ON STRONG ELLIPTICITY AND MONOTONICITY FOR IMPLICIT AND STRAIN-LIMITING THEORIES OF ELASTICITY

A Dissertation

by

NA TI MAI

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Chair of Committee,	Jay R. Walton
Committee Members,	Andrea Bonito
	Yalchin Efendiev
	Kumbakonam Rajagopal
Head of Department,	Emil Straube

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ABSTRACT

There has been increasing interest in the archival literature devoted to the study of implicit constitutive theories for non-dissipative materials generalizing the classical Green and Cauchy notions of elasticity, and for the special case of strain limiting models for which strains remain bounded, even infinitesimal, while stresses can become arbitrarily large. The first main part of this dissertation addresses the question of *strong ellipticity* for several classes of these models. A general approach for studying *strong ellipticity* for implicit theories is introduced and it is noted that there is a close connection between the questions of *strong ellipticity* and the existence of an equivalent Cauchy elastic formulation. For most of the models studied to date, it is shown that *strong ellipticity* holds if the Green-St.Venant strain is small enough, whereas it fails to hold for large strain. The large strain failure of *strong ellipticity* is generally associated with extreme compression.

Note that in the first main part of this dissertation, we study *strong ellipticity* for explicit strain-limiting theories of elasticity where the Green-St.Venant strain tensor is defined as a nonlinear response function of the second Piola-Kirchhoff stress tensor. The approach to *strong ellipticity* studied in the first main part of this dissertation requires that the Fréchet derivative of the response function be invertible as a fourth-order tensor. In the second main part of this dissertation, a weaker convexity notion is introduced in the case that the Fréchet derivative of the response function either fails to exist or is not invertible. We generalize the classical notion of *monotonicity* to a class of nonlinear strain-limiting models. It is shown that the generalized *monotonicity* holds for sufficiently small Green-St.Venant strains and fails (through demonstration by counterexample) when the small strain constraint is relaxed.

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I wish to thank everyone, everywhere, and everything!

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1. INTRODUCTION

At the beginning of the lectures on ‘‘Calculus of Variations and Applications to Elasticity’’, Wolfgang Reichel introduced convexity in the following way: *‘‘Convexity is truly a beautiful lady -- but do you also know her beautiful sisters?’’*

In this spirit, at Texas A&M Mathematical Physics, Harmonic Analysis, and Differential Equations Seminar on October 24, 2014, I introduced: *‘‘Convexity is a sweet lady -- but do you also know her sweet sisters?’’*

Answer by implying chain: convexity $\Rightarrow$ polyconvexity $\Rightarrow$ quasiconvexity $\Rightarrow$ rank-one convexity (or *strong ellipticity*) $\Rightarrow$ *monotonicity*. Over the past one and a half years, in our work [11], [12], the last two sisters (*strong ellipticity* and *monotonicity*) have made friends with Implicit and Strain-Limiting Theories of Elasticity.

A simple view of stress and strain is that stress ($\bar{\boldsymbol{S}}$ or $\boldsymbol{\sigma}$) as force per unit area causes strain ($\boldsymbol{E}$ or $\boldsymbol{\epsilon}$) as deformation per unit of the original length (Figure). An interpretation of *strong ellipticity* and *monotonicity* is that an increase in a component of strain ($\boldsymbol{E}$ or $\boldsymbol{\epsilon}$) should be accompanied by an increase in a corresponding component of stress ($\bar{\boldsymbol{S}}$ or $\boldsymbol{\sigma}$).

In classical elasticity, stress is a function of strain. In new theories of elasticity [15], stress and strain are in implicit relation $\mathbf{0} = \mathcal{F}(\boldsymbol{E}, \bar{\boldsymbol{S}})$, and in strain-limiting relation $\boldsymbol{E} = \mathcal{F}(\bar{\boldsymbol{S}})$, where $\mathcal{F}(\cdot)$ uniformly bounded in norm. In the context of Implicit and Strain-Limiting Theories of Elasticity, it was shown in our work [11] and [12] that *strong ellipticity* and *monotonicity* hold for sufficiently small strains and fail when the small strain constraint is relaxed. These results play key roles for stability of numerical approximations and for modeling waves.

In nonlinear elasticity, we adopt the *strong ellipticity condition* as our basic constitutive hypothesis, without endorsing it as the single way to mechanical truth [1].

All the main results (except Appendix B) in this dissertation have been presented in published papers, and at conferences and seminars, initiating and motivating from several collaborations developed during the Ph.D. studies.

In Chapter 2, we study *strong ellipticity* for implicit and strain-limiting theories of elasticity.

In Chapter 3, we investigate *monotonicity* for strain-limiting theories of elasticity.

Chapter 4 contains conclusions for the whole dissertation.

In Appendix A, we will have some comments on hyperelasticity of the main class of models in Chapter 3.

In Appendix B, we will examine the dynamical significance of the *strong ellipticity condition*.

In each Chapter and Appendix, we will introduce specific notation and preliminaries.

2. STRONG ELLIPTICITY FOR IMPLICIT AND STRAIN-LIMITING THEORIES OF ELASTICITY*

This Chapter introduces Strong Ellipticity for Implicit and Strain-Limiting Theories of Elasticity [12].

2.1 Motivation

In a trio of thought provoking papers [15, 18, 17], K. R. Rajagopal urges educators and researchers in mathematics and mechanics to broaden their traditional constructions for specifying the constitutive class of material bodies exhibiting elastic-like (non-dissipative) behavior to the setting of implicit constitutive theories. Among the compelling arguments offered by Rajagopal for use of implicit theories in elasticity, he emphasizes that they provide a framework for constructing logically consistent models for elastic-like material behavior that are both nonlinear and valid for infinitesimal strains. This feature is in marked contrast to the traditional Cauchy and Green approaches to modeling elasticity for which the derivation of logically consistent infinitesimal strain theories necessarily leads to linear models. Rajagopal and a number of co-authors have investigated a variety of implicit constitutive models of elastic bodies theoretically and numerically [2, 3, 4, 5, 14, 16, 19], and applied these to fracture [10, 20] and to coupled field applications [5, 6]. However, in none of these works has the issue of convexity for the various models been discussed.

The intention of this Chapter is to begin to address convexity for implicit elastic constitutive relations by considering *strong ellipticity* or rank-one convexity. While *strong ellipticity* is weaker than quasi-convexity (and hence also the stronger condition of poly-

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convexity) and is insufficient to prove existence of solutions to finite elastic boundary value problems, it is a key condition for stability of numerical approximations and for physically intuitive wave propagation behavior. In particular, *strong ellipticity* condition guarantees the hyperbolicity of an equilibrium dynamical system of partial differential equations, which means that the system admits the full range of wave-like behavior (see Appendix B). The traditional definition of *strong ellipticity* is set within the context of Cauchy and Green elasticity [1].

While many commonly used Cauchy or Green elastic models satisfy *strong ellipticity* for all allowable deformations and some fail to be strongly elliptic for any deformation, it is more typical that for a given constitutive model, one seeks to identify the class of deformations for which *strong ellipticity* holds and gives counterexamples for deformations for which it fails [9, 22, 23, 21]. In this spirit, after presenting an approach for studying *strong ellipticity* for general elastic implicit theories, we focus attention upon various classes of strain-limiting models. For all of the strain-limiting models considered, we show that *strong ellipticity* holds for deformations for which an appropriate strain measure is small enough in norm. Conversely, for these models we demonstrate that *strong ellipticity* fails when the small strain assumption is relaxed.

2.2 Notation and Preliminaries

Consider a deformation $\boldsymbol{f}(\cdot):\quad \mathcal{B} \longrightarrow \boldsymbol{f}(\mathcal{B})$ of a material body occupying configurations $\mathcal{B}$ and $\boldsymbol{f}(\mathcal{B})$ before and after deformation, respectively. Let $\boldsymbol{u}$ and $\boldsymbol{F}$ denote the displacement and deformation gradient through

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}\,, \tag{2.1}$$

and

$$\boldsymbol{F} := \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{X}}\,. \tag{2.2}$$

The left and right Cauchy-Green tensors $\boldsymbol{B}$ and $\boldsymbol{C}$ are defined by

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{T} \tag{2.3}$$

and

$$\boldsymbol{C} = \boldsymbol{F}^{T}\boldsymbol{F}\,, \tag{2.4}$$

respectively, and the Green-St.Venant tensor $\boldsymbol{E}$ is defined by

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}). \tag{2.5}$$

Let $\boldsymbol{T}$ denote the *Cauchy Stress Tensor*. Then the first and second *Piola-Kirchhoff Stress Tensors*, $\boldsymbol{S}$ and $\bar{\boldsymbol{S}}$, respectively, are defined by

$$\boldsymbol{S} := \boldsymbol{T}\boldsymbol{F}^{-\mathrm{T}}\det(\boldsymbol{F}), \qquad \bar{\boldsymbol{S}} := \boldsymbol{F}^{-1}\boldsymbol{S}. \tag{2.6}$$

A material body is said to be *Cauchy Elastic* if its constitutive class is determined by a response function of the form:

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{F}). \tag{2.7}$$

It is said to be *Green Elastic* (or equivalently *Hyperelastic*) if the stress response function is the gradient of a scalar valued potential

$$\hat{\boldsymbol{S}}(\boldsymbol{F}) = \partial_{\boldsymbol{F}}\hat{w}(\boldsymbol{F}). \tag{2.8}$$

In [15], Rajagopal considered implicit constitutive relations of the form:

$$\boldsymbol{0} = \mathcal{F}(\boldsymbol{B}, \boldsymbol{T}) \tag{2.9}$$

between the Cauchy stress and left Cauchy-Green tensors. He also considered the special case:

$$\boldsymbol{B} = \mathcal{F}(\boldsymbol{T}), \tag{2.10}$$

with special attention given to *strain limiting* theories for which the constitutive function

$\mathcal{F}(\cdot)$ in (2.10) is assumed to be uniformly bounded in norm. As Rajagopal notes, the form (2.10) provides a convenient framework for deriving non-linear, infinitesimal strain theories of the form:

$$\boldsymbol{\epsilon} = \mathcal{F}(\boldsymbol{T}) \tag{2.11}$$

in which $\boldsymbol{\epsilon}$ denotes the customary linearized strain tensor

$$\boldsymbol{\epsilon} := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}\right). \tag{2.12}$$

Also, as Rajagopal points out, a rich array of behaviors can be explored by considering the class of strain-limiting constitutive models of the form:

$$\mathcal{F}(\boldsymbol{T}) = \phi_0(\boldsymbol{T})\boldsymbol{I} + \phi_1(\boldsymbol{T})\boldsymbol{T} + \phi_2(\boldsymbol{T})\boldsymbol{T}^2, \tag{2.13}$$

in which $\boldsymbol{I}$ denotes the second-order identity tensor and each $\phi_j(\boldsymbol{T})$ is a scalar valued function with $|\phi_0(\boldsymbol{T})|$, $|\phi_1(\boldsymbol{T})||\boldsymbol{T}|$, and $|\phi_2(\boldsymbol{T})||\boldsymbol{T}^2|$ all uniformly bounded functions on the space of symmetric, second-order tensors.

Recall that isotropy is uniformity in all directions. Examples of isotropic materials are glass and metals. Recall also that anisotropy is the property of being directionally dependent. Examples of anisotropic materials are wood, laminated composites, and single crystal; they are stiffer when loaded along some material directions than others.

With an eye towards studying *strong ellipticity* for both isotropic and anisotropic implicit theories, we adopt the following constitutive relations as alternatives to (2.9) and (2.10)

$$\mathbf{0} = \mathcal{F}(\boldsymbol{E}, \bar{\boldsymbol{S}}) \tag{2.14}$$

$$\boldsymbol{E} = \mathcal{F}(\bar{\boldsymbol{S}}). \tag{2.15}$$

The present study focuses on the strain-limiting special case of (2.15) for which the response function $\mathcal{F}(\cdot)$ is uniformly bounded in norm. The analogue to (2.13) for the model (2.15) is given by:

$$\mathcal{F}(\bar{\boldsymbol{S}}) = \phi_0(\bar{\boldsymbol{S}})\boldsymbol{I} + \phi_1(\bar{\boldsymbol{S}})\bar{\boldsymbol{S}} + \phi_2(\bar{\boldsymbol{S}})\bar{\boldsymbol{S}}^2 \,. \tag{2.16}$$

One of the classes of models studied in this Chapter (see (2.25), (2.109) and (2.110) below), and that was motivated by examples considered by Rajagopal, has the form (2.16) with $\phi_2(\cdot) = 0$, $\phi_0(\bar{\boldsymbol{S}}) = \hat{\phi}_0(\bar{\boldsymbol{S}} \cdot \boldsymbol{I})$ and $\phi_1(\bar{\boldsymbol{S}}) = \hat{\phi}_1(|\bar{\boldsymbol{S}}|)$, and for which $\hat{\phi}_0(r)$ is a bounded, increasing function for $-\infty < r < \infty$ that vanishes for $r = 0$ and $r\hat{\phi}_1(r)$ is a bounded, increasing function for $0 < r$. These assumptions on the functions are quite natural since, for example, they give models satisfying intuitive *monotonicity* properties such as the Baker-Ericksen inequalities. If these *monotonicity* constraints on $\phi_0(\cdot)$ and $\phi_1(\cdot)$ are relaxed, one can readily construct models that, for example, predict volume increase under compressive stress.

The analysis below makes extensive use of the following notational conventions for performing tensoral manipulations. All tensor spaces are over the standard Euclidean vector space $\mathcal{E}^3$. First-order tensors are denoted by bold, lower-case letters, $\boldsymbol{a}$, the Euclidean inner-product two first-order tensors is denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and its accompanying norm is denoted $|\boldsymbol{a}| := \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$. Second-order tensors are denoted by bold, upper-case letters, $\boldsymbol{A}$, and the space of second-order tensors is equipped with the trace inner-product denoted by, $\boldsymbol{A} \cdot \boldsymbol{B} := \mathrm{tr}[\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}]$, where $\boldsymbol{A}^{\mathrm{T}}$ is the adjoint (transpose) of $\boldsymbol{A}$ viewed as a linear transformation on first-order tensors. The Frobenius norm is denoted by $|\boldsymbol{A}| := \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$, whereas the operator norm is denoted by $\|\boldsymbol{A}\| := \sup_{|\boldsymbol{a}|=1} |\boldsymbol{A}\boldsymbol{a}|$ with $\boldsymbol{A}\boldsymbol{a}$ denoting the action of the second-order tensor $\boldsymbol{A}$ on the first-order tensor $\boldsymbol{a}$. For first-order tensors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a} \otimes \boldsymbol{b}$ denotes the rank-one, second-order tensor defined by: $\boldsymbol{a} \otimes \boldsymbol{b}\,\boldsymbol{c} := \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c})$ for all first-order tensors $\boldsymbol{c}$. Similarly, given two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \otimes \boldsymbol{B}$ denotes the fourth-order, rank-1 tensor (viewed as a linear transformation on the space of second-order tensors) defined by: $\boldsymbol{A} \otimes \boldsymbol{B}\,\boldsymbol{C} := \boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})$ for all second-order tensors $\boldsymbol{C}$.

Definition of Fréchet derivative [7]: Let X, Y be Banach spaces. Let A be an open subset of X, and let F be an operator mapping A to Y. The *Fréchet derivative* of F at $a \in A$ is the bounded linear operator $\mathrm{D}F(a) : X \to Y$ which satisfies the following relation

$$\lim_{\|h\| \to 0} \frac{\|F(a+h) - F(a) - \mathrm{D}F(a)h\|}{\|h\|} = 0\,. \tag{2.17}$$

The limit being required to exist as $h \to 0$ in any manner.

Remark 1. The Fréchet derivative is a derivative defined on Banach spaces. It is used to generalize the derivative of a real-valued function of a single real variable to the case of a vector-valued function of multiple real variables, and to define the functional derivative used widely in the calculus of variations. Commonly, it is used to generalize the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ to the *Jacobian* of a function $f : \mathbb{R}^n \to \mathbb{R}^m$.

2.3 Strong Ellipticity - Definitions and General Observations

The classical notion of *strong ellipticity* in finite elasticity is typically formulated in the setting of Cauchy and Green elasticity by the requirement that for a given deformation gradient, $\boldsymbol{F}$, the following inequality holds [1]:

$$\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] > 0 \tag{2.18}$$

for all non-zero, rank-one tensors $\boldsymbol{H} := \boldsymbol{a} \otimes \boldsymbol{b}$. In (2.18), $\mathrm{D}_{\boldsymbol{F}}$ denotes Fréchet differentiation with $\mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}]$ denoting the fourth-order tensor $\mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\cdot]$ acting on the second-order, rank-one tensor $\boldsymbol{H}$. For later convenience, we assume $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$.

When the strain-energy function $\hat{w}(\cdot)$ in (2.8) is twice continuously differentiable, the *Strong Ellipticity condition* (2.18) reduces to the *Strong Legendre-Hadamard condition* for the *Euler-Lagrange system of equations* in Calculus of Variations.

The constitutive relation (2.14) can be viewed as defining a six-dimensional manifold in twelve-dimensional Euclidean space. In particular, the graph of (2.14) consists of a collection of surface elements defining branches on which $\bar{\boldsymbol{S}}$ is given as a function of $\boldsymbol{E}$. Appealing to the classical implicit function theorem then allows one to derive local (to a given branch of the graph of (2.14)) expressions for $\mathrm{D}_{\boldsymbol{F}}\hat{\boldsymbol{S}}(\boldsymbol{F})[\cdot]$ through implicit Fréchet differentiation of (2.14).

As a first step in extracting expressions for $\mathrm{D}_{\boldsymbol{F}}\hat{\boldsymbol{S}}(\boldsymbol{F})[\cdot]$ from (2.14), implicit Fréchet differentiation of (2.14) gives

$$\mathbf{0} = \mathrm{D}_{\boldsymbol{E}}\mathcal{F}(\bar{\boldsymbol{S}},\boldsymbol{E})[\boldsymbol{H}_s] = \partial_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}},\boldsymbol{E})\left[\mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}(\boldsymbol{E})[\boldsymbol{H}_s]\right] + \partial_{\boldsymbol{E}}\mathcal{F}(\bar{\boldsymbol{S}},\boldsymbol{E})[\boldsymbol{H}_s] \tag{2.19}$$

in which $\boldsymbol{H}_s$ denotes the symmetric part of $\boldsymbol{H}$, $\boldsymbol{H}_s := (\boldsymbol{H}+\boldsymbol{H}^{\mathrm{T}})/2$. If the fourth-order tensor $\partial_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}},\boldsymbol{E})[\cdot]$ is invertible (on a suitably restricted subset of second-order tensors), one can derive from (2.19) the following expression for $\mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}(\boldsymbol{E})[\boldsymbol{H}_s]$:

$$\mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}(\boldsymbol{E})[\boldsymbol{H}_s] = -(\partial_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}},\boldsymbol{E}))^{-1}\left[\partial_{\boldsymbol{E}}\mathcal{F}(\bar{\boldsymbol{S}},\boldsymbol{E})[\boldsymbol{H}_s]\right]. \tag{2.20}$$

We note that a straightforward calculation gives:

$$\mathrm{D}_{\boldsymbol{F}}\bar{\boldsymbol{S}}(\boldsymbol{E})[\boldsymbol{H}] = \mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}(\boldsymbol{E})\left[\mathrm{D}_{\boldsymbol{F}}\boldsymbol{E}[\boldsymbol{H}]\right] = \mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}(\boldsymbol{E})\left[(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s\right]. \tag{2.21}$$

Combining (2.20) and (2.21), one can derive the following notion of *strong ellipticity* for the implicit constitutive relation (2.14):

$$\begin{aligned}
\boldsymbol{H}\cdot\mathrm{D}_{\boldsymbol{F}}\hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] &= \boldsymbol{H}\cdot\left(\mathrm{D}_{\boldsymbol{F}}\left(\boldsymbol{F}\bar{\boldsymbol{S}}(\boldsymbol{E})\right)[\boldsymbol{H}]\right)\\
&= \boldsymbol{H}\cdot\left(\boldsymbol{H}\bar{\boldsymbol{S}}(\boldsymbol{E}) + \boldsymbol{F}\mathrm{D}_{\boldsymbol{F}}\bar{\boldsymbol{S}}(\boldsymbol{E})[\boldsymbol{H}]\right)\\
&= \boldsymbol{H}\cdot\left(\boldsymbol{H}\bar{\boldsymbol{S}}(\boldsymbol{E})\right) + \left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\cdot\mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}(\boldsymbol{E})\left[\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right]\\
&= \boldsymbol{H}\cdot\left(\boldsymbol{H}\bar{\boldsymbol{S}}(\boldsymbol{E})\right)\\
&\quad - \left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\cdot\left(\partial_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}},\boldsymbol{E})\right)^{-1}\left[\partial_{\boldsymbol{E}}\mathcal{F}(\bar{\boldsymbol{S}},\boldsymbol{E})\left[(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s\right]\right]\\
&> 0\,.
\end{aligned} \tag{2.22}$$

Applying (2.22) to the special case (2.15) gives the following requirement for *strong ellipticity*:

$$\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] = \boldsymbol{H} \cdot (\boldsymbol{H} \bar{\boldsymbol{S}}(\boldsymbol{E})) \\ - \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \cdot \left(\mathrm{D}_{\bar{\boldsymbol{S}}} \mathcal{F}(\bar{\boldsymbol{S}})\right)^{-1} \left[(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H})_s\right] > 0, \tag{2.23}$$

where $\left(\mathrm{D}_{\bar{\boldsymbol{S}}} \mathcal{F}(\bar{\boldsymbol{S}})\right)^{-1}[\cdot]$ denotes the inverse of the fourth-order tensor $\mathrm{D}_{\bar{\boldsymbol{S}}} \mathcal{F}(\bar{\boldsymbol{S}})[\cdot]$.

Remark 2. For a Cauchy elastic body whose First Piola-Kirchhoff response function $\hat{\boldsymbol{S}}(\cdot)$ is not differentiable at a deformation gradient $\boldsymbol{F}$, a weaker rank-1 convexity notion is provided by the *monotonicity* condition [1]:

$$(\hat{\boldsymbol{S}}(\boldsymbol{F} + \alpha \boldsymbol{H}) - \hat{\boldsymbol{S}}(\boldsymbol{F})) \cdot \boldsymbol{H} > 0 \tag{2.24}$$

for all norm-1, rank-1 tensors $\boldsymbol{H} = \boldsymbol{a} \otimes \boldsymbol{b}$ (with $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$) and $0 < \alpha \leq 1$. When the response function $\hat{\boldsymbol{S}}(\cdot)$ is differentiable, (2.24) implies a weaker form of (2.18) with ‘‘$>$’’ replaced by ‘‘$\geq$’’. The *monotonicity* condition (2.24) for Cauchy elastic bodies can be readily generalized to the implicit constitutive relation setting (2.14) by applying it locally on the graph of (2.14). More specifically, (2.24) is applied to a ‘‘local’’ First Piola-Kirchhoff response function $\hat{\boldsymbol{S}}(\boldsymbol{F}) := \boldsymbol{F} \bar{\boldsymbol{S}}(\boldsymbol{E})$, where the Second Piola-Kirchhoff response function $\bar{\boldsymbol{S}}(\cdot)$ is defined locally on a branch of the graph of (2.14). Studying this generalization of the *monotonicity* condition (2.24) to implicit constitutive relations of the form (2.14) will be addressed in Chapter 3.

The following subsections consider various classes of strain-limiting constitutive models and present results that guarantee *strong ellipticity* (typically if the tensor $\boldsymbol{E}$ is sufficiently small in norm). Subsequently, for the same classes of models, counterexamples to *strong ellipticity* are constructed (typically for deformations corresponding to sufficiently severe compression).

2.4 Invertibility and Strong Ellipticity

In this section, conditions guaranteeing *strong ellipticity* are derived for two classes of strain limiting models generalizing specific examples considered in the literature. More specifically, we study an isotropic model of the form:

$$\boldsymbol{E} = \mathcal{F}(\bar{\boldsymbol{S}}) := \phi_0\left(\bar{\boldsymbol{S}} \cdot \boldsymbol{I}\right)\boldsymbol{I} + \phi_1\left(\left|\bar{\boldsymbol{S}}\right|\right)\bar{\boldsymbol{S}}; \tag{2.25}$$

and an anisotropic model given by:

$$\boldsymbol{E} = \phi_1(|\mathbb{K}^{1/2}[\bar{\boldsymbol{S}}]|)\mathbb{K}[\bar{\boldsymbol{S}}]. \tag{2.26}$$

The constitutive functions $\phi_0(r)$ and $r\phi_1(r)$ are uniformly bounded and increasing while $\phi_1(r)$ is a decreasing function. Also, $\phi_1(r) > 0$, while $\phi_0(r) > 0$ if $r > 0$, and $\phi_0(r) < 0$ if $r < 0$. In (2.26), $\mathbb{K}^{1/2}[\cdot]$ denotes the positive-definite, symmetric (as a linear transformation on $\mathcal{S}$ym, the space of symmetric, second-order tensors) square-root (under composition) of the classical linear elastic compliance tensor $\mathbb{K}[\cdot]$. Thus, one has $\mathbb{K}^{1/2}\left[\mathbb{K}^{1/2}[\boldsymbol{S}]\right] = \mathbb{K}[\boldsymbol{S}]$ for all $\boldsymbol{S} \in \mathcal{S}$ym.

2.4.1 Invertibility

Prior to investigating *strong ellipticity*, we address the question of invertibility of the constitutive relations (2.25) and (2.26), with the first consideration given to (2.25).

One approach to studying the invertibility of the second-order tensoral equation (2.25), and many more general models, appeals to classical spectral theory for symmetric, second-order tensors. More specifically, since $\bar{\boldsymbol{S}}$ is symmetric, by the spectral theorem there exists an orthonormal basis, denoted by $\{\boldsymbol{a}_i\}_{i=1}^3$, for the space of first order tensors consisting entirely of eigenvectors of $\bar{\boldsymbol{S}}$. It is then obvious from (2.25) that the symmetric tensor $\boldsymbol{E}$ has the same eigen-basis as $\bar{\boldsymbol{S}}$. Let $\{\lambda_i\}_{i=1}^3$ and $\{\gamma_i\}_{i=1}^3$ denote the corresponding eigen-values of $\bar{\boldsymbol{S}}$, and $\boldsymbol{E}$, respectively. Then, we have the following spectral representations:

$$\bar{\boldsymbol{S}} = \sum_{i=1}^{3} \lambda_i \boldsymbol{a}_i \otimes \boldsymbol{a}_i, \quad \boldsymbol{E} = \sum_{i=1}^{3} \gamma_i \boldsymbol{a}_i \otimes \boldsymbol{a}_i, \quad \boldsymbol{I} = \sum_{i=1}^{3} \boldsymbol{a}_i \otimes \boldsymbol{a}_i. \tag{2.27}$$

To simplify notation in the following derivations, we introduce three first-order tensors:

$$\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \gamma_3)^{\mathrm{T}}, \quad \boldsymbol{\lambda} := (\lambda_1, \lambda_2, \lambda_3)^{\mathrm{T}}, \quad \mathbf{1} := (1, 1, 1)^{\mathrm{T}}$$

and for any first order tensor $\boldsymbol{v}$, let $|\boldsymbol{v}| := \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$. Substitution of (2.27) into (2.25) gives

$$\sum_{i=1}^{3} \gamma_i \boldsymbol{a}_i \otimes \boldsymbol{a}_i = \phi_0 \left(\boldsymbol{\lambda} \cdot \mathbf{1} \right) \left(\sum_{i=1}^{3} \boldsymbol{a}_i \otimes \boldsymbol{a}_i \right) + \phi_1 \left(|\boldsymbol{\lambda}| \right) \left(\sum_{i=1}^{3} \lambda_i \boldsymbol{a}_i \otimes \boldsymbol{a}_i \right) \tag{2.28}$$

and hence the equivalent 3×3 system of algebraic equations

$$\gamma_j = g\left(\lambda_j\right) := \phi_0 \left(\boldsymbol{\lambda} \cdot \mathbf{1} \right) + \lambda_j \phi_1 \left(|\boldsymbol{\lambda}| \right), \quad j = 1, 2, 3. \tag{2.29}$$

Thus, inverting the tensoral equation (2.25) is equivalent to solving the algebraic system (2.29) for the eigen-values of $\bar{\boldsymbol{S}}$, $\{\lambda_i\}_{i=1}^{3}$, in terms of the eigen-values of $\boldsymbol{E}$, $\{\gamma_i\}_{i=1}^{3}$.

A second approach to studying the invertibility of the constitutive model (2.25), as well as other more general models, is based upon the decomposition of a second-order, symmetric tensor into its deviatoric (trace-less) and trace parts. Specifically, one can decompose the symmetric tensor $\bar{\boldsymbol{S}}$ as the sum

$$\bar{\boldsymbol{S}} = \bar{\boldsymbol{S}}_0 + \bar{\sigma} \boldsymbol{I} \tag{2.30}$$

with

$$\bar{\sigma} := \frac{1}{3}(\bar{\boldsymbol{S}} \cdot \boldsymbol{I}), \qquad \bar{\boldsymbol{S}}_0 := \bar{\boldsymbol{S}} - \bar{\sigma} \boldsymbol{I}. \tag{2.31}$$

Since $\bar{\boldsymbol{S}}_0 \cdot \boldsymbol{I} = 0$, (2.30) gives an orthogonal decomposition for $\bar{\boldsymbol{S}}$ in the space of symmetric, second-order tensors equipped with the trace inner product. Decomposing $\boldsymbol{E}$ in similar fashion to (2.30) gives

$$\boldsymbol{E} = \boldsymbol{E}_0 + \bar{\gamma} \boldsymbol{I} \tag{2.32}$$

with $\bar{\gamma} := \frac{1}{3}(\boldsymbol{E} \cdot \boldsymbol{I})$ and $\boldsymbol{E}_0 := \boldsymbol{E} - \bar{\gamma}\boldsymbol{I}$, one sees that inverting the second-order tensoral equation (2.25) is equivalent to solving the 2×2 algebraic system:

$$|\boldsymbol{E}_0| = \phi_1(r)|\bar{\boldsymbol{S}}_0| \tag{2.33}$$

$$\bar{\gamma} = \phi_0(3\bar{\sigma}) + \phi_1(r)\bar{\sigma} \tag{2.34}$$

with $r := \sqrt{3\bar{\sigma}^2 + |\bar{\boldsymbol{S}}_0|^2}$.

Discussing the solvability of the algebraic system (2.33), (2.34) is facilitated by introducing the simplifying notation: $x := \bar{\sigma}$, $y := |\bar{\boldsymbol{S}}_0|$. By assumption: $\phi_1'(r) < 0$ and $\phi_1(r) + r\phi_1'(r) > 0$. The system (2.33), (2.34) now takes the form:

$$|\boldsymbol{E}_0| = \psi_1(y; x) := y\phi_1(\sqrt{3x^2 + y^2}) \tag{2.35}$$

$$\bar{\gamma} = \phi_0(3x) + x\phi_1(\sqrt{3x^2 + y^2}). \tag{2.36}$$

To show that the system (2.35), (2.36) is uniquely solvable for given $|\boldsymbol{E}_0|$ and $\bar{\gamma}$ (suitably restricted so that $\boldsymbol{E}$ is in the compact range of the constitutive response function in (2.25)), we first note that for a given x, the right-hand-side of (2.35) is an increasing function of $y > 0$ as follows from:

$$\begin{aligned}
\partial_y \psi_1(y; x) &= \phi_1(\sqrt{3x^2 + y^2}) + y^2 \frac{\phi_1'(\sqrt{3x^2 + y^2})}{\sqrt{3x^2 + y^2}} \\
&= \phi_1(r) + r\phi_1'(r) - \frac{3x^2}{r}\phi_1'(r) > 0.
\end{aligned}$$

Solving (2.35) for y gives:

$$y = \psi_1^{-1}(|\boldsymbol{E}_0|; x). \tag{2.37}$$

Substitution of (2.37) into (2.36) now gives the single equation for x:

$$\bar{\gamma} = \phi_0(3x) + x\phi_1(\sqrt{3x^2 + y^2}) \tag{2.38}$$

with y given by (2.37). To see that (2.38) is uniquely solvable for x, it suffices to show that the right-hand-side is an increasing function of x. By assumption, $\phi_0(3x)$ is increasing. So it suffices to show that $\psi_2(x) := x\phi_1(\sqrt{3x^2+y^2})$ is also increasing. To that end, a straightforward calculation gives:

$$\psi_2'(x) = \frac{r\phi_1(r)\left(\phi_1(r) + r\phi_1'(r)\right)}{r\phi_1(r) + y^2\phi_1'(r)} > 0,$$

where use has been made of the formula:

$$\frac{d}{dx}\psi_1^{-1}(|\boldsymbol{E}_0|; x) = \frac{-3x|y|\phi_1'(r)}{r\phi_1(r) + y^2\phi_1'(r)},$$

which follows from implicit partial-differentiation of (2.35) with respect to x. Indeed, from (2.35) and (2.37), we have

$$|\boldsymbol{E}_0| = \psi_1(\psi_1^{-1}(|\boldsymbol{E}_0|; x); x). \tag{2.39}$$

Differentiating both sides of (2.39) gives

$$0 = \psi_1'(y; x)\frac{d}{dx}\psi_1^{-1}(|\boldsymbol{E}_0|; x) + \partial_x\psi_1(\psi_1^{-1}(|\boldsymbol{E}_0|; x); x)\,. \tag{2.40}$$

Thus, by implicit partial-differentiation of (2.35) with respect to x, we get

$$\begin{aligned} y_x' = \frac{d}{dx}\psi_1^{-1}(|\boldsymbol{E}_0|; x) &= \frac{-\partial_x\left(y\phi_1\left(\sqrt{3x^2+y^2}\right)\right)}{\psi_1'\left(y\phi_1\left(\sqrt{3x^2+y^2}\right)\right)} \\ &= \frac{\dfrac{-3xy\phi_1'(r)}{r}}{\phi_1(r) + \dfrac{\phi_1'(r)y^2}{r}} \\ &= \frac{-3xy\phi_1'(r)}{r\phi_1(r) + \phi_1'(r)y^2}\,. \end{aligned} \tag{2.41}$$

Recall that $\psi_2(x) := x\phi_1\left(\sqrt{3x^2+y^2}\right)$, so

$$\psi_2'(x) = \phi_1(r) + x\phi_1'(r)\left(\frac{6x + 2yy_x'}{2r}\right)$$

$$
\begin{aligned}
&= \phi_1(r) + x\phi_1'(r)\left(\frac{3x + \dfrac{-3xy^2\phi_1'(r)}{r\phi_1(r)+\phi_1'(r)y^2}}{r}\right) \\
&= \phi_1(r) + x\phi_1'(r)\left(\frac{3xr\phi_1(r)}{r(r\phi_1(r)+y^2\phi_1'(r))}\right) \\
&= \phi_1(r) + \frac{3x^2\phi_1'(r)\phi_1(r)}{r\phi_1(r)+y^2\phi_1'(r)} \\
&= \frac{r\phi_1^2(r) + y^2\phi_1(r)\phi_1'(r) + 3x^2\phi_1'(r)\phi_1(r)}{r\phi_1(r)+y^2\phi_1'(r)} \\
&= \frac{r\phi_1^2(r) + r^2\phi_1(r)\phi_1'(r)}{r\phi_1(r)+y^2\phi_1'(r)} \\
&= \frac{r\phi_1(r)(\phi_1(r)+r\phi_1'(r))}{r\phi_1(r)+y^2\phi_1'(r)}\,. \qquad (2.42)
\end{aligned}
$$

Note that $r > 0, \phi_1(r) > 0, (r\phi_1(r))' = \phi_1(r) + r\phi_1'(r) > 0, \phi_1'(r) < 0$, and $r^2 = \left(\sqrt{3x^2+y^2}\right)^2 > y^2$. Therefore, in (2.42), the numerator

$$
r\phi_1(r)(\phi_1(r)+r\phi_1'(r)) > 0,
$$

and the denominator

$$
r\phi_1(r) + y^2\phi_1'(r) > r\phi_1(r) + r^2\phi_1'(r) = r(\phi_1(r)+r\phi_1'(r)) > 0.
$$

Hence, $\psi_2'(x) > 0$.

Thus, one sees that the constitutive relation (2.25) is uniquely invertible provided $\boldsymbol{E}$ lies within the compact range of the function on the right hand side of (2.25).

Proving invertibility for (2.26) is a simpler task than for (2.25). It is useful to introduce the linear elasticity tensor $\mathbb{E}[\cdot]$ inverse to the compliance tensor $\mathbb{K}[\cdot]$. Therefore, $\mathbb{E}[\mathbb{K}[\boldsymbol{S}]] = \boldsymbol{S}$ for all second order tensors $\boldsymbol{S} \in \mathcal{S}\text{ym}$. The constitutive relation (2.26) is readily inverted by first applying the operator $\mathbb{E}^{1/2}[\cdot]$ to both sides followed by taking the Frobenius norm giving:

$$|\mathbb{E}^{1/2}[\boldsymbol{E}]| = \psi_1(|\mathbb{K}^{1/2}[\bar{\boldsymbol{S}}]|) := \phi_1(|\mathbb{K}^{1/2}[\bar{\boldsymbol{S}}]|)|\mathbb{K}^{1/2}[\bar{\boldsymbol{S}}]|. \tag{2.43}$$

By assumption, $\psi_1(r) := r\phi_1(r)$ is an increasing function so that (2.43) can be solved to give

$$|\mathbb{K}^{1/2}[\bar{\boldsymbol{S}}]| = \psi_1^{-1}(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)$$

which upon substitution into (2.26) followed by a second application of the operator $\mathbb{E}^{1/2}[\cdot]$ gives

$$\bar{\boldsymbol{S}} = \tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)\mathbb{E}[\boldsymbol{E}], \tag{2.44}$$

where

$$\tilde{\phi}_1(r) := \frac{1}{\phi_1(\psi_1^{-1}(r))}. \tag{2.45}$$

These arguments show that both strain-limiting models (2.25) and (2.26) have equivalent Cauchy elastic formulations. In fact, as shown below, they are also Green elastic. This begs the question: What is gained from the use of the formulations (2.25) and (2.26)? One advantage of these formulations over the equivalent Cauchy formulations, as argued in [15, 18, 17], is that they lead to a logically consistent derivation of a nonlinear, infinitesimal strain theory. Another advantage derives from applications in which classical linearized elasticity predicts strain singularities, such as fracture, bodies with sharp re-entrant corners and punch problems with sharp-edged indenters. Studying such problems in the setting of strain-limiting models of the form (2.25) or (2.26), or their equivalent Cauchy elastic formulation, prevents singular strains, but leaves open the possibility of singular stresses. However, as argued in [20], studying the asymptotic behavior of stresses at crack edges or sharp re-entrant corners in such problems is much more convenient using the formulations (2.25), (2.26) rather than their equivalent Cauchy elastic forms.

2.4.1.1 Green Elasticity

Given the Cauchy elastic form (2.44), construct the strain energy potential:

$$\hat{w}(\boldsymbol{E}) := \tilde{w}(|\mathbb{E}^{1/2}[\boldsymbol{E}]|) \tag{2.46}$$

where

$$\tilde{w}(r) := \int r\tilde{\phi}_1(r)\, dr.$$

One can readily show that

$$\partial_{\boldsymbol{E}}\hat{w}(\boldsymbol{E}) = \partial_{\boldsymbol{E}}\tilde{w}(|\mathbb{E}^{1/2}[\boldsymbol{E}]|) = \tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)\mathbb{E}[\boldsymbol{E}] = \bar{\boldsymbol{S}}\,. \tag{2.47}$$

The proof of (2.47) is facilitated by appealing to the identity

$$|\mathbb{E}^{1/2}[\boldsymbol{E}]|^2 = \mathbb{E}[\boldsymbol{E}] \cdot \boldsymbol{E}, \tag{2.48}$$

and note that the Fréchet derivative of a continuous linear operator is the same operator. More specifically,

$$\begin{aligned}
\partial_{\boldsymbol{E}}\hat{w}(\boldsymbol{E}) \cdot \boldsymbol{H} &= \mathrm{D}_{\boldsymbol{E}}\tilde{w}(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)[\boldsymbol{H}] \\
&= |\mathbb{E}^{1/2}[\boldsymbol{E}]|\tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)[\mathrm{D}_{\boldsymbol{E}}\left((\mathbb{E}[\boldsymbol{E}] \cdot \boldsymbol{E})^{1/2}\right)[\boldsymbol{H}]] \\
&= \frac{1}{2}\tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)[\mathrm{D}_{\boldsymbol{E}}(\mathbb{E}[\boldsymbol{E}] \cdot \boldsymbol{E})[\boldsymbol{H}]] \\
&= \frac{1}{2}\tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)(\mathbb{E}[\boldsymbol{H}] \cdot \boldsymbol{E} + \mathbb{E}[\boldsymbol{E}] \cdot \boldsymbol{H}) \\
&= \frac{1}{2}\tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)(2\mathbb{E}[\boldsymbol{E}] \cdot \boldsymbol{H}) \\
&= \tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)\mathbb{E}[\boldsymbol{E}] \cdot \boldsymbol{H}\,.
\end{aligned} \tag{2.49}$$

A strain energy function for the model (2.25) is most easily derived by first constructing an associated complementary stress potential and then making use of the Legendre transform. More specifically, define the stress potential:

$$\hat{k}(\bar{\boldsymbol{S}}) := \tilde{k}_0(\bar{\boldsymbol{S}} \cdot \boldsymbol{I}) + \tilde{k}_1(|\bar{\boldsymbol{S}}|) \tag{2.50}$$

where

$$\tilde{k}_0(r) := \int \phi_0(r)\, dr, \qquad \tilde{k}_1(r) := \int r\phi_1(r)\, dr.$$

A direct calculation verifies that

$$\boldsymbol{E} = \partial_{\bar{\boldsymbol{S}}} \hat{k}(\bar{\boldsymbol{S}}) = \mathcal{F}(\bar{\boldsymbol{S}}) = \phi_0(\bar{\boldsymbol{S}} \cdot \boldsymbol{I})\boldsymbol{I} + \phi_1(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}} \tag{2.51}$$

as desired. Indeed,

$$\begin{aligned}
\boldsymbol{E} \cdot \boldsymbol{H} &= \partial_{\bar{\boldsymbol{S}}} \hat{k}(\bar{\boldsymbol{S}}) \cdot \boldsymbol{H} \\
&= (\partial_{\bar{\boldsymbol{S}}} \tilde{k}_0(\bar{\boldsymbol{S}} \cdot \boldsymbol{I})) \cdot \boldsymbol{H} + (\partial_{\bar{\boldsymbol{S}}} \tilde{k}_1(|\bar{\boldsymbol{S}}|)) \cdot \boldsymbol{H} \\
&= \phi_0(\bar{\boldsymbol{S}} \cdot \boldsymbol{I}) \mathrm{D}_{\bar{\boldsymbol{S}}}(\bar{\boldsymbol{S}} \cdot \boldsymbol{I})[\boldsymbol{H}] + \phi_1(|\bar{\boldsymbol{S}}|)|\bar{\boldsymbol{S}}| \mathrm{D}_{\bar{\boldsymbol{S}}}(|\bar{\boldsymbol{S}}|)[\boldsymbol{H}] \\
&= (\phi_0(\bar{\boldsymbol{S}} \cdot \boldsymbol{I})\boldsymbol{I} + \phi_1(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}}) \cdot \boldsymbol{H}\,.
\end{aligned} \tag{2.52}$$

Appealing to the classical Legendre transform, one defines the strain energy function

$$\hat{w}(\boldsymbol{E}) := -\hat{k}(\bar{\boldsymbol{S}}) + \bar{\boldsymbol{S}} \cdot \boldsymbol{E} \tag{2.53}$$

with

$$\bar{\boldsymbol{S}} = \mathcal{F}^{-1}(\boldsymbol{E}) \tag{2.54}$$

which is guaranteed to exist from the invertibility argument given above. One now readily verifies from (2.51), (2.53) and (2.54) that:

$$\partial_{\boldsymbol{E}} \hat{w}(\boldsymbol{E}) = \mathcal{F}^{-1}(\boldsymbol{E}) = \bar{\boldsymbol{S}}\,. \tag{2.55}$$

as required. Indeed,

$$\partial_{\boldsymbol{E}} \hat{w}(\boldsymbol{E}) \cdot \boldsymbol{H} = \mathrm{D}_{\boldsymbol{E}} \hat{w}(\boldsymbol{E})[\boldsymbol{H}]$$

$$
\begin{aligned}
&= \mathrm{D}_{\boldsymbol{E}}(-\hat{k}(\bar{\boldsymbol{S}}) + \bar{\boldsymbol{S}} \cdot \boldsymbol{E})[\boldsymbol{H}] \\
&= -\mathrm{D}_{\bar{\boldsymbol{S}}}\hat{k}(\bar{\boldsymbol{S}})[\mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}[\boldsymbol{H}]] + (\boldsymbol{E} \cdot \mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}[\boldsymbol{H}]) + \bar{\boldsymbol{S}} \cdot \boldsymbol{H}) \\
&= -\partial_{\bar{\boldsymbol{S}}}\hat{k}(\bar{\boldsymbol{S}}) \cdot \mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}[\boldsymbol{H}] + (\boldsymbol{E} \cdot \mathrm{D}_{\boldsymbol{E}}\bar{\boldsymbol{S}}[\boldsymbol{H}]) + \bar{\boldsymbol{S}} \cdot \boldsymbol{H}) \\
&= \bar{\boldsymbol{S}} \cdot \boldsymbol{H}\,. \qquad (2.56)
\end{aligned}
$$

2.4.2 Strong Ellipticity

We now proceed to establish conditions under which the models (2.25) and (2.26) satisfy *strong ellipticity*, and give counterexamples for which it fails, by making use of the general approach outlined in Section 2.3. Both of these models have been shown to have equivalent Green elastic formulations, but while an explicit form for the strain energy function was constructed for the anisotropic model (2.26), such was not the case for the isotropic model (2.25) for which only an abstract existence result was established. This fact suggests that proving *strong ellipticity* for these two models most naturally proceeds along somewhat different paths. To illustrate these two lines of argument, consider a general constitutive model of the form

$$
\boldsymbol{E} = \mathcal{F}(\bar{\boldsymbol{S}}). \qquad (2.57)
$$

Suppose that (2.57) can be explicitly inverted giving

$$
\bar{\boldsymbol{S}} = \mathcal{F}^{-1}(\boldsymbol{E}). \qquad (2.58)
$$

Then the inequality guaranteeing *strong ellipticity* (2.18) is readily shown to take the form:

$$
0 < \boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}}\hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] = \boldsymbol{H} \cdot (\boldsymbol{H}\bar{\boldsymbol{S}}) + (\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s \cdot \mathrm{D}_{\boldsymbol{E}}\mathcal{F}^{-1}(\boldsymbol{E})[(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s]. \qquad (2.59)
$$

On the other hand, if (2.57) cannot be explicitly inverted, then one can use the *strong ellipticity* condition in the form (2.23) which requires constructing the inverse $(\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}}))^{-1}[\cdot]$ of the fourth order tensor $\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})[\cdot]$. We illustrate these two approaches by applying

(2.59) to the anisotropic model (2.26) and applying (2.23) to the isotropic model (2.25).

Consider first the anisotropic model (2.26) written in the equivalent Cauchy elastic form (2.44). Then using the notation in (2.57), (2.58), a straightforward calculation gives for every second-order tensor $\boldsymbol{H}_s \in \mathcal{S}\text{ym}$:

$$\mathrm{D}_{\boldsymbol{E}}\mathcal{F}^{-1}(\boldsymbol{E})[\boldsymbol{H}_s] = \tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}])\mathbb{E}[\boldsymbol{H}_s] + \frac{\tilde{\phi}_1'(|\mathbb{E}^{1/2}[\boldsymbol{E}|)}{|\mathbb{E}^{1/2}[\boldsymbol{E}]|}\mathbb{E}[\boldsymbol{E}] \otimes \mathbb{E}[\boldsymbol{E}][\boldsymbol{H}_s] \tag{2.60}$$

where the tensor product $\mathbb{E}[\boldsymbol{E}] \otimes \mathbb{E}[\boldsymbol{E}][\cdot]$ is a fourth-order tensor viewed as a linear transformation on $\mathcal{S}\text{ym}$. Substitution of (2.60) into (2.59) gives:

$$\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}}\hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] = \boldsymbol{H} \cdot (\boldsymbol{H}\bar{\boldsymbol{S}}) \tag{2.61}$$

$$+ \tilde{\phi}_1(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s \cdot \mathbb{E}[(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s] \tag{2.62}$$

$$+ \frac{\tilde{\phi}_1'(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)}{|\mathbb{E}^{1/2}[\boldsymbol{E}]|}\left((\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s \cdot \mathbb{E}[\boldsymbol{E}]\right)^2. \tag{2.63}$$

Proving *strong ellipticity* requires giving conditions on $\boldsymbol{F}$ and the model (2.26) guaranteeing the positivity of $\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}}\hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}]$ for all rank-one, second-order tensors $\boldsymbol{H} := \boldsymbol{a} \otimes \boldsymbol{b}$ with $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$. The right-hand-side of (2.61) can be re-written in the form:

$$\boldsymbol{H} \cdot (\boldsymbol{H}\bar{\boldsymbol{S}}) = \boldsymbol{b} \cdot \bar{\boldsymbol{S}}\boldsymbol{b} \tag{2.64}$$

which need not be positive. On the other hand, both (2.62) and (2.63) are non-negative. However, for a given $\boldsymbol{F}$, one can choose $\boldsymbol{a}$ and $\boldsymbol{b}$ so that

$$\left((\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s \cdot \mathbb{E}[\boldsymbol{E}]\right)^2 = (\boldsymbol{a} \cdot (\boldsymbol{F}\mathbb{E}[\boldsymbol{E}]\boldsymbol{b})^2 = 0. \tag{2.65}$$

The constitutive assumptions for the model (2.26) guarantee that the term (2.62) is always strictly positive for all choices of $\boldsymbol{a}$ and $\boldsymbol{b}$. In particular, by assumption, $\mathbb{E}[\cdot]$ is a positive definite, self-adjoint linear transformation on $\mathcal{S}\text{ym}$. Hence, there exists $\lambda > 0$ satisfying

$$(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s \cdot \mathbb{E}[(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s] > \lambda\,(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s \cdot (\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s = \frac{\lambda}{2}\left(|\boldsymbol{F}^{\mathrm{T}}\boldsymbol{a}|^2 + (\boldsymbol{a} \cdot \boldsymbol{F}\boldsymbol{b})^2\right) \geq \frac{\lambda}{2}\left(|\boldsymbol{F}^{\mathrm{T}}\boldsymbol{a}|^2\right). \tag{2.66}$$

Now make use of the observation: if $|\boldsymbol{F}-\boldsymbol{I}| \to 0$ then $|\boldsymbol{E}| \to 0$ and $\bar{\boldsymbol{S}} \to 0$. Also, by assumption on the model (2.26), $\tilde{\phi}_1(0) > 0$. If follows that one can always choose $\delta > 0$ so that if $|\boldsymbol{F}-\boldsymbol{I}| < \delta$, then (2.62) dominates (2.61) in magnitude for all choices of $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus, *strong ellipticity* holds provided $\boldsymbol{F}$ is restricted to be close enough to the identity tensor $\boldsymbol{I}$, which implies that the Green-Lagrange tensor $\boldsymbol{E}$ has small enough norm.

Remark 3. Through a straightforward asymptotic argument, one can readily see that in the infinitesimal strain limit, the *strong ellipticity* condition (2.61) -- (2.63) takes the form:

$$\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] = \boldsymbol{H} \cdot \mathbb{E}[\boldsymbol{H}] > 0 \tag{2.67}$$

which is the classical result from the linearized theory of elasticity.

To demonstrate the loss of *strong ellipticity* for the model (2.26), it suffices to show the existence of deformation gradients $\boldsymbol{F}$ and unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ for which the quadratic form in (2.61), (2.62), (2.63) is negative. To that end, consider compressive deformations and stresses in the form

$$\boldsymbol{F} = \gamma \boldsymbol{I} \quad \text{and} \quad \bar{\boldsymbol{S}} = \sigma \boldsymbol{I} \tag{2.68}$$

with $0 < \gamma < 1$ and $\sigma < 0$. For this choice of $\boldsymbol{F}$, $\boldsymbol{E} = \xi \boldsymbol{I}$ with $\xi := (\gamma^2 - 1)/2$. We note that

$$-\frac{1}{2} < \xi < 0. \tag{2.69}$$

Substitution of (2.68) into (2.61), (2.62), (2.63) gives:.

$$\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}[\boldsymbol{H}] = \sigma + \tilde{\phi}_1(|\xi|\epsilon)\gamma^2(\boldsymbol{H}_s \cdot \mathbb{E}[\boldsymbol{H}_s]) + \frac{\tilde{\phi}_1'(|\mathbb{E}^{1/2}[\boldsymbol{E}]|)}{|\mathbb{E}^{1/2}[\boldsymbol{E}]|}(\gamma\xi)^2 \left(\boldsymbol{H} \cdot \mathbb{E}[\boldsymbol{I}]\right)^2 \tag{2.70}$$

with $\epsilon := |\mathbb{E}^{1/2}[\boldsymbol{I}]| > 0$, and with ξ and σ related through the constitutive relation (2.26) by

$$\xi = \sigma \phi_1(|\sigma|\kappa) \tag{2.71}$$

in which $\kappa := |\mathbb{K}^{1/2}[\boldsymbol{I}]| > 0$.

The first term on the right hand side of (2.70) is negative while the second and third terms are non-negative. Also, by assumption

$$\lim_{r\to\infty}\psi_1(r) := \lim_{r\to\infty} r\phi_1(r) = \frac{1}{\beta} < \infty \tag{2.72}$$

for some $\beta > 0$ from which it follows that

$$|\xi| < \frac{1}{\beta\kappa}. \tag{2.73}$$

To construct the desired counterexample to *strong ellipticity*, we first choose $\boldsymbol{a}$ and $\boldsymbol{b}$ so that $\boldsymbol{H}\cdot\mathbb{E}[\boldsymbol{I}] = 0$ making the third term on the right hand side of (2.70) vanish. The idea now is to choose $\gamma^2 << 1$ and $\sigma < 0$ so that the second term on the right hand side of (2.70) is smaller than $|\sigma|$, thereby making the right hand side of (2.70) negative. Toward that goal, note that from the fact that $\tilde{\phi}_1(r)$ is increasing and (2.73), the second term on the right hand side of (2.70) satisfies the bound

$$\tilde{\phi}_1(|\xi|\epsilon)\gamma^2(\boldsymbol{H}_s\cdot\mathbb{E}[\boldsymbol{H}_s]) < \gamma^2\tilde{\phi}_1(\epsilon/2)\Lambda \tag{2.74}$$

where Λ is an upper bound for the quadratic form $\boldsymbol{N}\cdot\mathbb{E}[\boldsymbol{N}]$ over all norm-1, second order tensors $\boldsymbol{N}$. Notice that as $\gamma\to 0$, $\xi\to -1/2$. Moreover, from (2.71) it follows that

$$|\sigma| = \frac{\psi_1^{-1}(|\xi|\kappa)}{\kappa}. \tag{2.75}$$

Also, if $\beta < 2/\kappa$, that is, the strain limit is not too small, then $\sigma < \xi < 0$. Therefore, one can choose $\gamma << 1$ so that the right hand side of (2.74) satisfies

$$\sigma < -\gamma^2\tilde{\phi}_1(\epsilon/2)\Lambda \tag{2.76}$$

and hence that (2.70) is negative thereby violating *strong ellipticity*.

The model (2.25) exhibits similar *strong ellipticity* behavior as (2.26), namely *strong ellipticity* holds for deformation gradients sufficiently near the identity (small strains) but fails if strains are not so severely limited. However, because there is not, in general, a closed-form expression for the inverse constitutive relation to (2.25), the method of proof of these *strong ellipticity* results given below is based upon (2.23) rather than (2.59) as was done above for the anisotropic model (2.26). This requires investigating the invertibility of the fourth order tensor $\mathrm{D}_{\bar{\boldsymbol{S}}}\boldsymbol{E}\left(\bar{\boldsymbol{S}}\right) = \mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}\left(\bar{\boldsymbol{S}}\right)$. Note first that

$$\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\left|\bar{\boldsymbol{S}}\right|\right)[\boldsymbol{H}] = \mathrm{D}_{\bar{\boldsymbol{S}}}\left(\bar{\boldsymbol{S}}\cdot\bar{\boldsymbol{S}}\right)^{\frac{1}{2}}[\boldsymbol{H}] = \frac{1}{2}\left(\bar{\boldsymbol{S}}\cdot\bar{\boldsymbol{S}}\right)^{-\frac{1}{2}}\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\bar{\boldsymbol{S}}\cdot\bar{\boldsymbol{S}}\right)[\boldsymbol{H}] = \frac{\boldsymbol{H}\cdot\bar{\boldsymbol{S}}}{\left|\bar{\boldsymbol{S}}\right|}. \tag{2.77}$$

Let

$$\boldsymbol{A} := \frac{\bar{\boldsymbol{S}}}{\left|\bar{\boldsymbol{S}}\right|},\ \text{and}\ \boldsymbol{B} := \frac{\boldsymbol{I}}{|\boldsymbol{I}|}, \tag{2.78}$$

so that

$$|\boldsymbol{A}| = |\boldsymbol{B}| = 1, \tag{2.79}$$

and let $\mathbb{I}$ denote the fourth order identity tensor. Fréchet differentiation of the right hand side of (2.25) gives:

$$\begin{aligned}
\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}\left(\bar{\boldsymbol{S}}\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] &= \phi_0'\left(\bar{\boldsymbol{S}}\cdot\boldsymbol{I}\right)\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\bar{\boldsymbol{S}}\cdot\boldsymbol{I}\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\right)\boldsymbol{I} \\
&+\ \phi_1'\left(\left|\bar{\boldsymbol{S}}\right|\right)\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\left|\bar{\boldsymbol{S}}\right|\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\right)\bar{\boldsymbol{S}} \\
&+\ \phi_1\left(\left|\bar{\boldsymbol{S}}\right|\right)\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\bar{\boldsymbol{S}}\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\right) \\
&= \phi_0'\left(\bar{\boldsymbol{S}}\cdot\boldsymbol{I}\right)\left(\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\cdot\boldsymbol{I}\right)\boldsymbol{I} \\
&+\ \phi_1'\left(\left|\bar{\boldsymbol{S}}\right|\right)\left(\frac{\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\cdot\bar{\boldsymbol{S}}}{\left|\bar{\boldsymbol{S}}\right|}\right)\bar{\boldsymbol{S}} \\
&+\ \phi_1\left(\left|\bar{\boldsymbol{S}}\right|\right)\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s \\
&= \left(3\phi_0'\left(\bar{\boldsymbol{S}}\cdot\boldsymbol{I}\right)(\boldsymbol{B}\otimes\boldsymbol{B}) + \left|\bar{\boldsymbol{S}}\right|\phi_1'\left(\left|\bar{\boldsymbol{S}}\right|\right)(\boldsymbol{A}\otimes\boldsymbol{A})\right. \\
&\left.+\,\phi_1\left(\left|\bar{\boldsymbol{S}}\right|\right)\mathbb{I}\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right].
\end{aligned} \tag{2.80}$$

Define

$$\tilde{\alpha} = \frac{|\bar{\boldsymbol{S}}|\phi_1'\left(|\bar{\boldsymbol{S}}|\right)}{\phi_1\left(|\bar{\boldsymbol{S}}|\right)},\ \tilde{\beta} = \frac{3\phi_0'\left(\bar{\boldsymbol{S}}\cdot\boldsymbol{I}\right)}{\phi_1\left(|\bar{\boldsymbol{S}}|\right)}\,, \tag{2.81}$$

and note that $\tilde{\alpha} < 0$ and $\tilde{\beta} > 0$. Then the right hand side of (2.80) becomes

$$\phi_1\left(|\bar{\boldsymbol{S}}|\right)\left(\tilde{\beta}\boldsymbol{B}\otimes\boldsymbol{B} + \tilde{\alpha}\boldsymbol{A}\otimes\boldsymbol{A} + \mathbb{I}\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\,. \tag{2.82}$$

At issue is the invertibility of the fourth order tensor

$$\mathbb{L} := \tilde{\beta}\boldsymbol{B}\otimes\boldsymbol{B} + \tilde{\alpha}\boldsymbol{A}\otimes\boldsymbol{A} + \mathbb{I}, \tag{2.83}$$

viewed as an operator on $\mathcal{S}$ym, the six-dimensional vector space of symmetric, second-order tensors. As is obvious from (2.83), $\mathbb{L}$ is self-adjoint, and hence its invertibility can be studied by appealing to spectral theory: $\mathbb{L}$ is invertible if and only if its spectrum does not contain zero. The spectrum of $\mathbb{L}$ is readily determined by constructing an eigen-basis. To that end, we note that the tensor products $\boldsymbol{B}\otimes\boldsymbol{B}$ and $\boldsymbol{A}\otimes\boldsymbol{A}$ are rank-one tensors over $\mathcal{S}$ym. Thus, an orthonormal eigen-basis for $\mathbb{L}$ consists of the union of an orthonormal basis for $\mathrm{Span}\{\boldsymbol{A},\boldsymbol{B}\}$ and an orthonormal basis for $\{\boldsymbol{A},\boldsymbol{B}\}^\perp$. The construction of an orthonormal basis for $\mathrm{Span}\{\boldsymbol{A},\boldsymbol{B}\}$ requires the consideration of cases: $\boldsymbol{A}\perp\boldsymbol{B}$, $\boldsymbol{A} = \pm\boldsymbol{B}$ or $0 < \boldsymbol{A}\cdot\boldsymbol{B} < 1$.

Case 1: $\boldsymbol{A}\perp\boldsymbol{B}$.

For this case, one can take for the spectral expansion of $\mathbb{L}$

$$\mathbb{L} = (1+\tilde{\alpha})\boldsymbol{A}\otimes\boldsymbol{A} + (1+\tilde{\beta})\boldsymbol{B}\otimes\boldsymbol{B} + (\mathbb{I} - \boldsymbol{A}\otimes\boldsymbol{A} - \boldsymbol{B}\otimes\boldsymbol{B}). \tag{2.84}$$

We note that

$$\phi_1\left(|\bar{\boldsymbol{S}}|\right)(1+\tilde{\alpha}) = \phi_1\left(|\bar{\boldsymbol{S}}|\right) + |\bar{\boldsymbol{S}}|\phi_1'\left(|\bar{\boldsymbol{S}}|\right) > 0, \tag{2.85}$$

and

$$\phi_1\left(|\bar{\boldsymbol{S}}|\right)\left(1+\tilde{\beta}\right) = \phi_1\left(|\bar{\boldsymbol{S}}|\right) + 3\phi_0'\left(\bar{\boldsymbol{S}}\cdot\boldsymbol{I}\right) > 0, \tag{2.86}$$

and hence that $\mathbb{L}$ is invertible, and

$$\mathbb{L}^{-1} = \frac{1}{1+\tilde{\alpha}}\boldsymbol{A}\otimes\boldsymbol{A} + \frac{1}{1+\tilde{\beta}}\boldsymbol{B}\otimes\boldsymbol{B} + (\mathbb{I} - \boldsymbol{A}\otimes\boldsymbol{A} - \boldsymbol{B}\otimes\boldsymbol{B})\ . \tag{2.87}$$

Case 2: $\boldsymbol{A} = \pm\boldsymbol{B}$.

In this case, $\boldsymbol{A}\otimes\boldsymbol{A} = \boldsymbol{B}\otimes\boldsymbol{B}$ giving the spectral expansion:

$$\mathbb{L} = (\tilde{\alpha}+\tilde{\beta})\boldsymbol{A}\otimes\boldsymbol{A} + (\mathbb{I} - \boldsymbol{A}\otimes\boldsymbol{A}). \tag{2.88}$$

Noting that

$$\tilde{\beta} + \tilde{\alpha} + 1 = \frac{3\phi_0'\left(\bar{\boldsymbol{S}}\cdot\boldsymbol{I}\right) + \phi_1\left(|\bar{\boldsymbol{S}}|\right) + |\bar{\boldsymbol{S}}|\phi_1'\left(|\bar{\boldsymbol{S}}|\right)}{\phi_1\left(|\bar{\boldsymbol{S}}|\right)} > 0\,, \tag{2.89}$$

we see that for this case also, $\mathbb{L}$ is invertible with

$$\mathbb{L}^{-1} = \frac{1}{\tilde{\beta}+\tilde{\alpha}+1}\boldsymbol{A}\otimes\boldsymbol{A} + (\mathbb{I} - \boldsymbol{A}\otimes\boldsymbol{A})\,. \tag{2.90}$$

Case 3: $0 < |\boldsymbol{A}\cdot\boldsymbol{B}| < 1$.

Let

$$\mathbb{T} := \tilde{\alpha}\boldsymbol{A}\otimes\boldsymbol{A} + \tilde{\beta}\boldsymbol{B}\otimes\boldsymbol{B}, \tag{2.91}$$

then the eigenvectors of $\mathbb{T}$ lie in span$\{\boldsymbol{A},\boldsymbol{B}\}$. Therefore, if $\boldsymbol{V}$ is an eigenvector of $\mathbb{T}$ with associated eigenvalue λ, then $\boldsymbol{V}$ must satisfy

$$\boldsymbol{V} = \gamma_1\boldsymbol{A} + \gamma_2\boldsymbol{B},$$

in which $\gamma_1\gamma_2 \neq 0$, and

$$\lambda = \tilde{\beta}\left(1 + \delta(\boldsymbol{A}\cdot\boldsymbol{B})\right), \tag{2.92}$$

with

$$\delta := \frac{\gamma_1}{\gamma_2}. \tag{2.93}$$

One readily shows that δ must satisfy

$$\delta^2 + \frac{\left(1 + \left|\frac{\tilde{\alpha}}{\tilde{\beta}}\right|\right)}{\boldsymbol{A}\cdot\boldsymbol{B}}\delta + \left|\frac{\tilde{\alpha}}{\tilde{\beta}}\right| = 0\,. \tag{2.94}$$

The two roots, $\delta_{1,2}$, of (2.94) are both positive if $\boldsymbol{A}\cdot\boldsymbol{B} < 0$ and both negative if $\boldsymbol{A}\cdot\boldsymbol{B} > 0$. In either case, it follows that one can number the associated eigenvalues $\lambda_{1,2}$ in order that they satisfy:

$$\lambda_{1,2} = \tilde{\beta}\,(1 + \delta_{1,2}(\boldsymbol{A}\cdot\boldsymbol{B})) \mathrel{\substack{>\\<}} 0. \tag{2.95}$$

Letting $\tilde{\boldsymbol{V}}_{1,2}$ denote the normalized eigenvectors corresponding to $\delta_{1,2}$, one concludes that the fourth-order tensor $\mathbb{L}[\cdot]$ has the spectral expansion:

$$\mathbb{L} = \Big((\lambda_1 + 1)\tilde{\boldsymbol{V}}_1 \otimes \tilde{\boldsymbol{V}}_1 + (\lambda_2 + 1)\tilde{\boldsymbol{V}}_2 \otimes \tilde{\boldsymbol{V}}_2 + \Big(\mathbb{I} - \tilde{\boldsymbol{V}}_1 \otimes \tilde{\boldsymbol{V}}_1 - \tilde{\boldsymbol{V}}_2 \otimes \tilde{\boldsymbol{V}}_2\Big)\Big)\,. \tag{2.96}$$

From (2.95), it follows that the eigenvalue $\lambda_1 + 1 > 0$. However, in general, the eigenvalue $\lambda_2 + 1$ can be positive, negative or zero. Thus, the fourth-order tensor (2.96) need not be invertible on $\mathcal{S}$ym, and if $\lambda_2 + 1 = 0$, one can always choose $\boldsymbol{a},\ \boldsymbol{b}$ and $\boldsymbol{F}$ so that $(\boldsymbol{F}^T\boldsymbol{H})_s$ is not orthogonal to $\boldsymbol{V}_2$, and hence not in the subspace on which (2.96) is invertible. However, we make use of the observation: if $|\bar{\boldsymbol{S}}| \to 0$ then $|\boldsymbol{E}| \to 0$ and $|\boldsymbol{F} - \boldsymbol{I}| \to 0$, so $\tilde{\alpha} \to 0$ and $\tilde{\beta} \to 3\phi_0'(0)/\phi_1(0) > 0$. One then concludes that $1 + \lambda_2 > 0$ if $|\bar{\boldsymbol{S}}|$ and $|\boldsymbol{E}|$ are sufficiently small, and hence $\mathbb{L}^{-1}$ can be expressed through the spectral expansion:

$$\mathbb{L}^{-1} = \frac{1}{\lambda_1 + 1}\tilde{\boldsymbol{V}}_1 \otimes \tilde{\boldsymbol{V}}_1 + \frac{1}{\lambda_2 + 1}\tilde{\boldsymbol{V}}_2 \otimes \tilde{\boldsymbol{V}}_2 + \Big(\mathbb{I} - \tilde{\boldsymbol{V}}_1 \otimes \tilde{\boldsymbol{V}}_1 - \tilde{\boldsymbol{V}}_2 \otimes \tilde{\boldsymbol{V}}_2\Big) \tag{2.97}$$

or in the equivalent form:

$$\mathbb{L}^{-1} = \mathbb{I} - \frac{\lambda_1}{\lambda_1 + 1}\tilde{\boldsymbol{V}}_1 \otimes \tilde{\boldsymbol{V}}_1 - \frac{\lambda_2}{\lambda_2 + 1}\tilde{\boldsymbol{V}}_2 \otimes \tilde{\boldsymbol{V}}_2. \tag{2.98}$$

We consider next *strong ellipticity* for the model (2.25), making use of the above three cases for the specific form the fourth-order tensor $\mathbb{L}^{-1}[\cdot]$ takes.

Case 1: $\boldsymbol{A} \perp \boldsymbol{B}$.

Combining (2.23) and (2.87), one obtains:

$$\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] = \boldsymbol{H} \cdot (\boldsymbol{H} \bar{\boldsymbol{S}}(\boldsymbol{E})) + \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \cdot \mathrm{D}_{\boldsymbol{E}} \bar{\boldsymbol{S}}(\boldsymbol{E}) \left[\left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s\right] \tag{2.99}$$

$$= \boldsymbol{b} \cdot \left(\bar{\boldsymbol{S}} \boldsymbol{b}\right) \tag{2.100}$$

$$+ \Big(\left| \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \right|^2$$

$$- \frac{\tilde{\beta}}{1+\tilde{\beta}} \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \left[\left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s\right] \tag{2.101}$$

$$- \frac{\tilde{\alpha}}{1+\tilde{\alpha}} \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \cdot (\boldsymbol{A} \otimes \boldsymbol{A}) \left[\left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s\right] \Big) \frac{1}{\phi_1\left(|\bar{\boldsymbol{S}}|\right)}. \tag{2.102}$$

Note that

$$\begin{aligned} \left| \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \left[\left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s\right] \right| = & \quad \left| (\boldsymbol{B} \otimes \boldsymbol{B}) \cdot \left(\left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \otimes \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s\right) \right| \\ \leq & \quad \left| \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \right|^2. \end{aligned} \tag{2.103}$$

Therefore,

$$\left| \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \right|^2 - \frac{\tilde{\beta}}{1+\tilde{\beta}} \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \left[\left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s\right] \geq \left(\frac{1}{1+\tilde{\beta}}\right) \left| \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}\right)_s \right|^2. \tag{2.104}$$

The key observation to make is that as $\bar{\boldsymbol{S}} \to \boldsymbol{0}$, then: $\boldsymbol{E} \to \boldsymbol{0}$, $|\boldsymbol{F} - \boldsymbol{I}| \to 0$, $\tilde{\alpha} \to 0$ and $\tilde{\beta} \to 3\phi_0'(0)/\phi_1(0) > 0$. It follows that for sufficiently small stresses and strains, the right-hand-side of (2.99) is positive for all rank-one tensors $\boldsymbol{H}$, and hence *strong ellipticity* holds.

Case 2: $\boldsymbol{A} = \pm \boldsymbol{B}$.

Combining (2.23) and (2.90) gives:

$$\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] = \boldsymbol{b} \cdot \left(\bar{\boldsymbol{S}} \boldsymbol{b}\right)$$

$$
\begin{aligned}
&+\frac{1}{\phi_1\left(\left|\bar{\boldsymbol{S}}\right|\right)}\left(\left|\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right|^2\right. \\
&\left.-\frac{\tilde{\alpha}+\tilde{\beta}}{1+\tilde{\alpha}+\tilde{\beta}}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\cdot\left(\boldsymbol{B}\otimes\boldsymbol{B}\right)\left[\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right]\right). \qquad (2.105)
\end{aligned}
$$

Appealing to the inequalities (2.103) and (2.104), we see by the same argument used in Case 1, that for sufficiently small stresses and strains, the right-hand-side of (2.105) is positive for all rank-one tensors $\boldsymbol{H}$, as required for *strong ellipticity*.

Case 3: $0 < |\boldsymbol{A}\cdot\boldsymbol{B}| < 1$.

Substitution of (2.98) into (2.23) gives:

$$
\begin{aligned}
\boldsymbol{H}\cdot\mathrm{D}_{\boldsymbol{F}}\hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] &= \boldsymbol{b}\cdot\left(\bar{\boldsymbol{S}}\boldsymbol{b}\right) \\
&+\frac{1}{\phi_1\left(\left|\bar{\boldsymbol{S}}\right|\right)}\left(\left|\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right|^2\right. \\
&\left.-\frac{\lambda_1}{1+\lambda_1}\left(\tilde{\boldsymbol{V}}_1\cdot\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right)^2-\frac{\lambda_2}{1+\lambda_2}\left(\tilde{\boldsymbol{V}}_2\cdot\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right)^2\right) \qquad (2.106) \\
&\geq \boldsymbol{b}\cdot\left(\bar{\boldsymbol{S}}\boldsymbol{b}\right) \\
&+\frac{1}{\phi_1\left(\left|\bar{\boldsymbol{S}}\right|\right)}\left(\left|\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right|^2-\frac{\lambda_1}{1+\lambda_1}\left(\tilde{\boldsymbol{V}}_1\cdot\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right)^2\right) \qquad (2.107) \\
&\geq \boldsymbol{b}\cdot\left(\bar{\boldsymbol{S}}\boldsymbol{b}\right)+\frac{1}{\phi_1\left(\left|\bar{\boldsymbol{S}}\right|\right)(1+\lambda_1)}\left|\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right|^2, \qquad (2.108)
\end{aligned}
$$

in which (2.107) follows from (2.95) and (2.108) makes use of:

$$
\left(\tilde{\boldsymbol{V}}_1\cdot\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right)^2 \leq \left|\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s\right|^2.
$$

Now the argument of Case 1 applies to conclude that for sufficiently small stress and strain, the first term on the right-hand-side of (2.108) can be made arbitrarily small in magnitude while the second term has a strictly positive lower bound for all rank-one tensors $\boldsymbol{H}$ thereby proving *strong ellipticity*.

2.5 Examples

We examine in this section examples inspired by models studied by Rajagopal and co-authors [10, 20, 16]. Consider first the model (2.25) with

$$\phi_0(r) := \alpha_0 \left(1 - \exp\left(\frac{-\beta_0 r}{1+\delta_0|r|}\right)\right) \tag{2.109}$$

and

$$\phi_1(r) := \frac{\alpha_1}{1+\beta_1 r}\,, \tag{2.110}$$

where α_0, β_0, δ_0, α_1, and β_1 are positive constants. Then one immediately verifies that: $\phi_1(r)$ is a positive, decreasing function while $r\phi_1(r)$ is a bounded, increasing function; $\phi_0(r)$ is a bounded, increasing function that is positive for $r > 0$ and negative for $r < 0$. The results in the previous section apply to conclude that this model satisfies *strong ellipticity* provided the sum of the upper bounds on $\phi_0(r)$ and $r\phi_1(r)$ are small enough. Of interest in this section is deriving explicit conditions under which the model (2.25), (2.109), (2.110) loses *strong ellipticity*.

We take

$$\bar{\boldsymbol{S}} = \mu \boldsymbol{I}, \tag{2.111}$$

and

$$\boldsymbol{F} = \gamma \boldsymbol{I}, \tag{2.112}$$

where μ and γ are constants with $\gamma^2 << 1$. Then

$$\boldsymbol{E} = \frac{1}{2}\left(\gamma^2 - 1\right)\boldsymbol{I} = \phi_0(3\mu)\boldsymbol{I} + \phi_1\left(\sqrt{3}|\mu|\right)\mu\boldsymbol{I}, \tag{2.113}$$

i.e.

$$\begin{aligned}\frac{1}{2}\left(\gamma^2 - 1\right) &= \phi_0(3\mu) + \mu\phi_1\left(\sqrt{3}|\mu|\right) \\ &= \alpha_0\left(1 - \exp\left(\frac{-3\beta_0\mu}{1+3\delta_0|\mu|}\right)\right) + \frac{\alpha_1\mu}{1+\sqrt{3}\beta_1|\mu|}\,. \end{aligned} \tag{2.114}$$

From (2.109), we note that $\phi_0(r) > 0$ when $r > 0$, and $\phi_0(r) < 0$ when $r < 0$. By (2.114), we must have $\mu < 0$ (because if $\mu > 0$ then both terms on the right hand side of (2.114) are positive, which contradicts to the hypothesis $\gamma^2 << 1$).

Define

$$\boldsymbol{A} = \frac{\bar{\boldsymbol{S}}}{|\bar{\boldsymbol{S}}|}, \text{ and } \boldsymbol{B} = \frac{\boldsymbol{I}}{|\boldsymbol{I}|} \tag{2.115}$$

and note that

$$|\boldsymbol{A}| = |\boldsymbol{B}| = 1, \text{ and } \boldsymbol{A} \otimes \boldsymbol{A} = \boldsymbol{B} \otimes \boldsymbol{B}. \tag{2.116}$$

Let $\mathbb{I}$ denote the fourth order identity tensor.

A lengthy but straightforward calculation gives:

$$\begin{aligned} D_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] &= \left(\left(3\boldsymbol{\phi}_0'(3\mu) + \sqrt{3}|\mu|\boldsymbol{\phi}_1'\left(\sqrt{3}|\mu|\right)\right)\boldsymbol{B}\otimes\boldsymbol{B}\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\ &+ \left(\boldsymbol{\phi}_1\left(\sqrt{3}|\mu|\right)\mathbb{I}\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]. \end{aligned} \tag{2.117}$$

Let

$$K := \boldsymbol{\phi}_1\left(\sqrt{3}|\mu|\right), \tag{2.118}$$

$$L := 3\boldsymbol{\phi}_0'(3\mu) + \sqrt{3}|\mu|\boldsymbol{\phi}_1'\left(\sqrt{3}|\mu|\right), \tag{2.119}$$

and note that $K > 0$ and $K + L > 0$. We then have the spectral decomposition

$$D_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] = \quad K\left(\left(\frac{L}{K}+1\right)(\boldsymbol{B}\otimes\boldsymbol{B}) + (\mathbb{I} - \boldsymbol{B}\otimes\boldsymbol{B})\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]. \tag{2.120}$$

Hence,

$$\begin{aligned} D_{\boldsymbol{E}}\bar{\boldsymbol{S}}(\boldsymbol{E})\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] &= \left(D_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\right)^{-1}\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\ &= \frac{1}{K}\left(\left(\frac{K}{L+K}\right)\boldsymbol{B}\otimes\boldsymbol{B} + (\mathbb{I} - \boldsymbol{B}\otimes\boldsymbol{B})\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]. \end{aligned} \tag{2.121}$$

Since $|\boldsymbol{H}| = 1$, it follows that

$$\begin{aligned}
\boldsymbol{H} \cdot D_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] &= \boldsymbol{H} \cdot (\boldsymbol{H} \bar{\boldsymbol{S}}(\boldsymbol{E})) + \left(\boldsymbol{F}^T \boldsymbol{H}\right)_s \cdot D_{\boldsymbol{E}} \bar{\boldsymbol{S}}(\boldsymbol{E}) \left[\left(\boldsymbol{F}^T \boldsymbol{H}\right)_s\right] \\
&= \mu + \gamma^2 \boldsymbol{H}_s \cdot \left(D_{\bar{\boldsymbol{S}}} \mathcal{F}(\bar{\boldsymbol{S}})\right)^{-1} [\boldsymbol{H}_s] \\
&= \mu + \frac{\gamma^2}{K} \left(\left(\frac{-L}{L+K} \right) (\boldsymbol{H}_s \cdot \boldsymbol{B})^2 + \boldsymbol{H}_s \cdot \boldsymbol{H}_s \right) \\
&= \mu + \frac{\gamma^2}{K} \left(\left(\frac{-L}{L+K} \right) \frac{(\boldsymbol{a} \cdot \boldsymbol{b})^2}{3} + \frac{1 + (\boldsymbol{a} \cdot \boldsymbol{b})^2}{2} \right). \qquad (2.122)
\end{aligned}$$

Notice that as $\mu \to -\infty$, the right-hand-side of (2.114) converges to:

$$-\xi_\infty := -\alpha_0 \left(\exp\left(\frac{\beta_0}{\delta_0} \right) - 1 \right) - \frac{\alpha_1}{\sqrt{3}\beta_1}. \qquad (2.123)$$

Thus, provided $\xi > 1/2$ in (2.123), as $\gamma \to 0$, then μ tends to a finite, negative limit and K is bounded away from zero. Hence, provided $\xi > 1/2$, the right-hand-side of (2.114) can be made negative by taking γ small enough.

We consider next the model (in dimensionless form):

$$\boldsymbol{E} = \frac{\bar{\boldsymbol{S}}}{1 + \beta\, |\bar{\boldsymbol{S}}|} \qquad (2.124)$$

that was studied by Rajagopal and co-authors in the setting of anti-plane shear, infinitesimal deformations [10, 20]. Here, (2.124) is considered for finite strain, simple shear deformations of the form:

$$\boldsymbol{F} = \boldsymbol{I} + \gamma\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \qquad (2.125)$$

where γ is a scalar, and $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthogonal unit vectors. For deformations of the form (2.125), one easily shows that the spectrum of $\boldsymbol{E}$ is $\{\gamma\lambda_{1,2}, 0\}$ with

$$\lambda_{1,2} := \frac{1}{4} \left(\gamma \pm \sqrt{4 + \gamma^2} \right). \qquad (2.126)$$

We note that $\lambda_1 \lambda_2 = -1/4$, and hence $\boldsymbol{E}$ has one negative eigenvalue.

The constitutive relation (2.124) can be readily inverted to give:

$$\bar{\boldsymbol{S}} = \frac{\boldsymbol{E}}{1 - \beta\,|\boldsymbol{E}|} =: \psi(|\boldsymbol{E}|)\boldsymbol{E}. \tag{2.127}$$

It follows that the quadratic form (2.59) takes the form:

$$\begin{aligned}
\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] &= (\boldsymbol{b} \cdot \bar{\boldsymbol{S}} \boldsymbol{b}) + \psi(|\boldsymbol{E})|(\boldsymbol{F}^T \boldsymbol{H})_s|^2 \\
&\quad + \psi(|\boldsymbol{E}|)^2 \left((\boldsymbol{F}^T \boldsymbol{H})_s \cdot \boldsymbol{E})^2 \frac{\beta}{|\boldsymbol{E}|} \right) \\
&= \psi(|\boldsymbol{E}|) \left((\boldsymbol{b} \cdot \boldsymbol{E} \boldsymbol{b}) + |(\boldsymbol{F}^T \boldsymbol{H})_s|^2 \right) && (2.128) \\
&\quad + \psi(|\boldsymbol{E}|)^2 \left((\boldsymbol{F}^T \boldsymbol{H})_s \cdot \boldsymbol{E})^2 \frac{\beta}{|\boldsymbol{E}|} \right). && (2.129)
\end{aligned}$$

We now record a few useful facts. First, note that if $\boldsymbol{H} := \boldsymbol{a} \otimes \boldsymbol{b}$, then

$$|(\boldsymbol{F}^T \boldsymbol{H})_s|^2 = \frac{1}{2} \left(|\boldsymbol{F}^T \boldsymbol{a}|^2 + (\boldsymbol{b} \cdot \boldsymbol{F}^T \boldsymbol{a})^2 \right). \tag{2.130}$$

Moreover, one readily shows that:

$$|\boldsymbol{F}^T \boldsymbol{a}|^2 = (\boldsymbol{a} \cdot \boldsymbol{F} \boldsymbol{F}^T \boldsymbol{a}) = 1 + 2(\boldsymbol{a} \cdot \boldsymbol{D} \boldsymbol{a}) \tag{2.131}$$

where

$$\boldsymbol{D} := \frac{1}{2}(\boldsymbol{F}\boldsymbol{F}^T - \boldsymbol{I}). \tag{2.132}$$

Note also that $\boldsymbol{D}$ and $\boldsymbol{E}$ have the same spectrum.

We first show that the model (2.124) is strongly elliptic for simple shear deformations of the form (2.125) provided $|\gamma| << 1$. To that end, note that $|\boldsymbol{E}| = |\boldsymbol{D}| = (|\gamma|/2)\sqrt{2 + \gamma^2} \to \boldsymbol{0}$ as $\gamma \to 0$. Making use of (2.130) and (2.131), one sees that

$$\boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \boldsymbol{S}(\boldsymbol{F})[\boldsymbol{H}] \geq \psi(|\boldsymbol{E}|) \left(\frac{1}{2} + (\boldsymbol{b} \cdot \boldsymbol{E} \boldsymbol{b}) + (\boldsymbol{a} \cdot \boldsymbol{D} \boldsymbol{a}) \right). \tag{2.133}$$

Thus, there exists $\gamma_0 > 0$ so that (2.133) is positive for all unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ provided $|\gamma| < \gamma_0$ proving *strong ellipticity*.

We now demonstrate the loss of *strong ellipticity* for β small enough. Note, from (2.124) it follows that $1/\beta$ is the upper-bound on $|\boldsymbol{E}|$, so that taking β small, raises the limiting strain upper-bound. The violation of *strong ellipticity* will be demonstrated by finding unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and scalar γ so that the sum of (2.128) and (2.129) is negative. To that end, express $\boldsymbol{a}$ and $\boldsymbol{b}$ as:

$$\boldsymbol{a} = \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2, \quad \boldsymbol{b} = \beta_1 \boldsymbol{e}_1 + \beta_2 \boldsymbol{e}_2 \tag{2.134}$$

with

$$\alpha_1^2 + \alpha_2^2 = \beta_1^2 + \beta_2^2 = 1. \tag{2.135}$$

Then (2.128) can be rewritten as:

$$\psi(|\boldsymbol{E}|)\left((\boldsymbol{b} \cdot \boldsymbol{E}\boldsymbol{b}) + |(\boldsymbol{F}^T \boldsymbol{H})_s|^2\right) = \frac{\psi(|\boldsymbol{E}|)}{2}(1 + g(\gamma; \alpha_1, \beta_1)) \tag{2.136}$$

in which

$$g(\gamma; \alpha_1, \beta_1) := A\left(\gamma + \frac{B}{A}\right)^2 + C - \frac{B^2}{A} \tag{2.137}$$

with

$$A(\alpha_1, \beta_1) = 1.0 + 2\alpha_1^2 - \beta_1^2 - \alpha_1^2\beta_1^2 \tag{2.138}$$

$$B(\alpha_1, \beta_1) = \alpha_1(2.0 - \beta_1^2)\sqrt{1.0 - \alpha_1^2} + \beta_1(1.0 + \alpha_1^2)\sqrt{1.0 - \beta_1^2} \tag{2.139}$$

$$C(\alpha_1, \beta_1) = \left(\alpha_1\beta_1 + \sqrt{(1.0 - \alpha_1^2)(1.0 - \beta_1^2)}\right)^2. \tag{2.140}$$

Indeed, we have

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}((\boldsymbol{I} + \gamma \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)(\boldsymbol{I} + \gamma \boldsymbol{e}_1 \otimes \boldsymbol{e}_2) - \boldsymbol{I})$$

$$= \frac{1}{2}(\gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + \gamma^2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)\,. \tag{2.141}$$

Now,

$$
\begin{aligned}
\boldsymbol{b} \cdot \boldsymbol{E}\boldsymbol{b} &= \frac{1}{2}(\beta_1 \boldsymbol{e}_1 + \beta_2 \boldsymbol{e}_2) \cdot (\gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + \gamma^2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)(\beta_1 \boldsymbol{e}_1 + \beta_2 \boldsymbol{e}_2) \\
&= \frac{1}{2}(\beta_1 \boldsymbol{e}_1 + \beta_2 \boldsymbol{e}_2) \cdot (\gamma \beta_2 \boldsymbol{e}_1 + \gamma \beta_1 \boldsymbol{e}_2 + \gamma^2 \beta_2 \boldsymbol{e}_2) \\
&= \frac{1}{2}(2\gamma \beta_1 \beta_2 + \gamma^2 \beta_2^2) \\
&= \frac{1}{2}\left(2\gamma \beta_1 \sqrt{1-\beta_1^2} + \gamma^2(1-\beta_1^2)\right) . \qquad (2.142)
\end{aligned}
$$

Next,

$$
\begin{aligned}
\boldsymbol{F}^T \boldsymbol{H} &= (\boldsymbol{I} + \gamma \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)((\alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2) \otimes (\beta_1 \boldsymbol{e}_1 + \beta_2 \boldsymbol{e}_2)) \\
&= (\boldsymbol{I} + \gamma \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)(\alpha_1 \beta_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \alpha_1 \beta_2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \alpha_2 \beta_1 \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \alpha_2 \beta_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) \\
&= \alpha_1 \beta_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \alpha_1 \beta_2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \alpha_2 \beta_1 \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \alpha_2 \beta_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \\
&+ \ \gamma \alpha_1 \beta_1 \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \gamma \alpha_1 \beta_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \\
&= \alpha_1 \beta_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \alpha_1 \beta_2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \\
&+ \ (\alpha_2 \beta_1 + \gamma \alpha_1 \beta_1) \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + (\alpha_2 \beta_2 + \gamma \alpha_1 \beta_2) \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \, . \qquad (2.143)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|(\boldsymbol{F}^T \boldsymbol{H})_s|^2 &= (\boldsymbol{F}^T \boldsymbol{H})_s \cdot (\boldsymbol{F}^T \boldsymbol{H})_s \\
&= \frac{1}{4}\,(\alpha_1 \beta_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \alpha_1 \beta_2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \\
&+ (\alpha_2 \beta_1 + \gamma \alpha_1 \beta_1) \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + (\alpha_2 \beta_2 + \gamma \alpha_1 \beta_2) \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \\
&+ \alpha_1 \beta_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \alpha_1 \beta_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 \\
&+ (\alpha_2 \beta_1 + \gamma \alpha_1 \beta_1) \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + (\alpha_2 \beta_2 + \gamma \alpha_1 \beta_2) \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\ \ (\alpha_1\beta_1\boldsymbol{e}_1\otimes\boldsymbol{e}_1+\alpha_1\beta_2\boldsymbol{e}_1\otimes\boldsymbol{e}_2\\
&+(\alpha_2\beta_1+\gamma\alpha_1\beta_1)\boldsymbol{e}_2\otimes\boldsymbol{e}_1+(\alpha_2\beta_2+\gamma\alpha_1\beta_2)\boldsymbol{e}_2\otimes\boldsymbol{e}_2\\
&+\alpha_1\beta_1\boldsymbol{e}_1\otimes\boldsymbol{e}_1+\alpha_1\beta_2\boldsymbol{e}_2\otimes\boldsymbol{e}_1\\
&+(\alpha_2\beta_1+\gamma\alpha_1\beta_1)\boldsymbol{e}_1\otimes\boldsymbol{e}_2+(\alpha_2\beta_2+\gamma\alpha_1\beta_2)\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\\
&=\frac{1}{4}\,(2\alpha_1\beta_1\boldsymbol{e}_1\otimes\boldsymbol{e}_1+(\alpha_1\beta_2+\alpha_2\beta_1+\gamma\alpha_1\beta_1)\boldsymbol{e}_1\otimes\boldsymbol{e}_2\\
&+(\alpha_1\beta_2+\alpha_2\beta_1+\gamma\alpha_1\beta_1)\boldsymbol{e}_2\otimes\boldsymbol{e}_1+2\beta_2(\alpha_2+\gamma\alpha_1)\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\\
&\cdot(2\alpha_1\beta_1\boldsymbol{e}_1\otimes\boldsymbol{e}_1+(\alpha_1\beta_2+\alpha_2\beta_1+\gamma\alpha_1\beta_1)\boldsymbol{e}_1\otimes\boldsymbol{e}_2\\
&+(\alpha_1\beta_2+\alpha_2\beta_1+\gamma\alpha_1\beta_1)\boldsymbol{e}_2\otimes\boldsymbol{e}_1+2\beta_2(\alpha_2+\gamma\alpha_1)\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\\
&=\frac{1}{4}\left(4\alpha_1^2\beta_1^2+2(\alpha_1\beta_2+\alpha_2\beta_1+\gamma\alpha_1\beta_1)^2+4\beta_2^2(\alpha_2+\gamma\alpha_1)^2\right)\\
&=\frac{1}{2}\left(2\alpha_1^2\beta_1^2+(\alpha_1^2\beta_2^2+\alpha_2^2\beta_1^2+\gamma^2\alpha_1^2\beta_1^2)\right.\\
&+2(\alpha_1\alpha_2\beta_1\beta_2+\gamma\alpha_1^2\beta_1\beta_2+\gamma\alpha_1\alpha_2\beta_1^2)\\
&\left.+2\beta_2^2(\alpha_2^2+2\gamma\alpha_1\alpha_2+\gamma^2\alpha_1^2)\right)\ . \qquad (2.144)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\boldsymbol{b}\cdot\boldsymbol{E}\boldsymbol{b}+|(\boldsymbol{F}^T\boldsymbol{H})_s|^2=&\ \frac{1}{2}\left(2\alpha_1^2\beta_1^2+\alpha_1^2(1-\beta_1^2)+(1-\alpha_1^2)\beta_1^2+\gamma^2\alpha_1^2\beta_1^2\right.\\
&+2\alpha_1\beta_1\sqrt{1-\alpha_1^2}\sqrt{1-\beta_1^2}+2\gamma\alpha_1^2\beta_1\sqrt{1-\beta_1^2}+2\gamma\alpha_1\beta_1^2\sqrt{1-\alpha_1^2}\\
&+2(1-\beta_1^2)(1-\alpha_1^2+2\gamma\alpha_1\sqrt{1-\alpha_1^2}+\gamma^2\alpha_1^2)\\
&\left.+2\gamma\beta_1\sqrt{1-\beta_1^2}+\gamma^2(1-\beta_1^2)\right)\\
=&\ \frac{1}{2}\left(2\alpha_1^2\beta_1^2+\alpha_1^2-\alpha_1^2\beta_1^2+\beta_1^2-\alpha_1^2\beta_1^2+\gamma^2\alpha_1^2\beta_1^2\right.
\end{aligned}
$$

$$+ 2\alpha_1\beta_1\sqrt{1-\alpha_1^2}\sqrt{1-\beta_1^2} + 2\gamma\alpha_1^2\beta_1\sqrt{1-\beta_1^2} + 2\gamma\alpha_1\beta_1^2\sqrt{1-\alpha_1^2}$$

$$+ 2(1-\beta_1^2)(1-\alpha_1^2 + 2\gamma\alpha_1\sqrt{1-\alpha_1^2} + \gamma^2\alpha_1^2)$$

$$+ 2 - 2\alpha_1^2 + 4\gamma\alpha_1\sqrt{1-\alpha_1^2} + 2\gamma^2\alpha_1^2$$

$$- 2\beta_1^2 + 2\alpha_1^2\beta_1^2 - 4\gamma\alpha_1\beta_1^2\sqrt{1-\alpha_1^2} - 2\gamma^2\alpha_1^2\beta_1^2$$

$$+ 2\gamma\beta_1\sqrt{1-\beta_1^2} + \gamma^2 - \gamma^2\beta_1^2\Big)$$

$$= \frac{1}{2}\Big(2 + 2\alpha_1^2\beta_1^2 - \alpha_1^2 - \beta_1^2 - \gamma^2\alpha_1^2\beta_1^2 + 2\alpha_1\beta_1\sqrt{1-\alpha_1^2}\sqrt{1-\beta_1^2}$$

$$+ 2\gamma\alpha_1^2\beta_1\sqrt{1-\beta_1^2} - 2\gamma\alpha_1\beta_1^2\sqrt{1-\alpha_1^2} + 4\gamma\alpha_1\sqrt{1-\alpha_1^2}$$

$$+ 2\gamma^2\alpha_1^2 + 2\gamma\beta_1\sqrt{1-\beta_1^2} + \gamma^2 - \gamma^2\beta_1^2\Big)$$

$$= \frac{1}{2}\Big(-\gamma^2\alpha_1^2\beta_1^2 + 2\gamma\alpha_1^2\beta_1\sqrt{1-\beta_1^2} - 2\gamma\alpha_1\beta_1^2\sqrt{1-\alpha_1^2} + 4\gamma\alpha_1\sqrt{1-\alpha_1^2}$$

$$+ 2\gamma^2\alpha_1^2 + 2\gamma\beta_1\sqrt{1-\beta_1^2} + \gamma^2 - \gamma^2\beta_1^2$$

$$+ 2 + 2\alpha_1^2\beta_1^2 - \alpha_1^2 - \beta_1^2 + 2\alpha_1\beta_1\sqrt{1-\alpha_1^2}\sqrt{1-\beta_1^2}\Big)$$

$$= \frac{1}{2}\left(1 + \left(\alpha_1\beta_1 + \sqrt{(1-\alpha_1^2)(1-\beta_1^2)}\right)^2\right.$$

$$+ \gamma^2(-\alpha_1^2\beta_1^2 + 2\alpha_1^2 + 1 - \beta_1^2)$$

$$\left. + 2\gamma(\alpha_1^2\beta_1\sqrt{1-\beta_1^2} - \alpha_1\beta_1^2\sqrt{1-\alpha_1^2} + 2\alpha_1\sqrt{1-\alpha_1^2} + \beta_1\sqrt{1-\beta_1^2}\right)$$

$$= \frac{1}{2}\left(1 + A\left(\gamma + \frac{B}{A}\right)^2 + C - \frac{B^2}{A}\right)\,. \tag{2.145}$$

From (2.137), it follows that the minimum value of $g(\gamma; \alpha_1, \beta_1)$ is given by

$$g(\gamma_{\mathrm{m}}; \alpha_1, \beta_1) = C - \frac{B^2}{A} \tag{2.146}$$

where

$$\gamma_{\mathrm{m}} := -\frac{B}{A}. \tag{2.147}$$

By direct calculation, one easily sees that (2.146) can take values less than -1. For example, letting $\alpha_1 = 0.5$ and $\beta_1 = 0.87$ gives $g(\gamma_{\mathrm{m}}; 0.5, 0.87) = -1.34$ with $\gamma_{\mathrm{m}} = -1.94$. Substituting these values into (2.136) gives a negative result. Next, ones observes that for γ fixed, $\psi(|\boldsymbol{E}|) \to 1.0$ and $\beta/|\boldsymbol{E}| \to 0$ as $\beta \to 0$, that is, as the upper bound on strains grows to infinity. It follows that (2.129) can be made arbitrarily small while (2.128) remains negative and uniformly bounded away from zero making the sum of (2.128) and (2.129) negative, thereby violating *strong ellipticity*.

2.6 Conclusions

This Chapter investigates the question of *strong ellipticity* (rank-1 convexity) for implicit constitutive and strain-limiting models of elastic-like (non-dissipative) material bodies. A general strategy is suggested for addressing the question for general implicit theories which is then applied to a variety of strain-limiting models inspired by examples considered recently by Rajagopal and a number of co-authors. It is shown for the classes of models studied herein, that *strong ellipticity* holds in the small strain limit and that it fails if strains can become sufficiently large. Failure of *strong ellipticity* was shown to occur for both purely compressive and simple shear deformations. It should be noted that most of the emphasis of Rajagopal and co-authors in strain-limiting theories of elastic material behavior has been directed to the infinitesimal strain setting because it provides a logically consistent means of deriving nonlinear, infinitesimal strain models from the corresponding finite strain formulation. As shown herein, it is in the small strain limit that *strong ellipticity* is guaranteed to hold for the strain-limiting models considered by Rajagopal and co-authors.

The present contribution should be viewed as a small early step in an extended program to study mathematical properties of implicit constitutive theories in elasticity. Emphasis here has been directed to a few relatively simple classes of strain-limiting models explored by Rajagopal and various co-authors. Of particular interest is extending the exploration of *strong ellipticity*, invertibility and equivalent Green elastic formulations begun herein to strain-limiting models with stronger nonlinearity of the form:

$$\boldsymbol{E} = \mathcal{F}(\bar{\boldsymbol{S}}) := \phi_0(\bar{\boldsymbol{S}})\boldsymbol{I} + \phi_1(\bar{\boldsymbol{S}})\bar{\boldsymbol{S}} + \phi_2(\bar{\boldsymbol{S}})\bar{\boldsymbol{S}}^2 \tag{2.148}$$

in which $\phi_j(\cdot),\ j = 1, 2, 3$ are scalar-valued functions. For this class of models, it is easy to produce physically meaningful examples for which (2.148) does not have a globally equivalent Cauchy elastic formulation, that is, the graph of the relation (2.148) is not also the graph of a constitutive relation of the form

$$\bar{\boldsymbol{S}} = \mathcal{F}^{-1}(\boldsymbol{E}). \tag{2.149}$$

Moreover, the class of models (2.148) also gives rise to examples for which the locally defined (that is, from a branch of the graph of (2.148)) response function $\hat{\boldsymbol{S}}(\boldsymbol{F})$ is not differentiable, so that the relevant rank-one convexity notion to consider is not (2.18), but rather the more general *monotonicity* condition (2.24). These issues will be addressed in Chapter 3.

3. MONOTONICITY FOR STRAIN-LIMITING THEORIES OF ELASTICITY*

This Chapter introduces Monotonicity for Strain-Limiting Theories of Elasticity [11].

3.1 Motivation

As we discussed in Chapter 2, in [12], the strain-limiting models have the Green-St.Venant strain written as a nonlinear response function of the second Piola-Kirchhoff stress, and it was assumed that the nonlinear response function has the Fréchet derivative invertible as a fourth-order tensor. However, in some important classes of models introduced by Rajagopal and co-authors, this invertibility condition fails. We investigate here the more general notion of *monotonicity* for such strain-limiting models.

The study of strong ellipticity in nonlinear elasticity typically involves determining, for a particular constitutive relation, classes of deformation gradients for which strong ellipticity holds and conversely classes of deformation gradients for which it fails [9, 13, 21, 22, 23]. Of particular interest in this context is the work of Merodio and Ogden [13]. For the class of strain limiting constitutive models considered herein, *monotonicity* is investigated for several classes of deformations including pure compression and simple shear. We show that *monotonicity* holds for deformations with (a suitable) strain having sufficiently small norm; whereas, counterexamples are constructed to demonstrate the failure of mononicity for appropriately chosen deformations.

3.2 Notation and Preliminaries

In this Chapter, we adopt the same notation and preliminaries from Chapter 2, and introduce some new ones. For a Cauchy Elastic body whose First Piola-Kirchhoff response

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function $\hat{\boldsymbol{S}}(\cdot)$ is not differentiable at some deformation gradient $\boldsymbol{F}$, we consider a weaker rank-1 convexity notion provided by the *monotonicity* condition [1]:

$$(\hat{\boldsymbol{S}}(\boldsymbol{F} + \alpha \boldsymbol{H}) - \hat{\boldsymbol{S}}(\boldsymbol{F})) \cdot \boldsymbol{H} > 0 \tag{3.1}$$

for all rank-1 tensors $\boldsymbol{H} = \boldsymbol{a} \otimes \boldsymbol{b}$ (with $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$) and $0 < \alpha \leq 1$ such that

$$\det(\boldsymbol{F} + \alpha \boldsymbol{H}) > 0. \tag{3.2}$$

In case the response function $\hat{\boldsymbol{S}}(\cdot)$ is differentiable, a weaker form of (2.18) is derived from (2.24) with ''$>$'' replaced by ''$\geq$''.

Let us recall from Chapter 2 that an interesting special class of strain-limiting constitutive relations introduced by Rajagopal and co-authors in [17, 18, 19] has the form:

$$\mathcal{F}(\boldsymbol{T}) = \phi_0(\boldsymbol{T})\boldsymbol{I} + \phi_1(\boldsymbol{T})\boldsymbol{T} + \phi_2(\boldsymbol{T})\boldsymbol{T}^2, \tag{3.3}$$

where $\boldsymbol{I}$ denotes the second-order identity tensor and each $\phi_j(\boldsymbol{T})$ is a scalar valued function of the isotropic invariants of $\boldsymbol{T}$ with $|\phi_0(\boldsymbol{T})|$, $|\phi_1(\boldsymbol{T})||\boldsymbol{T}|$, and $|\phi_2(\boldsymbol{T})||\boldsymbol{T}^2|$ all uniformly bounded functions on $\mathcal{S}$ym, the space of symmetric, second-order tensors. Strong ellipticity was investigated in [12] for models inspired by (3.3) for the special case of $\phi_2(\cdot) = 0$. The case of $\phi_2(\cdot) \neq 0$ is more difficult and is the subject of the present Chapter.

In the direction of studying strong ellipticity and relaxing the assumption of isotropy, we considered in [12] the following constitutive relations as substitutions for (2.9) and (2.10)

$$\boldsymbol{0} = \mathcal{F}(\boldsymbol{E}, \bar{\boldsymbol{S}}) \tag{3.4}$$

$$\boldsymbol{E} = \mathcal{F}(\bar{\boldsymbol{S}}). \tag{3.5}$$

The current study focuses on the special case of (2.15) in which the response function $\mathcal{F}(\cdot)$ is uniformly bounded in norm. Appealing to the classical Cayley-Hamilton theorem, the response function $\mathcal{F}(\bar{\boldsymbol{S}})$ has the general representation:

$$\mathcal{F}(\bar{\boldsymbol{S}}) = \phi_0(\bar{\boldsymbol{S}})\boldsymbol{I} + \phi_1(\bar{\boldsymbol{S}})\bar{\boldsymbol{S}} + \phi_2(\bar{\boldsymbol{S}})\bar{\boldsymbol{S}}^2, \tag{3.6}$$

where the coefficient functions $\phi_j(\cdot)$ are scalar valued, and in the strain limiting case, they satisfy the additional assumption that $|\phi_0(\bar{\boldsymbol{S}})|$, $|\phi_1(\bar{\boldsymbol{S}})||\bar{\boldsymbol{S}}|$ and $|\phi_2(\bar{\boldsymbol{S}})||\bar{\boldsymbol{S}}^2|$ are uniformly bounded. To date, in the various applications, analyses and numerical simulations of strain limiting models of the form (3.3) or (3.6) that have appeared in the literature, attention has been limited to the special case in which $\phi_2(\cdot)$ is identically zero. This is due primarily to two considerations, the first being that when $\phi_2(\cdot)$ is non-zero, analysis of (3.3) or (3.6) encounters significant added complexity, and the second being that even with $\phi_2(\cdot)$ identically zero, the models still exhibit a rich array of behaviors such as non-linear stress-strain response even in the infinitesimal strain regime.

An important issue with models of the form (3.6) is invertibility of the response function. Even when $\phi_2(\cdot)$ is identically zero, one can readily construct strain-limiting models in the class (3.6) for which $\mathcal{F}(\cdot)$ is not uniquely invertible, giving rise to stress-mediated bifurcations. However, for the analyses in [12], [20] and [8], attention was restricted to cases in which $\mathcal{F}(\cdot)$ is uniquely invertible. When $\phi_2(\cdot)$ is not identically zero, a set-valued inverse of (3.6) is the rule unless the third term in (3.6) is strongly dominated by the first two terms. Moreover, even when (3.6) is uniquely invertible, its Fréchet derivative need not be (when viewed as a linear transformation on the space of second-order tensors), thwarting attempts to generalize the approach utilized in [12] for studying strong ellipticity for (3.6) when $\phi_2(\cdot)$ is identically zero to the case when it is not. The primary purpose of the present contribution is to show that the approach taken in [12] for studying strong ellipticity for (3.6) when $\phi_2(\cdot)$ is identically zero can be generalized to the case $\phi_2(\cdot)$ not identically zero by consideration of the weaker notion of convexity, *monotonicity*, that does not require Fréchet differentiability of $\mathcal{F}(\cdot)$.

As mentioned above, an important feature of the class of models (3.6) is their ability to capture a nonlinear stress-strain response even in the infinitesimal strain limit. It is important to point out that the infinitesimal strain limit of (3.6) need not be a limiting strain theory. However, when it is, it provides an appealing framework in which to study brittle fracture as illustrated in [20] and [8]. These studies adopted a special case of (3.6) in which $\phi_0(\cdot)$ and $\phi_2(\cdot)$ both vanish identically and

$$\boldsymbol{E} = \phi_1(\bar{\boldsymbol{S}})\bar{\boldsymbol{S}} = \tilde{\phi}(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}} = \frac{\bar{\boldsymbol{S}}}{1+\beta|\bar{\boldsymbol{S}}|}\,. \tag{3.7}$$

In [12], it was shown that this model is hyperelastic and strongly elliptic in the small strain regime. It was also shown that strong ellipticity fails if the small strain assumption is relaxed. Moreover, through asymptotic arguments, it was demonstrated in [20] and [8] that stress as well as strain is controlled in the neighborhood of a crack tip. Additionally, recent direct numerical simulations support these asymptotic predictions and illustrate how the use of such strain limiting models in numerical simulations of brittle fracture obviate the need for extensive mesh refinement near a crack tip, or the introduction of cohesive or process zone crack tip models. These numerical simulations also show that the linearly elastic solution is recovered outside of a small neighborhood of a crack tip and globally as $\beta \to 0$.

It is straightforward to adapt the analyses in [20], [8] and [12] to more general models of the form $\boldsymbol{E} = \phi(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}}$ with $\phi(r)$ bounded, nonnegative and decreasing for $r > 0$, and satisfying $r\phi(r) \leq M < \infty$ uniformly for $0 < r < \infty$, along with mild smoothness assumptions. However, the analysis in these works was restricted to the simple model (3.7) to avoid unnecessary complication that might obscure the essence of the arguments.

In the spirit of these past investigations, the present contribution focusses on the sub-class of (3.6) taking the form:

$$\mathcal{F}(\bar{\boldsymbol{S}}) = \phi_1(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}} + \phi_2(|\bar{\boldsymbol{S}}|^2)\bar{\boldsymbol{S}}^2, \tag{3.8}$$

in which

$$\phi_1(r) := \frac{\alpha_1}{1+\beta_1 r}\,, \tag{3.9}$$

$$\phi_2(r) := \frac{\alpha_2}{1+\beta_2 r}\,, \tag{3.10}$$

where α_1, α_2, β_1 and β_2 are nonnegative constants. In order for the model (3.8)...(3.10) to be strain limiting, β_1 and β_2 are assumed to be strictly positive, and stability demands that α_1 also be strictly positive. However, both α_2 positive and negative lead to physically relevant models. In the analysis to follow, it will be assumed that the response function (3.8)...(3.10) is uniquely invertible, which can be guaranteed provided the function $r\phi_2(r)$ is dominated by the function $r\phi_1(r)$, in a sense to be made precise below. When such is the case, the convexity analysis presented below is valid for α_2 positive, negative and zero. Therefore, without loss of generality, we assume that α_2 is nonnegative. When the function $r\phi_2(r)$ is not suitably dominated by the function $r\phi_1(r)$ and the response function (3.8)...(3.10) has a multi-valued inverse, then one must take account of the sign of α_2 in investigations of convexity. This results in a more delicate analysis that will be the topic of a future contribution. Also, we note that a necessary condition for the model (3.8) to correspond to a hyperelastic material response is for the coefficient function $\phi_2(r)$ to be identically constant. In particular, no strain limiting model of the form (3.8) with $\phi_2(r)$ not identically zero can be hyperelastic. In the appendix, we provide a brief discussion of this issue. Finally, through nondimensionalization, we can set $\alpha_1 = 1$.

The following notations are used for the rest of this Chapter. We denote by $\boldsymbol{H}_s$ the symmetric part of $\boldsymbol{H}$, $\boldsymbol{H}_s := (\boldsymbol{H} + \boldsymbol{H}^{\mathrm{T}})/2$, for any second-order tensor $\boldsymbol{H}$. We notice and also introduce new notations as follows:

$$\mathrm{D}_{\bar{\boldsymbol{S}}}\left(|\bar{\boldsymbol{S}}|\right)[\boldsymbol{H}] = \mathrm{D}_{\bar{\boldsymbol{S}}}\left(\bar{\boldsymbol{S}}\cdot\bar{\boldsymbol{S}}\right)^{\frac{1}{2}}[\boldsymbol{H}] = \frac{1}{2}\left(\bar{\boldsymbol{S}}\cdot\bar{\boldsymbol{S}}\right)^{-\frac{1}{2}}\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\bar{\boldsymbol{S}}\cdot\bar{\boldsymbol{S}}\right)[\boldsymbol{H}] = \frac{\boldsymbol{H}\cdot\bar{\boldsymbol{S}}}{|\bar{\boldsymbol{S}}|}\,. \tag{3.11}$$

Let

$$\boldsymbol{A} := \frac{\bar{\boldsymbol{S}}}{|\bar{\boldsymbol{S}}|}, \ \boldsymbol{B} := \frac{\bar{\boldsymbol{S}}^2}{\left|\bar{\boldsymbol{S}}^2\right|}, \text{ and } \boldsymbol{P} := \frac{\boldsymbol{I}}{|\boldsymbol{I}|}, \tag{3.12}$$

so that

$$|\boldsymbol{A}| = |\boldsymbol{B}| = |\boldsymbol{P}| = 1, \tag{3.13}$$

and let $\mathbb{I}$ denote the fourth-order identity tensor.

3.3 Monotonicity - General Observation

As shown in [12], strong ellipticity for the class of strain-limiting models (2.15) requires:

$$\begin{aligned} \boldsymbol{H} \cdot \mathrm{D}_{\boldsymbol{F}} \hat{\boldsymbol{S}}(\boldsymbol{F})[\boldsymbol{H}] &= \boldsymbol{H} \cdot (\boldsymbol{H}\bar{\boldsymbol{S}}(\boldsymbol{E})) \\ &\quad - \left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}\right)_s \cdot \left(\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\right)^{-1}\left[(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H})_s\right] > 0 \end{aligned} \tag{3.14}$$

where $\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\right)^{-1}[\cdot]$ denotes the inverse of the fourth-order tensor $\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})[\cdot]$.

When the fourth-order tensor $\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\right)[\cdot]$ is not invertible, as will be shown to be the case for the models (3.6), we investigate *monotonicity* as a substitute for strong ellipticity. When (2.15) is uniquely invertible, we make use of an equivalent form of *monotonicity* (2.24):

$$(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}}) \cdot \boldsymbol{H} > 0, \tag{3.15}$$

where

$$\tilde{\boldsymbol{F}} = \boldsymbol{F} + \alpha\boldsymbol{H}, \tag{3.16}$$

$$\widehat{\boldsymbol{S}}(\boldsymbol{F}) = \boldsymbol{F}\widehat{\bar{\boldsymbol{S}}}(\boldsymbol{E}) = \boldsymbol{F}\bar{\boldsymbol{S}}, \tag{3.17}$$

and

$$\widehat{\boldsymbol{S}}(\boldsymbol{F} + \alpha\boldsymbol{H}) = \tilde{\boldsymbol{F}}\widehat{\bar{\boldsymbol{S}}}(\tilde{\boldsymbol{E}}) = \tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}}, \tag{3.18}$$

for all rank-1 tensors $\boldsymbol{H} = \boldsymbol{a} \otimes \boldsymbol{b}$ (with $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$) and $0 < \alpha \leq 1$. Note that when $\boldsymbol{E}, \bar{\boldsymbol{S}}$ satisfy (2.15), so do $\tilde{\boldsymbol{E}}, \tilde{\boldsymbol{S}}$.

3.4 Compression and Dilation

Consider a model of the form (3.8) with

$$\boldsymbol{F} = \gamma \boldsymbol{I}, \tag{3.19}$$

γ is a positive scalar. Compression and dilation correspond to $0 < \gamma < 1$ and $1 < \gamma$, respectively. Then, the Green-St.Venant strain is

$$\boldsymbol{E} = \frac{1}{2}(\gamma^2 - 1)\boldsymbol{I}, \tag{3.20}$$

and from the constitutive relation (3.8), the stress has the form

$$\bar{\boldsymbol{S}} = \bar{\sigma}\boldsymbol{I}, \tag{3.21}$$

where $\bar{\sigma}$ is a constant. Now, (3.8) becomes

$$\frac{1}{2}(\gamma^2 - 1) = g(\bar{\sigma}) := \frac{\bar{\sigma}}{1 + \sqrt{3}\beta_1|\bar{\sigma}|} + \frac{\alpha_2\bar{\sigma}^2}{1 + 3\beta_2\bar{\sigma}^2}\,. \tag{3.22}$$

We note that letting $\gamma \to 0+$ in (3.22) imposes a lower bound on compressive stresses $\bar{\sigma} < 0$ while letting $\bar{\sigma} \to \infty$ imposes an upper bound on dilatational strains $\gamma > 1$.

3.4.1 Invertibility

In this section, we investigate the questions of unique invertibility of the response function (3.8) and its Fréchet derivative (viewed as a linear transformation on the space of symmetric second-order tensors) for the special cases of pure compression and dilation. We show that (3.8) is uniquely invertible for dilation but can fail to be for compression except for sufficiently small strains. It is also shown that even when (3.8) is uniquely invertible, its Fréchet derivative need not be. As noted above, when this is the case, the approach to strong ellipticity utilized in [12] cannot be applied to (3.8) which motivates our consideration of *monotonicity* in the following section.

From (3.22), a straightforward calculation gives:

$$g'(\bar{\sigma}) = \frac{1}{(1+\sqrt{3}\beta_1|\bar{\sigma}|)^2} + \frac{2\alpha_2\bar{\sigma}}{(1+3\beta_2\bar{\sigma}^2)^2} \,. \tag{3.23}$$

Since $\alpha_2 \geq 0$, it follows that if $\bar{\sigma} > 0$ (dilation), then $g'(\bar{\sigma}) > 0$, and thus (3.8) is uniquely invertible under dilation. In case $\bar{\sigma} < 0$ (compression), we show below that there exists an interval of $\bar{\sigma} < 0$ in which $g'(\bar{\sigma}) > 0$, i.e. (3.8) is uniquely invertible.

We now consider the invertibility of the fourth-order tensor $\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\right)[\cdot]$ for models of the form (3.8). With the notations $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{P}$ as defined in (3.12) and (3.13), Fréchet differentiation of the right hand side of (3.8) gives:

$$\begin{aligned}
\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\mathcal{F}\left(\bar{\boldsymbol{S}}\right)\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] &= \mathrm{D}_{\bar{\boldsymbol{S}}}\left(\phi_1\left(|\bar{\boldsymbol{S}}|\right)\bar{\boldsymbol{S}} + \phi_2\left(|\bar{\boldsymbol{S}}|^2\right)\bar{\boldsymbol{S}}^2\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\
&= \phi_1'\left(|\bar{\boldsymbol{S}}|\right)\left(D_{\bar{\boldsymbol{S}}}\left(|\bar{\boldsymbol{S}}|\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\right)\bar{\boldsymbol{S}} + \phi_1\left(|\bar{\boldsymbol{S}}|\right)D_{\bar{\boldsymbol{S}}}\bar{\boldsymbol{S}}\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\
&+ \ \phi_2'\left(|\bar{\boldsymbol{S}}|^2\right)\left(D_{\bar{\boldsymbol{S}}}\left(|\bar{\boldsymbol{S}}|^2\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\right)\bar{\boldsymbol{S}}^2 \\
&+ \ \phi_2\left(|\bar{\boldsymbol{S}}|^2\right)D_{\bar{\boldsymbol{S}}}\left(\bar{\boldsymbol{S}}^2\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\
&= \phi_1'\left(|\bar{\boldsymbol{S}}|\right)\frac{\left(\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\cdot\bar{\boldsymbol{S}}\right)}{|\bar{\boldsymbol{S}}|}\bar{\boldsymbol{S}} + \phi_1\left(|\bar{\boldsymbol{S}}|\right)\left(\mathbb{I}\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\right) \\
&+ \ \phi_2'\left(|\bar{\boldsymbol{S}}|^2\right)\left(2\bar{\boldsymbol{S}}\cdot\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right)\bar{\boldsymbol{S}}^2 \\
&+ \ \phi_2\left(|\bar{\boldsymbol{S}}|^2\right)\left(\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\bar{\boldsymbol{S}} + \bar{\boldsymbol{S}}\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right) \\
&= \left(\left(\frac{-\beta_1|\bar{\boldsymbol{S}}|}{\left(1+\beta_1|\bar{\boldsymbol{S}}|\right)^2}\right)(\boldsymbol{A}\otimes\boldsymbol{A}) + \frac{1}{1+\beta_1|\bar{\boldsymbol{S}}|}\mathbb{I}\right. \\
&\quad + \left.\left(\frac{-2\alpha_2\beta_2\left|\bar{\boldsymbol{S}}^2\right||\bar{\boldsymbol{S}}|}{\left(1+\beta_2|\bar{\boldsymbol{S}}|^2\right)^2}\right)(\boldsymbol{B}\otimes\boldsymbol{A})\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\
&+ \left(\frac{\alpha_2}{1+\beta_2|\bar{\boldsymbol{S}}|^2}\right)\left(\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\bar{\boldsymbol{S}} + \bar{\boldsymbol{S}}\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right) \,.
\end{aligned} \tag{3.24}$$

In the case of compression, again with the notation $\boldsymbol{P}$ as defined in (3.12) and (3.13), we have

$$\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\mathcal{F}\left(\bar{\boldsymbol{S}}\right)\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] = \left(K(\boldsymbol{P}\otimes\boldsymbol{P}) + L(\mathbb{I})\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\,, \tag{3.25}$$

where

$$L = \frac{1}{1+\sqrt{3}\beta_1|\bar{\sigma}|} + \frac{2\alpha_2\bar{\sigma}}{1+3\beta_2\bar{\sigma}^2}\,, \tag{3.26}$$

$$K = \frac{-\beta_1\sqrt{3}\,|\bar{\sigma}|}{(1+\sqrt{3}\beta_1|\bar{\sigma}|)^2} + \frac{-6\alpha_2\beta_2|\bar{\sigma}|^3}{(1+3\beta_2\bar{\sigma}^2)^2}\,. \tag{3.27}$$

We now show that the fourth-order tensor $\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\mathcal{F}\left(\bar{\boldsymbol{S}}\right)\right)[\cdot]$ on the left hand side of (3.25) is not invertible. If $\bar{\sigma} > 0$, then $L > 0$. However, when $\alpha_2 = 3, \beta_1 = 1, \beta_2 = 4$, it follows that at

$$\bar{\sigma}_0 = \frac{-\alpha_2 - \sqrt{\alpha_2^2 - (3\beta_2 - 2\sqrt{3}\alpha_2\beta_1)}}{3\beta_2 - 2\sqrt{3}\alpha_2\beta_1} = -3.5572 < 0\,, \tag{3.28}$$

$L = 0$ as an eigenvalue of $\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\mathcal{F}\left(\bar{\boldsymbol{S}}\right)\right)[\cdot]$. Therefore, $\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\mathcal{F}\left(\bar{\boldsymbol{S}}\right)\right)[\cdot]$ is not invertible. From (3.22), we have $g(\bar{\sigma}) \geq -1/2$. We note that $g(\bar{\sigma}_0) = -0.2484 > -1/2,$ and $g'(\bar{\sigma}_0) = 0.0186 > 0$. Thus, (3.8) is uniquely invertible in some interval of $\bar{\sigma} < 0$ centered at $\bar{\sigma}_0$ by continuity of $g'(\bar{\sigma})$. In conclusion, the fourth-order tensor $\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\mathcal{F}\left(\bar{\boldsymbol{S}}\right)\right)[\cdot]$ on the left hand side of (3.25) is not invertible in an interval of $\bar{\sigma} < 0$ centered at $\bar{\sigma}_0$ even though the model (3.8) itself is uniquely invertible in that interval of $\bar{\sigma} < 0$.

3.4.2 Monotonicity

As shown in the previous subsection, the model (3.8), even when it has an equivalent Cauchy elastic formulation, can lack sufficient regularity to support strong ellipticity (as defined in [12]). For that reason, we now investigate the weaker convexity condition of *monotonicity* in the form (3.15). We first consider the case $\gamma = 1$, then use a continuity argument to generalize for all γ near 1.

When $\gamma = 1$, we have $\boldsymbol{F} = \boldsymbol{I}$, and thus:

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) = 0. \tag{3.29}$$

From (3.8), it follows that $\bar{\boldsymbol{S}} = 0$, and $\bar{\sigma} = 0$. Under the conditions $\boldsymbol{H} = \boldsymbol{a} \otimes \boldsymbol{b}$ (with $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$) and $0 < \alpha \leq 1$, we consider three cases: $\boldsymbol{a} = \boldsymbol{b}$, $\boldsymbol{a} \perp \boldsymbol{b}$, or $0 < |\boldsymbol{a} \cdot \boldsymbol{b}| < 1$.

Case 1: $\boldsymbol{a} = \boldsymbol{b}$. Then,

$$\tilde{\boldsymbol{F}} = \boldsymbol{F} + \alpha\boldsymbol{H} = \boldsymbol{I} + \alpha(\boldsymbol{a} \otimes \boldsymbol{a}). \tag{3.30}$$

Now,

$$\det(\boldsymbol{I} + \alpha(\boldsymbol{a} \otimes \boldsymbol{a})) = \det((\alpha + 1)(\boldsymbol{a} \otimes \boldsymbol{a}) + (\boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a})) > 0, \tag{3.31}$$

as $(\alpha + 1) > 0$. Hence, the condition (3.2) holds. Next,

$$\tilde{\boldsymbol{E}} = \frac{1}{2}(\tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}} - \boldsymbol{I}) = (\alpha + \frac{\alpha^2}{2})\boldsymbol{a} \otimes \boldsymbol{a}\,. \tag{3.32}$$

From the constitutive relation (3.8), it follows that

$$\tilde{\boldsymbol{S}} = \tilde{\sigma}(\boldsymbol{a} \otimes \boldsymbol{a}), \tag{3.33}$$

where $\tilde{\sigma}$ is a scalar. We also deduce that $|\tilde{\boldsymbol{S}}| = |\tilde{\sigma}|$. Then, the constitutive relation

$$\tilde{\boldsymbol{E}} = \phi_1(|\tilde{\boldsymbol{S}}|)\tilde{\boldsymbol{S}} + \phi_2(|\tilde{\boldsymbol{S}}|^2)\tilde{\boldsymbol{S}}^2 \tag{3.34}$$

becomes

$$\alpha + \frac{\alpha^2}{2} = g(\tilde{\sigma}) := \phi_1(\tilde{\sigma})\tilde{\sigma} + \phi_2(\tilde{\sigma}^2)\tilde{\sigma}^2\,. \tag{3.35}$$

Since the left hand side of (3.35) is always positive, it follows that $\tilde{\sigma} > 0$. Then, $g'(\tilde{\sigma}) > 0$ by an argument as for (3.23), and thus (3.34) is uniquely invertible.

Now, the left hand side of (3.15) becomes

$$(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}}) \cdot (\boldsymbol{a} \otimes \boldsymbol{a}) = ((\boldsymbol{I} + \alpha(\boldsymbol{a} \otimes \boldsymbol{a}))(\tilde{\sigma}(\boldsymbol{a} \otimes \boldsymbol{a}))) \cdot (\boldsymbol{a} \otimes \boldsymbol{a})$$

$$= (1+\alpha)\tilde{\sigma} > 0 \tag{3.36}$$

because $\tilde{\sigma} > 0$, and then *monotonicity* also holds for all γ belonging to an interval centered at $\gamma = 1$ by continuity, given α in $(0, 1]$.

Remark 4. In this case for (3.8), the *monotonicity* condition (3.36) implies the invertibility condition, but the reverse statement does not hold.

Case 2: $\boldsymbol{a} \perp \boldsymbol{b}$. We have

$$\tilde{\boldsymbol{F}} = \boldsymbol{I} + \alpha(\boldsymbol{a} \otimes \boldsymbol{b}), \tag{3.37}$$

and note that $\det(\boldsymbol{I} + \alpha(\boldsymbol{a} \otimes \boldsymbol{b})) > 0$ for α sufficiently near 0. Next,

$$\tilde{\boldsymbol{E}} = \frac{1}{2}(\tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}} - \boldsymbol{I}) = \frac{1}{2}(\alpha(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) + \alpha^2(\boldsymbol{b} \otimes \boldsymbol{b}))\,. \tag{3.38}$$

The eigenvectors of $\tilde{\boldsymbol{E}}$ associated with the nonzero eigenvalues of $\tilde{\boldsymbol{E}}$ then have the following form

$$\boldsymbol{v} = \eta_1\boldsymbol{a} + \eta_2\boldsymbol{b}, \tag{3.39}$$

corresponding to eigenvalues

$$\lambda = \frac{1}{2}\alpha\delta\,, \tag{3.40}$$

where $\eta_1\eta_2 \neq 0$, and

$$\delta := \frac{\eta_2}{\eta_1}\,. \tag{3.41}$$

One can easily verify that δ must satisfy

$$\delta^2 - \alpha\delta - 1 = 0, \tag{3.42}$$

and thus have the specific form

$$\delta_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2+4}}{2}, \tag{3.43}$$

in which $\delta_1 > 0, \delta_2 < 0$. Correspondingly,

$$\lambda_{1,2} = \frac{1}{4}\alpha\left(\alpha \pm \sqrt{\alpha^2+4}\right), \tag{3.44}$$

and $\lambda_1 > 0, \lambda_2 < 0$. Let

$$\delta_1 = \frac{\eta_2}{\eta_1},\ \delta_2 = \frac{\eta_2'}{\eta_1'}. \tag{3.45}$$

Without loss of generality, we can choose $\eta_1 = 1, \eta_1' = 1$. Denote by $\tilde{\boldsymbol{v}}_{1,2}$ the normalized eigenvectors associated with $\lambda_{1,2}$, and note that $\tilde{\boldsymbol{v}}_1 \perp \tilde{\boldsymbol{v}}_2$. Let $\tilde{\boldsymbol{v}}_3$ be an orthonormal vector to $\{\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2\}$, we then have the spectral decompositions

$$\tilde{\boldsymbol{E}} = \lambda_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + \lambda_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2), \tag{3.46}$$

and from the constitutive relation (3.8),

$$\tilde{\boldsymbol{S}} = c_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + c_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2), \tag{3.47}$$

$$\tilde{\boldsymbol{S}}^2 = c_1^2(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + c_2^2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2), \tag{3.48}$$

where c_1, c_2 are scalars. Since

$$\tilde{\boldsymbol{E}} = \phi_1(|\tilde{\boldsymbol{S}}|)\tilde{\boldsymbol{S}} + \phi_2(|\tilde{\boldsymbol{S}}|^2)\tilde{\boldsymbol{S}}^2, \tag{3.49}$$

we deduce that

$$\lambda_1 = \phi_1(|\tilde{\boldsymbol{S}}|)c_1 + \phi_2(|\tilde{\boldsymbol{S}}|^2)c_1^2, \tag{3.50}$$

$$\lambda_2 = \phi_1(|\tilde{\boldsymbol{S}}|)c_2 + \phi_2(|\tilde{\boldsymbol{S}}|^2)c_2^2. \tag{3.51}$$

Equivalently,

$$\lambda_1 = \frac{1}{4}\alpha\left(\alpha + \sqrt{\alpha^2+4}\right) = \frac{c_1}{1+\beta_1\sqrt{c_1^2+c_2^2}} + \frac{\alpha_2 c_1^2}{1+\beta_2(c_1^2+c_2^2)}, \tag{3.52}$$

$$\lambda_2 = \frac{1}{4}\alpha\left(\alpha - \sqrt{\alpha^2+4}\right) = \frac{c_2}{1+\beta_1\sqrt{c_1^2+c_2^2}} + \frac{\alpha_2 c_2^2}{1+\beta_2(c_1^2+c_2^2)}. \tag{3.53}$$

It follows that $c_1 > 0, c_2 < 0$.

We now investigate the invertibility of (3.52) and (3.53). Fixing c_2 in (3.52), then

$$\lambda_1 = g(c_1; c_2) := \frac{c_1}{1+\beta_1\sqrt{c_1^2+c_2^2}} + \frac{\alpha_2 c_1^2}{1+\beta_2(c_1^2+c_2^2)}\,. \tag{3.54}$$

Next,

$$\begin{aligned}\partial_{c_1} g(c_1; c_2) &= \frac{1}{\left(1+\beta_1\sqrt{c_1^2+c_2^2}\right)^2} + \frac{\beta_1 c_2^2}{\sqrt{c_1^2+c_2^2}\left(1+\beta_1\sqrt{c_1^2+c_2^2}\right)^2} \\ &\quad + \frac{2\alpha_2 c_1(1+\beta_2 c_2^2)}{(1+\beta_2(c_1^2+c_2^2))^2}\,. \end{aligned}\tag{3.55}$$

Letting $\alpha \to 0$ in (3.52) leads to $c_1 \to 0$. With $c_1 = 0$, we have

$$\partial_{c_1} g(0; c_2) = \frac{1}{1+\beta|c_2|} > 0\,, \tag{3.56}$$

and thus (3.54) is uniquely solvable for all α near 0 by continuity. Within such an interval of α, let

$$c_1 = g^{-1}(\lambda_1; c_2) := h(c_2; \lambda_1). \tag{3.57}$$

A substitution of (3.57) into (3.53) gives

$$\lambda_2 = k(c_2; \lambda_1) := \frac{c_2}{1+\beta_1\sqrt{c_1^2+c_2^2}} + \frac{\alpha_2 c_2^2}{1+\beta_2(c_1^2+c_2^2)}\,, \tag{3.58}$$

in which $c_1 = h(c_2; \lambda_1)$ is a function of c_2 as in (3.57). We now note that

$$\begin{aligned}\partial_{c_2} h(c_2; \lambda_1) &= \partial_{c_2} g^{-1}(\lambda_1; c_2) \\ &= \frac{-\partial_{c_2} g(g^{-1}(\lambda_1; c_2); c_2)}{\partial_{c_1} g(g^{-1}(\lambda_1; c_2); c_2)} \\ &= \frac{\dfrac{\beta_1 c_1 c_2}{\sqrt{c_1^2+c_2^2}\left(1+\sqrt{c_1^2+c_2^2}\right)^2} + \dfrac{2\alpha_2\beta_2 c_1^2 c_2}{\left(1+\beta_2(c_1^2+c_2^2)\right)^2}}{\partial_{c_1} g(c_1; c_2)}\,. \end{aligned}\tag{3.59}$$

Therefore,

$$
\begin{aligned}
\partial_{c_2} k(c_2;\lambda_1) &= \frac{\sqrt{c_1^2+c_2^2}+\beta_1 c_1(c_1-c_2\partial_{c_2}h(c_2;\lambda_1))}{\sqrt{c_1^2+c_2^2}\left(1+\beta_1\sqrt{c_1^2+c_2^2}\right)^2} \\
&+ \frac{2\alpha_2 c_2(1+\beta_2 c_1^2-\beta_2 c_1 c_2 \partial_{c_2}h(c_2;\lambda_1))}{\left(1+\beta_2(c_1^2+c_2^2)\right)^2} \\
&= \frac{1}{\left(1+\beta_1\sqrt{c_1^2+c_2^2}\right)^2} \\
&+ \frac{\beta_1 c_1\left(\frac{c_1}{\left(1+\beta_1\sqrt{c_1^2+c_2^2}\right)^2}+\frac{2\alpha_2 c_1^2}{\left(1+\beta_2(c_1^2+c_2^2)\right)^2}\right)}{\sqrt{c_1^2+c_2^2}\left(1+\beta_1\sqrt{c_1^2+c_2^2}\right)^2\partial_{c_1}g(c_1;c_2)} \\
&+ \frac{2\alpha_2 c_2\left((1+\beta_2 c_1^2)\sqrt{c_1^2+c_2^2}+\beta_1 c_2^2\right)}{\sqrt{c_1^2+c_2^2}\left(1+\beta_1\sqrt{c_1^2+c_2^2}\right)^2\left(1+\beta_2(c_1^2+c_2^2)\right)^2\partial_{c_1}g(c_1;c_2)} \\
&+ \frac{4\alpha_2^2 c_1 c_2(1+\beta_2 c_1^2+\beta_2 c_2^2)}{\left(1+\beta_2(c_1^2+c_2^2)\right)^2\partial_{c_1}g(c_1;c_2)}\,. \qquad (3.60)
\end{aligned}
$$

When $\alpha \to 0$ in both (3.50) and (3.51), we get $c_1 \to 0$, $c_2 \to 0$, which makes $\partial_{c_2}k(c_2;\lambda_1)$ in (3.60) positive. Hence, by a continuity argument, (3.58) is uniquely solvable, leading to (3.49) uniquely invertible in an interval of α near 0.

Now, the left hand side of the *monotonicity* condition (3.15) becomes

$$
\begin{aligned}
(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}}-\boldsymbol{F}\bar{\boldsymbol{S}})\cdot(\boldsymbol{a}\otimes\boldsymbol{b}) &= ((\boldsymbol{I}+\alpha(\boldsymbol{a}\otimes\boldsymbol{b}))(c_1(\tilde{\boldsymbol{v}}_1\otimes\tilde{\boldsymbol{v}}_1)+c_2(\tilde{\boldsymbol{v}}_2\otimes\tilde{\boldsymbol{v}}_2)))\cdot(\boldsymbol{a}\otimes\boldsymbol{b}) \\
&= c_1(\boldsymbol{a}\cdot\tilde{\boldsymbol{v}}_1)(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_1)+c_2(\boldsymbol{a}\cdot\tilde{\boldsymbol{v}}_2)(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_2)+\alpha c_1(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_1)^2+\alpha c_2(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_2)^2 \\
&= \frac{c_1}{|\boldsymbol{v}_1|^2}\eta_1\eta_2+\frac{c_2}{|\boldsymbol{v}_2|^2}\eta_1'\eta_2'+\frac{\alpha c_1}{|\boldsymbol{v}_1|^2}\eta_2^2+\frac{\alpha c_2}{|\boldsymbol{v}_2|^2}\eta_2'^2
\end{aligned}
$$

$$= \frac{c_1\eta_2}{1+\eta_2^2} + \frac{c_2\eta_2'}{1+\eta_2'^2} + \frac{\alpha c_1\eta_2^2}{1+\eta_2^2} + \frac{\alpha c_2\eta_2'^2}{1+\eta_2'^2}\,. \tag{3.61}$$

Dividing the right hand side of (3.61) by α, then letting $\alpha \to 0$, we have $c_1 \to 0, c_2 \to 0$ by the uniqueness of solution of the system (3.52) and (3.53); also, $\eta_2 \to 1$, $\eta_2' \to -1$, and thus

$$J := \lim_{\alpha\to 0} \frac{c_1}{2\alpha} + \left(-\lim_{\alpha\to 0}\frac{c_2}{2\alpha}\right)\,. \tag{3.62}$$

Let

$$m_1 = \lim_{\alpha\to 0}\frac{c_1}{2\alpha} > 0\,.$$

Then, there exists $\alpha_0 > 0$ such that

$$\frac{c_1}{2\alpha} > \frac{m_1}{2}$$

for all α in the interval $(0, \alpha_0)$. This result holds for $\gamma = 1$, so it must hold for an interval of γ by continuity of the given model (3.8). More specifically, there exist $\tilde{m}_1 > 0$ and $\gamma_0 > 0$ such that

$$\frac{c_1}{2\alpha} > \tilde{m}_1 > 0$$

for all $|\gamma - 1| < \gamma_0$. Applying this argument again to the second summand of J shows that J is uniformly bounded below by a positive constant. In summary, αJ is positive, and thus the right hand side of (3.61) is also positive so that *monotonicity* holds for the given model (3.8) in an interval of α sufficiently near 0, and in an interval of γ centered at $\gamma = 1$.

Case 3: $0 < |\boldsymbol{a}\cdot\boldsymbol{b}| < 1$. We still have

$$\tilde{\boldsymbol{F}} = \boldsymbol{I} + \alpha(\boldsymbol{a}\otimes\boldsymbol{b}), \tag{3.63}$$

and det$(\boldsymbol{I} + \alpha(\boldsymbol{a}\otimes\boldsymbol{b})) > 0$ for all α sufficiently near 0. Similar to Case 2,

$$\tilde{\boldsymbol{E}} = \frac{1}{2}(\tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}} - \boldsymbol{I}) = \frac{1}{2}(\alpha(\boldsymbol{a}\otimes\boldsymbol{b} + \boldsymbol{b}\otimes\boldsymbol{a}) + \alpha^2(\boldsymbol{b}\otimes\boldsymbol{b}))\,. \tag{3.64}$$

The eigenvectors of $\tilde{\boldsymbol{E}}$ corresponding to the nonzero eigenvalues of $\tilde{\boldsymbol{E}}$ then have the following form

$$\boldsymbol{v} = \eta_1 \boldsymbol{a} + \eta_2 \boldsymbol{b}, \tag{3.65}$$

associated with eigenvalues

$$\lambda = \frac{1}{2}\alpha(\boldsymbol{a}\cdot\boldsymbol{b} + \delta)\,, \tag{3.66}$$

where $\eta_1\eta_2 \neq 0$, and

$$\delta := \frac{\eta_2}{\eta_1}\,. \tag{3.67}$$

One can readily show that δ must satisfy

$$\delta^2 - \alpha\delta - 1 - \alpha(\boldsymbol{a}\cdot\boldsymbol{b}) = 0, \tag{3.68}$$

and thus have specific form

$$\delta_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 4\alpha(\boldsymbol{a}\cdot\boldsymbol{b}) + 4}}{2}, \tag{3.69}$$

in which $\delta_1 > 0, \delta_2 < 0$. Correspondingly,

$$\lambda_{1,2} = \frac{1}{4}\alpha\left(2(\boldsymbol{a}\cdot\boldsymbol{b}) + \alpha \pm \sqrt{\alpha^2 + 4\alpha(\boldsymbol{a}\cdot\boldsymbol{b}) + 4}\right)\,. \tag{3.70}$$

Since

$$\lambda_1\lambda_2 = \frac{1}{16}\alpha^2\left(4(\boldsymbol{a}\cdot\boldsymbol{b})^2 - 4\right) < 0\,, \tag{3.71}$$

it follows that $\lambda_1 > 0, \lambda_2 < 0$. Let

$$\delta_1 = \frac{\eta_2}{\eta_1},\ \delta_2 = \frac{\eta_2'}{\eta_1'}\,. \tag{3.72}$$

The constitutive relation

$$\tilde{\boldsymbol{E}} = \phi_1(|\tilde{\boldsymbol{S}}|)\tilde{\boldsymbol{S}} + \phi_2(|\tilde{\boldsymbol{S}}|^2)\tilde{\boldsymbol{S}}^2 \tag{3.73}$$

is then uniquely invertible in an interval of α near 0 by a similar continuity argument to Case 2.

Without loss of generality, we assume that $\eta_1 = \eta_1' = 1$. Using the same notations and argument as in Case 2, we deduce the expression for the left hand side of the *monotonicity* condition (3.15) as follows:

$$
\begin{aligned}
(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}}) \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) &= ((\boldsymbol{I} + \alpha(\boldsymbol{a} \otimes \boldsymbol{b}))(c_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + c_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2))) \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) \\
&= c_1(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_1)(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_1) + c_2(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_2)(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_2) + \alpha c_1(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_1)^2 + \alpha c_2(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_2)^2 \\
&= \frac{c_1}{|\boldsymbol{v}_1|^2}(\eta_1 + \eta_2(\boldsymbol{a} \cdot \boldsymbol{b}))(\eta_1(\boldsymbol{a} \cdot \boldsymbol{b}) + \eta_2) \\
&\quad + \frac{c_2}{|\boldsymbol{v}_2|^2}(\eta_1' + \eta_2'(\boldsymbol{a} \cdot \boldsymbol{b}))(\eta_1'(\boldsymbol{a} \cdot \boldsymbol{b}) + \eta_2') \\
&\quad + \frac{\alpha c_1}{|\boldsymbol{v}_1|^2}(\eta_1 + \eta_2(\boldsymbol{a} \cdot \boldsymbol{b}))^2 + \frac{\alpha c_2}{|\boldsymbol{v}_2|^2}(\eta_1' + \eta_2'(\boldsymbol{a} \cdot \boldsymbol{b}))^2 \,. \qquad (3.74)
\end{aligned}
$$

We now apply the same argument as in Case 2. Dividing the right hand side of (3.74) by α, then letting $\alpha \to 0$, we get $c_1 \to 0, c_2 \to 0$ by the uniqueness of solution of the system (3.52) and (3.53); also $\eta_2 \to 1$, $\eta_2' \to -1$, and thus

$$
R := \left(\lim_{\alpha \to 0} \frac{c_1}{2\alpha}\right)(1 + \boldsymbol{a} \cdot \boldsymbol{b})^2 + \left(-\lim_{\alpha \to 0} \frac{c_2}{2\alpha}\right)(1 - \boldsymbol{a} \cdot \boldsymbol{b})^2 \,,
$$

Since $0 < |\boldsymbol{a} \cdot \boldsymbol{b}| < 1$, it follows that $(1 + \boldsymbol{a} \cdot \boldsymbol{b}) \neq 0$, and $(1 - \boldsymbol{a} \cdot \boldsymbol{b}) \neq 0$. With this hypothesis, in R, the first summand is uniformly bounded below by a positive constant (involving $\boldsymbol{a} \cdot \boldsymbol{b}$), and the second summand is also uniformly bounded below by a positive constant (involving $\boldsymbol{a} \cdot \boldsymbol{b}$). Therefore, αR is positive, and thus the right hand side of (3.74) is positive also, leading to *monotonicity* holds for the given model (3.8) in an interval of α near 0, and in an interval of γ centered at $\gamma = 1$.

In summary, given α_2 nonnegative, *monotonicity* holds for the given model (3.8), in an interval of α which is the intersection of the three intervals of α near 0 in the above three cases, and in an interval of γ centered at $\gamma = 1$ which is also the intersection of the three intervals of γ in the above three cases.

3.5 Simple Shear

Now (3.8) is considered for simple shear deformations of the form:

$$\boldsymbol{F} = \boldsymbol{I} + \gamma \boldsymbol{e}_1 \otimes \boldsymbol{e}_2, \tag{3.75}$$

in which, γ is a scalar satisfying det$\boldsymbol{F} > 0$, and $\boldsymbol{e}_1, \boldsymbol{e}_2$ are orthonormal vectors. Then, the corresponding Green-St.Venant strain is

$$\boldsymbol{E} = \frac{1}{2}\gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \gamma \boldsymbol{e}_2 \otimes \boldsymbol{e}_2). \tag{3.76}$$

It follows that the eigenvalues of $\boldsymbol{E}$ are $\lambda_{1,2}, 0$ with

$$\lambda_{1,2} = \frac{1}{4}\gamma\left(\gamma \pm \sqrt{\gamma^2+4}\right), \tag{3.77}$$

and note that $\lambda_1 > 0, \lambda_2 < 0$. We denote by $\tilde{\boldsymbol{u}}_{1,2}$ the normalized eigenvectors correponding to the nonzero eigenvalues of $\boldsymbol{E}$. The strain then has the following form

$$\boldsymbol{E} = \lambda_1 \tilde{\boldsymbol{u}}_1 \otimes \tilde{\boldsymbol{u}}_1 + \lambda_2 \tilde{\boldsymbol{u}}_2 \otimes \tilde{\boldsymbol{u}}_2, \tag{3.78}$$

and from the constitutive relation (3.8), the stress is

$$\bar{\boldsymbol{S}} = c_1 \tilde{\boldsymbol{u}}_1 \otimes \tilde{\boldsymbol{u}}_1 + c_2 \tilde{\boldsymbol{u}}_2 \otimes \tilde{\boldsymbol{u}}_2, \tag{3.79}$$

where c_1, c_2 are eigenvalues of $\bar{\boldsymbol{S}}$. From the constitutive equation (3.8), we get

$$\lambda_1 = \phi_1(|\bar{\boldsymbol{S}}|)c_1 + \phi_2(|\bar{\boldsymbol{S}}|^2)c_1^2, \tag{3.80}$$

$$\lambda_2 = \phi_1(|\bar{\boldsymbol{S}}|)c_2 + \phi_2(|\bar{\boldsymbol{S}}|^2)c_2^2. \tag{3.81}$$

Equivalently,

$$\lambda_1 = \frac{1}{4}\gamma\left(\gamma + \sqrt{\gamma^2+4}\right) = \frac{c_1}{1+\beta_1\sqrt{c_1^2+c_2^2}} + \frac{\alpha_2 c_1^2}{1+\beta_2(c_1^2+c_2^2)}, \tag{3.82}$$

$$\lambda_2 = \frac{1}{4}\gamma\left(\gamma - \sqrt{\gamma^2+4}\right) = \frac{c_2}{1+\beta_1\sqrt{c_1^2+c_2^2}} + \frac{\alpha_2 c_2^2}{1+\beta_2(c_1^2+c_2^2)}, \tag{3.83}$$

where α_2, β_1, and β_2 are non-negative constants.

3.5.1 Invertibility

By a similar continuity argument to the system (3.52) and (3.53), we deduce that for the system (3.82) and (3.83), the model (3.8) is uniquely invertible within an interval of γ centered at $\gamma = 0$. Indeed, fixing c_2 in the equation (3.82), then

$$\lambda_1 = g(c_1; c_2) := \frac{c_1}{1 + \beta_1\sqrt{c_1^2 + c_2^2}} + \frac{\alpha_2 c_1^2}{1 + \beta_2(c_1^2 + c_2^2)} \,. \tag{3.84}$$

Next,

$$\begin{aligned}\partial_{c_1} g(c_1; c_2) = & \frac{1}{\left(1 + \beta_1\sqrt{c_1^2 + c_2^2}\right)^2} + \frac{\beta_1 c_2^2}{\sqrt{c_1^2 + c_2^2}\left(1 + \beta_1\sqrt{c_1^2 + c_2^2}\right)^2} \\ & + \frac{2\alpha_2 c_1(1 + \beta_2 c_2^2)}{(1 + \beta_2(c_1^2 + c_2^2))^2} \,. \end{aligned} \tag{3.85}$$

Letting $\gamma \to 0$ in (3.82) leads to $c_1 \to 0$. When $c_1 = 0$

$$\partial_{c_1} g(0; c_2) = \frac{1}{1 + \beta|c_2|} > 0 \,. \tag{3.86}$$

Thus, (3.84) is uniquely solvable in an interval of γ sufficiently close to 0. Within such an interval of γ, let

$$c_1 = g^{-1}(\lambda_1; c_2) := h(c_2; \lambda_1). \tag{3.87}$$

A substitution of (3.87) into (3.83) gives

$$\lambda_2 = k(c_2; \lambda_1) := \frac{c_2}{1 + \beta_1\sqrt{c_1^2 + c_2^2}} + \frac{\alpha_2 c_2^2}{1 + \beta_2(c_1^2 + c_2^2)} \,, \tag{3.88}$$

in which $c_1 = h(c_2; \lambda_1)$ is a function of c_2 as in (3.87). When $\gamma \to 0$ in (3.82) and (3.83), we get $c_1 \to 0$, $c_2 \to 0$, which make $\partial_{c_2} k(c_2; \lambda_1)$ positive, and (3.88) uniquely solvable, thus (3.8) is uniquely invertible in an interval of γ centered at $\gamma = 0$.

We next investigate the invertibility of the fourth-order tensor $\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\right)[\cdot]$ for the class of models (3.8). Consider the case $\boldsymbol{a} = \boldsymbol{b} \perp \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. Then,

$$\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s = \left((\boldsymbol{I} + \gamma \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)(\boldsymbol{a} \otimes \boldsymbol{a})\right)_s = \boldsymbol{a} \otimes \boldsymbol{a}. \tag{3.89}$$

When $c_1 = c_2$, we have $\bar{\boldsymbol{S}}^2 = \bar{\boldsymbol{S}}$, $|\bar{\boldsymbol{S}}| = \sqrt{2}$. With the notations of $\boldsymbol{A}, \boldsymbol{B}$ as defined in (3.12) and (3.13), the Fréchet derivative of (3.8) has the following expression:

$$\begin{aligned}
\mathrm{D}_{\bar{\boldsymbol{S}}}\left(\mathcal{F}\left(\bar{\boldsymbol{S}}\right)\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] &= \mathrm{D}_{\bar{\boldsymbol{S}}}\left(\phi_1\left(|\bar{\boldsymbol{S}}|\right)\bar{\boldsymbol{S}} + \phi_2\left(|\bar{\boldsymbol{S}}|^2\right)\bar{\boldsymbol{S}}^2\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\
&= \phi_1'\left(|\bar{\boldsymbol{S}}|\right)\left(D_{\bar{\boldsymbol{S}}}\left(|\bar{\boldsymbol{S}}|\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\right)\bar{\boldsymbol{S}} + \phi_1\left(|\bar{\boldsymbol{S}}|\right)D_{\bar{\boldsymbol{S}}}\bar{\boldsymbol{S}}\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\
&+ \phi_2'\left(|\bar{\boldsymbol{S}}|^2\right)\left(D_{\bar{\boldsymbol{S}}}\left(|\bar{\boldsymbol{S}}|^2\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\right)\bar{\boldsymbol{S}}^2 \\
&+ \phi_2\left(|\bar{\boldsymbol{S}}|^2\right)D_{\bar{\boldsymbol{S}}}\left(\bar{\boldsymbol{S}}^2\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\
&= \left(\left(\frac{-\beta_1|\bar{\boldsymbol{S}}|}{\left(1+\beta_1|\bar{\boldsymbol{S}}|\right)^2}\right)(\boldsymbol{A}\otimes\boldsymbol{A}) + \frac{1}{1+\beta_1|\bar{\boldsymbol{S}}|}\mathbb{I}\right. \\
&\quad + \left(\frac{-2\alpha_2\beta_2\left|\bar{\boldsymbol{S}}^2\right||\bar{\boldsymbol{S}}|}{\left(1+\beta_2|\bar{\boldsymbol{S}}|^2\right)^2}\right)(\boldsymbol{A}\otimes\boldsymbol{A}) \\
&+ \left.\frac{\alpha_2}{1+\beta_2|\bar{\boldsymbol{S}}|^2}\mathbb{I}\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right] \\
&= \left(K(\boldsymbol{A}\otimes\boldsymbol{A}) + L(\mathbb{I})\right)\left[\left(\boldsymbol{F}^T\boldsymbol{H}\right)_s\right]\,.
\end{aligned} \tag{3.90}$$

Let

$$L := \frac{\alpha_2}{1+2\beta_2} + \frac{1}{1+\sqrt{2}\beta_2} > 0\,, \tag{3.91}$$

$$K := -\frac{\sqrt{2}\beta_1}{(1+\sqrt{2}\beta_1)^2} - \frac{4\alpha_2\beta_2}{(1+2\beta_2)^2}\,. \tag{3.92}$$

When $\beta_1 = \beta_2 = 1$, and $\alpha_2 = 9/(3+2\sqrt{2})$, we have $K + L = 0$. Hence, the fourth-order tensor $\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\right)[\cdot]$ in (3.90) is not invertible, although the given model (3.8) is uniquely invertible for all γ in a neighborhood of $\gamma = 0$.

3.5.2 *Monotonicity*

Similar to the previous Section, we study *monotonicity* of (3.8) for $\gamma = 0$, then we generalize for all γ near 0 by a continuity argument. Since $\boldsymbol{H} = \boldsymbol{a} \otimes \boldsymbol{b}$, we consider several cases according to positions of vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{e}_1, \boldsymbol{e}_2$: $\boldsymbol{a} = \boldsymbol{b} \perp \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$; $\boldsymbol{a} = \boldsymbol{b}$ and $\angle(\boldsymbol{a}, \{\boldsymbol{e}_1, \boldsymbol{e}_2\}) = \theta, 0 \leq \theta < \pi/2$; and $0 < |\boldsymbol{a} \cdot \boldsymbol{b}| < 1$.

Case 1: $\boldsymbol{a} = \boldsymbol{b} \perp \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. Then,

$$\tilde{\boldsymbol{F}} = \boldsymbol{I} + \gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + \alpha(\boldsymbol{a} \otimes \boldsymbol{a}). \tag{3.93}$$

Also note that $\det(\boldsymbol{I} + \gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + \alpha(\boldsymbol{a} \otimes \boldsymbol{a})) > 0$. The Green-St.Venant strain associated with $\tilde{\boldsymbol{F}}$ has the form

$$\begin{aligned}\tilde{\boldsymbol{E}} &= \frac{1}{2}\left(\tilde{\boldsymbol{F}}^T \tilde{\boldsymbol{F}} - \boldsymbol{I}\right) \\ &= \frac{1}{2}(\gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + \gamma^2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + (\alpha^2 + 2\alpha)(\boldsymbol{a} \otimes \boldsymbol{a}))\,. \end{aligned} \tag{3.94}$$

Let

$$\boldsymbol{W} := \frac{1}{2}(\gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + \gamma^2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2))\,, \tag{3.95}$$

The eigenvectors corresponding to nonzero eigenvalues λ of $\boldsymbol{W}$ have the form:

$$\boldsymbol{v} = \eta_1 \boldsymbol{e}_1 + \eta_2 \boldsymbol{e}_2, \tag{3.96}$$

in which, $\eta_1 \eta_2 \neq 0$. By a similar argument to Case 1, we get

$$\lambda = \frac{1}{2}\gamma\delta\,, \tag{3.97}$$

where

$$\delta := \frac{\eta_2}{\eta_1}\,, \tag{3.98}$$

and

$$\delta_{1,2} = \frac{\gamma \pm \sqrt{\gamma^2 + 4}}{2}\,. \tag{3.99}$$

Correspondingly,

$$\lambda_{1,2} = \frac{1}{4}\gamma(\gamma \pm \sqrt{\gamma^2+4})\,. \tag{3.100}$$

Note that $\delta_1 > 0, \delta_2 < 0$, and $\lambda_1 > 0, \lambda_2 < 0$. Denote by $\tilde{\boldsymbol{v}}_{1,2}$ the normalized eigenvectors associated with $\lambda_{1,2}$, and note that $\tilde{\boldsymbol{v}}_1 \perp \tilde{\boldsymbol{v}}_2$, $\boldsymbol{a}$ is orthonormal to $\{\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2\}$, we then have the spectral decompositions

$$\boldsymbol{W} = \lambda_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + \lambda_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2), \tag{3.101}$$

$$\tilde{\boldsymbol{E}} = \lambda_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + \lambda_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + \left(\frac{\alpha^2}{2} + \alpha\right)(\boldsymbol{a} \otimes \boldsymbol{a})\,, \tag{3.102}$$

and by (3.8),

$$\tilde{\boldsymbol{S}} = d_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + d_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + d_3(\boldsymbol{a} \otimes \boldsymbol{a}), \tag{3.103}$$

and

$$\tilde{\boldsymbol{S}}^2 = d_1^2(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + d_2^2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + d_3^2(\boldsymbol{a} \otimes \boldsymbol{a}), \tag{3.104}$$

where d_1, d_2, d_3 are scalars as eigenvalues of $\tilde{\boldsymbol{S}}$. The constitutive equation (3.8) leads to

$$\lambda_1 = \frac{1}{4}\gamma\left(\gamma + \sqrt{\gamma^2+4}\right) = \frac{d_1}{1+\beta_1\sqrt{d_1^2+d_2^2+d_3^2}} + \frac{\alpha_2 d_1^2}{1+\beta_2(d_1^2+d_2^2+d_3^2)}\,, \tag{3.105}$$

$$\lambda_2 = \frac{1}{4}\gamma\left(\gamma - \sqrt{\gamma^2+4}\right) = \frac{d_2}{1+\beta_1\sqrt{d_1^2+d_2^2+d_3^2}} + \frac{\alpha_2 d_2^2}{1+\beta_2(d_1^2+d_2^2+d_3^2)}\,, \tag{3.106}$$

and

$$\frac{\alpha^2}{2} + \alpha = \frac{d_3}{1+\beta_1\sqrt{d_1^2+d_2^2+d_3^2}} + \frac{\alpha_2 d_3^2}{1+\beta_2(d_1^2+d_2^2+d_3^2)}\,. \tag{3.107}$$

It follows that $d_1 > 0, d_2 < 0$, and $d_3 > 0$ (as $\alpha > 0$).

We now investigate the invertibility of the system (3.105), (3.106), and (3.107). Fixing d_2 and d_3 in (3.105), then this equation is uniquely solvable within an interval of γ centered at $\gamma = 0$ by a similar continuity argument to the Invertibility Section. The solution of (3.105) gives d_1 as a function of d_2 and d_3, namely $d_1 = k(d_2; \lambda_1, d_3)$. Then we substitute this d_1 into (3.106), while fixing d_3. Again, (3.106) is uniquely solvable for all γ belonging to an interval centered at $\gamma = 0$. The solution of (3.106) derives d_2 as a function of d_3, say $d_2 = h(d_3; \lambda_1, \lambda_2)$. A substitution of this d_2 into (3.107) gives an equation of d_3 only, namely

$$\frac{\alpha^2}{2} + \alpha = l(d_3; \lambda_1, \lambda_2)\,.$$

This equation of d_3 has a unique solution for an interval of α near 0, and for an interval of α_2 near 0. Indeed, taking partial derivative of the function $l(d_3; \lambda_1, \lambda_2)$ with respect to d_3 then letting $\alpha_2 \to 0$ gives

$$\begin{aligned}
\partial_{d_3} l(d_3; \lambda_1, \lambda_2) = {} & \frac{1}{\left(1 + \beta_1\sqrt{d_1^2 + d_2^2 + d_3^2}\right)} \\
& - \frac{\beta_1 d_3 (d_1 \partial_{d_3} k(h(d_3; \lambda_1, \lambda_2); \lambda_1, d_3) + d_2 \partial_{d_3} h(d_3; \lambda_1, \lambda_2) + d_3)}{\sqrt{d_1^2 + d_2^2 + d_3^2}\left(1 + \beta_1\sqrt{d_1^2 + d_2^2 + d_3^2}\right)^2} \\
= {} & \frac{\sqrt{d_1^2 + d_2^2 + d_3^2} + \beta_1 (d_1^2 + d_2^2)}{\sqrt{d_1^2 + d_2^2 + d_3^2}\left(1 + \beta_1\sqrt{d_1^2 + d_2^2 + d_3^2}\right)^2} \\
& - \frac{\beta_1 d_3 (\partial_{d_3} k(h(d_3; \lambda_1, \lambda_2); \lambda_1, d_3) + d_2 \partial_{d_3} h(d_3; \lambda_1, \lambda_2))}{\sqrt{d_1^2 + d_2^2 + d_3^2}\left(1 + \beta_1\sqrt{d_1^2 + d_2^2 + d_3^2}\right)^2}\,.
\end{aligned} \tag{3.108}$$

Note that $d_3 > 0$ as $\alpha > 0$ in (3.107). Letting $\alpha \to 0$, it follows that $d_3 \to 0$ in (3.108), and thus $\partial_{d_3} l(d_3; \lambda_1, \lambda_2) > 0$, i.e. (3.107) is uniquely invertible. By continuity, (3.107) is invertible in an interval of α near 0, and in an interval of α_2 near 0. In summary, by continuity, the system (3.105), (3.106), and (3.107) is uniquely solvable in an interval of γ centered at $\gamma = 0$, in an interval of α near 0, and in an interval of α_2 near 0.

To investigate *monotonicity*, consider the left hand side of (3.15):

$$\begin{aligned}
(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}}) \cdot (\boldsymbol{a} \otimes \boldsymbol{a}) = {} & ((\boldsymbol{I} + \gamma \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \alpha \boldsymbol{a} \otimes \boldsymbol{a})(d_1 \tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1 + d_2 \tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2 + d_3 \boldsymbol{a} \otimes \boldsymbol{a}) \\
& - (\boldsymbol{I} + \gamma \boldsymbol{e}_1 \otimes \boldsymbol{e}_2)(c_1 \tilde{\boldsymbol{u}}_1 \otimes \tilde{\boldsymbol{u}}_1 + c_2 \tilde{\boldsymbol{u}}_2 \otimes \tilde{\boldsymbol{u}}_2)) \cdot (\boldsymbol{a} \otimes \boldsymbol{a}) \\
= {} & (d_1 (\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + d_2 (\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + (d_3 + \alpha d_3)(\boldsymbol{a} \otimes \boldsymbol{a})) \cdot (\boldsymbol{a} \otimes \boldsymbol{a})
\end{aligned}$$

$$+(-c_1(\tilde{\boldsymbol{u}}_1\otimes\tilde{\boldsymbol{u}}_1)-c_2(\tilde{\boldsymbol{u}}_2\otimes\tilde{\boldsymbol{u}}_2))\cdot(\boldsymbol{a}\otimes\boldsymbol{a})$$

$$=d_3+\alpha d_3=(1+\alpha)d_3. \tag{3.109}$$

This expression is positive because $d_3>0$, for all γ belonging to an interval centered at $\gamma=0$, for an interval of α near 0, and for an interval of α_2 near 0.

Remark 5. The model (3.8) in this case has the *monotonicity* condition (3.109) implying the invertibility condition, but the inverse statement does not hold generally.

Case 2: $\boldsymbol{a}=\boldsymbol{b}$ and $\angle(\boldsymbol{a},\{\boldsymbol{e}_1,\boldsymbol{e}_2\})=\theta, 0\leq\theta<\pi/2$. Let $\boldsymbol{e}_3$ be an orthonormal vector to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Then,

$$\boldsymbol{a}=a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3, \tag{3.110}$$

where a_1,a_2,a_3 are scalars. We have

$$\tilde{\boldsymbol{F}}=\boldsymbol{I}+\gamma(\boldsymbol{e}_1\otimes\boldsymbol{e}_2)+\alpha(\boldsymbol{a}\otimes\boldsymbol{a}), \tag{3.111}$$

and $\det(\boldsymbol{I}+\gamma(\boldsymbol{e}_1\otimes\boldsymbol{e}_2)+\alpha(\boldsymbol{a}\otimes\boldsymbol{a}))>0$ within an interval of γ centered at $\gamma=0$ and α is sufficiently near 0. Now,

$$\begin{aligned}\tilde{\boldsymbol{E}}&=\frac{1}{2}\left(\tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}}-\boldsymbol{I}\right)\\&=\frac{1}{2}(\gamma(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1+\gamma(\boldsymbol{e}_2\otimes\boldsymbol{e}_2))+\gamma\alpha a_1(\boldsymbol{a}\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{a}))\\&+\frac{1}{2}(\alpha^2+2\alpha)(\boldsymbol{a}\otimes\boldsymbol{a}).\end{aligned} \tag{3.112}$$

Let $\{\tilde{\boldsymbol{v}}_i\}_{i=1}^3$ be an orthonormal eigenbasis for $\mathbb{R}^3$ corresponding to nonzero eigenvalues ξ of $\tilde{\boldsymbol{E}}$, where $\tilde{\boldsymbol{v}}$ has the form

$$\tilde{\boldsymbol{v}}=\tau_1\boldsymbol{e}_1+\tau_2\boldsymbol{e}_2+\tau_3\boldsymbol{e}_3. \tag{3.113}$$

We then have the spectral decompositions

$$\tilde{\boldsymbol{E}} = \xi_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + \xi_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + \xi_3(\tilde{\boldsymbol{v}}_3 \otimes \tilde{\boldsymbol{v}}_3)\,, \tag{3.114}$$

and from the constitutive relation (3.8),

$$\tilde{\boldsymbol{S}} = d_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + d_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + d_3(\tilde{\boldsymbol{v}}_3 \otimes \tilde{\boldsymbol{v}}_3), \tag{3.115}$$

$$\tilde{\boldsymbol{S}}^2 = d_1^2(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + d_2^2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + d_3^2(\tilde{\boldsymbol{v}}_3 \otimes \tilde{\boldsymbol{v}}_3), \tag{3.116}$$

where d_1, d_2, d_3 are scalars as eigenvalues of $\tilde{\boldsymbol{S}}$.

The constitutive equation (3.8) is then equivalent to

$$\xi_1 = \phi_1(|\tilde{\boldsymbol{S}}|)d_1 + \phi_2(|\tilde{\boldsymbol{S}}|^2)d_1^2, \tag{3.117}$$

$$\xi_2 = \phi_1(|\tilde{\boldsymbol{S}}|)d_2 + \phi_2(|\tilde{\boldsymbol{S}}|^2)d_2^2, \tag{3.118}$$

$$\xi_3 = \phi_1(|\tilde{\boldsymbol{S}}|)d_3 + \phi_2(|\tilde{\boldsymbol{S}}|^2)d_3^2, \tag{3.119}$$

Now, note that when letting $\gamma \to 0$, we have

$$\tilde{\boldsymbol{E}} = \frac{1}{2}(\alpha^2 + 2\alpha)(\boldsymbol{a} \otimes \boldsymbol{a})\,, \tag{3.120}$$

with the following eigenvalues of $\tilde{\boldsymbol{E}}$: $\xi_1 = 0, \xi_2 = 0, \xi_3 = (\alpha^2 + 2\alpha)/2 > 0$. By similar continuity argument to Case 1, there exist an interval of γ centered at $\gamma = 0$, an interval of α near 0, and an interval of α_2 near 0, in which the system (3.117), (3.118) is uniquely solvable for d_1, d_2 as functions of d_3. A substitution of these d_1, d_2 into (3.119), where ξ_3 keeps its positive sign, gives unique solution for $d_3 > 0$, and thus the system (3.117), (3.118), (3.119) is uniquely solvable.

Finally, consider the left hand side of *monotonicity* condition (3.15)

$$\begin{aligned}(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}}) \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) &= (1+\alpha)(d_1(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_1)^2 + d_2(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_2)^2 + d_3(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_3)^2) \\ &\quad - (c_1(\boldsymbol{a} \otimes \tilde{\boldsymbol{u}}_1)^2 + c_2(\boldsymbol{a} \otimes \tilde{\boldsymbol{u}}_2)^2).\end{aligned} \tag{3.121}$$

Note that letting $\gamma \to 0$ it follows that c_1, c_2, d_1, d_2 all approach 0, which makes (3.121) become

$$(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}}) \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) = (1+\alpha)d_3(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_3)^2, \tag{3.122}$$

which is positive because $d_3 > 0$, and $(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_3) \neq 0)$. Thus, by continuity, *monotonicity* holds for the given model (3.8) in an interval of γ centered at $\gamma = 0$, in an interval of α near 0, and in an interval of α_2 near 0.

Case 3: $0 < |\boldsymbol{a} \cdot \boldsymbol{b}| < 1$. Let $\boldsymbol{e}_3$ be an orthonormal vector to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. We then have

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3, \tag{3.123}$$

$$\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3, \tag{3.124}$$

where a_i, b_i are scalars, $i = 1, 2, 3$. Now,

$$\tilde{\boldsymbol{F}} = \boldsymbol{I} + \gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + \alpha(\boldsymbol{a} \otimes \boldsymbol{b}), \tag{3.125}$$

and $\det(\boldsymbol{I} + \gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + \alpha(\boldsymbol{a} \otimes \boldsymbol{b})) > 0$ within an interval of γ centered at $\gamma = 0$, and in an interval of α near 0. The strain corresponding to $\tilde{\boldsymbol{F}}$ is

$$\begin{aligned}\tilde{\boldsymbol{E}} &= \frac{1}{2}\left(\tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}} - \boldsymbol{I}\right)\\ &= \frac{1}{2}(\gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \gamma(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2)) + \gamma\alpha(\boldsymbol{a} \cdot \boldsymbol{e}_1)(\boldsymbol{b} \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{b}))\\ &+ \frac{1}{2}(\alpha(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) + \alpha^2(\boldsymbol{b} \otimes \boldsymbol{b}))\,. \end{aligned}\tag{3.126}$$

Let $\{\tilde{\boldsymbol{v}}_i\}_{i=1}^3$ be an orthonormal eigenbasis for $\mathbb{R}^3$ associated with the nonzero eigenvalues ξ of $\tilde{\boldsymbol{E}}$, of the form

$$\tilde{\boldsymbol{v}} = \tau_1\boldsymbol{e}_1 + \tau_2\boldsymbol{e}_2 + \tau_3\boldsymbol{e}_3. \tag{3.127}$$

We then have the spectral decompositions

$$\tilde{\boldsymbol{E}} = \xi_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + \xi_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + \xi_3(\tilde{\boldsymbol{v}}_3 \otimes \tilde{\boldsymbol{v}}_3), \tag{3.128}$$

and by the constitutive relation (3.8),

$$\tilde{\boldsymbol{S}} = d_1(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + d_2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + d_3(\tilde{\boldsymbol{v}}_3 \otimes \tilde{\boldsymbol{v}}_3), \tag{3.129}$$

$$\tilde{\boldsymbol{S}}^2 = d_1^2(\tilde{\boldsymbol{v}}_1 \otimes \tilde{\boldsymbol{v}}_1) + d_2^2(\tilde{\boldsymbol{v}}_2 \otimes \tilde{\boldsymbol{v}}_2) + d_3^2(\tilde{\boldsymbol{v}}_3 \otimes \tilde{\boldsymbol{v}}_3), \tag{3.130}$$

where d_1, d_2, d_3 are scalars. From the constitutive equation (3.8), we get

$$\xi_1 = \frac{d_1}{1 + \beta_1\sqrt{d_1^2 + d_2^2 + d_3^2}} + \frac{\alpha_2 d_1^2}{1 + \beta_2(d_1^2 + d_2^2 + d_3^2)}\,, \tag{3.131}$$

$$\xi_2 = \frac{d_2}{1 + \beta_1\sqrt{d_1^2 + d_2^2 + d_3^2}} + \frac{\alpha_2 d_2^2}{1 + \beta_2(d_1^2 + d_2^2 + d_3^2)}\,, \tag{3.132}$$

and

$$\xi_3 = \frac{d_3}{1 + \beta_1\sqrt{d_1^2 + d_2^2 + d_3^2}} + \frac{\alpha_2 d_3^2}{1 + \beta_2(d_1^2 + d_2^2 + d_3^2)}\,. \tag{3.133}$$

Letting $\gamma \to 0$ gives

$$\tilde{\boldsymbol{E}} = \frac{1}{2}(\alpha(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) + \alpha^2(\boldsymbol{b} \otimes \boldsymbol{b})), \tag{3.134}$$

which is the same as $\tilde{\boldsymbol{E}}$ in Case 3 of the Compression Section. Thus, when letting $\gamma \to 0$, we have

$$\xi_1 = \frac{1}{4}\alpha\left(2(\boldsymbol{a} \cdot \boldsymbol{b}) + \alpha + \sqrt{\alpha^2 + 4\alpha(\boldsymbol{a} \cdot \boldsymbol{b}) + 4}\right) > 0\,, \tag{3.135}$$

$$\xi_1 = \frac{1}{4}\alpha\left(2(\boldsymbol{a} \cdot \boldsymbol{b}) + \alpha - \sqrt{\alpha^2 + 4\alpha(\boldsymbol{a} \cdot \boldsymbol{b}) + 4}\right) < 0\,, \tag{3.136}$$

$$\xi_3 = 0. \tag{3.137}$$

Hence, there exists an interval of γ centered at $\gamma = 0$ in which each of the values ξ_1, ξ_2 keeps the same sign and ξ_3 approaches 0, and thus, by a similar continuity argument to Case 1, the system (3.131), (3.132), (3.133) is uniquely solvable in an interval of α near 0, and in an interval of α_2 near 0.

Toward the study of *monotonicity*, we consider again the strain $\tilde{\boldsymbol{E}}$ when $\gamma = 0$. We thus have

$$\tilde{\boldsymbol{E}} = \frac{1}{2}(\alpha(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) + \alpha^2(\boldsymbol{b} \otimes \boldsymbol{b}))\,, \tag{3.138}$$

with orthonormal eigenvectors

$$\tilde{\boldsymbol{v}}_1 = \frac{1}{\sqrt{\nu_1^2 + \nu_2^2}}(\nu_1 \boldsymbol{a} + \nu_2 \boldsymbol{b})\,, \tag{3.139}$$

$$\tilde{\boldsymbol{v}}_2 = \frac{1}{\sqrt{\nu_1'^2 + \nu_2'^2}}(\nu_1' \boldsymbol{a} + \nu_2' \boldsymbol{b})\,. \tag{3.140}$$

The associated eigenvalues of $\tilde{\boldsymbol{E}}$ are

$$\xi_{1,2} = \frac{1}{2}\alpha(\boldsymbol{a} \cdot \boldsymbol{b} + \rho_{1,2})\,, \tag{3.141}$$

where

$$\rho_1 = \frac{\nu_2}{\nu_1},\ \rho_2 = \frac{\nu_2'}{\nu_1'}\,. \tag{3.142}$$

Without loss of generality, we can take $\nu_1 = \nu_1' = 1$. Then

$$\nu_2 = \rho_1 = \frac{\alpha + \sqrt{\alpha^2 + 4\alpha(\boldsymbol{a} \cdot \boldsymbol{b}) + 4}}{2}\,, \nu_2' = \rho_2 = \frac{\alpha - \sqrt{\alpha^2 + 4\alpha(\boldsymbol{a} \cdot \boldsymbol{b}) + 4}}{2}\,. \tag{3.143}$$

We now note from (3.82), (3.83) that

$$\lim_{\gamma \to 0} c_1 = 0,\ \lim_{\gamma \to 0} c_2 = 0,\ \lim_{\gamma \to 0} \frac{c_1}{\gamma} = \frac{1}{2}\,,\ \lim_{\gamma \to 0} \frac{c_2}{\gamma} = -\frac{1}{2}\,. \tag{3.144}$$

Writing α as $\alpha = \gamma\beta$, when $\gamma \to 0$, from (3.143), we have

$$\nu_2 = 1,\ \nu_2' = -1. \tag{3.145}$$

Thus, from (3.131), (3.132), (3.133), and (3.135), (3.136), (3.137), letting $\gamma \to 0$ leads to

$$\lim_{\gamma \to 0} d_1 = 0,\quad \lim_{\gamma \to 0} d_2 = 0,\ \ \lim_{\gamma \to 0} d_3 = 0, \tag{3.146}$$

and

$$\lim_{\gamma \to 0} \frac{d_1}{\gamma} = \frac{\beta}{2}\left(\boldsymbol{a} \cdot \boldsymbol{b} + 1\right)\,,\ \lim_{\gamma \to 0} \frac{d_2}{\gamma} = \frac{\beta}{2}\left(\boldsymbol{a} \cdot \boldsymbol{b} - 1\right)\,,\ \lim_{\gamma \to 0} \frac{d_3}{\gamma} = 0\,. \tag{3.147}$$

We next consider the left hand side of (3.15)

$$
\begin{aligned}
(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}}) \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) &= d_1(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_1)(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_1) + d_2(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_2)(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_2) + d_3(\boldsymbol{a} \cdot \tilde{\boldsymbol{v}}_3)(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_3) \\
&+ \gamma d_1(\boldsymbol{e}_2 \cdot \tilde{\boldsymbol{v}}_1)(\boldsymbol{e}_1 \cdot \boldsymbol{a})(\tilde{\boldsymbol{v}}_1 \cdot \boldsymbol{b}) + \gamma d_2(\boldsymbol{e}_2 \cdot \tilde{\boldsymbol{v}}_2)(\boldsymbol{e}_1 \cdot \boldsymbol{a})(\tilde{\boldsymbol{v}}_2 \cdot \boldsymbol{b}) \\
&+ \gamma d_3(\boldsymbol{e}_2 \cdot \tilde{\boldsymbol{v}}_3)(\boldsymbol{e}_1 \cdot \boldsymbol{a})(\tilde{\boldsymbol{v}}_3 \cdot \boldsymbol{b}) \\
&+ \gamma\beta d_1(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_1)^2 + \gamma\beta d_2(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_2)^2 + \gamma\beta d_3(\boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_3)^2 \\
&- c_1(\tilde{\boldsymbol{u}}_1 \cdot \boldsymbol{a})(\tilde{\boldsymbol{u}}_1 \cdot \boldsymbol{b}) - c_2(\tilde{\boldsymbol{u}}_2 \cdot \boldsymbol{a})(\tilde{\boldsymbol{u}}_2 \cdot \boldsymbol{b}) \\
&- \gamma c_1(\boldsymbol{e}_1 \cdot \tilde{\boldsymbol{u}}_1)(\boldsymbol{e}_1 \cdot \boldsymbol{a})(\tilde{\boldsymbol{u}}_1 \cdot \boldsymbol{b}) \\
&- \gamma c_2(\boldsymbol{e}_2 \cdot \tilde{\boldsymbol{u}}_2)(\boldsymbol{e}_1 \cdot \boldsymbol{a})(\tilde{\boldsymbol{u}}_2 \cdot \boldsymbol{b}). \qquad (3.148)
\end{aligned}
$$

Dividing (3.148) by γ, then letting $\gamma \to 0$, we get

$$
\begin{aligned}
(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}}) \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) &= \left(\lim_{\gamma\to 0} \frac{d_1}{\gamma}\right) \frac{1}{2(1+\boldsymbol{a}\cdot\boldsymbol{b})}(1+\boldsymbol{a}\cdot\boldsymbol{b})^2 \\
&+ \left(\lim_{\gamma\to 0} \frac{d_2}{\gamma}\right) \frac{1}{2(1-\boldsymbol{a}\cdot\boldsymbol{b})}(1-\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{a}\cdot\boldsymbol{b}-1) \\
&- \left(\lim_{\gamma\to 0} \frac{c_1}{\gamma}\right)(\tilde{\boldsymbol{u}}_1 \cdot \boldsymbol{a})(\tilde{\boldsymbol{u}}_1 \cdot \boldsymbol{b}) - \left(\lim_{\gamma\to 0} \frac{c_2}{\gamma}\right)(\tilde{\boldsymbol{u}}_2 \cdot \boldsymbol{a})(\tilde{\boldsymbol{u}}_2 \cdot \boldsymbol{b}) \\
&= \left(\frac{\beta(\boldsymbol{a}\cdot\boldsymbol{b}+1)}{2}\right)\left(\frac{(\boldsymbol{a}\cdot\boldsymbol{b}+1)}{2}\right) \\
&+ \left(\frac{\beta(\boldsymbol{a}\cdot\boldsymbol{b}-1)}{2}\right)\left(\frac{(\boldsymbol{a}\cdot\boldsymbol{b}-1)}{2}\right) \\
&- \frac{1}{2}(\tilde{\boldsymbol{u}}_1 \cdot \boldsymbol{a})(\tilde{\boldsymbol{u}}_1 \cdot \boldsymbol{b}) + \frac{1}{2}(\tilde{\boldsymbol{u}}_2 \cdot \boldsymbol{a})(\tilde{\boldsymbol{u}}_2 \cdot \boldsymbol{b}) \\
&> \frac{\beta}{2}(1+(\boldsymbol{a}\cdot\boldsymbol{b})^2) - 1 \\
&> \frac{\beta}{2} - 1\,. \qquad (3.149)
\end{aligned}
$$

Monotonicity holds provided $\beta > 2$, i.e. $0 < \gamma < 1/2$, in an interval of α near 0, and in an interval of α_2 near 0.

In conclusion, *monotonicity* holds for the model (3.8) in an interval of γ centered at $\gamma = 0$, in an interval of α near 0, and in an interval of α_2 near 0, resulting from intersecting all the three intervals of γ, the three intervals of α, and the three interval of α_2, respectively, in the three cases above.

3.5.3 Counterexample

We construct a counterexample demonstrating the failure of *monotonicity* for the given model (3.8). Consider Case 3 of Simple Shear. Taking $\alpha_2 = 0$, this choice of α_2 is based on the fact that in each of equations (3.82), (3.83), and (3.131), (3.132), (3.133), the linear term dominates the quadratic term. Note that $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$, thus from (3.123) and (3.124),

$$|a_3| = \sqrt{1 - a_1^2 - a_2^2},\ |b_3| = \sqrt{1 - b_1^2 - b_2^2}. \tag{3.150}$$

Now, from (3.82) and (3.83), it follows that

$$\sqrt{\lambda_1^2 + \lambda_2^2} = \frac{\sqrt{c_1^2 + c_2^2}}{1 + \beta_1\sqrt{c_1^2 + c_2^2}}\,, \tag{3.151}$$

and

$$\frac{c_2}{c_1} = \frac{\lambda_2}{\lambda_1}\,. \tag{3.152}$$

Hence,

$$\sqrt{c_1^2 + c_2^2} = \frac{\sqrt{\lambda_1^2 + \lambda_2^2}}{1 - \beta_1\sqrt{\lambda_1^2 + \lambda_2^2}}\,, \tag{3.153}$$

and

$$c_1 = \frac{\sqrt{\lambda_1^2 + \lambda_2^2}}{1 - \beta_1\sqrt{\lambda_1^2 + \lambda_2^2}}\left(\sqrt{1 + \left(\frac{\lambda_2}{\lambda_1}\right)^2}\right)^{-1}\,, \tag{3.154}$$

$$c_2 = \frac{\lambda_2}{\lambda_1}c_1\,. \tag{3.155}$$

Similarly, from (3.131), (3.132), (3.133), we have

$$d_1 = (\mathrm{sgn}\xi_1)\frac{\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}{1-\beta_1\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}\left(\sqrt{1+\left(\frac{\xi_2}{\xi_1}\right)^2+\left(\frac{\xi_3}{\xi_1}\right)^2}\right)^{-1}, \tag{3.156}$$

$$d_2 = \frac{\xi_2}{\xi_1}d_1,\ d_3 = \frac{\xi_3}{\xi_1}d_1\,. \tag{3.157}$$

Here, $\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2$ are two orthonormal eigenvectors corresponding to two nonzero eigenvalues λ_1, λ_2 of $\boldsymbol{E}$ in (3.78). Also, $\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2, \tilde{\boldsymbol{v}}_3$ are three orthonormal eigenvectors corresponding to three eigenvalues ξ_1, ξ_2, ξ_3 of $\tilde{\boldsymbol{E}}$ in (3.128).

Finally, we choose $\alpha = 0.1$, $\beta_1 = 0.2$, $\gamma = -2.5$, $a_1 = 0.5$, $a_2 = 0.1$, $b_1 = 0.6$, $b_2 = 0.2$. Here, $\gamma = -2.5$ belongs to the range of the respond function (3.8). Furthermore, by Matlab computation (including finding eigenvalues and eigenvectors process), the right hand side of (3.148) equals -0.1198, leading to the failure of *monotonicity* of the model (3.8).

3.6 General Models

For the model (3.8), consider a class of deformation gradients having the form

$$\boldsymbol{F} = \boldsymbol{I} + \gamma\tilde{\boldsymbol{U}}, \tag{3.158}$$

where γ is a scalar, $\tilde{\boldsymbol{U}}$ is a fixed, constant displacement gradient. Thus,

$$\boldsymbol{E} = \frac{1}{2}(\gamma(\tilde{\boldsymbol{U}}^T + \tilde{\boldsymbol{U}}) + \gamma^2\tilde{\boldsymbol{U}}^T\tilde{\boldsymbol{U}})\,. \tag{3.159}$$

Denote by $\{\boldsymbol{e}_i\}_{i=1}^3$ an orthonormal basis for $\mathbb{R}^3$ consisting entirely of eigenvectors of $\boldsymbol{E}$, and $\{\lambda_i\}_{i=1}^3$ the associated eigenvalues of $\boldsymbol{E}$, i.e.

$$\boldsymbol{E}\boldsymbol{e}_i = \lambda_i\boldsymbol{e}_i, \tag{3.160}$$

for $i = 1, 2, 3$. We then have the spectral decompositions of strain

$$\boldsymbol{E} = \lambda_1\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \lambda_2\boldsymbol{e}_2\otimes\boldsymbol{e}_2 + \lambda_3\boldsymbol{e}_3\otimes\boldsymbol{e}_3, \tag{3.161}$$

and stress (by constitutive equation (3.8))

$$\bar{\boldsymbol{S}} = c_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + c_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + c_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \tag{3.162}$$

where $\{c_i\}_{i=1}^3$ are scalars as eigenvalues of $\bar{\boldsymbol{S}}$. It follows from the constitutive relation (3.8) that

$$\lambda_1 = \frac{c_1}{1+\beta_1\sqrt{c_1^2+c_2^2}} + \frac{\alpha_2 c_1^2}{1+\beta_2(c_1^2+c_2^2)}\,, \tag{3.163}$$

$$\lambda_2 = \frac{c_2}{1+\beta_1\sqrt{c_1^2+c_2^2}} + \frac{\alpha_2 c_2^2}{1+\beta_2(c_1^2+c_2^2)}\,, \tag{3.164}$$

where α_2, β_1, and β_2 are non-negative constants.

3.6.1 Invertibility

From previous two Sections, we first note that the constitutive relation (3.8) is not always uniquely invertible. Second, in an interval of γ centered at $\gamma = 0$ the system (3.163) and (3.164) is uniquely solvable by similar argument in Simple Shear Section (because each of the eigenvalues of $\boldsymbol{E}$ is a multiple of γ), thus (3.8) is uniquely invertible. However, also through two previous Sections, even when the constitutive relation (3.8) is invertible uniquely, the invertibility of the fourth-order tensor $\left(\mathrm{D}_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})\right)[\cdot]$ for the models (3.8) is not always guaranteed. Thus, it is reasonable to investigate *monotonicity* instead of strong ellipticity for the present general models.

3.6.2 Monotonicity

We first have

$$\tilde{\boldsymbol{F}} = \boldsymbol{I} + \gamma\tilde{\boldsymbol{U}} + \alpha(\boldsymbol{a}\otimes\boldsymbol{b}). \tag{3.165}$$

Note that $\det(\boldsymbol{I}+\gamma\tilde{\boldsymbol{U}}+\alpha\boldsymbol{a}\otimes\boldsymbol{b}) > 0$ with γ in a neighborhood of $\gamma = 0$ and α is sufficiently near 0. The strain associated with $\tilde{\boldsymbol{F}}$ is

$$\tilde{\boldsymbol{E}} = \frac{1}{2}\left(\tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}} - \boldsymbol{I}\right)$$

$$= \frac{1}{2}\gamma\left(\tilde{\boldsymbol{U}} + \tilde{\boldsymbol{U}}^T + \tilde{\boldsymbol{U}}^T\tilde{\boldsymbol{U}} + \alpha\left(\left(\tilde{\boldsymbol{U}}^T\boldsymbol{a}\right)\otimes\boldsymbol{b} + \boldsymbol{b}\otimes\left(\tilde{\boldsymbol{U}}^T\boldsymbol{a}\right)\right)\right)$$

$$+ \frac{1}{2}(\alpha(\boldsymbol{a}\otimes\boldsymbol{b} + \boldsymbol{b}\otimes\boldsymbol{a}) + \alpha^2(\boldsymbol{b}\otimes\boldsymbol{b}))\,. \tag{3.166}$$

Denote by $\{\tilde{\boldsymbol{v}}_i\}_{i=1}^3$ an orthonormal eigenbasis for $\mathbb{R}^3$ corresponding to the eigenvalues $\{\xi_i\}_{i=1}^3$ of $\tilde{\boldsymbol{E}}$. Then,

$$\tilde{\boldsymbol{E}} = \xi_1(\tilde{\boldsymbol{v}}_1\otimes\tilde{\boldsymbol{v}}_1) + \xi_2(\tilde{\boldsymbol{v}}_2\otimes\tilde{\boldsymbol{v}}_2) + \xi_3(\tilde{\boldsymbol{v}}_3\otimes\tilde{\boldsymbol{v}}_3), \tag{3.167}$$

and from the constitutive relation (3.8),

$$\tilde{\boldsymbol{S}} = d_1(\tilde{\boldsymbol{v}}_1\otimes\tilde{\boldsymbol{v}}_1) + d_2(\tilde{\boldsymbol{v}}_2\otimes\tilde{\boldsymbol{v}}_2) + d_3(\tilde{\boldsymbol{v}}_3\otimes\tilde{\boldsymbol{v}}_3), \tag{3.168}$$

$$\tilde{\boldsymbol{S}}^2 = d_1^2(\tilde{\boldsymbol{v}}_1\otimes\tilde{\boldsymbol{v}}_1) + d_2^2(\tilde{\boldsymbol{v}}_2\otimes\tilde{\boldsymbol{v}}_2) + d_3^2(\tilde{\boldsymbol{v}}_3\otimes\tilde{\boldsymbol{v}}_3), \tag{3.169}$$

where d_1, d_2, d_3 are scalars as eigenvalues of $\tilde{\boldsymbol{S}}$. It follows by a similar continuity argument to Case 3 of Simple Shear Section that

$$\tilde{\boldsymbol{E}} = \phi_1(|\tilde{\boldsymbol{S}}|)\tilde{\boldsymbol{S}} + \phi_2(|\tilde{\boldsymbol{S}}|^2)\tilde{\boldsymbol{S}}^2 \tag{3.170}$$

is uniquely invertible in an interval of γ centered at $\gamma = 0$, in an interval of α near 0, and in an interval of α_2 near 0.

For *monotonicity*, we analyze the left hand side of (3.15) as follows

$$\begin{aligned}
(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}})\cdot(\boldsymbol{a}\otimes\boldsymbol{b}) = {} & d_1(\boldsymbol{a}\cdot\tilde{\boldsymbol{v}}_1)(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_1) + d_2(\boldsymbol{a}\cdot\tilde{\boldsymbol{v}}_2)(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_2) + d_3(\boldsymbol{a}\cdot\tilde{\boldsymbol{v}}_3)(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_3) \\
& + \gamma d_1((\tilde{\boldsymbol{U}}\boldsymbol{v}_1)\cdot\boldsymbol{a})(\boldsymbol{v}_1\cdot\boldsymbol{b}) + \gamma d_2((\tilde{\boldsymbol{U}}\boldsymbol{v}_2)\cdot\boldsymbol{a})(\boldsymbol{v}_2\cdot\boldsymbol{b}) \\
& + \gamma d_3((\tilde{\boldsymbol{U}}\boldsymbol{v}_3)\cdot\boldsymbol{a})(\boldsymbol{v}_3\cdot\boldsymbol{b}) \\
& + \gamma\beta d_1(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_1)^2 + \gamma\beta d_2(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_2)^2 + \gamma\beta d_3(\boldsymbol{b}\cdot\tilde{\boldsymbol{v}}_3)^2 \\
& - c_1(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{b}) - c_2(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{b}) - c_3(\tilde{\boldsymbol{e}}_3\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_3\cdot\boldsymbol{b})) \\
& - \gamma c_1((\tilde{\boldsymbol{U}}\boldsymbol{e}_1)\cdot\boldsymbol{a})(\boldsymbol{e}_1\cdot\boldsymbol{b}) - \gamma c_2((\tilde{\boldsymbol{U}}\boldsymbol{e}_2)\cdot\boldsymbol{a})(\boldsymbol{e}_2\cdot\boldsymbol{b})
\end{aligned}$$

$$- \gamma c_3((\tilde{\boldsymbol{U}}\boldsymbol{e}_3)\cdot\boldsymbol{a})(\boldsymbol{e}_3\cdot\boldsymbol{b}). \tag{3.171}$$

Since each of the eigenvalues of $\boldsymbol{E}$ is a multiple of γ, we rewrite them as the forms

$$\lambda_1 = \gamma\mu_1(\gamma),\ \lambda_2 = \gamma\mu_2(\gamma),\ \lambda_3 = \gamma\mu_3(\gamma). \tag{3.172}$$

Dividing (3.171) by γ, then letting $\gamma \to 0$, we get

$$\begin{aligned}
(\tilde{\boldsymbol{F}}\tilde{\boldsymbol{S}} - \boldsymbol{F}\bar{\boldsymbol{S}})\cdot(\boldsymbol{a}\otimes\boldsymbol{b}) &= \left(\lim_{\gamma\to 0}\frac{d_1}{\gamma}\right)\frac{1}{2(1+\boldsymbol{a}\cdot\boldsymbol{b})}(1+\boldsymbol{a}\cdot\boldsymbol{b})^2 \\
&+ \left(\lim_{\gamma\to 0}\frac{d_2}{\gamma}\right)\frac{1}{2(1-\boldsymbol{a}\cdot\boldsymbol{b})}(1-\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{a}\cdot\boldsymbol{b}-1) \\
&- \left(\lim_{\gamma\to 0}\frac{c_1}{\gamma}\right)(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{b}) - \left(\lim_{\gamma\to 0}\frac{c_2}{\gamma}\right)(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{b}) \\
&- \left(\lim_{\gamma\to 0}\frac{c_3}{\gamma}\right)(\tilde{\boldsymbol{e}}_3\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_3\cdot\boldsymbol{b}) \\
&= \left(\frac{\beta(\boldsymbol{a}\cdot\boldsymbol{b}+1)}{2}\right)\left(\frac{(\boldsymbol{a}\cdot\boldsymbol{b}+1)}{2}\right) \\
&+ \left(\frac{\beta(\boldsymbol{a}\cdot\boldsymbol{b}-1)}{2}\right)\left(\frac{(\boldsymbol{a}\cdot\boldsymbol{b}-1)}{2}\right) \\
&- (\lim_{\gamma\to 0}\mu_1(\gamma))(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_1\cdot\boldsymbol{b}) - (\lim_{\gamma\to 0}\mu_2(\gamma))(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_2\cdot\boldsymbol{b}) \\
&- (\lim_{\gamma\to 0}\mu_3(\gamma))(\tilde{\boldsymbol{e}}_3\cdot\boldsymbol{a})(\tilde{\boldsymbol{e}}_3\cdot\boldsymbol{b}) \\
&> \frac{\beta}{2}(1+(\boldsymbol{a}\cdot\boldsymbol{b})^2) - (|\lim_{\gamma\to 0}\mu_1(\gamma)| + |\lim_{\gamma\to 0}\mu_2(\gamma)| + |\lim_{\gamma\to 0}\mu_3(\gamma)|) \\
&> \frac{\beta}{2} - (|\lim_{\gamma\to 0}\mu_1(\gamma)| + |\lim_{\gamma\to 0}\mu_2(\gamma)| + |\lim_{\gamma\to 0}\mu_3(\gamma)|)\,.
\end{aligned} \tag{3.173}$$

Therefore, *monotonicity* of the model (3.8) holds provided

$$\beta > 2(|\lim_{\gamma\to 0}\mu_1(\gamma)| + |\lim_{\gamma\to 0}\mu_2(\gamma)| + |\lim_{\gamma\to 0}\mu_3(\gamma)|)\,, \tag{3.174}$$

i.e.,

$$0 < \gamma < 1/(2(|\lim_{\gamma\to 0}\mu_1(\gamma)| + |\lim_{\gamma\to 0}\mu_2(\gamma)| + |\lim_{\gamma\to 0}\mu_3(\gamma)|)) \,. \tag{3.175}$$

In summary, by continuity, in an interval of γ centered at $\gamma = 0$, in an interval of α near 0, and in an interval of α_2 near 0, *monotonicity* holds for the model (3.8).

3.7 Conclusions

The *monotonicity* we have studied in this Chapter is for a class of nonlinear strain-limiting models of elastic-like (non-dissipative) material bodies in the form

$$\boldsymbol{E} = \phi_1(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}} + \phi_2(|\bar{\boldsymbol{S}}|^2)\bar{\boldsymbol{S}}^2, \tag{3.176}$$

with

$$\phi_1(r) := \frac{1}{1+\beta_1 r}\,, \tag{3.177}$$

$$\phi_2(r) := \frac{\alpha_2}{1+\beta_2 r}\,, \tag{3.178}$$

where α_2 is a constant, and β_1, β_2 are non-negative constants. In this class of models, it can happen that the Fréchet derivative of the response function is not invertible as a fourth-order tensor even when the response function itself is uniquely invertible. The notion of strong ellipticity introduced in [12] is then no longer valid, leading to the introduction in this Chapter the *monotonicity* as a weaker convexity notion. For the class of models studied herein, it is shown that *monotonicity* holds for strains with sufficiently small norms, and fails (by constructed counterexample) when strain is large enough. These results are similar to the conditions on strain for strong ellipticity of implicit constitutive and strain-limiting models investigated in [12]. As we noted in the Compression Section and the Simple Shear Section, the *monotonicity* condition implies the invertibility of the considered class of models in some cases, but the reverse statement does not hold generally, that is the invertibility of this class of models does not guarantee the monotoncity. This observation emphasizes the independence between the invertibility notion and the *monotonicity* notion. As another note, in this Chapter, we restricted the study of *monotonicity* to the case when (3.176) is uniquely invertible at least for sufficiently small strain. In a future study, we will focus on a more general case when the inverse of (3.176) is a multivalued map, and investigate strong ellipticity as well as *monotonicity* on each branch of the graph of (3.176), i.e. where (3.176) is uniquely invertible. This issue has not been investigated in neither this Chapter nor Chapter 2.

Regarding hyperelasticity, as noted above and demonstrated in the following Appendix A, the class of models (3.8) does not arise as the gradient of a potential unless the function $\phi_2(|\bar{\boldsymbol{S}}|^2)$ is identically constant, and hence not strain limiting unless that constant is zero. However, a natural modification of (3.8) does produce hyperelastic models. Indeed, if the second term on the right-hand-side of (3.8) is replaced by

$$\mathcal{F}_2(\bar{\boldsymbol{S}}) := \phi_2(\det(\bar{\boldsymbol{S}}))\bar{\boldsymbol{S}}^{-1}, \tag{3.179}$$

it is shown in the following Appendix A that one can readily construct a potential for $\mathcal{F}_2(\bar{\boldsymbol{S}})$ in (3.179). Studying convexity for strain limiting models including a response function of the form (3.179) is beyond the scope of the dissertation and will be addressed in a future investigation.

4. CONCLUSIONS

We have investigated the question of *strong ellipticity* (rank-1 convexity) for implicit constitutive and strain-limiting models, and *monotonicity* for strain-limiting models of elastic-like (non-dissipative) material bodies. It was shown in our work [11] and [12] that *strong ellipticity* and *monotonicity* hold for sufficiently small strains and fail when the small strain constraint is relaxed. The dissertation should be viewed as small early steps in studying mathematical properties of the recently established theory of implicit constitutive relations in elasticity [15]. Being new and simple, the theory can be used to characterize a large spectrum of materials so that one can describe a wide range of material behavior. It also leads to new classes of interesting math problems.

We close with a real world application of the study of this dissertation. We wish to mention an example in biology about blast wave as the pressure and flow generated from an explosive core. Exposure to blast wave may result in brain injury and related neurological impairments. We hope our research in the context of implicit and strain-limiting theories of elasticity has a contribution to the study of the mechanism of blast wave and helmet design. For a broad view, in a brain as the Universe, nonlinear elasticity finds its many applications, from blast wave to music wave. In the Universe as a giant brain, nonlinear elasticity has applications in almost everywhere, everything, from safe aircraft to safe buildings.

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APPENDIX A

COMMENTS ON HYPERELASTICITY

The issue at hand concerns under what conditions the response function in (3.8) arises as the tensoral gradient of a scalar potential, that is:

$$\mathcal{F}(\bar{\boldsymbol{S}}) = \partial_{\bar{S}}\xi(\bar{\boldsymbol{S}}) \tag{A.1}$$

with $\xi(\cdot)$ being a real-valued function defined on $\mathcal{S}$ym, the space of symmetric, second-order tensors. The gradient operator $\partial_{\bar{\boldsymbol{S}}}$ in (A.1) is defined through the Riesz Representation Theorem. More specifically, $\partial_{\bar{\boldsymbol{S}}}\xi(\bar{\boldsymbol{S}})$ is defined to be the uniquely defined second-order tensor for which

$$\partial_{\bar{\boldsymbol{S}}}\xi(\bar{\boldsymbol{S}}) \cdot \boldsymbol{H} = D_{\bar{\boldsymbol{S}}}\xi(\bar{\boldsymbol{S}})[\boldsymbol{H}]$$

holds for all symmetric tensors $\boldsymbol{H}$ where $D_{\bar{\boldsymbol{S}}}$ denotes Fréchet differentiation. A well known necessary condition for (A.1) to hold is that the Fréchet derivative of the response function $\mathcal{F}(\cdot)$ define a symmetric fourth-order tensor on $\mathcal{S}$ym, that is,

$$\boldsymbol{H}_1 \cdot D_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})[\boldsymbol{H}_2] = \boldsymbol{H}_2 \cdot D_{\bar{\boldsymbol{S}}}\mathcal{F}(\bar{\boldsymbol{S}})[\boldsymbol{H}_1] \tag{A.2}$$

for all symmetric second-order tensors $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. Re-writing (3.8) as $\mathcal{F}(\bar{\boldsymbol{S}}) = \mathcal{F}_1(\bar{\boldsymbol{S}}) + \mathcal{F}_2(\bar{\boldsymbol{S}})$ with $\mathcal{F}_1(\bar{\boldsymbol{S}}) := \phi_1(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}}$ and $\mathcal{F}_2(\bar{\boldsymbol{S}}) := \phi_2(|\bar{\boldsymbol{S}}|^2)\bar{\boldsymbol{S}}^2$, we note first that in [12], it was shown that the first term on the right-hand-side of (3.8) corresponds to a hyperelastic response function. However, Fréchet differentiation of $\mathcal{F}_2(\bar{\boldsymbol{S}})$ gives:

$$D_{\bar{\boldsymbol{S}}}\mathcal{F}_2(\bar{\boldsymbol{S}})[\boldsymbol{H}] = \phi_2(|\bar{\boldsymbol{S}}|)(\bar{\boldsymbol{S}}\boldsymbol{H} + \boldsymbol{H}\bar{\boldsymbol{S}}) + 2(\bar{\boldsymbol{S}} \cdot \boldsymbol{H})\phi_2'(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}}^2$$

for all $\boldsymbol{H}$ in $\mathcal{S}$ym from which it follows that:

$$\boldsymbol{H}_1 \cdot D_{\bar{\boldsymbol{S}}}\mathcal{F}_2(\bar{\boldsymbol{S}})[\boldsymbol{H}_2] = \phi_2(|\bar{\boldsymbol{S}}|)\bar{\boldsymbol{S}} \cdot (\boldsymbol{H}_1\boldsymbol{H}_2 + \boldsymbol{H}_2\boldsymbol{H}_1) + 2\phi_2'(|\bar{\boldsymbol{S}}|)(\boldsymbol{H}_1 \cdot \bar{\boldsymbol{S}}^2)(\boldsymbol{H}_2 \cdot \bar{\boldsymbol{S}}). \tag{A.3}$$

While the first term on the right-hand-side of (A.3) is symmetric in $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$, the second term is clearly not unless $\phi_2'(\cdot)$ vanishes identically.

Finally, concerning hyperelasticity for response functions of the form (3.179), it is straight forward to verify that:

$$\phi_2(\det(\bar{\boldsymbol{S}}))\bar{\boldsymbol{S}}^{-1} = \partial_{\bar{\boldsymbol{S}}}\xi_2(\det(\bar{\boldsymbol{S}}))$$

where

$$\xi_2(r) := \int \phi_2(r)\frac{dr}{r}\,.$$

APPENDIX B

DYNAMICAL SIGNIFICANCE OF THE STRONG ELLIPTICITY CONDITION

This Appendix is based on our several discussions and [1].

Consider a displacement equation of motion (built on the basis of the balance of momentum):

$$\rho\ddot{u} = \mathrm{Div}\hat{\boldsymbol{S}}(\nabla u), \tag{B.1}$$

Let us now examine the dynamical significance of the *strong ellipticity condition* by classifying the type of (B.1) as a system of partial differential equations from the view point of wave propagation. Since such classifications are purely local, we study the behavior of a solution u for (X, t) near (X_0, t_0) by studying the linearization of (B.1) about an equilibrium state with constant deformation $\boldsymbol{F}$ for a homogeneous elastic body occupying all space $\mathcal{E}^3$ with constitutive function $\boldsymbol{F} \to \hat{\boldsymbol{S}}(\boldsymbol{F}, X_0)$ and constant density ρ.

For linearization purpose, consider a solution of (B.1) written in the form

$$u(X,t) = u_0(X_0, t_0) + \delta w(X,t), \tag{B.2}$$

where δ: small amplitude, X: material point, x: position point, b_0: body force. From (B.2) and (B.1), we have

$$\begin{aligned}
\rho\delta\ddot{w}(X,t) &= \rho\ddot{u}(X,t) - \rho\ddot{u}_0(X_0,t_0) \\
&= \rho\ddot{u}(X,t) \\
&= \mathrm{Div}\hat{\boldsymbol{S}}(\nabla u).
\end{aligned} \tag{B.3}$$

A substitution of (B.2) into (B.3) with a note that the gradient operator ∇ is linear gives

$$\rho\delta\ddot{w}(X,t) = \mathrm{Div}\hat{\boldsymbol{S}}(\nabla u_0(X_0) + \delta\nabla w(X,t)), \tag{B.4}$$

where we denote $\hat{\boldsymbol{S}}(\nabla u_0(X_0)) := \hat{\boldsymbol{S}}(\nabla u_0(X_0, t_0), X_0)$, and $\nabla u_0(X_0) := \nabla u_0(X_0, t_0)$.

Now, using Taylor expansion for $\hat{\boldsymbol{S}}(\cdot)$ about $\nabla u_0(X_0)$ in (B.4), and note that the operator Div is linear, we get

$$\begin{aligned} \rho\delta\ddot{w}(X,t) &= \mathrm{Div}(\hat{\boldsymbol{S}}(\nabla u_0(X_0)) + \mathrm{D}\hat{\boldsymbol{S}}(\nabla u_0(X_0))[\delta\nabla w]) \\ &= \mathrm{Div}\hat{\boldsymbol{S}}(\nabla u_0(X_0)) + \mathrm{Div}(\mathrm{D}\hat{\boldsymbol{S}}(\nabla u_0(X_0))[\delta\nabla w]). \end{aligned} \tag{B.5}$$

From (B.5), we have

$$\begin{aligned} \rho\ddot{w}(X,t) &= \mathrm{Div}(\mathrm{D}\hat{\boldsymbol{S}}(\nabla u_0(X_0))[\nabla w]) \\ &= \mathrm{Div}(\mathbb{E}_0[\nabla w]), \end{aligned} \tag{B.6}$$

where

$$\mathbb{E}_0[\nabla w] = \mathrm{D}\hat{\boldsymbol{S}}(\nabla u_0(X_0)). \tag{B.7}$$

We seek solution of (B.6) in the form of plane traveling waves in the direction ξ with speed c, i.e., solutions of the form

$$w(X,t) = \phi(X\cdot\xi - ct)a, \tag{B.8}$$

in which, a: amplitude, $|\xi| = 1$. Then

$$\begin{aligned} \nabla_X w(X,t) &= \phi(X\cdot\xi - ct)\nabla_X a + a\otimes\nabla_X\phi(X\cdot\xi - ct) \\ &= \phi'(X\cdot\xi - ct)(a\otimes\xi). \end{aligned} \tag{B.9}$$

From (B.6) and (B.8), we have

$$\begin{aligned} \rho c^2\phi''(X\cdot\xi - ct)a &= \mathrm{Div}((\phi'(X\cdot\xi - ct))(\mathbb{E}_0[a\otimes\xi])) \\ &= (\mathbb{E}_0[a\otimes\xi])(\nabla_X\phi'(X\cdot\xi - ct)) \\ &= (\mathbb{E}_0[a\otimes\xi])(\phi''(X\cdot\xi - ct))\xi\,. \end{aligned} \tag{B.10}$$

For a given ξ, this equation has a nontrivial solution $\phi''(X \cdot \xi - ct)$, i.e. (B.6) admits a traveling wave in direction ξ if and only if the acoustic tensor $\mathbb{A}_0$ (where $\rho c^2 a = \mathbb{A}_0 a = \mathbb{E}_0[a \otimes \xi]\xi$) has a positive eigenvalue $c^2\rho$, which satisfies

$$\det(\mathbb{A}_0 - c^2\rho\boldsymbol{I}) = 0\,. \tag{B.11}$$

We note that

$$\begin{aligned} a \cdot (\mathbb{A}_0 a) &= a \cdot \mathbb{E}_0[a \otimes \xi]\xi \\ &= (a \otimes \xi) \cdot \mathbb{E}_0[a \otimes \xi], \end{aligned} \tag{B.12}$$

$\mathbb{E}_0$ is called elastic tensor, and $\mathbb{A}_0$ is called acoustic tensor.

Now, $\mathbb{E}_0$ is strong elliptic if and only if $\boldsymbol{H} \cdot \mathbb{E}_0[\boldsymbol{H}] > 0 \;\forall\; \boldsymbol{H} = a \otimes \xi$, $a \neq 0$, $\xi \neq 0$, if and only if $(a \otimes \xi) \cdot \mathbb{E}_0[a \otimes \xi] > 0$, if and only if $a \cdot \mathbb{E}_0[a \otimes \xi]\xi > 0$, if and only if $a \cdot \mathbb{A}_0 a > 0$, if and only if $\mathbb{A}_0$ positive definite for each direction ξ.

The right hand side of (B.6) is said to be *elliptic* at any solution for which there is a positive solution $c^2\rho$ to (B.11). The *strong ellipticity condition* ensures that $\mathbb{A}_0$, which is not necessarily symmetric, is positive definite. Thus, there is a positive eigenvalue $c^2\rho$ with a corresponding eigenvector $\phi''(X \cdot \xi - ct) = \xi$. Hence, the *strong ellipticity condition* ensures the existence of longitudinal traveling wave. Equation (B.6) is *hyperbolic* if and only if for each ξ, all the eigenvalues of (B.10) are positive and if the corresponding eigenvectors span the Euclidean space $\mathcal{E}^3$. The *strong ellipticity condition* ensures the positivity of the eigenvalues. If $\mathbb{A}_0$ is symmetric for all ξ (so, by Spectral Theorem, there is an orthonormal basis for $\mathcal{E}^3$ consisting entirely of eigenvectors of $\mathbb{A}_0$), then the *strong ellipticity condition* also ensures the hyperbolicity of (B.6), which means that it admits the full range of wave-like behavior. (It can be shown that $\mathbb{A}_0$ is symmetric for all $\boldsymbol{F}$ and ξ if and only if the material is hyperelastic. [1])