CAPACITY APPROACHING STRATEGIES FOR THE UNCOORDINATED GAUSSIAN MULTIPLE ACCESS CHANNEL

A Thesis

by

SRIHARSHA MADALA

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Chair of Committee,	Krishna Narayanan
Co-Chair of Committee,	Henry Pfister
Committee Members,	Narasimha Reddy
	Swaroop Darbha
Head of Department,	Chanan Singh

December 2014

Major Subject: Electrical and Computer Engineering

ABSTRACT

The Gaussian Multiple access channel (GMAC) is a well-studied multiuser channel in information theory and communication theory. The capacity of the GMAC can be achieved in theory and closely approached with practical coding schemes when there is coordination between the transmitters and the base station. In recent years, there has been growing interest in the design of uncoordinated multiple access or random access schemes for the GMAC, where there is no coordination between the users and the base station. The performance of such random access schemes with iterative collision resolution or Successive Interference Cancellation (SIC) for the interference-limited channel has been studied in previous works and the results show that the throughput efficiency of random access schemes can be as high as those of coordinated multiple access schemes. However, these works do not consider transmit power constraints and additive white Gaussian noise at the receiver.

In this thesis, we consider the design of uncoordinate multiple access schemes that explicitly consider transmit power constraint and additive white Gaussian noise. We first show that direct extensions of the existing schemes, to the power constrained channel results in an inefficient scheme and we also show that using maximal ratio combining does not improve its performance significantly. Most importantly, we propose a novel uncoordinated multiple access scheme that allows each transmitter to pick the rate of transmission from a predetermined distribution. By selecting the rates from *corner points* of the achievable rate region, a SIC decoder can be used which has a single-user decoding complexity. We show that using this scheme we can achieve an absolute gap of the order $O(\log_2 \log_2 K)$ in the finite SNR regime and $O(1)$ in the infinitesimal SNR regime from the GMAC capacity, where K is the

number of active transmitters in the network. Thus the proposed scheme has a gap that is a function of both the SNR and the number of users, unlike some previous schemes whose performance depend only on the number of users. Apart from being optimal in the low SNR regime, this scheme has other advantages such as minimal latency, flexibility to be used with other iterative decoders, different channel models such as varying channel gains and variable power constraints.

DEDICATION

To my Mom, Dad and Sister,

for all the love and support.

NOMENCLATURE

AWGN	Additive White Gaussian Noise
DE	Density Evolution
GMAC	Gaussian Multiple-Access Channel
MAC	Medium Access Control
MIMO	Multi-Input Multi-Output
MRC	Maximal Ratio Combining
OQ	Optimal Quantization
PMF	Probability Mass Function
RA	Random Access
SA	Scheduled Access
SIC	Successive Interference Cancellation
SISO	Single-Input Single-Output
SNR	Signal to Noise Ratio
UQ	Uniform Quantization
URS	Uncoordinated Rate Selection

TABLE OF CONTENTS

Page

LIST OF FIGURES

FIGURE Page

LIST OF TABLES

1. INTRODUCTION

1.1 Overview

The Medium Access Control (MAC) layer plays the crucial role of determining how multiple nodes in a wireless network share the physical transmission medium and has a substantial impact on the throughput, latency and reliability of wireless networks. Traditional MAC layer protocols can be classified into two categories: Scheduled access (SA) and Random access (RA) [1]. In SA, typically a central node coordinates the transmission of all nodes and assigns resources such as time slots to the transmissions in a way that interference is avoided. In RA, there is no such central coordination and each node transmits information as soon as its available or at the beginning of the next time slot in an uncoordinated fashion. Although the scheduled access ensures very high reliability, there are some scenarios where RA is preferred over SA. In networks with large round trip times such as satellite networks, Wireless sensor networks where there is power constraint at the transmitter, and large wireless ad-hoc networks where scalability is required RA protocols have to be employed. In this work, we focus on such a situation and we consider well-known random access schemes such as slotted ALOHA.

In slotted ALOHA, each user transmits information as a packet at the beginning of the next time slot. Assuming all the nodes are synchronized and packets are fixed size, this scheme ensures minimal latency and minimal overhead in communicating the slot scheduling policy. The disadvantage however is that the transmitted packets can collide with very high probability. Traditionally, when transmissions overlap, the collided packets are discarded, thereby, resulting in a loss in the net throughput. It is well known that the throughput of slotted ALOHA is $1/e \approx 0.37$ [1]. Recently, there is growing interest in MAC protocols where the collided packets are not discarded, but kept in a buffer and then a successive interference cancellation (or, decoding) algorithm is applied to all the received transmissions.Schemes such as *Contention Resolution Diversity Slotted Aloha (CRDSA)*[**?**], *Irregular repetition Slotted Aloha (IRSA)* [2] were proposed to improve the performance of Slotted Aloha. In these works a connection has been established between SIC decoding and message-passing decoding on an equivalent bipartile graph. Using this idea, it has been shown that when there is no noise in the channel, i.e., only collisions or error-free transmission and when there is no power constraint to the transmissions, throughput arbitrarily close to 1 can be obtained by choosing soliton distribution for the choice of repetition rates. [3].

In this work, we consider the important extension to the case when there is noise in the channel in addition to collisions and when there are power constraints at the transmitter. The overarching goal of this thesis is to understand how much improvement in throughput can be obtained due to the enhanced decoding procedure compared to discarding the packets. This also helps us answer the fundamental question of whether uncoordinated transmission can be as good as coordinated transmission in wireless networks.

1.2 Prior Work

We are interested in a scenario where a large number of nodes, K, are trying to communicate to a single base station, B. This multiuser channel is commonly known as a Multiple-access channel. Let P be the maximum power available for each node to transmit its information. We assume the channel to have Additive White Gaussian Noise (AWGN) with zero mean and variance, σ^2 . This is the well known Gaussian Multiple Access Channel (GMAC).

The achievable rate region for this channel is given by the following set of K in-

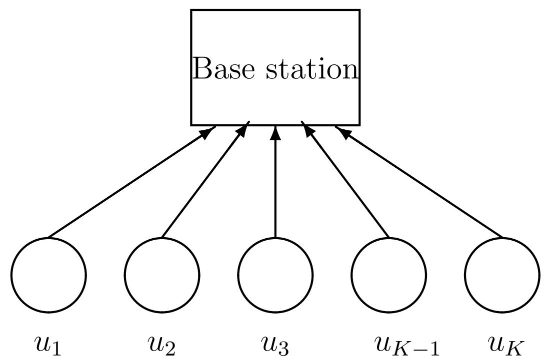

Figure 1.1: Gaussian Multiple-access channel with nodes $u_1, ..., u_K$ transmitting to the base station B

equalities [4]:

$$\sum_{i=1}^{k} r_i < \frac{1}{2} \log \left(1 + \frac{kP}{\sigma^2}\right) \qquad \text{for } k = 1, 2, ..., K \tag{1.1}$$

The achievable rate region is a subspace of $\mathbb{R}^K$. By picking a corner point of the subspace, MMSE decoder combined with Successive Interference Cancellation (SIC) can achieve sum rate optimality with the complexity of a single user decoder [4]. In [5], achievable rate region is derived taking into account the channel gains. The MMSE-SIC scheme is replaced by Successive Integer Forcing (SIF) scheme which uses a set of K nested Lattices. This scheme is shown to be sum rate optimal and has the advantage of achieving non-corner points of the rate region. With a more complicated decoder, it has been shown that the "equal-rate" point is also achievable [4] [6]. In [7], a new scheme called Rate Splitting Multiple Access (RSMA) is proposed to achieve any point in the achievable rate region with complexity similar to single user decoder by splitting M users into $2M - 1$ virtual users.

1.3 Contribution

The fundamental question our work aims to address is whether it is possible to achieve the capacity of Gaussian MAC channel without coordination among the nodes, which is the characteristic of Random Access MAC schemes. In [3] it has been shown that for an error free channel (only packet collisions), in the absence of coordination, *sumrate* equal to capacity can be obtained through Iterative Collision Resolution (ICR). We define *sumrate* as the sum of individual rates obtained by all the nodes transmitting at the same time. In chapter 2, we explore the possibility of extending this scheme to a Gaussian MAC scenario and the consequent pros and cons of this scheme.

In chapter 3, we propose an uncoordinated scheme, where the nodes choose their rates r_i without communicating with each other. Each node picks a rate from a rate distribution, $\mathbf{Q}$, which depends on the number of nodes, K, and the probability of iterative decoder failure tolerated, $\widehat{\epsilon}$. We obtain the rate distribution by formulating the problem as a series of optimization problems. Using concepts from information theory and convex optimization, we obtain a unique rate distribution for a given $(K, \widehat{\epsilon})$. In the asymptotic limits as $K \to \infty$, we find upper and lower bound for the gap between the sumrate achieved with this scheme and GMAC capacity.

In Chapter-4, we set up some simulation scenarios to observe the performance of our scheme under different constraints. We observe that the simulations follow the trends expected from our analysis in chapter-3. Finally in chapter-5, we conclude our analysis and suggest possible extensions for this work.

2. PERFORMANCE OF ICR FOR GMAC

Iterative Collision Resolution is known to be *sumrate* optimal for a noise-free channel [3]. It is possible to use the same scheme with small modifications to get good throughputs. Assuming there exists a capacity achieving code for a point-to-point AWGN channel, we can encode the information of each node using such a code and use ICR for accessing the channel. In this chapter we analyze the upper and lower bounds of the *sumrate* achieved using such a scheme.

2.1 System Model

Each user has k bits of information which is encoded using any capacity achieving code for a point-to-point AWGN channel. Time is divided into MAC frames, each of which is subdivided into M time slots. Figure 2.1 shows the time division into a MAC frame of M time slots. The resulting n length code word is transmitted in i time slots where $i = 2, 3, \ldots, N$, where N is chosen as the maximum repetition rate. Due to lack of coordination, each node choses the repetition rate, i independent of one another from a predetermined distribution. Each node then chooses a random subset of i time slots from the current MAC frame. The receiver then performs Successive Interference Cancellation decoding on the obtained MAC frame to iteratively decode the information from each node. The decoding process can be represented as a Tanner graph as shown in Figure 2.2.

In [3], it has been shown that the Soliton distribution is the optimal distribution. Particularly, it has been shown that as $M \to \infty$, the efficiency of this scheme, $\eta \to 1$

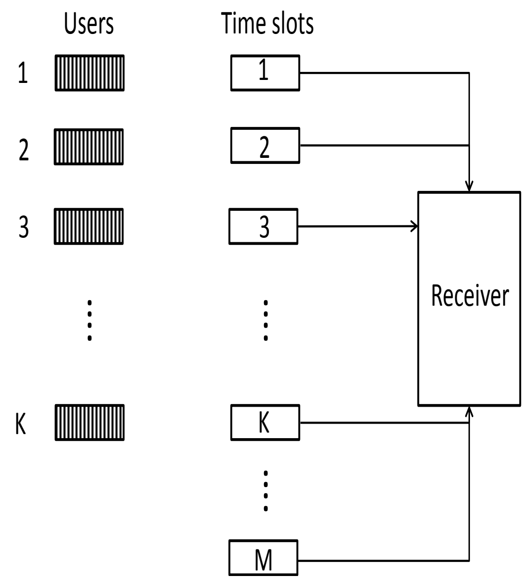

Figure 2.1: Time division of each MAC frame into M slots

as $N \to \infty$. Efficiency of this system is defined as:

$$\eta = \begin{cases} \frac{M}{K} & \text{if decoding is successful} \\ 0 & \text{otherwise} \end{cases}$$

2.2 Performance Analysis of ICR

Let us extend the same scheme to an AWGN channel instead of a simple BEC. Let us assume that each node has a power constraint of P' and the AWGN channel has mean 0 and variance σ^2. Figure 2.3 shows the time division at the receiver. We know from the results in [3] that in the limit as $K \to \infty$ with a Soliton distribution, the system has an efficiency of 1. Hence the *sumrate* is given by:

$$S_{ICR} = \frac{K \cdot \log_2\left(1 + \frac{P'}{\sigma^2}\right)}{M} \quad \text{bits/channel use}$$

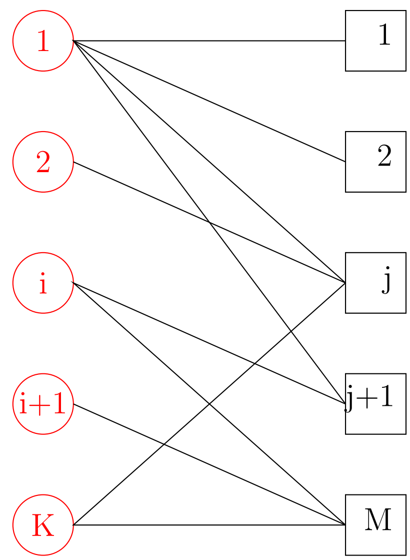

Figure 2.2: Tanner graph representation of ICR based slotted Aloha

$$\lim_{k\to\infty} S_{ICR} = \log_2\left(1+\frac{P^{'}}{\sigma^2}\right) \quad \text{bits/channel use}$$
$$\lim_{k\to\infty} S_{ICR} = \log_2\left(1+\frac{K\cdot P}{(\ln K)\cdot\sigma^2}\right) \quad \text{bits/channel use} \tag{2.1}$$

We assume that we require M time slots to decode K users successfully using ICR. Since the decoding is successful, each user gets one interference free time slot in which it can transmit at a rate of $\log_2\left(1+\frac{P^{'}}{\sigma^2}\right)$. Hence we get a total sum rate of $K\cdot\log_2\left(1+\frac{P^{'}}{\sigma^2}\right)$ in M channel uses. In the limit $K\to\infty$, we have $\frac{M}{K}\to 1$. To compare S_{ICR} with the capacity of GMAC, S_{cap}, we need to equate the average energy spent by all the users in both the schemes. The average degree of a soliton distribution with K nodes is $\ln K$. (2.1) is a result of this following substitution:

$$K\cdot P = (\ln K)\cdot P^{'} \tag{2.2}$$

Let us analyze the gap between S_{ICR} and S_{cap} in two different SNR regimes. We

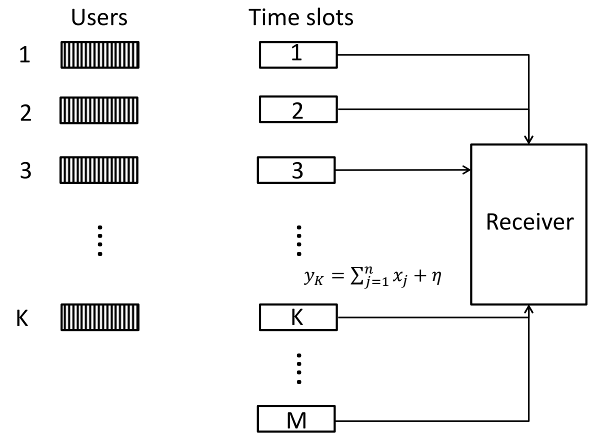

Figure 2.3: Time division of each MAC frame into M slots

divide the whole spectrum of SNRs into two regimes, (i) Finite SNR regime, where $\frac{P}{\sigma^2}$ is finite and (ii) Infinitesimal SNR regime, where $\frac{KP}{\sigma^2}$ is constant, which means the SNR goes to zero as K goes to ∞. The choice of such classification will be more evident in the following chapter, when we propose a different model of power constraint. We define *absolute gap* between S_{ICR} and S_{cap} as $g_a = S_{cap} - S_{ICR}$. We also define fractional gap between S_{ICR} and S_{cap} as $g_f = \frac{S_{ICR}}{S_{cap}}$.

2.2.1 ICR Analysis: Finite SNR regime

From the expression of *sumrate* obtained in (2.1) and capacity of GMAC, we calculate gaps in the finite SNR regime.

$$\lim_{k\to\infty} S_{ICR} = \log_2\left(1 + \frac{K \cdot P}{(\ln K)\cdot \sigma^2}\right)$$
$$\approx \log_2\left(\frac{K \cdot P}{(\ln K)\cdot \sigma^2}\right) \tag{2.3}$$

$$= \log_2\left(\frac{K \cdot P}{\sigma^2}\right) - \log_2(\ln K)$$

$$\approx S_{cap} - \log_2(\ln K) \tag{2.4}$$

Using the approximation $\ln(1+x) \approx \ln(x)$ for large x and the finite SNR regime definition, we get (2.3). Using the same approximation, we can write $S_{cap} = \log_2(\frac{KP}{\sigma^2})$. From (2.4), we observe that the absolute gap, g_a, is of the order, $O(\ln(\ln K))$ and fractional gap, $g_f \to 1$.

2.2.2 ICR Analysis: Infinitesimal SNR regime

In the infinitesimal SNR regime, instead of observing the absolute gap to capacity, $S_{cap} - S_{ICR}$ (which is trivially zero), we observe the fractional gap, S_{ICR}/S_{cap}. From the definition of infinitesimal SNR regime, let $\frac{KP}{\sigma^2} = s$, where s is constant. From the following analysis, we observe that the fractional gap tends to zero. Hence we can conclude that ICR is not capacity achieving in either of the SNR regimes.

$$\lim_{k\to\infty} \frac{S_{ICR}}{S_{cap}} = \frac{\log_2\left(1 + \frac{K \cdot P}{(\ln K)\cdot\sigma^2}\right)}{\log_2\left(1 + \frac{K\cdot P}{\sigma^2}\right)}$$

$$\approx \frac{\frac{K\cdot P}{(\ln K)\cdot\sigma^2}}{\log_2\left(1 + \frac{K\cdot P}{\sigma^2}\right)} \tag{2.5}$$

$$= \frac{s}{(\ln K)(\ln(1+s))}$$

$$= \left(\frac{s}{\ln(1+s)}\right) \cdot \left(\frac{1}{\ln K}\right) \tag{2.6}$$

(2.5) is a result of the approximation, $\ln(1+x) \approx x$ for $x \approx 0$. Applying the limit $K \to \infty$ for (2.6), gives the result that $g_f \to 0$.

2.3 Performance Analysis of ICR coupled with MRC

Looking at the ICR, we speculate that the gap to capacity is due to the suboptimal *Selection Combining* policy when using the SIC decoder. As we know, to get the maximum diversity, we need to employ Maximal Ratio Combining (MRC) at the decoder. In this section we analyze the performance of ICR when we MRC in addition to the SIC decoding. The coefficients for MRC are calculated in the following way:

$$\begin{aligned} y_{i1} &= x_i + \ldots + \eta \\ y_{i2} &= x_i + \ldots + \eta \\ &\vdots \\ y_{ik} &= x_i + \ldots + \eta \end{aligned} \tag{2.7}$$

Effective symbol transmitted by node i with degree k is calculated as:

$$y_{i,\text{eff}} = c_1 \cdot y_{i1} + \ldots + c_k \cdot y_{ik} \tag{2.8}$$

The choice of coefficients, $\{c_j\}$, is such that we obtain maximum SNR for $y_{i,\text{eff}}$. We intuitively understand that c_j is inversely related to the number of interfering symbols, n_j in the j^{th} time slot of node i. Note that this index is with respect to node i and not the overall slot index.

$$\text{SNR}_{i,\text{eff}} = \frac{(c_1 + \ldots + c_k)^2 \cdot P'}{(c_1^2 + \ldots + c_k^2) \cdot \sigma^2 + (c_1^2 \cdot n_1 + \ldots + c_k^2 \cdot n_k) \cdot P'}$$

$$\frac{\partial \text{SNR}_{i,\text{eff}}}{\partial c_j} = 0 \implies \frac{\left[\left(\sum_{m=1}^{k} c_m^2\right)\sigma^2 + \left(\sum_{m=1}^{k} c_m^2 n_m\right)P'\right]\cdot 2\left(\sum_{m=1}^{k} c_m\right)P'}{\left[\left(\sum_{m=1}^{k} c_m^2\right)\sigma^2 + \left(\sum_{m=1}^{k} c_m^2 n_m\right)P'\right]^2}$$

$$- \frac{\left(\sum_{m=1}^{k} c_m\right)^2 \cdot P'\left[2c_j\sigma^2 + 2c_j n_j P'\right]}{\left[\left(\sum_{m=1}^{k} c_m^2\right)\sigma^2 + \left(\sum_{m=1}^{k} c_m^2 n_m\right)P'\right]^2} = 0$$

$$\implies c_j\left[\sigma^2 + n_j P'\right] = \frac{\left(\sum_{m=1}^{k} c_m^2\right)\sigma^2 + \left(\sum_{m=1}^{k} c_m^2 n_m\right)P'}{\left(\sum_{m=1}^{k} c_m\right)} \tag{2.9}$$

$$\implies c_j = \frac{1}{\sigma^2 + n_j P'} \qquad j = 1, 2, \ldots, k \tag{2.10}$$

With the choice of coefficients given in (2.10), let us analyze the performance this hybrid ICR-MRC scheme. Just as in the ICR, the decoder looks for an interference free time slot, performs MRC for the node i transmitting in that time slot. After successful decoding of user i, all the edges corresponding to that node are removed form the graph and the decoding continues until it no longer finds an interference free time slot (decoding failure) or all the users have been decodes (decoding successful). Each node chooses the degree distribution from Soliton, which was demonstrated to guarantee successful decoding with efficiency of one in the asymptotic regime. This directly applies to our hybrid scheme. However, the users can now transmit at rates, $\log_2(1 + \mathbb{E}[\text{SNR}_{i,\text{eff}}])$. We first demonstrate these rates are higher than the constant rate, $r_j = \log_2(1 + \frac{P'}{\sigma^2})$, obtained in ICR. Consider the effective SNR of node i with a repetition rate of k,

$$\text{SNR}_{i,\text{eff}} = \frac{\left(\sum_{m=1}^{k} c_m\right)^2 \cdot P'}{\sum_{m=1}^{k}\left(c_m^2 \cdot [\sigma^2 + n_m P']\right)}$$

$$= \frac{\left(\sum_{m=1}^{k} \frac{1}{\sigma^2+n_m P'}\right)^2 \cdot P'}{\sum_{m=1}^{k} \left(\frac{1}{\sigma^2+n_m P'}\right)^2 \cdot [\sigma^2 + n_m P']}$$

$$= \frac{P'}{\sigma^2} \left[\sum_{m=1}^{k} \frac{\sigma^2}{\sigma^2 + n_m P'}\right]$$

$$\mathbb{E}[\mathrm{SNR}_{i,\mathrm{eff}}] = \frac{P'}{\sigma^2} \left(\sum_{m=1}^{k} \mathbb{E}\left[\frac{\sigma^2}{\sigma^2 + n_m P'}\right]\right) \tag{2.11}$$

If each of the users chooses the time slots uniformly at random, we have a check node distribution that is Poisson distribution in the asymptotic limit as $K \to \infty$. Hence the number of interfering symbols in each time slot, n_j, is a random variable that has Poisson distribution.

$$\mathbb{E}\left[\frac{\sigma^2}{\sigma^2 + n_j P'}\right] = \sum_{n_j=0}^{\infty} \left[\frac{\sigma^2}{\sigma^2 + n_j P'}\right] \left[\frac{\exp^{-\lambda} \lambda^{n_j+1}}{(n_j+1)!\,(1-\exp^{-\lambda})}\right] \tag{2.12}$$

We choose $\lambda = \frac{K}{M} \cdot L_{\mathrm{avg}}$, which is the average degree from node perspective for the time slots. Note that as the decoding proceeds, λ decreases and the effective SNR increases. The exact evaluation of the summation in (2.12) is hard and it does not have an intuitive closed form expression which requires a more rigorous study which is beyond the scope of this work. Instead, we analyze the sumrate by considering the upper bound of the effective SNR and show that the absolute gap is a function of K. Thus we derive valid upper and lower bounds as follows:

$$\mathbb{E}\left[\frac{\sigma^2}{\sigma^2 + n_j P'}\right] \geq \left(\frac{\sigma^2}{\sigma^2 + \mathbb{E}[n_j] \cdot P'}\right) \tag{2.13}$$

$$= \left(\frac{\sigma^2}{\sigma^2 + \left(\frac{\lambda}{1-\exp^{-\lambda}} - 1\right) \cdot P'}\right) \tag{2.14}$$

(2.13) is a result of application Jensen's inequality for the convex function, $\frac{\sigma^2}{\sigma^2+n_jP'}$. Random variable n_j is the number of interfering symbols given there is at least one incoming edge for the time slot. The expected value is simply the mean of the Poisson distribution normalized by $(1 - \exp^{-\lambda})$, which is the probability that there is at least one user transmitting in that time slot.

The upper bound for the summation in (2.12) is obtained by substituting $n_j = 0$, since the number of interfering symbols is greater than or equal to zero. Thus the upper bound for the summation is 1. This upper bound is tight only when $\frac{P'}{\sigma^2} \ll 1$, which falls in the infinitesimal SNR regime. From (2.14) and the above upper bound, we can say that:

$$k \cdot \frac{P'}{\sigma^2}\left(\frac{\sigma^2}{\sigma^2 + \left(\frac{\lambda}{1-\exp^{-\lambda}} - 1\right) \cdot P'}\right) \leq \mathbb{E}[\mathrm{SNR}_{i,\mathrm{eff}}] \leq k \cdot \frac{P'}{\sigma^2} \tag{2.15}$$

Intuitively the upper bound in (2.15) simply means that any node that has degree k can expect to have an effective SNR that is at most k-times the SNR in an interference free slot. It is also clear that such a situation occurs when there are no interfering signals in any of the k slots it transmits in. In the following sections we analyze the gap to capacity with our new ICR-MRC scheme in both the SNR regimes.

2.3.1 ICR-MRC in finite SNR regime

The exact computation of $S_{ICR-MRC}$ is beyond the scope of this work. Hence we use the upper bound in (2.15) to analyze the gap to capacity with the most optimal (even if it is not feasible) ICR-MRC implementation. This approach serves the purpose of refuting the claim that with ICR-MRC, we can approach the capacity of GMAC. In the following analysis, using the upper bound, we show that ICR-MRC scheme has a $O(\ln \ln K)$ gap to capacity in the finite SNR regime. Let p_k be the probability that a node chooses repetition rate of k.

$$\begin{aligned} S_{ICR-MRC} &\leq \sum_{k=1}^{K} p_k \log_2 \left(1 + \frac{k \cdot P'}{\sigma^2}\right) \\ &\approx \sum_{k=1}^{K} p_k \log_2 \left(\frac{k \cdot P'}{\sigma^2}\right) \\ &= \sum_{k=1}^{K} p_k \log_2 \left(\frac{P'}{\sigma^2}\right) + \sum_{k=1}^{K} p_k \log_2(k) \\ &= \log_2 \left(\frac{P'}{\sigma^2}\right) + \sum_{k=1}^{K} p_k \log_2(k) \end{aligned}$$

From [3], we know that for an induced Poisson distribution on the right, Soliton is the unique distribution on the left. In a soliton distribution, probability that a node picks a repetition rate of k is $p_k = \frac{1}{k(k-1)}$. This result coupled with the equivalent power constraint relation in (2.2), we have,

$$S_{ICR-MRC} \leq \log_2 \left(\frac{K \cdot P}{\sigma^2}\right) - \log_2(\ln K) + \sum_{k=1}^{K} \frac{\log_2(k)}{k(k-1)} \tag{2.16}$$

$$= S_{cap} - \log_2(\ln K) + O(1) \tag{2.17}$$

The summation series in (2.16) is a convergent series. It can be shown that $\sum_{k=1}^{\infty} \frac{\log_2(k)}{k(k-1)} \approx 2.3731$. Please refer to Appendix for the proof. This concludes that even if we are able to exploit all the available diversity using MRC, there is only a constant gain in the *sumrate*. This is still not sufficient to approach the capacity of GMAC channel in the finite SNR regime and the absolute gap. g_a is still of the same order $O(\ln \ln K)$.

2.3.2 ICR-MRC in infinitesimal SNR regime

Analyzing (2.15) in the infinitesimal SNR regime gives an interesting result. We observe that the ratio of lower and upper bounds tend to one, which implies that the bounds are tight. Intuitively, this means that the effective SNR of a node transmitting in k slots is k-times the SNR in an interference free slot. This is due to the fact that at such low SNRs, the signal distortion is dominated by AWGN noise compared to the interference from the other symbols. Hence we can say that,

$$\begin{aligned}\mathbb{E}[\mathrm{SNR}_{i,\mathrm{eff}}] &\approx k \cdot \frac{P'}{\sigma^2} \\ S_{ICR-MRC} &\approx \sum_{k=1}^{K} p_k \log_2\left(1 + \frac{k \cdot P'}{\sigma^2}\right) \\ &= \sum_{k=1}^{K} p_k \log_2\left(1 + \frac{k \cdot K \cdot P}{(\ln K)\sigma^2}\right) \end{aligned} \tag{2.18}$$

Using the upper and lower bounds, $\frac{2x}{2+x} \leq \ln(1+x) \leq x$, we evaluate the upper and lower bounds of (2.18). From the following analysis, it is evident that the fractional gap $\frac{S_{ICR-MRC}}{S_{cap}}$ may not tend to one.In this regime, $S_{cap} = \frac{KP}{\sigma^2}$. The lower and upper bounds do not converge. Hence tightness of either bounds cannot be established. So we use the Taylors series expansion of $\ln(1+x)$ to evaluate the fractional gap. Let $B = \frac{K \cdot P}{(\ln K)\sigma^2}$.

$$\begin{aligned}\sum_{k=2}^{K} p_k \left(\frac{2\frac{k\cdot K\cdot P}{(\ln K)\sigma^2}}{2 + \frac{k\cdot K\cdot P}{(\ln K)\sigma^2}}\right) \leq \quad & S_{ICR-MRC} \leq \sum_{k=2}^{K} p_k \left(\frac{k \cdot K \cdot P}{(\ln K)\sigma^2}\right) \\ \sum_{k=2}^{K} \left(\frac{1}{k(k-1)}\right)\left(\frac{2\frac{k\cdot K\cdot P}{(\ln K)\sigma^2}}{2 + \frac{k\cdot K\cdot P}{(\ln K)\sigma^2}}\right) \leq \quad & S_{ICR-MRC} \leq \sum_{k=2}^{K} \left(\frac{1}{k(k-1)}\right)\left(\frac{k \cdot K \cdot P}{(\ln K)\sigma^2}\right) \\ \left[\frac{2K \cdot P}{(\ln K)\sigma^2}\right] \sum_{k=2}^{K} \left(\frac{1}{(k-1)(2 + \frac{k\cdot K\cdot P}{(\ln K)\sigma^2})}\right) \leq \quad & S_{ICR-MRC} \leq \left[\frac{K \cdot P}{(\ln K)\sigma^2}\right] \sum_{k=2}^{K} \left(\frac{1}{(k-1)}\right)\end{aligned}$$

$$\left[\frac{2K\cdot P}{(\ln K)\sigma^2}\right]\sum_{k=2}^{K}\left(\frac{1}{(k-1)(k+\frac{k\cdot K\cdot P}{(\ln K)\sigma^2})}\right)\leq \quad S_{ICR-MRC}\leq\left[\frac{K\cdot P}{(\ln K)\sigma^2}\right]\cdot(\ln(K-1))$$

$$\left[\frac{2K\cdot P}{(\ln K)(1+\frac{K\cdot P}{(\ln K)\sigma^2})\sigma^2}\right]\sum_{k=2}^{K}\left(\frac{1}{k(k-1)}\right)\leq \quad S_{ICR-MRC}\leq\left[\frac{K\cdot P\cdot\ln(K-1)}{(\ln K)\sigma^2}\right]$$

$$\left[\frac{2K\cdot P}{\ln K\sigma^2}\right]\leq \quad S_{ICR-MRC}\leq\left[\frac{K\cdot P}{\sigma^2}\right]$$

$$\begin{aligned}
S_{ICR-MRC} &= \sum_{k=1}^{K} p_k\log_2(1+k\cdot B)\\
&= \sum_{k=2}^{K}\left(\frac{1}{k(k-1)}\right)\log_2(1+k\cdot B) && (2.19)\\
&\approx \sum_{k=2}^{K}\left(\frac{1}{k(k-1)}\right)\left[k\cdot B-\frac{(k\cdot B)^2}{2}+\frac{(k\cdot B)^3}{3}-\dots\right] && (2.20)\\
&= \sum_{k=2}^{K}\left(\frac{B}{k-1}\right)-\sum_{k=2}^{K}\left(\frac{k\cdot B^2}{2(k-1)}\right)+\sum_{k=2}^{K}\left(\frac{k^2\cdot B^3}{3(k-1)}\right)-\dots\\
&= B\cdot\ln(K-1)-\frac{B^2}{2}\left[K-1+\ln(K-1)\right]\\
&\quad+\frac{B^3}{3}\left[\frac{K(K+1)}{2}+K-1+\ln(K-1)\right]\\
&\quad-\frac{B^4}{4}\left[\frac{K(K+1)(2K+1)}{6}+\frac{K(K+1)}{2}+K-1+\ln(K-1)\right]+\dots\\
\lim_{\substack{K\to\infty\\ \frac{KP}{\sigma^2}\to 0}} S_{ICR-MRC} &= B\cdot\ln(K-1)-\frac{B^2K}{2\cdot 1}+\frac{B^3K^2}{3\cdot 2}-\frac{B^4K^3}{4\cdot 3}\\
&= \frac{1}{K}\left[B\cdot K\cdot\ln(K-1)-\frac{(BK)^2}{2\cdot 1}+\frac{(BK)^3}{3\cdot 2}-\frac{(BK)^4}{4\cdot 3}\right]\\
&= \frac{1}{K}\left[B\cdot K\cdot\ln(K-1)-\int\ln(1+B\cdot K)\,d(BK)\right]\\
&= \frac{1}{K}\left[B\cdot K\cdot\ln(K-1)-(1+B\cdot K)\ln(1+B\cdot K)+B\cdot K+\text{const}\right]\\
\lim_{\substack{K\to\infty\\ \frac{KP}{\sigma^2}\to 0}}\frac{S_{ICR-MRC}}{S_{cap}} &= \frac{\frac{1}{K}\left[B\cdot K\cdot\ln(K-1)-(1+B\cdot K)\ln(1+B\cdot K)+B\cdot K+\text{const}\right]}{\log_2(1+\frac{KP}{\sigma^2})}
\end{aligned}$$

$$
\begin{aligned}
&\approx \frac{\frac{1}{K}\left[B\cdot K\cdot\ln(K-1)-(1+B\cdot K)\ln(1+B\cdot K)+B\cdot K+\text{const}\right]}{\frac{B\ln K}{\ln 2}}\\
&=\lim_{\substack{K\to\infty\\ \frac{KP}{\sigma^2}\to 0}}\left[\frac{B\ln(K-1)\ln 2}{B\ln K}-\frac{\ln 2(1+BK)\ln(1+BK)}{BK\ln K}+\frac{BK\ln 2}{BK\ln K}\right]\\
&=\ln 2-\ln 2=0
\end{aligned}
$$

Equation (2.19) is from the result that soliton distribution is uniquely optimal for ICR decoding. (2.20) is due to Taylor series expansion of $\ln(1+x)$. From the above analysis, it is evident that the fractional gap in the infinitesimal SNR regime and as $K\to\infty$, goes to zero. This concludes our intuition that MRC is not a sufficient addition to ICR to approach the capacity of GMAC. At this point, we note that one possible reason for the non-constant absolute gap or non-unity fractional gap is the inherent sub-optimality of the repetition coding schemes. In the following chapter we propose a novel Random Access MAC scheme that promises to be better than the previous schemes.

3. OPTIMAL RATE SELECTION FOR UNCOORDINATED GAUSSIAN MULTIPLE-ACCESS CHANNEL

As we noted in the previous chapter the suboptimality of the previous schemes lie in the fact that we use repetition coding at the transmitter. Hence we propose a new paradigm in which the transmitter does not repeat information. This lost degree of freedom is compensated by the ability of transmitters to choose their rate of transmission according to a predefined policy. In the following sections we describe the proposed paradigm in great detail.

3.1 System Model

Let us assume there are K nodes communicating with the base station, B. Let P be the maximum power available at each node to transmit its information. We assume the channel to be AWGN with zero mean and σ^2 variance. The achievable rate region for a Gaussian multiple-access channel (GMAC) with power constraints is given by the following set of K inequalities [4]:

$$\sum_{i=1}^{k} r_i \leq \frac{1}{2}\log_2\left(1+\frac{kP}{\sigma^2}\right) \qquad k = 1, 2, ..., K \tag{3.1}$$

The rate region is a subspace of $\mathbb{R}^K$. A point in $\mathbb{R}^K$ represents a K-tuple of rates chosen by all the nodes. Some points in this rate region are easily achieved compared to the others. For example the following set of points are achievable by a simple interference cancellation decoder, which has the complexity of a single user decoder.

$$r_i \leq \frac{1}{2}\log_2\left(1+\frac{P_i}{\sigma^2+\sum\limits_{j<i} P_j}\right) \qquad i = 1, 2, ..., K \tag{3.2}$$

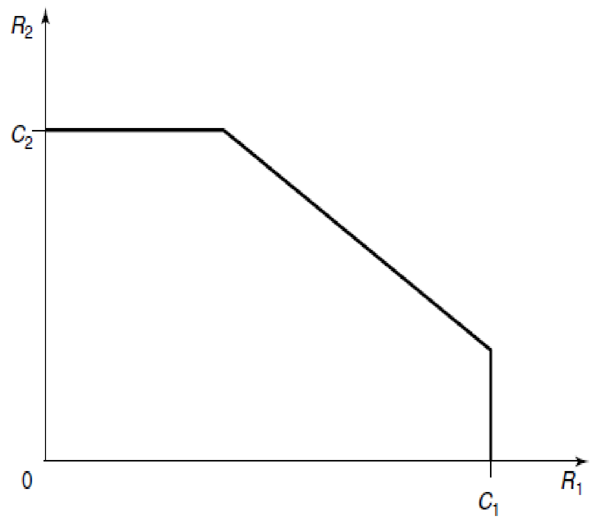

Figure 3.1: Achievable rate region for the two-user GMAC

where P_i is the power constraint at each node and σ^2 is the variance of the AWGN channel. In the following scheme each user independently chooses a rate from the above set, which corresponds to the *corner points* of the rate region. Figure 3.1 shows the rate region for a two-user scenario.

Each node uses a random codebook of rate chosen from (3.2), with every codeword being a sequence of i.i.d random variable with $\mathcal{N}(0, P_i)$. This is a necessary condition for the codebook to be capacity achieving for a Gaussian channel. For rates chosen according to (3.2), this $K \times 1$ MIMO channel is transformed into K level SISO channels. The decoding starts with the node whose rate is the lowest. It decodes the codeword treating all other codewords as Gaussian noise. After successful decoding, the codeword is subtracted from the received word and the process continues to the higher levels. This idea is described in detail in [8]. This kind of decoding process is often referred to as *Onion peeling*, *Successive Interference Cancellation* and *Successive cancellation*.

One important characteristic of the GMAC that we are interested in is that it is dynamically configured. It means only a subset of all available nodes are active in any time slot and nodes switch between *idle* and *active* modes. This is a more general configuration and the the results obtained in this work are valid for a static configuration as well.

3.2 Problem Formulation

The proposed random access scheme for GMAC is based on the same idea of dividing time into *time slots* as commonly used in slotted ALOHA. This inherently assumes that nodes and the base station have synchronized clocks. There are K nodes active in the network, trying to communicate with the base station. Each node at the beginning of the time slots transmits its data in packets of fixed length. Let each node maintain a set of L codebooks, each with rates $r_1, \ldots, r_L$. Length of codewords in all the codebooks must be the same. Nodes must choose the rates (codebooks) according to some predefined policy. Since the scheme is RA in nature, the nodes cannot coordinate among themselves to pick the rates. As described earlier rate tuples given by (3.2) are easily achievable using a SIC decoder. This scheme targets to achieve the same rate tuples without coordination. We assume that each node knows the number of active users in the network. This is a reasonable assumption since the base station can broadcast this number which is a negligible overhead.

3.3 The Uncoordinated Rate Selection (URS) Scheme

In the proposed scheme, each node that has information to transmit to the base station will choose a rate from a predetermined rate PMF. The message is encoded using the codebook of chosen rate. The SIC decoder iteratively decodes each message transmitted in that time slot. We define *sumrate* as the sum of rates of all the nodes that were successfully decoded at the base station. This scheme is said to be capacity achieving if the *sumrate* is equal to the capacity of GMAC, which is equal to $\frac{1}{2}\log_2\left(1+\frac{KP}{\sigma^2}\right)$.

The proposed scheme can be broadly divided into three stages. In the first stage, the set of K rates is quantized into L rates. This is called *Rate Quantization*. The rate PMF is obtained after completing this stage. In the second stage the rate PMF is biased such that the probability of failure of the SIC decoder is bounded. This is called *Rate Biasing*. At the end of these two stages, we have a rate distribution function the nodes use to pick a rate. Using this distribution function, the probability of decoder failure is bounded at the base station. Figure 3.2 summarizes the various stages of URS.

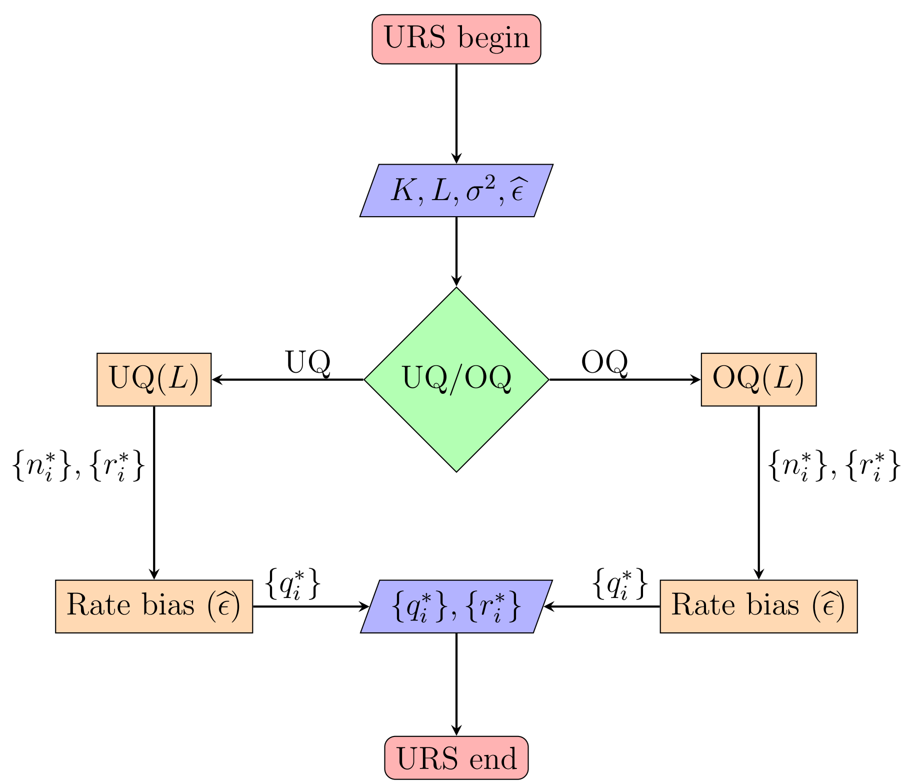

Figure 3.2: Flowchart summarizing the operations of URS

3.3.1 Rate Quantization

Each node maintains a set of L codebooks. Ideally L must be equal to K to fully operate at any point in the rate region. But L is chosen to be less than K for two reasons. Firstly, maintaining K codebooks is not feasible since the number of nodes in the network, K can potentially be very large (of the order of 10^4). Secondly, it provides additional degree of freedom in the asymptotic analysis, which we will see in more detail in the subsequent sections. In this work two ways of rate quantization are proposed namely, Uniform quantization and Optimal quantization. The former scheme is more suitable for the asymptotic analysis while the latter achieves higher sum rates for finite K. In the asymptotic limit the performance of both the schemes converge.

3.3.1.1 Uniform Quantization

In this scheme, the number of users picking each of the quantized rates is assumed to be uniform. Let n_i be the number of users picking rate r_i. Uniform quantization implies:

$$n_1 = n_2 = \ldots = n_L = \frac{K}{L}$$

For the given $\{n_i\}$, we can calculate the rates $\{r_i\}$ such that they satisfy the constraints imposed by the SIC decoder:

$$r_1 = \log_2\left(1 + \frac{P}{(K-1)P + \sigma^2}\right) \tag{3.3}$$

$$r_i = \log_2\left(1 + \frac{P}{\left(K - \sum_{j=1}^{i-1} n_j - 1\right) P + \sigma^2}\right) \qquad i = 2, ..., L$$

3.3.1.2 Optimal Quantization

The objective of rate quantization is to maximize the *sumrate*, which is the sum of rates of all the nodes transmitting in the current time slot. The constraints are due to the choice of rates that are easily decoded using SIC (3.2). Each node with rate r_i has interference from all the other nodes with rates $r_j < r_i$, which can be treated as noise by the SIC decoder. Hence we can formulate this quantization operation as the following optimization problem:

$$\begin{aligned}
\underset{n_i, r_i \forall i \in [1,L]}{\text{maximize}} \quad & \sum_{i=1}^{L} n_i r_i \\
\text{subject to} \quad & \sum_{i=1}^{L} n_i = K \\
& r_1 \leq \log_2 \left(1 + \frac{P}{(K-1)P + \sigma^2}\right) \\
& r_i \leq \log_2 \left(1 + \frac{P}{\left(K - \sum_{j=1}^{i-1} n_j - 1\right) P + \sigma^2}\right) \quad i = 2, ..., L \\
& r_i \geq 0 \quad i = 1, ..., L \\
& n_i \geq 0 \quad i = 1, ..., L
\end{aligned} \tag{3.4}$$

The above problem is non-convex in n_i and r_i. However a suboptimal solution can be obtained by finding a local maximum. With the motivation of obtaining a simple closed form solution for the above optimization problem, an approximation is used for each of the inequality constraints. This approximation is only valid when the following condition is satisfied:

$$\frac{P}{\left(K - \sum_{j=1}^{L-1} n_j - 1\right) P + \sigma^2} \ll 1 \tag{3.5}$$

This condition is true when SNR is low (ie., $\frac{\sigma^2}{P} \gg 1$). The deviation of the true value from the approximation is less than 10^{-3} for SNR $\leq$ -15 dB. From the Maclaurin series, we have,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \approx x \qquad \text{for } x \ll 1 \tag{3.6}$$

Using the above approximation and the condition in (3.5), the rate constraint is modified as:

$$\log_2\left(1 + \frac{P}{\left(K - \sum_{j=1}^{i-1} n_j - 1\right) P + \sigma^2}\right) \approx \frac{1}{\ln 2}\left(\frac{P}{\left(K - \sum_{j=1}^{i-1} n_j - 1\right) P + \sigma^2}\right) \tag{3.7}$$

Using the method of Lagrange multipliers, a closed form solution for the modified convex optimization problem is derived. The solution is given below and the proof is described in the appendix.

$$
\begin{aligned}
n_i^* &= \frac{1}{(A\lambda)^{L-i-1}}\left[\frac{1}{A\lambda^2}-\frac{1}{\lambda}\right] \qquad \text{for } i=1,2,...L-1 \\
n_L^* &= K-\sum_{j=1}^{L-1} n_j^* \\
r_1^* &= \frac{1}{\ln 2}\left(\frac{P}{(K-1)P+\sigma^2}\right) \\
r_i^* &= \frac{1}{\ln 2}\left(\frac{P}{\left(K-\sum_{j=1}^{i-1} n_j^* - 1\right)P+\sigma^2}\right) \qquad \text{for } i=2,...,L \\
A &= \frac{\sigma^2}{P}, \lambda = \left(\frac{1}{A^{L-1}(K-1+A)}\right)^{1/L}
\end{aligned}
\tag{3.8}
$$

3.3.2 Rate Biasing

Having obtained the optimal $\{r_i^*\}$ and $\{n_i^*\}$, the next step is to obtain the PMF, $\mathbf{Q}$, according to which each node chooses its rate. Thus the average sum rate obtained is $\sum_{i=1}^{K} q_i r_i$. The objective is to maximize the expected sum rate subject to a bound on the probability of iterative decoding failure. Iterative decoder proceeds by identifying the nodes that transmit at the lowest rate among all the undecoded nodes. After decoding all the users at that rate, it repeats the whole process. Thus a maximum of L stages are possible for the decoder. Iterative decoding fails at stage m ($m \leq L$), when

$$
e_1 + e_2 + \ldots + e_m < \sum_{j=1}^{m} n_j \tag{3.9}
$$

where e_i is the fraction of nodes picking rate r_i. Looking at $\{e_i\}$ as a vector, $\mathbf{E}$ can be helpful in understanding the following analysis. Another way to look at $\{e_i\}$ is that they form an empirical PMF. (3.9) simply means that there are not enough number of lower rate nodes to decode to proceed to stage $m+1$ ($\sum_{j=1}^{m} n_j$ is the minimum required number). We can formulate the above as an optimization problem where one of the constraints is a bound on the probability of decoder failure.

$$\begin{aligned} \underset{q_i \forall i \in [1,L]}{\text{maximize}} \quad & \sum_{i=1}^{L} q_i r_i \\ \text{subject to} \quad & \sum_{i=1}^{L} q_i = 1 \\ & P_{fail} \leq \widehat{\epsilon} \end{aligned} \tag{3.10}$$

The probability of iterative decoding failure is analyzed using the Sanov's theorem[**?**], which gives a bound on the probability of observing an atypical sequence of samples. It states that if $X_1, X_2, ...X_n$ are i.i.d. $\sim \mathbf{Q}$ and E is any set of empirical distributions, then

$$Q^n(E) \leq (n+1)^{|\chi|} 2^{-nD(\mathbf{E}_i^* || \mathbf{Q})} \tag{3.11}$$

where $Q^n(E)$ is the probability that an empirical distribution in the set E results by taking n samples following the probability distribution $\mathbf{Q}$ and $\mathbf{E}^*$ is the empirical distribution in E that is closest to $\mathbf{Q}$ in terms of Kulback-Liebler distance.

E is defined as a set of all the distributions in the n-dimensional space that cause the iterative decoding to fail. Thus $Q^n(E)$ is the probability of decoding failure. We define E as the union of each set of empirical distribution that causes a particular mode of failure. Failure mode m occurs when the iterative decoding fails in stage m whose condition is given by (3.9). P_{fail} for mode m can be estimated using Sanov's theorem by calculating $\mathbf{E}_m^*$, which is the empirical distribution in the set of E that has the failure mode of m that is closest to $\mathbf{Q}$. Finding the closest distribution that is in the failure set is formulated as the following optimization problem:

$$\begin{aligned} \underset{e_i \forall i \in [1,L]}{\text{minimize}} \quad & D(\mathbf{E}||\mathbf{Q}) \\ \text{subject to} \quad & \sum_{j=1}^{L} e_j = 1 \\ & e_1 + \ldots + e_m \leq \sum_{j=1}^{m} n_j^* \end{aligned} \tag{3.12}$$

The value of the objective function of the above optimization problem gives $D(\mathbf{E}_m^*||\mathbf{Q})$. The closed form solution can be obtained using the method of Lagrange multipliers.

$$\mathbf{E}_m^*(j) = \begin{cases} \dfrac{q_j \sum\limits_{i=1}^{m} n_i^*}{\sum\limits_{i=1}^{m} q_i} & \text{for } j = 1, 2, ..., m \\ \dfrac{q_j \sum\limits_{i=m+1}^{L} n_i^*}{\sum\limits_{i=m+1}^{L} q_m} & \text{for } j = m+1, m+2, ..., L \end{cases} \tag{3.13}$$

The constraint $P_{fail} \leq \widehat{\epsilon}$ is equivalent to $D(\mathbf{E}_m^*||\mathbf{Q}) > \epsilon$ for $m = 1, 2, \ldots, L$. Intuitively, it means that if each set of empirical distributions that result in decoding failure (stages $1, 2, ...L$) have a minimum distance of ϵ from $\mathbf{Q}$, we can certify that the iterative decoding succeeds with at least a probability of $(1 - \widehat{\epsilon})$. The relation between them is derived in the following theorem.

Theorem 3.3.1. *Let $\widehat{\epsilon}$ be the upper bound on the probability of decoder failure. The smallest distance between any empirical distribution $\boldsymbol{E}$, resulting in decoder failure and $\boldsymbol{Q}$, ϵ, is given by:*

$$\epsilon \geq \frac{1}{K}\left[\log_2(L-1) - \log_2 \widehat{\epsilon} + L \log_2(K+1)\right]$$

Proof. Let $P_{f,m}$ be the probability that the iterative decoder fails in mode m, the condition for which is given in (3.9). From Sanov's theorem, we have,

$$P_{f,m} \leq (K+1)^{L}\, 2^{-KD(\mathbf{E}_m^*||\mathbf{Q})} \tag{3.14}$$

By solving (3.12), we find $\mathbf{E}_m^*$ that is closes to $\mathbf{Q}$ such that $D\left(\mathbf{E}_m^*||\mathbf{Q}\right) \geq \epsilon$. Combining this and the above equation, we have,

$$\begin{aligned}
P_f &\leq \widehat{\epsilon} \\
\sum_{i=1}^{L-1} P_{f,i} &\leq \widehat{\epsilon} \\
\sum_{i=1}^{L-1} \frac{(K+1)^L}{2^{K\epsilon}} &\leq \widehat{\epsilon} \\
\frac{(L-1)\,(K+1)^L}{2^{K\epsilon}} &\leq \widehat{\epsilon} \\
\epsilon &\geq \frac{1}{K}\log_2 \frac{(L-1)(K+1)^L}{\widehat{\epsilon}} \qquad (3.15) \\
\epsilon &\geq \frac{1}{K}\left[\log_2(L-1) - \log_2 \widehat{\epsilon} + L\log_2(K+1)\right] \qquad \square
\end{aligned}$$

Note that the upper bound of $P_{f,i}$ is the same for all $i = 1, 2, \ldots, L$. This relation is fundamental in understanding the asymptotic regime behavior of the rate biasing operation. Figure 3.3 shows how the distribution is biased for a particular choice of ϵ

The relation between $\mathbf{E}_m^*$ and $\mathbf{Q}$ is obtained by solving (3.12). The closed form solution for this optimization problem is given in (3.13). Hence the condition $D(\mathbf{E}_m^*||\mathbf{Q}) > \epsilon$ can be equivalently written as:

$$sq(m)^{sn(m)}\left(1 - sq(m)\right)^{1-sn(m)} \leq sn(m)^{sn(m)}\left(1 - sn(m)\right)^{1-sn(m)} \tag{3.16}$$

where $sn(m) = \sum_{i=1}^{m} n_j^*$ and $sq(m) = \sum_{i=1}^{m} q_j$ are the cumulative sums of vectors $\mathbf{N}$ and $\mathbf{Q}$. It can be shown that equation in (3.16) has only two real roots. The proof is given in the appendix. Solving equation (3.16) for $sq(m)$ gives two conditions:

$$sq(m) \leq a_{m1} \text{ or } sq(m) \geq a_{m2} \tag{3.17}$$

where a_{m1}, a_{m2} are the two roots of the above equation. This discontinuous choice of Q-region is difficult to handle while solving the optimization problem in (3.10). But it turns out that the condition $sq(m) \leq a_{m1}$ results in a $\mathbf{Q}$ that is already in E, the set of distributions that cause decoding failure. Thus the constraint $D(\mathbf{E}_m^*||\mathbf{Q}) > \epsilon$ is equivalent to $sq(m) \geq a_{m2}$, which makes the following optimization problem a Linear problem.

$$\begin{aligned} \underset{q_i \forall i \in [1,L]}{\text{maximize}} \quad & \sum_{i=1}^{L} q_i r_i \\ \text{subject to} \quad & \sum_{i=1}^{L} q_i = 1 \\ & \sum_{i=1}^{m} q_i \geq a_{m2} \qquad m = 1, 2, \ldots, L-1 \end{aligned} \tag{3.18}$$

3.4 Asymptotic Analysis

In this section, the performance of the proposed scheme is analyzed for two different models. The two models have a similar topology, encoding and decoding schemes but different power constraints. We use the same criteria defined in chapter-2 to measure the performance relative to GMAC capacity such as absolute gap tending to 0 and fractional gap tending to 1.

1. *Model-1, Power constraint only*: Let P be the maximum power used by each

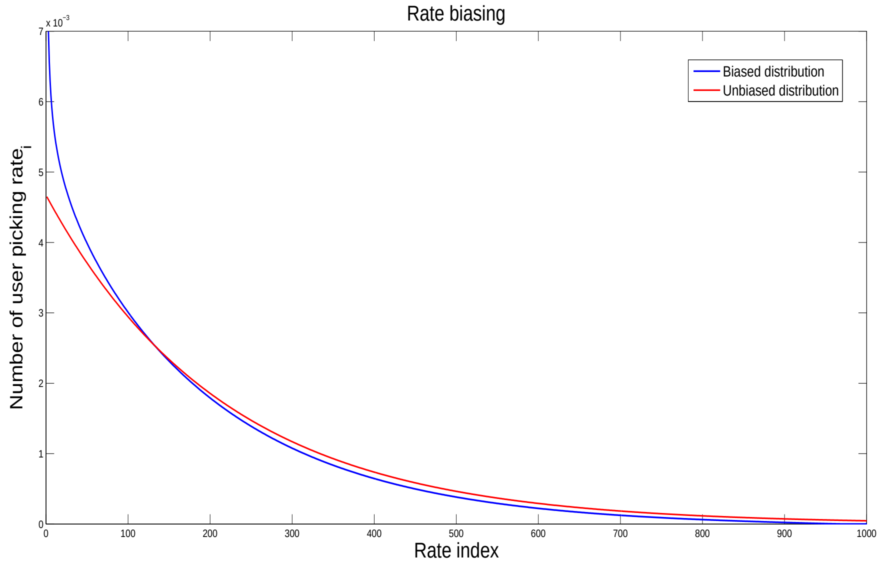

Figure 3.3: Rate biasing in action; Model-1, $K = 10^4, L = 10^3$ $\epsilon = 0.01$

node. The sumrate capacity is $\log_2\left(1 + \frac{KP}{\sigma^2}\right)$ bits/channel use. This belongs to the Finite SNR regime defined in chapter-2.

2. *Model-2, Power & Energy constraint*: An additional constraint that each node should operate in M time slots is imposed. To compare two schemes, the energy used by each node must be the same. Hence in this model each node can operate at a maximum power of $\frac{P}{M}$ per time slot. The sum rate capacity is $\log_2\left(1 + \frac{KP}{M\sigma^2}\right)$ bits/channel use. We assume that M grows at the same rate as K. This belongs to the Infinitesimal SNR regime defined in chapter-2.

3.4.1 Asymptotic Analysis in the Finite SNR Regime

Let S_{uq} be the theoretical sumrate obtained through uniform quantization. Similarly S_{oq} is the sumrate obtained through Optimal rate quantization, S_{cap} is the capacity of GMAC and S_{sim} is the sumrate obtained by simulating the decoder after the rate biasing operation. In the proposed scheme, loss in sum rate occurs in both the stages of our scheme. In the first stage of rate quantization, since $L \neq K$, $S_{uq} \leq S_{cap}$. In the second stage due to the biasing operation, for any $\epsilon \geq 0$, we have $S_{sim} \leq S_{uq}$. In the following theorem the dependence of loss in rate quantization stage on the number of quantization levels, L is derived.

Theorem 3.4.1. *In a finite SNR regime, for a fixed L, as $K \to \infty$, the sum rate achieved through uniform quantization is of the order $O(\log_2 L)$.*

Proof. : Let $A = SNR^{-1}$

$$S_{uq} = \frac{K}{L}\log_2\left(1 + \frac{1}{K-1+A}\right) + \sum_{i=2}^{L}\frac{K}{L}\log_2\left(1 + \frac{1}{K-(i-1)\frac{K}{L}-1+A}\right)$$

Using the approximation $\ln(1+x) \approx x$ when $x \to 0$, the individual terms in the above equation can be approximated as follows:

$$\begin{aligned} \lim_{K\to\infty}\frac{K}{L}\log_2\left(1 + \frac{1}{K-1+A}\right) &= \frac{1}{L\ln 2} \\ \lim_{K\to\infty}\frac{K}{L}\log_2\left(1 + \frac{1}{K-(i-1)\frac{K}{L}-1+A}\right) &= \frac{1}{L\ln 2}\left(\frac{1}{1-\left(\frac{i-1}{L}\right)}\right) \end{aligned} \tag{3.19}$$

Using the following upper bound for the summation of harmonic series an upper bound is derived for S_{uq}. Here γ is the EulerMascheroni constant and $\epsilon_k \approx \frac{1}{2k}$

$$\sum_{n=1}^{k}\frac{1}{n} = \ln k + \gamma + \epsilon_k < \ln k + 1 \tag{3.20}$$

Using the above upper bound and (3.19), the order of S_{uq} is calculated as follows:

$$\begin{aligned}\lim_{K\to\infty} S_{uq} &= \frac{1}{L\ln 2} + \sum_{i=2}^{L} \frac{1}{L\ln 2}\left(\frac{1}{1-\left(\frac{j-1}{L}\right)}\right) \\ &< \frac{1}{L\ln 2} + \frac{1}{L\ln 2}\left(L\ln L\right) \\ &= \mathrm{O}(\log_2 L) \end{aligned} \tag{3.21}$$

Since S_{cap} is $\mathrm{O}(\log_2 K)$, we can conclude that for a fixed L, the absolute gap to capacity or the loss in rate quantization is increasing logarithmically. □

Similarly, we analyze the dependence of loss in sumrate due to the rate biasing operation on L. In the following theorem, we show that for a fixed L as $K \to \infty$, the second loss goes to zero.

Theorem 3.4.2. *In the finite SNR regime, for a fixed L, as $K \to \infty$, the sumrate obtained after the rate biasing operation, $S_{sim} \to S_{uq}$*

Proof. Let us recall the result of Theorem 3.2.1 summarized in (3.15).

$$\epsilon \geq \frac{1}{K}\left[\log_2(L-1) - \log_2\hat{\epsilon} + L\log_2(K+1)\right]$$

For a fixed L and $K \to \infty$, there exists an arbitrarily small δ such that $\epsilon \to \delta$ can guarantee arbitrarily small decoder failure probability, γ ($\hat{\epsilon} \to \gamma$). Since $\epsilon \to 0$ implies that the biasing operation can be bypassed, the sumrate obtained through simulation is equal to the sumrate expected from quantizing the rate distribution. Hence $S_{uq} - S_{sim} \to 0$. □

From Theorems 3.3.1, it is evident that for a fixed L, the proposed scheme does not perform well in the asymptotic regime. From Theorem 3.3.2, it can be seen that when L increases at a certain rate relative to K, the rate loss in biasing operation can still go to zero. For example, when $L = \log_2 K$, $\epsilon \to 0$. Hence L should grow as a function of K such that the loss in sumrate in both the operations is minimal. From the expressions in theorems 3.3.1 and 3.3.2, it can be inferred that the best performance is achieved when $L = \frac{K}{(\log_2 K)^{1+\delta}}$ for an arbitrarily small $\delta > 0$. This is the reason behind the rate quantization operation. For this choice of L, the following theorems give upper and lower bounds for S_{uq}, absolute gap and fractional gap.

Theorem 3.4.3. *Let S_{uq} be the sum rate obtained through uniform quantization using L levels, where $L = \frac{K}{(\log_2 K)^{1+\delta}}$. The upper and lower bounds for S_{uq} are:*

$$\begin{aligned}
&\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1\right) - \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-L+1\right) - \log_2 \mathrm{e} \\
&\leq S_{uq} \leq \log_2\left(\frac{L}{K}\cdot(K-1+A)-1\right) + \log_2 \mathrm{e}
\end{aligned}$$

Proof. Let L, the number of quantization levels that grow with the number of nodes, K as $L = \frac{K}{(\log_2 K)^{1+\delta}}$ for some arbitrarily small $\delta \approx 0$.

$$\begin{aligned}
S_{uq} &= \sum_{i=1}^{L} n_i^* r_i^* \\
S_{uq} &= \frac{K}{L}\log_2\left(1+\frac{1}{K-1+A}\right) + \sum_{i=2}^{L}\frac{K}{L}\log_2\left(1+\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A}\right)
\end{aligned} \tag{3.22}$$

We calculate the upper and lower bounds for each of the two parts separately. Upper bound for the first term:

$$
\begin{aligned}
&= \frac{K}{L} \log_2 \left(1 + \frac{1}{K - 1 + A} \right) \\
&= \log_2 \left(1 + \frac{1}{K - 1 + A} \right)^{\frac{K}{L}} \\
&\leq \log_2 \left(\mathrm{e}^{\frac{K}{L} \cdot \frac{1}{K-1+A}} \right) \\
&= \left(\frac{K}{L} \right) \cdot \left(\frac{1}{K - 1 + A} \right) \cdot (\log_2 \mathrm{e})
\end{aligned} \tag{3.23}
$$

Lower bound for the first part:

$$
\begin{aligned}
&= \frac{K}{L} \log_2 \left(1 + \frac{1}{K - 1 + A} \right) \\
&= \log_2 \left(1 + \frac{1}{K - 1 + A} \right)^{\frac{K}{L}} \\
&\geq \log_2 \left(\mathrm{e}^{\frac{K}{L} \cdot \frac{1}{K-\frac{1}{2}+A}} \right) \\
&= \left(\frac{K}{L} \right) \cdot \left(\frac{1}{K - \frac{1}{2} + A} \right) \cdot (\log_2 \mathrm{e})
\end{aligned} \tag{3.24}
$$

(3.23) and (3.24) are a result of the application of the following bound, which is true for all $x \geq 0$.

$$
\frac{2x}{2 + x} \leq \ln(1 + x) \leq x \tag{3.25}
$$

Bounds for the second term in (3.22) are evaluated as follows:

$$
\begin{aligned}
&= \sum_{i=2}^{L} \frac{K}{L} \log_2 \left(1 + \frac{1}{K - 1 - (i-1) \cdot \frac{K}{L} + A}\right) \\
&= \frac{K}{L} \log_2 \prod_{i=2}^{L} \left(1 + \frac{1}{K - 1 - (i-1) \cdot \frac{K}{L} + A}\right) \\
&= \log_2 \left[\prod_{i=2}^{L} \left(1 + \frac{1}{K - 1 - (i-1) \cdot \frac{K}{L} + A}\right)\right]^{\frac{K}{L}}
\end{aligned}
$$

Using the same inequality in (3.25), the upper and lower bounds for the above term are obtained. Upper bound:

$$
\begin{aligned}
&\leq \log_2 \left[\prod_{i=2}^{L} \mathrm{e}^{\frac{K}{L} \cdot \frac{1}{K-1-(i-1)\cdot \frac{K}{L} + A}}\right] && (3.26) \\
&= \log_2 \left[\mathrm{e}^{\sum\limits_{i=2}^{L} \frac{K}{L} \cdot \frac{1}{K-1-(i-1)\cdot \frac{K}{L} + A}}\right] \\
&= \log_2 \left[\mathrm{e}^{\sum\limits_{i=2}^{L} \frac{K}{L} \cdot \frac{1}{K-1-(i-1)\cdot \frac{K}{L} + A}}\right] \\
&\leq \log_2 \left[\mathrm{e}^{\ln\left(\frac{L}{K}\cdot(K-1+A)-1\right)+1}\right] && (3.27) \\
&= \log_2 \left[\left(\frac{L}{K} \cdot (K - 1 + A) - 1\right) \cdot \mathrm{e}\right] \\
&= \log_2 \left(\frac{L}{K} \cdot (K - 1 + A) - 1\right) + \log_2 \mathrm{e} && (3.28)
\end{aligned}
$$

(3.26) is a result of the application of upper bound of $\ln(1+x)$ given in (3.25). (3.27) is the upper bound of the summation of the harmonic series. Let $\frac{L}{K} \cdot (K-1+A) = x$. This upper bound is derived as follows:

$$
\sum_{i=2}^{L} \frac{K}{L} \cdot \frac{1}{K - 1 - (i-1) \cdot \frac{K}{L} + A} = \sum_{i=2}^{L} \frac{1}{\frac{L}{K} \cdot (K - 1 + A) - (i - 1)}
$$

$$= \sum_{i=2}^{L} \frac{1}{x-(i-1)}$$
$$= \frac{1}{x-L+1} + \ldots + \frac{1}{x-1}$$

Using the known bounds for the summation of harmonic series, (3.20), we get:

$$\ln k < \sum_{n=1}^{k} \frac{1}{n} < \ln k + 1 \tag{3.29}$$

$$\ln\left(\frac{x-1}{x-L+1}\right) - 1 < \sum_{i=2}^{L} \frac{1}{x-(i-1)} < \ln(x-1) + 1 \tag{3.30}$$

Similarly, the lower bound is calculated as follows:

$$\geq \log_2 \left[\prod_{i=2}^{L} \mathrm{e}^{\frac{K}{L} \cdot \frac{\frac{2}{K-1-(i-1)\cdot\frac{K}{L}+A}}{2+\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A}}} \right] \tag{3.31}$$

$$= \log_2 \left[\mathrm{e}^{\sum_{i=2}^{L} \frac{K}{L} \cdot \frac{2}{2\cdot(K-1-(i-1)\cdot\frac{K}{L}+A)+1}} \right]$$

$$= \log_2 \left[\mathrm{e}^{\sum_{i=2}^{L} \frac{1}{\frac{L}{K}\cdot(K-\frac{1}{2}+A)-(i-1)}} \right]$$

$$\leq \log_2 \left[\mathrm{e}^{\ln\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1\right)-\ln\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-L+1\right)-1} \right] \tag{3.32}$$

$$= \log_2 \left[\frac{\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1\right)\cdot \mathrm{e}}{\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-L+1\right)} \right]$$

$$= \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1\right) - \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-L+1\right) - \log_2 \mathrm{e} \tag{3.33}$$

(3.31) is a result of the application of lower bound of $\ln(1+x)$ given in (3.25). (3.32) is the lower bound of the summation of the harmonic series given in (3.30). Thus from (3.28) and (3.33) we have:

$$\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1\right)-\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-L+1\right)-\log_2\mathrm{e}$$
$$\leq S_{uq}\leq\log_2\left(\frac{L}{K}\cdot(K-1+A)-1\right)+\log_2\mathrm{e} \qquad \square$$

From the following analysis, it can be seen that asymptotically there is a constant gap between the upper and lower bounds of S_{uq}, which can be evaluated as follows:

$$=\lim_{K\to\infty}\left[\log_2\left(\frac{L}{K}\cdot(K-1+A)-1\right)+\log_2\mathrm{e}-\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1\right)\right.$$
$$\left.+\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-L+1\right)+\log_2\mathrm{e}\right]$$
$$=2\log_2\mathrm{e}$$

since, for the choice of $L=\frac{K}{(\log_2 K)^{(1+\delta)}}$, we have, $\lim_{K\to\infty}\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-L+1\right)=0$ and $\lim_{K\to\infty}\log_2\left(\frac{\frac{L}{K}\cdot(K-1+A)-1}{\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1}\right)=0$.

Theorem 3.4.4. *Let S_{uq} be the sum rate obtained through uniform quantization using L levels, where $L=\frac{K}{(\log_2 K)^{1+\delta}}$. Asymptotically, the absolute gap from capacity is of the order $O(\ln\ln K)$ and the fractional gap tends to 1.*

$$\lim_{K\to\infty} g_a = \mathrm{O}(\log_2\log_2 K)$$
$$\lim_{K\to\infty} g_f = 1$$

Proof. let us begin the proof with the fractional gap. Using the upper bound for S_{uq} derived in (3.28), the upper bound for g_f is derived as follows:

$$\lim_{K\to\infty} g_f \leq \lim_{K\to\infty}\frac{\log_2\left(\frac{L}{K}\cdot(K-1+A)-1\right)+\log_2\mathrm{e}}{\log_2\left(1+\frac{K}{A}\right)}$$

$$
\begin{aligned}
&= \lim_{K\to\infty} \frac{\left[\frac{1}{\frac{L}{K}\cdot(K-1+A)-1}\right]\left[\frac{(\log_2 K)^{1+\delta}-(K-1+A)\cdot(1+\delta)(\log_2 K)^{\delta}\frac{1}{K\ln 2}}{(\log_2 K)^{2+2\delta}}\right]}{\frac{1}{1+K/A}\cdot\frac{1}{A}} \\
&= \lim_{K\to\infty} \frac{(K+A)\cdot K}{(\log_2 K)^{1+\delta}\left(L(K-1+A)-K\right)} \\
&\qquad - \lim_{K\to\infty} \frac{(K+A)(K-1+A)(1+\delta)}{(L(K-1+A)-K)\cdot\ln 2\cdot(\log_2 K)^{2+\delta}} \\
&= \lim_{K\to\infty} \frac{(K+A)}{(\log_2 K)^{1+\delta}\left(\frac{L}{K}\cdot(K-1+A)-1\right)} \\
&\qquad - \lim_{K\to\infty} \left(\frac{1+\delta}{\ln 2}\right)\left[\frac{(K+A)\cdot(K-1+A)}{K\left(\frac{L}{K}(K-1+A)-1\right)\cdot(\log_2 K)^{2+\delta}}\right] \\
&= \lim_{K\to\infty} \frac{(K+A)}{(\log_2 K)^{1+\delta}((\log_2 K)^{-1-\delta}\cdot(K-1+A)-1)} \\
&\qquad - \lim_{K\to\infty} \left(\frac{1+\delta}{\ln 2}\right)\left[\frac{(K+A)\cdot(K-1+A)}{K\left[(\log_2 K)^{-1-\delta}\cdot(K-1+A)-1\right]\cdot(\log_2 K)^{2+\delta}}\right] \\
&= \lim_{K\to\infty} \frac{1}{(\log_2 K)^{1+\delta}\left((\log_2 K)^{-1-\delta}\cdot(K-1+A)-1\right)} \\
&\qquad - \lim_{K\to\infty} \left(\frac{1+\delta}{\ln 2}\right)\left[\frac{(K+A)\cdot(K-1+A)}{K\left[(K-1+A)\cdot(\log_2 K)-(\log_2 K)^{2+\delta}\right]}\right] \\
&= \lim_{K\to\infty} \frac{1}{\frac{K-1+A}{K+A}-\frac{(\log_2 K)^{1+\delta}}{K+A}} \\
&\qquad - \lim_{K\to\infty} \left(\frac{1+\delta}{\ln 2}\right)\left[\frac{1}{\left(\frac{K}{K+A}\right)\cdot\left[\log_2 K-\frac{(\log_2 K)^{2+\delta}}{(K-1+A)}\right]}\right] \\
&= 1-0=1
\end{aligned}
$$

Similarly, using the lower bound for S_{uq} derived in (3.28), it can be shown that the lower bound for g_f is also 1. Since both the bounds converge, we can say that, as $K\to\infty$ for this particular choice of L, fractional gap, g_f tends to 1.

Similar to the above approach, we calculate upper and lower bounds for the absolute gap, g_a. The lower bound for g_a is derived as follows:

$$
\begin{aligned}
\lim_{K\to\infty} S_{uq} - S_{cap} &\leq \lim_{K\to\infty} \log_2\left(\frac{L}{K}\cdot(K-1+A)-1\right) + \log_2 \mathrm{e} - \log_2\left(1+\frac{K}{A}\right) \\
&= \lim_{K\to\infty} \log_2\left(\frac{\frac{L}{K}\cdot(K-1+A)-1}{\left(1+\frac{K}{A}\right)}\right) + \log_2 \mathrm{e} \\
&= \lim_{K\to\infty} \log_2\left(\frac{(\log_2 K)^{-1-\delta}\cdot(K-1+A)-1}{\left(1+\frac{K}{A}\right)}\right) + \log_2 \mathrm{e} \\
&= \lim_{K\to\infty} \log_2\left(\frac{(K-1+A)-(\log_2 K)^{1+\delta}}{\left(1+\frac{K}{A}\right)}\right) + \log_2 \mathrm{e} - \log_2\left((\log_2 K)^{1+\delta}\right) \\
&= \log_2\left(\frac{A}{A+1}\right) + \log_2 \mathrm{e} - \log_2\left((1+\delta)\cdot\log_2 K\right) \qquad (3.34)
\end{aligned}
$$

The upper bound for g_a is calculated as follows:

$$
\begin{aligned}
S_{uq} - S_{cap} &\geq \lim_{K\to\infty} \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1\right) \\
&\quad - \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A)-L+1\right) - \log_2 \mathrm{e} - \log_2\left(1+\frac{K}{A}\right) \\
&= \lim_{K\to\infty} \log_2\left(\frac{(K-\frac{1}{2}+A)-(\log_2 K)^{1+\delta}}{(K-\frac{1}{2}+A)-K+(\log_2 K)^{1+\delta}}\right) - \log_2 \mathrm{e} - \log_2\left(1+\frac{K}{A}\right) \\
&= \lim_{K\to\infty} \log_2\left(\frac{(K-\frac{1}{2}+A)-(\log_2 K)^{1+\delta}}{\left(1+\frac{K}{A}\right)}\right) \\
&\quad - \log_2\left((\log_2 K)^{1+\delta}-\frac{1}{2}+A\right) - \log_2 \mathrm{e} \\
&= \log_2\left(\frac{A}{A+1}\right) - \log_2 \mathrm{e} - \log_2\left((1+\delta)\cdot\log_2 K + A - \frac{1}{2}\right) \qquad (3.35)
\end{aligned}
$$

From (3.34) and (3.35) we have the upper and lower bounds for g_a.

$$
\begin{aligned}
&\log_2\left((1+\delta)\cdot\log_2 K\right) - \log_2 \mathrm{e} - \log_2\left(\frac{A}{A+1}\right) \\
&\leq g_a \leq \log_2\left((1+\delta)\cdot\log_2 K + A - \frac{1}{2}\right) + \log_2 \mathrm{e} - \log_2\left(\frac{A}{A+1}\right) \qquad \square
\end{aligned}
$$

It can be seen from the above equation that both the bounds converge. Hence the absolute gap, g_a is of the order, $O(\log_2 \log_2 K)$. Note that the g_a is also a function of A and hence a function of the signal to noise ratio.

3.4.2 Asymptotic Analysis in the Infinitesimal Regime

The analysis of bounds on S_{uq}, g_a and g_f in the infinitesimal regime is similar to the analysis in the finite SNR regime. The fundamental difference in the following proofs is that A is no longer constant. This model assumes that the SNR goes to zero at the same rate as K goes to infinity. Hence A tends to infinity at the same rate.

Theorem 3.4.5. *Let S_{uq} be the sum rate obtained using L levels, where $L = \frac{K}{(\log_2 K)^{1+\delta}}$ and A' be the inverse SNR. The upper and lower bounds for S_{uq} in the infinitesimal SNR range are:*

$$\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-1\right)-\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-L\right)-\log_2 \mathrm{e}$$
$$\leq S_{uq} \leq \log_2\left(\frac{L}{K}\cdot(K-1+A^{'})-1\right)-\log_2\left(\frac{L}{K}\cdot(K-1+A^{'})-L\right)+\log_2 \mathrm{e}$$

Proof.

$$\begin{aligned} S_{uq} &= \sum_{i=1}^{L} n_i^* r_i^* \\ &= \frac{K}{L}\log_2\left(1+\frac{1}{K-1+A'}\right)+\sum_{i=2}^{L}\frac{K}{L}\log_2\left(1+\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A'}\right) \end{aligned} \tag{3.36}$$

We calculate the upper and lower bounds for each of the two parts in (3.36) separately. The bounds for the first part are calculated the same way as in Theorem

3.3.3.

$$\frac{K}{L}\left(\frac{1}{K-\frac{1}{2}+A'}\right)\log_2 \mathrm{e} \le \frac{K}{L}\log_2\left(1+\frac{1}{K-1+A'}\right) \le \frac{K}{L}\left(\frac{1}{K-1+A'}\right)\log_2 \mathrm{e} \tag{3.37}$$

The second term in the equation (3.37) is evaluated as follows:

$$\begin{aligned}
&= \sum_{i=2}^{L}\frac{K}{L}\log_2\left(1+\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A'}\right)\\
&= \frac{K}{L}\log_2\prod_{i=2}^{L}\left(1+\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A'}\right)\\
&= \log_2\left[\prod_{i=2}^{L}\left(1+\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A'}\right)\right]^{\frac{K}{L}} \qquad (3.38)
\end{aligned}$$

We calculate an upper bound and a lower bound for (3.38). Upper bound:

$$\begin{aligned}
&\le \log_2\left[\prod_{i=2}^{L}\mathrm{e}^{\frac{K}{L}\cdot\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A'}}\right] \qquad (3.39)\\
&= \log_2\left[\mathrm{e}^{\sum_{i=2}^{L}\frac{K}{L}\cdot\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A'}}\right]\\
&= \log_2\left[\mathrm{e}^{\sum_{i=2}^{L}\frac{K}{L}\cdot\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A'}}\right]\\
&\le \log_2\left[\mathrm{e}^{\ln\left(\frac{L}{K}\cdot(K-1+A')-1\right)+1-\ln\left(\frac{L}{K}\cdot(K-1+A')-L\right)}\right] \qquad (3.40)\\
&= \log_2\left[\mathrm{e}^{\ln\left(\frac{L}{K}\cdot(K-1+A')-1\right)}\cdot\mathrm{e}\cdot\mathrm{e}^{-\ln\left(\frac{L}{K}\cdot(K-1+A')-L\right)}\right]\\
&= \log_2\left(\frac{L}{K}\cdot(K-1+A')-1\right)-\log_2\left(\frac{L}{K}\cdot(K-1+A')-L\right)+\log_2\mathrm{e} \qquad (3.41)
\end{aligned}$$

(3.39) is from the approximation given in (3.25). (3.40) is obtained by evaluating the summation of the harmonic series similar to (3.27) with minor changes. Since

$x - L$ does not go to zero, we find tighter upper and lower bounds.

$$
\begin{aligned}
\sum_{i=2}^{L} \frac{K}{L} \cdot \frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A} &= \sum_{i=2}^{L} \frac{1}{\frac{L}{K}\cdot(K-1+A)-(i-1)} \\
&= \sum_{i=2}^{L} \frac{1}{x-(i-1)} \\
&= \frac{1}{x-L+1} + \ldots + \frac{1}{x-1} \\
&= \left(1+\frac{1}{2}+\ldots\frac{1}{x-1}\right) - \left(1+\frac{1}{2}+\ldots+\frac{1}{x-L}\right)
\end{aligned}
$$

$$
\ln\left(\frac{x-1}{x-L}\right) - 1 < \sum_{i=2}^{L} \frac{1}{x-(i-1)} < \ln\left(\frac{x-1}{x-L}\right) + 1 \tag{3.42}
$$

We can calculate the lower bound for the second term in (3.38) as follows:

$$
\geq \log_2\left[\prod_{i=2}^{L} \mathrm{e}^{\frac{K}{L}\cdot\frac{\frac{2}{K-1-(i-1)\cdot\frac{K}{L}+A'}}{2+\frac{1}{K-1-(i-1)\cdot\frac{K}{L}+A'}}}\right] \tag{3.43}
$$

$$
\begin{aligned}
&= \log_2\left[\mathrm{e}^{\sum_{i=2}^{L}\frac{K}{L}\cdot\frac{2}{2\cdot(K-1-(i-1)\cdot\frac{K}{L}+A')+1}}\right] \\
&= \log_2\left[\mathrm{e}^{\sum_{i=2}^{L}\frac{1}{\frac{L}{K}\cdot(K-\frac{1}{2}+A')-(i-1)}}\right]
\end{aligned}
$$

$$
\leq \log_2\left[\mathrm{e}^{\ln\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A')-1\right)-\ln\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A')-L\right)-1}\right] \tag{3.44}
$$

$$
= \log_2\left[\frac{\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A')-1\right)}{\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A')-L\right)\cdot\mathrm{e}}\right]
$$

$$
= \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A')-1\right) - \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A')-L\right) - \log_2\mathrm{e} \tag{3.45}
$$

(3.43) is from is from the approximation given in (3.25). (3.44) is from the upper bound in (3.42). Thus from (3.41) and (3.45) we have:

$$\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-1\right)-\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-L\right)-\log_2 \mathrm{e} \qquad \square$$
$$\leq S_{uq}\leq \log_2\left(\frac{L}{K}\cdot(K-1+A^{'})-1\right)-\log_2\left(\frac{L}{K}\cdot(K-1+A^{'})-L\right)+\log_2 \mathrm{e}$$

We can see that asymptotically there is a constant gap between the upper and lower bounds, which can be evaluated as follows:

$$\begin{aligned}\lim_{K\to\infty} g = \lim_{K\to\infty}\Bigg[&\log_2\left(\frac{L}{K}\cdot(K-1+A^{'})-1\right)-\log_2\left(\frac{L}{K}\cdot(K-1+A^{'})-L\right)+\log_2 \mathrm{e}\\ &-\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-1\right)+\log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-L\right)-\log_2 \mathrm{e}\Bigg]\\ &=2\log_2 \mathrm{e}\end{aligned}$$

since $\lim_{K\to\infty}\log_2\left(\frac{\frac{L}{K}\cdot(K-1+A^{'})-L}{\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-L}\right)=0$ and $\lim_{K\to\infty}\log_2\left(\frac{\frac{L}{K}\cdot(K-1+A)-1}{\frac{L}{K}\cdot(K-\frac{1}{2}+A)-1}\right)=0$.

Theorem 3.4.6. *Let S_{uq} be the sum rate obtained using L quantization levels, where $L=\frac{K}{(\log_2 K)^{1+\delta}}$. Asymptotically, the absolute gap from capacity is of constant order, $O(1)$ and the fractional gap tends to 1.*

$$\lim_{K\to\infty} g_a = 0$$
$$\lim_{K\to\infty} g_f = 1$$

Proof. Let us first evaluate the upper and lower bounds for the absolute gap. The lower bound is calculated as follows:

$$\lim_{K\to\infty} S_{uq} - S_{cap} \le \lim_{K\to\infty} \log_2\left(\frac{L}{K}\cdot(K-1+A^{'})-1\right) - \log_2\left(\frac{L}{K}\cdot(K-1+A^{'})-L\right)$$
$$+ \log_2 \mathrm{e} - \log_2\left(1+\frac{K}{A'}\right)$$
$$= \lim_{K\to\infty} \log_2\left(\frac{\frac{L}{K}\cdot(K-1+K\cdot A)-1}{\frac{L}{K}\cdot(K-1+K\cdot A)-L}\right) + \log_2 \mathrm{e} - \log_2\left(1+\frac{1}{A}\right)$$
$$= \lim_{K\to\infty} \log_2\left(\frac{(K-1+K\cdot A)-\frac{K}{L}}{(K-1+K\cdot A)-K}\right) + \log_2 \mathrm{e} - \log_2\left(1+\frac{1}{A}\right)$$
$$= \log_2\left(\frac{A+1}{A}\right) + \log_2 \mathrm{e} - \log_2\left(1+\frac{1}{A}\right)$$
$$= \log_2 \mathrm{e} \tag{3.46}$$

Similarly the upper bound for the absolute gap is:

$$\lim_{K\to\infty} S_{uq} - S_{cap} \ge \lim_{K\to\infty} \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-1\right) - \log_2\left(\frac{L}{K}\cdot(K-\frac{1}{2}+A^{'})-L\right)$$
$$- \log_2 \mathrm{e} - \log_2\left(1+\frac{K}{A'}\right)$$
$$= \lim_{K\to\infty} \log_2\left(\frac{\frac{L}{K}\cdot(K-\frac{1}{2}+K\cdot A)-1}{\frac{L}{K}\cdot(K-\frac{1}{2}+K\cdot A)-L}\right) - \log_2 \mathrm{e} - \log_2\left(1+\frac{1}{A}\right)$$
$$= \lim_{K\to\infty} \log_2\left(\frac{(K-\frac{1}{2}+K\cdot A)-\frac{K}{L}}{(K-\frac{1}{2}+K\cdot A)-K}\right) - \log_2 \mathrm{e} - \log_2\left(1+\frac{1}{A}\right)$$
$$= \log_2\left(\frac{A+1}{A}\right) - \log_2 \mathrm{e} - \log_2\left(1+\frac{1}{A}\right)$$
$$= -\log_2 \mathrm{e} \tag{3.47}$$

Since the absolute gap to capacity, is always positive, zero is a trivial lower bound for g_a. Since zero is tighter than the estimated lower bound in (3.46), it is used. Combining this with (3.47), we have:

$$0 \leq g_a \leq \log_2 \mathrm{e} \qquad \square$$

4. SIMULATION RESULTS

In this chapter, five different simulations are conducted. Scenarios I and II help in understanding the operations of rate quantization and rate biasing. Scenarios III, IV and V validate our theoretical analysis in chapter-3. Although the simulations are constrained by the computational resources required for large values of K, they present trends which closely match the theoretical analysis. We use *bits/channel use* as the standard unit for GMAC capacity and the sumrates in all the following results.

4.1 Scenario-I: Role of ϵ- Model 1

In this scenario, $K = 10^4$, $L = 10^2$. Simulations are run for different values of ϵ, which is related to the probability of decoding failure. The following figure

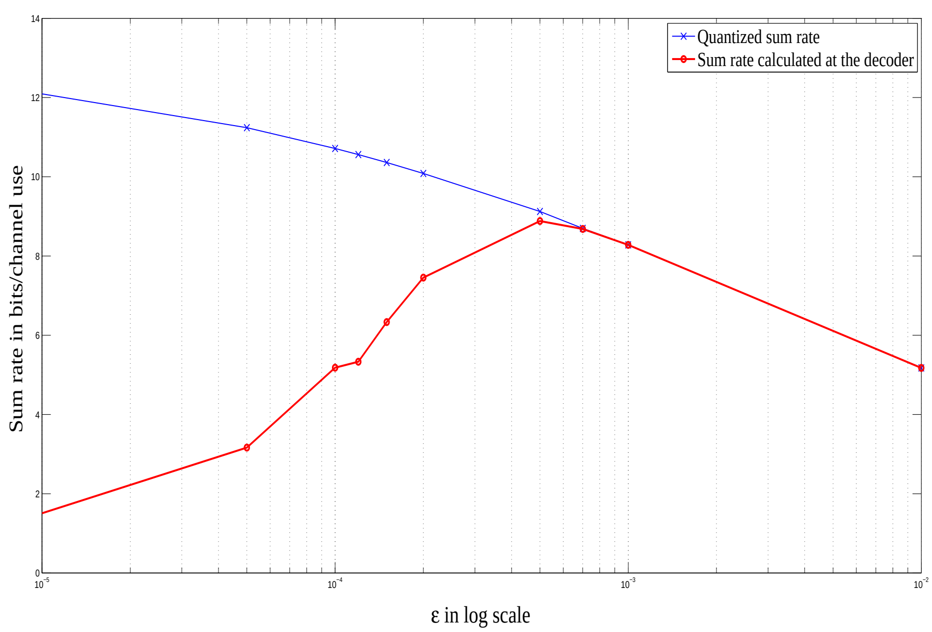

Figure 4.1: $K = 10^4, L = 10^2$ GMAC capacity=12.9557

shows the quantized sum rate, S_q, which is the sum rate obtained if the decoding is successful and the average sum rate obtained, S_{avg}, which is the sum rate obtained by implementing the decoder against ϵ.

From Fig 4.1, it is observed that as ϵ decreases, probability of decoder failure increases and hence we get a lower sum rate ($S_{avg} \leq S_q$). Thus there exists an optimal value for the biasing parameter, ϵ that achieves maximum sum rate for a given K, L.

4.2 Scenario-II: Uniform Quantization vs Optimal Quantization

In this scenario we simulate the two types of quantizations suggested earlier. We observe that the optimal quantization outperforms uniform quantization.

L	$S_{uq,unbiased}$	$S_{oq,unbiased}$	$S_{uq,sim}$	$S_{oq,sim}$
10	4.2088	7.3657	3.7729	4.6621
100	7.3153	10.0988	5.5257	6.0682
1000	9.6354	10.4469	6.0508	6.2844

Table 4.1: $K = 1000$, GMAC capacity=9.6354

From Table 4.1, it is interesting to note that $S_{oq_unbiased}$, which is the sum rate obtained from optimal quantization, is greater than the capacity. This happens because the decision variables in the optimization problem n_i is constrained to be an integer, but solving a mixed integer problem is hard. Hence we relax the integer constraint. Thus solving this problem gives a sub-optimal solution. It is of significant interest to find a global optimum for the quantization problem with integer constraints on n_i.

4.3 Scenario-III: Asymptotic Behavior

We study the asymptotic behavior of this scheme by constructing a sub scenarios. We fix L and increase K to observe the gap to the capacity.

From Table 4.2, we can validate Theorems 1 and 2. Since K is increasing exponentially, Theorem 1 predicts that the gap between S_{cap} and S_{uq} increases in the order of $O(\log_2 K)$ and hence linearly in this case. It can be seen from columns 2 and 3 that this is true. Theorem-2 also predicts that ϵ goes to zero as $K \to \infty$, which can be seen as true from column 5. Figure 4.2 shows the closeness of $S_{\mathrm{oq,sim}}$ to the GMAC capacity.

K	GMAC capacity	S_{oq}	$S_{oq,sim}$	ϵ
10^2	0.0144	0.0144	0.0143	9×10^{-2}
10^3	0.1375	0.1374	0.1368	1×10^{-2}
10^4	1	0.9966	0.9857	1×10^{-3}
10^5	3.4594	3.3710	3.3760	5×10^{-5}
10^6	6.6582	6.4328	6.4355	11×10^{-6}
5×10^6	8.9687	8.6956	8.6175	1.5×10^{-6}

Table 4.2: SNR=-40dB, L=100, Asymptotic behavior of K

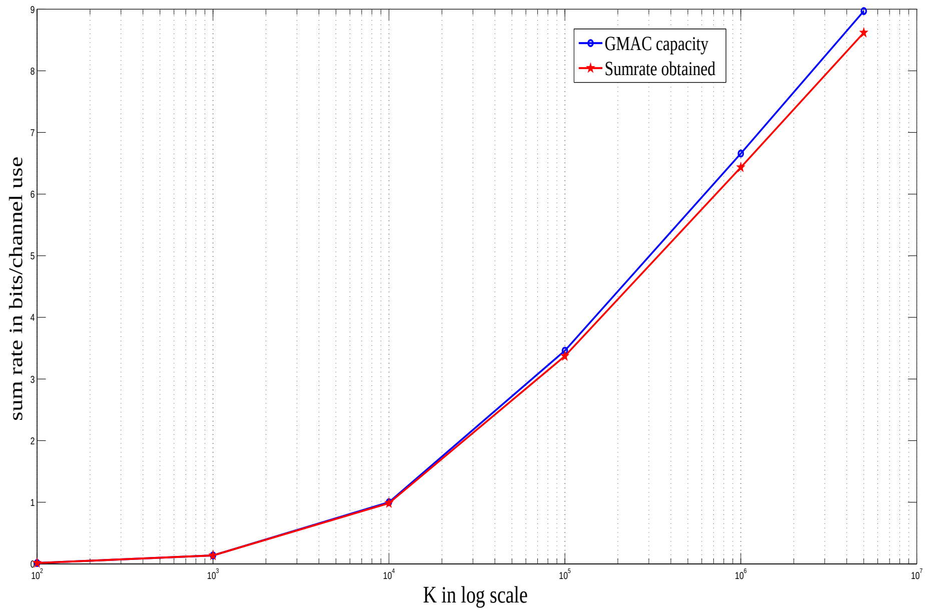

Figure 4.2: Asymptotic behavior of K. SNR=-40dB, L=100

4.4 Scenario-IV: Model-1- Finite SNR Regime

Instead of a fixed L in scenarios 1,2 and 3, we increase L as a function of K. We observe the absolute and fractional gaps to validate Theorems 3 and 4.

From table 4.3, column 5, we observe that $\epsilon \to 0$ hence $S_{sim} \to S_{uq}$. We also observe that the fractional gap progresses as $\{0.31, 0.39, 0.47, 0.54, 0.56, 0.58\}$. This data is far from conclusive that $g_f \to 1$ as $K \to \infty$. But it quickly becomes infeasible to simulate for such large values of K. The absolute gap progresses as $\{4.359, 5.87, 6.87, 7.52, 7.72, 7.98\}$. It can be seen that this sequence is logarithmically increasing when K is increasing exponentially. This conforms with the analysis in theorem 3.3.4, which states that $\lim_{K\to\infty} g_a = \text{O}(\log_2 \log_2 K)$.

K	L	GMAC capacity	S_{uq}	S_{sim}	ϵ
10^2	2	6.3297	2.1371	1.9711	3×10^{-2}
10^3	10	9.6354	4.2088	3.7644	5×10^{-3}
10^4	57	12.9557	6.6681	6.0898	1×10^{-3}
10^5	362	16.2780	9.3281	8.7594	11×10^{-5}
3×10^5	906	17.8624	10.6515	10.1390	3×10^{-5}
6×10^5	1629	18.8624	11.4981	10.8846	1.2×10^{-5}

Table 4.3: Asymptotic behavior for increasing L. SNR=-1dB, $L = \frac{K}{(\log_2 K)^2}$

4.5 Scenario-V: Model-2: Infinitesimal SNR Regime

This scenario is similar to scenario IV except we impose the constraints for Model 2. We observe the absolute and fractional gaps to validate Theorems 5 and 6.

From table 4.4, column 5, we observe that $\epsilon \to 0$ hence $S_{sim} \to S_{uq}$. We also observe that the fractional gap progresses as $\{0.8544, 0.9630, 0.9921, 0.9983\}$, which supports out claim that $g_f \to 1$ as $K \to \infty$. The absolute gap progresses as $\{0.1228, 0.03120.0067, 0.0014\}$. This validates theorem 3.3.6, which states that $\lim_{K\to\infty} g_a = \text{O}(1) = 0$.

K	L	GMAC capacity	S_{uq}	S_{sim}	ϵ
10^2	2	0.8434	0.7313	0.7206	1×10^{-2}
10^3	10	0.8434	0.8123	0.8122	1×10^{-3}
10^4	57	0.8434	0.8390	0.8367	1×10^{-4}
10^5	362	0.8434	0.8427	0.8420	1×10^{-5}

Table 4.4: Asymptotic behavior for increasing L. SNR=-1dB, $L = \frac{K}{(\log_2 K)^2}$

5. CONCLUSION

5.1 Contribution

This work primarily addressed the question if uncoordinated MAC schemes could perform as good as their counterparts. In [3] it has been shown that for a noise less channel at the cost of $(\ln K)$ times additional transmit power an efficiency of 1 can be achieved asymptotically. In section-2.2 it is shown that extending this scheme to a gaussian MAC has an absolute gap to capacity of the order $O(\ln \ln K)$ in the finite SNR regime and a fractional gap tending to zero in the infinitesimal SNR regime. In section-2.3 it is shown that adding MRC at the decoder only improves the sumrate by a constant order and hence is suboptimal in both the SNR regimes. From these results, it is inferred that the suboptimality lies in repetitive coding.

In chapter 3, a new paradigm is proposed to tackle the limitation of repetitive based random access MAC schemes. In this scheme every active node transmits in the current time slot by picking a rate from a predetermined distribution, which depends only on the number of active users in the network. In section-3.2, operations such as rate quantization and rate biasing are described and formulated as optimization problems. In section-3.3 an asymptotic analysis is presented for this scheme in both the finite and infinitesimal SNR regimes. From theorems 3.3.1 and 3.3.2 it is inferred that the number of quantization levels, L should increase with K as $\frac{K}{\log_2 K^{(1+\delta)}}$ to obtain maximum sumrate. In theorem 3.3.3 upper and lower bounds for the sumrate in the finite SNR regime were derived, which are valid for all K and L. Using these bounds, it is shown in theorem 3.3.4 that the absolute gap to capacity is of the order $O(\log_2 \log_2 K)$ and the fractional gap tends to one. Similarly in theorems 3.3.5 and 3.3.6 valid bounds and the order of absolute gap and fractional gap are derived. It is shown that the absolute gap is a constant order away from GMAC capacity.

In chapter-4 five simulation scenarios were setup to better understand URS. Scenario-I demonstrates that for any finite K and L there exists an optimal biasing parameter, ϵ, that gives the maximum S_{sim}. In scenario-II, it is shown that optimal rate quantization outperforms uniform rate quantization but their performances converge asymptotically. Scenario-III validated theorems 3.3.1 and 3.3.2 by observing that the gap to capacity for fixed L and increasing K increases in the order of $O(\log_2 K)$. Scenarios-IV and V have a similar setup but the SNRs are in finite and infinitesimal regimes respectively. The trends of absolute and fractional gaps are observed to be following the results from the above theorems closely.

URS has several advantages over ICR. It is optimal in the infinitesimal SNR regime. It has minimal latency of one time slot instead of K time slots. It can be used with other iterative decoders such as ZF-SIC, MMSE-SIC or SIF [5]. No modifications are necessary to include more complicated models such as variable channel gains and different power constraints for transmitters. The proposed scheme simply enables us to operate at a point on the achievable rate region that is sumrate optimal and decodable using an iterative decoder without coordination among the nodes and with single user encoding and decoding complexities. Hence this scheme is applicable as long as such a point exists in the assumed channel model.

5.2 Future Work

The proposed scheme is optimal in the infinitesimal SNR regime but not in the finite SNR regime. It is of interest to determine if there exist fundamental limits on the performance that can be achieved with uncoordinated multiple access schemes. We would also investigate the use of raptor codes to see if we can avoid the $\ln \ln K$ gap in the finite SNR regime.

REFERENCES

[1] A. Leon-Garcia and I. Widjaja, *Communication networks.* McGraw-Hill, Inc., 2003.

[2] G. Liva, "Graph-based analysis and optimization of contention resolution diversity slotted aloha," *Communications, IEEE Transactions on*, vol. 59, no. 2, pp. 477–487, 2011.

[3] K. R. Narayanan and H. D. Pfister, "Iterative collision resolution for slotted aloha: An optimal uncoordinated transmission policy," in *Turbo Codes and Iterative Information Processing (ISTC), 2012 7th International Symposium on*, pp. 136–139, IEEE, 2012.

[4] T. Cover and J. Thomas, *Elements of Information Theory.* John Wiley and sons, 2006.

[5] O. Ordentlich, U. Erez, and B. Nazer, "Successive integer-forcing and its sum-rate optimality," *arXiv preprint arXiv:1307.2105*, 2013.

[6] A. Yedla, P. Nguyen, H. Pfister, and K. Narayanan, "Universal codes for the Gaussian mac via spatial coupling," in *Proc. Allerton Conf. Communications Control and Computing*, pp. 1801–1808, IEEE, 2011.

[7] B. Rimoldi and R. Urbanke, "A rate-splitting approach to the gaussian multiple-access channel," *Information Theory, IEEE Transactions on*, vol. 42, no. 2, pp. 364–375, 1996.

[8] P. Bergmans and T. M. Cover, "Cooperative broadcasting," *Information Theory, IEEE Transactions on*, vol. 20, no. 3, pp. 317–324, 1974.

APPENDIX

Closed form solution of Optimal Rate Quantization problem

The rate quantization problem proposed in (3.4) is modified by applying the approximation $\ln(1+x) \approx x$, which is valid for the low SNR regime. This approximation is only to obtain an intuitive closed form solution.

$$\begin{aligned}
\underset{n_i, r_i \forall i \in [1,L]}{\text{maximize}} \quad & \sum_{i=1}^{L} n_i r_i \\
\text{subject to} \quad & \sum_{i=1}^{L} n_i = K \\
& r_1 \leq \frac{1}{\ln 2}\left(\frac{P}{(K-1)P+\sigma^2}\right) \\
& r_i \leq \frac{1}{\ln 2}\left(\frac{P}{\left(K-\sum_{j=1}^{i-1} n_j - 1\right)P+\sigma^2}\right) \qquad i = 2, ..., L \\
& r_i \geq 0 \qquad i = 1, ..., L \\
& n_i \geq 0 \qquad i = 1, ..., L
\end{aligned} \tag{5.1}$$

The objective function is nonconvex in n_i, r_i. We find the lower bound of our objective function by solving its dual problem using the Lagrange multipliers. The dual problem is guaranteed to be a convex problem. There is a finite gap between the optimal value of the primal and dual objective functions. The Lagrange function for the optimization problem in (5.1) can be written as:

$$\begin{aligned}\Lambda(n_1,..,n_L,r_1,..,r_L,\gamma,\lambda_1,..,\lambda_L) =& \sum_{i=1}^{L} n_i r_i + \gamma\left(\sum_{i=1}^{L} n_i - K\right)\\ &+ \lambda_1\left(r_1 - \frac{1}{\ln 2}\left(\frac{P}{(K-1)P+\sigma^2}\right)\right)\\ &+ \sum_{i=2}^{L}\lambda_i\left(r_i - \frac{1}{\ln 2}\left(\frac{P}{\left(K-\sum_{j=1}^{i-1} n_j - 1\right)P+\sigma^2}\right)\right)\end{aligned}$$

To find the optimal n_i^*, r_i^* we solve the set of equations obtained by equating the partial derivatives of Λ to zero.

$$\begin{aligned}&\frac{\partial\Lambda}{\partial\lambda_1} = 0 \implies r_1^* = \frac{1}{\ln 2}\left(\frac{P}{(K-1)P+\sigma^2}\right)\\ &\frac{\partial\Lambda}{\partial\lambda_i} = 0 \implies r_i^* = \frac{1}{\ln 2}\left(\frac{P}{\left(K-\sum_{j=1}^{i-1} n_j - 1\right)P+\sigma^2}\right) \qquad i = 2,...,L\\ &\frac{\partial\Lambda}{\partial\gamma} = 0 \implies \sum_{i=1}^{L} n_i^* = K\end{aligned} \tag{5.2}$$

Let $\frac{\sigma^2}{P} = A$. The following partial derivatives use (5.2):

$$\begin{aligned}\frac{\partial\Lambda}{\partial n_L} = 0 &\implies \sum_{j=1}^{L-1} n_i^* = K - 1 + A - \frac{1}{\gamma\ln 2}\\ \frac{\partial\Lambda}{\partial n_{L-1}} = 0 &\implies \sum_{j=1}^{L-2} n_i^* = K - 1 + A - \frac{1}{\gamma^2\ln 2^2\,(A-1)}\end{aligned}$$

$$\Longrightarrow \quad n^*_{L-1} = \frac{1}{\gamma^2 \ln 2^2 (A-1)} - \frac{1}{\gamma \ln 2}$$

$$\vdots$$

$$\frac{\partial \Lambda}{\partial n_i} = 0 \Longrightarrow \sum_{j=1}^{i-1} n^*_i = K - 1 + A - \frac{1}{(\gamma \ln 2)^{L-i+1} (A-1)^{L-i}} \quad \forall i = 1, \ldots, L-1$$

$$\Longrightarrow \quad n^*_i = \frac{1}{(\gamma \ln 2)^{L-i+1} (A-1)^{L-i}} - \frac{1}{(\gamma \ln 2)^{L-i} (A-1)^{L-i-1}}$$

$$\gamma \ln 2 = \frac{1}{(A-1)^{L-1} (K-1+A)} \tag{5.3}$$

The closed form solution to (5.1) is derived from the results in (5.2) and (5.3) and summarized as follows:

$$\begin{aligned}
n^*_i &= \frac{1}{(A\lambda)^{L-i-1}} \left[\frac{1}{A\lambda^2} - \frac{1}{\lambda} \right] \qquad \text{for } i = 1, 2, ...L-1 \\
n^*_L &= K - \sum_{j=1}^{L-1} n^*_j \\
r^*_1 &= \frac{1}{\ln 2} \left(\frac{P}{(K-1)P + \sigma^2} \right) \\
r^*_i &= \frac{1}{\ln 2} \left(\frac{P}{\left(K - \sum_{j=1}^{i-1} n^*_j - 1 \right) P + \sigma^2} \right) \qquad \text{for } i = 2, ..., L \\
A &= \frac{\sigma^2}{P}, \lambda = \left(\frac{1}{A^{L-1}(K-1+A)} \right)^{1/L}
\end{aligned} \tag{5.4}$$

Closed form solution of Optimal biased Rate distribution

Lagrange function for (3.12) is given by:

$$\begin{aligned}\Lambda(e_{m,1},\ldots,e_{m,L},\gamma,\lambda) = &\sum_{j=1}^{L} e_{m,j}\log_2\frac{e_{m,j}}{q_j} + \lambda\left(\sum_{j=1}^{L} e_{m,j} - 1\right)\\ &+ \gamma\left(e_{m,j} + \ldots + e_{m,m} - \sum_{j=1}^{m} n_j\right)\end{aligned} \tag{5.5}$$

To find the optimal $\mathbf{E}_m^*$ in terms of $\mathbf{Q}$, we equate the partial derivatives of the Lagrangian function w.r.t $e_{m,j}$ to zero. The result is:

$$\begin{aligned}\frac{\partial\Lambda}{\partial e_{m,j}} = 0 &\implies \quad e_{m,j}^* = q_j 2^{-(1+\lambda+\gamma)} \quad j = 1,\ldots,m\\ \frac{\partial\Lambda}{\partial e_{m,j}} = 0 &\implies \quad e_{m,j}^* = q_j 2^{-(1+\lambda)} \quad j = m+1,\ldots,L\\ \frac{\partial\Lambda}{\partial\gamma} = 0 &\implies \sum_{j=1}^{i} e_{m,j}^* = \sum_{j=1}^{m} n_j\\ \frac{\partial\Lambda}{\partial\lambda} = 0 &\implies \sum_{j=1}^{L} e_{m,j}^* = 1\end{aligned} \tag{5.6}$$

Solving for $\mathbf{E}_m^*$ from (5.5) gives the following relation between $\mathbf{E}_m^*$ and $\mathbf{Q}$:

$$e_{m,j}^* = \begin{cases} q_j\left(\frac{\sum_{i=1}^{m} n_i}{\sum_{i=1}^{m} q_i}\right) & \text{for } j = 1,2,\ldots,m\\ q_j\left(\frac{\sum_{i=m+1}^{L} n_i}{\sum_{i=m+1}^{L} q_i}\right) & \text{for } j = m+1,m+2,\ldots,L\end{cases} \tag{5.7}$$

let $sn(m) = \sum_{i=1}^{m} n_i$ and $sq(m) = \sum_{i=1}^{m} q_i$. The condition $D(\mathbf{E}_m^*||\mathbf{Q}) \geq \epsilon$ implies:

$$
\begin{aligned}
&\implies \sum_{j=1}^{L} \mathbf{E}_m^*(j) \log \frac{\mathbf{E}_m^*(j)}{\mathbf{Q}(j)} \geq \epsilon \\
&\implies \sum_{j=1}^{m} q_j \frac{sn(m)}{sq(m)} \left(\log \frac{sn(m)}{sq(m)} \right) + \sum_{j=m+1}^{L} q_j \left(\frac{1-sn(m)}{1-sq(m)} \right) \left(\log \frac{1-sn(m)}{1-sq(m)} \right) \geq \epsilon \\
&\implies sn(m) \log \frac{sn(m)}{sq(m)} + (1-sn(m)) \log \frac{1-sn(m)}{1-sq(m)} \geq \epsilon \\
&\implies \frac{sn(m)^{sn(m)} \left(1-sn(m)\right)^{1-sn(m)}}{sq(m)^{sn(m)} \left(1-sq(m)\right)^{1-sn(m)}} \geq 2^{\epsilon} \\
&\implies sq(m)^{sn(m)} \left(1-sq(m)\right)^{1-sn(m)} \leq 2^{-\epsilon} \left(sn(m)^{sn(m)} \left(1-sn(m)\right)^{1-sn(m)} \right)
\end{aligned}
\tag{5.8}
$$

Proof that equation (5.8) has exactly two roots

Let us assume, $x = sq(m)$ and $A = sn(m)$.

$$
\begin{aligned}
F &= x^A (1-x)^{1-A} \\
\frac{dF}{dx} &= A x^{A-1} (1-x)^{1-A} - x^A (1-A)(1-x)^{-A} \\
&= x^{A-1}(1-x)^{-A} \left[A(1-x) - (1-A)x \right] \\
&= \frac{x^{A-1}}{(1-x)^A} \left[A - x \right]
\end{aligned}
\tag{5.9}
$$

For $0 \leq x \leq 1, 0 \leq A \leq 1$, $\frac{dF}{dx} = 0$ has exactly one root at $x = A$. Thus $F = B$, where B is some constant has at most 2 real roots.

Choice of rates for SIC decoder is optimal

In chapter-3, we stated that the following choice of rates is optimal for GMAC and are easily decodable with a SIC decoder:

$$
r_i = \log_2 \left(1 + \frac{P}{(K-i)P + \sigma^2} \right) \qquad \text{for} \qquad i = 1, 2, \ldots, K \tag{5.10}
$$

In the following analysis we show that the sumrate obtained is equal to the capacity of GMAC. Let $A = \sigma^2/P$

$$\begin{aligned}
\sum_{i=1}^{K} r_i &= \sum_{i=1}^{K} \log_2 \left(1 + \frac{1}{(K-i)+A}\right) \\
&= \log_2 \prod_{i=1}^{K} \left(\frac{K-i+A+1}{K-i+A}\right) \\
&= \log_2 \left(\frac{K+A}{K+A-1}\right)\left(\frac{K+A-1}{K+A-2}\right)\cdots\left(\frac{A+1}{A}\right) \\
&= \log_2 \left(\frac{K+A}{A}\right) = \log_2 \left(1 + \frac{KP}{\sigma^2}\right)
\end{aligned}$$