EPIC: A NEW AND ADVANCED NONLINEAR PARABOLIZED STABILITY EQUATION SOLVER

A Thesis

by

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ABSTRACT

Recent years have witnessed the linear and nonlinear parabolized stability equations (PSE) become a quintessential component toward understanding boundary-layer laminar-to-turbulent transition. Because of the abundant benefits an accurate and trustworthy computational analysis can provide, wind tunnel experiments are commonly supplemented with such studies. Prompted by the rising need to develop a fast, modern, intuitive, and user-friendly PSE code, this work describes the development, validation, and verification of EPIC.

EPIC is a new Nonlinear Parabolized Stability Equation (NPSE) solver developed in-house in our Computational Stability and Transition (CST) lab that will aid in the study, understanding, and prediction of laminar-to-turbulent boundary layer transition problems. This entirely new code is an improvement upon and is intended to replace CST's prior NPSE solver, called JoKHeR. PSE results computed for the NASA Langley 93-10 flared cone, Purdue compression cone, and SWIFTER airfoil are compared and show successful agreement with published computational and experimental results. It is expected that further application of a physics-based approach such as EPIC will lead to more accurate prediction, smaller and more manageable uncertainties in design, and an improved fundamental understanding of the laminar-turbulent transition process that will lead to efficient control strategies.

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NOMENCLATURE

VARIABLES

α	Disturbance Streamwise Complex Wavenumber
β	Disturbance Spanwise Complex Wavenumber
β_{AZ}	Azimuthal Beta
η	Uniform Normal Grid; $\eta \in [0, 1]$
γ	Ratio of Specific Heats
κ	Thermal Conductivity
λ	Second Coefficient of Viscosity or Bulk Viscosity
μ	Dynamic Coefficient Viscosity
Ω	Pressure Gradient Coefficient
ω	Disturbance Frequency
$\partial\xi$	Step Size
Φ	Eigenvector for LST; $[\hat{u}; \hat{v}; \hat{w}; \hat{T}; \hat{\rho}; \hat{\alpha u}; \hat{\alpha v}; \hat{\alpha w}; \hat{\alpha T}]$
ϕ	Flow variables $[u, v, w, T, \rho]$
Ψ	Normalization Parameter Function
ρ	Density
θ_{ki}	Disturbance Growth Direction
θ_k	Phase Angle
ξ	Uniform Streamwise Grid; $\xi \in [Xs_0, Xs_{end}]$
a	Computational Normal Grid Coefficient
b	Computational Normal Grid Coefficient
C_1	Sutherland's Law Constant
C_2	Sutherland's Law Constant
C_p	Specific Heat for Constant Pressure
C_v	Specific Heat for Constant Volume

h_1, h_2, h_3	Streamwise Marching, Wall Normal, and Spanwise Curvilinear Metric Coefficient, respectively
i	$= \sqrt{-1}$
L	Boundary-Layer Reference Length
M	Mach Number
Nx	Number of Streamwise Points
Ny	Number of Normal Points
P	Pressure
Pr	Prandtl Number
R_c	Radius of Curvature
R_g	Specific Gas Constant
Re	Reynolds Number
S_κ	Sutherland's κ Reference Constant
S_μ	Sutherland's μ Reference Constant
T	Temperature
t	Time
u, v, w	Streamwise Marching, Wall Normal, and Spanwise Velocity, respectively
x, y, z	Streamwise Marching, Wall Normal, and Spanwise Coordinate Direction, respectively
Xs	Streamwise Surface Distance
y_{crit}	Used to Determine Computational Normal Grid Clustering
yc	Clustered Computational Normal Grid; $yc \in [0, y_{max}]$
Z	Compressibility Factor
SUBSCRIPTS	
(n, k)	Mode Number, where n and k are integer coefficients of the fundamental ω_0 and β_0 respectively
0	Initial or Fundamental Value
e	Edge or Reference Value

i	Imaginary Value
j	The jth point of an array
r	Real Value

SUPERSCRIPTS

$'$	Unsteady Disturbance Quantity
$\dagger$	Complex Conjugate
$*$	Dimensional Value
-	Steady Basic-State Term
~	Slow-Mode

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I. INTRODUCTION

I.1 Historical Background

Edward Norton Lorenz, a chaos theory pioneer, once summarized his findings as "Chaos: When the present determines the future, but the approximate present does not approximately determine the future" [7]. Thus, it is fitting that while laminar flow is characterized by a smooth and uninterrupted stream, turbulent flow is its chaotic antithesis. As history has taught us, there could not be a more apt description of turbulence and its related processes than Lorenz's own definition.

Though keen observers had previously documented the differences between laminar and turbulent flow, Osborne Reynolds was one of the first to publish about the phenomenon known as laminar-to-turbulent transition. In perhaps his most famous experiment, Reynolds studied water with streaks of color flowing through small glass tubes [38]. He observed that the colored bands remained in the given streak pattern under laminar flow conditions, but the streaks would diffuse and blend together under turbulent conditions. Reynolds noted during his tests that the resulting flow was not determined by just one property. By varying his initial flow conditions, Reynolds used a dimensionless quantity that combined all relevant flow properties, which he penned as Reynolds Number, to help predict flow characteristics. Reynolds noted the baseline criterion for when to expect laminar-to-turbulent transition to occur was anywhere between $Re = 2000 - 13000$ depending on how much care was given to the initial conditions.

While studying the same pipe flow as Reynolds (later coined as Poiseuille flow), William McFadden Orr [32, 33] and Arnold Sommerfeld [42] both independently developed what would eventually be known as the Orr-Sommerfeld equation. This equation attempted to prove that a critical Reynolds number could be solved for, suggesting that any portion of a flow could theoretically be determined as laminar or turbulent based on its characteristics. The Orr-Sommerfeld equation assumes that a small disturbance, possibly originating from an irregularity in the flow or wall roughness, acts upon a laminar flow and can be modeled as a perturbation. If this perturbation is shown to grow, the flow is determined to be unstable and turbulence is assumed to occur. Despite intricate calculations, the problem would remain unsolved for a number of years.

In 1905, Ludwig Prandtl revolutionized the fluid mechanics field with his concept of the boundary layer. A boundary layer is the small fluid layer nearest to the surface that experiences substantial viscosity effects and is responsible for the majority of drag. Prandtl theorized that one could study the flow of a fluid by separating it into two layers. Viscosity was crucial for the flow within the boundary layer, but the outside layer could be treated as inviscid, vastly simplifying the Navier-Stokes equations associated with both flows. This revelation, however, did little more at the time than provide plausibility answers to many of the prevalent questions and paradoxes. The first mathematical application of the boundary layer was in 1908 when Prandtl's first student, Paul Richard Heinrich Blasius, used it to justify solving the Navier-Stokes equations with an order-of-magnitude analysis [5]. The resulting Blasius boundary layer describes the steady, two-dimensional boundary layer of a semi-infinite flat plate parallel to a constant unidirectional flow.

After over a decade of studies in boundary-layer drag and turbulent flow characteristics, Prandtl and his doctoral student, Oskar Tietjans, presented their controversial findings at the Jena physics conference in 1921. Tietjans' doctoral work had been the calculation of boundary layer motion with Rayleigh oscillations, but his findings contradicted previous presumptions. Thus far, all theories failed to yield a possible transition between laminar and turbulent flow, despite higher Reynolds numbers and experiments that proved the opposite. Prandtl and Tietjans' calculations suggested that even the slightest disturbance at the lowest Reynolds number resulted in transition. Tietjans postulated that his theory failed because it was based on unrealistic velocity profiles composed of straight lines with arbitrary kinks. Shortly after this, Werner Heisenberg investigated the stability of the Poiseuille flow, a parabolic velocity profile between two parallel plates commonly observed in a pipe. Heisenberg's doctoral thesis [10] provided a limit of stability, but his approximation methods were impossible to justify at the time. Despite this, Prandtl was again motivated to solve the transition problem and he assigned a new dissertation topic to his next student, Walter Tollmien.

In 1929, Walter Tollmien completed his doctoral dissertation and produced the first successful stability phase diagram. By applying the Orr-Sommerfeld equation to the Blasius boundary layer profile, Tollmien could determine if a system was stable or unstable by the wavelength of the assumed disturbance and the Reynolds number of the flow. Hermann Schlichting expanded on Tollmien's results by explicitly solving for the neutral stability location. He also extended Tollmien's stability phase diagram to account for the pressure gradient of the profile, revealing that the unstable region would grow dramatically for an increasing positive pressure gradient. Finally, Schlichting determined that the onset of turbulence did not occur immediately upon entrance into the unstable region, but that it was dependent on the disturbance wave amplitude. The longer a disturbance wave remained in an unstable state, the more amplified it became, eventually reaching a high enough level to instigate turbulence. Based upon transition locations documented in experiments and his new stability phase diagrams, Schlichting calculated that the natural logarithm of amplitude ratio necessary for turbulence was about nine. This relation is commonly referred to as the N-factor or e^N method, after van Ingen (1956) performed extensive calculations with it.

The theoretical results of Tollmien and Schlichting could not be proven until 1943 (published in 1947 after the war) when Schubauer and Skramstad confirmed their findings with experimental results [41]. Until this time, experiments had been conducted in wind tunnels with high free stream turbulence. By running their experiment in a new, low disturbance wind tunnel, Schubauer and Skramstad saw oscillatory waves as they grew downstream and broke down into turbulent flow. Their experimental agreement validated the linear stability theory efforts of the past 40 years and confirmed that the long-sought critical Reynolds number only determined where the transition would begin, as opposed to the transition location. The two-dimensional (2-D) waves theorized by Tollmien and

Schlichting and witnessed by Schubauer and Skramstad are now referred to as Tollmien-Schlichting waves, or TS waves.

The first attempt at a linear stability theory for compressible flow was made in 1938 by D. Küchemann, a student of none other than Walter Tollmien. Lester Lees and C.C. Lin followed this with a much more in depth application of linear stability theory on a compressible flow in 1946 (published 1947) [24]. Most of their report was focused on the inviscid theory, but an asymptotic viscous theory was included as well. Lees & Lin used a system of sixth-order ordinary differential equations and assumed locally parallel flow in order to derive two-dimensional disturbances in a perfect gas. These methods involved massive hand calculations, and thus disturbance amplitudes were not initially included. Through their results, Lees & Lin expanded upon Rayleigh's inflection theory by confirming that a generalized inflection point ($D\left(\rho D U\right) = 0$, where D is $\frac{\partial}{\partial y}$, ρ is the density, and U is the mean velocity) is a necessary and sufficient condition for neutral stability in a case where the phase speed is less than the freestream velocity. It was also falsely concluded that higher Mach numbers had a stabilizing effect on flows with an adiabatic-wall, based on the fact that the minimum critical Reynolds number decreased; the opposite was actually proved after the disturbance amplitudes were calculated.

The Dunn-Lin theory [8] attempted to develop a better viscous compressible stability theory by removing the largest restrictions from Lees & Lin's theory: the idea that phase speed must be small and that disturbances could only be 2-D. However, the full 3-D stability equations resulted in a system of eighth-order ordinary differential equations that could not be reduced or solved at the time. To circumvent this, the Dunn-Lin theory drops the dissipation terms in the energy equation, thus permitting an order reduction by way of the Squire transformation. As pointed out by Dunn & Lin, this limits the validity of the their theory to studies below Mach 2.0. Their studies concluded that at speeds between Mach 1.0 and Mach 2.0, oblique 3-D disturbances begin to play a large role in the general instability of a boundary layer, most notably because they cannot be fully stabilized through wall cooling like the 2-D disturbances. Soon after, Reshotko [37] and Lees & Reshotko [25] were able to add the dissipation terms back in to the theory, but they still relied on an asymptotic theory to obtain a final solution. These results produced strange multiple neutral curves at high Mach numbers and showed large discrepancies when compared with experimental results from Laufer and Vrebalovich [23].

The utilization of high speed computers in the 1960s finally allowed for direct solutions to linear stability theories. Mack's in depth documentation of this is perhaps the most significant contribution to the stability problem in decades [29]. Mack thoroughly studied supersonic boundary-layer flows over flat plates at speeds up to Mach 10 and noted that supersonic boundary layer disturbances have unique features not seen in their subsonic counterparts. His initial direct solution results compared well to experimental values and confirmed that the asymptotic solution methods attempted by Dunn-Lin and Lees-Reshotko failed to produce accurate neutral stability curves above speeds of Mach 1.6.

Mack summarized his compressible linear stability findings as a couple of key points: 1) For all supersonic mean flows, the first mode is most unstable as an oblique wave, or 3-D disturbance. 2) A region of supersonic mean flow relative to the disturbance phase speed results in an infinite number of additional unstable modes, referred to as acoustic modes, not observed in subsonic mean flows. The first of these acoustic modes, known as the "second mode" or "Mack mode," is the most unstable. All of these additional modes are most unstable as a 2-D wave disturbance. Converse to first mode disturbances, the additional modes are destabilized by wall cooling.

Mack followed up his findings with an update to his 3-D material in 1984 [30]. In addition, this report summarizes and pays homage to previous contributions to the linear stability problem dating all the way back to Rayleigh in the 1800s. Many today still consider this to be the de facto guide to linear stability theory.

For a period following Mack's studies, linear stability theory remained relatively unchanged. It would seem that many no longer viewed it as a problem to be solved, but as a tool to be utilized, and so focus was instead directed towards better implementation. As computers grew more powerful, new and more robust solution methods were developed.

While Mack's approach utilized an initial value method (IVM), a shooting method by way of Runge-Kutta integration, Malik employed a boundary value method (BVM) in 1990 [31]. The BVM reduces the ordinary differential equations into a linear algebraic system and yields a solution without any prior knowledge of the problem or the expected solution.

Although the linear stability theory compared well to experimental values for a flat plate, more accurate formulations were needed for more complex geometries; researchers agreed that the next step was to account for upstream history of disturbances and eliminate both the linearization and the parallel flow approximation. Two distinct methods successfully achieved this goal: the parabolized stability equations and direct numerical simulation of the Navier-Stokes equations.

"Direct numerical simulation" refers to solving for "the numerical solution of the full, nonlinear, time-dependent Navier-Stokes equations without any empirical closure assumptions for prescribed initial and boundary conditions" [18]. This implies that all relevant time and space scales must be resolved. Coincidentally, while DNS is capable of providing the most accurate solutions and remains accurate through transition into turbulent flow, the associated time scales between different regimes vary by orders of magnitude. To further complicate the problem, this in turn necessitates the use of an astronomical number of grid points. DNS analyses are severely inhibited by available algorithms and computational resources. In 2001, Joslin estimated that while DNS of an atmospheric boundary layer is theoretically possible, it would require on the order of 10^{18} grid points, 10^{19} Mwords of memory, 10^{23} operations per second, and about 10 million years of continuous computing time (at 330 Mflops) [17].

Despite its limitations, a growing abundance of DNS studies are published every year. Aided by continual technological advancement, a multitude of research consistently provides explosive growth in the applicability, use, and development of DNS methods. Because PSE methods are the main focus of this paper, we list here only a limited scope of the considerable amount of work and publications contributing to DNS development for interested readers to explore. Kleiser and Zang's 1991 annual review [18] remains a great introduction, Fasel et al. [9] introduces 3-D temporal DNS formulations, Reed [36] extends DNS to the spatial regime, Joslin [17] focuses on the application to laminar flow control, and Zhong [26] and Subbareddy and Candler [43] detail DNS methods appropriate for hypersonic studies. Each of these papers contains further troves of related DNS work.

The parabolized stability equations (PSE) are a complementary method to the DNS that help to facilitate the study of transitional flow by taking advantage of physically accurate assumptions and simplifying equations when able. Developed primarily by Herbert and Bertolotti [3, 4, 11], PSE exists in both a linear (LPSE) and nonlinear (NPSE) variant. It has been proven to accurately predict wave evolution along a predefined path within the shear layer of a configuration for a wide variety of operating conditions. Satisfying the appropriate assumptions to justify marching, PSE methods run in a fraction of the time and at a fraction of the computational cost required for DNS methods. Although the PSE assumptions break down in the later stages of transition and are subject to imposed limitations and assumptions throughout the scheme, excellent validation with DNS and experiments have been achieved from laminar flow through the early stages of transition. For this reason, PSE methods are constantly evolving to become more applicable and capable. The full PSE derivations and related numerical method implementations will be explored during the course of this paper; limitations and assumptions are elaborated upon as they arise.

While much progress has been made over the past century, the stability problem rightly remains one of great interest. The many advances and solution methods have certainly illuminated a great deal, but much remains to be understood. Validation and verification still require that initial conditions be treated with extreme care, as small differences can lead to vastly different results. Laminar flow control appears promising in simpler geometries, but our limited understanding of how different transition mechanisms act in accord with each other have also led to inconsistent transition-control attempts on more complicated geometries. However, the progress made has led to regular use of transition analysis in industry, thus spawning extra motivation for increasingly accurate and practical methods.

I.2 Motivation and Objective

Accurate laminar-to-turbulent transition modeling and prediction is an important problem for nearly every flight regime. Usually through a stability analysis, ongoing research in this area is focused on the physical understanding of transition so as to better capture the appropriate mechanisms in prediction. With better understanding and prediction also come more efficient means of control, whether the desire is to delay or advance a transition location. It has been shown that being able to delay transition and extend the laminar flow regime over commercial aircraft wing surfaces can reduce fuel expenses by 20 or 30 percent. Alternatively, promoting early turbulent flow would extend the flight envelopes for advanced performance aircraft or extremely low speed flight vehicles by delaying or preventing the onset of stall. Accurate transition prediction is especially enabling for hypersonic and reentry vehicles for which the overwhelming heat associated with turbulent flows at high speeds is a considerable design constraint. The ability to extend laminar flow or accurately predict transition location would potentially allow the use of lighter materials and structures, and positively affect range, accuracy predictions, and aerostability controls for high-speed, exoatmospheric-, and space-flight regimes. Furthermore, because turbulent breakdown has been shown to vary greatly with different operating conditions, precise modeling of the transition process and location provides accurate upstream conditions that are of great benefit to the turbulence community.

The present work was begun under the auspices of the Air Force Office of Scientific Research (AFOSR) and the National Aeronautics and Space Administration (NASA) as part of the National Center for Hypersonic Laminar-Turbulent Transition Research, and continues even after the Center has ended. The main objective of the research, consistent with Center goals, is to extend and enhance the theoretical framework to include relevant hypersonic physics and identify dominant instability modes for both two-dimensional (2-D) and three-dimensional (3-D) flowfields. In the process, comprehensive validation with fundamental stability experiments, also a goal of the Center, is being completed. Although especially vital for hypersonic research, transition modeling and prediction research performed is also very relevant to other low speed applications.

EPIC (Euonymous Parabolized Instability Code) is a new Nonlinear Parabolized Stability Equation (NPSE) solver developed in-house in our Computational Stability and Transition (CST) lab that will aid in the study, understanding, and prediction of laminar-to-turbulent boundary layer transition problems. For hypersonic applications, two particular PSE research codes in use today are NASA's LASTRAC (NPSE/LPSE; [6]) and GoHypersonic's STABL (LPSE only; [16]). While both are capable of performing various stability analyses and each has its own appealing and unique features, the objective here is to build a research code that can be promptly used and modified in our own lab. It should be apparent how advantageous it is to possess a well documented, flexible, fully accessible, and fully modifiable code in a rapidly evolving research environment. This entirely new code is an improvement upon and is intended to replace CSTs prior NPSE solver, called JoKHeR. Previously, JoKHeR had been used to successfully validate and improve wind tunnel experiments [14, 19], and help advance stability studies by pioneering new solution techniques for upstream conditions and the modeling of downstream disturbance evolution [21, 20]. EPIC has been designed from the ground up to be modular, more user friendly and intuitive, more accurate and robust, and more easily upgraded as new physical understanding is gained.

I.3 Outline of Thesis

Chapter II presents the full derivation for PSE and the governing basic-state equations. Chapter III will describe the numerical methods taken to implement the aforementioned equations. Chapters IV-VI will focus on verification and validation results of EPIC for the 93-10 Langley flared cone, Purdue compression cone, and SWIFTER wing glove, respectively. Finally, chapter VII will supply a summary of key results. Because of the length and nature of the equations solved, many of the detailed expansions will be found in the appendix.

II. GOVERNING EQUATIONS

Transition from laminar to turbulent flow in flight is known to occur because of the unbounded growth of disturbances within the boundary layer. This process is studied by perturbing a steady basic-state with an initial disturbance. The stability of the flow is then determined based on whether these disturbances grow or decay. Despite being initially infinitesimal in magnitude and too small to accurately measure, a disturbance in an unstable environment can grow large enough to lead to breakdown to turbulent flow.

The governing equations for the basic-state and for the perturbed flow are the Navier-Stokes equations. The basic-state is by itself a solution to these equations, so the perturbed terms remaining constitute the disturbance equations. It is then determined if these perturbations grow or decay downstream. The flow is stable if all of the perturbations are shown to decay, and unstable if at least one element grows. Presented in this chapter are the derivations for the basic state and disturbance governing equations.

II.1 Coordinate System

Considering that the PSE equations march along a path, a Cartesian coordinate system aligned with this marching path is appropriate. The x coordinate will always represent the streamwise direction, the y coordinate is the normal to the surface, and the z coordinate is in the mutually orthogonal spanwise direction. For the rest of this document, we interpret "streamwise direction," and by extension the $x-$axis, to mean "in the marching direction." This is to account for paths that curve or bend downstream. The u, v, and w velocities are the directional velocities oriented in the $x-$, $y-$, and $z-$ directions respectively.

Past studies have proven that boundary-layer stability is very sensitive to a multitude of factors, one of which is the surface geometry. Many real world surfaces are not flat and different surface curvatures will influence pressure gradients, boundary-layer height, and other flow aspects. Curvature effects can be especially crucial in hypersonic regimes because the frequency of the most amplified second mode disturbance is highly tuned to the boundary-layer height.

Thus we must account for these curvature effects by transforming our system into a curvilinear Cartesian system and introducing the necessary metric coefficients. The metric coefficients, h_1, h_2, and h_3, will represent the curvature in the $x-$ (streamwise), $y-$ (normal), and $z-$ (spanwise) directions respectively. These terms are defined as

$$h_{1,3} = \frac{\partial\xi + \frac{y \cdot \partial\xi}{R_c}}{2R_c \cdot \sin\left(\frac{\partial\xi}{2R_c}\right)}. \tag{II.1}$$

R_c is the radius of curvature measured in the streamwise marching direction for h_1 and the spanwise direction for h_3. A negative R_c denotes concave curvatures; positive is convex. Finally, y is the normal distance away from the surface and $\partial\xi$ is the constant streamwise step size. The derivation for these metric terms comes from the definition of a curvilinear coordinate system. If each curve is considered in its own planar direction, the metric coefficient represents the distance traveled along the curved path (arc length) over the equivalent straight line distance. In the absence of any curvature, these lengths are equal and the ratio reduces to 1. An exaggerated visual example can be seen in figure II.1. This formulation breaks down in the limit of a straight line because $R_c \to \infty$ and the metric coefficient must be assigned the correct value of 1.

Note that h_2 does not have a curvature; this will be the wall normal grid. In the interest of demonstrating a complete derivation for a general 3D curvilinear system h_2 will remain in the final equations, but for our purposes $h_2 = 1$.

II.2 Basic-State Equations

Foregoing the curvilinear terms for a moment, the dimensional governing equations for a thermally perfect gas in Cartesian coordinates are listed below.

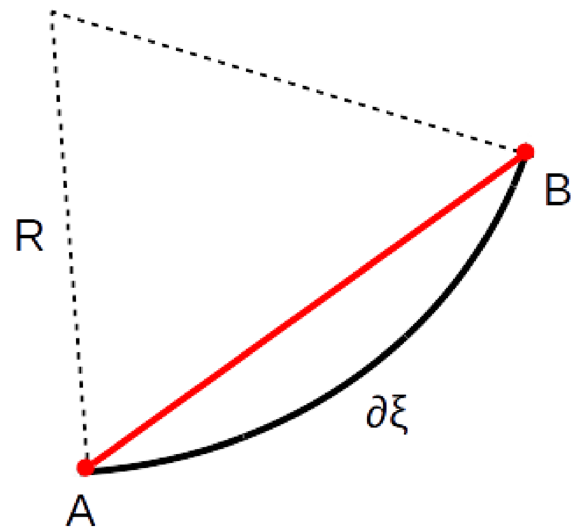

Figure II.1: Exaggerated visualization of curvilinear transformation. Let points A and B represent two sequential points along the surface. The curved surface, black line, possesses a radius of curvature R, which we assume remains constant over this entire step. Traveling from point A to B, the black line $\widehat{\mathrm{AB}}$ represents the physical path and the red line $\overline{AB}$ represents the computational path. The curvature terms account for this.

$$\rho^* \left(\frac{\partial \vec{u^*}}{\partial t^*} + \vec{u^*} \bullet \nabla \vec{u^*} \right) = \nabla \left[-P^* + \mu^* \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right) + \delta_{ij} \lambda^* \nabla \frac{\partial u_k^*}{\partial x_k^*} \right], \qquad i = 1, 2, 3 \tag{II.2}$$

$$\rho^* C_p^* \frac{DT^*}{Dt} = \frac{DP^*}{Dt} + (\nabla \bullet \kappa^* \nabla) T^* + \frac{\mu^*}{2} \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right)^2 + \lambda^* \left(\nabla \bullet \vec{u^*} \right)^2 \tag{II.3}$$

$$\frac{\partial \rho^*}{\partial t^*} + \nabla \bullet \left(\rho^* \vec{u^*} \right) = 0 \tag{II.4}$$

$$P^* = \rho^* R_g^* T^* \tag{II.5}$$

The Navier-Stokes equations, represented as equation (II.2), apply the principles of

X-, Y-, and Z- momentum conservation of a compressible fluid. Equations (II.3)-(II.5) define energy conservation, mass continuity, and the equation of state for a thermally perfect gas, respectively. These equations are then expanded and converted to a general 3D curvilinear coordinate systems as stated in the above section. For brevity, the results will be shown after we nondimensionalize. By using these equations, we have made the following assumptions:

1. The fluid is a Newtonian fluid.
2. There is no body force.
3. There are no chemical reactions.
4. Heat transfer and thermal conductivity follows Fourier's Law.
5. The pressure and temperature are in ranges that allow us to accurately model the gas as a thermally perfect ideal gas.

II.2.1 Thermodynamic Properties

By making the thermally perfect ideal gas assumption, listed above as assumption (5), we are forcing our compressibility factor $Z = \frac{P^*}{\rho^* R_g^* T^*} = 1$. This holds very well if the absolute pressure is near 0 atm abs or if the temperature is above the critical temperature, 133 K for air. It should be observed that at extreme temperatures this assumption will break down. Vibrational excitation begins to take place around a pressure and temperature combination of 1 atm abs and 800 K. Furthermore, oxygen begins to dissociate around 2500 K, forcing the inclusion of chemical nonequilibrium reactions. We can successfully achieve hypersonic speeds that avoid necessitating these reactions as long as care is taken when determining the freestream conditions, but there certainly exists an upper bound where our governing equations will no longer apply. The following mathematical methods used to study stability can be applied, with care, to a different set of governing equations that would account for phenomena such as thermal and chemical nonequilibrium, however we will save those derivations for future research.

For our present situation, a thermally perfect ideal gas model allows us to establish transport quantities as constants or functions of temperature (T) only. Specific heat for constant pressure (C_p), specific heat for constant volume (C_v), and ratio of specific heats (γ) will be held constant, based on an initialized reference point. Thermal conductivity (κ) and dynamic coefficient of viscosity (μ) will be defined using Sutherland's Law,

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S_\mu}{T + S_\mu} \tag{II.6}$$

$$\kappa = \kappa_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S_\kappa}{T + S_\kappa} \tag{II.7}$$

which can also be represented as

$$\mu = \frac{C_1 T^{3/2}}{T + S_\mu} \tag{II.8}$$

$$\kappa = \frac{C_2 T^{3/2}}{T + S_\kappa} \tag{II.9}$$

where

$$\begin{aligned} C_1 &= \frac{\mu_0}{T_0^{3/2}}(T_0 + S_\mu) = 1.458e^{-6} \frac{kg}{m \cdot s \cdot K^{1/2}}, & S_\mu &= 110.40K \\ C_2 &= \frac{\kappa_0}{T_0^{3/2}}(T_0 + S_\kappa) = 2.49e^{-3} \frac{kg \cdot m}{s^3 \cdot K^{3/2}}, & S_\kappa &= 194.00K \end{aligned}$$

are the values used for perfect air. This formulation allows for the easy inclusion of any ideal gas with Sutherland constants. Finally, thermodynamic equilibrium allows the use of Stokes' Hypothesis to set

$$\lambda = -\frac{2}{3}\mu. \tag{II.10}$$

II.3 Nondimensionalization

All variables in equations (II.2)-(II.5) are nondimensionalized by their relative edge quantities, denoted by a subscript e. The edge refers to a location where perturbations have died out and only basic-state quantities remain. It is typically defined as the edge of the boundary layer, but can also be defined by freestream values, sometimes denoted as a "reference" quantity. Pressure is nondimensionalized by dynamic pressure, $\rho_e^* U_e^{*2}$, both viscosity coefficients by μ_e^*, and time by $\frac{L^*}{U_e^*}$. All dimensionless quantities are shown below.

$$u = \frac{u^*}{U_e^*} \quad v = \frac{v^*}{U_e^*} \quad w = \frac{\tilde{w}^*}{U_e^*} \quad x = \frac{x^*}{L^*} \quad y = \frac{y^*}{L^*} \quad z = \frac{z^*}{L^*}$$

$$T = \frac{T^*}{T_e^*} \quad \rho = \frac{\rho^*}{\rho_e^*} \quad \mu = \frac{\mu^*}{\mu_e^*} \quad \kappa = \frac{\kappa^*}{\kappa_e^*} \quad C_p = \frac{C_p^*}{C_{p_e}^*} = 1 \quad C_v = \frac{C_v^*}{C_{v_e}^*} = 1$$

$$t = \frac{t^* L^*}{U_e^*} \quad P = \frac{P^*}{\rho_e^* U_e^{*2}} \quad R_g = \frac{R_g^*}{R_{ge}^*} = 1 \quad \lambda = \frac{\lambda^*}{\mu_e^*} \quad L^* \equiv \sqrt{\frac{\mu_e^* x^*}{\rho_e^* U_e^*}}$$

We choose to define our length quantity, L^*, as the boundary-layer reference length. We also define the following dimensionless quantities.

$$Re \equiv \frac{\rho_e^* U_e^* L^*}{\mu_e^*} \quad Pr \equiv \frac{C_p^* \mu_e^*}{\kappa_e^*} \quad \gamma \equiv \frac{C_p^*}{C_v^*} \quad M^2 \equiv \frac{U_e^{*2}}{\gamma R_g^* T_e^*}$$

The equations below are the result of equations (II.2)-(II.5) in nondimensional 3-D curvilinear coordinates.

X-Momentum

$$
\begin{aligned}
&\rho\left[\frac{\partial u}{\partial t}+\frac{u}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_2}\frac{\partial u}{\partial y}+\frac{w}{h_3}\frac{\partial u}{\partial z}\right.\\
&\qquad\left.-v\left(\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)+w\left(\frac{u}{h_3h_1}\frac{\partial h_1}{\partial z}-\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\right]\\
&\qquad=-\frac{1}{h_1}\frac{\partial P}{\partial x}+\frac{1}{Re}\frac{1}{h_1}\frac{\partial}{\partial x}\left[\frac{\lambda}{h_1h_2h_3}\left(\frac{\partial h_2h_3u}{\partial x}+\frac{\partial h_1h_3v}{\partial y}+\frac{\partial h_1h_2w}{\partial z}\right)\right]\\
&\qquad+\frac{1}{Re}\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x}\left[2\mu h_2h_3\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\right]\\
&\qquad+\frac{1}{Re}\frac{1}{h_1h_2h_3}\frac{\partial}{\partial y}\left[\mu h_1h_3\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u}{h_1}\right)\right)\right]\\
&\qquad+\frac{1}{Re}\frac{1}{h_1h_2h_3}\frac{\partial}{\partial z}\left[\mu h_1h_2\left(\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w}{h_3}\right)\right)\right]\\
&\qquad+\frac{\mu}{Re}\frac{1}{h_1h_2}\frac{\partial h_1}{\partial y}\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u}{h_1}\right)\right)\\
&\qquad+\frac{\mu}{Re}\frac{1}{h_3h_1}\frac{\partial h_1}{\partial z}\left(\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w}{h_3}\right)\right)\\
&\qquad-\frac{2\mu}{Re}\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x}\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_3h_2}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&\qquad-\frac{2\mu}{Re}\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x}\left(\frac{1}{h_3}\frac{\partial w}{\partial z}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_3h_2}\frac{\partial h_3}{\partial y}\right)
\end{aligned}
\tag{II.11}
$$

Y-Momentum

$$
\begin{aligned}
\rho \Bigg[\frac{\partial v}{\partial t} + \frac{u}{h_1}\frac{\partial v}{\partial x} + \frac{v}{h_2}\frac{\partial v}{\partial y} + \frac{w}{h_3}\frac{\partial v}{\partial z} & \\
-w\left(\frac{w}{h_3h_2}\frac{\partial h_3}{\partial y} - \frac{v}{h_3h_2}\frac{\partial h_2}{\partial z}\right) + u\left(\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x} - \frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\Bigg] & \\
= -\frac{1}{h_2}\frac{\partial P}{\partial y} + \frac{1}{Re}\frac{1}{h_2}\frac{\partial}{\partial y}\left[\frac{\lambda}{h_1h_2h_3}\left(\frac{\partial h_2h_3u}{\partial x} + \frac{\partial h_1h_3v}{\partial y} + \frac{\partial h_1h_2w}{\partial z}\right)\right] & \\
+ \frac{1}{Re}\frac{1}{h_1h_2h_3}\frac{\partial}{\partial y}\left[2\mu h_1h_3\left(\frac{1}{h_2}\frac{\partial v}{\partial y} + \frac{w}{h_3h_2}\frac{\partial h_2}{\partial z} + \frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\right] & \\
+ \frac{1}{Re}\frac{1}{h_1h_2h_3}\frac{\partial}{\partial z}\left[\mu h_1h_2\left(\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v}{h_2}\right)\right)\right] & \\
+ \frac{1}{Re}\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x}\left[\mu h_2h_3\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u}{h_1}\right)\right)\right] & \\
+ \frac{\mu}{Re}\frac{1}{h_3h_2}\frac{\partial h_2}{\partial z}\left(\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v}{h_2}\right)\right) & \\
+ \frac{\mu}{Re}\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x}\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u}{h_1}\right)\right) & \\
- \frac{2\mu}{Re}\frac{1}{h_3h_2}\frac{\partial h_3}{\partial y}\left(\frac{1}{h_3}\frac{\partial w}{\partial z} + \frac{u}{h_1h_3}\frac{\partial h_3}{\partial x} + \frac{v}{h_3h_2}\frac{\partial h_3}{\partial y}\right) & \\
- \frac{2\mu}{Re}\frac{1}{h_1h_2}\frac{\partial h_1}{\partial y}\left(\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{v}{h_1h_2}\frac{\partial h_1}{\partial y} + \frac{w}{h_3h_1}\frac{\partial h_1}{\partial z}\right) & \qquad \text{(II.12)}
\end{aligned}
$$

Z-Momentum

$$
\begin{aligned}
\rho\Bigg[\frac{\partial w}{\partial t} &+ \frac{u}{h_1}\frac{\partial w}{\partial x} + \frac{v}{h_2}\frac{\partial w}{\partial y} + \frac{w}{h_3}\frac{\partial w}{\partial z} \\
&\qquad -v\left(\frac{v}{h_3 h_2}\frac{\partial h_2}{\partial z} - \frac{w}{h_3 h_2}\frac{\partial h_3}{\partial y}\right) + u\left(\frac{w}{h_1 h_3}\frac{\partial h_3}{\partial x} - \frac{u}{h_3 h_1}\frac{\partial h_1}{\partial z}\right)\Bigg] \\
&= -\frac{1}{h_3}\frac{\partial P}{\partial z} + \frac{1}{Re}\frac{1}{h_3}\frac{\partial}{\partial z}\left[\frac{\lambda}{h_1 h_2 h_3}\left(\frac{\partial h_2 h_3 u}{\partial x} + \frac{\partial h_1 h_3 v}{\partial y} + \frac{\partial h_1 h_2 w}{\partial z}\right)\right] \\
&\quad + \frac{1}{Re}\frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial z}\left[2\mu h_1 h_2\left(\frac{1}{h_3}\frac{\partial w}{\partial z} + \frac{v}{h_3 h_2}\frac{\partial h_3}{\partial y} + \frac{u}{h_1 h_3}\frac{\partial h_3}{\partial x}\right)\right] \\
&\quad + \frac{1}{Re}\frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial y}\left[\mu h_1 h_3\left(\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v}{h_2}\right) + \frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w}{h_3}\right)\right)\right] \\
&\quad + \frac{1}{Re}\frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x}\left[\mu h_2 h_3\left(\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w}{h_3}\right) + \frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u}{h_1}\right)\right)\right] \\
&\quad + \frac{\mu}{Re}\frac{1}{h_3 h_2}\frac{\partial h_3}{\partial y}\left(\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v}{h_2}\right) + \frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w}{h_3}\right)\right) \\
&\quad + \frac{\mu}{Re}\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x}\left(\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w}{h_3}\right) + \frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u}{h_1}\right)\right) \\
&\quad - \frac{2\mu}{Re}\frac{1}{h_3 h_2}\frac{\partial h_2}{\partial z}\left(\frac{1}{h_2}\frac{\partial v}{\partial y} + \frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x} + \frac{w}{h_3 h_2}\frac{\partial h_2}{\partial z}\right) \\
&\quad - \frac{2\mu}{Re}\frac{1}{h_3 h_1}\frac{\partial h_1}{\partial z}\left(\frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{w}{h_3 h_1}\frac{\partial h_1}{\partial z} + \frac{v}{h_1 h_2}\frac{\partial h_1}{\partial y}\right) \qquad \text{(II.13)}
\end{aligned}
$$

Energy

$$
\begin{aligned}
\rho\left(\frac{\partial T}{\partial t}+\frac{u}{h_1}\frac{\partial T}{\partial x}+\frac{v}{h_2}\frac{\partial T}{\partial y}+\frac{w}{h_3}\frac{\partial T}{\partial z}\right)&=(\gamma-1)\,M^2\left(\frac{\partial P}{\partial t}+\frac{u}{h_1}\frac{\partial P}{\partial x}+\frac{v}{h_2}\frac{\partial P}{\partial y}+\frac{w}{h_3}\frac{\partial P}{\partial z}\right)\\
&+\frac{1}{PrRe}\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x}\left(\frac{\kappa h_2h_3}{h_1}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\kappa h_1h_3}{h_2}\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\kappa h_1h_2}{h_3}\frac{\partial T}{\partial z}\right)\right]\\
&+\frac{(\gamma-1)\,M^2}{Re}\Bigg\{\\
&\left[(2\mu+\lambda)\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_3h_1}\frac{\partial h_1}{\partial z}\right)^2+\mu\left(\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v}{h_2}\right)\right)^2\right]\\
&+\left[(2\mu+\lambda)\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_3h_2}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)^2+\mu\left(\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w}{h_3}\right)+\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u}{h_1}\right)\right)^2\right]\\
&+\left[(2\mu+\lambda)\left(\frac{1}{h_3}\frac{\partial w}{\partial z}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_3h_2}\frac{\partial h_3}{\partial y}\right)^2+\mu\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u}{h_1}\right)\right)^2\right]\\
&+2\lambda\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_3h_2}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&+2\lambda\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_3}\frac{\partial w}{\partial z}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_3h_2}\frac{\partial h_3}{\partial y}\right)\\
&+2\lambda\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_3h_2}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_3}\frac{\partial w}{\partial z}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_3h_2}\frac{\partial h_3}{\partial y}\right)\Bigg\}
\end{aligned}
\tag{II.14}
$$

Continuity

$$
\frac{\partial\rho}{\partial t}+\frac{1}{h_1h_2h_3}\left(\frac{\partial(h_2h_3\rho u)}{\partial x}+\frac{\partial(h_1h_3\rho v)}{\partial y}+\frac{\partial(h_1h_2\rho w)}{\partial z}\right)=0 \tag{II.15}
$$

Equation of State

$$
\gamma M^2P=\rho T \tag{II.16}
$$

II.4 Disturbance Equations

In order to formulate the disturbance equations, a first order perturbation is superposed upon each flow variable. We let ϕ with no superscript represent the total instantaneous value of our flow variables (u, v, w, T, ρ, P, μ, λ, and κ), while $\overline{\phi}$ and ϕ' will represent the steady basic-state and unsteady disturbance quantities respectively.

$$\phi = \overline{\phi}\,(x, y) + \phi'\,(x, y, z, t)\,, \qquad \phi' \ll \overline{\phi} \tag{II.17}$$

Note that the thermodynamic quantities are only a function of T and must be related to x, y, and z. The perturbations of these quantities will be modeled by Taylor Expansion derivatives, resulting in the following relations:

$$\mu' = \frac{\partial\overline{\mu}}{\partial T}T', \qquad \lambda' = \frac{\partial\overline{\lambda}}{\partial T}T', \qquad \kappa' = \frac{\partial\overline{\kappa}}{\partial T}T', \qquad \frac{\partial\overline{\lambda}}{\partial T} = \frac{\overline{\lambda}}{\overline{\mu}}\frac{\partial\overline{\mu}}{\partial T}. \tag{II.18}$$

Adding the steady and unsteady parts results in the total instantaneous value. By substituting equations (II.17) and (II.18) into (II.11)-(II.16), the full governing equations for the instantaneous value results. The basic state by itself is still a solution to these governing equations. This allows for the subtraction of equations (II.11)-(II.16) from the instantaneous result, culminating in the isolation of the disturbance equations.

Pressure can be eliminated from the problem by using the perturbed form of equation (II.16) to give a final disturbance equation in the form

$$\begin{split}\mathcal{B}_0\frac{\partial\phi'}{\partial t} + \mathcal{B}_1\frac{\partial\phi'}{\partial x} + \mathcal{B}_2\frac{\partial\phi'}{\partial y} + \mathcal{B}_3\frac{\partial\phi'}{\partial z} + \mathcal{C}_1\frac{\partial^2\phi'}{\partial x^2} + \mathcal{C}_2\frac{\partial^2\phi'}{\partial y^2} + \mathcal{C}_3\frac{\partial^2\phi'}{\partial z^2} \\ + \mathcal{D}_1\frac{\partial^2\phi'}{\partial x\partial y} + \mathcal{D}_2\frac{\partial^2\phi'}{\partial x\partial z} + \mathcal{D}_3\frac{\partial^2\phi'}{\partial y\partial z} + \mathcal{F}_0\phi' = \mathcal{NL}\end{split} \tag{II.19}$$

where $\phi' = [u', v', w', T', \rho']^T$ and $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{F}_0$ are 5×5 matrices containing only basic-state terms. The entire left hand side of the equation is linear; $\mathcal{NL}$ represents a 5×1

column of nonlinear terms. The next objective is to solve for the disturbance quantities, ϕ'. These quantities must be real in order to solve the governing equations given earlier.

II.4.1 Boundary Conditions

In order to formulate a numerical solution, a set of boundary conditions must be applied.

$$
\begin{aligned}
y_{wall} = 0, \qquad & \begin{cases} u' = v' = w' = T' = 0 & \text{Constant Wall Temperature} \\ u' = v' = w' = \frac{\partial T'}{\partial y} = 0 & \text{Adiabatic Wall} \end{cases} \\
y \to \infty, \qquad & \begin{cases} u' = v' = w' = T' = \rho' = 0 & \text{Subsonic} \\ \frac{\partial u'}{\partial y} = \frac{\partial v'}{\partial y} = \frac{\partial w'}{\partial y} = \frac{\partial T'}{\partial y} = \frac{\partial \rho'}{\partial y} = 0 & \text{Supersonic} \end{cases}
\end{aligned}
\tag{II.20}
$$

Equation (II.20) portrays the boundary conditions applied to the disturbance quantities throughout the various solution methods. These conditions are representative of what we expect from a disturbance in the boundary-layer. At the wall, the no-slip condition demands the total instantaneous values $u = v = w = 0$, thus requiring $u' = v' = w' = 0$. The basic-state values must already independently fulfill the no-slip condition, accounting for $\overline{u} = \overline{v} = \overline{w} = 0$. If given a non-adiabatic wall condition, this also applies to $T\left(y_{wall}\right)$. We ensure y_{max} is sufficiently far away from the wall (but still within the shock if one is present) so that the other half of the boundary conditions can be safely applied. This simply declares y_{max} as the location where the perturbations have died out. The subsonic and supersonic conditions can be interchanged, but this setup provides the conditions that we have found to give us the most clear and consistent results.

II.5 Disturbance Quantity Formulation

Thus far no assumptions about what form the disturbance quantities take have been imposed. The following sections will address three different approximations we can make in order to solve for these quantities. Chapter III will formulate numerical methods to solve each of the three methods.

II.5.1 Linear Stability Theory

Linear stability theory (LST) will be used to generate initial conditions for the more accurate parabolized stability equations below. LST is based upon three main assumptions:

1. The basic state is "locally parallel."

2. Disturbances are small enough to eliminate nonlinear interactions.

3. Unsteady disturbances take the form

$$\phi'(x, y, z, t) \equiv \hat{\phi}(y)\, e^{i(\alpha x + \beta z - \omega t)} + c.c. \tag{II.21}$$

Assumption (1) states that there can be no flow in the basic-state wall normal direction, $\overline{v} \equiv 0$, and that the other basic-state quantities are only functions of y, such that $\overline{u} = \overline{u}(y)$, $\overline{w} = \overline{w}(y)$, $\overline{T} = \overline{T}(y)$, and $\overline{\rho} = \overline{\rho}(y)$. Assumption (3), a wave equation, results from applying Fourier transformations in x and z and a Laplace transformation in t. The wave amplitude ($\hat{\phi}$) is complex, which necessitates the addition of the complex conjugate (*c.c.*) because the disturbance (ϕ') must remain real. The unsteady disturbance amplitude is a function of only y, similar to the steady basic state, and the phase is a function of x, z, t.

LST can be solved as a temporal or spatial problem. Due to its role in the following solution methods, only the *spatial stability* problem will be addressed here. We force ω to be real and allow α and β to be complex, thus allowing our wave disturbance amplitude to grow or decay exponentially in space. α_r and β_r represent the nondimensional streamwise and spanwise wave number respectively ($\lambda_x = \frac{2\pi}{\alpha_r}$ and $\lambda_z = \frac{2\pi}{\beta_r}$) while ω is the nondimensional frequency ($\omega = \frac{2f^*\pi L^*}{U_e^*}$, f^* is frequency in Hz). Applying the above assumptions and (II.21) to equation (II.19) results in

$$\mathcal{A}\frac{\partial^2\hat{\phi}}{\partial y^2} + \mathcal{B}\frac{\partial\hat{\phi}}{\partial y} + \mathcal{C}\hat{\phi} = 0 \tag{II.22}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are 5×5 linear matrices based on the parameters $\left(\alpha, \beta, \omega, \overline{\phi}\right)$ for each y point at a specific x location. These matrices can be seen in full in appendix B. The remaining relations needed to solve the problem are

$$\theta_k = \arctan\left(\frac{\beta_r}{\alpha_r}\right) \qquad \text{and} \qquad \theta_{ki} = \arctan\left(\frac{\beta_i}{\alpha_i}\right),$$

which define the phase angle and the disturbance growth direction respectively. Given a specified ω, β_r, and β_i at our streamwise location, a solution can be found for α_r and α_i. Typically, β_i will be defined as 0 to define the disturbance growth in the marching direction. Then the sign of α_i will determine the stability of the given frequency at the specified x location.

$$\begin{aligned}
&\alpha_i < 0, && \text{amplified disturbances; unstable}\\
&\alpha_i = 0, && \text{no change in space; neutral}\\
&\alpha_i > 0, && \text{damped disturbances; stable}
\end{aligned}$$

II.5.2 Linear Parabolized Stability Equations

The parabolized stability equations have become a popular method for stability analysis because of their improvements over LST. The linear parabolized stability equations (LPSE) eliminate the "locally parallel" assumption that LST requires. In doing this, LPSE also delivers a marching solution that reflects upstream influences. In order to derive the LPSE disturbance form, we take advantage of the fact that basic-state quantities change rapidly in the surface normal direction as compared to the surface streamwise direction. This allows us to use a WKB approximation to decompose our disturbance into a rapidly varying "wave function" and a slowly varying "shape function."

$$\phi'(x,y,z,t) \equiv \underbrace{\hat{\phi}(\tilde{x},y)}_{\text{Shape Function}} \underbrace{e^{i\left(\int_{\tilde{x}_0}^{\tilde{x}} \alpha(x)\partial x + \beta z - \omega t\right)}}_{\text{Wave Function}} + c.c. \tag{II.23}$$

We relate our slow and fast scales through $\tilde{x} = \frac{x}{Re}$. Our streamwise derivatives now take the form

$$\frac{\partial \phi'}{\partial x} = \left(\frac{1}{Re}\frac{\partial \hat{\phi}}{\partial \tilde{x}} + i\alpha\hat{\phi} \right) e^{i\left(\int_{\tilde{x}_0}^{\tilde{x}} \alpha(x)\partial x + \beta z - \omega t\right)} + c.c. \tag{II.24}$$

$$\frac{\partial^2 \phi'}{\partial x^2} = \left[\frac{1}{Re^2}\frac{\partial^2 \hat{\phi}}{\partial \tilde{x}^2} + \frac{i2\alpha}{Re}\frac{\partial \hat{\phi}}{\partial \tilde{x}} + \frac{i\hat{\phi}}{Re}\frac{\partial \alpha}{\partial \tilde{x}} - \alpha^2\hat{\phi} \right] e^{i\left(\int_{\tilde{x}_0}^{\tilde{x}} \alpha(x)\partial x + \beta z - \omega t\right)} + c.c. \tag{II.25}$$

We notice that there is an elliptic term in equation (II.25), but that it is $\mathcal{O}\left(\frac{1}{Re^2}\right)$. By an order of magnitude analysis, we choose to neglect the term $\frac{\partial^2 \hat{\phi}}{\partial x^2}$, resulting in a parabolic equation instead. As numerous papers have previously shown [4, 3, 11], this is a good approximation because most of the ellipticity is captured in the combination of the $i\alpha\hat{\phi}$ and $\frac{\partial \hat{\phi}}{\partial x}$ terms. Note that the basic state is assumed to be in the "fast scale" so when performing the calculations we do not actually perform the $\frac{\partial}{\partial x} = \frac{1}{Re}\frac{\partial}{\partial \tilde{x}}$ substitution.

Finally, substituting equation (II.23) into equation (II.19) and dropping the $\frac{\partial^2}{\partial x^2}$ terms results in

$$\mathcal{A}\frac{\partial^2 \hat{\phi}}{\partial y^2} + \mathcal{B}\frac{\partial^2 \hat{\phi}}{\partial x \partial y} + \mathcal{C}\frac{\partial \hat{\phi}}{\partial y} + \mathcal{D}\frac{\partial \hat{\phi}}{\partial x} + \mathcal{E}\hat{\phi} = 0. \tag{II.26}$$

Once again, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are all 5×5 linear matrices and our problem statement is now parabolic. These matrices have been fully detailed in appendix C. Finding a solution will require an initial condition (LST result) along with the boundary conditions (II.20), as well as a marching scheme and normalization parameter. This and more will be discussed in our problem formulation in the following chapter.

II.5.3 Nonlinear Parabolized Stability Equations

The nonlinear parabolized stability equations (NPSE) are derived in a similar manner as LPSE, but employ a finite-amplitude disturbance instead of the infinitesimally small amplitudes assumed in the previous two methods. By eliminating this approximation, nonlinear disturbances come into play and must be accounted for. As in LPSE, the total disturbance is still assumed periodic in the temporal and spanwise directions, so again a Fourier transformation is utilized.

$$\phi'(x,y,z,t) \equiv \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} A_{0(n,k)} \underbrace{\hat{\phi}_{(n,k)}(\tilde{x},y)}_{\text{Shape Function}} \underbrace{e^{i\int_{\tilde{x}_0}^{\tilde{x}} \alpha_{(n,k)}(x)\partial x} e^{i(k\beta_0 z - n\omega_0 t)}}_{\text{Wave Function}} \tag{II.27}$$

However, for NPSE, the transformation is applied to each mode, represented by (n,k). $A_{0(n,k)}$ is the initial amplitude being applied to each particular mode. Operations are applied the same to NPSE as they were to LPSE, including the parabolization technique. Inserting (II.27) into (II.19) and performing a harmonic balance leads to a system of equations

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left\{ \left[\mathcal{A}\frac{\partial^2 \hat{\phi}}{\partial y^2} + \mathcal{B}\frac{\partial^2 \hat{\phi}}{\partial x \partial y} + \mathcal{C}\frac{\partial \hat{\phi}}{\partial y} + \mathcal{D}\frac{\partial \hat{\phi}}{\partial x} + \mathcal{E}\hat{\phi} \right]_{(n,k)} A_{0(n,k)} e^{i\int_{\tilde{x}_0}^{\tilde{x}} \alpha_{(n,k)}(x)\partial x} e^{i(k\beta_0 z - n\omega_0 t)} \right\} = \mathcal{NL}_{(n,k)} \tag{II.28}$$

where each (n,k) mode corresponds to an individual system of equations. The left hand operators in brackets are the same as in the LPSE equation (II.26) except that each mode has its own particular $\alpha_{(n,k)}$ and $\hat{\phi}_{(n,k)}$. Additionally, ω and β must be replaced with $n\omega_0$ and $k\beta_0$ respectively.

The $\mathcal{NL}$ right-hand side contains the 5×1 array of nonlinear terms. Our harmonic balance ensures that the $\mathcal{NL}$ terms will be of the form

$$
\begin{aligned}
\mathcal{NL}_{(n,k)} = \sum_{n_1}\sum_{n_2}\sum_{k_1}\sum_{k_2} \Big\{ & A_{0(n_1,k_1)} A_{0(n_2,k_2)} \mathcal{NL}^{(quad)}_{(n,k)} \\
& e^{i\int_{x_0}^{x} \alpha_{(n_1,k_1)}(x)\partial x} e^{i\int_{x_0}^{x} \alpha_{(n_2,k_2)}(x)\partial x} e^{i((k_1+k_2)\beta_0 z + (n_1+n_2)\omega_0 t)} \Big\} \\
+ \sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{k_1}\sum_{k_2}\sum_{k_3} \Big\{ & A_{0(n_1,k_1)} A_{0(n_2,k_2)} A_{0(n_3,k_3)} \mathcal{NL}^{(cubic)}_{(n,k)} \\
e^{i\int_{x_0}^{x} \alpha_{(n_1,k_1)}(x)\partial x} e^{i\int_{x_0}^{x} \alpha_{(n_2,k_2)}(x)\partial x} & e^{i\int_{x_0}^{x} \alpha_{(n_3,k_3)}(x)\partial x} e^{i((k_1+k_2+k_3)\beta_0 z + (n_1+n_2+n_3)\omega_0 t)} \Big\}
\end{aligned}
\quad \text{(II.29)}
$$

where n_1, n_2,..., k_3 are all summed from $-\infty$ to ∞, $n_1+n_2\,(+n_3) = n$, and $k_1+k_2\,(+k_3) = k$, thus giving a matching phase speed with the linear terms on the left hand side. The unique system of equations for each (n,k) mode are now coupled by the nonlinear terms. The full nonlinear matrix is expanded in appendix D.

Because the disturbances must still be real, the solution requires the use of complex conjugates ($\hat{\phi}^{\dagger}$). These are accounted for through symmetry properties.

$$
\begin{aligned}
\alpha^{\dagger}_{(n,k)} &= -\alpha_{(-n,-k)} & \beta^{\dagger}_{0(n,k)} &= \beta_{0(-n,-k)} \\
A^{\dagger}_{0(n,k)} &= A_{0(-n,-k)} & \hat{u}^{\dagger}_{(n,k)} &= \hat{u}_{(-n,-k)} \\
\hat{v}^{\dagger}_{(n,k)} &= \hat{v}_{(-n,-k)} & \hat{w}^{\dagger}_{(n,k)} &= \hat{w}_{(-n,-k)} \\
\hat{T}^{\dagger}_{(n,k)} &= \hat{T}_{(-n,-k)} & \hat{\rho}^{\dagger}_{(n,k)} &= \hat{\rho}_{(-n,-k)}
\end{aligned}
\quad \text{(II.30)}
$$

If the basic state exhibits a z-direction symmetry, we can additionally set the following properties. The z-symmetry does not apply for modes with $n = 0$, as these are already covered by (II.30).

$$
\begin{aligned}
\alpha_{(n,k)} &= \alpha_{(n,-k)} & \beta_{0(n,k)} &= \beta_{0(n,-k)} \\
A_{0(n,k)} &= A_{0(n,-k)} & \hat{u}_{(n,k)} &= \hat{u}_{(n,-k)} \\
\hat{v}_{(n,k)} &= \hat{v}_{(n,-k)} & \hat{w}_{(n,k)} &= -\hat{w}_{(n,-k)} \\
\hat{T}_{(n,k)} &= \hat{T}_{(n,-k)} & \hat{\rho}_{(n,k)} &= \hat{\rho}_{(n,-k)}
\end{aligned}
\tag{II.31}
$$

Note that because the complex conjugate is required to formulate a real disturbance, the initial amplitude a particular mode experiences will be double $(A_{0(n,k)} + A^{\dagger}_{0(n,k)})$. For clarity, when applying an initial amplitude of A_0 to a mode, we actually apply $A_0/2$ to the mode and its complex conjugate, (n,k) and $(-n,-k)$.

Finally, the unique mode $(0,0)$, the mean flow distortion, will be addressed in chapter III.

III. NUMERICAL FORMULATION

This chapter focuses on formulating a numerical solution for the equations derived in the previous chapter. The solution for the linear stability problem (II.22) will follow Malik's BVM method [31]. Solutions for both the linear and nonlinear parabolized stability equations (II.26 and II.28) will be aided by Herbert and Bertolotti's methods [4, 11].

III.1 Computational Grid

We begin by reducing the system of second-order differential equations into a system of algebraic equations by way of finite-difference methods. In order to perform an accurate finite differentiation, the first step is to discretize our data; we create a grid with uniform x (ξ) and y (η) to perform our stability calculations on.

Uniform x is a straightforward discretization using surface distance (Xs) of the starting and ending point along with the desired number of streamwise marching points (Nx).

$$\partial\xi = \frac{Xs_{end} - Xs_0}{Nx} \tag{III.1}$$

This gives a constant step size, which is cumulatively added to Xs_0 to build uniform ξ.

Because the stability calculations require a high resolution in the boundary layer, the uniform normal (η) grid will be treated differently. If η were formed in the same manner as ξ, achieving the accuracy required in the boundary layer would require an extremely large number of points. In order to save computation time and minimize the number of normal points (Ny), a uniform normal grid (η) is created and algebraically mapped to a wall clustered computational normal grid (yc) defined by

$$yc = \frac{a\eta}{b - \eta}, \tag{III.2}$$

where

$$\eta \in [0,1]\,, \qquad\qquad yc \in [0, y_{max}]$$
$$a = \frac{y_{max} y_{crit}}{y_{max} - 2y_{crit}}, \qquad\qquad b = 1 + \frac{a}{y_{max}}.$$

This relation puts half of our normal points (Ny) between $yc = 0$ and $yc = y_{crit}$, where y_{crit} is preselected accordingly. y_{max} is defined as far enough away from the wall to apply the $y \to \infty$ boundary conditions from equation (II.20). Finite differences in the normal direction can now be calculated with a constant $\Delta\eta$ spacing and then related to the new computational grid through

$$\begin{aligned} \frac{\partial}{\partial yc} &= \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial yc} \\ \frac{\partial^2}{\partial yc^2} &= \frac{\partial^2}{\partial \eta^2}\left(\frac{\partial \eta}{\partial yc}\right)^2 + \frac{\partial}{\partial \eta}\frac{\partial^2 \eta}{\partial yc^2} \end{aligned} \tag{III.3}$$

where $\frac{\partial \eta}{\partial yc}$ and $\frac{\partial^2 \eta}{\partial yc^2}$ can be calculated directly from equation (III.2).

Note that for ease of use, transformation (III.3) may be applied by declaring our previously unused $h2$ term as

$$h_2 = 1/\frac{\partial \eta}{\partial yc}, \qquad \frac{\partial h_2}{\partial y} = -h_2^3 \frac{\partial^2 \eta}{\partial yc^2}. \tag{III.4}$$

All other h_2 derivatives should remain 0, and then $\Delta\eta$ is used as the spacing for y-derivative finite differences. This can be seen from the following relationship:

	Grid Transformation	Curvilinear Derivative
$\frac{\partial \phi}{\partial yc} =$	$\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial yc}$	$\frac{1}{h_2}\frac{\partial \phi}{\partial \eta}$
$\frac{\partial^2 \phi}{\partial yc^2} =$	$\frac{\partial^2 \phi}{\partial \eta^2}\left(\frac{\partial \eta}{\partial yc}\right)^2 + \frac{\partial \phi}{\partial \eta}\frac{\partial^2 \eta}{\partial yc^2}$	$\frac{1}{h_2^2}\frac{\partial^2 \phi}{\partial \eta^2} - \frac{1}{h_2^3}\frac{\partial h_2}{\partial yc}\frac{\partial \phi}{\partial \eta}$

All of the following calculations are performed on the computational grid (ξ, yc). For simplicity only, the rest of the equations will refer to the (x, y) grid (e.g. the term $\frac{\partial u}{\partial x}$ actually refers to $\frac{\partial u}{\partial \xi}$ and $\Delta\xi$ should be used in finite derivatives, NOT Δx).

III.2 Finite-Difference Method

By mapping to a uniform computational grid, we permit the use of standard finite-difference methods. Wall normal derivatives will be solved with a fourth order central finite difference scheme,

$$
\begin{aligned}
\frac{\partial \hat{\phi}_j}{\partial y} &= \frac{-\hat{\phi}_{j+2} + 8\hat{\phi}_{j+1} - 8\hat{\phi}_{j-1} + \hat{\phi}_{j-2}}{12\Delta y} \\
\frac{\partial^2 \hat{\phi}_j}{\partial y^2} &= \frac{-\hat{\phi}_{j+2} + 16\hat{\phi}_{j+1} - 30\hat{\phi}_j + 16\hat{\phi}_{j-1} - \hat{\phi}_{j-2}}{12\Delta y^2},
\end{aligned}
\tag{III.5}
$$

where applicable. Boundaries will utilize second order left- or right-sided scheme and one point off from boundaries will utilize a central second order scheme.

Because the LPSE and NPSE equations are parabolized and no longer elliptical, we must used a different scheme for the streamwise derivatives. By utilizing a second order left-sided finite difference scheme,

$$\frac{\partial \hat{\phi}_j}{\partial x} = \frac{3\hat{\phi}_j - 4\hat{\phi}_{j-1} + \hat{\phi}_{j-2}}{2\Delta x}, \tag{III.6}$$

our equations will be influenced by upstream values, but immune to the downstream effects. Again, accuracy must be decreased to a first order scheme when one point off of a boundary.

III.3 Local Eigenvalue Solution

In order to solve the LST equation (II.22), an eigenvalue problem approach will be used. By applying a fourth order central finite difference scheme (III.5), the problem simplifies to a system of algebraic equations with five unknowns ($\hat{\phi}_j = \left[\hat{u}_j, \hat{v}_j, \hat{w}_j, \hat{T}_j, \hat{\rho}_j\right]^T$) at each y location. Boundary conditions (II.20) are then applied to y_1 and y_{max}. In addition we are left with an unknown global complex α that is independent of the y location.

This system's solution can be obtained by treating the complex α as an eigenvalue and the associating vector of $\hat{\phi}$s as the corresponding eigenvector. To account for the nonlinearity of α that occurs in the viscous terms, the following transformation is applied.

$$\hat{\Phi}_j = \begin{bmatrix} \hat{u}_j \\ \hat{v}_j \\ \hat{w}_j \\ \hat{T}_j \\ \hat{\rho}_j \\ \alpha\hat{u}_j \\ \alpha\hat{v}_j \\ \alpha\hat{w}_j \\ \alpha\hat{T}_j \end{bmatrix} \tag{III.7}$$

The eigenvalue problem now takes the form

$$\mathbf{A}\hat{\Phi} = \alpha\mathbf{B}\hat{\Phi}. \tag{III.8}$$

$\mathbf{A}$ and $\mathbf{B}$ now constitute $9Ny \times 9Ny$ matrices; correspondingly $\hat{\Phi}$ is a $9Ny \times 1$ eigenvector. Matrix $\mathbf{A}$ is built by assuming $\alpha = 0$, conversely matrix $\mathbf{B}$ will contain only the α and α^2 coefficients. Some basic identity formulas relating $\hat{\Phi}$ values ($\hat{\Phi}_j(1) = \alpha\hat{\Phi}_j(6)$) round out the final equations needed in order to solve the entire system.

EPIC utilizes a QZ algorithm to solve equation (III.8), resulting in the full eigenvalue spectrum of $9Ny$ results. Many of these will be spurious results. At this point, filters must be applied in order to pick the most unstable (most negative α_i), physical eigensolution.

The linear stability theory provides a localized stability solution that carries the assumptions addressed in chapter II. As mentioned earlier, one unstable location is not indicative of laminar-to-turbulent transition. LST can be performed along a path and the collected growth rates can be integrated to ascertain if and how much a disturbance grows downstream, however EPIC is designed to use the LST solution as an initial value for the more accurate PSE methods.

III.4 Linear Marching Procedure

The LPSE solution method employs a marching scheme to solve the boundary value problem (BVP) at each x location, one step at a time. Instead of creating an eigenvalue problem at each step, the previous steps' solutions formulate an initial guess that justifies solving the BVP in an iterative fashion, reducing its computational expense. LST cannot adopt this method unless an approximate solution is already known.

A fourth order central finite difference scheme (III.5) is again applied in the normal direction. Due to the parabolic nature of the problem, a left-sided finite difference scheme (III.6) is implemented for the streamwise derivatives instead of a central scheme. This permits us to forgo involving downstream values to solve our current step while simultaneously accounting for upstream influences. Upon applying the finite difference schemes, the resulting algebraic system is arranged to form a pentadiagonal-block matrix on the left hand side.

$$
\begin{bmatrix}
C_1 & D'_1 & E'_1 & A'_1 & & & & \\
B'_2 & C_2 & D'_2 & & & & & \\
A_3 & B_3 & C_3 & D_3 & E_3 & & & \\
 & & & \ddots & & & & \\
 & & & & \ddots & & & \\
 & & A_{Ny-2} & B_{Ny-2} & C_{Ny-2} & D_{Ny-2} & E_{Ny-2} & \\
 & & & & B'_{Ny-1} & C_{Ny-1} & D'_{Ny-1} & \\
 & & & E'_{Ny} & A'_{Ny} & B'_{Ny} & C_{Ny} &
\end{bmatrix}
\begin{bmatrix}
\hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \\ \vdots \\ \vdots \\ \hat{\phi}_{Ny-2} \\ \hat{\phi}_{Ny-1} \\ \hat{\phi}_{Ny}
\end{bmatrix}
=
\begin{bmatrix}
RHS_1 \\ RHS_2 \\ RHS_3 \\ \vdots \\ \vdots \\ RHS_{Ny-2} \\ RHS_{Ny-1} \\ RHS_{Ny}
\end{bmatrix}
\tag{III.9}
$$

The A_j, B_j, C_j, D_j, and E_j terms represent 5×5 matrices at that location. Each $\hat{\phi}_j$ and RHS_j are 5×1 vectors. Matrices near the boundary using the different finite differencing stencils (as mentioned previously) are denoted by $'$. Furthermore, the equations denoted by blocks $j = 1$ and $j = Ny$ will reflect the imposed boundary conditions (II.20).

The coefficient matrices are based on $\overline{\phi}_x$, α_x, β, and ω, where x denotes our current step location. Additionally, the right hand side (RHS) vectors include $\hat{\phi}_{x-1}$ and $\hat{\phi}_{x-2}$. From a marching standpoint, the equation (III.9) can also be expressed as

$$
\mathcal{A}_x \hat{\phi}_x = \mathcal{B}_x \hat{\phi}_{x-1} + \mathcal{C}_x \hat{\phi}_{x-2}, \tag{III.10}
$$

where $\mathcal{A}_x$ is the $5Ny \times 5Ny$ pentadiagonal-block matrix and the rest of the components are $5Ny \times 1$ vectors.

As briefly mentioned earlier, the LST solution makes an appropriate initial condition to formulate the RHS in equation (III.9). If provided an α_x, then the vector $\hat{\phi}_x$ can be best solved with a simple LU decomposition scheme. The result is then used to formulate an error, which is iteratively driven toward zero. In our tests, a simple Newton-Raphson method proved to be more than sufficient; it yielded accurate convergence with minimal time and computational costs. α is independent of the normal direction, thus we expect that α will change slowly in the streamwise direction and we formulate our initial guess by assuming no change from the previous step.

$$\frac{\partial \alpha_x}{\partial x} = \frac{\alpha_x - \alpha_{x-1}}{\Delta x} = 0 \tag{III.11}$$

III.4.1 Normalization Condition

Our iterative solver still requires a solution condition in order to converge. To ensure that the shape function is slowly varying in the streamwise direction, a normalization condition is imposed upon it that will become the solution condition. A standard normalization condition has the form

$$\int_0^\infty \frac{\partial}{\partial x} \Psi \partial y = 0 \tag{III.12}$$

where Ψ is the parameter to be normalized. Because our shape function is a complex value $(\hat{\phi}_r + i\hat{\phi}_i)$, both its magnitude $(\|\hat{\phi}\|)$ and its phase $\left(\arctan\left(\frac{\hat{\phi}_i}{\hat{\phi}_r}\right)\right)$ must be normalized.

In simplifying both of the normalization parameters,

$$\int_0^\infty \frac{\partial}{\partial x}\left(\|\hat{\phi}\|\right)\partial y \xrightarrow{\text{reduces to}} 2\int_0^\infty \left(\hat{\phi}_r \frac{\partial \hat{\phi}_r}{\partial x} + \hat{\phi}_i \frac{\partial \hat{\phi}_i}{\partial x}\right)\partial y \tag{III.13}$$

$$\int_0^\infty \frac{\partial}{\partial x}\left(arctan\left(\frac{\hat{\phi}_i}{\hat{\phi}_r}\right)\right)\partial y \xrightarrow{\text{reduces to}} \int_0^\infty \left(\frac{\hat{\phi}_r \frac{\partial \hat{\phi}_i}{\partial x} - \hat{\phi}_i \frac{\partial \hat{\phi}_r}{\partial x}}{\hat{\phi}_r^2 + \hat{\phi}_i^2}\right)\partial y \tag{III.14}$$

the driving values to be normalized can be reduced to $\hat{\phi}_r \frac{\partial \hat{\phi}_r}{\partial x} + \hat{\phi}_i \frac{\partial \hat{\phi}_i}{\partial x}$ and $\hat{\phi}_r \frac{\partial \hat{\phi}_i}{\partial x} - \hat{\phi}_i \frac{\partial \hat{\phi}_r}{\partial x}$. This conveniently allows us to formulate one solution condition,

$$\int_0^\infty \hat{\phi}^\dagger \frac{\partial \hat{\phi}}{\partial x} \partial y = err_r + ierr_i \tag{III.15}$$

resulting in a complex error value. The real error will constrain the magnitude while the imaginary error constrains the phase. This error is subsequently used in the Newton-Raphson iterations to adjust the complex α accordingly. The resulting $\hat{\phi}$s from solving (III.9) with new αs will eventually reduce both normalization parameters to within a predefined tolerance. Once this tolerance is reached, the α and $\hat{\phi}$s for that streamwise location are saved, a step is taken in the marching direction, and the process is repeated downstream.

Remember however that $\hat{\phi}$ represents five distinct variables ($\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{T}$, $\hat{\rho}$), whereas α can only be refined by one single quantity. Instead of trying to select the most crucial term, we opt to combine all five terms. This will ensure that all aspects of the flow are evolving downstream as they should. Furthermore, a Pythagorean normalization parameter is applied to our solution condition so that modes comprised of very small magnitudes do not bypass the tolerance. Our final normalization parameter takes the form

$$\frac{\int_0^\infty \left(\hat{u}^\dagger \frac{\partial \hat{u}}{\partial x} + \hat{v}^\dagger \frac{\partial \hat{v}}{\partial x} + \hat{w}^\dagger \frac{\partial \hat{w}}{\partial x} + \hat{T}^\dagger \frac{\partial \hat{T}}{\partial x} + \hat{\rho}^\dagger \frac{\partial \hat{\rho}}{\partial x} \right) \partial y}{max \left(\hat{\phi} * \hat{\phi}^\dagger \right)} = err_r + ierr_i. \qquad \text{(III.16)}$$

III.5 Nonlinear Marching Procedure

The NPSE problem is reminiscent of solving multiple, coupled LPSE problems. Each mode (n, k) has a unique equation (II.28) that is coupled through the nonlinear terms. In the interest of finding a numerical solution, analysis is restricted to $-N \leq n \leq N$ and $-K \leq k \leq K$. If we have a wave with frequency F and initial amplitude A, we expect that the harmonics $2F$, $3F$, $4F$, ... will possess initial amplitudes of A^2, A^3, A^4, ..., thus eliminating the extremes is not a severe restriction. More computational costs can be saved by exercising the symmetry laws shown previously (equations II.30 and II.31) whenever possible. All modes will exhibit real physical disturbances, meaning only modes ($0 \leq n \leq N$, $-K \leq k \leq K$) must be calculated. Furthermore, only modes ($0 \leq n \leq N$, $0 \leq k \leq K$) need be solved if a geometry exhibits z-axis symmetry.

Applying the same finite difference schemes and boundary conditions as exercised in the LPSE method results in a similar pentadiagonal-block matrix system for each mode (n, k) with the addition of the relevant nonlinear terms on the right hand side. These nonlinear terms will contain unknown $\hat{\phi}_x$ terms, and so we must incorporate a nonlinear convergence loop into our marching. When guessing an α_x (equation III.11), we will also make an initial guess for $\hat{\phi}_x$, again expecting only slow changes based on our normalization parameter. In our experience, it was found that setting the second derivative to zero tended to give the fastest convergence.

$$\frac{\partial^2 \hat{\phi}_x}{\partial x^2} = \frac{\hat{\phi}_x - 2\hat{\phi}_{x-1} + \hat{\phi}_{x-2}}{\Delta x^2} = 0 \tag{III.17}$$

Applying our α_x guess to all equations as before and using our $\hat{\phi}_x$ guess to compute only the nonlinear right hand side, we once again have a system of the form $Ax = B$ that can be solved with an LU decomposition. Before beginning the α_x convergence, an implicit "nonlinear convergence" loop is needed to converge the nonlinear RHS;

$$LHS\left(\alpha_x\right)\hat{\phi}_x^{n+1} = RHS\left(\alpha_x, \hat{\phi}_x^n\right). \tag{III.18}$$

Here, n represents the iterations of the "nonlinear convergence" loop. Once $\left|\hat{\phi}_x^{n+1} - \hat{\phi}_x^n\right|$ is less than a predefined tolerance for every mode (n, k), the normalization condition is then applied to each α_x and the entire process is repeated until all values are sufficiently converged.

III.5.1 Mean Flow Distortion

As mentioned briefly at the end of chapter II, there exists a real mode $(0, 0)$ that warrants special attention. This mode, referred to as the mean flow distortion (MFD), is a nonlinear disturbance driven by the interactions between itself and all other modes in the system. It is further differentiated from all other modes by possessing no complex conjugate. Combined with the requirement that all disturbances be real, this dictates that $\hat{\phi}_{(0,0)}$ must be purely real and $\alpha_{(0,0)}$ must be purely imaginary ($A_{0(0,0)}$ will always equal 1).

This can be proven with a rather simple, albeit lengthy, expansion of the NPSE modal disturbance, equation II.28, in conjunction with the symmetry laws II.30 for the MFD mode. If this expansion is performed, it should be obvious that if an imaginary $\hat{\phi}_{(0,0)}$ or real $\alpha_{(0,0)}$ are ever introduced, they will continue grow and potentially impact the solution.

Because we can mathematically prove that imaginary $\hat{\phi}_{(0,0)}$ and real $\alpha_{(0,0)}$ only originate through numerical error, we opt to eliminate them. The least invasive method to accomplish this is to always ensure that $\mathcal{NL}_{(0,0)}$ (equation II.29) is purely real before solving for $\hat{\phi}_{(0,0)}$. In practice, our tests verify that the final results are measurably identical while convergence routines are notably more expeditious.

III.5.2 Step Size Limitation

In the interest of stabilizing the PSE approximation, Li and Malik performed a numerical study in 1996 [27]. They concluded that the equations are not completely parabolized because ellipticity is introduced through the pressure gradient term $\frac{\partial \hat{P}}{\partial x}$. This ellipticity leads to the step size limitation,

$$\partial \xi_{min} = \frac{1}{|\alpha_r|} \tag{III.19}$$

such that too small of a step size will cause the solution to diverge. Although data is given to show that the nature of the PSE approximation responds well to large step sizes, such that equation III.19 is not typically a limitation, there do exist zero frequency modes, the MFD $(0,0)$ and the longitudinal vortex modes $(0,k)$, that typically express very small α_r values.

Upon further investigation, Li and Malik find that dropping the pressure gradient $\frac{\partial \hat{P}}{\partial x}$, while not completely eliminating the ellipticity, does reduce the minimum step size requirement by "an order of magnitude" [27]. Because most of the pressure gradient is absorbed by the $i\alpha\hat{P}$ term, dropping $\frac{\partial \hat{P}}{\partial x}$ was shown to have very minor, if any, effect on the final solution. Further attempts to eliminate all ellipticity could not be consistently implemented without resulting in a final solution of reduced accuracy.

To ensure a more reliable and robust marching scheme for both PSE schemes, we follow the recommendations of other authors and add a coefficient, Ω, to the front of our pressure gradient. The full pressure gradient disturbance now takes the form of equation III.20.

$$\frac{\partial P'}{\partial x} = \left(\Omega\frac{\partial \hat{P}}{\partial x} + i\alpha\hat{P}\right) e^{i\left(\int_{x_0}^{x} \alpha(x)\partial x + \beta z - \omega t\right)} + c.c. \qquad \begin{cases} \Omega = 1 & \text{if } \omega \neq 0 \\ \Omega = 0 & \text{if } \omega = 0 \end{cases} \tag{III.20}$$

III.6 Result Analysis Methods

The majority of results in the following chapters will be presented in the form of two common methods: either an N-factor or an amplitude plot. Computational stability results, similar to experimental stability results, are dependent on a multitude of factors, thus the following sections seek to clarify in detail how our results are calculated and presented.

III.6.1 N-factor Analysis

The N-factor or e^N method, first used by Hermann Schlichting in 1933 and later popularized by van Ingen in 1956, is the most popular transition-prediction technique. As Schubauer and Skramstad confirmed in their experiments [41], transition is not an instantaneous phenomenon found at the first unstable disturbance, but is instead a process governed by the relative growth of said disturbance during its unstable regime.

The quantity N is simply a ratio of amplitude growth, such that $N = \ln\left(\frac{A}{A_0}\right)$, where A_0 here is the first neutral-stability point. As Reed, Saric, and Arnal explain, "as long as laminar flow is maintained and the disturbances remain linear, the e^N method contains all of the necessary physics to accurately predict disturbance behavior" [35]. However, Reed et al. express caution in treating the N-factor as an authoritative value. The initial disturbance amplitude, the crucial factor in receptivity studies, is not accounted for by this method. Even in the most applicable cases, N is still a correlation; it is not solely indicative of transition. To minimize error, comparisons should be restricted to experiments of identical conditions whenever possible. Nevertheless, N-factors are a critical calculation when used correctly.

Original N-factors calculated the growth rate using α_i.

$$N = \ln\left(\frac{A}{A_0}\right) = \int_{x_0}^{x} -\alpha_i \partial x \tag{III.21}$$

Supersonic wind tunnel experiments typically measure growth rate as a measure of the mass flux fluctuation, $(\rho u)' = \rho' \overline{u} + \overline{\rho} u'$. Therefore, N-factors calculated using mass flux as the growth rate are

$$N = \ln\left(\frac{A}{A_0}\right), \quad \text{where } A = max\left[\left(\overline{u}\hat{\rho} + \overline{\rho}\hat{u}\right) e^{i\int_{x_0}^{x} \alpha \partial x} + \left(\overline{u}\hat{\rho}^{\dagger} + \overline{\rho}\hat{u}^{\dagger}\right) e^{-i\int_{x_0}^{x} \alpha^{\dagger} \partial x}\right]. \tag{III.22}$$

Unless noted differently, all our N-factor results presented in this paper use the mass flux calculation (equation III.22).

III.6.2 Amplitude Analysis

Results not represented as N-factors will be presented as an amplitude analysis. Amplitude analyses do not correlate to a transition location like the N-factor does, but instead give a raw display of the disturbance modes' maximum amplitudes. This is useful for visualizing how disturbances react in nonlinear regimes, respond to different initial amplitudes, and interact with other modes present.

Our results present the maximum u-velocity disturbance, u'. Recall that in NPSE, when an initial amplitude A_0 is given, we apply $\frac{A_0}{2}$ to both the mode and its complex conjugate. A singular mode (other than the MFD) expresses a complex disturbance, thus the real amplitude results presented are the full disturbance (equation III.23).

$$u'_{max}(x) = max\left[\frac{A_0}{2}(x)\,\hat{u}(x)\, e^{i\int_{x_0}^{x} \alpha(x)\partial x} + \frac{A_0^{\dagger}}{2}(x)\,\hat{u}^{\dagger}(x)\, e^{-i\int_{x_0}^{x} \alpha^{\dagger}(x)\partial x}\right] \tag{III.23}$$

The u' disturbances plotted have been nondimensionalized by U_e. Unless stated otherwise, all initial amplitude A_0 values given are in terms of the u-velocity perturbation.

IV. LANGLEY 93-10 RESULTS

The Langley 93-10 flared cone was a focus of the hypersonic stability community for a time. In an attempt to better understand and predict laminar-to-turbulent prediction, effort was focused on studying a geometry that would undergo transition in a quiet wind tunnel. Schneider notes that transition is not observed on straight cones under these conditions due to experimental size restrictions [40]. Flared cones (compression cones) create an adverse pressure gradient that causes disturbances to grow faster, thereby shifting the laminar-to-turbulent transition point upstream. In addition, Saric points out that a concave flare can also induce Görtler (centrifugal) instabilities [39]. Although the Langley 93-10 still does not experience transition in a quiet tunnel, it demonstrates a much larger instability than previous models and still makes for an academic case study.

Computational results from JoKHeR were published [19] in support of experimental and validation efforts in the NASA Langley Mach 6 Quiet Tunnel (M6QT) located at Texas A&M University [12, 14]. The computational basic-state used for these results provide an excellent and convenient verification scenario for EPIC. Validation with additional experimental results will also be presented.

IV.1 Geometry

We make use of the same basic-state data that Kocian et al. utilized with JoKHeR [19]. Thus this section detailing the geometry and freestream conditions is adopted from the cited paper.

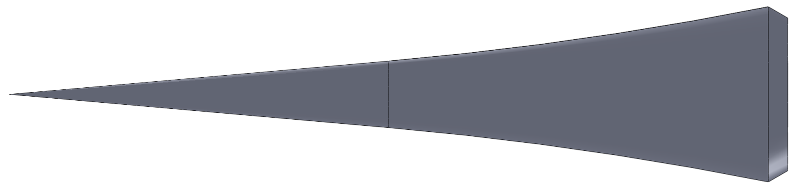

Figure IV.1: Langley 93-10 flared cone.

The Langley 93-10 flared cone model under consideration is 0.508 m in length and consists of a nose tip with a radius of 38 microns. See figure IV.1. The nose was modeled using the modified-super-ellipse equation in order to eliminate discontinuities in slope and curvature at the juncture [28]. This method is shown in equation IV.1

$$\left(\frac{a-x}{a}\right)^{m(x)} + \left(\frac{y}{b}\right)^2 = 1 \tag{IV.1}$$

where $m(x) = 2 + \left(\frac{x}{a}\right)^2$, a is the major axis, and b is the minor axis. For modeling the Langley 93-10 flared cone, a and b are set to be equal to more closely resemble a circular nose tip. See figure IV.2. The geometry transitions from a 5° half-angle cone to a flare at 0.254 m. The flare has a radius of curvature of 2.364 m and extends to the base of the cone, which is 0.1168 m in diameter.

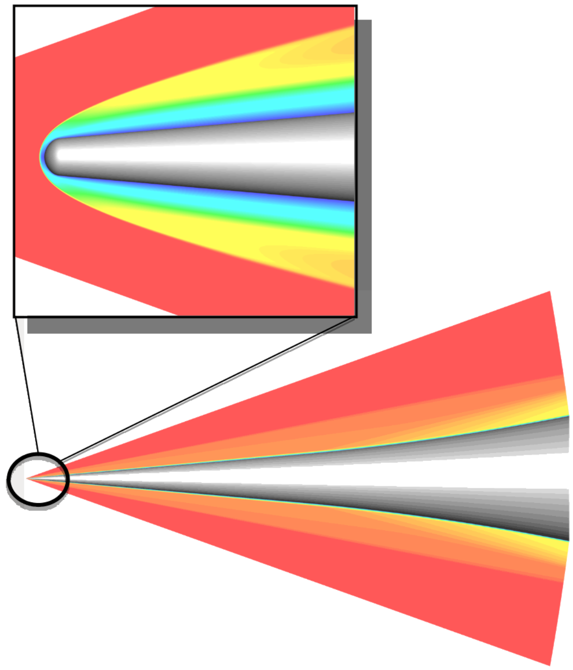

Figure IV.2: Nose tip blended into straight portion of cone with modified super ellipse.

Run conditions for the computations were matched to test conditions in the M6QT. These consisted of a freestream Mach number $M_\infty = 5.9$, unit Reynolds number $Re' = 9.764 \times 10^6$ per meter, freestream static pressure $P_\infty = 620$ Pa abs, and freestream temperature $T_\infty = 54.38$ K. The M6QT does not run long enough to establish adiabatic-wall conditions. Over the course of a run with the 93-10 cone, temperatures vary between 403 K and 386 K for a variety of locations along the cone and throughout the run. So, while the actual wind tunnel model had a small temperature variation on the body as time passed, the computational model uses a constant wall temperature 398 K to represent an average value (figure IV.3). Moreover Hofferth et al. [13, 14] noted the difficulty in obtaining an exactly 0° AoA during a typical experiment and its significant effect on second mode frequency. The 0° AoA case is considered here as part of the computational study.

A stability analysis requires high-fidelity, undisturbed basic-state calculations, which themselves satisfy the Navier-Stokes equations. The steady, laminar basic-state solution

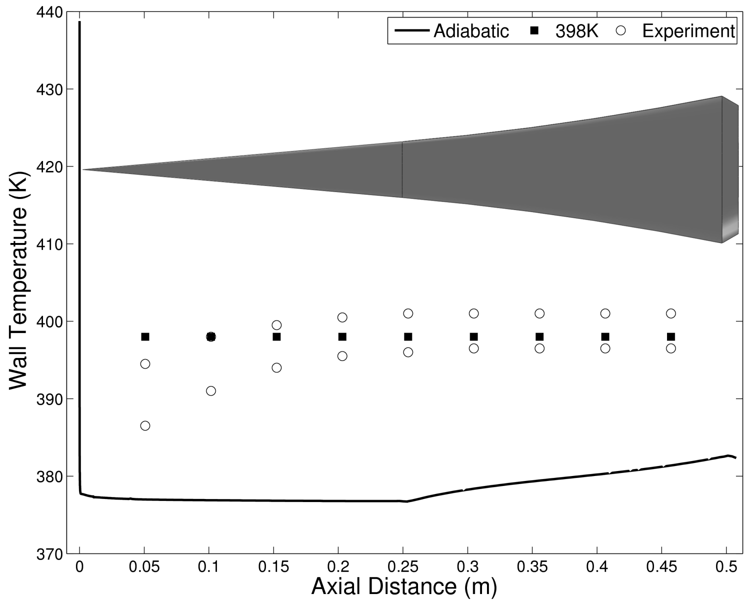

Figure IV.3: Wall temperature distributions for the Langley 93-10 flared cone, showing the experimental conditions in the M6QT [14], the adiabatic distribution, and the computational model of 398K.

is computed using GASP (General Aerodynamic Simulation Program), which solves the unsteady Navier-Stokes equations using a cell-centered finite-volume scheme. A 3rd-order Roe with Harten solving scheme, with a Van Albada limiter equal to 0.3333, was used for the flared-cone geometry due to its reliability in finding stationary discontinuities, low dissipation compared to other methods, and its entropy fix to counter the Carbuncle effect.

The grids used in these basic state calculations were generated using Pointwise. The undisturbed basic-state flow is axisymmetric, thus a 2-D grid is sufficient. The 2-D mesh is composed of two main parts in the wall normal direction: a band capturing the shock and a high-resolution shock layer. See figure IV.4. The final grid used for this case contained 912 points between the shock and the body and 392 points to resolve the shock. In the streamwise direction, 599 points were used. A comparison of stability results were used to confirm convergence for this configuration [34].

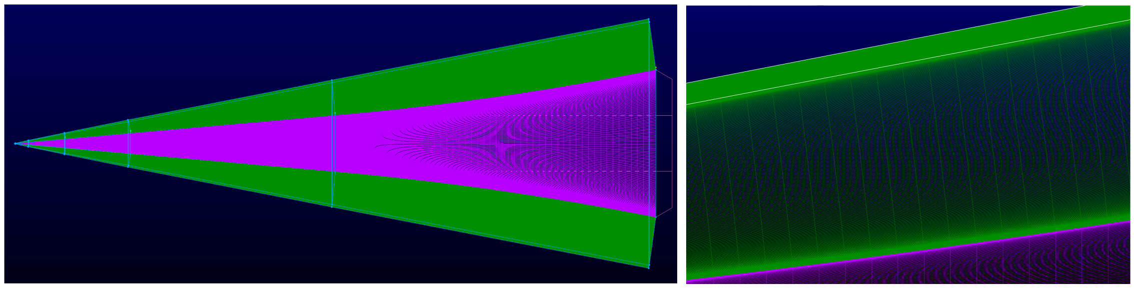

Figure IV.4: Langley 93-10 basic-state grid composed of two main parts in the wall normal direction: a band capturing the shock and a high-resolution shock layer.

IV.2 JoKHeR Verification

LPSE was performed with EPIC to find the most unstable frequency at the conditions previously mentioned. In perfect agreement with JoKHeR, the most amplified frequency at an axial distance of 0.495 m was found to be 234 kHz (shown in figure IV.5). JoKHeR's results are shown in figure IV.6 for comparison. The most amplified frequency at the very back of the cone was found to be 235 kHz with both codes (comparison shown in figures IV.7 and IV.8). The notable difference between codes thus far is that EPIC finds higher N-factors for all disturbances, with the peak disturbance having a max N-factor about 1.5 higher than what was found with JoKHeR. EPIC appears to agree extremely well for this case.

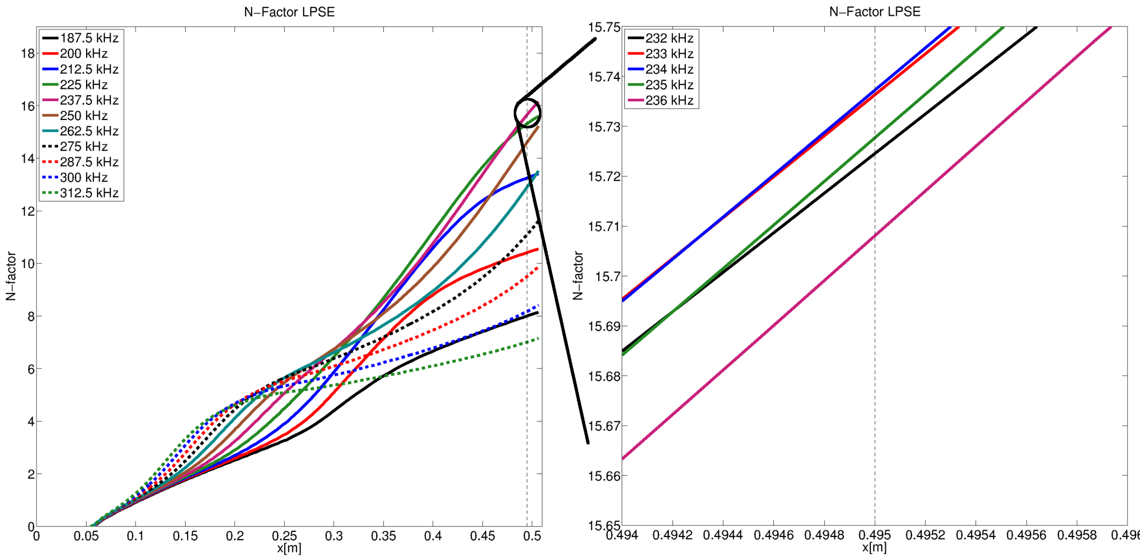

Figure IV.5: EPIC calculated LPSE N-factors for Langley 93-10 flared cone, featuring a zoom at axial location x=0.495 m

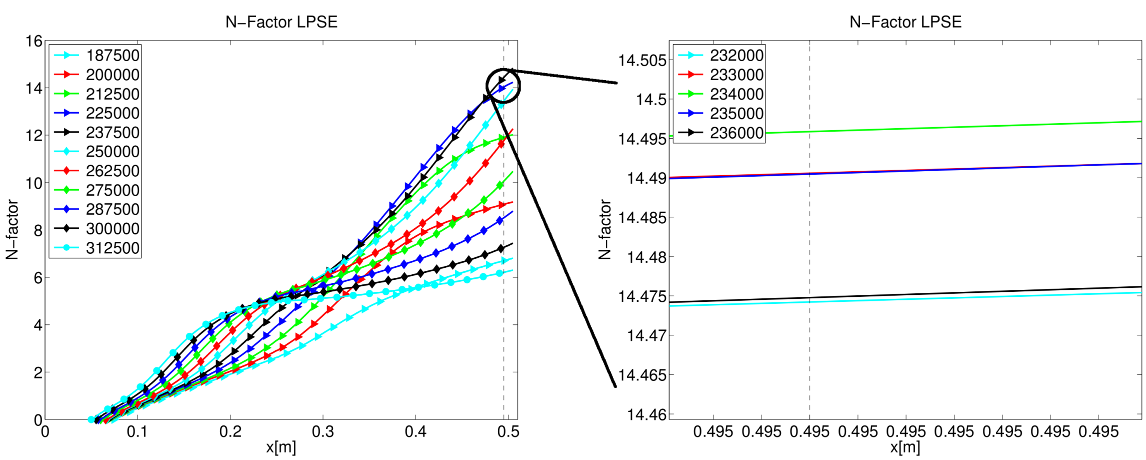

Figure IV.6: JoKHeR calculated LPSE N-factors for Langley 93-10 flared cone, featuring a zoom at axial location x=0.495 m [19]

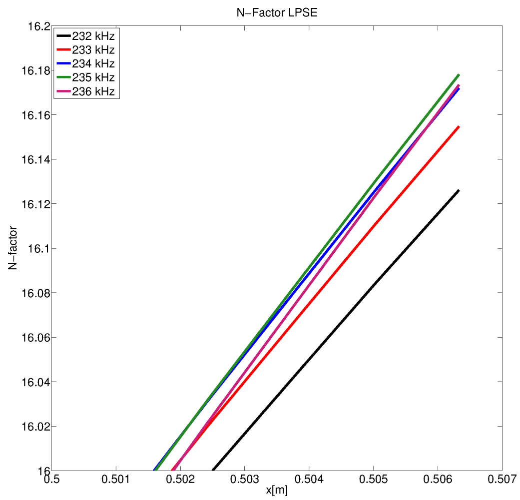

Figure IV.7: EPIC calculated LPSE N-Factors for Langley 93-10 flared cone, zoomed at back of cone.

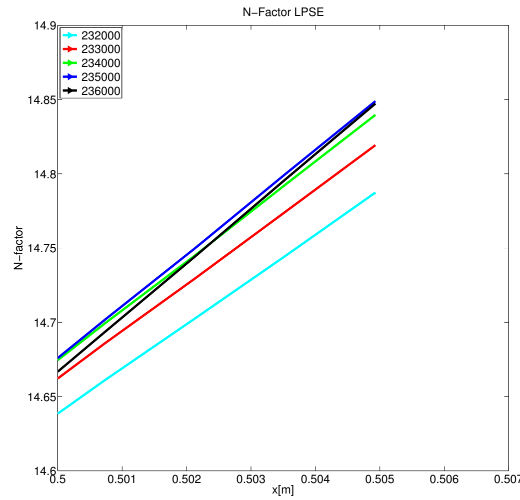

Figure IV.8: JoKHeR calculated LPSE N-Factors for Langley 93-10 flared cone, zoomed at back of cone.

NPSE analyses were conducted in order to consider the effects of a finite frequency distribution and the interplay between multiple modal disturbances. The pure mode disturbances used in LPSE are not truly representative of experimental initial conditions.

The single discrete case considers a single primary second-mode with two additional harmonics and mean flow distortion. Primary mode frequencies included 220 kHz, 227.5 kHz, 235 kHz, 242.5 kHz, and 250 kHz. Conversely, the multiple discrete case considers all of the primary second-modes from the discrete case, two additional "harmonic tiers", and the mean flow distortion. A "harmonic tier" includes harmonics that result purely from primary mode interplay, in addition to the direct harmonics of the primary modes. If harmonic tier 1 is our primary second-mode frequencies (220 kHz, 227.5 kHz, 235 kHz, 242.5 kHz, and 250 kHz), then harmonic tier 2 will consist of 440 kHz, 447.5 kHz, 455 kHz, 462.5 kHz, 470 kHz, 477.5 kHz, 485 kHz, 492.5 kHz, and 500 kHz. In an effort to validate with JoKHeR runs from Kocian et al., both single and multiple cases were given initial amplitudes based on temperature disturbance of $A_0 = 2e - 7$ and $A_0 = 10e - 7$. Results can be seen in figure IV.9. As a proof a concept, figure IV.10 shows that the NPSE single and multiple cases will recover the original linear solution with a low enough initial amplitude.

As expected, the energy exchange between the primary and harmonic modes has a stabilizing effect, and this effect is greater in every multiple mode case than its corresponding single mode case. This stabilizing effect shows evidence of "nonlinear saturation" being reached, a situation where the rate of energy transfer from the basic state to the primary mode is surpassed by the transfer rate of energy away from the primary mode and into the harmonics.

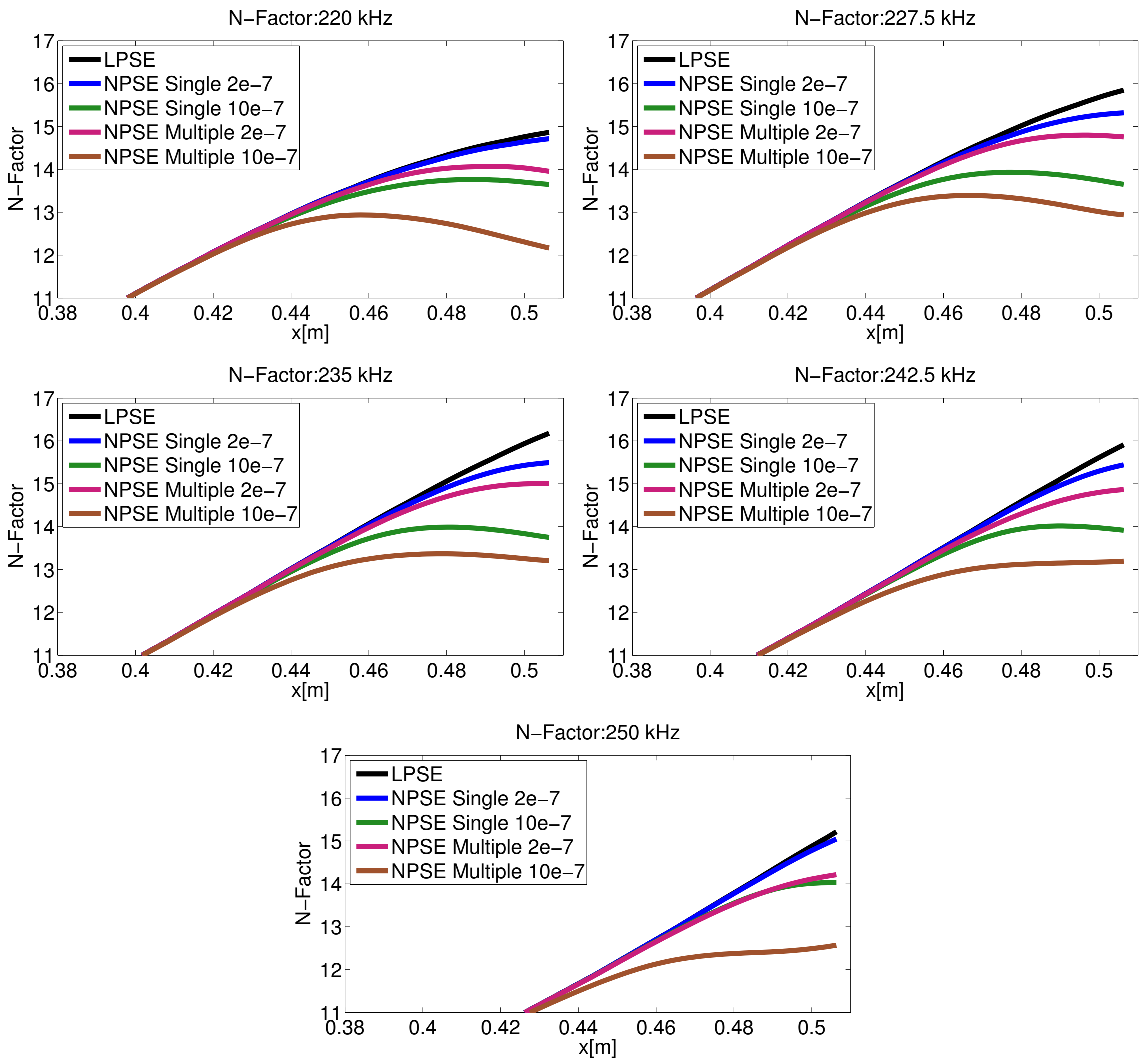

Figure IV.9: Comparison of single and broadband NPSE for Langley 93-10 flared cone. Initial amplitude given in terms of temperature perturbation.

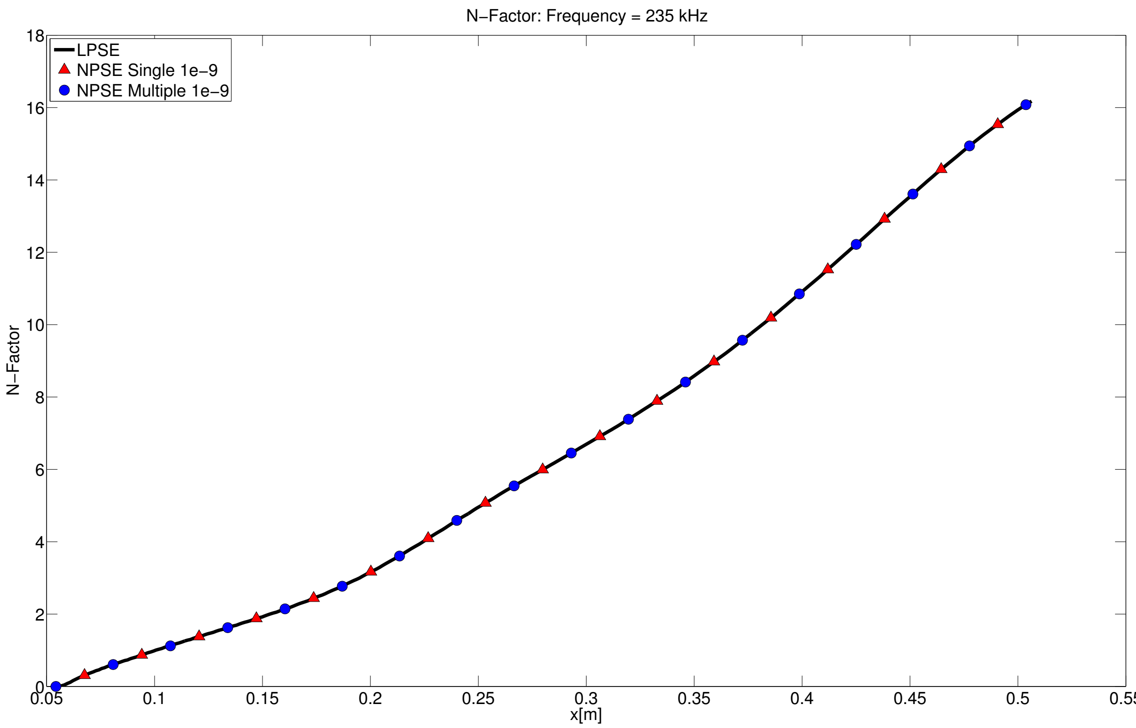

Figure IV.10: The single and multiple NPSE cases will recover the same solution as the LPSE case if given a small enough initial amplitude. Initial amplitude given in terms of temperature perturbation.

IV.3 Validation

Table IV.1 compares EPIC's results with other published computations and experiments. It should be noted that most of the other runs being compared to used an adiabatic T_{wall}, however the adiabatic wall temperature acheived should be very near the constant 398 K T_{wall} used in our computations. The difference in nose radii and unit Reynold's number are more likely to account for different results. When accounting for the condition differences, we find these results very agreeable and within an acceptable error range. Most importantly, it is evident that the flare destabilizes the second mack-mode disturbances within the 220-240 kHz range.

Authors	**Method**	**Nose Radius (mm)**	**T_{wall} (K)**	**Mach**	**Re' (1/m)**	**X-Location (mm)**	**Frequency (kHz)**
Horvath et al. [15]	Experiment and LST	0.00254	adiabatic	6.00	8.95e6	311	230
Balakumar & Kegerise [1]	DNS	0.01270	adiabatic	6.00	8.95e6	400	220
Lachowicz et al. [22]	LST	0.00254	adiabatic	5.91	9.25e6	444	220
	Experiment	0.00254	adiabatic	5.91	9.25e6	482	226
Balakumar & Malik [2]	Computational	0.00305	adiabatic	6.00	8.95e6	508	230
Hofferth & Saric [14]	Experiment	0.03800	Fig IV.3	5.91	9.764e6	495	250[1]
Oliviero	LPSE	0.03800	398	5.91	9.764e6	311	239
						400	225
						444	228
						482	232
						508	235

[1] Approximate frequency after adjusting for AoA offset

Table IV.1: Langley 93-10 flared cone validation comparisons

V. PURDUE COMPRESSION CONE RESULTS

The Purdue compression cone (PCC) [45] was designed to exhibit the highest possible N-factor while still starting in a wind tunnel. The constant flare creates an adverse streamwise pressure gradient and maintains a near-constant-thickness boundary layer, making it an ideal model for studying second-mode instabilities. Surprisingly, although higher N-factors were calculated, the PCC still did not undergo transition in a quiet tunnel.

V.1 Geometry

The PCC is a circular-base cone 0.49 m in total length and begins with a nose tip radius of 0.001 m. It maintains a flare with constant radius of curvature of 3.0 m along the entire length of the body, concluding with a base diameter of 0.11684 m. Run conditions are as follows: Mach number $M_\infty = 6$, freestream temperature $T_\infty = 52.8$ K, freestream pressure $P_\infty = 610.7752$ Pa abs, unit Reynolds number $Re' = 10.2834 \times 10^6$ 1/m, and constant wall temperature $T_{wall} = 300$ K.

The PCC case is run at 0° AoA, permitting the use of a 2-D axisymmetric grid. The basic-state solution was calculated using GASP. The grid consisted of 901 points between the shock and the body, 401 in the shock-capture layer, and 731 in the streamwise direction. This basic state was used in previous computational studies with JoKHeR and is confirmed to be converged [34]. The model can be viewed in figure V.1.

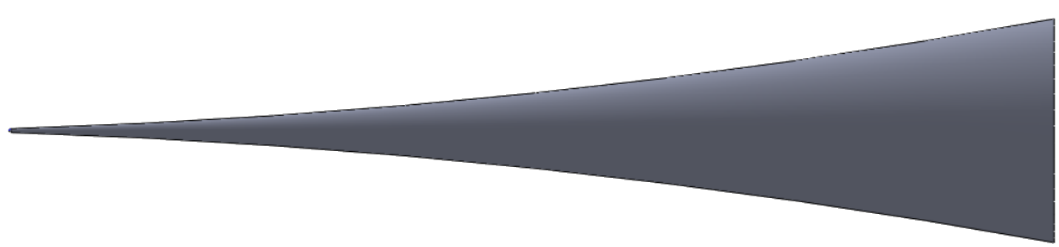

Figure V.1: Purdue compression cone.

V.2 LPSE Results and Validation

Our PCC analysis begins with a multitude of EPIC LPSE results. It was first confirmed that the 2-D second-mode is the most dominant instability present. After performing a fine sweep of frequencies, the most amplified second-mode disturbance frequency was found to be 285 kHz at axial location x=0.45m, shown in figure V.2.

Despite not being the most unstable, there do exist 3-D disturbances of significant value on this geometry. Azimuthal beta β_{AZ} relates the azimuthal wavelength λ_z to the radius of the cone by representing the number of azimuthal waves that fit around the body. Keeping this number constant allows λ_z to grow downstream as the cone widens, a more physical reaction than forcing λ_z constant and having more waves appear downstream. The 3-D oblique second-modes, figure V.3, confirm that the 2-D second-mode is the strongest instability mechanism at play, as expected. Phase angles for the oblique modes are shown in figure V.4. The constant concave geometry also produces strong streamwise counter-rotating streaks, a fully 3-D disturbance, as seen in figure V.5.

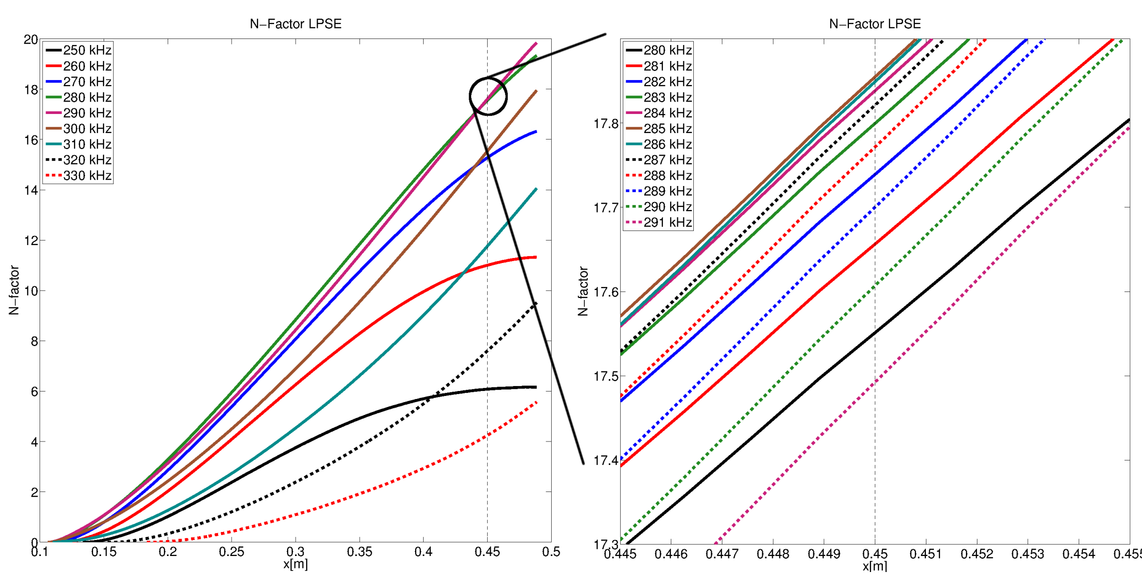

Figure V.2: EPIC calculated LPSE N-factors for Purdue Compression Cone, featuring a zoom at axial location x=0.45 m

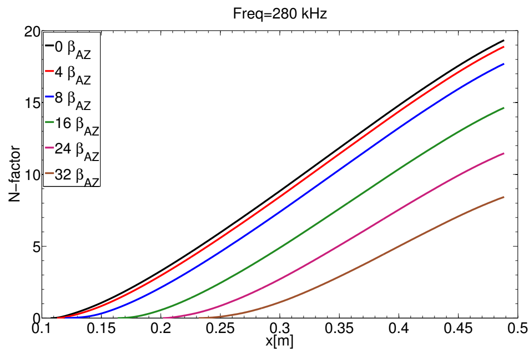

Figure V.3: Oblique mode LPSE N-Factors for Purdue compression cone.

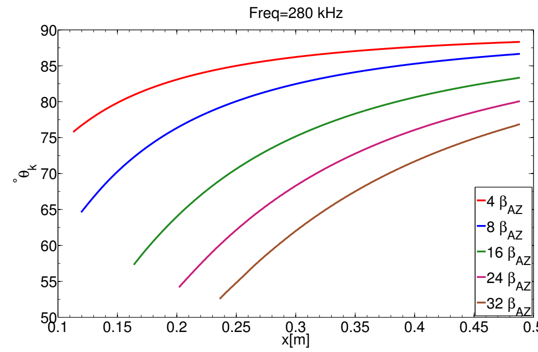

Figure V.4: Purdue compression cone oblique mode phase angles along axial distance. The 0 β_{AZ} mode has a constant phase angle of 90°.

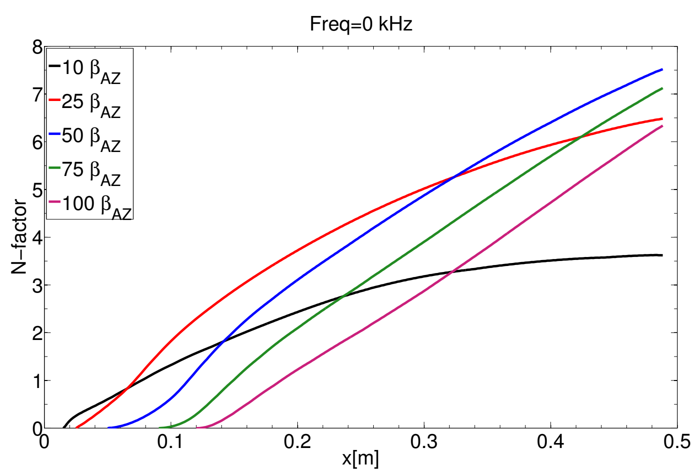

Figure V.5: LPSE N-Factors for streamwise counter-rotating streaks on Purdue compression cone.

Authors	Method	X-Location (mm)	Frequency (kHz)
Balakumar & Kegerise [1]	LST	400	279.0
	LPSE	400	286.0
Wheaton et al. [45]	LST	450	279.0
	LPSE	450	286.0
	Computational	200	287.0
		300	285.0
		400	285.0
	BAM6QT	200	294.9
		300	293.0
		400	294.9
Oliviero	LPSE	200	283.0
		300	282.0
		400	285.0
		450	285.0

Table V.1: Purdue compression cone validation comparisons. BAM6QT stands for "Boeing/AFOSR Mach-6 Quiet Tunnel"

These results agree very well with the previously published computational results shown in table V.1. Note that if comparing plots, our variation of the cone has an axial length of 0.49 m instead of 0.45 m. Otherwise each of these cases was confirmed to run at near identical operating conditions.

V.3 Bandwidth NPSE Results

While there is excellent agreement among computational results, Table V.1 also shows that experimental results were about 10 kHz higher. This is in the same range as the 15 kHz difference between our computational comparison with Hofferth et. al [14] on the Langley cone. In reaction to this, finite-bandwidth effects were studied on the second-mode instability of the PCC as well. In order to compare to figures 3-6 of Kuehl et al. [20], 1-, 3-, 5-, 7-, 9-, and 11-mode NPSE cases were calculated via EPIC. All cases were centered around 287 kHz (the most amplified at axial location x=0.49 m) with a 2 kHz interval between neighboring modes. Each case was also run with a second "harmonic tier" and each primary mode was given a very minimal initial amplitude of 2.0×10^{-8} (non-dimensionalized by the temperature perturbation). Figures V.6 and V.7 show the first harmonic of the 1-, 3-, 5- and 7-, 9-, 11-mode cases respectively. Figures V.8 and V.9 show the second harmonic tier.

Many of the effects observed with JoKHeR are also present when calculated with EPIC, but with more clarity. EPIC had no issue marching to the back of the cone for all cases and shows a more consistent and smooth progression between cases. As more modes are introduced, the peak amplitude begins to lower for the primary modes, but rise slightly for the harmonics. More modes are also shown to push the onset of nonlinear saturation further upstream. Once this nonlinear saturation begins, the higher frequency disturbances continue to grow while the lower ones begin to decay. These trends hint that if nonlinear saturation began earlier, perhaps in response to a higher initial amplitude, that a second peak of disturbed frequencies closer to the range observed in experiments would appear in response. However, because this presentation is focused on the verification and validation of a new code, we will save newer results for a later presentation.

Despite the differences shown between JoKHeR and EPIC's NPSE results, we are more inclined to believe EPIC. Considering how well the LPSE results matched up and that the newer NPSE results appear more physical, these differences actually inspire confidence in EPIC's increased capabilities over JoKHeR.

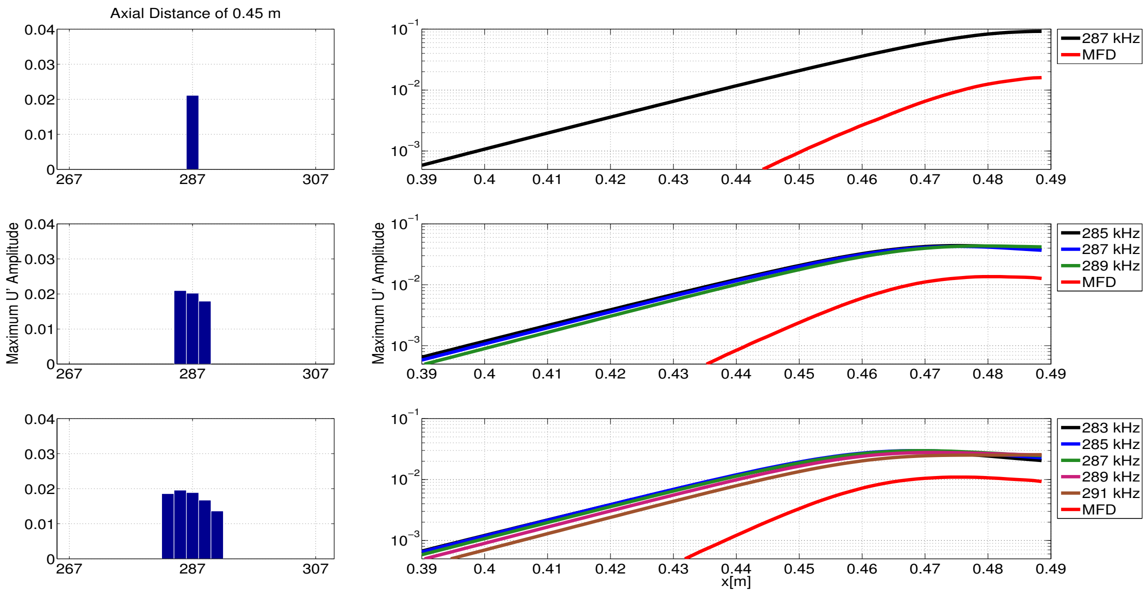

Figure V.6: First harmonic of EPIC calculated 1-, 3-, and 5-mode bandwidth NPSE amplitudes for Purdue compression cone.

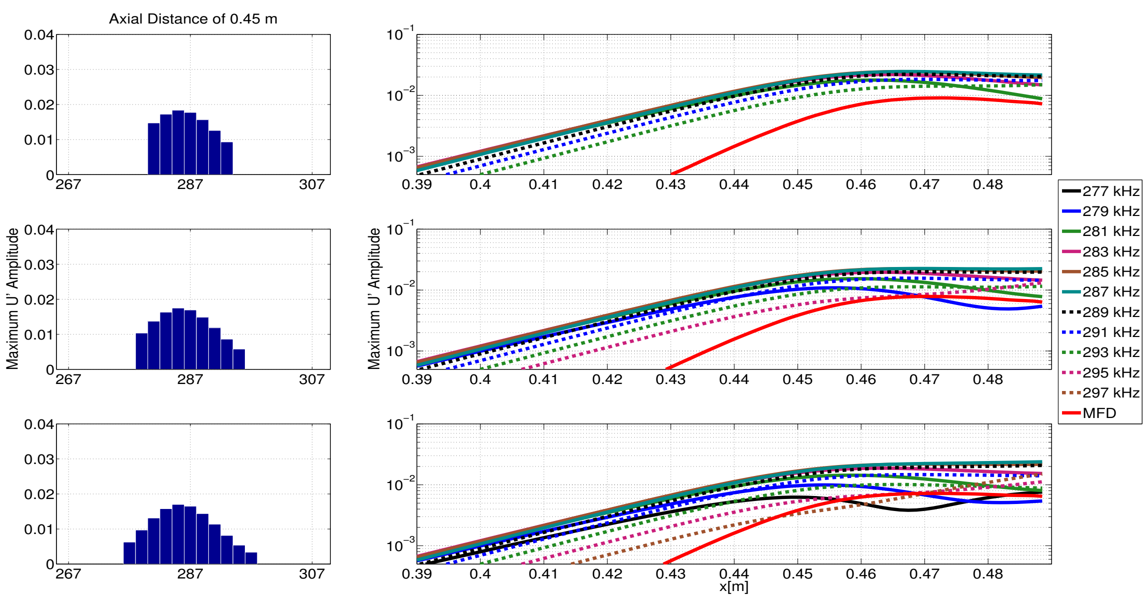

Figure V.7: First harmonic of EPIC calculated 7-, 9-, and 11-mode bandwidth NPSE amplitudes for Purdue compression cone.

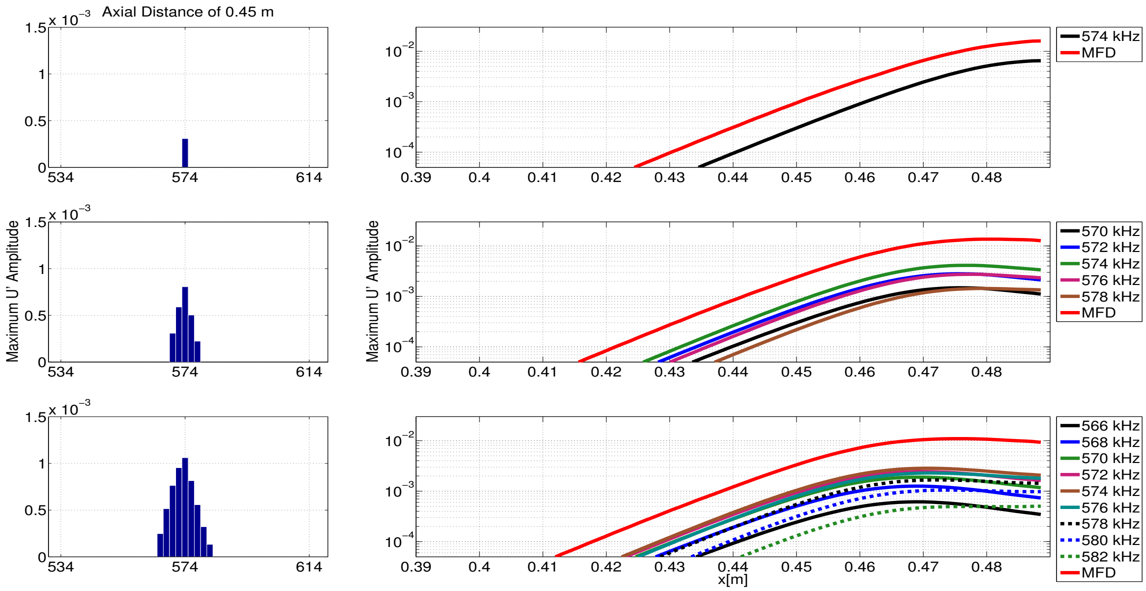

Figure V.8: Second harmonic of EPIC calculated 1-, 3-, and 5-mode bandwidth NPSE amplitudes for Purdue compression cone.

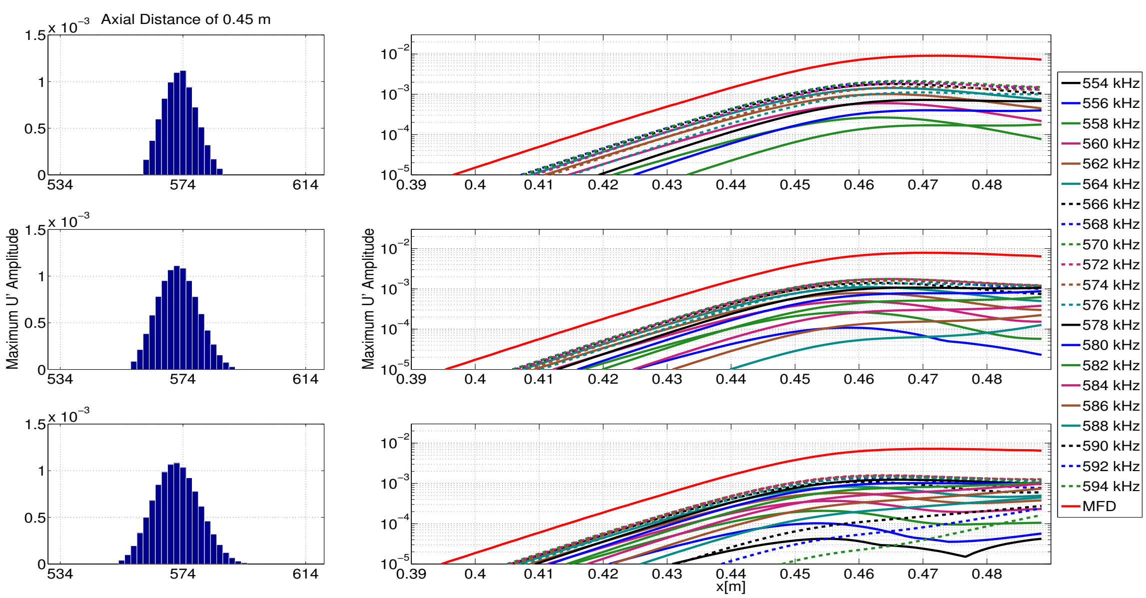

Figure V.9: Second harmonic of EPIC calculated 7-, 9-, and 11-mode bandwidth NPSE amplitudes for Purdue compression cone.

VI. SWIFTER RESULTS

As a final verification, the case presented in this chapter is of a significantly different geometry and flow condition than the previously explored experiments. Furthermore, SWIFTER exhibits a stationary-crossflow instability [44], a notably dissimilar instability mechanism than the second-mode that dominates the 2-D boundary layers of the unyawed hypersonic cones displayed formerly. LPSE results via NASA's LASTRAC program [6] provide the basis for this verification.

VI.1 Geometry

The Swept-Wing In-Flight Testing Excrescence Research (SWIFTER) model is an airfoil glove tested at Texas A&M University [44]. The article is a subsonic, spanwise-invariant, swept-wing that has undergone extensive in-flight and wind tunnel experiments. SWIFTER features a 30° sweep with a swept chord length of 1.37 m and span of 1.07 m. The test side is comprised of only convex curvature, resulting in a favorable streamwise pressure gradient up until the pressure minimum, located at 70% x/c.

VI.2 LASTRAC Verification

Supporting computational stability analyses (both LST and LPSE) were calculated via NASA's LASTRAC program, creating the beneficial opportunity to verify EPIC with a third-party source. The original SWIFTER LASTRAC results, produced by Ph.D. student and colleague Matthew Tufts, can be seen in his publication [44]. In order to verify that identical data was being used, Mr. Tufts graciously agreed to run a new set of results with LASTRAC to compare with EPIC. Figure VI.1 shows LPSE results obtained with EPIC with LASTRAC's results overlaid on top for a $-6.5°$ AoA case. As done in LASTRAC, EPIC N-factors for this case are calculated with α_i values (III.21). Stationary-crossflow disturbances consist of a 0.0 Hz frequency and a beta of varying range of spanwise wavelengths. The most amplified disturbance consisted of a spanwise wavelength of 4.00 mm and all N-factor results compare exceeding well with LASTRAC results.

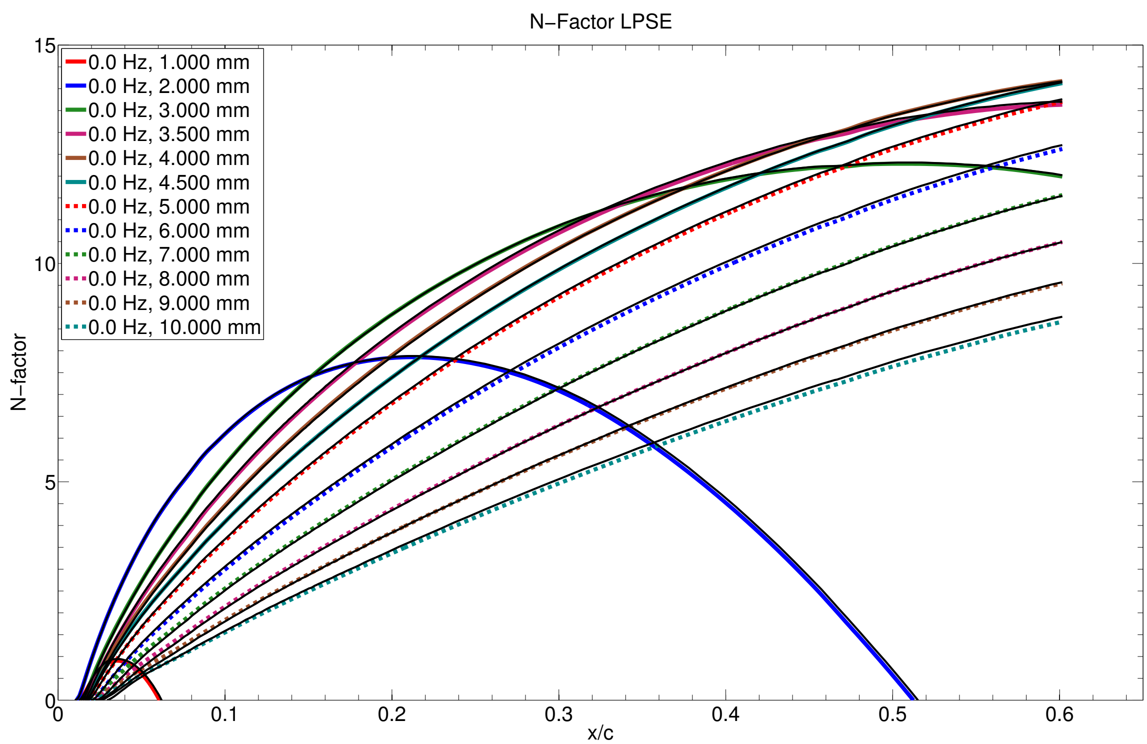

Figure VI.1: EPIC LPSE N-factors (colored lines) for SWIFTER wing glove with LASTRAC's calculations (black lines) overlaid on top.

VII. SUMMARY

With an increased demand lately for more aggressive flight envelopes that require more advanced flight technology, it is evident that a more complete understanding of the different laminar-to-turbulent transition mechanisms will be of great benefit. Despite pervasive use in today's aerospace industry, transition prediction tools are still inadequate for the type of transition modeling we envision. True to its defintion, turbulence, and by extension turbulent transition, are chaotic in nature and not easily understood or solved. This is not to say that progress has not been made, as it most definitely has. If we have learned anything from history, it is that research into this century-old problem will continue to yield new and beneficial revelations, ultimately transforming our lofty aspirations into reality.

As new wind tunnel and computational experiments are performed, it is equally essential that our analytical tools are easy to use and reflect the current knowledge base of the problem. Being a proponent of NPSE analysis, the Euonymous (meaning appropriately named) Parabolized Instability Code is a crucial tool for our CST lab for a myriad of reasons:

- EPIC is faster, more robust, and easier to use than its predecessor.
- EPIC's modular design allows for easy and convenient future modifications to stay up-to-date.
- Most importantly, as displayed in this presentation, EPIC's accuracy compares well to preexisting computational and experimental results.

LPSE results on the Langley 93-10 cone proved to be in line with similar experimental and computational results. Comparatively, all disturbances exhibited the expected behavior, culminating with the most amplified disturbance being found within the expected range. NPSE tests showed evidence of shared energy and nonlinear saturation and successfully recovered the linear solutions given a small enough initial amplitude.

The Purdue Compression Cone results demonstrated EPIC's capability to handle different instability mechanisms. Oblique modes and 3-D disturbances behaved as expected under LPSE analysis while the second-mode instability again matched well-documented computational and experimental results. Bandwidth NPSE results showed promising improvement and clarity over JoKHeR's previous results.

Finally, EPIC results corresponded correctly with LASTRAC on the subsonic SWIFTER wing glove, further validating EPIC's capabilities to analyze the crossflow instability mechanism and subsonic flight regimes.

With these preliminary comparisons successfully completed, EPIC has already begun to see daily use in the CST lab. Future stability research will continue to utilize EPIC's effectiveness and hopefully provide the answers that will bring researchers ever closer to predicting and understanding the phenomenon of laminar-to-turbulent transition.

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APPENDIX A

BASIC-STATE EQUATIONS

This section will demonstrate the derivation for the fully expanded and perturbed basic-state equations that will serve as the starting point for the following derivations. We will begin from the nondimensional, curvilinear, basic-state equations (equations II.11-II.16). Each steady, basic-state quantity is perturbed with an unsteady disturbance quantity (equation II.17). For convenience, this and the thermodynamic perturbation relations are listed here.

$$\phi = \overline{\phi}\,(x, y) + \phi'\,(x, y, z, t)\,, \qquad \phi' \ll \overline{\phi} \tag{A.1}$$

$$\mu' = \frac{\partial \overline{\mu}}{\partial T} T', \qquad \lambda' = \frac{\partial \overline{\lambda}}{\partial T} T', \qquad \kappa' = \frac{\partial \overline{\kappa}}{\partial T} T', \qquad \frac{\partial \overline{\lambda}}{\partial T} = \frac{\overline{\lambda}}{\overline{\mu}} \frac{\partial \overline{\mu}}{\partial T}. \tag{A.2}$$

Applying equations A.1 and A.2 to equations II.11-II.16, subtracting out the purely steady terms, and collecting the nonlinear terms to the right give the following result. These equations will be the starting part for the LST and PSE formulations in the following appendices.

X-Momentum

$$
\begin{aligned}
&\overline{\rho}\left[\frac{\partial u'}{\partial t}+\frac{u'}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{u}}{h_1}\frac{\partial u'}{\partial x}+\frac{v'}{h_2}\frac{\partial \overline{u}}{\partial y}+\frac{\overline{v}}{h_2}\frac{\partial u'}{\partial y}+\frac{w'}{h_3}\frac{\partial \overline{u}}{\partial z}+\frac{\overline{w}}{h_3}\frac{\partial u'}{\partial z}\right.\\
&\qquad -v'\left(\frac{\overline{v}}{h_2h_1}\frac{\partial h_2}{\partial x}-\frac{\overline{u}}{h_2h_1}\frac{\partial h_1}{\partial y}\right)-\overline{v}\left(\frac{v'}{h_2h_1}\frac{\partial h_2}{\partial x}-\frac{u'}{h_2h_1}\frac{\partial h_1}{\partial y}\right)\\
&\qquad \left.+w'\left(\frac{\overline{u}}{h_1h_3}\frac{\partial h_1}{\partial z}-\frac{\overline{w}}{h_1h_3}\frac{\partial h_3}{\partial x}\right)+\overline{w}\left(\frac{u'}{h_1h_3}\frac{\partial h_1}{\partial z}-\frac{w'}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\right]\\
&\qquad +\rho'\left[\frac{\partial \overline{u}}{\partial t}+\frac{\overline{u}}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_2}\frac{\partial \overline{u}}{\partial y}+\frac{\overline{w}}{h_3}\frac{\partial \overline{u}}{\partial z}\right.\\
&\qquad \left.-\overline{v}\left(\frac{\overline{v}}{h_2h_1}\frac{\partial h_2}{\partial x}-\frac{\overline{u}}{h_2h_1}\frac{\partial h_1}{\partial y}\right)+\overline{w}\left(\frac{\overline{u}}{h_1h_3}\frac{\partial h_1}{\partial z}-\frac{\overline{w}}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\right]\\
&\qquad +\frac{1}{h_1}\frac{\partial P'}{\partial x}-\frac{1}{Re}\left\{\frac{1}{h_1}\frac{\partial}{\partial x}\left[\frac{\overline{\lambda}}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3u'\right)}{\partial x}+\frac{\partial\left(h_1h_3v'\right)}{\partial y}+\frac{\partial\left(h_1h_2w'\right)}{\partial z}\right)\right.\right.\\
&\qquad \left.+\frac{\lambda'}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3\overline{u}\right)}{\partial x}+\frac{\partial\left(h_1h_3\overline{v}\right)}{\partial y}+\frac{\partial\left(h_1h_2\overline{w}\right)}{\partial z}\right)\right]\\
&\qquad +\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x}\left[2\mu'h_2h_3\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right.\\
&\qquad \left.+2\overline{\mu}h_2h_3\left(\frac{1}{h_1}\frac{\partial u'}{\partial x}+\frac{v'}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w'}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right]\\
&\qquad +\frac{1}{h_1h_2h_3}\frac{\partial}{\partial y}\left[\overline{\mu}h_1h_3\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v'}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u'}{h_1}\right)\right)\right.\\
&\qquad \left.+\mu'h_1h_3\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{v}}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{u}}{h_1}\right)\right)\right]\\
&\qquad +\frac{1}{h_1h_2h_3}\frac{\partial}{\partial z}\left[\overline{\mu}h_1h_2\left(\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u'}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w'}{h_3}\right)\right)\right.\\
&\qquad \left.+\mu'h_1h_2\left(\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{u}}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{w}}{h_3}\right)\right)\right]\\
&+\frac{\overline{\mu}}{h_1h_2}\frac{\partial h_1}{\partial y}\left[\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v'}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u'}{h_1}\right)\right]+\frac{\mu'}{h_1h_2}\frac{\partial h_1}{\partial y}\left[\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{v}}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{u}}{h_1}\right)\right]\\
&+\frac{\overline{\mu}}{h_1h_3}\frac{\partial h_1}{\partial z}\left[\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w'}{h_3}\right)+\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u'}{h_1}\right)\right]+\frac{\mu'}{h_1h_3}\frac{\partial h_1}{\partial z}\left[\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{w}}{h_3}\right)+\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{u}}{h_1}\right)\right]\\
&-\frac{2\overline{\mu}}{h_1h_2}\frac{\partial h_2}{\partial x}\left[\frac{1}{h_2}\frac{\partial v'}{\partial y}+\frac{w'}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u'}{h_1h_2}\frac{\partial h_2}{\partial x}\right]-\frac{2\mu'}{h_1h_2}\frac{\partial h_2}{\partial x}\left[\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right]\\
&\qquad -\frac{2\overline{\mu}}{h_1h_3}\frac{\partial h_3}{\partial x}\left[\frac{1}{h_3}\frac{\partial w'}{\partial z}+\frac{u'}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v'}{h_2h_3}\frac{\partial h_3}{\partial y}\right]\\
&\qquad \left.-\frac{2\mu'}{h_1h_3}\frac{\partial h_3}{\partial x}\left[\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right]\right\}=NL_x
\end{aligned}
\tag{A.3}
$$

Y-Momentum

$$
\begin{aligned}
\overline{\rho}\Bigg[\frac{\partial v'}{\partial t} + \frac{u'}{h_1}\frac{\partial \overline{v}}{\partial x} + \frac{\overline{u}}{h_1}\frac{\partial v'}{\partial x} + \frac{v'}{h_2}\frac{\partial \overline{v}}{\partial y} + \frac{\overline{v}}{h_2}\frac{\partial v'}{\partial y} + \frac{w'}{h_3}\frac{\partial \overline{v}}{\partial z} + \frac{\overline{w}}{h_3}\frac{\partial v'}{\partial z} \\
-w'\left(\frac{\overline{w}}{h_2h_3}\frac{\partial h_3}{\partial y} - \frac{\overline{v}}{h_2h_3}\frac{\partial h_2}{\partial z}\right) - \overline{w}\left(\frac{w'}{h_2h_3}\frac{\partial h_3}{\partial y} - \frac{v'}{h_2h_3}\frac{\partial h_2}{\partial z}\right) \\
+u'\left(\frac{\overline{v}}{h_1h_2}\frac{\partial h_2}{\partial x} - \frac{\overline{u}}{h_1h_2}\frac{\partial h_1}{\partial y}\right) + \overline{u}\left(\frac{v'}{h_1h_2}\frac{\partial h_2}{\partial x} - \frac{u'}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\Bigg] \\
+\rho'\Bigg[\frac{\partial \overline{v}}{\partial t} + \frac{\overline{u}}{h_1}\frac{\partial \overline{v}}{\partial x} + \frac{\overline{v}}{h_2}\frac{\partial \overline{v}}{\partial y} + \frac{\overline{w}}{h_3}\frac{\partial \overline{v}}{\partial z} \\
-\overline{w}\left(\frac{\overline{w}}{h_2h_3}\frac{\partial h_3}{\partial y} - \frac{\overline{v}}{h_2h_3}\frac{\partial h_2}{\partial z}\right) + \overline{u}\left(\frac{\overline{v}}{h_1h_2}\frac{\partial h_2}{\partial x} - \frac{\overline{u}}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\Bigg] \\
+\frac{1}{h_2}\frac{\partial P'}{\partial y} - \frac{1}{Re}\Bigg\{\frac{1}{h_2}\frac{\partial}{\partial y}\Bigg[\frac{\overline{\lambda}}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3u'\right)}{\partial x} + \frac{\partial\left(h_1h_3v'\right)}{\partial y} + \frac{\partial\left(h_1h_2w'\right)}{\partial z}\right) \\
+\frac{\lambda'}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3\overline{u}\right)}{\partial x} + \frac{\partial\left(h_1h_3\overline{v}\right)}{\partial y} + \frac{\partial\left(h_1h_2\overline{w}\right)}{\partial z}\right)\Bigg] \\
+\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x}\Bigg[\overline{\mu}h_2h_3\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v'}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u'}{h_1}\right)\right) \\
+\mu'h_2h_3\left(\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{v}}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{u}}{h_1}\right)\right)\Bigg] \\
+\frac{1}{h_1h_2h_3}\frac{\partial}{\partial y}\Bigg[2\mu'h_1h_3\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y} + \frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z} + \frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right) \\
+2\overline{\mu}h_1h_3\left(\frac{1}{h_2}\frac{\partial v'}{\partial y} + \frac{w'}{h_2h_3}\frac{\partial h_2}{\partial z} + \frac{u'}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\Bigg] \\
+\frac{1}{h_1h_2h_3}\frac{\partial}{\partial z}\Bigg[\overline{\mu}h_1h_2\left(\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w'}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v'}{h_2}\right)\right) \\
+\mu'h_1h_2\left(\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{w}}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{v}}{h_2}\right)\right)\Bigg] \\
+\frac{\overline{\mu}}{h_2h_3}\frac{\partial h_2}{\partial z}\left[\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w'}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v'}{h_2}\right)\right] + \frac{\mu'}{h_2h_3}\frac{\partial h_2}{\partial z}\left[\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{w}}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{v}}{h_2}\right)\right] \\
+\frac{\overline{\mu}}{h_1h_2}\frac{\partial h_2}{\partial x}\left[\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v'}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u'}{h_1}\right)\right] + \frac{\mu'}{h_1h_2}\frac{\partial h_2}{\partial x}\left[\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{v}}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{u}}{h_1}\right)\right] \\
-\frac{2\overline{\mu}}{h_2h_3}\frac{\partial h_3}{\partial y}\left[\frac{1}{h_3}\frac{\partial w'}{\partial z} + \frac{u'}{h_1h_3}\frac{\partial h_3}{\partial x} + \frac{v'}{h_2h_3}\frac{\partial h_3}{\partial y}\right] \\
-\frac{2\mu'}{h_2h_3}\frac{\partial h_3}{\partial y}\left[\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z} + \frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x} + \frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right] \\
-\frac{2\overline{\mu}}{h_1h_2}\frac{\partial h_1}{\partial y}\left[\frac{1}{h_1}\frac{\partial u'}{\partial x} + \frac{v'}{h_1h_2}\frac{\partial h_1}{\partial y} + \frac{w'}{h_1h_3}\frac{\partial h_1}{\partial z}\right] \\
-\frac{2\mu'}{h_1h_2}\frac{\partial h_1}{\partial y}\left[\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x} + \frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y} + \frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right]\Bigg\} = NL_y \quad \text{(A.4)}
\end{aligned}
$$

Z-Momentum

$$
\begin{aligned}
&\overline{\rho}\left[\frac{\partial w'}{\partial t}+\frac{u'}{h_1}\frac{\partial \overline{w}}{\partial x}+\frac{\overline{u}}{h_1}\frac{\partial w'}{\partial x}+\frac{v'}{h_2}\frac{\partial \overline{w}}{\partial y}+\frac{\overline{v}}{h_2}\frac{\partial w'}{\partial y}+\frac{w'}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{w}}{h_3}\frac{\partial w'}{\partial z}\right.\\
&\qquad -u'\left(\frac{\overline{u}}{h_1h_3}\frac{\partial h_1}{\partial z}-\frac{\overline{w}}{h_1h_3}\frac{\partial h_3}{\partial x}\right)-\overline{u}\left(\frac{u'}{h_1h_3}\frac{\partial h_1}{\partial z}-\frac{w'}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\\
&\qquad \left.+v'\left(\frac{\overline{w}}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{\overline{v}}{h_2h_3}\frac{\partial h_2}{\partial z}\right)+\overline{v}\left(\frac{w'}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{v'}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\right]\\
&\qquad +\rho'\left[\frac{\partial \overline{w}}{\partial t}+\frac{\overline{u}}{h_1}\frac{\partial \overline{w}}{\partial x}+\frac{\overline{v}}{h_2}\frac{\partial \overline{w}}{\partial y}+\frac{\overline{w}}{h_3}\frac{\partial \overline{w}}{\partial z}\right.\\
&\qquad \left.-\overline{u}\left(\frac{\overline{u}}{h_1h_3}\frac{\partial h_1}{\partial z}-\frac{\overline{w}}{h_1h_3}\frac{\partial h_3}{\partial x}\right)+\overline{v}\left(\frac{\overline{w}}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{\overline{v}}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\right]\\
&\qquad +\frac{1}{h_3}\frac{\partial P'}{\partial z}-\frac{1}{Re}\left\{\frac{1}{h_3}\frac{\partial}{\partial z}\left[\frac{\overline{\lambda}}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3u'\right)}{\partial x}+\frac{\partial\left(h_1h_3v'\right)}{\partial y}+\frac{\partial\left(h_1h_2w'\right)}{\partial z}\right)\right.\right.\\
&\qquad \left.+\frac{\lambda'}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3\overline{u}\right)}{\partial x}+\frac{\partial\left(h_1h_3\overline{v}\right)}{\partial y}+\frac{\partial\left(h_1h_2\overline{w}\right)}{\partial z}\right)\right]\\
&\qquad +\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x}\left[\overline{\mu}h_2h_3\left(\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u'}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w'}{h_3}\right)\right)\right.\\
&\qquad \left.+\mu' h_2h_3\left(\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{u}}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{w}}{h_3}\right)\right)\right]\\
&\qquad +\frac{1}{h_1h_2h_3}\frac{\partial}{\partial y}\left[\overline{\mu}h_1h_3\left(\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w'}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v'}{h_2}\right)\right)\right.\\
&\qquad \left.+\mu' h_1h_3\left(\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{w}}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{v}}{h_2}\right)\right)\right]\\
&\qquad +\frac{1}{h_1h_2h_3}\frac{\partial}{\partial z}\left[2\mu' h_1h_2\left(\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right.\\
&\qquad \left.+2\overline{\mu}h_1h_2\left(\frac{1}{h_3}\frac{\partial w'}{\partial z}+\frac{u'}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v'}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]\\
&+\frac{\overline{\mu}}{h_1h_3}\frac{\partial h_3}{\partial x}\left[\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u'}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w'}{h_3}\right)\right]+\frac{\mu'}{h_1h_3}\frac{\partial h_3}{\partial x}\left[\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{u}}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{w}}{h_3}\right)\right]\\
&+\frac{\overline{\mu}}{h_2h_3}\frac{\partial h_3}{\partial y}\left[\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w'}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v'}{h_2}\right)\right]+\frac{\mu'}{h_2h_3}\frac{\partial h_3}{\partial y}\left[\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{w}}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{v}}{h_2}\right)\right]\\
&-\frac{2\overline{\mu}}{h_1h_3}\frac{\partial h_1}{\partial z}\left[\frac{1}{h_1}\frac{\partial u'}{\partial x}+\frac{v'}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w'}{h_1h_3}\frac{\partial h_1}{\partial z}\right]-\frac{2\mu'}{h_1h_3}\frac{\partial h_1}{\partial z}\left[\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right]\\
&\qquad -\frac{2\overline{\mu}}{h_2h_3}\frac{\partial h_2}{\partial z}\left[\frac{1}{h_2}\frac{\partial v'}{\partial y}+\frac{w'}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u'}{h_1h_2}\frac{\partial h_2}{\partial x}\right]\\
&\qquad \left.-\frac{2\mu'}{h_2h_3}\frac{\partial h_2}{\partial z}\left[\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right]\right\}=NL_z \qquad \text{(A.5)}
\end{aligned}
$$

Energy

$$
\begin{aligned}
&\overline{\rho}\left[\frac{\partial T'}{\partial t}+\frac{u'}{h_1}\frac{\partial \overline{T}}{\partial x}+\frac{\overline{u}}{h_1}\frac{\partial T'}{\partial x}+\frac{v'}{h_2}\frac{\partial \overline{T}}{\partial y}+\frac{\overline{v}}{h_2}\frac{\partial T'}{\partial y}+\frac{w'}{h_3}\frac{\partial \overline{T}}{\partial z}+\frac{\overline{w}}{h_3}\frac{\partial T'}{\partial z}\right]\\
&\qquad+\rho'\left[\frac{\partial \overline{T}}{\partial t}+\frac{\overline{u}}{h_1}\frac{\partial \overline{T}}{\partial x}+\frac{\overline{v}}{h_2}\frac{\partial \overline{T}}{\partial y}+\frac{\overline{w}}{h_3}\frac{\partial \overline{T}}{\partial z}\right]\\
&-(\gamma-1)M^2\left[\frac{\partial P'}{\partial t}+\frac{u'}{h_1}\frac{\partial \overline{P}}{\partial x}+\frac{\overline{u}}{h_1}\frac{\partial P'}{\partial x}+\frac{v'}{h_2}\frac{\partial \overline{P}}{\partial y}+\frac{\overline{v}}{h_2}\frac{\partial P'}{\partial y}+\frac{w'}{h_3}\frac{\partial \overline{P}}{\partial z}+\frac{\overline{w}}{h_3}\frac{\partial P'}{\partial z}\right]\\
&-\frac{1}{PrRe}\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x}\left(\frac{\overline{\kappa}h_2h_3}{h_1}\frac{\partial T'}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\kappa' h_2h_3}{h_1}\frac{\partial \overline{T}}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\overline{\kappa}h_1h_3}{h_2}\frac{\partial T'}{\partial y}\right)\right.\\
&\qquad\left.+\frac{\partial}{\partial y}\left(\frac{\kappa' h_1h_3}{h_2}\frac{\partial \overline{T}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\overline{\kappa}h_1h_2}{h_3}\frac{\partial T'}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\kappa' h_1h_2}{h_3}\frac{\partial \overline{T}}{\partial z}\right)\right]\\
&\qquad-\frac{(\gamma-1)M^2}{Re}\left\{(2\mu'+\lambda')\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right)^2\right.\\
&+2\left(2\overline{\mu}+\overline{\lambda}\right)\left(\frac{1}{h_1}\frac{\partial u'}{\partial x}+\frac{v'}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w'}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\\
&\qquad+\mu'\left[\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{w}}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{v}}{h_2}\right)\right]^2\\
&\qquad+2\overline{\mu}\left[\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{w}}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{v}}{h_2}\right)\right]\left[\frac{h_3}{h_2}\frac{\partial}{\partial y}\left(\frac{w'}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v'}{h_2}\right)\right]\\
&\qquad+(2\mu'+\lambda')\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)^2\\
&+2\left(2\overline{\mu}+\overline{\lambda}\right)\left(\frac{1}{h_2}\frac{\partial v'}{\partial y}+\frac{w'}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u'}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&\qquad+\mu'\left[\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{w}}{h_3}\right)+\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{u}}{h_1}\right)\right]^2\\
&\qquad+2\overline{\mu}\left[\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{w}}{h_3}\right)+\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{\overline{u}}{h_1}\right)\right]\left[\frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w'}{h_3}\right)+\frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u'}{h_1}\right)\right]\\
&\qquad+(2\mu'+\lambda')\left(\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)^2\\
&+2\left(2\overline{\mu}+\overline{\lambda}\right)\left(\frac{1}{h_3}\frac{\partial w'}{\partial z}+\frac{u'}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v'}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\left(\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&\qquad+\mu'\left[\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{v}}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{u}}{h_1}\right)\right]^2\\
&\qquad+2\overline{\mu}\left[\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{\overline{v}}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{\overline{u}}{h_1}\right)\right]\left[\frac{h_2}{h_1}\frac{\partial}{\partial x}\left(\frac{v'}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial y}\left(\frac{u'}{h_1}\right)\right]\\
&\quad+2\lambda'\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\overline{\lambda}\left(\frac{1}{h_1}\frac{\partial u'}{\partial x}+\frac{v'}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w'}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&+2\overline{\lambda}\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_2}\frac{\partial v'}{\partial y}+\frac{w'}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u'}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&+2\lambda'\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&+2\overline{\lambda}\left(\frac{1}{h_1}\frac{\partial u'}{\partial x}+\frac{v'}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w'}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&+2\overline{\lambda}\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_3}\frac{\partial w'}{\partial z}+\frac{u'}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v'}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&+2\lambda'\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&+2\overline{\lambda}\left(\frac{1}{h_2}\frac{\partial v'}{\partial y}+\frac{w'}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u'}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_3}\frac{\partial \overline{w}}{\partial z}+\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&+2\overline{\lambda}\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_3}\frac{\partial w'}{\partial z}+\frac{u'}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v'}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\Bigg\}=NL_e
\end{aligned}
\tag{A.6}
$$

Continuity

$$
\begin{aligned}
&\frac{\partial \rho'}{\partial t}+\left[\frac{1}{h_1}\left(\frac{\partial\overline{\rho}}{\partial x}u'+\frac{\partial \rho'}{\partial x}\overline{u}+\frac{\partial \overline{u}}{\partial x}\rho'+\frac{\partial u'}{\partial x}\overline{\rho}\right)+\left(\overline{\rho}u'+\overline{u}\rho'\right)\left(\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\right]\\
&+\left[\frac{1}{h_2}\left(\frac{\partial\overline{\rho}}{\partial y}v'+\frac{\partial \rho'}{\partial y}\overline{v}+\frac{\partial \overline{v}}{\partial y}\rho'+\frac{\partial v'}{\partial y}\overline{\rho}\right)+\left(\overline{\rho}v'+\overline{v}\rho'\right)\left(\frac{1}{h_3h_2}\frac{\partial h_3}{\partial y}+\frac{1}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right]\\
&+\left[\frac{1}{h_3}\left(\frac{\partial\overline{\rho}}{\partial z}w'+\frac{\partial \rho'}{\partial z}\overline{w}+\frac{\partial \overline{w}}{\partial z}\rho'+\frac{\partial w'}{\partial z}\overline{\rho}\right)+\left(\overline{\rho}w'+\overline{w}\rho'\right)\left(\frac{1}{h_3h_2}\frac{\partial h_2}{\partial z}+\frac{1}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right]=NL_m
\end{aligned}
\tag{A.7}
$$

Equation of State

$$
P'=\frac{\rho'\overline{T}+\overline{\rho}T'}{\gamma M^2}
\tag{A.8}
$$

APPENDIX B

LST FORMULATION

The full LST equations and formulations will be displayed here. Beginning from equations A.3-A.8, the following assumptions are imposed:

- $\overline{\phi} = \overline{\phi}\,(y)$
- $\overline{v} = 0$
- $\mathcal{NL} = 0$
- $\phi' = \hat{\phi}\,(y)\,e^{i(\alpha x+\beta z-\omega t)}$

Pressure is eliminated through use of the equation of state (equation A.8), thus reducing the set of equations to five. The result is arranged in the format

$$\mathcal{A}\frac{\partial^2\hat{\phi}}{\partial y^2}+\mathcal{B}\frac{\partial\hat{\phi}}{\partial y}+\mathcal{C}\hat{\phi}=0 \tag{B.1}$$

where $\hat{\phi} = \left[\hat{u},\hat{v},\hat{w},\hat{T},\hat{\rho}\right]$. **A**, **B**, and **C** are 5x5 matrices at each point in the normal direction. These are expanded below. All terms should be basic-state quantities, as the disturbance quantities and derivatives are accounted for in equation B.1. X-momentum, Y-momentum, Z-momentum, energy, and mass continuity are represented by rows 1-5 respectively. Similarly, columns 1-5 are the coefficients of $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{T}$, $\hat{\rho}$, and their y-derivatives.

A-Matrix

$$A = \begin{pmatrix} -\frac{\mu}{Reh_2^2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{Reh_2^2}(\lambda + 2\mu) & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu}{Reh_2^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\kappa}{PrReh_2^2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

B-Matrix

$$B_{1,1} = \frac{\mu}{Reh_2^3}\frac{\partial h_2}{\partial y} - \frac{1}{Reh_1 h_2^2 h_3}\left(h_1 h_3 \frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} + h_1 \mu \frac{\partial h_3}{\partial y} + h_3 \mu \frac{\partial h_1}{\partial y}\right)$$

$$\begin{aligned} B_{1,2} = &\frac{\lambda h_3}{Re}\left(\frac{1}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1 h_2^2 h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\right) \\ &- \frac{\lambda}{Reh_1^2 h_2 h_3}\left(h_1 \frac{\partial h_3}{\partial x} + h_3 \frac{\partial h_1}{\partial x} + i\alpha h_1 h_3\right) + \frac{3\mu}{Reh_1 h_2^2}\frac{\partial h_2}{\partial x} - \frac{i\alpha\mu}{Reh_2 h_1} \end{aligned}$$

$$B_{1,3} = 0$$

$$B_{1,4} = -\frac{1}{Reh_2^2}\frac{\partial \mu}{\partial T}\left(\frac{\partial u}{\partial y} - \frac{u}{h_1}\frac{\partial h_1}{\partial y}\right)$$

$$B_{1,5} = 0$$

$$
\begin{aligned}
B_{2,1} =& -\frac{\lambda}{Reh_1h_2^2h_3}\left(i\alpha h_2h_3 + h_2\frac{\partial h_3}{\partial x} + h_3\frac{\partial h_2}{\partial x}\right) - \frac{\mu}{Reh_1h_2h_3}\left(i\alpha h_3 + \frac{\partial h_3}{\partial x}\right)\\
& - \frac{3\mu}{Reh_1h_2^2}\frac{\partial h_2}{\partial x}\\
B_{2,2} =& \left[\frac{\lambda}{Reh_2}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right) - \frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\right]h_1h_3\\
& - \frac{2\lambda}{Reh_1h_2^2h_3}\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y}\right) - \frac{2}{Reh_2^2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\\
& - \frac{2\mu}{Reh_1h_2^2h_3}\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y} - \frac{h_1h_3}{h_2}\frac{\partial h_2}{\partial y}\right)\\
B_{2,3} =& -\frac{\lambda}{Reh_1h_2^2h_3}\left(i\beta h_1h_2 + h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right) - \frac{\mu}{Reh_1h_2h_3}\left(i\beta h_1 + \frac{\partial h_1}{\partial z}\right)\\
& - \frac{3\mu}{Reh_2^2h_3}\frac{\partial h_2}{\partial z}\\
B_{2,4} =& \frac{\rho}{h_2\gamma M^2} - \frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(uh_2\frac{\partial h_3}{\partial x} + uh_3\frac{\partial h_2}{\partial x} + wh_1\frac{\partial h_2}{\partial z} + wh_2\frac{\partial h_1}{\partial z}\right)\\
& - \frac{2}{Reh_1h_2^2h_3}\frac{\partial \mu}{\partial T}\left(wh_1\frac{\partial h_2}{\partial z} + uh_3\frac{\partial h_2}{\partial x}\right)\\
B_{2,5} =& \frac{T}{h_2\gamma M^2}
\end{aligned}
$$

$$
\begin{aligned}
B_{3,1} =& 0\\
B_{3,2} =& -\frac{\lambda}{Reh_1h_2h_3^2}\left(i\beta h_1h_3 + h_1\frac{\partial h_3}{\partial z} + h_3\frac{\partial h_1}{\partial z}\right)\\
& + \frac{\lambda h_1}{Re}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right) + \frac{3\mu}{Reh_2^2h_3}\frac{\partial h_2}{\partial z} - \frac{i\beta\mu}{Reh_2h_3}\\
B_{3,3} =& -\frac{1}{Reh_2^2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} - \frac{\mu}{Reh_1h_2h_3}\left(\frac{h_1}{h_2}\frac{\partial h_3}{\partial y} + \frac{h_3}{h_2}\frac{\partial h_1}{\partial y} - \frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\right)\\
B_{3,4} =& -\frac{1}{Reh_2^2h_3}\frac{\partial \mu}{\partial T}\left(h_3\frac{\partial w}{\partial y} - w\frac{\partial h_3}{\partial y}\right)\\
B_{3,5} =& 0
\end{aligned}
$$

$$B_{4,1} = -\frac{2\mu(\gamma-1)M^2}{Reh_2}\left(\frac{1}{h_2}\frac{\partial u}{\partial y} - \frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)$$

$$B_{4,2} = -\frac{2(\gamma-1)M^2}{Reh_2}\left[(2\mu+\lambda)\left(\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z} + \frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right) + \lambda\left(\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z} + \frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\right]$$

$$B_{4,3} = -\frac{2\mu(\gamma-1)M^2}{Reh_2}\left(\frac{1}{h_2}\frac{\partial w}{\partial y} - \frac{w}{h_2h_3}\frac{\partial h_3}{\partial y}\right)$$

$$B_{4,4} = -\frac{1}{PrReh_1h_2^2h_3}\left[\kappa\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y} - \frac{h_1h_3}{h_2}\frac{\partial h_2}{\partial y}\right) + 2h_1h_3\frac{\partial \kappa}{\partial T}\frac{\partial T}{\partial y}\right]$$

$$B_{4,5} = 0$$

$$B_{5,1} = 0$$

$$B_{5,2} = \frac{\rho}{h_2}$$

$$B_{5,3} = 0$$

$$B_{5,4} = 0$$

$$B_{5,5} = 0$$

C-Matrix

$$\begin{aligned}
C_{1,1} =& \rho\left(-i\omega + \frac{u}{h_1}i\alpha + \frac{w}{h_3}i\beta + \frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\right) \\
&- \frac{\lambda}{Reh_1^2h_2h_3}\left(-\alpha^2h_2h_3 + i2\alpha h_2\frac{\partial h_3}{\partial x} + i2\alpha h_3\frac{\partial h_2}{\partial x} + 2\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x} + h_2\frac{\partial^2 h_3}{\partial x^2} + h_3\frac{\partial^2 h_2}{\partial x^2}\right) \\
&+ \frac{\lambda}{Reh_1}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial x}\right)\left(i\alpha h_2h_3 + h_2\frac{\partial h_3}{\partial x} + h_3\frac{\partial h_2}{\partial x}\right) \\
&- \frac{i2\alpha\mu}{Reh_1h_2h_3}\left(\frac{h_3}{h_1}\frac{\partial h_2}{\partial x} + \frac{h_2}{h_1}\frac{\partial h_3}{\partial x} - \frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\right) + \frac{2\alpha^2\mu}{Reh_1^2} \\
&+ \frac{1}{Reh_1^2h_2^2h_3}\frac{\partial h_1}{\partial y}\left(h_1h_3\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} + h_1\mu\frac{\partial h_3}{\partial y} + h_3\mu\frac{\partial h_1}{\partial y}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mu}{Reh_1 h_2^2}\left(\frac{1}{h_2}\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial y} + \frac{1}{h_1}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial y} - \frac{\partial^2 h_1}{\partial y^2}\right)\\
&- \frac{\mu}{Reh_1 h_2 h_3}\left(\frac{i\beta}{h_3} - \frac{1}{h_1 h_3}\frac{\partial h_1}{\partial z}\right)\left(h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right)\\
&+ \frac{\mu}{Reh_3}\left(\frac{\beta^2}{h_3} + \frac{i\beta}{h_3^2}\frac{\partial h_3}{\partial z} + \frac{i\beta}{h_1 h_3}\frac{\partial h_1}{\partial z} - \frac{1}{h_1^2 h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_1}{\partial z} - \frac{1}{h_1 h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_1}{\partial z} + \frac{1}{h_1 h_3}\frac{\partial^2 h_1}{\partial z^2}\right)\\
&+ \frac{\mu}{Reh_1^2 h_2^2}\left[\left(\frac{\partial h_1}{\partial y}\right)^2 + 2\left(\frac{\partial h_2}{\partial x}\right)^2\right]\\
&+ \frac{\mu}{Reh_1 h_3}\left[\frac{2}{h_1 h_3}\left(\frac{\partial h_3}{\partial x}\right)^2 - \frac{\partial h_1}{\partial z}\left(\frac{i\beta}{h_3} - \frac{1}{h_1 h_3}\frac{\partial h_1}{\partial z}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
C_{1,2} =&\rho\left(\frac{1}{h_2}\frac{\partial u}{\partial y} + \frac{u}{h_1 h_2}\frac{\partial h_1}{\partial y}\right)\\
&- \frac{\lambda}{Reh_1^2 h_2 h_3}\left(i\alpha h_1\frac{\partial h_3}{\partial y} + \frac{\partial h_1}{\partial x}\frac{\partial h_3}{\partial y} + h_1\frac{\partial^2 h_3}{\partial x\partial y} + i\alpha h_3\frac{\partial h_1}{\partial y} + \frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y} + h_3\frac{\partial^2 h_1}{\partial x\partial y}\right)\\
&+ \frac{\lambda}{Reh_1}\left(\frac{1}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1 h_2^2 h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\right)\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y}\right)\\
&- \frac{2\mu}{Reh_1 h_2 h_3}\left(i\alpha\frac{h_3}{h_1}\frac{\partial h_1}{\partial y} + \frac{1}{h_1}\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial x} - \frac{h_3}{h_1^2}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial x} + \frac{h_3}{h_1}\frac{\partial^2 h_1}{\partial y\partial x}\right)\\
&- \frac{1}{Reh_1^2 h_2 h_3}\left(h_1 h_3\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial y} + h_1\mu\frac{\partial h_3}{\partial y} + h_3\mu\frac{\partial h_1}{\partial y}\right)\left(i\alpha - \frac{1}{h_2}\frac{\partial h_2}{\partial x}\right)\\
&+ \frac{\mu}{Reh_2}\left(\frac{i\alpha}{h_1^2}\frac{\partial h_1}{\partial y} + \frac{1}{h_1 h_2}\frac{\partial^2 h_2}{\partial x\partial y} - \frac{1}{h_1^2 h_2}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial y} - \frac{1}{h_1 h_2^2}\frac{\partial h_2}{\partial x}\frac{\partial h_2}{\partial y}\right)\\
&- \frac{\mu}{Reh_1^2 h_2}\frac{\partial h_1}{\partial y}\left(i\alpha - \frac{1}{h_2}\frac{\partial h_2}{\partial x}\right) + \frac{2\mu}{Reh_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
C_{1,3} =&\frac{\rho}{h_1 h_3}\left(u\frac{\partial h_1}{\partial z} - 2w\frac{\partial h_3}{\partial x}\right) - \frac{\lambda}{Reh_1^2 h_2 h_3}\left(-\alpha\beta h_1 h_2 + i\beta h_1\frac{\partial h_2}{\partial x} + i\beta h_2\frac{\partial h_1}{\partial x}\right.\\
&\left.+i\alpha h_1\frac{\partial h_2}{\partial z} + \frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial z} + h_1\frac{\partial^2 h_2}{\partial x\partial z} + i\alpha h_2\frac{\partial h_1}{\partial z} + \frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} + h_2\frac{\partial^2 h_1}{\partial x\partial z}\right)\\
&+ \frac{\lambda}{Reh_1}\left(\frac{1}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1 h_2^2 h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\right)\left(i\beta h_1 h_2 + h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right)\\
&- \frac{2\mu}{Reh_1^2 h_2 h_3}\left(i\alpha h_2\frac{\partial h_1}{\partial z} + \frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial x} + h_2\frac{\partial^2 h_1}{\partial x\partial z} - \frac{h_2}{h_1}\frac{\partial h_1}{\partial z}\frac{\partial h_1}{\partial x}\right)\\
&- \frac{\mu}{Reh_1 h_2 h_3}\left(\frac{i\alpha}{h_1} - \frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x}\right)\left(h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right)\\
&+ \frac{\mu}{Reh_3}\left(\frac{\alpha\beta}{h_1} + \frac{i\alpha}{h_1^2}\frac{\partial h_1}{\partial z} + \frac{i\beta}{h_1 h_3}\frac{\partial h_3}{\partial x} + \frac{1}{h_1^2 h_3}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial z} + \frac{1}{h_1 h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial z} - \frac{1}{h_1 h_3}\frac{\partial^2 h_3}{\partial x\partial z}\right)\\
&- \frac{\mu}{Reh_1^2 h_3}\frac{\partial h_1}{\partial z}\left(i\alpha - \frac{1}{h_3}\frac{\partial h_3}{\partial x}\right) + \frac{2\mu}{Reh_1 h_2^2 h_3}\frac{\partial h_2}{\partial x}\frac{\partial h_2}{\partial z} + \frac{i2\mu\beta}{Reh_1 h_3^2}\frac{\partial h_3}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
C_{1,4} =& \frac{i\alpha\rho}{h_1\gamma M^2} - \frac{1}{Re}\frac{\lambda}{\mu}\frac{\partial\mu}{\partial T}\left\{\left(uh_2\frac{\partial h_3}{\partial x} + uh_3\frac{\partial h_2}{\partial x} + wh_1\frac{\partial h_2}{\partial z} + wh_2\frac{\partial h_1}{\partial z}\right)\right.\\
&\left.\left[\frac{i\alpha}{h_1^2h_2h_3} - \frac{1}{h_1}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial x}\right)\right]\right\}\\
&- \frac{1}{Reh_1^2h_2h_3}\frac{\lambda}{\mu}\frac{\partial\mu}{\partial T}\left(2u\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x} + uh_2\frac{\partial^2 h_3}{\partial x^2} + uh_3\frac{\partial^2 h_2}{\partial x^2}\right.\\
&\left.+w\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial z} + wh_1\frac{\partial^2 h_2}{\partial x\partial z} + w\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} + wh_2\frac{\partial^2 h_1}{\partial x\partial z}\right)\\
&- \frac{2w}{Reh_1^2h_2h_3}\frac{\partial\mu}{\partial T}\left(i\alpha h_2\frac{\partial h_1}{\partial z} + \frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial x} - \frac{h_2}{h_1}\frac{\partial h_1}{\partial z}\frac{\partial h_1}{\partial x} + h_2\frac{\partial^2 h_1}{\partial x\partial z}\right)\\
&- \frac{1}{Reh_1h_2h_3}\frac{\partial\mu}{\partial T}\left(\frac{1}{h_2}\frac{\partial u}{\partial y} - \frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y}\right)\\
&- \frac{1}{Reh_2}\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial y}\left(\frac{1}{h_2}\frac{\partial u}{\partial y} - \frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\\
&- \frac{1}{Reh_2}\frac{\partial\mu}{\partial T}\left(\frac{1}{h_2}\frac{\partial^2 u}{\partial y^2} - \frac{1}{h_2^2}\frac{\partial u}{\partial y}\frac{\partial h_2}{\partial y} - \frac{1}{h_1h_2}\frac{\partial u}{\partial y}\frac{\partial h_1}{\partial y}\right.\\
&\left.-\frac{u}{h_1h_2}\frac{\partial^2 h_1}{\partial y^2} + \frac{u}{h_1^2h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial y} + \frac{u}{h_1h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial y}\right)\\
&+ \frac{1}{Reh_1h_2h_3}\frac{\partial\mu}{\partial T}\left(\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z} + \frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\left(i\beta h_1h_2 + h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right)\\
&+ \frac{1}{Reh_3}\frac{\partial\mu}{\partial T}\left(\frac{u}{h_1h_3}\frac{\partial^2 h_1}{\partial z^2} - \frac{u}{h_1^2h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_1}{\partial z} - \frac{u}{h_1h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_1}{\partial z} + \frac{w}{h_1h_3}\frac{\partial^2 h_3}{\partial x\partial z}\right.\\
&\left.-\frac{w}{h_1^2h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial x} - \frac{w}{h_1h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial x}\right) - \frac{1}{Reh_1h_2^2}\frac{\partial\mu}{\partial T}\frac{\partial h_1}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{u}{h_1}\frac{\partial h_1}{\partial y}\right)\\
&+ \frac{1}{Reh_1^2h_3^2}\frac{\partial\mu}{\partial T}\frac{\partial h_1}{\partial z}\left(w\frac{\partial h_3}{\partial x} + u\frac{\partial h_1}{\partial z}\right) + \frac{2}{Reh_1h_2^2}\frac{\partial\mu}{\partial T}\frac{\partial h_2}{\partial x}\left(\frac{w}{h_3}\frac{\partial h_2}{\partial z} + \frac{u}{h_1}\frac{\partial h_2}{\partial x}\right)\\
&+ \frac{2u}{Reh_1^2h_3^2}\frac{\partial\mu}{\partial T}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial x}\\
C_{1,5} =& \frac{w}{h_1h_3}\left(u\frac{\partial h_1}{\partial z} - w\frac{\partial h_3}{\partial x}\right) + \frac{i\alpha T}{h_1\gamma M^2}
\end{aligned}
$$

$$
\begin{aligned}
C_{2,1} =& -\frac{2\rho u}{h_1h_2}\frac{\partial h_1}{\partial y} + \left\{\left[\frac{\lambda}{Reh_2}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right)\right.\right.\\
&\left.\left.-\frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial y}\right]\left(i\alpha h_2h_3 + h_2\frac{\partial h_3}{\partial x} + h_3\frac{\partial h_2}{\partial x}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\lambda}{Reh_1h_2^2h_3}\left(i\alpha h_2\frac{\partial h_3}{\partial y} + i\alpha h_3\frac{\partial h_2}{\partial y} + \frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial x} + h_2\frac{\partial^2 h_3}{\partial x\partial y} + \frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x} + h_3\frac{\partial^2 h_2}{\partial x\partial y}\right)\\
&+ \frac{\mu}{Reh_1^2h_2h_3}\left(i\alpha h_3\frac{\partial h_1}{\partial y} + \frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y} + h_3\frac{\partial^2 h_1}{\partial x\partial y} - \frac{h_3}{h_1}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial y}\right)\\
&- \frac{2}{Reh_1h_2^2}\frac{\partial h_2}{\partial x}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} - \frac{2\mu}{Reh_1h_2^2h_3}\left(\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x} + h_3\frac{\partial^2 h_2}{\partial x\partial y} - \frac{h_3}{h_2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial x}\right)\\
&+ \frac{\mu}{Reh_1^2h_2^2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial x} + \frac{2\mu}{Reh_1h_2h_3^2}\frac{\partial h_3}{\partial y}\frac{\partial h_3}{\partial x} + \frac{i2\alpha\mu}{Reh_1^2h_2}\frac{\partial h_1}{\partial y}\\
C_{2,2} =&\rho\left[-i\omega + \frac{u}{h_1}i\alpha + \frac{w}{h_3}i\beta + \frac{\partial h_2}{\partial x}\left(\frac{w}{h_2h_3} + \frac{u}{h_1h_2}\right)\right]\\
&+ \left\{\left[\frac{\lambda}{Reh_2}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right) - \frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\right]\right.\\
&\left.\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y}\right)\right\} - \frac{\lambda}{Reh_1h_2^2h_3}\left(2\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial y} + h_1\frac{\partial^2 h_3}{\partial y^2} + h_3\frac{\partial^2 h_1}{\partial y^2}\right)\\
&- \frac{\mu}{Reh_1^2h_2h_3}\left(i\alpha h_2\frac{\partial h_3}{\partial x} - \alpha^2h_2h_3 - \frac{i\alpha h_2h_3}{h_1}\frac{\partial h_1}{\partial x} - \frac{\partial h_3}{\partial x}\frac{\partial h_2}{\partial x}\right.\\
&\left.-h_3\frac{\partial^2 h_2}{\partial x^2} + \frac{h_3}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial x}\right)\\
&- \frac{\mu}{Reh_1h_2h_3^2}\left(i\beta h_2\frac{\partial h_1}{\partial z} - \beta^2h_1h_2 - \frac{i\beta h_1h_2}{h_3}\frac{\partial h_3}{\partial z} - \frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z}\right.\\
&\left.-h_1\frac{\partial^2 h_2}{\partial z^2} + \frac{h_1}{h_3}\frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial z}\right)\\
&- \frac{\mu}{Reh_2h_3}\frac{\partial h_2}{\partial z}\left(\frac{i\beta}{h_3} - \frac{1}{h_2h_3}\frac{\partial h_2}{\partial z}\right) - \frac{\mu}{Reh_1h_2}\frac{\partial h_2}{\partial x}\left(\frac{i\alpha}{h_1} - \frac{1}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&+ \frac{2\mu}{Reh_2^2h_3^2}\frac{\partial h_3}{\partial y}\frac{\partial h_3}{\partial y} + \frac{2\mu}{Reh_1^2h_2^2}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial y}\\
C_{2,3} =&- 2\frac{\rho w}{h_2h_3}\frac{\partial h_3}{\partial y} + \left\{\left[\frac{\lambda}{Reh_2}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right)\right.\right.\\
&\left.\left.-\frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\right]\left(i\beta h_1h_2 + h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right)\right\}\\
&- \frac{\lambda}{Reh_1h_2^2h_3}\left(i\beta h_1\frac{\partial h_2}{\partial y} + i\beta h_2\frac{\partial h_1}{\partial y} + \frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}\right.\\
&\left.+h_1\frac{\partial^2 h_2}{\partial y\partial z} + \frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial z} + h_2\frac{\partial^2 h_1}{\partial y\partial z}\right) - \frac{2}{Reh_2^2h_3}\frac{\partial h_2}{\partial z}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\\
&- \frac{2\mu}{Reh_1h_2^2h_3}\left(\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z} + h_1\frac{\partial^2 h_2}{\partial y\partial z} - \frac{h_1}{h_2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z}\right)\\
&+ \frac{\mu}{Reh_1h_2h_3^2}\left(i\beta h_1\frac{\partial h_3}{\partial y} + \frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y} + h_1\frac{\partial^2 h_3}{\partial y\partial z} - \frac{h_1}{h_3}\frac{\partial h_3}{\partial y}\frac{\partial h_3}{\partial z}\right)\\
&+ \frac{\mu}{Reh_2^2h_3^2}\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial y} + \frac{i2\beta\mu}{Reh_2h_3^2}\frac{\partial h_3}{\partial y} + \frac{2\mu}{Reh_1^2h_2h_3}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
C_{2,4} =& \frac{1}{h_2\gamma M^2}\frac{\partial \rho}{\partial y} + \left\{ \left[\frac{1}{Reh_2}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y} \right) \right.\right. \\
& \left.\left. -\frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial^2 \mu}{\partial T^2}\frac{\partial T}{\partial y} \right] \left(uh_2\frac{\partial h_3}{\partial x} + uh_3\frac{\partial h_2}{\partial x} + wh_1\frac{\partial h_2}{\partial z} + wh_2\frac{\partial h_1}{\partial z} \right) \right\} \\
& - \frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(h_2\frac{\partial h_3}{\partial x}\frac{\partial u}{\partial y} + u\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial x} + uh_2\frac{\partial^2 h_3}{\partial x\partial y} + h_3\frac{\partial h_2}{\partial x}\frac{\partial u}{\partial y} \right. \\
& +u\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x} + uh_3\frac{\partial^2 h_2}{\partial x\partial y} + h_1\frac{\partial h_2}{\partial z}\frac{\partial w}{\partial y} + w\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z} + wh_1\frac{\partial^2 h_2}{\partial y\partial z} \\
& \left. +h_2\frac{\partial h_1}{\partial z}\frac{\partial w}{\partial y} + w\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial z} + wh_2\frac{\partial^2 h_1}{\partial y\partial z} \right) - \frac{i\alpha}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(h_3\frac{\partial u}{\partial y} - \frac{uh_3}{h_1}\frac{\partial h_1}{\partial y} \right) \\
& - \frac{1}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{\partial u}{\partial y}\frac{\partial h_3}{\partial x} - \frac{u}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y} - \frac{uh_3}{h_1}\frac{\partial^2 h_1}{\partial x\partial y} + \frac{uh_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial y} \right) \\
& - \frac{2}{Reh_1h_2h_3}\frac{\partial^2 \mu}{\partial T^2}\frac{\partial T}{\partial y}\left(\frac{wh_1}{h_2}\frac{\partial h_2}{\partial z} + \frac{uh_3}{h_2}\frac{\partial h_2}{\partial x} \right) \\
& - \frac{2}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{h_1}{h_2}\frac{\partial w}{\partial y}\frac{\partial h_2}{\partial z} + \frac{w}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z} + \frac{wh_1}{h_2}\frac{\partial^2 h_2}{\partial y\partial z} - \frac{wh_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z} \right. \\
& \left. +\frac{h_3}{h_2}\frac{\partial u}{\partial y}\frac{\partial h_2}{\partial x} + \frac{u}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x} + \frac{uh_3}{h_2}\frac{\partial^2 h_2}{\partial x\partial y} - \frac{uh_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial x} \right) \\
& - \frac{i\beta}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(h_1\frac{\partial w}{\partial y} - \frac{wh_1}{h_3}\frac{\partial h_3}{\partial y} \right) \\
& - \frac{1}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{\partial h_1}{\partial z}\frac{\partial w}{\partial y} - \frac{w}{h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y} - \frac{wh_1}{h_3}\frac{\partial^2 h_3}{\partial y\partial z} + \frac{wh_1}{h_3^2}\frac{\partial h_3}{\partial y}\frac{\partial h_3}{\partial z} \right) \\
& - \frac{1}{Reh_2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_2}{\partial z}\left(\frac{1}{h_2}\frac{\partial w}{\partial y} - \frac{w}{h_2h_3}\frac{\partial h_3}{\partial y} \right) - \frac{1}{Reh_1h_2}\frac{\partial \mu}{\partial T}\frac{\partial h_2}{\partial x}\left(\frac{1}{h_2}\frac{\partial u}{\partial y} - \frac{u}{h_1h_2}\frac{\partial h_1}{\partial y} \right) \\
& + \frac{2u}{h_1h_2h_3^2}\frac{\partial \mu}{\partial T}\frac{\partial h_3}{\partial y}\frac{\partial h_3}{\partial x} + \frac{2w}{h_1^2h_2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial z} \\
C_{2,5} =& -\left(\frac{w^2}{h_2h_3}\frac{\partial h_3}{\partial y} + \frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial y} \right) + \frac{1}{h_2\gamma M^2}\frac{\partial T}{\partial y}
\end{aligned}
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$$
\begin{aligned}
C_{3,1} =& \frac{\rho}{h_1 h_3}\left(w\frac{\partial h_3}{\partial x} - 2u\frac{\partial h_1}{\partial z}\right) - \frac{\lambda}{Reh_1h_2h_3^2}\left(-\alpha\beta h_2h_3 + i\alpha h_2\frac{\partial h_3}{\partial z} + i\alpha h_3\frac{\partial h_2}{\partial z}\right.\\
&\left. + i\beta h_2\frac{\partial h_3}{\partial x} + \frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x} + h_2\frac{\partial^2 h_3}{\partial x\partial z} + i\beta h_3\frac{\partial h_2}{\partial x} + \frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial x} + h_3\frac{\partial^2 h_2}{\partial x\partial z}\right)\\
&+ \frac{\lambda}{Reh_3}\left(i\alpha h_2h_3 + h_2\frac{\partial h_3}{\partial x} + h_3\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right)\\
&- \frac{\mu}{Reh_1h_2h_3}\left(-\alpha\beta h_2 + i\beta\frac{\partial h_2}{\partial x} - \frac{i\alpha h_2}{h_1}\frac{\partial h_1}{\partial z} - \frac{1}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} - \frac{h_2}{h_1}\frac{\partial^2 h_1}{\partial x\partial z} + \frac{h_2}{h_1^2}\frac{\partial h_1}{\partial z}\frac{\partial h_1}{\partial x}\right)\\
&- \frac{2\mu}{Reh_1h_2h_3}\left(\frac{i\beta h_2}{h_3}\frac{\partial h_3}{\partial x} + \frac{1}{h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x} + \frac{h_2}{h_3}\frac{\partial^2 h_3}{\partial x\partial z} - \frac{h_2}{h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial z}\right)\\
&- \frac{\mu}{Reh_1h_3^2}\frac{\partial h_3}{\partial x}\left(i\beta - \frac{1}{h_1}\frac{\partial h_1}{\partial z}\right) + \frac{i2\mu\alpha}{Reh_1^2h_3}\frac{\partial h_1}{\partial z} + \frac{2\mu}{Reh_1h_2^2h_3}\frac{\partial h_2}{\partial x}\frac{\partial h_2}{\partial z}\\
C_{3,2} =& \frac{\rho}{h_2}\left(\frac{\partial w}{\partial y} + \frac{w}{h_3}\frac{\partial h_3}{\partial y}\right) - \frac{\lambda}{Reh_1h_2h_3^2}\left(i\beta h_1\frac{\partial h_3}{\partial y} + \frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y} + h_1\frac{\partial^2 h_3}{\partial y\partial z}\right.\\
&\left. + i\beta h_3\frac{\partial h_1}{\partial y} + \frac{\partial h_3}{\partial z}\frac{\partial h_1}{\partial y} + h_3\frac{\partial^2 h_1}{\partial y\partial z}\right)\\
&+ \frac{\lambda}{Reh_3}\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y}\right)\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right)\\
&- \frac{1}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\left(i\beta h_1 - \frac{h_1}{h_2}\frac{\partial h_2}{\partial z}\right)\\
&- \frac{\mu}{Reh_1h_2h_3}\left(i\beta\frac{\partial h_1}{\partial y} - \frac{1}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z} - \frac{h_1}{h_2}\frac{\partial^2 h_2}{\partial y\partial z} + \frac{h_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z}\right)\\
&- \frac{2\mu}{Reh_1h_2h_3}\left(\frac{i\beta h_1}{h_3}\frac{\partial h_3}{\partial y} + \frac{1}{h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y} + \frac{h_1}{h_3}\frac{\partial^2 h_3}{\partial y\partial z} - \frac{h_1}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial y}\right)\\
&- \frac{\mu}{Reh_2h_3^2}\frac{\partial h_3}{\partial y}\left(i\beta - \frac{1}{h_2}\frac{\partial h_2}{\partial z}\right) + \frac{2\mu}{Reh_1^2h_2h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_1}{\partial y}\\
C_{3,3} =& \rho\left(-i\omega + \frac{iu\alpha}{h_1} + \frac{iw\beta}{h_3} + \frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\\
&- \frac{\lambda}{Reh_1h_2h_3^2}\left(-\beta^2h_1h_2 + i2\beta h_1\frac{\partial h_2}{\partial z} + i2\beta h_2\frac{\partial h_1}{\partial z} + 2\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z} + h_1\frac{\partial^2 h_2}{\partial z^2} + h_2\frac{\partial^2 h_1}{\partial z^2}\right)\\
&+ \frac{\lambda}{Reh_3}\left(i\beta h_1h_2 + h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right)\\
&+ \frac{\mu}{Reh_1h_2h_3}\left(\frac{\alpha^2h_2h_3}{h_1} - \frac{i\alpha h_3}{h_1}\frac{\partial h_2}{\partial x} + \frac{i\alpha h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\right.\\
&\left. + \frac{1}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x} + \frac{h_2}{h_1}\frac{\partial^2 h_3}{\partial x^2} - \frac{h_2}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_3}{\partial x}\right)\\
&+ \frac{1}{Reh_2^2h_3}\frac{\partial h_3}{\partial y}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} + \frac{\mu}{Reh_1h_2h_3}\left(\frac{1}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial y} + \frac{h_1}{h_2}\frac{\partial^2 h_3}{\partial y^2} - \frac{h_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial y}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2\mu}{Reh_1h_2h_3}\left(-\frac{\beta^2 h_1h_2}{h_3} + \frac{i\beta h_1}{h_3}\frac{\partial h_2}{\partial z} + \frac{i\beta h_2}{h_3}\frac{\partial h_1}{\partial z} - \frac{i\beta h_1 h_2}{h_3^2}\frac{\partial h_3}{\partial z}\right)\\
&- \frac{\mu}{Reh_1^2h_3}\frac{\partial h_3}{\partial x}\left(i\alpha - \frac{1}{h_3}\frac{\partial h_3}{\partial x}\right) + \frac{\mu}{Reh_2^2h_3^2}\left[\left(\frac{\partial h_3}{\partial y}\right)^2 + 2\left(\frac{\partial h_2}{\partial z}\right)^2\right]\\
&+ \frac{2\mu}{Reh_1^2h_3^2}\left(\frac{\partial h_1}{\partial z}\right)^2\\
C_{3,4} =& \frac{i\beta\rho}{h_3\gamma M^2} - \left[\frac{1}{Reh_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(\frac{i\beta}{h_1h_2h_3} - \frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} - \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} - \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right)\right.\\
&\left.\left(uh_2\frac{\partial h_3}{\partial x} + uh_3\frac{\partial h_2}{\partial x} + wh_1\frac{\partial h_2}{\partial z} + wh_2\frac{\partial h_1}{\partial z}\right)\right]\\
&- \frac{1}{Reh_1h_2h_3^2}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(u\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x} + uh_2\frac{\partial^2 h_3}{\partial x\partial z} + u\frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial x}\right.\\
&\left. + uh_3\frac{\partial^2 h_2}{\partial x\partial z} + 2w\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z} + wh_1\frac{\partial^2 h_2}{\partial z^2} + wh_2\frac{\partial^2 h_1}{\partial z^2}\right)\\
&+ \frac{1}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{i\alpha h_2}{h_1} + \frac{1}{h_1}\frac{\partial h_2}{\partial x} - \frac{h_2}{h_1^2}\frac{\partial h_1}{\partial x}\right)\left(u\frac{\partial h_1}{\partial z} + w\frac{\partial h_3}{\partial x}\right)\\
&+ \frac{1}{Reh_1^2h_3}\frac{\partial \mu}{\partial T}\left(u\frac{\partial^2 h_1}{\partial x\partial z} + w\frac{\partial^2 h_3}{\partial x^2}\right) - \frac{1}{Reh_2^2h_3}\frac{\partial \mu}{\partial T}\left(h_3\frac{\partial^2 w}{\partial y^2} - w\frac{\partial^2 h_3}{\partial y^2}\right)\\
&- \frac{1}{Reh_1h_2h_3}\left(\frac{\partial^2 \mu}{\partial T^2}\frac{h_1}{h_2}\frac{\partial T}{\partial y} + \frac{1}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial \mu}{\partial T} - \frac{h_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial \mu}{\partial T}\right)\left(h_3\frac{\partial w}{\partial y} - w\frac{\partial h_3}{\partial y}\right)\\
&- \frac{2u}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{i\beta h_2}{h_3}\frac{\partial h_3}{\partial x} + \frac{1}{h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x} + \frac{h_2}{h_3}\frac{\partial^2 h_3}{\partial x\partial z} - \frac{h_2}{h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial z}\right)\\
&+ \frac{1}{Reh_1^2h_3^2}\frac{\partial \mu}{\partial T}\frac{\partial h_3}{\partial x}\left(u\frac{\partial h_1}{\partial z} + w\frac{\partial h_3}{\partial x}\right) - \frac{1}{Reh_2^2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_3}{\partial y}\left(\frac{\partial w}{\partial y} - \frac{w}{h_3}\frac{\partial h_3}{\partial y}\right)\\
&+ \frac{2}{Re}\frac{\partial \mu}{\partial T}\left[\frac{w}{h_1^2h_3^2}\left(\frac{\partial h_1}{\partial z}\right)^2 + \frac{w}{h_2^2h_3^2}\left(\frac{\partial h_2}{\partial z}\right)^2 + \frac{u}{h_1h_2^2h_3}\frac{\partial h_2}{\partial x}\frac{\partial h_2}{\partial z}\right]\\
C_{3,5} =& - \frac{u}{h_1h_3}\left(u\frac{\partial h_1}{\partial z} - w\frac{\partial h_3}{\partial x}\right) + \frac{i\beta T}{h_3\gamma M^2}
\end{aligned}
$$

$$
\begin{aligned}
C_{4,1} =& -\frac{(\gamma-1)\,M^2}{Re}\left\{\frac{i2\alpha w}{h_1^2 h_3}\frac{\partial h_1}{\partial z}(2\mu+\lambda)+\frac{2}{h_1 h_2}\frac{\partial h_2}{\partial x}(2\mu+\lambda)\left(\frac{w}{h_2 h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)\right.\\
&-2\mu\left(\frac{w}{h_1 h_3}\frac{\partial h_3}{\partial x}+\frac{u}{h_1 h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{i\beta}{h_3}-\frac{1}{h_1 h_3}\frac{\partial h_1}{\partial z}\right)+(2\mu+\lambda)\frac{2u}{h_1^2 h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial x}\\
&-\frac{2\mu}{h_1 h_2}\frac{\partial h_1}{\partial y}\left(\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1 h_2}\frac{\partial h_1}{\partial y}\right)+\frac{i2\lambda\alpha}{h_1}\left(\frac{w}{h_2 h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)\\
&+\frac{2\lambda w}{h_1^2 h_2 h_3}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z}+\frac{i2\lambda\alpha u}{h_1^2 h_3}\frac{\partial h_3}{\partial x}+\frac{2\lambda w}{h_1^2 h_3^2}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial x}\\
&\left.+\frac{2\lambda u}{h_1^2 h_2 h_3}\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x}+\frac{2\lambda}{h_1 h_3}\frac{\partial h_3}{\partial x}\left(\frac{w}{h_2 h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)\right\}\\
C_{4,2} =& \frac{\rho}{h_2}\frac{\partial T}{\partial y}-\frac{(\gamma-1)}{\gamma h_2}\left(T\frac{\partial \rho}{\partial y}+\rho\frac{\partial T}{\partial y}\right)-\frac{(\gamma-1)\,M^2}{Re}\left\{(2\mu+\lambda)\frac{2w}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial z}\right.\\
&+2\mu\left(\frac{1}{h_2}\frac{\partial w}{\partial y}-\frac{w}{h_2 h_3}\frac{\partial h_3}{\partial y}\right)\left(\frac{i\beta}{h_3}-\frac{1}{h_2 h_3}\frac{\partial h_2}{\partial z}\right)+(2\mu+\lambda)\frac{2u}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial y}\\
&+2\mu\left(\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1 h_2}\frac{\partial h_1}{\partial y}\right)\left(\frac{i\alpha}{h_1}-\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)+\frac{2\lambda}{h_1 h_2}\frac{\partial h_1}{\partial y}\left(\frac{w}{h_2 h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)\\
&\left.\frac{2\lambda u}{h_1^2 h_2 h_3}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y}+\frac{2\lambda w}{h_1 h_2 h_3^2}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y}+\frac{2\lambda}{h_2 h_3}\frac{\partial h_3}{\partial y}\left(\frac{w}{h_2 h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)\right\}\\
C_{4,3} =& -\frac{(\gamma-1)\,M^2}{Re}\left\{(2\mu+\lambda)\frac{2w}{h_1^2 h_3^2}\frac{\partial h_1}{\partial z}\frac{\partial h_1}{\partial z}-\frac{2\mu}{h_2 h_3}\frac{\partial h_3}{\partial y}\left(\frac{1}{h_2}\frac{\partial w}{\partial y}-\frac{w}{h_2 h_3}\frac{\partial h_3}{\partial y}\right)\right.\\
&+(2\mu+\lambda)\frac{2}{h_2 h_3}\frac{\partial h_2}{\partial z}\left(\frac{w}{h_2 h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)\\
&-2\mu\left(\frac{w}{h_1 h_3}\frac{\partial h_3}{\partial x}+\frac{u}{h_1 h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{i\alpha}{h_1}-\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x}\right)\\
&+(2\mu+\lambda)\frac{i2\beta u}{h_1 h_3^2}\frac{\partial h_3}{\partial x}+\frac{2\lambda}{h_1 h_3}\frac{\partial h_1}{\partial z}\left(\frac{w}{h_2 h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)+\frac{2\lambda w}{h_1 h_2 h_3^2}\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z}\\
&\left.+\frac{2\lambda u}{h_1^2 h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial z}+\frac{i2\lambda\beta w}{h_1 h_3^2}\frac{\partial h_1}{\partial z}+\frac{2\lambda u}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_2}{\partial z}+\frac{i2\lambda\beta}{h_3}\left(\frac{w}{h_2 h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1 h_2}\frac{\partial h_2}{\partial x}\right)\right\}\\
C_{4,4} =& \rho\left(-i\omega+\frac{iu\alpha}{h_1}+\frac{iw\beta}{h_3}\right)-\frac{(\gamma-1)\,\rho}{\gamma}\left(-i\omega+\frac{iu\alpha}{h_1}+\frac{iw\beta}{h_3}\right)\\
&-\frac{1}{PrReh_1 h_2 h_3}\left[\kappa\left(\frac{i\alpha h_2}{h_1}\frac{\partial h_3}{\partial x}+\frac{i\alpha h_3}{h_1}\frac{\partial h_2}{\partial x}-\frac{\alpha^2 h_2 h_3}{h_1}-\frac{i\alpha h_2 h_3}{h_1^2}\frac{\partial h_1}{\partial x}\right.\right.\\
&\left.+\frac{i\beta h_2}{h_3}\frac{\partial h_1}{\partial z}+\frac{i\beta h_1}{h_3}\frac{\partial h_2}{\partial z}-\frac{\beta^2 h_1 h_2}{h_3}-\frac{i\beta h_1 h_2}{h_3^2}\frac{\partial h_3}{\partial z}\right)\\
&\left.+\frac{\partial \kappa}{\partial T}\left(\frac{h_1}{h_2}\frac{\partial T}{\partial y}\frac{\partial h_3}{\partial y}+\frac{h_3}{h_2}\frac{\partial T}{\partial y}\frac{\partial h_1}{\partial y}-\frac{h_1 h_3\frac{\partial T}{\partial y}}{h_2^2}\frac{\partial h_2}{\partial y}+\frac{h_1 h_3\frac{\partial^2 T}{\partial y^2}}{h_2}\right)+\frac{\partial^2\kappa}{\partial T^2}\left(\frac{\partial T}{\partial y}\right)^2\frac{h_1 h_3}{h_2}\right]\\
&-\frac{(\gamma-1)\,M^2}{Re}\frac{\partial \mu}{\partial T}\left\{\left(2+\frac{\lambda}{\mu}\right)\left(\frac{w}{h_1 h_3}\frac{\partial h_1}{\partial z}\right)^2+\left(\frac{1}{h_2}\frac{\partial w}{\partial y}-\frac{w}{h_2 h_3}\frac{\partial h_3}{\partial y}\right)^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(2+\frac{\lambda}{\mu}\right)\left(\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)^2+\left(\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z}\right)^2\\
&+\left(2+\frac{\lambda}{\mu}\right)\left(\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}\right)^2+\left(\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)^2\\
&+2\frac{\lambda}{\mu}\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\left(\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)+2\frac{\lambda}{\mu}\frac{uw}{h_1^2h_3^2}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial x}\\
&+2\frac{\lambda}{\mu}\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}\left(\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\Bigg\}\\
C_{4,5}=&-\frac{(\gamma-1)\,T}{\gamma}\left(-i\omega+\frac{i\alpha u}{h_1}+\frac{i\beta w}{h_3}\right)
\end{aligned}
$$

$$
\begin{aligned}
C_{5,1}=&\frac{\rho}{h_1}\left(i\alpha+\frac{1}{h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_3}\frac{\partial h_3}{\partial x}\right)\\
C_{5,2}=&\frac{1}{h_2}\left[\frac{\partial \rho}{\partial y}+\rho\left(\frac{1}{h_3}\frac{\partial h_3}{\partial y}+\frac{1}{h_1}\frac{\partial h_1}{\partial y}\right)\right]\\
C_{5,3}=&\frac{\rho}{h_3}\left(i\beta+\frac{1}{h_2}\frac{\partial h_2}{\partial z}+\frac{1}{h_1}\frac{\partial h_1}{\partial z}\right)\\
C_{5,4}=&0\\
C_{5,5}=&-i\omega+\frac{u}{h_1}\left(i\alpha+\frac{1}{h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_3}\frac{\partial h_3}{\partial x}\right)+\frac{w}{h_3}\left(i\beta+\frac{1}{h_2}\frac{\partial h_2}{\partial z}+\frac{1}{h_1}\frac{\partial h_1}{\partial z}\right)
\end{aligned}
$$

APPENDIX C

LPSE FORMULATION

The full LPSE equations and formulations will be displayed here. Beginning from equations A.3-A.8, the wave format C.1 is imposed while following the details in chapter II to eliminate ellipitcal terms. Nonlinear terms are set to zero.

$$\phi'\left(x,y,z,t\right) \equiv \hat{\phi}\left(x,y\right) e^{i\left(\int_{x_0}^{x} \alpha(x)\partial x+\beta z-\omega t\right)} \tag{C.1}$$

Pressure is eliminated through use of the equation of state (equation A.8), thus reducing the set of equations to five. The result is arranged into the format

$$\mathcal{A}\frac{\partial^2\hat{\phi}}{\partial y^2} + \mathcal{B}\frac{\partial^2\hat{\phi}}{\partial x\partial y} + \mathcal{C}\frac{\partial\hat{\phi}}{\partial y} + \mathcal{D}\frac{\partial\hat{\phi}}{\partial x} + \mathcal{E}\hat{\phi} = 0 \tag{C.2}$$

where $\hat{\phi} = \left[\hat{u}, \hat{v}, \hat{w}, \hat{T}, \hat{\rho}\right]$.

Matrices **A** through **E** are 5x5 matrices at each unique normal and streamwise location. These are expanded below. All terms should be basic-state quantities, as the disturbance quantities and derivatives are accounted for in equation C.2. X-momentum, Y-momentum, Z-momentum, energy, and mass continuity are represented by rows 1-5 respectively. Similarly, columns 1-5 are the coefficients of $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{T}$, $\hat{\rho}$, and their x- and y-derivatives.

A-Matrix

$$A = \begin{pmatrix} -\frac{\mu}{Reh_2^2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{Reh_2^2}\left(\lambda + 2\mu\right) & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu}{Reh_2^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\kappa}{PrReh_2^2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

B-Matrix

$$B = \begin{pmatrix} 0 & -\frac{1}{Reh_1h_2}(\lambda+\mu) & 0 & 0 & 0 \\ -\frac{1}{Reh_1h_2}(\lambda+\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

C-Matrix

$$C_{1,1} = -\frac{1}{Reh_2^2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} - \frac{\mu}{Reh_2^2h_3}\left(\frac{\partial h_3}{\partial y} - \frac{h_3}{h_2}\frac{\partial h_2}{\partial y} + \frac{h_3}{h_1}\frac{\partial h_1}{\partial y}\right) + \frac{\rho v}{h_2}$$

$$\begin{aligned} C_{1,2} =& \frac{\lambda h_3}{Re}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial x}\right) - \frac{1}{Reh_1h_2}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x} \\ &- \frac{\lambda}{Reh_1^2h_2h_3}\left(h_1\frac{\partial h_3}{\partial x} + h_3\frac{\partial h_1}{\partial x} + i\alpha h_1h_3\right) + \frac{3\mu}{Reh_1h_2^2}\frac{\partial h_2}{\partial x} - \frac{i\alpha\mu}{Reh_1h_2} \end{aligned}$$

$$C_{1,3} = 0$$

$$C_{1,4} = -\frac{1}{Reh_1h_2}\frac{\partial \mu}{\partial T}\left(\frac{\partial v}{\partial x} - \frac{v}{h_2}\frac{\partial h_2}{\partial x} + \frac{h_1}{h_2}\frac{\partial u}{\partial y} - \frac{u}{h_2}\frac{\partial h_1}{\partial y}\right)$$

$$C_{1,5} = 0$$

$$
\begin{aligned}
C_{2,1} =& -\frac{\lambda}{Reh_1h_2^2h_3}\left(i\alpha h_2h_3 + h_2\frac{\partial h_3}{\partial x} + h_3\frac{\partial h_2}{\partial x}\right) - \frac{1}{Reh_1h_2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x} \\
& - \frac{\mu}{Reh_1h_2h_3}\left(\frac{\partial h_3}{\partial x} + h_3 i\alpha\right) - \frac{3\mu}{Reh_1h_2^2}\frac{\partial h_2}{\partial x} \\
C_{2,2} =& \frac{\rho v}{h_2} - \frac{h_1h_3}{Re}\left[\frac{1}{h_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} - \frac{\lambda}{h_2}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right)\right] \\
& - \frac{2\lambda}{Reh_1h_2^2h_3}\left(h_3\frac{\partial h_1}{\partial y} + h_1\frac{\partial h_3}{\partial y}\right) - \frac{2}{Reh_2^2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} \\
& - \frac{2\mu}{Reh_1h_2^2h_3}\left(h_3\frac{\partial h_1}{\partial y} + h_1\frac{\partial h_3}{\partial y} - \frac{h_1h_3}{h_2}\frac{\partial h_2}{\partial y}\right) \\
C_{2,3} =& -\frac{\lambda}{Reh_1h_2^2h_3}\left(i\beta h_1h_2 + h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right) \\
& - \frac{\mu}{Reh_1h_2h_3}\left(\frac{\partial h_1}{\partial z} + h_1 i\beta\right) - \frac{3\mu}{Reh_2^2h_3}\frac{\partial h_2}{\partial z} \\
C_{2,4} =& \frac{\rho}{h_2\gamma M^2} - \frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(h_2h_3\frac{\partial u}{\partial x} + uh_2\frac{\partial h_3}{\partial x} + uh_3\frac{\partial h_2}{\partial x} + h_1h_3\frac{\partial v}{\partial y} + vh_3\frac{\partial h_1}{\partial y}\right. \\
& \left. + vh_1\frac{\partial h_3}{\partial y} + wh_1\frac{\partial h_2}{\partial z} + wh_2\frac{\partial h_1}{\partial z}\right) - \frac{2}{Reh_1h_2^2h_3}\frac{\partial \mu}{\partial T}\left(h_1h_3\frac{\partial v}{\partial y} + wh_1\frac{\partial h_2}{\partial z} + uh_3\frac{\partial h_2}{\partial x}\right) \\
C_{2,5} =& \frac{T}{h_2\gamma M^2}
\end{aligned}
$$

$$
\begin{aligned}
C_{3,1} =& 0 \\
C_{3,2} =& \frac{\lambda h_1}{Re}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right) \\
& - \frac{\lambda}{Reh_1h_2h_3^2}\left(i\beta h_1h_3 + h_1\frac{\partial h_3}{\partial z} + h_3\frac{\partial h_1}{\partial z}\right) + \frac{3\mu}{Reh_2^2h_3}\frac{\partial h_2}{\partial z} - \frac{i\beta\mu}{Reh_2h_3} \\
C_{3,3} =& \frac{\rho v}{h_2} - \frac{1}{Reh_2^2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} - \frac{\mu}{Reh_1h_2^2}\left(\frac{\partial h_1}{\partial y} - \frac{h_1}{h_2}\frac{\partial h_2}{\partial y} + \frac{h_1}{h_3}\frac{\partial h_3}{\partial y}\right) \\
C_{3,4} =& -\frac{1}{Reh_2^2h_3}\frac{\partial \mu}{\partial T}\left(h_3\frac{\partial w}{\partial y} - w\frac{\partial h_3}{\partial y} - v\frac{\partial h_2}{\partial y}\right) \\
C_{3,5} =& 0
\end{aligned}
$$

$$
\begin{aligned}
C_{4,1} =& -\frac{2\left(\gamma-1\right)M^2\mu}{Reh_2}\left(\frac{1}{h_1}\frac{\partial v}{\partial x}-\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\\
C_{4,2} =& -\frac{-2\left(\gamma-1\right)M^2}{Reh_2}\left[\left(2\mu+\lambda\right)\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\right.\\
&\left.+\frac{\lambda}{h_3}\left(\frac{u}{h_1}\frac{\partial h_3}{\partial x}+\frac{v}{h_2}\frac{\partial h_3}{\partial y}\right)+\frac{\lambda}{h_1}\left(\frac{\partial u}{\partial x}+\frac{v}{h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_3}\frac{\partial h_1}{\partial z}\right)\right]\\
C_{4,3} =& -\frac{2\left(\gamma-1\right)M^2\mu}{Reh_2}\left(\frac{1}{h_2}\frac{\partial w}{\partial y}-\frac{w}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{v}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\\
C_{4,4} =& \frac{\rho v}{h_2}-\frac{\left(\gamma-1\right)}{\gamma}\frac{v\rho}{h_2}-\frac{1}{PrReh_1h_2^2h_3}\left(2\frac{\partial\kappa}{\partial T}\frac{\partial T}{\partial y}h_1h_3+\kappa h_1\frac{\partial h_3}{\partial y}+\kappa h_3\frac{\partial h_1}{\partial y}-\frac{\kappa h_1h_3}{h_2}\frac{\partial h_2}{\partial y}\right)\\
C_{4,5} =& -\frac{\left(\gamma-1\right)}{\gamma}\frac{vT}{h_2}
\end{aligned}
$$

$$
\begin{aligned}
C_{5,1} =& 0\\
C_{5,2} =& \frac{\rho}{h_2}\\
C_{5,3} =& 0\\
C_{5,4} =& 0\\
C_{5,5} =& \frac{v}{h_2}
\end{aligned}
$$

$$
\begin{aligned}
D_{1,1} =& \frac{\rho u}{h_1} - \frac{2\lambda}{Re h_1^2 h_2 h_3}\left(i\alpha h_2 h_3 + h_2 \frac{\partial h_3}{\partial x} + h_3 \frac{\partial h_2}{\partial x}\right) \\
&+ \frac{\lambda h_2 h_3}{Re h_1}\left(\frac{1}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1 h_2^2 h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\right) \\
&- \frac{2\mu}{Re h_1^2 h_2 h_3}\left(h_2 \frac{\partial h_3}{\partial x} + h_3 \frac{\partial h_2}{\partial x} - \frac{h_2 h_3}{h_1}\frac{\partial h_1}{\partial x} + i2\alpha h_2 h_3\right) \\
D_{1,2} =& - \frac{\lambda}{Re h_1^2 h_2 h_3}\left(h_1 \frac{\partial h_3}{\partial y} + h_3 \frac{\partial h_1}{\partial y}\right) - \frac{3\mu}{Re h_1^2 h_2}\frac{\partial h_1}{\partial y} - \frac{1}{Re h_1 h_2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} - \frac{\mu}{Re h_1 h_2 h_3}\frac{\partial h_3}{\partial y} \\
D_{1,3} =& - \frac{\lambda}{Re h_1^2 h_2 h_3}\left(i\beta h_1 h_2 + h_1 \frac{\partial h_2}{\partial z} + h_2 \frac{\partial h_1}{\partial z}\right) - \frac{3\mu}{Re h_1^2 h_3}\frac{\partial h_1}{\partial z} \\
&- \frac{i\beta\mu}{Re h_1 h_3} - \frac{\mu}{Re h_1 h_2 h_3}\frac{\partial h_2}{\partial z} \\
D_{1,4} =& \Omega \frac{\rho}{h_1 \gamma M^2} - \frac{1}{Re h_1^2 h_2 h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(u h_2 \frac{\partial h_3}{\partial x} + u h_3 \frac{\partial h_2}{\partial x} + v h_1 \frac{\partial h_3}{\partial y} + v h_3 \frac{\partial h_1}{\partial y} + h_1 h_3 \frac{\partial v}{\partial y}\right. \\
&\left. + w h_1 \frac{\partial h_2}{\partial z} + w h_2 \frac{\partial h_1}{\partial z}\right) - \frac{2}{Re h_1 h_2 h_3}\frac{\partial \mu}{\partial T}\left(\frac{v h_3}{h_1}\frac{\partial h_1}{\partial y} + \frac{w h_2}{h_1}\frac{\partial h_1}{\partial z}\right) \\
D_{1,5} =& \Omega \frac{T}{h_1 \gamma M^2}
\end{aligned}
$$

$$
\begin{aligned}
D_{2,1} =& - \frac{h_2 h_3}{Re}\left[\frac{1}{h_1 h_2^2 h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y} - \frac{\lambda}{h_2}\left(\frac{1}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial y} + \frac{1}{h_1 h_2^2 h_3}\frac{\partial h_2}{\partial y} + \frac{1}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial y}\right)\right] \\
&- \frac{\lambda}{Re h_1 h_2^2 h_3}\left(h_3 \frac{\partial h_2}{\partial y} + h_2 \frac{\partial h_3}{\partial y}\right) + \frac{3\mu}{Re h_1^2 h_2}\frac{\partial h_1}{\partial y} \\
D_{2,2} =& \frac{\rho u}{h_1} - \frac{\mu}{Re h_1^2 h_2 h_3}\left(i2\alpha h_2 h_3 + h_2 \frac{\partial h_3}{\partial x} + h_3 \frac{\partial h_2}{\partial x} - \frac{h_2 h_3}{h_1}\frac{\partial h_1}{\partial x}\right) \\
D_{2,3} =& 0 \\
D_{2,4} =& - \frac{1}{Re h_1^2 h_2 h_3}\frac{\partial \mu}{\partial T}\left(-v h_3 \frac{\partial h_2}{\partial x} + \frac{h_3}{h_1}\frac{\partial u}{\partial y} - u h_3 \frac{\partial h_1}{\partial y}\right) \\
D_{2,5} =& 0
\end{aligned}
$$

$$
\begin{aligned}
D_{3,1} =& \frac{\lambda h_2}{Re}\left(\frac{1}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial z}+\frac{1}{h_1 h_2^2 h_3}\frac{\partial h_2}{\partial z}+\frac{1}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial z}\right)\\
&-\frac{\lambda}{Reh_1h_2h_3^2}\left(i\beta h_2 h_3+h_2\frac{\partial h_3}{\partial z}+h_3\frac{\partial h_2}{\partial z}\right)+\frac{3\mu}{Reh_1^2h_3}\frac{\partial h_1}{\partial z}-\frac{i\beta\mu}{Reh_1h_3}\\
D_{3,2} =& 0\\
D_{3,3} =& \frac{\rho u}{h_1}-\frac{\mu}{Reh_1^2h_2h_3}\left(i2\alpha h_2h_3-\frac{h_2h_3}{h_1}\frac{\partial h_1}{\partial x}+h_3\frac{\partial h_2}{\partial x}+h_2\frac{\partial h_3}{\partial x}\right)\\
D_{3,4} =& -\frac{1}{Reh_1^2h_2h_3}\frac{\partial \mu}{\partial T}\left(-uh_2\frac{\partial h_1}{\partial z}-wh_2\frac{\partial h_3}{\partial x}\right)\\
D_{3,5} =& 0
\end{aligned}
$$

$$
\begin{aligned}
D_{4,1} =& -\frac{2\left(\gamma-1\right)M^2}{Reh_1}\left[\left(2\mu+\lambda\right)\left(\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\right)+\frac{\lambda}{h_2}\left(\frac{\partial v}{\partial y}+\frac{w}{h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1}\frac{\partial h_2}{\partial x}\right)\right.\\
&\left.+\frac{\lambda}{h_3}\left(\frac{u}{h_1}\frac{\partial h_3}{\partial x}+\frac{v}{h_2}\frac{\partial h_3}{\partial y}\right)\right]\\
D_{4,2} =& -\frac{2\mu\left(\gamma-1\right)M^2}{Reh_1}\left(-\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\\
D_{4,3} =& -\frac{2\mu\left(\gamma-1\right)M^2}{Reh_1}\left(-\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}-\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\\
D_{4,4} =& \frac{u\rho}{h_1}-\Omega\frac{\left(\gamma-1\right)u\rho}{\gamma h_1}-\frac{1}{PrReh_1^2h_2h_3}\left(\kappa h_2\frac{\partial h_3}{\partial x}+\kappa h_3\frac{\partial h_2}{\partial x}-\frac{\kappa h_2h_3}{h_1}\frac{\partial h_1}{\partial x}+i2\alpha\kappa h_2h_3\right)\\
D_{4,5} =& -\Omega\frac{\left(\gamma-1\right)uT}{\gamma h_1}
\end{aligned}
$$

$$
\begin{aligned}
D_{5,1} =& \frac{\rho}{h_1}\\
D_{5,2} =& 0\\
D_{5,3} =& 0\\
D_{5,4} =& 0\\
D_{5,5} =& \frac{u}{h_1}
\end{aligned}
$$

$$
\begin{aligned}
E_{1,1} =& \rho\left(-i\omega + \frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{iu\alpha}{h_1} + \frac{iw\beta}{h_3} + \frac{v}{h_1 h_2}\frac{\partial h_1}{\partial y} + \frac{w}{h_1 h_3}\frac{\partial h_1}{\partial z}\right) \\
& - \frac{\lambda}{Reh_1^2 h_2 h_3}\left(i2\alpha h_2\frac{\partial h_3}{\partial x} + i2\alpha h_3\frac{\partial h_2}{\partial x} - \alpha^2 h_2 h_3 + ih_2 h_3\frac{\partial \alpha}{\partial x}\right. \\
& \left. +2\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x} + h_2\frac{\partial^2 h_3}{\partial x^2} + h_3\frac{\partial^2 h_2}{\partial x^2}\right) \\
& - \frac{1}{Re}\left\{\left[\frac{1}{h_1^2 h_2 h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x} - \frac{\lambda}{h_1}\left(\frac{1}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1 h_2^2 h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\right)\right]\right. \\
& \left.\left(i\alpha h_2 h_3 + h_2\frac{\partial h_3}{\partial x} + h_3\frac{\partial h_2}{\partial x}\right)\right\} - \frac{i2\alpha}{Reh_1^2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x} \\
& - \frac{2\mu}{Reh_1 h_2 h_3}\left[i\alpha\left(\frac{h_3}{h_1}\frac{\partial h_2}{\partial x} + \frac{h_2}{h_1}\frac{\partial h_3}{\partial x} - \frac{h_2 h_3}{h_1^2}\frac{\partial h_1}{\partial x}\right) + i\frac{h_2 h_3}{h_1}\frac{\partial \alpha}{\partial x} - \alpha^2\frac{h_2 h_3}{h_1}\right] \\
& + \frac{1}{Reh_1 h_2^2}\frac{\partial \mu}{\partial T}\frac{\partial h_1}{\partial y}\frac{\partial T}{\partial y} + \frac{\mu}{Reh_1 h_2 h_3}\left(\frac{h_3}{h_2}\frac{\partial^2 h_1}{\partial y^2} + \frac{1}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_1}{\partial y} - \frac{h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial y}\right) \\
& - \frac{\mu}{Reh_1 h_2 h_3}\left(i\beta\frac{h_1}{h_3}\frac{\partial h_2}{\partial z} - i\beta\frac{h_1 h_2}{h_3^2}\frac{\partial h_3}{\partial z} - \beta^2\frac{h_1 h_2}{h_3}\right. \\
& \left. -\frac{1}{h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_1}{\partial z} - \frac{h_2}{h_3}\frac{\partial^2 h_1}{\partial z^2} + \frac{h_2}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_1}{\partial z}\right) + \frac{\mu}{Reh_1^2 h_2^2}\left(\left(\frac{\partial h_1}{\partial y}\right)^2 + 2\left(\frac{\partial h_2}{\partial x}\right)^2\right) \\
& - \frac{\mu}{Reh_1 h_3^2}\frac{\partial h_1}{\partial z}\left(i\beta - \frac{1}{h_1}\frac{\partial h_1}{\partial z}\right) + \frac{2\mu}{Reh_1^2 h_3^2}\left(\frac{\partial h_3}{\partial x}\right)^2 \\
E_{1,2} =& \rho\left[\frac{1}{h_2}\frac{\partial u}{\partial y} - \frac{1}{h_1 h_2}\left(2v\frac{\partial h_2}{\partial x} - u\frac{\partial h_1}{\partial y}\right)\right] \\
& - \frac{\lambda}{Reh_1^2 h_2 h_3}\left(i\alpha h_1\frac{\partial h_3}{\partial y} + \frac{\partial h_1}{\partial x}\frac{\partial h_3}{\partial y} + h_1\frac{\partial^2 h_3}{\partial x \partial y} + i\alpha h_3\frac{\partial h_1}{\partial y} + \frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y} + h_3\frac{\partial^2 h_1}{\partial x \partial y}\right) \\
& - \frac{1}{Re}\left\{\left[\frac{1}{h_1^2 h_2 h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x} - \frac{\lambda}{h_1}\left(\frac{1}{h_1^2 h_2 h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1 h_2^2 h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\right)\right]\right. \\
& \left.\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y}\right)\right\} - \frac{2}{Reh_1^2 h_2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x}\frac{\partial h_1}{\partial y} \\
& - \frac{2\mu}{Reh_1 h_2 h_3}\left(i\alpha\frac{h_3}{h_1}\frac{\partial h_1}{\partial y} + \frac{1}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y} + \frac{h_3}{h_1}\frac{\partial^2 h_1}{\partial x \partial y} - \frac{h_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial y}\right) \\
& - \frac{1}{Reh_1 h_2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\left(i\alpha - \frac{1}{h_2}\frac{\partial h_2}{\partial x}\right) \\
& - \frac{\mu}{Reh_1 h_2 h_3}\left(i\alpha\frac{\partial h_3}{\partial y} - \frac{1}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x} - \frac{h_3}{h_2}\frac{\partial^2 h_2}{\partial x \partial y} + \frac{h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial x}\right) \\
& - \frac{\mu}{Reh_1^2 h_2}\frac{\partial h_1}{\partial y}\left(i\alpha - \frac{1}{h_2}\frac{\partial h_2}{\partial x}\right) + \frac{2\mu}{Reh_1 h_2 h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
E_{1,3} =& \rho\left(\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z} - \frac{2w}{h_1h_3}\frac{\partial h_3}{\partial x}\right) - \frac{\lambda}{Reh_1^2h_2h_3}\left(i\beta h_1\frac{\partial h_2}{\partial x} + i\beta h_2\frac{\partial h_1}{\partial x} - \alpha\beta h_1h_2 + i\alpha h_1\frac{\partial h_2}{\partial z}\right.\\
&\left. + \frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial z} + h_1\frac{\partial^2 h_2}{\partial x\partial z} + i\alpha h_2\frac{\partial h_1}{\partial z} + \frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} + h_2\frac{\partial^2 h_1}{\partial x\partial z}\right) - \frac{1}{Re}\left\{\left[\frac{1}{h_1^2h_2h_3}\frac{\lambda}{\mu}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial x}\right.\right.\\
&\left.\left. -\frac{\lambda}{h_1}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial x}\right)\right]\left(i\beta h_1h_2 + h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right)\right\}\\
&- \frac{2}{Reh_1^2h_3}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial x}\frac{\partial h_1}{\partial z} - \frac{2\mu}{Reh_1^2h_2h_3}\left(\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} + h_2\frac{\partial^2 h_1}{\partial x\partial z} - \frac{h_2}{h_1}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial z} + i\alpha h_2\frac{\partial h_1}{\partial z}\right)\\
&- \frac{\mu}{Reh_1h_2h_3}\left(i\alpha\frac{\partial h_2}{\partial z} - h_2\alpha\beta - \frac{i\beta h_2}{h_3}\frac{\partial h_3}{\partial x} - \frac{1}{h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x} - \frac{h_2}{h_3}\frac{\partial^2 h_3}{\partial x\partial z} + \frac{h_2}{h_3^2}\frac{\partial h_3}{\partial x}\frac{\partial h_3}{\partial z}\right)\\
&- \frac{\mu}{Reh_1^2h_3}\frac{\partial h_1}{\partial z}\left(i\alpha - \frac{1}{h_3}\frac{\partial h_3}{\partial x}\right) + \frac{2\mu}{Reh_1h_2^2h_3}\frac{\partial h_2}{\partial x}\frac{\partial h_2}{\partial z} + \frac{i2\mu\beta}{Reh_1h_3^2}\frac{\partial h_3}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
E_{1,4} =& \frac{1}{h_1\gamma M^2}\left(i\alpha\rho + \Omega\frac{\partial\rho}{\partial x}\right) - \frac{1}{Re}\left\{\left[\frac{1}{h_1^2h_2h_3}\frac{\lambda}{\mu}\left(\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial x} + i\alpha\frac{\partial\mu}{\partial T}\right)\right.\right.\\
&\left. -\frac{1}{h_1}\frac{\lambda}{\mu}\frac{\partial\mu}{\partial T}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial x} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial x} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial x}\right)\right]\left(uh_2\frac{\partial h_3}{\partial x} + uh_3\frac{\partial h_2}{\partial x}\right.\\
&\left.\left. +h_2h_3\frac{\partial u}{\partial x} + vh_1\frac{\partial h_3}{\partial y} + vh_3\frac{\partial h_1}{\partial y} + h_1h_3\frac{\partial v}{\partial y} + wh_1\frac{\partial h_2}{\partial z} + wh_2\frac{\partial h_1}{\partial z}\right)\right\}\\
&- \frac{1}{Reh_1^2h_2h_3}\frac{\lambda}{\mu}\frac{\partial\mu}{\partial T}\left(2h_2\frac{\partial u}{\partial x}\frac{\partial h_3}{\partial x} + 2u\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x} + uh_2\frac{\partial^2 h_3}{\partial x^2} + 2h_3\frac{\partial u}{\partial x}\frac{\partial h_2}{\partial x}\right.\\
&+uh_3\frac{\partial^2 h_2}{\partial x^2} + h_1\frac{\partial v}{\partial x}\frac{\partial h_3}{\partial y} + v\frac{\partial h_1}{\partial x}\frac{\partial h_3}{\partial y} + vh_1\frac{\partial^2 h_3}{\partial x\partial y} + h_3\frac{\partial v}{\partial x}\frac{\partial h_1}{\partial y} + v\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y}\\
&+vh_3\frac{\partial^2 h_1}{\partial x\partial y} + h_3\frac{\partial h_1}{\partial x}\frac{\partial v}{\partial y} + h_1\frac{\partial h_3}{\partial x}\frac{\partial v}{\partial y} + h_1h_3\frac{\partial^2 v}{\partial x\partial y} + h_1\frac{\partial w}{\partial x}\frac{\partial h_2}{\partial z} + w\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial z}\\
&\left. +wh_1\frac{\partial^2 h_2}{\partial x\partial z} + h_2\frac{\partial w}{\partial x}\frac{\partial h_1}{\partial z} + w\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} + wh_2\frac{\partial^2 h_1}{\partial x\partial z}\right)\\
&- \frac{2}{Reh_1h_2h_3}\left(i\alpha\frac{\partial\mu}{\partial T} + \frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial x}\right)\left(\frac{h_2h_3}{h_1}\frac{\partial u}{\partial x} + \frac{vh_3}{h_1}\frac{\partial h_1}{\partial y} + \frac{wh_2}{h_1}\frac{\partial h_1}{\partial z}\right)\\
&- \frac{2}{Reh_1h_2h_3}\frac{\partial\mu}{\partial T}\left(\frac{h_3}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial u}{\partial x} + \frac{h_2}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial u}{\partial x} - \frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial u}{\partial x} + \frac{v}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y}\right.\\
&+\frac{h_3}{h_1}\frac{\partial v}{\partial x}\frac{\partial h_1}{\partial y} + \frac{vh_3}{h_1}\frac{\partial^2 h_1}{\partial x\partial y} - \frac{vh_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial y} + \frac{w}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} + \frac{h_2}{h_1}\frac{\partial w}{\partial x}\frac{\partial h_1}{\partial z}\\
&\left. +\frac{wh_2}{h_1}\frac{\partial^2 h_1}{\partial x\partial z} - \frac{wh_2}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial z}\right)\\
&- \frac{1}{Reh_1h_2h_3}\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial y}\left(h_3\frac{\partial v}{\partial x} - \frac{vh_3}{h_2}\frac{\partial h_2}{\partial x} + \frac{h_1h_3}{h_2}\frac{\partial u}{\partial y} - \frac{uh_3}{h_2}\frac{\partial h_1}{\partial y}\right)\\
&- \frac{1}{Reh_1h_2h_3}\frac{\partial\mu}{\partial T}\left(\frac{\partial h_3}{\partial y}\frac{\partial v}{\partial x} + h_3\frac{\partial^2 v}{\partial x\partial y} - \frac{h_3}{h_2}\frac{\partial v}{\partial y}\frac{\partial h_2}{\partial x} - \frac{v}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x} - \frac{vh_3}{h_2}\frac{\partial^2 h_2}{\partial x\partial y}\right.\\
&+\frac{vh_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial x} + \frac{h_1}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial u}{\partial y} + \frac{h_1h_3}{h_2}\frac{\partial^2 u}{\partial y^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial u}{\partial y}-\frac{u}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_1}{\partial y}-\frac{uh_3}{h_2}\frac{\partial^2 h_1}{\partial y^2}+\frac{uh_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial y}\bigg)\\
&-\frac{i\beta}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(h_2\frac{\partial w}{\partial x}-\frac{uh_2}{h_3}\frac{\partial h_1}{\partial z}-\frac{wh_2}{h_3}\frac{\partial h_3}{\partial x}\right)\\
&-\frac{1}{Reh_1h_2}\frac{\partial \mu}{\partial T}\frac{\partial h_1}{\partial y}\left(\frac{1}{h_1}\frac{\partial v}{\partial x}-\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\\
&-\frac{1}{Reh_1h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_1}{\partial z}\left(\frac{1}{h_1}\frac{\partial w}{\partial x}-\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}-\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\\
&+\frac{2}{Reh_1h_2}\frac{\partial \mu}{\partial T}\frac{\partial h_2}{\partial x}\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&+\frac{2}{Reh_1h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_3}{\partial x}\left(\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&+\frac{1}{Reh_1^2}\frac{\lambda}{\mu}\frac{\partial^2 \mu}{\partial T^2}\frac{\partial T}{\partial x}\frac{\partial u}{\partial x}+\frac{2}{Reh_1^2}\frac{\partial^2 \mu}{\partial T^2}\frac{\partial T}{\partial x}\frac{\partial u}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
E_{1,5}=&\frac{u}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_2}\frac{\partial u}{\partial y}-\frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial x}-\frac{w^2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{vu}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{wu}{h_1h_3}\frac{\partial h_1}{\partial z}\\
&+\frac{1}{h_1\gamma M^2}\left(i\alpha T+\Omega\frac{\partial T}{\partial x}\right)
\end{aligned}
$$

$$
\begin{aligned}
E_{2,1}=&\rho\left(\frac{1}{h_1}\frac{\partial v}{\partial x}-\frac{2u}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}\right)-\frac{1}{Re}\left\{\left[\frac{1}{h_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\right.\right.\\
&\left.\left.-\frac{\lambda}{h_2}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y}+\frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y}+\frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right)\right]\left(i\alpha h_2h_3+h_2\frac{\partial h_3}{\partial x}+h_3\frac{\partial h_2}{\partial x}\right)\right\}\\
&-\frac{\lambda}{Reh_1h_2^2h_3}\left(i\alpha h_2\frac{\partial h_3}{\partial y}+i\alpha h_3\frac{\partial h_2}{\partial y}+\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial x}+h_2\frac{\partial^2 h_3}{\partial x\partial y}+\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x}+h_3\frac{\partial^2 h_3}{\partial x\partial y}\right)\\
&+\frac{1}{Reh_1^2h_2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x}\frac{\partial h_1}{\partial y}+\frac{\mu}{Reh_1^2h_2h_3}\left(i\alpha h_3\frac{\partial h_1}{\partial y}+\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y}+h_3\frac{\partial^2 h_1}{\partial x\partial y}-\frac{h_3}{h_1}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial y}\right)\\
&-\frac{2}{Reh_1h_2^2}\frac{\partial \mu}{\partial T}\frac{\partial h_2}{\partial x}\frac{\partial T}{\partial y}-\frac{2\mu}{Reh_1h_2^2h_3}\left(\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial y}+h_3\frac{\partial^2 h_2}{\partial x\partial y}-\frac{h_3}{h_2}\frac{\partial h_2}{\partial x}\frac{\partial h_2}{\partial y}\right)\\
&+\frac{\mu}{Reh_1^2h_2^2}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial y}+\frac{2\mu}{Reh_1h_2h_3^2}\frac{\partial h_3}{\partial y}\frac{\partial h_3}{\partial x}+\frac{i2\mu\alpha}{Reh_1^2h_2}\frac{\partial h_1}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
E_{2,2}=&\rho\left(-i\omega+\frac{iu\alpha}{h_1}+\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{iw\beta}{h_3}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)-\frac{1}{Re}\left\{\left[\frac{1}{h_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\right.\right.\\
&\left.\left.-\frac{\lambda}{h_2}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y}+\frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y}+\frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right)\right]\left(h_1\frac{\partial h_3}{\partial y}+h_3\frac{\partial h_1}{\partial y}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\lambda}{Reh_1h_2^2h_3}\left(2\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial y}+h_1\frac{\partial^2 h_3}{\partial y^2}+h_3\frac{\partial^2 h_1}{\partial y^2}\right)-\frac{1}{Reh_1^2h_2}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x}\left(i\alpha h_2-\frac{\partial h_2}{\partial x}\right)\\
&- \frac{\mu}{Reh_1^2h_2h_3}\left(i\alpha h_2\frac{\partial h_3}{\partial x}-\frac{i\alpha h_2h_3}{h_1}\frac{\partial h_1}{\partial x}+ih_2h_3\frac{\partial \alpha}{\partial x}-\alpha^2h_2h_3\right.\\
&\left.-\frac{\partial h_3}{\partial x}\frac{\partial h_2}{\partial x}-h_3\frac{\partial^2 h_2}{\partial x^2}+\frac{h_3}{h_1}\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial x}\right)-\frac{\mu}{Reh_1h_2h_3^2}\left(i\beta h_2\frac{\partial h_1}{\partial z}-\beta^2h_1h_2\right.\\
&\left.-\frac{i\beta h_1h_2}{h_3}\frac{\partial h_3}{\partial z}-\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z}-h_1\frac{\partial^2 h_2}{\partial z^2}+\frac{h_1}{h_3}\frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial z}\right)-\frac{\mu}{Reh_2h_3^2}\frac{\partial h_2}{\partial z}\left(i\beta-\frac{1}{h_2}\frac{\partial h_2}{\partial z}\right)\\
&- \frac{\mu}{Reh_1^2h_2}\frac{\partial h_2}{\partial x}\left(i\alpha-\frac{1}{h_2}\frac{\partial h_2}{\partial x}\right)+\frac{2\mu}{Reh_2^2h_3^2}\left(\frac{\partial h_3}{\partial y}\right)^2+\frac{2\mu}{Reh_1^2h_2^2}\left(\frac{\partial h_1}{\partial y}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
E_{2,3}=&\rho\left(\frac{v}{h_2h_3}\frac{\partial h_2}{\partial z}-\frac{2w}{h_2h_3}\frac{\partial h_3}{\partial y}\right)-\frac{1}{Re}\left\{\left[\frac{1}{h_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\right.\right.\\
&\left.\left.-\frac{\lambda}{h_2}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y}+\frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y}+\frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right)\right]\left(i\beta h_1h_2+h_1\frac{\partial h_2}{\partial z}+h_2\frac{\partial h_1}{\partial z}\right)\right\}\\
&- \frac{\lambda}{Reh_1h_2^2h_3}\left(i\beta h_1\frac{\partial h_2}{\partial y}+i\beta h_2\frac{\partial h_1}{\partial y}+\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}+h_1\frac{\partial^2 h_2}{\partial y\partial z}+\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial z}+h_2\frac{\partial^2 h_1}{\partial y\partial z}\right)\\
&- \frac{2}{Reh_2^2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_2}{\partial z}\frac{\partial T}{\partial y}-\frac{2\mu}{Reh_1h_2^2h_3}\left(\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}+h_1\frac{\partial^2 h_2}{\partial y\partial z}-\frac{h_1}{h_2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z}\right)\\
&+ \frac{\mu}{Reh_1h_2h_3^2}\left(3i\beta h_1\frac{\partial h_3}{\partial y}+\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y}+h_1\frac{\partial^2 h_3}{\partial y\partial z}-\frac{h_1}{h_3}\frac{\partial h_3}{\partial y}\frac{\partial h_3}{\partial z}\right)+\frac{\mu}{Reh_2^2h_3^2}\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial y}\\
&+ \frac{2\mu}{Reh_1^2h_2h_3}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
E_{2,4}=&\frac{1}{h_2\gamma M^2}\frac{\partial \rho}{\partial y}-\frac{1}{Re}\left\{\left[\frac{1}{h_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial^2 \mu}{\partial T^2}\frac{\partial T}{\partial y}\right.\right.\\
&\left.-\frac{1}{h_2}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial y}+\frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial y}+\frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial y}\right)\right]\left(h_2h_3\frac{\partial u}{\partial x}\right.\\
&\left.\left.+uh_2\frac{\partial h_3}{\partial x}+uh_3\frac{\partial h_2}{\partial x}+h_1h_3\frac{\partial v}{\partial y}+vh_3\frac{\partial h_1}{\partial y}+vh_1\frac{\partial h_3}{\partial y}+wh_1\frac{\partial h_2}{\partial z}+wh_2\frac{\partial h_1}{\partial z}\right)\right\}\\
&- \frac{1}{Reh_1h_2^2h_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(h_2h_3\frac{\partial^2 u}{\partial x\partial y}+h_2\frac{\partial h_3}{\partial y}\frac{\partial u}{\partial x}+h_3\frac{\partial h_2}{\partial y}\frac{\partial u}{\partial x}+h_2\frac{\partial u}{\partial y}\frac{\partial h_3}{\partial x}+u\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial x}\right.\\
&+uh_2\frac{\partial^2 h_3}{\partial x\partial y}+h_3\frac{\partial u}{\partial y}\frac{\partial h_2}{\partial x}+u\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x}+uh_3\frac{\partial^2 h_2}{\partial x\partial y}\\
&+h_1h_3\frac{\partial^2 v}{\partial y^2}+2h_1\frac{\partial h_3}{\partial y}\frac{\partial v}{\partial y}+2h_3\frac{\partial h_1}{\partial y}\frac{\partial v}{\partial y}+2v\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial y}+vh_3\frac{\partial^2 h_1}{\partial y^2}+vh_1\frac{\partial^2 h_3}{\partial y^2}\\
&\left.+h_1\frac{\partial w}{\partial y}\frac{\partial h_2}{\partial z}+w\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}+wh_1\frac{\partial^2 h_2}{\partial y\partial z}+h_2\frac{\partial w}{\partial y}\frac{\partial h_1}{\partial z}+w\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial z}+wh_2\frac{\partial^2 h_1}{\partial y\partial z}\right)\\
&- \frac{1}{Reh_1h_2h_3}\left(\frac{\partial^2 \mu}{\partial T^2}\frac{\partial T}{\partial x}+i\alpha\frac{\partial \mu}{\partial T}\right)\left(\frac{h_2h_3}{h_1}\frac{\partial v}{\partial x}-\frac{vh_3}{h_1}\frac{\partial h_2}{\partial x}+h_3\frac{\partial u}{\partial y}-\frac{uh_3}{h_1}\frac{\partial h_1}{\partial y}\right)\\
&- \frac{1}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{h_2}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial v}{\partial x}-\frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial v}{\partial x}-\frac{v}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x}-\frac{vh_3}{h_1}\frac{\partial^2 h_2}{\partial x^2}\right.
\end{aligned}
$$

$$+\frac{vh_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial x}+\frac{\partial h_3}{\partial x}\frac{\partial u}{\partial y}+h_3\frac{\partial^2 u}{\partial x\partial y}-\frac{h_3}{h_1}\frac{\partial h_1}{\partial y}\frac{\partial u}{\partial x}-\frac{u}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y}-\frac{uh_3}{h_1}\frac{\partial^2 h_1}{\partial x\partial y}$$

$$+\frac{uh_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial y}\Bigg)-\frac{2}{Reh_1h_2h_3}\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial y}\left(\frac{h_1h_3}{h_2}\frac{\partial v}{\partial y}+\frac{wh_1}{h_2}\frac{\partial h_2}{\partial z}+\frac{uh_3}{h_2}\frac{\partial h_2}{\partial x}\right)$$

$$-\frac{2}{Reh_1h_2h_3}\frac{\partial\mu}{\partial T}\Bigg(\frac{h_1h_3}{h_2}\frac{\partial^2 v}{\partial y^2}+\frac{h_1}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial v}{\partial y}+\frac{h_3}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial v}{\partial y}-\frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial v}{\partial y}+\frac{h_1}{h_2}\frac{\partial h_2}{\partial z}\frac{\partial w}{\partial y}$$

$$+\frac{w}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}+\frac{wh_1}{h_2}\frac{\partial^2 h_2}{\partial y\partial z}-\frac{wh_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z}+\frac{h_3}{h_2}\frac{\partial h_2}{\partial x}\frac{\partial u}{\partial y}+\frac{u}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x}$$

$$+\frac{uh_3}{h_2}\frac{\partial^2 h_2}{\partial x\partial y}-\frac{uh_3}{h_2^2}\frac{\partial h_2}{\partial x}\frac{\partial h_2}{\partial y}\Bigg)-\frac{i\beta}{Reh_1h_2h_3}\frac{\partial\mu}{\partial T}\left(h_1\frac{\partial w}{\partial y}-\frac{wh_1}{h_3}\frac{\partial h_3}{\partial y}-\frac{vh_1}{h_3}\frac{\partial h_2}{\partial z}\right)$$

$$-\frac{1}{Reh_1h_2h_3}\frac{\partial\mu}{\partial T}\Bigg(\frac{\partial h_1}{\partial z}\frac{\partial w}{\partial y}-\frac{w}{h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y}-\frac{wh_1}{h_3}\frac{\partial^2 h_3}{\partial y\partial z}+\frac{wh_1}{h_3^2}\frac{\partial h_3}{\partial y}\frac{\partial h_3}{\partial z}$$

$$-\frac{v}{h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z}-\frac{vh_1}{h_3}\frac{\partial^2 h_2}{\partial z^2}+\frac{vh_1}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial z}\Bigg)$$

$$-\frac{1}{Reh_2^2h_3}\frac{\partial\mu}{\partial T}\frac{\partial h_2}{\partial z}\left(\frac{\partial w}{\partial y}-\frac{w}{h_3}\frac{\partial h_3}{\partial y}-\frac{v}{h_3}\frac{\partial h_2}{\partial z}\right)$$

$$-\frac{1}{Reh_1h_2}\frac{\partial\mu}{\partial T}\frac{\partial h_2}{\partial x}\left(\frac{1}{h_1}\frac{\partial v}{\partial x}-\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)$$

$$+\frac{2}{Reh_2h_3^2}\frac{\partial\mu}{\partial T}\frac{\partial h_3}{\partial y}\left(\frac{u}{h_1}\frac{\partial h_3}{\partial x}+\frac{v}{h_2}\frac{\partial h_3}{\partial y}\right)+\frac{2}{Reh_1^2h_2}\frac{\partial\mu}{\partial T}\frac{\partial h_1}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{v}{h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_3}\frac{\partial h_1}{\partial z}\right)$$

$$+\frac{1}{Reh_1^2}\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial x}\frac{\partial v}{\partial x}$$

$$E_{2,5}=\frac{u}{h_1}\frac{\partial v}{\partial x}+\frac{v}{h_2}\frac{\partial v}{\partial y}-w\left(\frac{w}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{v}{h_2h_3}\frac{\partial h_2}{\partial z}\right)+u\left(\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)$$

$$+\frac{1}{h_2\gamma M^2}\frac{\partial T}{\partial y}$$

$$E_{3,1}=\rho\left(\frac{1}{h_1}\frac{\partial w}{\partial x}-\frac{2u}{h_1h_3}\frac{\partial h_1}{\partial z}+\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}\right)-\frac{\lambda}{Reh_1h_2h_3^2}\Bigg(i\alpha h_2\frac{\partial h_3}{\partial z}+i\alpha h_3\frac{\partial h_2}{\partial z}-\alpha\beta h_2h_3$$

$$+i\beta h_2\frac{\partial h_3}{\partial x}+\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x}+h_2\frac{\partial^2 h_3}{\partial x\partial z}+i\beta h_3\frac{\partial h_2}{\partial x}+\frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial x}+h_3\frac{\partial^2 h_2}{\partial x\partial z}\Bigg)$$

$$+\frac{\lambda}{Reh_3}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z}+\frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z}+\frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right)\left(i\alpha h_2h_3+h_2\frac{\partial h_3}{\partial x}+h_3\frac{\partial h_2}{\partial x}\right)$$

$$-\frac{1}{Reh_1h_3}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial x}\left(i\beta-\frac{1}{h_1}\frac{\partial h_1}{\partial z}\right)$$

$$
\begin{aligned}
&- \frac{\mu}{Reh_1h_2h_3}\left(i\beta\frac{\partial h_2}{\partial x} - \alpha\beta h_2 - \frac{i\alpha h_2}{h_1}\frac{\partial h_1}{\partial z} - \frac{1}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} - \frac{h_2}{h_1}\frac{\partial^2 h_1}{\partial x\partial z} + \frac{h_2}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial z}\right)\\
&- \frac{2\mu}{Reh_1h_2h_3^2}\left(i\beta h_2\frac{\partial h_3}{\partial x} + \frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x} + h_2\frac{\partial^2 h_3}{\partial x\partial z} - \frac{h_2}{h_3}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial x}\right)\\
&- \frac{\mu}{Reh_1h_3^2}\frac{\partial h_3}{\partial x}\left(i\beta - \frac{1}{h_1}\frac{\partial h_1}{\partial z}\right) + \frac{i2\mu\alpha}{Reh_1^2h_3}\frac{\partial h_1}{\partial z} + \frac{2\mu}{Reh_1h_2^2h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_2}{\partial x}\\
E_{3,2} =&\rho\left(\frac{1}{h_2}\frac{\partial w}{\partial y} - \frac{2v}{h_2h_3}\frac{\partial h_2}{\partial z} + \frac{w}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&- \frac{\lambda}{Reh_1h_2h_3^2}\left(i\beta h_1\frac{\partial h_3}{\partial y} + \frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y} + h_1\frac{\partial^2 h_3}{\partial y\partial z} + i\beta h_3\frac{\partial h_1}{\partial y} + \frac{\partial h_3}{\partial z}\frac{\partial h_1}{\partial y} + h_3\frac{\partial^2 h_1}{\partial y\partial z}\right)\\
&+ \frac{\lambda}{Reh_3}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right)\left(h_1\frac{\partial h_3}{\partial y} + h_3\frac{\partial h_1}{\partial y}\right)\\
&- \frac{1}{Reh_2h_3}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\left(i\beta - \frac{1}{h_2}\frac{\partial h_2}{\partial z}\right)\\
&- \frac{\mu}{Reh_1h_2h_3}\left(i\beta\frac{\partial h_1}{\partial y} - \frac{1}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z} - \frac{h_1}{h_2}\frac{\partial^2 h_2}{\partial y\partial z} + \frac{h_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z}\right)\\
&- \frac{2\mu}{Reh_1h_2h_3^2}\left(i\beta h_1\frac{\partial h_3}{\partial y} + \frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y} + h_1\frac{\partial^2 h_3}{\partial y\partial z} - \frac{h_1}{h_3}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial y}\right)\\
&- \frac{\mu}{Reh_2h_3^2}\frac{\partial h_3}{\partial y}\left(i\beta - \frac{1}{h_2}\frac{\partial h_2}{\partial z}\right) + \frac{2\mu}{Reh_1^2h_2h_3}\frac{\partial h_1}{\partial y}\frac{\partial h_1}{\partial z}\\
E_{3,3} =&\rho\left(-i\omega + \frac{iu\alpha}{h_1} + \frac{iw\beta}{h_3} + \frac{v}{h_2h_3}\frac{\partial h_3}{\partial y} + \frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\\
&- \frac{\lambda}{Reh_1h_2h_3^2}\left(i2\beta h_1\frac{\partial h_2}{\partial z} + i2\beta h_2\frac{\partial h_1}{\partial z} - \beta^2 h_1h_2 + 2\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z} + h_1\frac{\partial^2 h_2}{\partial z^2} + h_2\frac{\partial^2 h_1}{\partial z^2}\right)\\
&+ \frac{\lambda}{Reh_3}\left(\frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} + \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} + \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right)\left(i\beta h_1h_2 + h_1\frac{\partial h_2}{\partial z} + h_2\frac{\partial h_1}{\partial z}\right)\\
&- \frac{1}{Reh_1^2h_3}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial x}\left(ih_3\alpha - \frac{\partial h_3}{\partial x}\right) - \frac{\mu}{Reh_1^2h_2h_3}\left(i\frac{\partial \alpha}{\partial x}h_2h_3 - \alpha^2 h_2h_3 + i\alpha h_3\frac{\partial h_2}{\partial x}\right.\\
&\left.-\frac{i\alpha h_2h_3}{h_1}\frac{\partial h_1}{\partial x} - \frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x} - h_2\frac{\partial^2 h_3}{\partial x^2} + \frac{h_2}{h_1}\frac{\partial h_1}{\partial x}\frac{\partial h_3}{\partial x}\right)\\
&+ \frac{1}{Reh_2^2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_3}{\partial y}\frac{\partial T}{\partial y} + \frac{\mu}{Reh_1h_2^2h_3}\left(\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial y} + h_1\frac{\partial^2 h_3}{\partial y^2} - \frac{h_1}{h_2}\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial y}\right)\\
&- \frac{2\mu}{Reh_1h_2h_3^2}\left(i\beta h_1\frac{\partial h_2}{\partial z} + i\beta h_2\frac{\partial h_1}{\partial z} - \beta^2 h_1h_2 - \frac{i\beta h_1h_2}{h_3}\frac{\partial h_3}{\partial z}\right) - \frac{i\alpha\mu}{Reh_1^2h_3}\frac{\partial h_3}{\partial x}\\
&+ \frac{\mu}{Reh_1^2h_3^2}\left(\frac{\partial h_3}{\partial x}\right)^2 + \frac{\mu}{Reh_2^2h_3^2}\left(\frac{\partial h_3}{\partial y}\right)^2 + \frac{2\mu}{Reh_1^2h_3^2}\left(\frac{\partial h_1}{\partial z}\right)^2 + \frac{2\mu}{Reh_2^2h_3^2}\left(\frac{\partial h_2}{\partial z}\right)^2\\
E_{3,4} =&\frac{i\beta\rho}{h_3\gamma M^2} - \left[\frac{1}{Reh_3}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(\frac{i\beta}{h_1h_2h_3} - \frac{1}{h_1^2h_2h_3}\frac{\partial h_1}{\partial z} - \frac{1}{h_1h_2^2h_3}\frac{\partial h_2}{\partial z} - \frac{1}{h_1h_2h_3^2}\frac{\partial h_3}{\partial z}\right)\right.\\
&\left(h_2h_3\frac{\partial u}{\partial x} + uh_2\frac{\partial h_3}{\partial x} + uh_3\frac{\partial h_2}{\partial x} + h_1h_3\frac{\partial v}{\partial y} + vh_1\frac{\partial h_3}{\partial y} + vh_3\frac{\partial h_1}{\partial y}\right.
\end{aligned}
$$

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\begin{aligned}
&+wh_1\frac{\partial h_2}{\partial z}+wh_2\frac{\partial h_1}{\partial z}\bigg)\bigg]-\frac{1}{Reh_1h_2h_3^2}\frac{\lambda}{\mu}\frac{\partial \mu}{\partial T}\left(h_2\frac{\partial h_3}{\partial z}\frac{\partial u}{\partial x}+h_3\frac{\partial h_2}{\partial z}\frac{\partial u}{\partial x}+u\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x}\right.\\
&+uh_2\frac{\partial^2 h_3}{\partial x\partial z}+u\frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial x}+uh_3\frac{\partial^2 h_2}{\partial x\partial z}+h_1\frac{\partial h_3}{\partial z}\frac{\partial v}{\partial y}+h_3\frac{\partial h_1}{\partial z}\frac{\partial v}{\partial y}+v\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y}\\
&\left.+vh_1\frac{\partial^2 h_3}{\partial y\partial z}+v\frac{\partial h_3}{\partial z}\frac{\partial h_1}{\partial y}+vh_3\frac{\partial^2 h_1}{\partial y\partial z}+2w\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z}+wh_1\frac{\partial^2 h_2}{\partial z^2}+wh_2\frac{\partial^2 h_1}{\partial z^2}\right)\\
&-\frac{1}{Reh_1h_2h_3}\left(\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial x}+i\alpha\frac{\partial \mu}{\partial T}\right)\left(\frac{h_2h_3}{h_1}\frac{\partial w}{\partial x}-\frac{uh_2}{h_1}\frac{\partial h_1}{\partial z}-\frac{wh_2}{h_1}\frac{\partial h_3}{\partial x}\right)\\
&-\frac{1}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{h_3}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial w}{\partial x}-\frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial w}{\partial x}-\frac{h_2}{h_1}\frac{\partial h_1}{\partial z}\frac{\partial u}{\partial x}-\frac{u}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z}\right.\\
&\left.-\frac{uh_2}{h_1}\frac{\partial^2 h_1}{\partial x\partial z}+\frac{uh_2}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial z}-\frac{wh_2}{h_1}\frac{\partial^2 h_3}{\partial x^2}-\frac{w}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x}+\frac{wh_2}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_3}{\partial x}\right)\\
&-\frac{1}{Reh_1h_2^2h_3}\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial y}\left(h_1h_3\frac{\partial w}{\partial y}-wh_1\frac{\partial h_3}{\partial y}-vh_1\frac{\partial h_2}{\partial z}\right)\\
&-\frac{1}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{h_1h_3}{h_2}\frac{\partial^2 w}{\partial y^2}+\frac{h_3}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial w}{\partial y}-\frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial w}{\partial y}-\frac{w}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial y}\right.\\
&\left.-\frac{wh_1}{h_2}\frac{\partial^2 h_3}{\partial y^2}+\frac{wh_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial y}-\frac{h_1}{h_2}\frac{\partial h_2}{\partial z}\frac{\partial v}{\partial y}-\frac{v}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}-\frac{vh_1}{h_2}\frac{\partial^2 h_2}{\partial y\partial z}+\frac{vh_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z}\right)\\
&-\frac{i2\beta}{Reh_1h_2h_3^2}\frac{\partial \mu}{\partial T}\left(uh_2\frac{\partial h_3}{\partial x}+vh_1\frac{\partial h_3}{\partial y}\right)-\frac{2}{Reh_1h_2h_3}\frac{\partial \mu}{\partial T}\left(\frac{uh_2}{h_3}\frac{\partial^2 h_3}{\partial x\partial z}+\frac{u}{h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x}\right.\\
&\left.-\frac{uh_2}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial x}+\frac{vh_1}{h_3}\frac{\partial^2 h_3}{\partial y\partial z}+\frac{v}{h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y}-\frac{vh_1}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial y}\right)\\
&-\frac{1}{Reh_1^2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_3}{\partial x}\left(\frac{\partial w}{\partial x}-\frac{u}{h_3}\frac{\partial h_1}{\partial z}-\frac{w}{h_3}\frac{\partial h_3}{\partial x}\right)\\
&-\frac{1}{Reh_2^2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_3}{\partial y}\left(\frac{\partial w}{\partial y}-\frac{w}{h_3}\frac{\partial h_3}{\partial y}-\frac{v}{h_3}\frac{\partial h_2}{\partial z}\right)\\
&+\frac{2}{Reh_1^2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_1}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{v}{h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_3}\frac{\partial h_1}{\partial z}\right)\\
&+\frac{2}{Reh_2^2h_3}\frac{\partial \mu}{\partial T}\frac{\partial h_2}{\partial z}\left(\frac{\partial v}{\partial y}+\frac{w}{h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1}\frac{\partial h_2}{\partial x}\right)+\frac{1}{Reh_1^2}\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial x}\frac{\partial w}{\partial x}\\
E_{3,5}=&\frac{u}{h_1}\frac{\partial w}{\partial x}+\frac{v}{h_2}\frac{\partial w}{\partial y}-u\left(\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z}-\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}\right)+v\left(\frac{w}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{v}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\\
&+\frac{i\beta T}{h_3\gamma M^2}
\end{aligned}
$$

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\begin{aligned}
E_{4,1} =& \frac{\rho}{h_1}\frac{\partial T}{\partial x} - \frac{(\gamma-1)}{\gamma h_1}\left(T\frac{\partial\rho}{\partial x}+\rho\frac{\partial T}{\partial x}\right) \\
&- \frac{(\gamma-1)\,M^2}{Re}\Bigg\{\frac{i2\alpha}{h_1}\,(2\mu+\lambda)\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\right) \\
&+\frac{2}{h_1h_2}\frac{\partial h_2}{\partial x}\,(2\mu+\lambda)\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right) \\
&+2\mu\left(\frac{i\beta}{h_3}-\frac{1}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_1}\frac{\partial w}{\partial x}-\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}-\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z}\right) \\
&+\frac{2}{h_1h_3}\frac{\partial h_3}{\partial x}\Bigg[(2\mu+\lambda)\left(\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right) \\
&+\lambda\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}+\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\Bigg] \\
&+\frac{i2\lambda\alpha}{h_1}\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right) \\
&+\frac{2\lambda}{h_1h_2}\frac{\partial h_2}{\partial x}\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right) \\
&-\frac{2\mu}{h_1h_2}\frac{\partial h_1}{\partial y}\left(\frac{1}{h_1}\frac{\partial v}{\partial x}-\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\Bigg\} \\
E_{4,2} =& \frac{\rho}{h_2}\frac{\partial T}{\partial y} - \frac{(\gamma-1)}{h_2\gamma}\left(T\frac{\partial\rho}{\partial y}+\rho\frac{\partial T}{\partial y}\right) \\
&- \frac{(\gamma-1)\,M^2}{Re}\Bigg\{\frac{2}{h_1h_2}\frac{\partial h_1}{\partial y}\Bigg[(2\mu+\lambda)\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\right) \\
&+\lambda\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\Bigg] \\
&+2\mu\Bigg[\left(\frac{1}{h_2}\frac{\partial w}{\partial y}-\frac{w}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{v}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\left(\frac{i\beta}{h_3}-\frac{1}{h_2h_3}\frac{\partial h_2}{\partial z}\right) \\
&+\left(\frac{1}{h_1}\frac{\partial v}{\partial x}-\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\left(\frac{i\alpha}{h_1}-\frac{1}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\Bigg] \\
&+\frac{2}{h_2h_3}\frac{\partial h_3}{\partial y}\Bigg[(2\mu+\lambda)\left(\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right) \\
&+\lambda\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}+\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\Bigg]\Bigg\} \\
E_{4,3} =& -\frac{(\gamma-1)\,M^2}{Re}\Bigg\{\frac{2}{h_1h_3}\frac{\partial h_1}{\partial z}\Bigg[(2\mu+\lambda)\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\right) \\
&+\lambda\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\Bigg] \\
&-\frac{2\mu}{h_2h_3}\frac{\partial h_3}{\partial y}\left(\frac{1}{h_2}\frac{\partial w}{\partial y}-\frac{w}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{v}{h_2h_3}\frac{\partial h_2}{\partial z}\right) \\
&+\frac{2}{h_2h_3}\frac{\partial h_2}{\partial z}\Bigg[(2\mu+\lambda)\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)
\end{aligned}
$$

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\begin{aligned}
&+\lambda\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}+\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\Bigg]\\
&+2\mu\left(\frac{1}{h_1}\frac{\partial w}{\partial x}-\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}-\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{i\alpha}{h_1}-\frac{1}{h_1h_3}\frac{\partial h_3}{\partial x}\right)\\
&+\frac{i2\lambda\beta}{h_3}\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}+\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&+\frac{i2\beta}{h_3}\left(2\mu+\lambda\right)\left(\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\Bigg\}\\
E_{4,4}=&\frac{i\rho}{\gamma}\left(-\omega+\frac{u\alpha}{h_1}+\frac{w\beta}{h_3}\right)-\frac{(\gamma-1)}{\gamma}\left(\Omega\frac{u}{h_1}\frac{\partial\rho}{\partial x}+\frac{v}{h_2}\frac{\partial\rho}{\partial y}\right)\\
&-\frac{1}{PrReh_1h_2h_3}\Bigg\{\frac{i\alpha}{h_1}\left(h_2h_3\frac{\partial\kappa}{\partial T}\frac{\partial T}{\partial x}+\kappa h_2\frac{\partial h_3}{\partial x}+\kappa h_3\frac{\partial h_2}{\partial x}-\frac{\kappa h_2h_3}{h_1}\frac{\partial h_1}{\partial x}\right)\\
&+i\frac{\partial\alpha}{\partial x}\frac{\kappa h_2h_3}{h_1}-\frac{\alpha^2\kappa h_2h_3}{h_1}+\frac{i\beta}{h_3}\left(\kappa h_1\frac{\partial h_2}{\partial z}+\kappa h_2\frac{\partial h_1}{\partial z}-\frac{\kappa h_1h_2}{h_3}\frac{\partial h_3}{\partial z}\right)-\frac{\beta^2\kappa h_1h_2}{h_3}\\
&+\left(\frac{\partial^2\kappa}{\partial T^2}\frac{\partial T}{\partial x}+i\alpha\frac{\partial\kappa}{\partial T}\right)\frac{h_2h_3}{h_1}\frac{\partial T}{\partial x}+\frac{\partial^2\kappa}{\partial T^2}\frac{h_1h_3}{h_2}\left(\frac{\partial T}{\partial y}\right)^2\\
&+\frac{\partial\kappa}{\partial T}\Bigg(\frac{h_2}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial T}{\partial x}+\frac{h_3}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial T}{\partial x}-\frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial T}{\partial x}+\frac{h_1h_3}{h_2}\frac{\partial^2T}{\partial y^2}\\
&+\frac{h_1}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial T}{\partial y}+\frac{h_3}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial T}{\partial y}-\frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial T}{\partial y}\Bigg)\Bigg\}\\
&-\frac{(\gamma-1)\,M^2}{Re}\frac{\partial\mu}{\partial T}\Bigg\{\left(2+\frac{\lambda}{\mu}\right)\Bigg[\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\right)^2\\
&+\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)^2+\left(\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right)^2\Bigg]\\
&+\left(\frac{1}{h_2}\frac{\partial w}{\partial y}-\frac{w}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{v}{h_2h_3}\frac{\partial h_2}{\partial z}\right)^2+\left(\frac{1}{h_1}\frac{\partial w}{\partial x}-\frac{w}{h_1h_3}\frac{\partial h_3}{\partial x}-\frac{u}{h_1h_3}\frac{\partial h_1}{\partial z}\right)^2\\
&+\left(\frac{1}{h_1}\frac{\partial v}{\partial x}-\frac{v}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial u}{\partial y}-\frac{u}{h_1h_2}\frac{\partial h_1}{\partial y}\right)^2\\
&+2\frac{\lambda}{\mu}\Bigg[\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&+\left(\frac{1}{h_1}\frac{\partial u}{\partial x}+\frac{v}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{w}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\left(\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&+\left(\frac{1}{h_2}\frac{\partial v}{\partial y}+\frac{w}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{u}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{u}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{v}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\Bigg]\Bigg\}\\
&+\frac{1}{PrReh_1^2}\frac{\partial^2\kappa}{\partial T^2}\frac{\partial T}{\partial x}\frac{\partial T}{\partial x}\\
&+\frac{(\gamma-1)\,M^2}{Re}\frac{\partial\mu}{\partial T}\left[\left(2+\frac{\lambda}{\mu}\right)\left(\frac{1}{h_1}\frac{\partial u}{\partial x}\right)^2+\left(\frac{1}{h_1}\frac{\partial v}{\partial x}\right)^2+\left(\frac{1}{h_1}\frac{\partial w}{\partial x}\right)^2\right]
\end{aligned}
$$

$$E_{4,5} = \frac{u}{h_1}\frac{\partial T}{\partial x} + \frac{v}{h_2}\frac{\partial T}{\partial y} - \frac{(\gamma-1)}{\gamma}\left[iT\left(-\omega + \frac{\alpha u}{h_1} + \frac{\beta w}{h_3}\right) + \Omega\frac{u}{h_1}\frac{\partial T}{\partial x} + \frac{v}{h_2}\frac{\partial T}{\partial y}\right]$$

$$\begin{aligned}
E_{5,1} =& \frac{1}{h_1}\frac{\partial \rho}{\partial x} + \frac{\rho}{h_1}\left(i\alpha + \frac{1}{h_2}\frac{\partial h_2}{\partial x} + \frac{1}{h_3}\frac{\partial h_3}{\partial x}\right)\\
E_{5,2} =& \frac{1}{h_2}\frac{\partial \rho}{\partial y} + \frac{\rho}{h_2}\left(\frac{1}{h_3}\frac{\partial h_3}{\partial y} + \frac{1}{h_1}\frac{\partial h_1}{\partial y}\right)\\
E_{5,3} =& \frac{\rho}{h_3}\left(i\beta + \frac{1}{h_2}\frac{\partial h_2}{\partial z} + \frac{1}{h_1}\frac{\partial h_1}{\partial z}\right)\\
E_{5,4} =& 0\\
E_{5,5} =& -i\omega + \frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{u}{h_1}\left(i\alpha + \frac{1}{h_2}\frac{\partial h_2}{\partial x} + \frac{1}{h_3}\frac{\partial h_3}{\partial x}\right) + \frac{1}{h_2}\frac{\partial v}{\partial y} + \frac{v}{h_2}\left(\frac{1}{h_3}\frac{\partial h_3}{\partial y} + \frac{1}{h_1}\frac{\partial h_1}{\partial y}\right)\\
&+ \frac{w}{h_3}\left(i\beta + \frac{1}{h_2}\frac{\partial h_2}{\partial z} + \frac{1}{h_1}\frac{\partial h_1}{\partial z}\right)
\end{aligned}$$

APPENDIX D

NPSE FORMULATION

$$\phi'(x,y,z,t) \equiv \sum_{n=-N}^{N} \sum_{k=-K}^{K} A_{0(n,k)} \hat{\phi}_{(n,k)}(x,y)\, e^{i\int_{x_0}^{x} \alpha_{(n,k)}(x)\partial x} e^{i(k\beta_0 z - n\omega_0 t)} \tag{D.1}$$

Equation D.1 is the wave format for NPSE disturbances. By following the same steps associated with the LPSE problem formulation (appendix C), the result can be arranged into

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left\{ \left[\mathcal{A}\frac{\partial^2 \hat{\phi}}{\partial y^2} + \mathcal{B}\frac{\partial^2 \hat{\phi}}{\partial x \partial y} + \mathcal{C}\frac{\partial \hat{\phi}}{\partial y} + \mathcal{D}\frac{\partial \hat{\phi}}{\partial x} + \mathcal{E}\hat{\phi} \right]_{(n,k)} A_{0(n,k)} e^{i\int_{x_0}^{x} \alpha_{(n,k)}(x)\partial x} e^{i(k\beta_0 z - n\omega_0 t)} \right\} = \mathcal{NL}_{(n,k)} \tag{D.2}$$

where $\hat{\phi} = \left[\hat{u}, \hat{v}, \hat{w}, \hat{T}, \hat{\rho}\right]$. Note that the nonlinear terms have not been eliminated this time around. Matrices **A** through **E** are the same 5x5 matrices from the LPSE formulation once ω and β are given the coefficients n and k respectively.

The expanded nonlinear terms take the form

$$\begin{aligned} \mathcal{NL}_{(n,k)} = & \sum_{n_1} \sum_{n_2} \sum_{k_1} \sum_{k_2} \Big[A_{0(n_1,k_1)} A_{0(n_2,k_2)} \mathcal{NL}^{(quad)}_{(n,k)} \\ & e^{i\int_{x_0}^{x} \alpha_{(n_1,k_1)}(x)\partial x} e^{i\int_{x_0}^{x} \alpha_{(n_2,k_2)}(x)\partial x} e^{i((k_1+k_2)\beta_0 z + (n_1+n_2)\omega_0 t)} \Big] \\ & + \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{k_1} \sum_{k_2} \sum_{k_3} \Big[A_{0(n_1,k_1)} A_{0(n_2,k_2)} A_{0(n_3,k_3)} \mathcal{NL}^{(cubic)}_{(n,k)} \\ & e^{i\int_{x_0}^{x} \alpha_{(n_1,k_1)}(x)\partial x} e^{i\int_{x_0}^{x} \alpha_{(n_2,k_2)}(x)\partial x} e^{i\int_{x_0}^{x} \alpha_{(n_3,k_3)}(x)\partial x} e^{i((k_1+k_2+k_3)\beta_0 z + (n_1+n_2+n_3)\omega_0 t)} \Big] \end{aligned} \tag{D.3}$$

where n_1, n_2,..., k_3 are each summed from $-N$ or $-K$ to N or K, $n_1 + n_2\,(+n_3) = n$, and $k_1 + k_2\,(+k_3) = k$. $\mathcal{NL}^{(quad)}$ and $\mathcal{NL}^{(cubic)}$ are both 5x1 vectors. These will align with the rows of the LPSE matrices such that $\mathcal{NL}_1 \equiv \mathcal{NL}_x$, $\mathcal{NL}_2 \equiv \mathcal{NL}_y$,..., $\mathcal{NL}_5 \equiv \mathcal{NL}_m$ from equations A.3-A.8. The full expansion of the nonlinear terms is given below. Note that subscripts on $\hat{\phi}$ and α denote which mode (n,k) that term belongs to (e.g. $\hat{u}_2 \equiv \hat{u}_{(n_2,k_2)}$).

X-Momentum Quadratic Nonlinear

$$
\begin{aligned}
NL_x^{quad} = \hat{\rho}_1 \Bigg[& i\,(n_2\omega_0)\,\hat{u}_2 - \frac{\hat{u}_2}{h_1}\frac{\partial \overline{u}}{\partial x} - \frac{\overline{u}}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2\hat{u}_2\right) - \frac{\hat{v}_2}{h_2}\frac{\partial \overline{u}}{\partial y} - \frac{\overline{v}}{h_2}\frac{\partial \hat{u}_2}{\partial y} \\
& - \frac{\overline{w}}{h_3} i\,(k_2\beta_0)\,\hat{u}_2 + \frac{2\hat{v}_2\overline{v}}{h_1 h_2}\frac{\partial h_2}{\partial x} - \frac{\hat{v}_2\overline{u}}{h_1 h_2}\frac{\partial h_1}{\partial y} - \frac{\overline{v}\hat{u}_2}{h_1 h_2}\frac{\partial h_1}{\partial y} - \frac{\hat{w}_2\overline{u}}{h_1 h_3}\frac{\partial h_1}{\partial z} + \frac{2\hat{w}_2\overline{w}}{h_1 h_3}\frac{\partial h_3}{\partial x} - \frac{\hat{u}_2\overline{w}}{h_1 h_3}\frac{\partial h_1}{\partial z}\Bigg] \\
& + \overline{\rho}\left[-\frac{\hat{u}_1}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2\hat{u}_2\right) - \frac{\hat{v}_1}{h_2}\frac{\partial \hat{u}_2}{\partial y} - \frac{\hat{w}_1}{h_3} i\,(k_2\beta_0)\,\hat{u}_2\right. \\
& \left. + \frac{\hat{v}_1\hat{v}_2}{h_1 h_2}\frac{\partial h_2}{\partial x} - \frac{\hat{v}_1\hat{u}_2}{h_1 h_2}\frac{\partial h_1}{\partial y} - \frac{\hat{w}_1\hat{u}_2}{h_1 h_3}\frac{\partial h_1}{\partial z} + \frac{\hat{w}_1\hat{w}_2}{h_1 h_3}\frac{\partial h_3}{\partial x}\right] \\
& - \frac{1}{h_1\gamma M^2}\left[\hat{T}_1\left(\Omega\frac{\partial \hat{\rho}_2}{\partial x} + i\alpha_2\hat{\rho}_2\right) + \hat{\rho}_2\left(\Omega\frac{\partial \hat{T}_1}{\partial x} + i\alpha_1\hat{T}_1\right)\right] \\
& + \frac{\overline{\lambda}}{Re h_1^2 h_2 h_3 \overline{\mu}}\Bigg\{\Bigg[\hat{T}_1\left(\frac{\partial^2 \overline{\mu}}{\partial T^2}\frac{\partial \overline{T}}{\partial x} - \frac{\partial \overline{\mu}}{\partial T}\left(\frac{1}{h_1}\frac{\partial h_1}{\partial x} + \frac{1}{h_2}\frac{\partial h_2}{\partial x} + \frac{1}{h_3}\frac{\partial h_3}{\partial x}\right)\right) \\
& + \frac{\partial \overline{\mu}}{\partial T}\left(\frac{\partial \hat{T}_1}{\partial x} + i\alpha_1\hat{T}_1\right)\Bigg]\Bigg(h_2 h_3\left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2\hat{u}_2\right) + \hat{u}_2 h_3\frac{\partial h_2}{\partial x} + \hat{u}_2 h_2\frac{\partial h_3}{\partial x} + h_1 h_3\frac{\partial \hat{v}_2}{\partial y} \\
& + \hat{v}_2 h_3\frac{\partial h_1}{\partial y} + \hat{v}_2 h_1\frac{\partial h_3}{\partial y} + h_1 h_2 i\,(k_2\beta_0)\,\hat{w}_2 + \hat{w}_2 h_2\frac{\partial h_1}{\partial z} + \hat{w}_2 h_1\frac{\partial h_2}{\partial z}\Bigg) \\
& + \frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\Bigg[h_2 h_3\left(i2\alpha_2\frac{\partial \hat{u}_2}{\partial x} + i\hat{u}_2\frac{\partial \alpha_2}{\partial x} - \alpha_2^2\hat{u}_2\right) + 2h_2\frac{\partial h_3}{\partial x}\left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2\hat{u}_2\right) \\
& + 2h_3\frac{\partial h_2}{\partial x}\left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2\hat{u}_2\right) + 2\hat{u}_2\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x} + \hat{u}_2 h_2\frac{\partial^2 h_3}{\partial x^2} + \hat{u}_2 h_3\frac{\partial^2 h_2}{\partial x^2} \\
& + h_1 h_3\left(\frac{\partial^2 \hat{v}_2}{\partial x\partial y} + i\alpha_2\frac{\partial \hat{v}_2}{\partial y}\right) + h_1\frac{\partial h_3}{\partial x}\frac{\partial \hat{v}_2}{\partial y} + h_3\frac{\partial h_1}{\partial x}\frac{\partial \hat{v}_2}{\partial y} + \hat{v}_2\frac{\partial h_1}{\partial x}\frac{\partial h_3}{\partial y} \\
& + h_1\frac{\partial h_3}{\partial y}\left(\frac{\partial \hat{v}_2}{\partial x} + i\alpha_2\hat{v}_2\right) + \hat{v}_2 h_1\frac{\partial^2 h_3}{\partial x\partial y} + \hat{v}_2\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y} + h_3\frac{\partial h_1}{\partial y}\left(\frac{\partial \hat{v}_2}{\partial x} + i\alpha_2\hat{v}_2\right) \\
& + \hat{v}_2 h_3\frac{\partial^2 h_1}{\partial x\partial y} + h_1 h_2\left(i\,(k_2\beta_0)\,\frac{\partial \hat{w}_2}{\partial x} - \alpha_2\,(k_2\beta_0)\,\hat{w}_2\right) + i\,(k_2\beta_0)\,h_1\hat{w}_2\frac{\partial h_2}{\partial x} \\
& + i\,(k_2\beta_0)\,h_2\hat{w}_2\frac{\partial h_1}{\partial x} + \hat{w}_2\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial z} + h_1\frac{\partial h_2}{\partial z}\left(\frac{\partial \hat{w}_2}{\partial x} + i\alpha_2\hat{w}_2\right) \\
& + \hat{w}_2 h_1\frac{\partial^2 h_2}{\partial x\partial z} + \hat{w}_2\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} + h_2\frac{\partial h_1}{\partial z}\left(\frac{\partial \hat{w}_2}{\partial x} + i\alpha_2\hat{w}_2\right) + \hat{w}_2 h_2\frac{\partial^2 h_1}{\partial x\partial z}\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{Reh_1h_2h_3}\left\{\left[\left(\frac{\partial\overline{\mu}}{\partial T}\left(\frac{\partial\hat{T}_1}{\partial x}+i\alpha_1\hat{T}_1\right)+\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial\overline{T}}{\partial x}\hat{T}_1\right)\left(\frac{h_2h_3}{h_1}\left(\frac{\partial\hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2h_3}{h_1}\frac{\partial h_1}{\partial y}\right.\right.\right.\\
&\qquad\left.\left.+\frac{\hat{w}_2h_2}{h_1}\frac{\partial h_1}{\partial z}\right)\right]+\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[\frac{h_2h_3}{h_1}\left(i2\alpha_2\frac{\partial\hat{u}_2}{\partial x}+i\hat{u}_2\frac{\partial\alpha_2}{\partial x}-\alpha_2^2\hat{u}_2\right)\right.\\
&\qquad+\left(\frac{h_3}{h_1}\frac{\partial h_2}{\partial x}+\frac{h_2}{h_1}\frac{\partial h_3}{\partial x}-\frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\right)\left(\frac{\partial\hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)\\
&\qquad+\frac{\hat{v}_2h_3}{h_1}\frac{\partial^2h_1}{\partial x\partial y}+\frac{\hat{v}_2}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y}-\frac{\hat{v}_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial y}+\frac{h_3}{h_1}\frac{\partial h_1}{\partial y}\left(\frac{\partial\hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)\\
&\qquad\left.\left.+\frac{\hat{w}_2h_2}{h_1}\frac{\partial^2h_1}{\partial x\partial z}+\frac{\hat{w}_2}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z}-\frac{\hat{w}_2h_2}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial z}+\frac{h_2}{h_1}\frac{\partial h_1}{\partial z}\left(\frac{\partial\hat{w}_2}{\partial x}+i\alpha_2\hat{w}_2\right)\right]\right\}\\
&+\frac{1}{Reh_1h_2h_3}\left\{\left[\left(\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial\overline{T}}{\partial y}\hat{T}_1+\frac{\partial\overline{\mu}}{\partial T}\frac{\partial\hat{T}_1}{\partial y}\right)\left(h_3\left(\frac{\partial\hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)-\frac{\hat{v}_2h_3}{h_2}\frac{\partial h_2}{\partial x}+\frac{h_1h_3}{h_2}\frac{\partial\hat{u}_2}{\partial y}\right.\right.\right.\\
&\qquad\left.\left.-\frac{\hat{u}_2h_3}{h_2}\frac{\partial h_1}{\partial y}\right)\right]+\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[\frac{\partial h_3}{\partial y}\left(\frac{\partial\hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)+h_3\left(\frac{\partial^2\hat{v}_2}{\partial x\partial y}+i\alpha_2\frac{\partial\hat{v}_2}{\partial y}\right)-\frac{h_3}{h_2}\frac{\partial\hat{v}_2}{\partial y}\frac{\partial h_2}{\partial x}\right.\\
&\qquad-\frac{\hat{v}_2}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x}+\frac{\hat{v}_2h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial x}-\frac{\hat{v}_2h_3}{h_2}\frac{\partial^2h_2}{\partial x\partial y}+\frac{h_1h_3}{h_2}\frac{\partial^2\hat{u}_2}{\partial y^2}\\
&\qquad\left.\left.+\frac{h_1}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial\hat{u}_2}{\partial y}-\frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial\hat{u}_2}{\partial y}-\frac{\hat{u}_2}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_1}{\partial y}-\frac{\hat{u}_2h_3}{h_2}\frac{\partial^2h_1}{\partial y^2}+\frac{\hat{u}_2h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial y}\right]\right\}\\
&+\frac{1}{Reh_1h_2h_3}\left\{\left[i\left(k_1\beta_0\right)\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left(\frac{h_1h_2}{h_3}i\left(k_2\beta_0\right)\hat{u}_2-\frac{\hat{u}_2h_2}{h_3}\frac{\partial h_1}{\partial z}+h_2\left(\frac{\partial\hat{w}_2}{\partial x}+i\alpha_2\hat{w}_2\right)\right.\right.\right.\\
&\qquad\left.\left.-\frac{\hat{w}_2h_2}{h_3}\frac{\partial h_3}{\partial x}\right)\right]+\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[i\left(k_2\beta_0\right)\frac{\hat{u}_2h_1}{h_3}\frac{\partial h_2}{\partial z}-\left(k_2\beta_0\right)^2\frac{\hat{u}_2h_1h_2}{h_3}-i\left(k_2\beta_0\right)\frac{\hat{u}_2h_1h_2}{h_3^2}\frac{\partial h_3}{\partial z}\right.\\
&\qquad-\frac{\hat{u}_2h_2}{h_3}\frac{\partial^2h_1}{\partial z^2}-\frac{\hat{u}_2}{h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_1}{\partial z}+\frac{\hat{u}_2h_2}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_1}{\partial z}+\frac{\partial h_2}{\partial z}\left(\frac{\partial\hat{w}_2}{\partial x}+i\alpha_2\hat{w}_2\right)\\
&\qquad+h_2\left(i\left(k_2\beta_0\right)\frac{\partial\hat{w}_2}{\partial x}-\alpha_2\left(k_2\beta_0\right)\hat{w}_2\right)-\frac{i\left(k_2\beta_0\right)\hat{w}_2h_2}{h_3}\frac{\partial h_3}{\partial x}\\
&\qquad\left.\left.-\frac{\hat{w}_2h_2}{h_3}\frac{\partial^2h_3}{\partial x\partial z}-\frac{\hat{w}_2}{h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x}+\frac{\hat{w}_2h_2}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial x}\right]\right\}\\
&+\frac{\hat{T}_1}{Reh_1h_2}\frac{\partial h_1}{\partial y}\frac{\partial\overline{\mu}}{\partial T}\left(\frac{1}{h_1}\left(i\alpha_2\hat{v}_2+\frac{\partial\hat{v}_2}{\partial x}\right)-\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial\hat{u}_2}{\partial y}-\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\\
&+\frac{\hat{T}_1}{Reh_1h_3}\frac{\partial h_1}{\partial z}\frac{\partial\overline{\mu}}{\partial T}\left(\frac{1}{h_1}\left(i\alpha_2\hat{w}_2+\frac{\partial\hat{w}_2}{\partial x}\right)-\frac{\hat{w}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{i\left(k_2\beta_0\right)\hat{u}_2}{h_3}-\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\\
&-\frac{2\hat{T}_1}{Reh_1h_2}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_2}{\partial x}\left(\frac{1}{h_2}\frac{\partial\hat{v}_2}{\partial y}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\\
&-\frac{2\hat{T}_1}{Reh_1h_3}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_3}{\partial x}\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&-\left(2+\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{1}{Reh_1^2}\frac{\partial\hat{u}_2}{\partial x}\right)\left(\hat{T}_1\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial\overline{T}}{\partial x}+\frac{\partial\overline{\mu}}{\partial T}\frac{\partial\hat{T}_1}{\partial x}\right)
\end{aligned}
$$

Y-Momentum Quadratic Nonlinear

$$
\begin{aligned}
NL_y^{quad} = &-\overline{\rho}\left[\frac{\hat{u}_1}{h_1}\left(\frac{\partial \hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)+\frac{\hat{v}_1}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{i\left(k_2\beta_0\right)\hat{w}_1\hat{v}_2}{h_3}\right.\\
&\left.-\hat{w}_1\left(\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_2}{\partial z}\right)+\hat{u}_1\left(\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_2}{\partial x}-\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right]\\
&-\hat{\rho}_1\left[-i\left(n_2\omega_0\right)\hat{v}_2+\frac{\overline{u}}{h_1}\left(\frac{\partial \hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)+\frac{\hat{u}_2}{h_1}\frac{\partial \overline{v}}{\partial x}+\frac{\overline{v}}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{v}_2}{h_2}\frac{\partial \overline{v}}{\partial y}\right.\\
&+\frac{i\left(k_2\beta_0\right)\overline{w}\hat{v}_2}{h_3}-\hat{w}_2\left(\frac{\overline{w}}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{\overline{v}}{h_2h_3}\frac{\partial h_2}{\partial z}\right)-\overline{w}\left(\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\\
&\left.+\hat{u}_2\left(\frac{\overline{v}}{h_1h_2}\frac{\partial h_2}{\partial x}-\frac{\overline{u}}{h_1h_2}\frac{\partial h_1}{\partial y}\right)+\overline{u}\left(\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_2}{\partial x}-\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right]-\frac{1}{h_2\gamma M^2}\left(\hat{\rho}_1\frac{\partial \hat{T}_2}{\partial y}+\hat{T}_2\frac{\partial \hat{\rho}_1}{\partial y}\right)\\
&+\frac{\overline{\lambda}}{Reh_1h_2^2h_3\overline{\mu}}\left\{\left[\hat{T}_1\left(\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial \overline{T}}{\partial y}-\frac{\partial \overline{\mu}}{\partial T}\left(\frac{1}{h_1}\frac{\partial h_1}{\partial y}+\frac{1}{h_2}\frac{\partial h_2}{\partial y}+\frac{1}{h_3}\frac{\partial h_3}{\partial y}\right)\right)+\frac{\partial \overline{\mu}}{\partial T}\frac{\partial \hat{T}_1}{\partial y}\right]\right.\\
&\left(h_2h_3\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\hat{u}_2h_3\frac{\partial h_2}{\partial x}+\hat{u}_2h_2\frac{\partial h_3}{\partial x}+h_1h_3\frac{\partial \hat{v}_2}{\partial y}+\hat{v}_2h_3\frac{\partial h_1}{\partial y}+\hat{v}_2h_1\frac{\partial h_3}{\partial y}\right.\\
&\left.+h_1h_2i\left(k_2\beta_0\right)\hat{w}_2+\hat{w}_2h_2\frac{\partial h_1}{\partial z}+\hat{w}_2h_1\frac{\partial h_2}{\partial z}\right)+\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left[h_2h_3\left(\frac{\partial^2\hat{u}_2}{\partial x\partial y}+i\alpha_2\frac{\partial \hat{u}_2}{\partial y}\right)\right.\\
&+\left(h_2\frac{\partial h_3}{\partial y}+h_3\frac{\partial h_2}{\partial y}\right)\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+h_2\frac{\partial \hat{u}_2}{\partial y}\frac{\partial h_3}{\partial x}+\hat{u}_2\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial x}+\hat{u}_2h_2\frac{\partial^2 h_3}{\partial x\partial y}\\
&+h_3\frac{\partial \hat{u}_2}{\partial y}\frac{\partial h_2}{\partial x}+\hat{u}_2\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x}+\hat{u}_2h_3\frac{\partial^2 h_2}{\partial x\partial y}+h_1h_3\frac{\partial^2\hat{v}_2}{\partial y^2}+2h_1\frac{\partial h_3}{\partial y}\frac{\partial \hat{v}_2}{\partial y}+2h_3\frac{\partial h_1}{\partial y}\frac{\partial \hat{v}_2}{\partial y}\\
&+2\hat{v}_2\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial y}+\hat{v}_2h_1\frac{\partial^2 h_3}{\partial y^2}+\hat{v}_2h_3\frac{\partial^2 h_1}{\partial y^2}+i\left(k_2\beta_0\right)h_1h_2\frac{\partial \hat{w}_2}{\partial y}+i\left(k_2\beta_0\right)h_2\hat{w}_2\frac{\partial h_1}{\partial y}\\
&+i\left(k_2\beta_0\right)h_1\hat{w}_2\frac{\partial h_2}{\partial y}+h_1\frac{\partial \hat{w}_2}{\partial y}\frac{\partial h_2}{\partial z}+\hat{w}_2\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}+\hat{w}_2h_1\frac{\partial^2 h_2}{\partial y\partial z}\\
&\left.\left.+h_2\frac{\partial \hat{w}_2}{\partial y}\frac{\partial h_1}{\partial z}+\hat{w}_2\frac{\partial h_2}{\partial y}\frac{\partial h_1}{\partial z}+\hat{w}_2h_2\frac{\partial^2 h_1}{\partial y\partial z}\right]\right\}\\
&+\frac{1}{Reh_1h_2h_3}\left\{\left[\left(\frac{\partial \overline{\mu}}{\partial T}\left(\frac{\partial \hat{T}_1}{\partial x}+i\alpha_1\hat{T}_1\right)+\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial \overline{T}}{\partial x}\hat{T}_1\right)\left(\frac{h_2h_3}{h_1}\left(\frac{\partial \hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)-\frac{\hat{v}_2h_3}{h_1}\frac{\partial h_2}{\partial x}\right.\right.\right.\\
&\left.\left.+h_3\frac{\partial \hat{u}_2}{\partial y}-\frac{\hat{u}_2h_3}{h_1}\frac{\partial h_1}{\partial y}\right)\right]+\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left[\frac{h_2h_3}{h_1}\left(i2\alpha_2\frac{\partial \hat{v}_2}{\partial x}+i\hat{v}_2\frac{\partial \alpha_2}{\partial x}-\alpha_2^2\hat{v}_2\right)\right.\\
&+\left(\frac{h_2}{h_1}\frac{\partial h_3}{\partial x}-\frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\right)\left(\frac{\partial \hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)-\frac{\hat{v}_2}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial h_2}{\partial x}-\frac{\hat{v}_2h_3}{h_1}\frac{\partial^2 h_2}{\partial x^2}+\frac{\hat{v}_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial x}
\end{aligned}
$$

$$
+\frac{\partial h_3}{\partial x}\frac{\partial \hat{u}_2}{\partial y}+h_3\left(\frac{\partial^2 \hat{u}_2}{\partial x\partial y}+i\alpha_2\frac{\partial \hat{u}_2}{\partial y}\right)-\frac{h_3}{h_1}\frac{\partial h_1}{\partial y}\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)
$$

$$
\left.\left.-\frac{\hat{u}_2 h_3}{h_1}\frac{\partial^2 h_1}{\partial x\partial y}-\frac{\hat{u}_2}{h_1}\frac{\partial h_3}{\partial x}\frac{\partial h_1}{\partial y}+\frac{\hat{u}_2 h_3}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial y}\right]\right\}
$$

$$
+\frac{2}{Reh_1h_2h_3}\left\{\left[\left(\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial \overline{T}}{\partial y}\hat{T}_1+\frac{\partial \overline{\mu}}{\partial T}\frac{\partial \hat{T}_1}{\partial y}\right)\left(\frac{h_1h_3}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{w}_2h_1}{h_2}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2h_3}{h_2}\frac{\partial h_2}{\partial x}\right)\right]\right.
$$

$$
+\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[\frac{h_1h_3}{h_2}\frac{\partial^2 \hat{v}_2}{\partial y^2}+\frac{h_1}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial \hat{v}_2}{\partial y}+\frac{h_3}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial \hat{v}_2}{\partial y}-\frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial \hat{v}_2}{\partial y}\right.
$$

$$
+\frac{h_1}{h_2}\frac{\partial \hat{w}_2}{\partial y}\frac{\partial h_2}{\partial z}+\frac{\hat{w}_2}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}+\frac{\hat{w}_2h_1}{h_2}\frac{\partial^2 h_2}{\partial y\partial z}-\frac{\hat{w}_2h_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z}
$$

$$
\left.\left.+\frac{h_3}{h_2}\frac{\partial \hat{u}_2}{\partial y}\frac{\partial h_2}{\partial x}+\frac{\hat{u}_2}{h_2}\frac{\partial h_3}{\partial y}\frac{\partial h_2}{\partial x}+\frac{\hat{u}_2h_3}{h_2}\frac{\partial^2 h_2}{\partial x\partial y}-\frac{\hat{u}_2h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial x}\right]\right\}
$$

$$
+\frac{1}{Reh_1h_2h_3}\left\{\left[i\left(k_1\beta_0\right)\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left(h_1\frac{\partial \hat{w}_2}{\partial y}-\frac{\hat{w}_2h_1}{h_3}\frac{\partial h_3}{\partial y}+\frac{h_1h_2}{h_3}i\left(k_2\beta_0\right)\hat{v}_2-\frac{\hat{v}_2h_1}{h_3}\frac{\partial h_2}{\partial z}\right)\right]\right.
$$

$$
+\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[\frac{\partial h_1}{\partial z}\frac{\partial \hat{w}_2}{\partial y}+i\left(k_2\beta_0\right)h_1\frac{\partial \hat{w}_2}{\partial y}-i\left(k_2\beta_0\right)\frac{\hat{w}_2h_1}{h_3}\frac{\partial h_3}{\partial y}-\frac{\hat{w}_2h_1}{h_3}\frac{\partial^2 h_3}{\partial y\partial z}-\frac{\hat{w}_2}{h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y}\right.
$$

$$
+\frac{\hat{w}_2h_1}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial y}-\left(k_2\beta_0\right)^2\frac{\hat{v}_2h_1h_2}{h_3}+i\left(k_2\beta_0\right)\frac{\hat{v}_2h_2}{h_3}\frac{\partial h_1}{\partial z}-i\left(k_2\beta_0\right)\frac{\hat{v}_2h_1h_2}{h_3^2}\frac{\partial h_3}{\partial z}
$$

$$
\left.\left.-\frac{\hat{v}_2h_1}{h_3}\frac{\partial^2 h_2}{\partial z^2}-\frac{\hat{v}_2}{h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z}+\frac{\hat{v}_2h_1}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial z}\right]\right\}
$$

$$
+\frac{\hat{T}_1}{Reh_2h_3}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_2}{\partial z}\left(\frac{1}{h_2}\frac{\partial \hat{w}_2}{\partial y}-\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_3}{\partial y}+\frac{i\left(k_2\beta_0\right)\hat{v}_2}{h_3}-\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_2}{\partial z}\right)
$$

$$
+\frac{\hat{T}_1}{Reh_1h_2}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_2}{\partial x}\left(\frac{1}{h_1}\left(\frac{\partial \hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)-\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial \hat{u}_2}{\partial y}-\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_1}{\partial y}\right)
$$

$$
-\frac{2\hat{T}_1}{Reh_2h_3}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_3}{\partial y}\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)
$$

$$
-\frac{2\hat{T}_1}{Reh_1h_2}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_1}{\partial y}\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_2}{h_1h_3}\frac{\partial h_1}{\partial z}\right)
$$

$$
-\left(\frac{1}{Reh_1^2}\frac{\partial \hat{v}_2}{\partial x}\right)\left(\hat{T}_1\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial \overline{T}}{\partial x}+\frac{\partial\overline{\mu}}{\partial T}\frac{\partial \hat{T}_1}{\partial x}\right)
$$

$$
\begin{aligned}
NL_z^{quad} = -\overline{\rho}\left[\frac{\hat{u}_1}{h_1}\left(\frac{\partial \hat{w}_2}{\partial x} + i\alpha_2\hat{w}_2\right) + \frac{\hat{v}_1}{h_2}\frac{\partial \hat{w}_2}{\partial y} + \frac{i\left(k_2\beta_0\right)\hat{w}_1\hat{w}_2}{h_3}\right.\\
\left. -\hat{v}_1\left(\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_2}{\partial z} - \frac{\hat{w}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right) + \hat{u}_1\left(\frac{\hat{w}_2}{h_1h_3}\frac{\partial h_3}{\partial x} - \frac{\hat{u}_2}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right]\\
-\hat{\rho}_1\left[-i\left(n_2\omega_0\right)\hat{w}_2 + \frac{\hat{u}_2}{h_1}\frac{\partial \overline{w}}{\partial x} + \frac{\overline{u}}{h_1}\left(\frac{\partial \hat{w}_2}{\partial x} + i\alpha_2\hat{w}_2\right) + \frac{\hat{v}_2}{h_2}\frac{\partial \overline{w}}{\partial y} + \frac{\overline{v}}{h_2}\frac{\partial \hat{w}_2}{\partial y} + \frac{\overline{w}}{h_3}i\left(k_2\beta_0\right)\hat{w}_2\right.\\
\left. -2\frac{\overline{u}\hat{u}_2}{h_1h_3}\frac{\partial h_1}{\partial z} - 2\frac{\overline{v}\hat{v}_2}{h_2h_3}\frac{\partial h_2}{\partial z} + \frac{1}{h_2h_3}\frac{\partial h_3}{\partial y}\left(\hat{w}_2\overline{v} + \hat{v}_2\overline{w}\right) + \frac{1}{h_1h_3}\frac{\partial h_3}{\partial x}\left(\hat{u}_2\overline{w} + \hat{w}_2\overline{u}\right)\right]\\
-\frac{i\left(k_1\beta_0 + k_2\beta_0\right)\hat{\rho}_1\hat{T}_2}{h_3\gamma M^2} + \frac{\overline{\lambda}\hat{T}_1}{Reh_1h_2h_3^2\overline{\mu}}\frac{\partial \overline{\mu}}{\partial T}\left\{\left[i\left(k_1\beta_0\right) - \left(\frac{1}{h_1}\frac{\partial h_1}{\partial z} + \frac{1}{h_2}\frac{\partial h_2}{\partial z} + \frac{1}{h_3}\frac{\partial h_3}{\partial z}\right)\right]\right.\\
\left(h_2h_3\left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2\hat{u}_2\right) + \hat{u}_2h_3\frac{\partial h_2}{\partial x} + \hat{u}_2h_2\frac{\partial h_3}{\partial x} + h_1h_3\frac{\partial \hat{v}_2}{\partial y} + \hat{v}_2h_3\frac{\partial h_1}{\partial y} + \hat{v}_2h_1\frac{\partial h_3}{\partial y}\right.\\
\left. +i\left(k_2\beta_0\right)h_1h_2\hat{w}_2 + \hat{w}_2h_2\frac{\partial h_1}{\partial z} + \hat{w}_2h_1\frac{\partial h_2}{\partial z}\right) + \left[h_2h_3\left(i\left(k_2\beta_0\right)\frac{\partial \hat{u}_2}{\partial x} - \alpha_2\left(k_2\beta_0\right)\hat{u}_2\right)\right.\\
+\left(h_2\frac{\partial h_3}{\partial z} + h_3\frac{\partial h_2}{\partial z}\right)\left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2\hat{u}_2\right) + i\left(k_2\beta_0\right)h_2\hat{u}_2\frac{\partial h_3}{\partial x} + \hat{u}_2\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x} + \hat{u}_2h_2\frac{\partial^2 h_3}{\partial x\partial z}\\
+i\left(k_2\beta_0\right)h_3\hat{u}_2\frac{\partial h_2}{\partial x} + \hat{u}_2\frac{\partial h_3}{\partial z}\frac{\partial h_2}{\partial x} + \hat{u}_2h_3\frac{\partial^2 h_2}{\partial x\partial z} + i\left(k_2\beta_0\right)h_1h_3\frac{\partial \hat{v}_2}{\partial y} + h_1\frac{\partial h_3}{\partial z}\frac{\partial \hat{v}_2}{\partial y}\\
+h_3\frac{\partial h_1}{\partial z}\frac{\partial \hat{v}_2}{\partial y} + \hat{v}_2\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y} + i\left(k_2\beta_0\right)\hat{v}_2h_1\frac{\partial h_3}{\partial y} + \hat{v}_2h_1\frac{\partial^2 h_3}{\partial y\partial z} + \hat{v}_2\frac{\partial h_3}{\partial z}\frac{\partial h_1}{\partial y}\\
+i\left(k_2\beta_0\right)\hat{v}_2h_3\frac{\partial h_1}{\partial y} + \hat{v}_2h_3\frac{\partial^2 h_1}{\partial y\partial z} - \left(k_2\beta_0\right)^2h_1h_2\hat{w}_2 + i2\left(k_2\beta_0\right)h_1\hat{w}_2\frac{\partial h_2}{\partial z}\\
\left.\left. +i2\left(k_2\beta_0\right)h_2\hat{w}_2\frac{\partial h_1}{\partial z} + 2\hat{w}_2\frac{\partial h_1}{\partial z}\frac{\partial h_2}{\partial z} + \hat{w}_2h_1\frac{\partial^2 h_2}{\partial z^2} + \hat{w}_2h_2\frac{\partial^2 h_1}{\partial z^2}\right]\right\}\\
+\frac{1}{Reh_1h_2h_3}\left\{\left[\left(\frac{\partial \overline{\mu}}{\partial T}\left(\frac{\partial \hat{T}_1}{\partial x} + i\alpha_1\hat{T}_1\right) + \frac{\partial^2 \overline{\mu}}{\partial T^2}\frac{\partial \overline{T}}{\partial x}\hat{T}_1\right)\left(\frac{h_2h_3}{h_1}\left(\frac{\partial \hat{w}_2}{\partial x} + i\alpha_2\hat{w}_2\right)\right.\right.\right.\\
\left.\left. -\frac{\hat{w}_2h_2}{h_1}\frac{\partial h_3}{\partial x} + i\left(k_2\beta_0\right)h_2\hat{u}_2 - \frac{\hat{u}_2h_2}{h_1}\frac{\partial h_1}{\partial z}\right)\right]\\
+\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left[\frac{h_2h_3}{h_1}\left(i2\alpha_2\frac{\partial \hat{w}_2}{\partial x} + i\hat{w}_2\frac{\partial \alpha_2}{\partial x} - \alpha_2^2\hat{w}_2\right) + \left(\frac{h_3}{h_1}\frac{\partial h_2}{\partial x} - \frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\right)\left(\frac{\partial \hat{w}_2}{\partial x} + i\alpha_2\hat{w}_2\right)\right.\\
-\frac{\hat{w}_2}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_3}{\partial x} - \frac{\hat{w}_2h_2}{h_1}\frac{\partial^2 h_3}{\partial x^2} + \frac{\hat{w}_2h_2}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_3}{\partial x} + i\left(k_2\beta_0\right)\frac{\partial h_2}{\partial x}\hat{u}_2\\
+h_2\left(i\left(k_2\beta_0\right)\frac{\partial \hat{u}_2}{\partial x} - \alpha_2\left(k_2\beta_0\right)\hat{u}_2\right) - \frac{h_2}{h_1}\frac{\partial h_1}{\partial z}\left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2\hat{u}_2\right)\\
\left.\left. -\frac{\hat{u}_2h_2}{h_1}\frac{\partial^2 h_1}{\partial x\partial z} - \frac{\hat{u}_2}{h_1}\frac{\partial h_2}{\partial x}\frac{\partial h_1}{\partial z} + \frac{\hat{u}_2h_2}{h_1^2}\frac{\partial h_1}{\partial x}\frac{\partial h_1}{\partial z}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{Reh_1h_2h_3}\left\{\left[\left(\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial\overline{T}}{\partial y}\hat{T}_1+\frac{\partial\overline{\mu}}{\partial T}\frac{\partial\hat{T}_1}{\partial y}\right)\left(i\left(k_2\beta_0\right)h_1\hat{v}_2-\frac{\hat{v}_2h_1}{h_2}\frac{\partial h_2}{\partial z}+\frac{h_1h_3}{h_2}\frac{\partial\hat{w}_2}{\partial y}\right.\right.\right.\\
&\left.\left.-\frac{\hat{w}_2h_1}{h_2}\frac{\partial h_3}{\partial y}\right)\right]+\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[i\left(k_2\beta_0\right)\hat{v}_2\frac{\partial h_1}{\partial y}+i\left(k_2\beta_0\right)h_1\frac{\partial\hat{v}_2}{\partial y}-\frac{h_1}{h_2}\frac{\partial\hat{v}_2}{\partial y}\frac{\partial h_2}{\partial z}\right.\\
&-\frac{\hat{v}_2}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_2}{\partial z}-\frac{\hat{v}_2h_1}{h_2}\frac{\partial^2 h_2}{\partial y\partial z}+\frac{\hat{v}_2h_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_2}{\partial z}+\frac{h_1h_3}{h_2}\frac{\partial^2\hat{w}_2}{\partial y^2}+\frac{h_3}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial\hat{w}_2}{\partial y}\\
&\left.\left.-\frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial\hat{w}_2}{\partial y}-\frac{\hat{w}_2}{h_2}\frac{\partial h_1}{\partial y}\frac{\partial h_3}{\partial y}-\frac{\hat{w}_2h_1}{h_2}\frac{\partial^2 h_3}{\partial y^2}+\frac{\hat{w}_2h_1}{h_2^2}\frac{\partial h_2}{\partial y}\frac{\partial h_3}{\partial y}\right]\right\}\\
&+\frac{2}{Reh_1h_2h_3}\left\{\left[i\left(k_1\beta_0\right)\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2h_1h_2}{h_3}+\frac{\hat{v}_2h_1}{h_3}\frac{\partial h_3}{\partial y}+\frac{\hat{u}_2h_2}{h_3}\frac{\partial h_3}{\partial x}\right)\right]\right.\\
&+\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[\frac{i\left(k_2\beta_0\right)\hat{w}_2h_2}{h_3}\frac{\partial h_1}{\partial z}+\frac{i\left(k_2\beta_0\right)\hat{w}_2h_1}{h_3}\frac{\partial h_2}{\partial z}-\frac{i\left(k_2\beta_0\right)\hat{w}_2h_1h_2}{h_3^2}\frac{\partial h_3}{\partial z}-\frac{\left(k_2\beta_0\right)^2\hat{w}_2h_1h_2}{h_3}\right.\\
&+\frac{i\left(k_2\beta_0\right)\hat{v}_2h_1}{h_3}\frac{\partial h_3}{\partial y}+\frac{\hat{v}_2h_1}{h_3}\frac{\partial^2 h_3}{\partial y\partial z}+\frac{\hat{v}_2}{h_3}\frac{\partial h_1}{\partial z}\frac{\partial h_3}{\partial y}-\frac{\hat{v}_2h_1}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial y}\\
&\left.\left.+\frac{i\left(k_2\beta_0\right)\hat{u}_2h_2}{h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{u}_2h_2}{h_3}\frac{\partial^2 h_3}{\partial x\partial z}+\frac{\hat{u}_2}{h_3}\frac{\partial h_2}{\partial z}\frac{\partial h_3}{\partial x}-\frac{\hat{u}_2h_2}{h_3^2}\frac{\partial h_3}{\partial z}\frac{\partial h_3}{\partial x}\right]\right\}\\
&+\frac{\hat{T}_1}{Reh_2h_3}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_3}{\partial y}\left(\frac{i\left(k_2\beta_0\right)\hat{v}_2}{h_3}-\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{1}{h_2}\frac{\partial\hat{w}_2}{\partial y}-\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\\
&+\frac{\hat{T}_1}{Reh_1h_3}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_3}{\partial x}\left(\frac{1}{h_1}\left(\frac{\partial\hat{w}_2}{\partial x}+i\alpha_2\hat{w}_2\right)-\frac{\hat{w}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{i\left(k_2\beta_0\right)\hat{u}_2}{h_3}-\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\\
&-\frac{2\hat{T}_1}{Reh_2h_3}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_2}{\partial z}\left(\frac{1}{h_2}\frac{\partial\hat{v}_2}{\partial y}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\\
&-\frac{2\hat{T}_1}{Reh_1h_3}\frac{\partial\overline{\mu}}{\partial T}\frac{\partial h_1}{\partial z}\left(\frac{1}{h_1}\left(\frac{\partial\hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_2}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\\
&-\left(\frac{1}{Reh_1^2}\frac{\partial\hat{w}_1}{\partial x}\right)\left(\hat{T}_1\frac{\partial^2\overline{\mu}}{\partial T^2}\frac{\partial\overline{T}}{\partial x}+\frac{\partial\overline{\mu}}{\partial T}\frac{\partial\hat{T}_1}{\partial x}\right)
\end{aligned}
$$

$$
\begin{aligned}
NL_e^{quad} = &-\overline{\rho}\left[\frac{\hat{u}_1}{h_1}\left(\frac{\partial \hat{T}_2}{\partial x}+i\alpha_2\hat{T}_2\right)+\frac{\hat{v}_1}{h_2}\frac{\partial \hat{T}_2}{\partial y}+\frac{i\left(k_2\beta_0\right)\hat{w}_1\hat{T}_2}{h_3}\right] \\
&-\hat{\rho}_1\left[-i\left(n_2\omega_0\right)\hat{T}_2+\frac{\hat{u}_2}{h_1}\frac{\partial \overline{T}}{\partial x}+\frac{\overline{u}}{h_1}\left(\frac{\partial \hat{T}_2}{\partial x}+i\alpha_2\hat{T}_2\right)+\frac{\hat{v}_2}{h_2}\frac{\partial \overline{T}}{\partial y}+\frac{\overline{v}}{h_2}\frac{\partial \hat{T}_2}{\partial y}+\frac{i\left(k_2\beta_0\right)\overline{w}\hat{T}_2}{h_3}\right] \\
&+\frac{(\gamma-1)}{\gamma}\left\{-i\left(n_1\omega_0+n_2\omega_0\right)\hat{\rho}_1\hat{T}_2+\frac{\overline{u}}{h_1}\left[\hat{\rho}_2\left(\Omega\frac{\partial \hat{T}_1}{\partial x}+i\alpha_1\hat{T}_1\right)+\hat{T}_1\left(\Omega\frac{\partial \hat{\rho}_2}{\partial x}+i\alpha_2\hat{\rho}_2\right)\right]\right. \\
&\quad+\frac{\hat{u}_1}{h_1}\left[\overline{\rho}\left(\Omega\frac{\partial \hat{T}_2}{\partial x}+i\alpha_2\hat{T}_2\right)+\overline{T}\left(\Omega\frac{\partial \hat{\rho}_2}{\partial x}+i\alpha_2\hat{\rho}_2\right)+\Omega\frac{\partial \overline{\rho}}{\partial x}\hat{T}_2+\Omega\frac{\partial \overline{T}}{\partial x}\hat{\rho}_2\right] \\
&\quad+\frac{\overline{v}}{h_2}\left(\hat{\rho}_1\frac{\partial \hat{T}_2}{\partial y}+\hat{T}_2\frac{\partial \hat{\rho}_1}{\partial y}\right)+\frac{\hat{v}_1}{h_2}\left(\overline{\rho}\frac{\partial \hat{T}_2}{\partial y}+\overline{T}\frac{\partial \hat{\rho}_2}{\partial y}+\hat{\rho}_2\frac{\partial \overline{T}}{\partial y}+\hat{T}_2\frac{\partial \overline{\rho}}{\partial y}\right) \\
&\quad\left.+\frac{i\left(k_1\beta_0+k_2\beta_0\right)\overline{w}\hat{\rho}_1\hat{T}_2}{h_3}+\frac{i\left(k_2\beta_0\right)\hat{w}_1}{h_3}\left(\overline{\rho}\hat{T}_2+\hat{\rho}_1\overline{T}\right)\right\} \\
&+\frac{1}{PrReh_1h_2h_3}\left\{\left[\frac{\partial^2\overline{\kappa}}{\partial T^2}\frac{\partial \overline{T}}{\partial x}\hat{T}_1+\frac{\partial \overline{\kappa}}{\partial T}\left(\frac{\partial \hat{T}_1}{\partial x}+i\alpha_1\hat{T}_1\right)\right]\left[\frac{h_2h_3}{h_1}\left(\frac{\partial \hat{T}_2}{\partial x}+i\alpha_2\hat{T}_2\right)\right]\right. \\
&+\frac{h_2h_3}{h_1}\frac{\partial \overline{\kappa}}{\partial T}\hat{T}_1\left(i2\alpha_2\frac{\partial \hat{T}_2}{\partial x}+i\hat{T}_2\frac{\partial \alpha_2}{\partial x}-\alpha_2^2\hat{T}_2\right)+\left[\frac{\partial \overline{\kappa}}{\partial T}\hat{T}_1\left(\frac{\partial \hat{T}_2}{\partial x}+i\alpha_2\hat{T}_2\right)\right]\left(\frac{h_3}{h_1}\frac{\partial h_2}{\partial x}\right. \\
&\left.+\frac{h_2}{h_1}\frac{\partial h_3}{\partial x}-\frac{h_2h_3}{h_1^2}\frac{\partial h_1}{\partial x}\right)+\left(\frac{\partial^2\overline{\kappa}}{\partial T^2}\frac{\partial \overline{T}}{\partial y}\hat{T}_1+\frac{\partial \overline{\kappa}}{\partial T}\frac{\partial \hat{T}_1}{\partial y}\right)\left(\frac{h_1h_3}{h_2}\frac{\partial \hat{T}_2}{\partial y}\right)+\frac{h_1h_3}{h_2}\frac{\partial \overline{\kappa}}{\partial T}\hat{T}_1\frac{\partial^2\hat{T}_2}{\partial y^2} \\
&+\left(\frac{\partial \overline{\kappa}}{\partial T}\hat{T}_1\frac{\partial \hat{T}_2}{\partial y}\right)\left(\frac{h_3}{h_2}\frac{\partial h_1}{\partial y}+\frac{h_1}{h_2}\frac{\partial h_3}{\partial y}-\frac{h_1h_3}{h_2^2}\frac{\partial h_2}{\partial y}\right)-\frac{\partial \overline{\kappa}}{\partial T}\left(k_1\beta_0\right)\left(k_2\beta_0\right)\hat{T}_1\hat{T}_2\frac{h_1h_2}{h_3} \\
&\left.-\frac{\partial \overline{\kappa}}{\partial T}\left(k_2\beta_0\right)^2\hat{T}_1\hat{T}_2\frac{h_1h_2}{h_3}+\left(i\left(k_2\beta_0\right)\frac{\partial \overline{\kappa}}{\partial T}\hat{T}_1\hat{T}_2\right)\left(\frac{h_2}{h_3}\frac{\partial h_1}{\partial z}+\frac{h_1}{h_3}\frac{\partial h_2}{\partial z}-\frac{h_1h_2}{h_3^2}\frac{\partial h_3}{\partial z}\right)\right\} \\
&+\frac{(\gamma-1)M^2}{Re}\left\{\right. \\
&\left[\left(2\overline{\mu}+\overline{\lambda}\right)\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_1}{\partial x}+i\alpha_1\hat{u}_1\right)+\frac{\hat{v}_1}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_1}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\right. \\
&\left.\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_2}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\right] \\
&+2\left[\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left(2+\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_2}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\right. \\
&\left.\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\right] \\
&+\left[\overline{\mu}\left(\frac{1}{h_2}\frac{\partial \hat{w}_1}{\partial y}-\frac{\hat{w}_1}{h_2h_3}\frac{\partial h_3}{\partial y}+\frac{i\left(k_1\beta_0\right)\hat{v}_1}{h_3}-\frac{\hat{v}_1}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\right.
\end{aligned}
$$

$$
\left.\left(\frac{1}{h_2}\frac{\partial \hat{w}_2}{\partial y}-\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_3}{\partial y}+\frac{i\left(k_2\beta_0\right)\hat{v}_2}{h_3}-\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\right]
$$

$$
+2\left[\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left(\frac{1}{h_2}\frac{\partial \hat{w}_2}{\partial y}-\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_3}{\partial y}+\frac{i\left(k_2\beta_0\right)\hat{v}_2}{h_3}-\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\right.
$$

$$
\left.\left(\frac{1}{h_2}\frac{\partial \overline{w}}{\partial y}-\frac{\overline{w}}{h_2h_3}\frac{\partial h_3}{\partial y}-\frac{\overline{v}}{h_2h_3}\frac{\partial h_2}{\partial z}\right)\right]
$$

$$
+\left[\left(2\overline{\mu}+\overline{\lambda}\right)\left(\frac{1}{h_2}\frac{\partial \hat{v}_1}{\partial y}+\frac{\hat{w}_1}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_1}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\right]
$$

$$
+2\left[\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left(2+\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{1}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\right]
$$

$$
+\left[\overline{\mu}\left(\frac{1}{h_1}\left(\frac{\partial \hat{w}_1}{\partial x}+i\alpha_1\hat{w}_1\right)-\frac{\hat{w}_1}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{i\left(k_1\beta_0\right)\hat{u}_1}{h_3}-\frac{\hat{u}_1}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right.
$$

$$
\left.\left(\frac{1}{h_1}\left(\frac{\partial \hat{w}_2}{\partial x}+i\alpha_2\hat{w}_2\right)-\frac{\hat{w}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{i\left(k_2\beta_0\right)\hat{u}_2}{h_3}-\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right]
$$

$$
+2\left[\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left(\frac{1}{h_1}\left(\frac{\partial \hat{w}_2}{\partial x}+i\alpha_2\hat{w}_2\right)-\frac{\hat{w}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{i\left(k_2\beta_0\right)\hat{u}_2}{h_3}-\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right.
$$

$$
\left.\left(\frac{1}{h_1}\frac{\partial \overline{w}}{\partial x}-\frac{\overline{w}}{h_1h_3}\frac{\partial h_3}{\partial x}-\frac{\overline{u}}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right]
$$

$$
+\left[\left(2\overline{\mu}+\overline{\lambda}\right)\left(\frac{i\left(k_1\beta_0\right)\hat{w}_1}{h_3}+\frac{\hat{u}_1}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_1}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right.
$$

$$
\left.\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]
$$

$$
+2\left[\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left(2+\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\left(\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]
$$

$$
+\left[\overline{\mu}\left(\frac{1}{h_1}\left(\frac{\partial \hat{v}_1}{\partial x}+i\alpha_1\hat{v}_1\right)-\frac{\hat{v}_1}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial \hat{u}_1}{\partial y}-\frac{\hat{u}_1}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right.
$$

$$
\left.\left(\frac{1}{h_1}\left(\frac{\partial \hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)-\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial \hat{u}_2}{\partial y}-\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right]
$$

$$
+2\left[\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left(\frac{1}{h_1}\left(\frac{\partial \hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)-\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial \hat{u}_2}{\partial y}-\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right.
$$

$$
\left.\left(\frac{1}{h_1}\frac{\partial \overline{v}}{\partial x}-\frac{\overline{v}}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial \overline{u}}{\partial y}-\frac{\overline{u}}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right]
$$

$$
+\left[2\overline{\lambda}\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_1}{\partial x}+i\alpha_1\hat{u}_1\right)+\frac{\hat{v}_1}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_1}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\right]
$$

$$
+2\frac{\overline{\lambda}}{\overline{\mu}}\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left[\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_2}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\right.
$$

$$
\left(\frac{1}{h_2}\frac{\partial \overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)
$$

$$
\left.+\left(\frac{1}{h_1}\frac{\partial \overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\right]
$$

$$+\left[2\overline{\lambda}\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_1}{\partial x}+i\alpha_1\hat{u}_1\right)+\frac{\hat{v}_1}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_1}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\right.$$

$$\left.\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]$$

$$+2\frac{\overline{\lambda}}{\overline{\mu}}\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_2}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\left(\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right.$$

$$\left.+\left(\frac{1}{h_1}\frac{\partial\overline{u}}{\partial x}+\frac{\overline{v}}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\overline{w}}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]$$

$$+\left[2\overline{\lambda}\left(\frac{1}{h_2}\frac{\partial \hat{v}_1}{\partial y}+\frac{\hat{w}_1}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_1}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]$$

$$+2\frac{\overline{\lambda}}{\overline{\mu}}\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[\left(\frac{1}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{\overline{u}}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\overline{v}}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right.$$

$$\left.\left.+\left(\frac{1}{h_2}\frac{\partial\overline{v}}{\partial y}+\frac{\overline{w}}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\overline{u}}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]\right\}$$

$$-\left(\frac{1}{PrReh_1^2}\frac{\partial\hat{T}_2}{\partial x}\right)\left(\hat{T}_1\frac{\partial^2\overline{\kappa}}{\partial T^2}\frac{\partial\overline{T}}{\partial x}+\frac{\partial\overline{\kappa}}{\partial T}\frac{\partial\hat{T}_1}{\partial x}\right)$$

$$-\frac{\left(\gamma-1\right)M^2}{Re}\left\{\left(2\overline{\mu}+\overline{\lambda}\right)\left(\frac{1}{h_1}\frac{\partial\hat{u}_1}{\partial x}\right)\left(\frac{1}{h_1}\frac{\partial\hat{u}_2}{\partial x}\right)\right.$$

$$+\overline{\mu}\left(\frac{1}{h_1}\frac{\partial\hat{v}_1}{\partial x}\right)\left(\frac{1}{h_1}\frac{\partial\hat{v}_2}{\partial x}\right)+\overline{\mu}\left(\frac{1}{h_1}\frac{\partial\hat{w}_1}{\partial x}\right)\left(\frac{1}{h_1}\frac{\partial\hat{w}_2}{\partial x}\right)$$

$$\left.+2\frac{\partial\overline{\mu}}{\partial T}\hat{T}_1\left[\left(2+\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{1}{h_1^2}\frac{\partial\overline{u}}{\partial x}\frac{\partial\hat{u}_2}{\partial x}\right)+\left(\frac{1}{h_1^2}\frac{\partial\overline{v}}{\partial x}\frac{\partial\hat{v}_2}{\partial x}\right)+\left(\frac{1}{h_1^2}\frac{\partial\overline{w}}{\partial x}\frac{\partial\hat{w}_2}{\partial x}\right)\right]\right\}$$

Continuity Quadratic Nonlinear

$$NL_m^{quad}=-\frac{1}{h_1h_2}\left(\frac{\partial h_2}{\partial x}\hat{\rho}_1\hat{u}_2+\frac{\partial h_1}{\partial y}\hat{\rho}_1\hat{v}_2\right)-\frac{1}{h_1h_3}\left(\frac{\partial h_3}{\partial x}\hat{\rho}_1\hat{u}_2+\frac{\partial h_1}{\partial z}\hat{\rho}_1\hat{w}_2\right)$$

$$-\frac{1}{h_2h_3}\left(\frac{\partial h_3}{\partial y}\hat{\rho}_1\hat{v}_2+\frac{\partial h_2}{\partial z}\hat{\rho}_1\hat{w}_2\right)-\frac{1}{h_1}\left[\left(\frac{\partial\hat{\rho}_1}{\partial x}+i\alpha_1\hat{\rho}_1\right)\hat{u}_2+\hat{\rho}_1\left(\frac{\partial\hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)\right]$$

$$-\frac{1}{h_2}\left[\frac{\partial\hat{\rho}_1}{\partial y}\hat{v}_2+\hat{\rho}_1\frac{\partial\hat{v}_2}{\partial y}\right]-\frac{1}{h_3}\left[i\hat{\rho}_1\hat{w}_2\left(k_1\beta_0+k_2\beta_0\right)\right]$$

X-Momentum Cubic Nonlinear

$$NL_x^{cubic} = -\hat{\rho}_1 \left[\frac{\hat{u}_2}{h_1} \left(\frac{\partial \hat{u}_3}{\partial x} + i\alpha_3 \hat{u}_3 \right) + \frac{\hat{v}_2}{h_2} \frac{\partial \hat{u}_3}{\partial y} + \frac{i \left(k_3 \beta_0 \right) \hat{w}_2 \hat{u}_3}{h_3} \right.$$
$$\left. - \hat{v}_2 \left(\frac{\hat{v}_3}{h_1 h_2} \frac{\partial h_2}{\partial x} - \frac{\hat{u}_3}{h_1 h_2} \frac{\partial h_1}{\partial y} \right) + \hat{w}_2 \left(\frac{\hat{u}_3}{h_1 h_3} \frac{\partial h_1}{\partial z} - \frac{\hat{w}_3}{h_1 h_3} \frac{\partial h_3}{\partial x} \right) \right]$$

Y-Momentum Cubic Nonlinear

$$NL_y^{cubic} = -\hat{\rho}_1 \left[\frac{\hat{u}_2}{h_1} \left(\frac{\partial \hat{v}_3}{\partial x} + i\alpha_3 \hat{v}_3 \right) + \frac{\hat{v}_2}{h_2} \frac{\partial \hat{v}_3}{\partial y} + \frac{i \left(k_3 \beta_0 \right) \hat{w}_2 \hat{v}_3}{h_3} \right.$$
$$\left. - \hat{w}_2 \left(\frac{\hat{w}_3}{h_2 h_3} \frac{\partial h_3}{\partial y} - \frac{\hat{v}_3}{h_2 h_3} \frac{\partial h_2}{\partial z} \right) + \hat{u}_2 \left(\frac{\hat{v}_3}{h_1 h_2} \frac{\partial h_2}{\partial x} - \frac{\hat{u}_3}{h_1 h_2} \frac{\partial h_1}{\partial y} \right) \right]$$

Z-Momentum Cubic Nonlinear

$$NL_z^{cubic} = -\hat{\rho}_1 \left[\frac{\hat{u}_2}{h_1} \left(\frac{\partial \hat{w}_3}{\partial x} + i\alpha_3 \hat{w}_3 \right) + \frac{\hat{v}_2}{h_2} \frac{\partial \hat{w}_3}{\partial y} + \frac{i \left(k_3 \beta_0 \right) \hat{w}_2 \hat{w}_3}{h_3} \right.$$
$$\left. + \hat{v}_2 \left(\frac{\hat{w}_3}{h_2 h_3} \frac{\partial h_3}{\partial y} - \frac{\hat{v}_3}{h_2 h_3} \frac{\partial h_2}{\partial z} \right) - \hat{u}_2 \left(\frac{\hat{u}_3}{h_1 h_3} \frac{\partial h_1}{\partial z} - \frac{\hat{w}_3}{h_1 h_3} \frac{\partial h_3}{\partial x} \right) \right]$$

Energy Cubic Nonlinear

$$NL_e^{cubic} = -\hat{\rho}_1 \left[\frac{\hat{u}_2}{h_1} \left(\frac{\partial \hat{T}_3}{\partial x} + i\alpha_3 \hat{T}_3 \right) + \frac{\hat{v}_2}{h_2} \frac{\partial \hat{T}_3}{\partial y} + \frac{i \left(k_3 \beta_0 \right) \hat{w}_2 \hat{T}_3}{h_3} \right]$$
$$+ \frac{(\gamma - 1)}{\gamma} \left[\frac{\hat{u}_1 \hat{\rho}_2}{h_1} \left(\Omega \frac{\partial \hat{T}_3}{\partial x} + i\alpha_3 \hat{T}_3 \right) + \frac{\hat{u}_1 \hat{T}_3}{h_1} \left(\Omega \frac{\partial \hat{\rho}_2}{\partial x} + i\alpha_2 \hat{\rho}_2 \right) + \frac{\hat{v}_1}{h_2} \left(\hat{\rho}_2 \frac{\partial \hat{T}_3}{\partial y} + \hat{T}_3 \frac{\partial \hat{\rho}_2}{\partial y} \right) \right.$$
$$\left. + \frac{i \left(k_2 \beta_0 + k_3 \beta_0 \right) \hat{w}_1 \hat{\rho}_2 \hat{T}_3}{h_3} \right] + \frac{(\gamma - 1) M^2}{Re} \frac{\partial \overline{\mu}}{\partial T} \hat{T}_1 \Bigg\{$$
$$\left[\left(2 + \frac{\overline{\lambda}}{\overline{\mu}} \right) \left(\frac{1}{h_1} \left(\frac{\partial \hat{u}_2}{\partial x} + i\alpha_2 \hat{u}_2 \right) + \frac{\hat{v}_2}{h_1 h_2} \frac{\partial h_1}{\partial y} + \frac{\hat{w}_2}{h_3 h_1} \frac{\partial h_1}{\partial z} \right) \right.$$
$$\left. \left(\frac{1}{h_1} \left(\frac{\partial \hat{u}_3}{\partial x} + i\alpha_3 \hat{u}_3 \right) + \frac{\hat{v}_3}{h_1 h_2} \frac{\partial h_1}{\partial y} + \frac{\hat{w}_3}{h_3 h_1} \frac{\partial h_1}{\partial z} \right) \right]$$
$$+ \left[\left(\frac{1}{h_2} \frac{\partial \hat{w}_2}{\partial y} - \frac{\hat{w}_2}{h_2 h_3} \frac{\partial h_3}{\partial y} + \frac{i \left(k_2 \beta_0 \right) \hat{v}_2}{h_3} - \frac{\hat{v}_2}{h_2 h_3} \frac{\partial h_2}{\partial z} \right) \right.$$
$$\left. \left(\frac{1}{h_2} \frac{\partial \hat{w}_3}{\partial y} - \frac{\hat{w}_3}{h_2 h_3} \frac{\partial h_3}{\partial y} + \frac{i \left(k_3 \beta_0 \right) \hat{v}_3}{h_3} - \frac{\hat{v}_3}{h_2 h_3} \frac{\partial h_2}{\partial z} \right) \right]$$

$$
\begin{aligned}
&+\left[\left(2+\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{1}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{1}{h_2}\frac{\partial \hat{v}_3}{\partial y}+\frac{\hat{w}_3}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_3}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\right]\\
&+\left[\left(\frac{1}{h_1}\left(\frac{\partial \hat{w}_2}{\partial x}+i\alpha_2\hat{w}_2\right)-\frac{\hat{w}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{i\left(k_2\beta_0\right)\hat{u}_2}{h_3}-\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right.\\
&\left.\left(\frac{1}{h_1}\left(\frac{\partial \hat{w}_3}{\partial x}+i\alpha_3\hat{w}_3\right)-\frac{\hat{w}_3}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{i\left(k_3\beta_0\right)\hat{u}_3}{h_3}-\frac{\hat{u}_3}{h_1h_3}\frac{\partial h_1}{\partial z}\right)\right]\\
&+\left[\left(2+\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{i\left(k_2\beta_0\right)\hat{w}_2}{h_3}+\frac{\hat{u}_2}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_2}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right.\\
&\left.\left(\frac{i\left(k_3\beta_0\right)\hat{w}_3}{h_3}+\frac{\hat{u}_3}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_3}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]\\
&+\left[\left(\frac{1}{h_1}\left(\frac{\partial \hat{v}_2}{\partial x}+i\alpha_2\hat{v}_2\right)-\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial \hat{u}_2}{\partial y}-\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right.\\
&\left.\left(\frac{1}{h_1}\left(\frac{\partial \hat{v}_3}{\partial x}+i\alpha_3\hat{v}_3\right)-\frac{\hat{v}_3}{h_1h_2}\frac{\partial h_2}{\partial x}+\frac{1}{h_2}\frac{\partial \hat{u}_3}{\partial y}-\frac{\hat{u}_3}{h_1h_2}\frac{\partial h_1}{\partial y}\right)\right]\\
&+\left[\frac{2\overline{\lambda}}{\overline{\mu}}\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_2}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\left(\frac{1}{h_2}\frac{\partial \hat{v}_3}{\partial y}+\frac{\hat{w}_3}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_3}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\right]\\
&+\left[\frac{2\overline{\lambda}}{\overline{\mu}}\left(\frac{1}{h_1}\left(\frac{\partial \hat{u}_2}{\partial x}+i\alpha_2\hat{u}_2\right)+\frac{\hat{v}_2}{h_1h_2}\frac{\partial h_1}{\partial y}+\frac{\hat{w}_2}{h_3h_1}\frac{\partial h_1}{\partial z}\right)\right.\\
&\left.\left(\frac{i\left(k_3\beta_0\right)\hat{w}_3}{h_3}+\frac{\hat{u}_3}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_3}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]\\
&+\left.\left[\frac{2\overline{\lambda}}{\overline{\mu}}\left(\frac{1}{h_2}\frac{\partial \hat{v}_2}{\partial y}+\frac{\hat{w}_2}{h_2h_3}\frac{\partial h_2}{\partial z}+\frac{\hat{u}_2}{h_1h_2}\frac{\partial h_2}{\partial x}\right)\left(\frac{i\left(k_3\beta_0\right)\hat{w}_3}{h_3}+\frac{\hat{u}_3}{h_1h_3}\frac{\partial h_3}{\partial x}+\frac{\hat{v}_3}{h_2h_3}\frac{\partial h_3}{\partial y}\right)\right]\right\}\\
&-\frac{\left(\gamma-1\right)M^2}{Re}\frac{\partial \overline{\mu}}{\partial T}\hat{T}_1\left[\left(2+\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{1}{h_1}\frac{\partial \hat{u}_2}{\partial x}\right)\left(\frac{1}{h_1}\frac{\partial \hat{u}_3}{\partial x}\right)\right.\\
&\left.+\left(\frac{1}{h_1}\frac{\partial \hat{v}_2}{\partial x}\right)\left(\frac{1}{h_1}\frac{\partial \hat{v}_3}{\partial x}\right)+\left(\frac{1}{h_1}\frac{\partial \hat{w}_2}{\partial x}\right)\left(\frac{1}{h_1}\frac{\partial \hat{w}_3}{\partial x}\right)\right]
\end{aligned}
$$

Continuity Cubic Nonlinear

$$NL_m^{cubic}=0$$