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QUALITY OF MAXIMUM LIKELIHOOD ESTIMATES OF PARAMETERS IN A LOG-LINEAR RATE MODEL

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1. OVERVIEW

Most longitudinal analyses in the social sciences can be grouped into one of two distinct traditions. The first tradition, typified by the work of Lazarsfeld (1948), Pelz and Andrews (1964), Duncan (1969, 1972), and Blalock (1970), is explicitly multivariate and causal. But it does not utilize the full potential of longitudinal data. Though the models used (difference equations) imply dynamic behavior, this tradition restricts itself to static inferences. The second tradition involves the application of stochastic processes, especially Markov models, to such social science problems as occupational mobility and learning (Blumen et al., 1955; Atkinson et al., 1965). This tradition concentrates on dynamic behavior. But it is largely acausal and univariate.

Recently some investigators have attempted to combine these two traditions (e.g., Spilerman (1972), Sørensen (1975), and Tuma (1976)) by formulating stochastic process models that are explicitly causal. For instance, Tuma (1976) estimated models in which the instantaneous rate of job leaving depends on a set of observable individual characteristics. A similar model has been used in analyzing the effects of experimental income maintenance schemes on marital formation and dissolution (Hannan et al., 1977). This type of model can be applied to any problem involving qualitative dependent variables in which changes of state occur stochastically and at any moment within a continuous time span.

Tuma's approach utilizes Maximum Likelihood Estimation (MLE) to obtain estimates of causal parameters. These estimators are asymptotically consistent, efficient, and normally distributed under fairly weak regularity conditions on the probability distribution function of the dependent variable (see, e.g., Dhrymes, 1970). However, their small-sample properties when applied to instantaneous rate models have not been completely determined. Mendenhall and Lehman (1960) have determined these analytically for the case in which the rate is constant across individuals.

As social researchers shift increasingly to multivariate dynamic models, it becomes important to understand the small-sample properties of MLE for a variety of models and complications likely to be encountered in practice. Here we concern ourselves with four complications. The first is censoring of observations. Sample censoring arises when the length of the observation period (length of time between first and last observations) is too short for a change to have occurred for every case. Consequently, the time of changes on the dependent variable is unknown for those cases in which a change has not occurred before the end of the observation period. This problem characterizes much research on relatively rare events, e.g., marriage, job change, failure of an organization, etc. The proportion of the sample who have not experienced a change determines the degree of censoring. We need to know, then, how the quality of MLE depends on levels of censoring.

Small sample size is another important problem frequently faced by social scientists. Thus we also wish to know how the quality of estimates of parameters of rate models depends on sample size. We also investigate how censoring and small sample size jointly affect the quality of maximum likelihood estimates. Finally we add two more complications: collinearity among causal variables and improper specification of the causal structure. What happens to the quality of the estimates under combinations of collinearity, small sample size, censoring, and model misspecification?

This paper reports the results of Monte Carlo simulations designed to answer these questions. In particular we study how the four complications influence the quality of ML estimates of parameters in a log-linear rate model when data consist of the lengths of time between events. The properties of estimators that we consider are bias, variance, and mean squared error.

Section 2 outlines the model studied. Section 3 formally presents the method of maximum likelihood estimation. Section 4 briefly discusses some previous findings on the quality of ML estimates of parameters of rate models. Section 5 describes the method of generating the Monte Carlo data. We present our results in Section 6 and discuss our conclusions in Section 7.

2. THE MODEL

An instantaneous rate of change is similar to the conditional probability of a change from one state to another within a momentary unit of time. Like a probability of change, an instantaneous rate of change cannot be observed directly. However, a rate model can be used to generate predictions about a variety of observable variables such as length of time between successive changes of state, the number of changes of state within a given time interval, and the states occupied at a series of points in time. Measurements on these observable variables can then be used to estimate parameters in the model (see Tuma and Crockford, 1976).

The probability of leaving a state (such as a job or a marital status) on or before t is denoted as $F(t)$. The instantaneous rate at which such an event (e.g., a job change) occurs, $r(t)$, is defined as follows:

$$r(t)\,\Delta t = \frac{\Pr(\text{event occurs between } t \text{ and } t + \Delta t)}{\Pr(\text{event has not occurred before } t)}\,, \tag{1}$$

or

$$r\,\Delta t = \frac{F(t + \Delta t) - F(t)}{1-F(t)}\,, \tag{2}$$

where Δt represents a nonnegative time interval. In the job change example $r\,\Delta t$ would be the proportion of those who have not changed jobs before time t, but who then leave their jobs between t and $t + \Delta t$. Dividing (2) by Δt and letting $\Delta t \to 0$, we have

$$r(t) = \lim_{\Delta t \to 0} r = \frac{\frac{dF(t)}{dt}}{1-F(t)} = \frac{f(t)}{1-F(t)} \tag{3}$$

where $f(t)$ is the probability density function of the length of time between events. Equation (3) is a differential equation that may be solved for F in terms of r:

$$F(t) = 1 - \exp\left(-\int_0^t r(u)du\right). \tag{4}$$

If r is constant over time, $r(u) = r$, then

$$F(t) = 1 - e^{-rt}\,. \tag{5}$$

Since we analyze only constant rate models, (5) is the basic stochastic equation of the model investigated.

As in Hannan et al. (1977), r is assumed to be an exponential function of exogenous variables. In our experiments r is a log-linear function of two causal variables and a constant term:

$$\ln r = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 \quad , \tag{6a}$$

or

$$r = \exp(\alpha_0 + \alpha_1 X_1 + \alpha_2 X_2) \quad . \tag{6b}$$

We assume that X_1 and X_2 are joint normally distributed. Since we are also interested in the effect of omitted variables on the quality of the estimates produced by MLE, we compare results from (6b) with the estimates of the model that incorrectly excludes X_2:

$$r = \exp(\alpha_0 + \alpha_1 X_1) \quad . \tag{7}$$

3. METHOD OF ESTIMATION

Maximum likelihood estimation has several advantages over the more commonly used method of ordinary least squares for the model stated above. Given the assumptions about rates of change and the type of data used in estimating the model, the relationship between the expected values of observable dependent variables and exogenous variables is usually nonlinear.[1] Least squares estimation of rate models requires more costly and complex iterative procedures. More importantly, when the expected value of the dependent variable is nonlinear in the parameters, we cannot rely on the Gauss-Markov Theorem concerning the desirability of the properties of the least squares estimators. ML estimators, however, are asymptotically consistent, efficient, and normally distributed under fairly weak regularity conditions on the probability distribution function specified by the model (see Dhrymes, 1970). Finally, there is no OLS method analogous to the ML method developed by Bartholomew (1957) for utilizing the information contained in censored observations. Thus MLE appears to be the more desirable method of estimation.

We work with the following data structure. For simplicity assume that all individuals enter the state in question at time zero. We observe each individual i at some later date, T_i. At that time we record either that they have left the state prior to T_i or that they are still in the original state. For those individuals who have changed state we record the date of the change, t_i. We also define an indicator variable, y_i, that takes the value of one for those for whom a change of state is observed and zero for censored observations.

Next we define a likelihood function in terms of these three observable variables, T_i, t_i, and y_i. The likelihood function is the joint probability of the sample observations. For those for whom an event is observed the probability is $f(t_i)$; for censored observations the probability is $1-F(T_i)$, the probability of not observing an event by T_i. If the sample observations are independent, the likelihood function is

$$L = \prod_{i=1}^{N} \left[\frac{dF(t_i|\underline{X}_i)}{dt_i}\right]^{y_i} \cdot \left[1-F(T_i|\underline{X}_i)\right]^{(1-y_i)} \qquad (8)$$

where N is the number of observations and $\underline{X}_i$ is the vector of exogenous variables.

Maximum likelihood estimates of the parameters of the model used in these experiments were produced by the FORTRAN program RATE, developed by Tuma and Crockford (1976). This program is capable of estimating a variety of general causal models of rates. The iterative procedure used by RATE is a variant of Newton's method developed by Gill and Murray (1972) and co-workers (Gill et al., 1972a; 1972b). This method is faster than the more widely used Fletcher-Powell algorithm; it compares favorably in terms of reliability and accuracy of convergence to other widely used algorithms (Gill and Murray, 1972).

4. PREVIOUS FINDINGS

Investigators in the fields of labor turnover and product life-testing (see Bartholomew, 1957, 1959, 1963; Mendenhall and Lehman, 1960) have investigated MLE for models in which the rate is the same for all individuals. Bartholomew (1957, 1963) found that the ML estimate of the rate was slightly upwardly biased in small samples. Mendenhall and Lehman (1960) developed small-sample approximations to the mean and variance of the ML estimator for a constant rate model. Their approximations are poor for small samples ($N \sim 50$) but improve as sample size increases (e.g., when $N=100$; see Mendenhall and Lehman 1960, pp. 238). Furthermore, they report that the sampling distributions differ in large and small samples.

Keeley (1975) investigated the properties of ML estimates of a model in which the rate was a log-linear function of uniformly-distributed independent variables. He used samples with $N=500$. He found that the estimated coefficients were within one standard deviation of the true values, and that for uncorrelated independent variables, the estimates were insensitive to model misspecification. These results have limited value, however, due to the fact that a uniform distribution is rather implausible in most social scientific applications.

5. DATA GENERATION

In Section 2 we assumed the rate is a log-linear function of X_1, X_2 and a constant term (see equation 6). In this investigation X_1 and X_2 are normally-distributed variables with mean μ of zero and variance σ^2 of 1. A log-linear rate model with normally-distributed independent variables implies a log-normal distribution of rates and a log-normal distribution of t, the length of time between events. Previous investigators have found that a log-normal distribution of t gives good predictions of the observed time between job changes, length of service phenomena, etc. (Bartholomew, 1973; Lane and Andrew, 1955; Young, 1971).

We generated random normal variates by Marsaglia's rectangle-wedge-tail algorithmn.[2] Two sample sizes (N=50 and N=100), and three levels of collinearity ($\rho \equiv \rho_{X_1X_2} = 0, 0.5, -0.5$) were studied. Using the Marsaglia method we produced 100 independent samples for each of the six combinations.

For each of the six conditions, we created three levels of censoring:

1) no censoring - individuals are observed until all experience a change;
2) 60% censoring - individuals are observed until 40% are expected to have experienced a change (60% have no change);
3) 80% censoring - individuals are observed until 20% are expected to have experienced a change (80% have no change).

The values of the parameters used in generating the data were: $\alpha_0 = -4$; $\alpha_1 = 2$; and $\alpha_2 = 2$. Given these values of α_0, α_1, α_2 and ρ, the expected value of the rate equals 1 when ρ is zero, exp[2] when ρ is 0.5, and exp[-2] when ρ is -0.5. The expected time until a change of state equals exp[8], exp[10], and exp[6] when ρ is 0, 0.5, and -0.5, respectively.

6. RESULTS

We evaluate the effects of censoring, sample size, and collinearity in terms of percent-bias, variance, and mean squared error (MSE) of estimates. We report percent-bias[3] rather than raw deviations of the mean of estimates from parameters to facilitate comparisons across different parameter values. Results on absolute bias are given in Appendices A and B.

Our results are presented in Tables 1 through 6. Tables 1, 2, and 3 show the MSE, percent-bias and variance, respectively, of the ML estimates of the correctly-specified model, equation (6b). Tables 4, 5, and 6 give the MSE, percent-bias and variance of the estimates of the misspecified model, equation (7).

In this section we first discuss the MSE, percent-bias and variance of the estimates for the correctly-specified model. Under each of these three evaluation criteria we examine the effects of sample size, censoring, and collinearity. This order of discussion is then repeated for the estimates from the misspecified model.

Correctly-Specified Model

Mean Squared Error. In all cases the larger sample size decreases the MSE of the estimates, thereby improving the quality of estimates

(Table 1). Since similar improvements in quality with greater sample size generally hold for both percent-bias and variance, the effect of sample size is not considered again until the discussion of the results for the misspecified model.

(Table 1 about here)

For each value of ρ (the correlation between the two exogenous variables) MSE is small (less than .05) for uncensored observations and increases with censoring. The magnitude of the increases in MSE with greater censoring is fairly consistent across all three values of ρ.

For each level of censoring, the correlation between the two causal variables has comparatively small effects on the MSE of estimates. When the level of censoring and the correlation of the independent variables are considered jointly, we see that the MSE of all three estimates (α_0, α_1, α_2) is largest for the case with the greatest level of censoring (80%) and a positive correlation (0.5) between the independent variables.

We conclude, therefore, that for a correctly-specified model: (1) a larger sample size reduces MSE of estimates, (2) greater censoring markedly increases MSE, and (3) the correlation between the causal variables has relatively little effect on MSE. Since our smallest sample size (50) is considered quite small by most social scientists and produces a very small MSE in the uncensored case, our results suggest that the level of censoring is the greatest practical problem insofar as MSE of ML estimates of parameters in a log-linear rate model are concerned.

Percent-Bias. Bias of these estimators appears responsive to censoring but only slightly sensitive to correlation between the exogenous variables

(Table 2). When observations are uncensored, the estimators have a small positive bias (always less than 5%). The bias is similar for all values of ρ. Overall we find that the effects of sample size, censoring, and correlation between the independent variables are small in comparison with their effects on MSE.

(Table 2 about here)

Variance. The variances of estimates (Table 3) are extremely similar to the mean squared errors (Table 1). In other words, for the correctly-specified model, bias of estimates is always so small that MSE is almost totally determined by the variance.

(Table 3 about here)

For all three levels of ρ the variance of estimates is small when data are uncensored. Efficiency declines with greater censoring, as indicated by the increase in the variance from uncensored observations to 60% and 80% levels of censoring. Correlation of the causal variables has little effect on efficiency.

In general the maximum likelihood estimates of parameters of a correctly-specified rate model are of good quality for both sample sizes and all three correlations between exogenous variables. Bias is consistently small and usually positive. Variances are also small, meaning that estimates are reasonably efficient, except when censoring is extreme. Censoring appears to be a more serious impediment to correct inference than either sample size or correlation among causal variables.

The Misspecified Model

We now turn to results on the estimates of the parameters of the misspecified model, which excludes X_2. We expect that misspecification will adversely affect estimator quality. We are interested in determining how reduction in quality depends on sample size, censoring, and collinearity of the causal variables.

Mean Squared Error. MSE tends to be greatly affected by the omission of X_2 from the estimation equation (Table 4). Most values of MSE are considerably greater than comparable entries for the correctly-specified model (Table 1), showing that misspecification can markedly reduce the quality of estimates, as anticipated.

(Table 4 about here)

More importantly, we find that the effects of sample size, censoring, and collinearity on MSE for the misspecified model differ considerably from those for the correctly-specified model. In the correctly-specified model the larger sample size leads to smaller MSE without exception. In the misspecified model the larger sample size reduces the MSE in 55 percent of the cases considered; however, this is so close to one-half that it may be due to chance. Thus the results for the misspecified model do not convincingly indicate that a larger sample size improves estimator quality.

The relationship between MSE and level of censoring for the misspecified model also differs from (and is less clear than) that for the correctly-specified model. When both causal variables are (correctly) included, MSE increases as censoring increases. This monotonic pattern is not repeated in the misspecified model. For all three values of ρ, the MSE of the constant term (α_0) is much smaller for 60% censored observations than for either uncensored or 80% censored data. Identifying the relationship between censoring and the MSE of α_1 is still more difficult, as it depends both on ρ and occasionally on the sample size. When ρ equals 0.0 or -0.5, the MSE of α_1 is usually smallest for uncensored observations. But when ρ equals 0.5, it is smallest for 60% censoring.

In the correctly-specified model MSE is only slightly affected by the correlation between the exogenous variables. Again the misspecified model shows a different pattern. When ρ equals -0.5, the MSE of the constant term is smallest while the MSE of the coefficient of the causal variable is greatest. Though the MSE of estimates for ρ equal to 0.0 and 0.5 are not always similar, which level of correlation produces a smaller MSE varies with both sample size and level of censoring. The effects of collinearity are much greater than in the correctly-specified model, even though they cannot be summarized in a simple way.

The results for the dependence of MSE on sample size, censoring, and collinearity in a misspecified model are complex. Examination of the results for percent-bias and variance of estimates in the misspecified model may help explain them.

Percent-Bias. The biases for the misspecified model (Table 5) are much larger than those for the correctly-specified model (Table 1). As anticipated, omitting a causal variable from the estimation equation distorts estimates of the constant term α_0 and of the effect of the included variable α_1.

(Table 5 about here)

In all but one case the larger sample size actually leads to a greater bias in estimates. However, the differences for the two sample sizes are small compared to those produced by variation in censoring and in the correlation between the independent variables.

For each value of ρ , the constant term is underestimated when observations are uncensored but is overestimated for the two levels of censoring. Moreover, the positive bias in α_0 increases with the degree of censoring. The situation for estimates of the causal parameter α_1 is more complicated. Level of censoring, correlation among independent variables, and (to a lesser extent) sample size apparently interact in affecting bias. When the exogenous variables are negatively correlated, the bias in estimates of α_1 is always negative and relatively unaffected by sample size. When they are positively correlated, bias is usually positive and decreases as censoring increases. When ρ is zero, bias is slightly positive in the uncensored case and quite negative in the censored cases.

Variance. The variance of estimates is not affected as much by misspecification (Table 6) as is the bias of estimates. Only in the case of uncensored observations is the variance of the constant term α_0 greater in the misspecified model than in the correctly-specified one. However, the variance of the coefficient of X_1 is almost always higher in the misspecified model than in the correctly-specified one.

(Table 6 about here)

In both correctly and incorrectly specified models, a larger sample size reduces the variance of estimates. However, effects of censoring and of correlation between the causal variables are quite different in the two models. In the correctly-specified model variance increases with censoring and is relatively unaffected by the correlation between the independent variables. In the misspecified model, the variance of estimates is usually least when there is an intermediate level of censoring (60%) and a negative correlation between the causal variables.

7. CONCLUSION

Our Monte Carlo experiments indicate that the quality of maximum likelihood estimates of a correctly-specified log-linear rate model is generally good. This applies to small samples (N = 50 and 100) and positively, negatively, and uncorrelated causal variables (ρ = 0.5, -0.5, and 0.0, respectively). Bias is usually positive but consistently small (under 5%). Variance is also small, except when censoring is extreme. This means that the estimators are reasonably efficient. Somewhat surprisingly (for those accustomed to least squares estimation of linear regression equations), efficiency is only slightly affected by collinearity between the independent variables. For the correctly-specified model, censoring seems to be the most serious impediment to correct inference, but can be compensated for by increasing the sample size.

As anticipated, misspecification noticeably reduces the quality of estimates. In addition, we find that sample size, censoring, and collinearity affect quality quite differently in the misspecified and correctly-specified models. In the misspecified model, the net effect of sample size on quality (as measured by mean squared error) is ambiguous; a larger sample size increases bias but decreases variance. We also find that in the misspecified model uncensored observations do not always produce the best estimates. Furthermore, the effects of censoring depend on the parameter considered, the correlation between the included and omitted variables, and sample size. Consequently there does not appear to be any simple rule about the effects of censoring when specification error is present. The correlation between the causal variables also affects quality. The coefficient of the included causal variable is estimated most accurately when its correlation with the omitted variable is zero, but the constant term is estimated best when the two causal variables have a negative correlation. The overall complexity of results in the presence of specification error only adds weight to the usual conclusion that problems abound when the model is wrong.

FOOTNOTES

1. If it is assumed that $r_j = \exp(\underline{\alpha}_j \cdot \underline{X}_j)$, then $E(t-t') = \exp(-\underline{\alpha} \cdot \underline{X}_j)$ and $\ln(E(t-t')) = -\underline{\alpha}_j \cdot \underline{X}_j$. In this case, the parameters $\underline{\alpha}_j$ can be estimated by least squares regression of the logarithmn of $(t-t')$ on the exogenous variables $\underline{X}_j$. The appropriate form of the linear regression when some data are censored, as is typically the case, is unclear.

2. A "fast normal random deviate generator" was used to produce single-precision pseudo-normal (0,1) random numbers. This method follows Marsaglia's rectangle-wedge-tail algorithmn as described in Knuth (1969). The Marsaglia method uses the following distribution:

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-v^2/2} dv \qquad x \geq 0$$

which gives the distribution of the absolute value of a normal deviate.

Following Knuth (1969), two standard normal variables, X_1 and X_2, were generated with correlation coefficient ρ set at 0.0, 0.5, and -0.5, using the equation

$$X_2 = \rho X_1 + (\sqrt{1-\rho_2})\, Y$$

where X_1 and Y are both standard normal variables.

The time of a change t was generated as follows:

$$t = \frac{-\log(U(0,1))}{e^{\alpha_0 + \alpha_1 X_1 + \alpha_2 X_2}}$$

where $U(0,1)$ represents a uniformly distributed variable on the range

0 to 1. For censored data the times at which a change occurs were calculated; however, here t_i or T_i (which ever was the smallest) was generated. For example, if $t_i > T_i$, then we would not have observed a change of state; therefore T_i (the ending time) would be used.

3. Percent-bias is calculated as $100 \cdot [\frac{\bar{\hat{\theta}}-\theta}{\theta}]$

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Table 1. Mean Squared Error of Estimates: Correctly-Specified Model
(All entries based on 100 observations;)

		α_0		α_1		α_2	
		N=50	N=100	N=50	N=100	N=50	N=100
	Uncensored	.024	.012	.024	.015	.030	.013
ρ=0.0	60% cens.	.109	.061	.105	.058	.077	.039
	80% cens.	.606	.237	.222	.095	.297	.127
	Uncensored	.026	.012	.032	.020	.039	.017
ρ=0.5	60% cens.	.198	.064	.124	.058	.121	.045
	80% cens.	1.324	.650	.390	.159	.395	.186
	Uncensored	.024	.012	.029	.017	.040	.017
ρ=−0.5	60% cens.	.071	.044	.095	.061	.092	.052
	80% cens.	.373	.153	.291	.126	.369	.152

Table 2. Percent-Bias of Estimates: Correctly-Specified Model
(All entries based on 100 observations.)

		α_0		α_1		α_2	
		N=50	N=100	N=50	N=100	N=50	N=100
	Uncensored	1.0%	.8%	2.4%	2.0%	-0-	.1%
$\rho=0.0$	60% cens.	-.3%	-.5%	5.8%	4.9%	1.3%	1.6%
	80% cens.	-3.5%	.2%	7.6%	3.1%	6.8%	2.3%
	Uncensored	1.3%	.7%	2.7%	1.6%	-.6%	.6%
$\rho=0.5$	60% cens.	.5%	-.2%	4.4%	3.2%	1.5%	1.9%
	80% cens.	4.5%	3.9%	-1.4%	1.2%	3.2%	-1.4%
	Uncensored	1.0%	.7%	2.4%	2.2%	-0-	.6%
$\rho=-0.5$	60% cens.	.3%	.5%	4.7%	3.1%	1.8%	1.4%
	80% cens.	-3.6%	-.7%	9.8%	4.7%	7.5%	3.4%

Table 3. Variance of Estimates: Correctly-Specified Model
(All entries based on 100 observations.)

		α_0		α_1		α_2	
		N=50	N=100	N=50	N=100	N=50	N=100
	Uncensored	.022	.011	.022	.013	.030	.013
ρ=0.0	60% cens.	.109	.061	.091	.049	.076	.038
	80% cens.	.586	.237	.199	.091	.279	.125
	Uncensored	.023	.011	.029	.019	.039	.017
ρ=0.5	60% cens.	.198	.064	.116	.054	.120	.044
	80% cens.	1.291	.626	.389	.158	.391	.185
	Uncensored	.022	.011	.027	.015	.040	.017
ρ=−0.5	60% cens.	.071	.044	.086	.057	.091	.051
	80% cens.	.351	.152	.252	.117	.346	.147

Table 4. Mean Squared Error of Estimates: Misspecified Model (All entries based on 100 observations.)

		α_0		α_1	
		N=50	N=100	N=50	N=100
	Uncensored	3.219	3.550	.344	.205
$\rho=0.0$	60% cens.	.135	.101	.547	.511
	80% cens.	1.230	1.304	.584	.125
	Uncensored	2.280	2.109	.943	1.251
$\rho=0.5$	60% cens.	.297	.108	.457	.275
	80% cens.	2.123	1.299	.580	.312
	Uncensored	1.967	2.108	1.090	1.029
$\rho=-0.5$	60% cens.	.086	.074	1.765	1.779
	80% cens.	.539	.565	1.597	1.660

Table 5. Percent-Bias of Estimates: Misspecified Model (All entries based on 100 observations.)

		α_0		α_1	
		N=50	N=100	N=50	N=100
	Uncensored	-42.1%	-45.4%	4.3%	4.3%
ρ=0.0	60% cens.	5.4%	6.1%	-28.9%	-31.4%
	80% cens.	25.1%	27.7%	-28.2%	-33.3%
	Uncensored	-32.5%	-34.8%	27.6%	52.7%
ρ=0.5	60% cens.	8.1%	4.8%	-4.5%	14.2%
	80% cens.	22.0%	25.9%	-3.4%	12.6%
	Uncensored	-32.5%	-34.8%	-46.0%	-47.2%
ρ=-0.5	60% cens.	4.9%	5.5%	-64.5%	-65.7%
	80% cens.	16.1%	17.9%	-60.1%	-63.2%

Table 6. Variance of Estimates: Misspecified Model (All entries based on 100 observations.)

		α_0		α_1	
		N=50	N=100	N=50	N=100
	Uncensored	.387	.256	.337	.198
$\rho=0.0$	60% cens.	.089	.041	.212	.155
	80% cens.	.224	.076	.266	.121
	Uncensored	.590	.171	.639	.138
$\rho=0.5$	60% cens.	.192	.071	.449	.195
	80% cens.	1.347	.222	.575	.248
	Uncensored	.274	.170	.244	.138
$\rho=-0.5$	60% cens.	.048	.026	.103	.055
	80% cens.	.126	.054	.155	.065

Appendix A. Bias of Estimates: Correctly-Specified Model (All entries based on 100 observations.)

		α_0		α_1		α_2	
		N=50	N=100	N=50	N=100	N=50	N=100
	Uncensored	.041	.030	.048	.040	-0-	.003
$\rho=0.0$	60% cens.	-.013	-.019	.177	.097	.026	.032
	80% cens.	-.140	.008	.152	.062	.136	.046
	Uncensored	.052	.027	.054	.032	-.012	.012
$\rho=0.5$	60% cens.	.022	-.009	.087	.063	.029	.038
	80% cens.	.181	.155	-.027	.024	.064	-.029
	Uncensored	.041	.027	.048	.044	-0-	.012
$\rho=-0.5$	60% cens.	.011	.020	.093	.062	.035	.028
	80% cens.	-.145	-.027	.196	.094	.150	.068

Appendix B. Bias of Estimates: Misspecified Model (All entries based on 100 observations.)

		α_0		α_1	
		N=50	N=100	N=50	N=100
	Uncensored	-1.683	-1.815	.086	.086
$\rho=0.0$	60% cens.	.215	.244	-.579	-.629
	80% cens.	1.003	1.108	-.564	-.667
	Uncensored	-1.300	-1.392	.551	1.055
$\rho=0.5$	60% cens.	.324	.193	-.089	.283
	80% cens.	.881	1.038	-.068	.253
	Uncensored	-1.301	-1.392	-.920	-.944
$\rho=-0.5$	60% cens.	.196	.218	-1.289	-1.313
	80% cens.	.643	.715	-1.201	-1.263