Samoilovych using a mixture of theoretical work and computation. so a natural next step would be to classify the topological finite generation and Hausdorff dimension for finitely constrained groups of binary tree automorphisms defined by patterns of size 5. Since enumerating all subgroups of $\mathrm{Aut}(X^{[5]})$ is not feasible at the present time, studying these groups will require an approach besides exhaustive enumeration of the possibilities.

5.2 Essential pattern groups and finitely constrained groups

Our study of finitely constrained groups of large Hausdorff dimension in Section 3 relied on a complete description of the essential pattern groups which could be used to define them. As we discussed in Section 3, the patterns of these groups for the case $p = 2$ are easily visualized. Moreover, we could use a certain pattern structure to prove that some finitely constrained groups were not topologically finitely generated.

At this moment, we are not sure how approachable is the task of classifying all finite essential pattern groups. What combinatorial properties do their patterns exhibit, and what relationship, if any, do these combinatorial properties have to algebraic properties of the finite groups and topological properties of the finitely constrained groups they define? This is an exceedingly broad question, so it seems natural to begin by examining a few specialized classes. We defined *full pattern groups of size d* and *linearly constrained groups* in Section 3, and the investigation of these classes seems somewhat approachable. The full pattern groups of size d can be investigated algebraically as extensions of one finite group by another, where the isomorphism class of each finitely constrained group is known. *Linearly constrained groups* are simple to understood both visually and algebraically, and classifying the groups in this class would shed light on the general case.

There is a lot of work left to do in this area. Recall that although we understand all essential pattern groups of binary tree automorphisms with pattern size d which define finitely constrained groups with Hausdorff dimension $1 - \frac{2}{2^{d-1}}$, we can only determine for some of them whether or not the finitely constrained group they define is topologically finitely generated. This leads to the following questions.

Question 5.2.1. For a given d, how many of the 2^{2d-3} finitely constrained groups of binary tree automorphisms with Hausdorff dimension $1 - \frac{2}{2^{d-1}}$ are topologically finitely generated?

Question 5.2.2. For a given d and a given k with $1 \leq k \leq 2^{d-1} - 1$, how many essential pattern groups of binary tree automorphisms with pattern size d define finitely constrained groups with Hausdorff dimension $\frac{k}{2^{d-1}}$? How many of these essential pattern groups are full pattern groups of size d? How many are linearly constrained?

Question 5.2.3. For any finite 2-group H, is there an essential pattern group P isomorphic to H?

Question 5.2.4. For a given d, can we give any quantititative information about the number of essential pattern groups of pattern size d? How does it grow asymptotically as a function of d? Are upper and lower bounds attainable?

Again, any solutions to these questions seem likely to require tools from both the theory of finite p-groups and self-similar groups.

5.3 Computational aspects of self-similar groups

As we discussed in both Section 2 and Section 4, the theory of self-similar groups naturally overlaps with symbolic dynamics and computation on trees. In Section 4, we discussed a hierarchy of tree languages, and gave examples of self-similar groups whose portraits fell into the various classes. It is fairly clear, however, that the examples we gave are not topologically finitely generated. This motivates the following question.

Question 5.3.1. Is there an example of a *topologically finitely generated* self-similar group whose portraits form a Rabin-recognizable language but do not form a Büchi-recognizable language? Is there an example of a *topologically finitely generated* self-similar group whose portraits form a sofic tree shift group, but do not form a Buchi recognizable language?

Recall also that in Section 4, we are not able to say whether or not the class of finitely constrained groups and the class of sofic tree shift groups coincide. This question is very interesting to us.

Question 5.3.2. Does there exist a self-similar group whose portraits form a sofic tree shift, but do not form a tree shift of finite type?

On a somewhat more concrete level, we recall also that in Section 4, we showed that a sofic tree shift accepted by an unrestricted Rabin automaton with N states is a finitely constrained group with patterns of size $2N-1$. This suggests a connection between pattern size and the number of states in an unrestricted Rabin automaton, and we are interested in exploring this connection.

Question 5.3.3. Let G be a finitely constrained group defined by patterns of size d. What is the smallest number of states (as a function of d) in an unrestricted Rabin automaton $\mathcal{A}$ which accepts the portraits of G?

5.4 Conclusion

In this work, we have discussed self-similar groups and finitely constrained groups from many different perspectives. We have tried to emphasize the connections between these areas, as in our view these groups are interesting not only as profinite groups, but also for their overlap with topics in finite groups, symbolic dynamics, and the theory of computation. Although we have been able to answer some questions, it is clear that there is much left to do, and many interesting avenues for future exploration.

REFERENCES

[1] A. Abercrombie. Subgroups and subrings of profinite rings. *Math. Proc. Cambridge Philos. Soc.*, 116(2):209–222, 1994.

[2] M. Abért and B. Virág. Dimension and randomness in groups acting on rooted trees. *J. Amer. Math. Soc.*, 18(1):157–192 (electronic), 2005.

[3] Goulnara Arzhantseva, Pierre de la Harpe, Delaram Kahrobaei, and Zoran uni. The true prosoluble completion of a group: Examples and open problems. *Geometriae Dedicata*, 124(1):5–26, 2007.

[4] N. Aubrun and M. Béal. Tree-shifts of finite type. *Theoret. Comput. Sci.*, (459):16–25, 2012.

[5] N. Aubrun and M. Béal. Sofic tree-shifts. *Theory of Computing Systems*, (53):621–644, 2013.

[6] Y. Barnea and A. Shalev. Hausdorff dimension, pro-p groups, and Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 349(12):5073–5091, 1997.

[7] L. Bartholdi. Branch rings, thinned rings, tree enveloping rings. *Israel Journal of Mathematics*, (154):93–109, 2006.

[8] L. Bartholdi. Fr – a {GAP} package "computations with functionally recursive groups", 2014. Version 2.1.1, available at `http://www.gap-system.org/Packages/fr.html` Accessed June 1, 2015.

[9] L. Bartholdi and R. Grigorchuk. Lie methods in growth of groups and groups of finite width. In *Computational and geometric aspects of modern algebra (Edinburgh, 1998)*, volume 275 of *London Math. Soc. Lecture Note Ser.*, pages 1–27. Cambridge Univ. Press, 2000.

[10] L. Bartholdi, R. Grigorchuk, and Z. Šunik. Branch groups. In *Handbook of Algebra*, pages 989–1112. North-Holland, 2003.

[11] L. Bartholdi and V. Nekrashevych. Iterated monodromy groups of quadratic polynomials, I. *Group Geom Dynam*, 2(3):309–336, 2008.

[12] L. Bartholdi and Z. Šunik. On the word and period growth of some groups of tree automorphisms. *Comm. Algebra*, 29(11):4923–4964, 2001.

[13] G. Baumslag. *Topics in Combinatorial Group Theory*. Springer Basel AG, 1993.

[14] I. Bondarenko, R. Grigorchuk, R. Kravchenko, Y. Muntyan, Volodymyr Nekrashevych, Dmytro Savchuk, and Zoran Šunić. On classification of groups generated by 3-state automata over a 2-letter alphabet. *Algebra Discrete Math.*, 1(1):1–163, 2008.

[15] I. Bondarenko and I. Samoilovych. On finite generation of self-similar groups of finite type. *Internat. J. Algebra Comput.*, 23(1):69–79, 2013.

[16] G. Bredon. *Topology and Geometry*. Number 139 in Graduate Texts in Mathematics. Springer, 1993.

[17] T. Ceccherini-Silberstein, M. Coornaert, F. Fiorenzi, and Z. Šunić. Cellular automata between sofic tree shifts. *Theoret. Comput. Sci.*, (506):79–101, 2013.

[18] T. Ceccherini-Silberstein and M. Coorneart. *Cellular Automata and Groups*. Springer, 2009.

[19] T. Ceccherini-Silberstein, Y. Leonov, F. Scarabotti, and F. Tolli. Generalized Kaloujnine groups, uniseriality, and the height of automorphisms. *Internat. J. Algebra Comput*, 15(3):503–527, 2005.

[20] M. Coornaert and A. Papadopolous. *Symbolic dynamics and hyperbolic groups*. Springer, 1993.

[21] P. de la Harpe. *Topics in Geometric Group Theory.* The University of Chicago Press, 2000.

[22] G. Edgar. *Measure, Topology, and Fractal Geometry.* Springer, 2008.

[23] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications.* Wiley, 2003.

[24] G. Fernández-Alcober and A. Zugadi-Reizabal. GGS-groups: Order of congruence quotients and Hausdorff dimension. *Trans. Amer. Math. Soc.*, (366):1993–2017, 2014.

[25] H. Furstenburg. Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation. *Mathematical Systems Theory*, 1(1):1–49, 1967.

[26] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.8*, 2015.

[27] R. Grigorchuk. Just infinite branch groups. In M. du Sautoy, D. Segal, and A. Shalev, editors, *New horizons in pro-p groups*, pages 121–179. Birkhäuser Boston, 2000.

[28] R. Grigorchuk. Solved and unsolved problems around one group. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math*, pages 117–218. Birkhäuser, Basel, 2005.

[29] R. Grigorchuk, Y. Leonov, V. Nekrashevych, and V. Sushchansky. Self-similar groups, automatic sequences, and unitriangular representations, 2014. Preprint. arXiv:1409.5027.

[30] R. Grigorchuk, V. Nekrashevych, and V. Suschanskii. Automata, dynamical systems, and groups. *Proceedings of the Steklov Institute of Mathematics*, (231):128–203, 2000.

[31] N. Gupta and S. Sidki. On the Burnside problem for periodic groups. *Math. Z.*, (182):385–388, 1983.

[32] T. Hungerford. *Algebra*, volume 73 of *Graduate Texts in Mathematics*. Springer, 2003.

[33] D. Johnson. *Presentation of Groups*, volume 22 of *London Math. Soc. Lecture Note Ser.* Cambridge University Press, 1976.

[34] L. Kaloujnine. La structure des p-groupes de Sylow des groupes symétriques finis. *Ann. Sci. École Norm. Sup.*, 3(65):239–276, 1948.

[35] B. Khoussainov and A. Nerode. *Automata Theory and Its Applications.* Birkhäuser, 2001.

[36] B. Kitchens. Expansive dynamics on zero-dimensional groups. *Ergodic Theory and Dynamical Systems*, 7(2):249–261, 1987.

[37] B. Kitchens. *Symbolic Dynamics: One-Sided, Two-Sided, and Countable State Markov Shifts.* Universitext, 1998.

[38] Y. Lavreniuk and V. Nekrashevych. Rigidity of branch groups acting on rooted trees. *Geometriae Dedicata*, 89(1):155–175, 2002.

[39] C. Leedham-Green and S. McKay. *The structure of groups of prime power order.* Oxford University Press, 2002.

[40] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, 1995.

[41] W. Magnus, A. Karass, and D. Solitar. *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations.* Dover Publications, 1976.

[42] J. Meier. *Groups, Graphs, and Trees: An Introduction to the Geometry of Infinite Groups*, volume 73 of *London Math. Soc. Student Texts*. Cambridge University Press, 2008.

[43] Y. Muntyan and D. Savchuk. Automgrp - a `{GAP}` package for computations in self-similar groups and semigroups, 2014. Version 1.2.4, Available at `http://www.gap-system.org/Packages/automgrp` Accessed June 1, 2015.

[44] V. Nekrashevych. *Self-Similar groups*, volume 117 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2005.

[45] A. Penland and Z. Šunić. Finitely constrained groups of maximal Hausdorff dimension. *J. Aus. Math. Soc.*, Accepted.

[46] R. Pink. Profinite iterated monodromy groups arising from quadratic polynomials, 2013. Preprint. arXiv:1307.5678.

[47] W. Plesken. Permutation groups with uniserial modules. *J. London Math. Soc.*, 15(1):51–55, 1983.

[48] M. Rabin. Decidability of second-order theories and automata on infinite trees. *Trans. Am. Math. Soc.*, (141):1–35, 1969.

[49] M. Rabin. Weakly definable relations and special automata. *Studies in Logic and the Foundations of Mathematics*, (59):1–23, 1970.

[50] L. Ribes and P. Zalesskii. *Profinite Groups*, volume 40 of *A Series of Modern Surveys in Mathematics*. Springer, 2000.

[51] S. Sidki. Automorphisms of one-rooted trees: Growth, circuit structure, and acyclicity. *Journal of Mathematical Sciences*, 100(1):1925–1943, 2000.

[52] O. Siegenthaler. Hausdorff dimension of some groups acting on the binary tree. *J. Group Theory*, 11(4):555–567, 2008.

[53] O. Siegenthaler. *Discrete and Profinite Groups Acting on Regular Rooted Trees.* PhD thesis, Georg-August-Universitat Göttingen, 2009.

[54] O. Siegenthaler and A. Zugadi-Reizabal. The equations satisfied by GGS-groups and the abelian group structure of the gupta-sidki group. *European Journal of Combinatorics*, 33(7):1672–1690, 2012.

[55] S. Simpson. Symbolic dynamics: entropy = dimension = complexity. *Theory of Computing Systems*, 56(3):527–548, 2015.

[56] Z. Šunić. Hausdorff dimension in a family of self-similar groups. *Geom. Dedicata*, (124):213–236, 2007.

[57] Z. Šunić. Pattern closure of groups of tree automorphisms. *Bull. Math. Sci.*, 1(1):115–127, 2011.

[58] W. Thomas. Automata on infinite objects. In *Handbook of theoretical computer science (vol. B)*, pages 133–191. MIT Press, 1990.

[59] B. Weiss. Sofic groups and dynamical systems. *Sankhya: The Indian Journal of Statistics, Series A*, 62(3):350–359, 2000.