FACETS FOR CONTINUOUS MULTI-MIXING SET AND ITS GENERALIZATIONS: STRONG CUTS FOR MULTI-MODULE CAPACITATED LOT-SIZING PROBLEM

A Dissertation

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ABSTRACT

The research objective of this dissertation is to develop new facet-defining valid inequalities for several new multi-parameter multi-constraint mixed integer sets. These valid inequalities result in cutting planes that significantly improve the efficiency of algorithms for solving mixed integer programming (MIP) problems involving multi-module capacity constraints. These MIPs arise in many classical and modern applications ranging from production planning to cloud computing. The research in this dissertation generalizes cut-generating methods such as mixed integer rounding (MIR), mixed MIR, continuous mixing, n-step MIR, mixed n-step MIR, migling, and n-step mingling, along with various well-known families of cuts for problems such as multi-module capacitated lot-sizing (MMLS), multi-module capacitated facility location (MMFL), and multi-module capacitated network design (MMND) problems.

More specifically, in the first step, we introduce a new generalization of the continuous mixing set, referred to as the continuous multi-mixing set, where the coefficients satisfy certain conditions. For each $n' \in \{1, \ldots, n\}$, we develop a class of valid inequalities for this set, referred to as the n'-step cycle inequalities, and present their facet-defining properties. We also present a compact extended formulation for this set and an exact separation algorithm to separate over the set of all n'-step cycle inequalities for a given $n' \in \{1, \ldots, n\}$.

In the next step, we extend the results of the first step to the case where conditions on the coefficients of the continuous multi-mixing set are relaxed. This leads to an extended formulation and a generalization of the n-step cycle inequalities, $n \in \mathbb{N}$, for the continuous multi-mixing set with general coefficients. We also show that these inequalities are facet-defining in many cases.

In the third step, we further generalize the continuous multi-mixing set (where no conditions are imposed on the coefficients) by incorporating upper bounds on the integer variables. We introduce a compact extended formulation and new families of multi-row cuts for this set, referred to as the mingled n-step cycle inequalities ($n \in \mathbb{N}$), through a generalization of the n-step mingling. We also provide an exact separation algorithm to separate over a set of all these inequalities. Furthermore, we present the conditions under which a subset of the mingled n-step cycle inequalities are facet-defining for this set.

Finally, in the fourth step, we utilize the results of first step to introduce new families of valid inequalities for MMLS, MMFL, and MMND problems. Our computational results show that the developed cuts are very effective in solving the MMLS instances with two capacity modules, resulting in considerable reduction in the integrality gap, the number of nodes, and total solution time.

DEDICATION

To My Teachers

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CHAPTER I

INTRODUCTION

Mixed integer programming (MIP) is a major optimization technique to solve a wide variety of real-world problems involving decisions of discrete nature [81, 111]. In general, MIPs are NP-hard to solve [43]. The branch-and-cut algorithm [83] is among the most successful algorithms used to solve MIPs. Branch-and-cut is a branch-and-bound algorithm [67, 81] in which cutting planes are used to tighten the formulations of node problems and hence achieve better bounds (refer to Section II.1.2 for details). As a result, developing strong valid inequalities as cutting planes is crucial for effectiveness of the branch-and-cut algorithm. To this end, studying the polyhedral structure of mixed integer "base" sets which constitute well-structured relaxations of important MIP problems is a promising approach. This is because oftentimes one can develop procedures in which the valid inequalities (or facets) developed for the base set are used to generate valid inequalities (or facets) for the original MIPs (see [6, 16, 15, 14, 36, 51, 62, 96, 111] for a few examples among many others). *Mixed integer rounding* (MIR) [82, 111] is one of the most basic procedures for deriving cuts for MIPs which utilizes the facet of a single-constraint two-variable mixed integer base set. Several important generalizations of MIR (shown in Fig. 1), including mixed MIR [51], continuous mixing [105], n-step MIR [62], mingling [6], mixed n-step MIR [96], and n-step mingling [7], are derived by studying the polyhedral structure of more complex mixed integer base sets (see Sections I.1, I.2, and I.3 for details).

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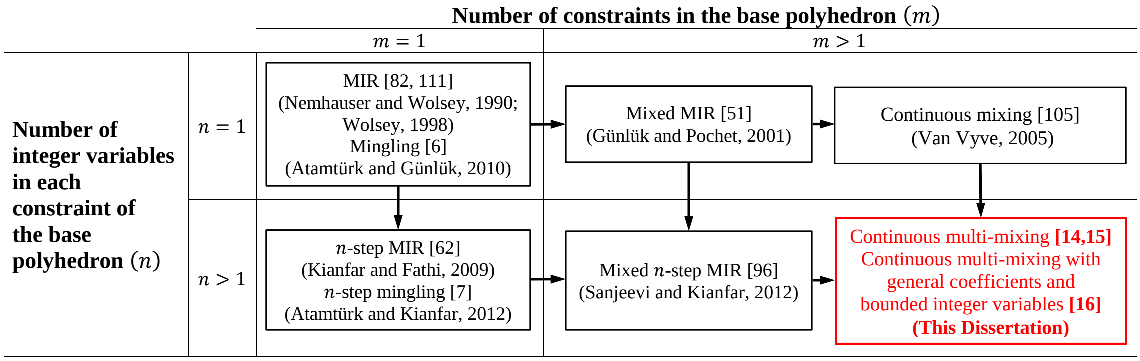

Figure 1: Generalizations of Mixed Integer Rounding (MIR)

Many well-known families of valid inequalities developed for MIP problems such as knapsack set, lot-sizing (production planning), facility location, and network design, are (or can be) derived using MIR and its aforementioned generalizations (see Table 1 for details).

As shown in Figure 1, in this dissertation, we generalize the aforementioned cut-generating procedures by developing facet-defining valid inequalities for the following generalizations of the well-studied continuous mixing set [105] (a single-parameter multi-constraint mixed integer set): (1) *Continuous multi-mixing set* (a multi-parameter multi-constraint mixed integer set) with certain conditions on the coefficients, (2) Continuous multi-mixing set with *general coefficients*, and (3) Continuous multi-mixing set with general coefficients and *bounded integer variables*. We also present compact extended formulations for these sets and an exact separation algorithm to separate over each family of valid inequalities developed for these sets (see Sections I.1, I.2, and I.3 for details). These results provide a knowledge base for developing new families of cutting planes for MIP problems involving "*multi-modularity capacity constraints*" (MMCCs).

Existence of multiple modularities (module sizes) of (production/service/process-

Problem type	**Inequalities in literature**	**Are/can be developed by**
Knapsack Set	Continuous cover [73]	2-step mingling
	Cover and pack [10, 11]	2-step mingling
	n-step mingling [6, 7]	n-step mingling
Lot-Sizing	(k, l, S, I) [87]	Mixing
	Mixed (k, l, S, I) [51]	Mixed MIR
	Multi-module (k, l, S, I) [96]	Mixed n-step MIR
Facility Location	Flow cover [84]	MIR
	Arc residual [68]	MIR
	(k, l, S, I) [2, 3, 1]	Mixed MIR
	Mixed (k, l, S, I) [51]	Mixed MIR
	Multi-module (k, l, S, I)	Mixed n-step MIR
Network Design	(2-Modularity) cut-set [70]	(2-step) MIR
	Flow cut-set [19]	MIR
	Cut-set [9]	MIR
	Mixed partition [52]	Mixed MIR
	Partition [89]	n-step MIR

Table 1: Relation between known inequalities and procedures in literature

ing/transmission/transportation/storage/power generation) capacity is inherent to many classical and modern applications. One can easily find evidence of this fact in the literature of applications such as data centers [58, 97, 110, 114], cloud computing [27, 47, 55], (survivable fiber-optic) communication networks [8, 17, 18, 19, 20, 32, 48, 49, 50, 52, 72, 115], batteries for electric vehicles/wind turbines/solar panels [23, 38, 46, 64, 101], semiconductor manufacturing [44, 53, 54, 60, 91], power/energy/smart grid systems [40, 57, 86, 104, 117], on-shore and off-shore construction in oil industry [41, 80], offshore natural gas/oil pipeline systems [22, 69, 93, 94], pharmaceutical manufacturing facilities [98, 102, 103], regional wastewater treatment systems [56], chemical processes [95], bioreactors [109], transportation systems [4, 42, 65, 66, 76, 85, 107, 108], and production systems [90]. Nevertheless, the MIP cutting plane literature to date has almost entirely focused on problems with single-modularity capacity constraints. We introduce new classes of multi-row cuts for the MIP problems with MMCCs, in particular multi-module capacitated lot-sizing (MMLS), multi-module capacitated facility location (MMFL), and multi-module capacitated network design (MMND). These inequalities generalize various well-known families of cuts (mentioned in Table 1) for MMLS, MMFL, and MMND problems. Our computational results show that these cutting planes significantly improve the efficiency of algorithms for solving the MMLS problem with(out) backlogging. See Section I.4 for details. In the following sections, we present brief summary of our research contribution.

I.1 Continuous Multi-Mixing Set

A well-known mixed integer base set is the continuous mixing set

$$Q := \{(y, v, s) \in \mathbb{Z}^m \times \mathbb{R}_+^{m+1} : y^i + v_i + s \geq \beta_i, i = 1, \dots, m\},$$

where $\beta_i \in \mathbb{R}, i = 1, \dots, m$ [105]. This set is a generalization of the well-studied mixing set $\{(y, s) \in \mathbb{Z}^m \times \mathbb{R}_+ : y^i + s \geq \beta_i, i = 1, \dots, m\}$ [51], which itself is a multi-constraint generalization of the base set $\{(y, s) \in \mathbb{Z} \times \mathbb{R}_+ : y + s \geq \beta\}$ that leads to the well-known mixed integer rounding (MIR) inequality (page 127 of [111]). In all these base sets each constraint has only one integer variable. Fig. 1 presents a summary of the generalization relationship between these base sets and other base sets of interest in this dissertation. The set Q arises as a substructure in relaxations of problems such as lot-sizing (production planning) with backlogging [78], lot-sizing with stochastic demand [5], capacitated facility location [2], and capacitated network design [50]. Miller and Wolsey [77] presented an extended formulation for $conv(Q)$ with $O(m^2)$ variables and $O(m^2)$ constraints. Later, Van Vyve [105] gave a compact and tight extended formulations with $O(m)$ variables and $O(m^2)$ constraints for

$conv(Q)$ and its relaxation to the case where $s \in \mathbb{R}$. He also introduced the so-called cycle inequalities (called *1-step* cycle inequalities in this dissertation) for these sets and showed that these inequalities along with bound constraints are sufficient to describe the convex hulls of these sets. The MIR inequalities (called *1-step* MIR inequalities in this dissertation) of Nemhauser and Wolsey [82, 111] and the mixed (1-step) MIR inequalities of Günlük and Pochet [51] are special cases of the 1-step cycle inequalities for Q (Fig. 1). It is important to note that the 1-step MIR cuts are equivalent to split cuts of Cook et al. [31] and Gomory mixed integer cuts [92], and are a special case of the disjunctive cuts [12, 13] (also see [21, 37]). Zhao and Farias [116] showed that the optimization over the relaxation of Q in which $s \in \mathbb{R}$ can be performed in $O(m \log m)$ time. Furthermore, Conforti et al. [30] studied two generalizations of Q: first, the intersection of several continuous mixing sets with distinct s variables and common y and v variables, and second, the continuous mixing set with flows. They introduced two extended formulations for the convex hull of each of these sets.

In another direction (Fig. 1), Kianfar and Fathi [62] generalized the 1-step MIR inequalities [82] and developed the n-step MIR inequalities for the mixed integer knapsack set by studying the base set

$$Q_0^{1,n} = \Big\{(y, s) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+ : \sum_{t=1}^{n} \alpha_t y_t + s \geq \beta\Big\},$$

where $\alpha_t \in \mathbb{R}_+\backslash\{0\}$, $t = 1, \ldots, n$ and $\beta \in \mathbb{R}$. Note that this base set has a single constraint and n integer variables in this constraint. The n-step MIR inequalities are valid and facet-defining for the base set $Q_0^{1,n}$ if α_t's and β satisfy the so-called

n-step MIR conditions, i.e.

$$\alpha_t \left\lceil \beta^{(t-1)} / \alpha_t \right\rceil \leq \alpha_{t-1}, \quad t = 2, \ldots, n. \tag{1}$$

However, n-step MIR inequalities can also be generated for a mixed integer constraint with no conditions imposed on the coefficients. In that case, the external parameters used in generating the inequality are picked such that they satisfy the n-step MIR conditions (see [62] for more details). The n-step MIR inequalities are facet-defining for the mixed integer knapsack set in many cases [7, 62]. The Gomory mixed integer cut [92] and the 2-step MIR inequalities [35, 36] are the special cases of n-step MIR inequalities, corresponding to $n = 1, 2$, respectively. Kianfar and Fathi [62, 63] showed that the n-step MIR inequalities define new families of facets for the finite and infinite group problems.

Recently, Sanjeevi and Kianfar [96] showed that the procedure proposed by Günlük and Pochet [51] to mix 1-step MIR inequalities can be generalized and used to mix the n-step MIR inequalities [62] (Fig. 1). As a result, they developed the mixed n-step MIR inequalities for a generalization of the mixing set called the n-mixing set, i.e.

$$Q_0^{m,n} = \left\{ (y, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+ : \sum_{t=1}^{n} \alpha_t y_t^i + s \geq \beta_i, i = 1, \ldots, m \right\},$$

where $\alpha_t \in \mathbb{R}_+ \backslash \{0\}$, $t = 1, \ldots, n$, and $\beta_i \in \mathbb{R}$, $i = 1, \ldots, m$, such that α_t and β_i satisfy the n-step MIR conditions in each constraint. Note that this is a multi-constraint base set with n integer variables in each constraint and a continuous variable which is common among all constraints. The mixed n-step MIR inequalities are valid for $Q_0^{m,n}$ and under certain conditions, these inequalities are also facet defining for the convex hull of $Q_0^{m,n}$.

In the first step of this dissertation, we generalize the concepts of continuous mixing [105] and mixed n-step MIR [96] by introducing a more general base set referred to as the *continuous multi-mixing set* which we define as

$$Q^{m,n} := \Big\{(y, v, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1} : \sum\nolimits_{t=1}^{n} \alpha_t y_t^i + v_i + s \geq \beta_i, i = 1, \ldots, m\Big\},$$

where $\alpha_t > 0$, $t = 1, \ldots, n$ and $\beta_i \in \mathbb{R}$, $i = 1, \ldots, m$ such that α_t and β_i satisfy the n-step MIR conditions (which are automatically satisfied if the parameters $\alpha_1, \ldots, \alpha_n$ are divisible) in each constraint (see Fig. 1). Note that this set has multiple (m) constraints with multiple (n) integer variables in each constraint; but it is more general than the n-mixing set because in addition to the common continuous variable s, each constraint has a continuous variable v_i of its own. The continuous mixing set Q is the special case of $Q^{m,n}$, where $n = 1$ and $\alpha_1 = 1$, and the n-mixing set of Sanjeevi and Kianfar [96] is the projection of $Q^{m,n} \cap \{v = 0\}$ on (y, s). The continuous multi-mixing set arises as a substructure in relaxations of multi-module capacitate lot-sizing (MMLS) with(out) backlogging, MMLS with stochastic demand, multi-module capacitated facility location (MMFL), and multi-module capacitated network design (MMND) problems (we will describe these problems in Section I.4). For each $n' \in \{1, \ldots, n\}$, we develop a class of valid inequalities for $Q^{m,n}$ which we refer to as n'-step cycle inequalities, and obtain conditions under which these inequalities are facet-defining for $conv(Q^{m,n})$. We discuss how the n-step MIR inequalities [62] and the mixed n-step MIR inequalities [96] are special cases of the n-step cycle inequalities. We also introduce a compact extended formulation for $Q^{m,n}$ and an efficient exact separation algorithm to separate over the set of all n'-step cycle inequalities, $n' \in \{1, \ldots, n\}$, for set $Q^{m,n}$.

I.2 Continuous Multi-Mixing Set with General Coefficients

In the next step, we relax the n-step MIR conditions on the coefficients of $Q^{m,n}$ and consider the continuous multi-mixing set with general coefficients, denoted by

$$Y^m := \left\{(y, v, s) \in \mathbb{Z}_+^{m\times N} \times \mathbb{R}_+^m \times \mathbb{R}_+ : \sum_{t=1}^{N} a_{it} y_t^i + v_i + s \geq b_i, i = 1, \ldots, m\right\}$$

where $a \in \mathbb{R}^{mN}$ and $b \in \mathbb{R}^m$. As mentioned before, Kianfar and Fathi [62] showed that, for each $n \in \mathbb{N}$, the n-step MIR facet of $Q_0^{1,n}$ can be used to generate a family of valid inequalities for the mixed integer knapsack set which is same as $Proj_{y,s}(Y^1 \cap \{v = 0\})$. Later Atamtürk and Kianfar [7] showed that these inequalities define facets for this set under certain conditions. In this dissertation, we generalize the n-step cycle inequalities to develop valid inequalities for Y^m and show that they are facet-defining for $conv(Y^m)$ in many cases.

I.3 Continuous Multi-Mixing Set with Bounded Integer Variables

Despite the effectiveness of MIR inequalities to solve MIPs with unbounded integer variables, cutting planes based on lifting techniques appear to be more effective for MIPs with bounded integer variables [6, 74]. This is because, unlike lifting techniques, the MIR procedure does not explicitly use bounds on integer variables. To overcome this drawback, Atamtürk and Günlük [6] introduced a simple procedure (called "*mingling*") which incorporates the variable bound information into MIR and gives stronger valid inequalities. They first developed the so-called mingling (and 2-step mingling) inequalities for the mixed integer knapsack set and then showed that the facets of this set derived earlier by superadditive lifting techniques are special cases of mingling or 2-step mingling inequalities. In particular, these inequalities subsume the continuous cover and reverse continuous cover inequalities of Marchand

and Wolsey [73] as well as the continuous integer knapsack cover and pack inequalities of Atamtürk [10, 11]. Recently, Atamtürk and Kianfar [7] generalized the mingling procedure of Atamtürk and Günlük [6] and introduced a variant of the n-step MIR inequalities [62] (which they call n-step mingling inequalities) for the mixed-integer knapsack set with bounded integer variables. Unlike n-step MIR inequalities, the n-step mingling inequalities utilize the information of bounds on integer variables to give stronger valid inequalities, which are facet-defining in many cases [7]. In addition, they used n-step mingling inequalities to develop new valid inequalities and facets based on covers and packs defined for mixed integer knapsack sets.

The third step of this dissertation is to unify the concepts of continuous multi-mixing and n-step mingling by incorporating upper bounds on the integer variables of the continuous multi-mixing set (where no conditions are imposed on the coefficients) and developing new families of valid inequalities for this set (which we refer to as the mingled n-step cycle inequalities). We denote this new generalization of continuous multi-mixing set by

$$Z^m := \Big\{ (y, v, s) \in \mathbb{Z}_+^{m\times N} \times \mathbb{R}_+^m \times \mathbb{R}_+ : \\ \sum_{t\in T} a_t y_t^i + \sum_{k\in K} a_k y_k^i + v_i + s \geq b_i, y^i \leq u^i, i = 1, \ldots, m \Big\}$$

where (T, K) is a partitioning of $\{1, \ldots, N\}$ with $a_t > 0$ for $t \in T$, $a_k < 0$ for $k \in K$, and $u^i \in \mathbb{Z}_+^N$ for $i \in \{1, \ldots, m\}$. We develop a compact extended formulation for Z^m and provide a separation algorithm to separate over the set of all mingled n-step cycle inequalities for a given $n \in \mathbb{N}$. Furthermore, we obtain the conditions under which a special case of mingled n-step cycle inequalities (referred to as the mingled n-step mixing inequalities) are facet-defining for $conv(Z^m)$.

I.4 Cuts for MMLS, MMFL, and MMND Problems

The objective of this step of dissertation is to utilize the n-step cycle inequalities to develop a new family of valid inequalities for MIPs involving "*multi-modularity capacity constraints*". In particular, we focus on the multi-modularity generalizations (where capacity can be composed of discrete units of multiple differentially-sized modularities) of three following high-impact classes of capacitated MIPs: lot-sizing (LS), facility location (FL), and network design (ND) problems. Over the years a large volume of the MIP cutting plane research has been dedicated to single modularity or constant-capacity versions of the LS [78, 87, 88, 90, 106, 112], FL [1, 2, 51], and ND [9, 26, 51, 70, 71] problems.

Recently, Sanjeevi and Kianfar [96] generalized the lot-sizing problem with constant batches [87] (where the capacity in each period can be some integer multiple of a single capacity module with a given size) and introduced the multi-module capacitated lot-sizing (MMLS) problem. In this problem, the total production capacity in each period can be the summation of some integer multiples of several capacity modules of different sizes. They showed that the mixed n-step MIR inequalities can be used to generate valid inequalities for the MMLS problem *without backlogging* (which we denote by MML-WB). They referred to these inequalities as the multi-module (k, l, S, I) inequalities. These inequalities generalize the (k, l, S, I) inequalities and mixed MIR inequalities which were introduced for the lot-sizing problem with constant batches by Pochet and Wolsey [87] and Günlük and Pochet [51], respectively. Similarly, they introduced multi-module capacitated facility location (MMFL) problem (a generalization of the capacitated facility location problem) and used mixed n-step MIR inequalities to develop valid inequalities for this problem. These inequalities generalize the mixed MIR [51] and (k, l, S, I) based [2, 3] inequalities for constant capacity facility location problem.

In literature, the cutting planes have been derived for multi-module capacitated network design (MMND) problem and its special cases [9, 19, 52, 61, 70, 72, 89]. Interestingly, the cuts developed in [19, 70, 72] for two-modularity ND with divisible capacities (2MND-DC) and in [9] for MMND can be derived just using 1-step MIR procedure. The fact that the problem is multi-modularity, is not used in developing potentially many more classes of cuts. The same is true for the mixed partition inequalities for 2MND-DC [52], which can be derived just using mixed MIR procedure. To our knowledge, the only classes of cuts derived by actually exploiting the existence of multiple modularities are the two-modularity cut-set inequalities for 3MND-DC [70] (which do not exploit the third modularity) and the partition inequalities for the single-arc MMND-DC [89]. The former can be derived using the 2-step MIR [36, 62], and the n-step MIR not only generates the latter but also generalizes them to non-divisible capacities [61].

In this dissertation, we introduce MMLS *with backlogging* (MML-B) and use n-step cycle inequalities to develop a new family of cutting planes for MML-(W)B, MMFL, and MMND problems which subsume valid inequalities introduced in [51, 87, 96] for LS problems, [2, 51, 96] for FL problems, and [9, 19, 51, 52, 61, 70, 72, 89] for ND problems, respectively. We also computationally evaluate the effectiveness of the n-step cycle inequalities for the MML-(W)B problem using our separation algorithm.

I.4.1 Computational Results

Our computational results on applying 2-step cycle inequalities using our separation algorithm show that our cuts are very effective in solving MML-WB and MML-B with two capacity modules, resulting in considerable reduction in the integrality gap (on average 85.90% for MML-WB and 86.32% for MML-B) and the number of nodes (on average 132 times for MML-WB and 31 times for MML-B). Also, the total time taken to solve an instance (which also includes the cut generation time) is in average 58.3 times (for MML-WB) and 9.9 times (for MML-B) smaller than the time taken by CPLEX with default settings (except for very easy instances). More interestingly, in these instances adding cuts by applying 2-step cycle inequalities over 1-step cycle inequalities has improved the closed gap (on average 19.47% for MML-WB and 15.96% for MML-B), the number of nodes (on average 43 times for MML-WB and 14 times for MML-B), and the total solution time (on average 18 times for MML-WB and 4 times for MML-B).

I.5 Dissertation Structure

The dissertation is organized as follows: In Chapter II, we present a brief introduction to mixed integer programming and review some fundamental definitions, concepts, and theorems in MIP and polyhedra to the extent required as background for the results in this dissertation. We present our research on continuous multi-mixing set, continuous multi-mixing set with general coefficients, continuous multi-mixing set with bounded integer variables, and cuts for MMLS, MMFL, and MMND problems in Chapters III, IV, V, and VI, respectively. We provide a conclusion in Chapter VII along with some future research plans.

CHAPTER II

MIXED INTEGER PROGRAMMING, POLYHEDRAL THEORY, AND GENERALIZATIONS OF MIXED INTEGER ROUNDING

This chapter presents an introduction to mixed integer programming and a theory of valid inequalities for mixed integer linear sets to the extent required as background for the results in this dissertation. In Section II.1, we define general (mixed) integer program, briefly discuss their importance and applications, and review three algorithms used to solve them (i.e. branch-and-bound, cutting plane, and branch-and-cut algorithms). We also reproduce the concept of extended formulation along with some fundamental definitions and theorems in polyhedral theory. In Section II.2, we review the MIR cut-generating procedure [81, 111] and its various generalizations (in particular, continuous mixing [105], n-step MIR [62], mixed n-step MIR [96], and n-step mingling [6, 7]).

II.1 Mixed Integer Programming

Mixed Integer Programming is a powerful method to formulate and solve optimization problems containing discrete decision variables with numerous applications in business, science, and engineering. In general, MIPs are NP-hard problems. Therefore, it is challenging to improve the existing algorithms (or develop new efficient algorithms) for solving MIP problems arising in applications such as production and distribution planning, facility location, telecommunication, transportation, airline crew scheduling, electricity generation planning, molecular biology, VLSI, and many more [81, 111].

A mixed integer program (MIP) can be written as

$$\min cv + hy$$
$$Av + Gy \leq b$$
$$y \in \mathbb{Z}^n, v \in \mathbb{R}^p$$

where A is an m by n matrix, G is an m by p matrix, c and h are row-vectors of dimensions n and p, respectively, and v, y are the decision variables. In this formulation, if $p = 0$, i.e. all variables are integer, we get the pure integer program

$$\min\{hy : Gy \leq b, y \in \mathbb{Z}^n\}$$

and if all variables are binary, we have the binary integer program

$$\min\{hy : Gy \leq b, y \in \{0, 1\}^n\}.$$

Furthermore, the linear problem obtained by dropping the integrality restrictions on decision variables of a MIP is called the linear relaxation of the MIP.

II.1.1 Some Definitions and Theoretical Results in Polyhedral Theory

In this section, some definitions and fundamental theoretical results in polyhedral theory are replicated from [81, 111] to the extent required to present our research results. We also define the concepts of extended formulation and projection (see [28, 29, 34, 113] for more details).

Definition 1. *The feasible region of a MIP (denoted by $P_{MIP} \subseteq \mathbb{Z}^n \times \mathbb{R}^p$) is the set* *of points $(y, v) \in \mathbb{Z}^n \times \mathbb{R}^p$ which satisfy its constraints:*

$$P_{MIP} := \{(y, v) \in \mathbb{Z}^n \times \mathbb{R}^p : Av + Gy \geq b\}.$$

Definition 2. *A subset of $\mathbb{R}^p$ described by a finite set of linear constraints $P = \{v \in \mathbb{R}^p : Av \geq b\}$ is a **polyhedron**.*

Definition 3. *Given a set $X \subseteq \mathbb{R}^n$, the convex hull of X, denoted $conv(X)$, is defined as: $conv(X) = \{x : x = \sum_{i=1}^{t} \lambda_i x^i,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_i \geq 0$ for $i = 1, \ldots, t$ over all finite subsets $\{x^1, \ldots, x^t\}$ of $X\}$.*

Theorem 1. *$conv(P_{MIP})$ is a polyhedron, if the data A, G, b is rational.*

The proof of Theorem 1 is provided in [81].

Definition 4. *An inequality $\pi x \leq \pi_0$ is a **valid inequality** for $X \subseteq \mathbb{R}^n$ if $\pi x \leq \pi_0$ for all $x \in X$.*

Theorem 2. *[81] If $\pi x \leq \pi_0$ is valid for $X \subseteq \mathbb{R}^n$, it is also valid for $conv(X)$.*

Definition 5. *If $\pi x \leq \pi_0$ and $\mu x \leq \mu_0$ are two valid inequalities for $P \subseteq \mathbb{R}^n_+$, $\pi x \leq \pi_0$ dominates $\mu x \leq \mu_0$ if there exists $u > 0$ such that $\pi \geq u\mu$ and $\pi_0 \leq u\mu_0$ and $(\pi, \pi_0) \neq (u\mu, u\mu_0)$.*

Observation 1. *If $\pi x \leq \pi_0$ dominates $\mu x \leq \mu_0$, then $\{x \in \mathbb{R}^n_+ : \pi x \leq \pi_0\} \subseteq \{x \in \mathbb{R}^n_+ : \mu x \leq \mu_0\}$.*

Definition 6. *The points $x^1, \ldots, x^k \in \mathbb{R}^n$ are affinely independent if the $k-1$ directions $x^2 - x^1, \ldots, x^k - x^1$ are linearly independent, or alternatively the k vectors $(x^1, 1), \ldots, (x^k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.*

Definition 7. *The **dimension** of P, denoted $dim(P)$, is one less than the maximum number of affinely independent points in P.*

Definition 8. *F defines a **face** of the polyhedron P if $F = \{x \in P : \pi x = \pi_0\}$ for some valid inequality $\pi x \geq \pi_0$ of P.*

Definition 9. *F is a **facet** of P if F is a face of P and $dim(F) = dim(P) - 1$.*

Definition 10. *If F is a face of P with $F = \{x \in P : \pi x = \pi_0\}$, the valid inequality $\pi_x \geq \pi_0$ is said to **represent** or **define** the face.*

Definition 11. *Given a polyhedron $P \subseteq (\mathbb{R}^n \times \mathbb{R}^p)$, the **projection** of P onto the space $\mathbb{R}^n$, denoted by $Proj_x(P)$, is defined as*

$$Proj_x(P) := \{x \in \mathbb{R}^n : (x, w) \in P \text{ for some } w \in \mathbb{R}^p\}.$$

Definition 12. *Given a set $X \subseteq \mathbb{R}^n$ and a polyhedron $P := \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Bw \leq b\}$ such that $conv(X) \subseteq Proj_x(P)$, the system $Ax + Bw \leq b$ provides an **extended formulation** for the set X.*

i) *In case $Proj_x(P) = conv(X)$, we call the extended formulation is **tight**.*

ii) *An extended formulation is **compact** if the addition of polynomial number of extra variables results in a formulation with a polynomial number of inequalities.*

II.1.2 Algorithms for Solving MIP Problems

Branch-and-cut algorithm is among the most successful algorithms used to solve MIPs. Branch-and-cut is a branch-and-bound algorithm in which cutting planes are used to tighten the formulations of node problems and hence achieve better bounds. This algorithm was first introduced by Padberg and Rinaldi [83], and today most of the commercial and non-commercial MIP solvers use it. This is because it combines the advantages of both branch-and-bound and cutting plane algorithms, and hence overcomes the drawbacks associated with each of those algorithms.

Branch-and-bound (BB) was first proposed by Land and Doig [67] for integer programming. The idea behind the BB algorithm for a maximization problem is as follows: The algorithm starts at the root node. The BB is done over a BB tree. Each node in the tree corresponds to a subset of the solution space. At each node, the upper bound for the best solution value obtainable in the solution space corresponding to the node is calculated. This is done by solving the linear relaxation (or any other easily solvable relaxation) of the MIP. Based on the upper bound at the node and best known feasible solution value (i.e. best lower bound of the problem), the node is either pruned or branched. A node can be pruned for two reasons: 1) if the upper bound value on that node is smaller than the best feasible solution value found so far. In this case there is no point in searching the node for optimal solution anymore (this is the main idea behind BB). 2) if a solution is found, the lower bound will be updated if this solution has a larger objective value. On the other hand, if a node cannot be pruned, the solution space of the node is subdivided into two or more subspaces (by generating child nodes). This action is known as branching. There are different problem dependent strategies for choosing the branching scheme in a node and also for choosing the next node in the tree. While solving the MIP, one commonly used branching strategy at a given node is to create two child nodes by adding the constraint ($y_i \leq \lfloor y_i^* \rfloor$ for first node and $y_i \geq \lceil y_i^* \rceil$ for second node, where y_i is an integer variable with the fractional LP solution y_i^*) to the linear relaxation at this node. The problem is solved when all nodes are pruned and the best lower bound will be the optimal value. The efficiency of the method depends strongly on the branching (node-splitting procedure) and on the upper and lower bound estimators. In order to solve minimization problem using BB, interchange the lower bound and upper bound in the description above. More details and references can be found in [81, 111].

Gomory [45, 92] presented the cutting plane algorithm to solve (M)IPs. In [45], he showed how a modified version of the simplex algorithm provides a finite algorithm to solve pure integer programs. This algorithm utilizes valid inequalities (referred to as the *cuts* or *cutting planes*) that are violated by the optimal solution of the current linear program, but satisfy all integral solutions. The algorithm in [92] is an extension of the cutting plane algorithm for pure integer programs [45] to MIPs. The basic idea behind this algorithm is as follows: Given a MIP, we solve its LP relaxation (LPR), generate a "strong" cut that is violated by the optimal solution of LPR (in case it does not satisfy integrality constraints), and add the cut to the LPR which tighten its feasible region without changing the feasible region of MIP. Then we re-solve LPR and repeat the procedure until all integer constraints are satisfied. Note that a cutting plane is called "stronger" than others if it cuts off bigger portion from the feasible region of the LPR, in comparison to others. Therefore, *facets* of the convex hull of integer solutions are the strongest possible cuts. The major advantage of this algorithm is that it can solve a pure integer program to optimality in finite number of steps. Despite that this approach on its own is not very effective in practice because of the so-called *tailing-off phenomenon* [24], i.e. after some steps the portion cuts off from the feasible region of the LPR by each cut becomes very small.

In branch-and-cut algorithm, the cutting planes are utilized to provide a tighter formulation of node problems and whenever the tailing-off begins (due to the addition of cutting planes) branching is used to create new nodes (see [39, 59, 75, 79] for surveys on different aspects of branch-and-cut algorithm). As a result, developing strong valid inequalities as cutting planes is crucial for effectiveness of the branch-and-cut algorithm. This fact is the major motivation for the research in the area of cutting planes.

II.2 Generalizations of Mixed Integer Rounding

Studying the polyhedral structure of mixed integer base sets which constitute well-structured relaxations of important MIP problems is a promising approach. This is because oftentimes one can develop procedures in which the valid inequalities (or facets) developed for the base set are used to generate valid inequalities (or facets) for the original MIPs (see [6, 36, 51, 62, 96, 111] for a few examples among many others). In this section, we briefly review the mixed integer rounding (MIR) cut-generating procedure [81, 111] and its various generalizations (in particular, continuous mixing [105], n-step MIR [62], mixed n-step MIR [96], and n-step mingling [6, 7]).

II.2.1 Mixed Integer Rounding (MIR)

One fundamental procedure to develop cuts for general MIPs is the *MIR procedure* [82, 111] which utilizes the facet of a single-constraint mixed integer base set,

$$Q_0^{1,1} := \{(y, s) \in \mathbb{Z} \times \mathbb{R}_+ : \alpha_1 y + s \geq \beta\}$$

where $\alpha_1 > 0$ and $\beta \in \mathbb{R}$, referred to as the (1-step) MIR facet (page 127 of [111]). It is interesting to note that all the facets of a general 0-1 MIP can be generated using MIR [82] and for general MIP, MIR can be used to obtain strong valid inequalities based on 1-row relaxations [74]. Furthermore, the 1-step MIR cuts are equivalent to split cuts of Cook et al. [31] and Gomory mixed integer cuts [92], and are a special case of the disjunctive cuts [12, 13] (also see [21, 37]). Because of computational effectivenes, the MIR procedure is being used in many MIP solvers today.

Theorem 3. *[111] The inequality (1-step MIR facet)*

$$y_1 + \frac{v}{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor} \geq \left\lceil \frac{\beta}{\alpha_1} \right\rceil, \tag{2}$$

is valid and facet-defining for $conv(Q_0^{1,1})$.

In a general setting, the 1-step MIR facet (2) for $conv(Q_0^{1,1})$ can be used to generate strong valid inequalities for a single-constraint mixed integer knapsack set with general coefficients. We define this set as follows:

$$Y_0^1 := \{(y,s) \in \mathbb{Z}_+^N \times \mathbb{R}_+ : \sum_{t=1}^{N} a_t y_t + s \geq b\}$$

where the coefficients $a_t, t = 1, \ldots, N$ and b are real numbers (no conditions imposed on them). Note that $Y_0^1 = \mathrm{Proj}_{y,s}(Y^1 \cap \{v = 0\})$. By choosing a parameter $\alpha_1 > 0$ such that $b^{(1)} = b - \alpha_1 \lfloor b/\alpha_1 \rfloor > 0$, the defining inequality of Y_0^1 can be relaxed to

$$\sum_{t \in J_0} \alpha_1 \left\lceil \frac{a_t}{\alpha_1} \right\rceil y_t + \sum_{t \in J_1} \left(\left\lfloor \frac{a_j}{\alpha_1} \right\rfloor + a_j^{(1)} \right) y_j + s \geq b \tag{3}$$

by partitioning $\{1, \ldots, N\}$ into two disjoint subsets J_0, J_1, relaxing a_t in the defining inequality of Y_0^1 to $\alpha_1 \lceil a_t/\alpha_1 \rceil (\geq a_t)$ for $t \in J_0$, and replacing a_t in the defining inequality of Y_0^1 by $\lfloor a_j/\alpha_1 \rfloor + a_j^{(1)} (= a_t)$ for $t \in J_1$. This is a relaxation because $y_t \geq 0$, $t \in J_0$. Observe that the terms in inequality (3) can be rearranged to have a structure similar to the defining inequality of $Q_0^{1,1}$, i.e. inequality (3) can be written as

$$\alpha_1 \left(\sum_{t \in J_0} \left\lceil \frac{a_t}{\alpha_1} \right\rceil y_t + \sum_{t \in J_1} \left\lfloor \frac{a_t}{\alpha_1} \right\rfloor y_t \right) + \left(\sum_{t \in J_1} a_t^{(1)} y_t + s \right) \geq b. \tag{4}$$

Setting

$$y := \sum_{t \in J_0} \left\lceil \frac{a_t}{\alpha_1} \right\rceil y_t + \sum_{t \in J_1} \left\lfloor \frac{a_t}{\alpha_1} \right\rfloor y_t \text{ and } \bar{s} := \sum_{t \in J_1} a_t^{(1)} y_t + s, \tag{5}$$

inequality (4) becomes of the same form as the defining inequality of $Q_0^{1,1}$ (notice that $\bar{s} \in \mathbb{R}_+$ and $y \in \mathbb{Z}$). Therefore the MIR inequality for (4), given by

$$b^{(1)} \left(\sum_{t \in J_0} \left\lceil \frac{a_t}{\alpha_1} \right\rceil y_t + \sum_{t \in J_1} \left\lfloor \frac{a_t}{\alpha_1} \right\rfloor y_t \right) + \left(\sum_{t \in J_1} a_t^{(1)} y_t + s \right) \geq b^{(1)} \left\lceil \frac{b}{\alpha_1} \right\rceil, \tag{6}$$

is valid for Y_0^1. Interestingly, inequality (6) becomes the Gomory Mixed Integer (GMI) cut [92] when $\alpha_1 = 1$. In a compact form, the MIR inequality (6) for Y_0^1 can be written as follows:

$$\sum_{t=1}^{N} \mu_{\alpha_1,b}^1(a_t) y_t + s \geq \mu_{\alpha_1,b}^1(b), \tag{7}$$

where $\mu_{\alpha_1,b}^1 = b^{(1)} \lfloor t/\alpha_1 \rfloor + min\{b^{(1)}, t^{(1)}\}$ is referred to as the 1-step MIR function.

II.2.2 Continuous Mixing

Van Vyve [105] generated the cycle inequalities for the continuous mixing set Q as follows: Define $\beta_0 := 0$, $f_i := \beta_i - \lfloor \beta_i \rfloor, i \in \{0, \ldots, m\}$ and without loss of generality assume that $f_{i-1} \leq f_i, i = 1, \ldots, m$. Let $G := (V, A)$ be a directed graph, where $V := \{0, 1, \ldots, m\}$ and $A := \{(i,j) : i,j \in V, f_i \neq f_j\}$. Note that G is a complete graph except for the arcs (i,j) where $f_i = f_j$. An arc $(i,j) \in A$ is called a forward arc if $i < j$ and a backward arc if $i > j$. To each arc $(i,j) \in A$, associate a linear function $\psi_{ij}(y, v, s)$ defined as

$$\psi_{ij}(y,v,s) := \begin{cases} s + v_i + (f_i - f_j + 1)(y^i - \lfloor \beta_i \rfloor) - f_j & \text{if } (i,j) \text{ is a forward arc,} \\ v_i + (f_i - f_j)(y^i - \lfloor \beta_i \rfloor) & \text{if } (i,j) \text{ is a backward arc,} \end{cases}$$

where $v_0 = y^0 = 0$. See Fig. 2.

Theorem 4 ([105]). *Given an elementary cycle $C = (V_C, A_C)$ in the graph G, the*

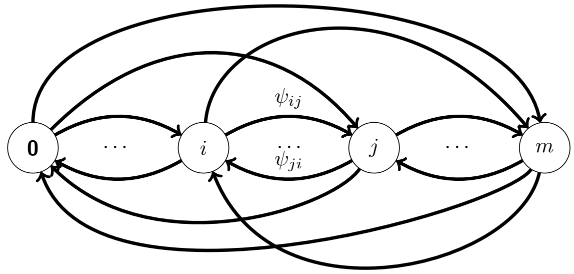

Figure 2: Each cycle in graph G gives rise to a cycle inequality.

inequality

$$\sum_{(i,j)\in A_C} \psi_{ij}(y, v, s) \geq 0, \tag{8}$$

referred to as the cycle inequality, is valid for Q. □

In [105], the validity of the cycle inequality (8) was proved indirectly through the following extended formulation for Q:

$$\begin{aligned} Q^\delta = \big\{(y, v, s, \delta) \in \mathbb{R}^m \times \mathbb{R}_+^{m+1} \times \mathbb{R}^{m+1} : \\ \psi_{ij}(y, v, s) \geq \delta_i - \delta_j \text{ for all } (i, j) \in A, \\ y^i + v_i + s \geq \beta_i, i = 1, \ldots, m\big\}. \end{aligned}$$

Note that the set of all original inequalities, all cycle inequalities, along with the bound constraints $v, s \geq 0$, define $\text{Proj}_{y,v,s}(Q^\delta)$. Van Vyve [105] showed that for every extreme point (or extreme ray) of Q, there exists a point (or a ray) in its extended formulation Q^δ. This implies $Q \subseteq \text{Proj}_{y,v,s}(Q^\delta)$, and hence, the cycle inequalities are valid for Q. Furthermore, it was shown in [105] that $conv(Q) = \text{Proj}_{y,v,s}(Q^\delta)$ and the separation over $conv(Q)$ can be performed in $O(m^3)$ time by finding a negative weight cycle in G. Similar results were presented for the relaxation of Q to the case where $s \in \mathbb{R}$.

II.2.3 n-step MIR Inequalities

In another direction, Kianfar and Fathi [62] developed the n-step MIR inequalities (a generalization of MIR inequalities [82, 111]) for the base set

$$Q_0^{1,n} = \Big\{(y,s) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+ : \sum_{t=1}^{n} \alpha_t y_t + s \geq \beta\Big\},$$

where $\alpha_t \in \mathbb{R}_+ \backslash \{0\}$, $t = 1, \dots, n$, $\beta \in \mathbb{R}$, and α_t's and β satisfy the so-called n-step MIR conditions, i.e.

$$\alpha_t \left\lceil \beta^{(t-1)}/\alpha_t \right\rceil \leq \alpha_{t-1}, \quad t = 2, \dots, n. \tag{9}$$

Note that $Q_0^{1,n} = \text{Proj}_{y,s}\big(Q^{1,n} \cap \{v = 0\}\big)$. The n-step MIR inequality for this set is

$$s \geq \beta^{(n)} \left(\prod_{l=1}^{n} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil - \beta^{(n)} \sum_{t=1}^{n} \prod_{l=t+1}^{n} \left\lceil \frac{\beta^{(l-1)}}{\alpha_l} \right\rceil y_t \right), \tag{10}$$

where the recursive remainders $\beta^{(t)}$ are defined as

$$\beta^{(t)} := \beta^{(t-1)} - \alpha_t \left\lfloor \beta^{(t-1)}/\alpha_t \right\rfloor, \quad t = 1, \dots, n, \tag{11}$$

and $\beta^{(0)} := \beta$ (note that $0 \leq \beta^{(t)} < \alpha_t$ for $t = 1, \dots, n$). By definition if $a > b$, then $\sum_a^b(.) = 0$ and $\prod_a^b(.) = 1$. For inequality (10) to be non-trivial, we assume that $\beta^{(t-1)}/\alpha_t \notin \mathbb{Z}, t = 1, \dots, n$. Kianfar and Fathi [62] showed that the n-step MIR inequality (10) is valid and facet-defining for the convex hull of $Q_0^{1,n}$. In a more general setting, Kianfar and Fathi [62] used n-step MIR facets of $Q_0^{1,n}$ to generate n-step MIR inequalities for Y_0^1, a single-constraint mixed integer knapsack set with general coefficients. Recall that $Y_0^1 = \text{Proj}_{y,s}(Y^1 \cap \{v = 0\})$. For each $n \in \mathbb{N}$, by choosing a parameter vector $\alpha = (\alpha_1, \ldots, \alpha_n) > 0$ that satisfy the n-step MIR conditions,

$$\alpha_t \left\lceil b^{(t-1)}/\alpha_t \right\rceil \leq \alpha_{t-1}, \quad t = 2, \ldots, n, \tag{12}$$

they introduced the so-called n-step MIR function to generate an n-step MIR inequality for Y_0^1. The n-step MIR function is defined as follows:

$$\mu^n_{\alpha,b}(x) = \begin{cases} \sum_{q=1}^{g} \prod_{l=q+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{x^{(q-1)}}{\alpha_q} \right\rfloor b^{(n)} + \prod_{l=g+2}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{x^{(g)}}{\alpha_{g+1}} \right\rceil b^{(n)} & \text{if } x \in \mathcal{I}^n_g, g = 0, \ldots, n-1 \\ \sum_{q=1}^{n} \prod_{l=q+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{x^{(q-1)}}{\alpha_q} \right\rfloor b^{(n)} + x^{(n)} & \text{if } x \in \mathcal{I}^n_n \end{cases}$$

where for $g = 0, \ldots, n-1$,

$$\mathcal{I}^n_g := \{x \in \mathbb{R} : x^{(q)} < b^{(q)}, q = 1, \ldots, g, x^{(g+1)} \geq b^{(g+1)}\};$$
$$\mathcal{I}^n_n := \{x \in \mathbb{R} : x^{(q)} < b^{(q)}, q = 1, \ldots, n\}.$$

The n-step MIR inequality for Y_0^1 is then

$$\sum_{t=1}^{N} \mu^n_{\alpha,b}(a_t) y_t + s \geq \mu^n_{\alpha,b}(b). \tag{13}$$

Kianfar and Fathi [62] proved that, for $n \in \mathbb{N}$, inequality (13) is valid for Y_0^1, and later, Atamtürk and Kianfar [7] showed that these inequalities also have facet-defining properties in several cases. Please refer to [7, 62] for more details.

II.2.4 n-step Mingling Inequalities

Atamtürk and Günlük [6] and Atamtürk and Kianfar [7] considered the mixed-integer knapsack set with bounded integer variables

$$Z_0^1 := \Big\{(y,s) \in \mathbb{Z}_+^N \times \mathbb{R}_+ : \sum_{t\in T} a_t y_t + \sum_{k\in K} a_k y_k + s \geq b, y \leq u\Big\},$$

where (T,K) is a partitioning of $\{1,\ldots,N\}$ with $a_t > 0$ for $t \in T$, $a_k < 0$ for $k \in K$, and $u \in \mathbb{Z}_+^N$. Atamtürk and Günlük [6] introduced (1-step) mingling and 2-step mingling inequalities for Z_0^1 which are generalized by Atamtürk and Kianfar [7] to n-step mingling inequalities, $n \in \mathbb{N}$, for Z_0^1. Unlike n-step MIR inequality (13), the n-step mingling inequality utilizes the information about the bounds and is derived as follows [6, 7]. Assuming $b \geq 0$, let $T^+ := \{1,\ldots,n^+\} \subseteq \{t \in T : a_t > b\}$ and $\bar{K} := \{k \in K : a_k + \sum_{t\in T^+} a_t u_t < 0\}$. We index T^+ in non-increasing order of a_t's. For $k \in K \setminus \bar{K}$, we define a set T_k, an integer l_k, and the numbers $\bar{u}_{tk}$ such that $u_{tk} \leq u_t$ for $t \in T_k$ as follows:

$$\begin{aligned}
T_k &:= \{1,\ldots,q(k)\}, \text{ where } q(k) := \min\Big\{q \in T^+ : a_k + \sum_{t=1}^{q} a_t u_t \geq 0\Big\};\\
l_k &:= \min\Big\{l \in \mathbb{Z}_+ : a_k + \sum_{t=1}^{q(k)-1} a_t u_t + a_{q(k)} l \geq 0\Big\}; \text{ and}\\
\bar{u}_{tk} &:= \begin{cases} u_t, & \text{if } t < q(k),\\ l_k, & \text{if } t = q(k). \end{cases}
\end{aligned}$$

Now for $k \in \bar{K}$, let $T_k := T^+, q(k) := n^+, l_k := u_{n^+}$, and $\bar{u}_{tk} := u_t$ for $t \in T_k$. We also define $K_t := \{k \in K : k \in T_k\}$; as a result, for $t \in T \setminus T^+, K_t = \emptyset$. Also for $k \in K$, let $\tau_k := \min\Big\{b, a_k + \sum_{t\in T_k} a_t \bar{u}_{tk}\Big\}$, and therefore, $0 \leq \tau_k \leq b$ for $k \in K \setminus \bar{K}$ and $\tau_k < 0$ for $k \in \bar{K}$. Using the $(n-1)$-step MIR function, they then proved that for $n \in \mathbb{N}$, the n-step mingling inequality

$$\begin{aligned}
&\sum_{t\in T^+} \mu^{n-1}_{\alpha,b}(b)\left[y_t - \sum_{k\in K_t} \bar{u}_{tk} y_k\right] + \sum_{t\in T\backslash T^+} \mu^{n-1}_{\alpha,b}(a_t) y_t \\
&+ \sum_{k\in K} \mu^{n-1}_{\alpha,b}(\tau_k) y_k + s \geq \mu^{n-1}_{\alpha,b}(b)
\end{aligned} \tag{14}$$

is valid for Z_0^1 for a parameter vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) > 0$ that satisfy the $(n-1)$-step MIR conditions (12). Note that for $n = 1$, we define $\mu^{n-1}_{\alpha,b}(x) = x$. These inequalities are used when integer variables are bounded from both sides. The n-step mingling utilizes the bounds on integer variables to give stronger inequalities, which are facet-defining in many cases [7]. Atamtürk and Günlük [6] proved that the 1-step mingling inequalities are facet-defining for $conv(Z_0^1)$ if $b - \min\{\tau_k : k \in \bar{K}\} \geq \max\{a_i : a_i > b, i \in T\backslash T^+\}$. For $n \geq 2$, Atamtürk and Kianfar [7] proved that the n-step mingling inequalities are facet-defining for $conv(Z_0^1)$ if the following conditions are satisfied (Theorem 2 in [7]):

i) $b^{(n-1)} > 0$ and $\alpha_d = a_{i_d}$ where $i_d \in T\backslash T^+$ for $k = 1, \ldots, n-1$;

ii) $T^+ = \{i \in I : a_i \geq \alpha_1 \lceil b/\alpha_1 \rceil\}$ and $\alpha_{d-1} \geq \alpha_d \left\lceil b^{(d-1)}/\alpha_d \right\rceil$ for $d = 2, \ldots, n-1$;

iii) $u_{t_1} \geq \left\lceil \frac{b}{\alpha_1} \right\rceil - \left\lceil \frac{\min\{\tau_k : k\in\bar{K}\}}{\alpha_1} \right\rceil$ and $u_{t_d} \geq \left\lceil \frac{b^{(d-1)}}{\alpha_d} \right\rceil$ for $d = 2, \ldots, n-1$.

It is important to note that for $T^+ = \emptyset$, the 1-step mingling inequality reduces to the base inequality and for $n \geq 2$, the n-step mingling inequality reduces to the $(n-1)$-step MIR inequality (13). Also, for $n > 1$, the n-step mingling inequality (14) dominates the inequality obtained by applying the $(n-1)$-step MIR procedure on 1-step mingling inequality [7]. Moreover, the facet-defining continuous integer cover inequality [10] (obtained by superadditive lifting) for Z_0^1 is a special case of inequality (14) for $n = 2$, $b > 0$, $\bar{K} = \emptyset$, $T^+ = \{t \in T : a_t \geq \alpha_1 \lceil b/\alpha_1 \rceil\}$, and $\alpha_1 = \alpha_d$ for some $d \in T$. Please refer to [6, 7] for more details.

II.2.5 Mixed n-step MIR Inequalities

As mentioned in Chapter I, Sanjeevi and Kianfar [96] generalized the MIR mixing procedure of Günlük and Pochet [51] to the case of n-step MIR and developed the mixed n-step MIR inequalities for the n-mixing set $Q_0^{m,n}$. Note that $Q_0^{m,n} = \text{Proj}_{y,s}\big(Q^{m,n} \cap \{v = 0\}\big)$. These inequalities are generated as follows: Without loss of generality, we assume $\beta_{i-1}^{(n)} \leq \beta_i^{(n)}, i = 2, \ldots, m$. Let $\hat{K} := \{i_1, \ldots, i_{|K|}\}$, where $i_1 < i_2 < \cdots < i_{|\hat{K}|}$, be a non-empty subset of $\{1, \ldots, m\}$. If the n-step MIR conditions (9) hold for each constraint $i \in \hat{K}$, i.e. $\alpha_t \left\lceil \beta_i^{(t-1)}/\alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n$, then the inequalities

$$s \geq \sum_{p=1}^{|\hat{K}|} \left(\beta_{i_p}^{(n)} - \beta_{i_{p-1}}^{(n)}\right) \phi_{i_p}^n(y^{i_p}) \tag{15}$$

$$s \geq \sum_{p=1}^{|\hat{K}|} \left(\beta_{i_p}^{(n)} - \beta_{i_{p-1}}^{(n)}\right) \phi_{i_p}^n(y^{i_p}) + \left(\alpha_n - \beta_{i_{|\hat{K}|}}^{(n)}\right) \left(\phi_{i_1}^n(y^{i_1}) - 1\right), \tag{16}$$

are valid for $Q_0^{m,n}$, where $\beta_{i_0}^{(n)} = 0$ and

$$\phi_i^n(y^i) := \prod_{l=1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{t=1}^{n} \prod_{l=t+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_t^i \tag{17}$$

for $i \in \hat{K}$. Inequalities (15) and (16) are referred to as the type I and type II mixed n-step MIR inequalities, respectively. Inequality (15) is shown to be facet-defining for $Q_0^{m,n}$. Inequality (16) also defines a facet for $Q_0^{m,n}$ if some additional conditions are satisfied (see [96] for details). Note that the function $\phi_i^n(y^i)$ has the same form as the multiple of $\beta^{(n)}$ in the right-hand side of the n-step MIR inequality (10). This function can alternatively be written as follows (see proof of Lemma 10 in [96]):

$$\phi_i^n(y^i) := 1 + \sum_{t=1}^{n} \prod_{l=t+1}^{n} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lfloor \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rfloor - y_t^i \right). \tag{18}$$

CHAPTER III

CONTINUOUS MULTI-MIXING SET

In this chapter, we introduce a multi-parameter multi-constraint mixed integer base set referred to as the *continuous multi-mixing set* which we define as

$$Q^{m,n} := \Big\{(y, v, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1} : \sum_{t=1}^{n} \alpha_t y_t^i + v_i + s \geq \beta_i, i = 1, \ldots, m\Big\},$$

where $\alpha_t > 0$, $t = 1, \ldots, n$ and $\beta_i \in \mathbb{R}$, $i = 1, \ldots, m$ such that the n-step MIR conditions for $i \in \{1, \ldots, m\}$ hold, i.e.

$$\alpha_t \left\lceil \beta_i^{(t-1)}/\alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n, i \in \{1, \ldots, m\}. \tag{19}$$

These n-step MIR conditions are automatically satisfied if the parameters $\alpha_1, \ldots, \alpha_n$ are divisible. The polyhedral study of this set generalizes the concepts of MIR [81, 111], mixed MIR [51], continuous mixing [105], n-step MIR [62], and mixed n-step MIR [96] (see Fig. 1). Note that this set has multiple (m) constraints with multiple (n) integer variables in each constraint; but it is more general than the n-mixing set (discussed in Chapter II) because in addition to the common continuous variable s, each constraint has a continuous variable v_i of its own. The continuous mixing set Q is the special case of $Q^{m,n}$, where $n = 1$ and $\alpha_1 = 1$, and the n-mixing set of Sanjeevi and Kianfar [96] is the projection of $Q^{m,n} \cap \{v = 0\}$ on (y, s). The continuous multi-mixing set arises as a substructure in relaxations of MML-WB, MML *with*

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backlogging (MML-B), MML with stochastic demand, multi-module facility location problem, and multi-module capacitated network design problem. In Section III.1, for each $n' \in \{1, \ldots, n\}$, we develop a class of valid inequalities for $Q^{m,n}$ which we refer to as n'-step cycle inequalities, and discuss how the n-step MIR inequalities [62] and the mixed n-step MIR inequalities [96] are special cases of the n-step cycle inequalities. We also introduce a compact extended formulation for $Q^{m,n}$. In Section III.2, we obtain conditions under which n'-step cycle inequalities are facet-defining for $conv(Q^{m,n})$. In Section III.3, we present an efficient exact separation algorithm to separate over the set of all n'-step cycle inequalities, $n' \in \{1, \ldots, n\}$, for set $Q^{m,n}$.

III.1 Valid Inequalities and Extended Formulation

In this section, we show that for each $n' \in \{1, \ldots, n\}$, there exist a family of valid inequalities for $Q^{m,n}$, which we refer to as the *n'-step cycle inequalities*. In proving the validity of these inequalities, Theorem 4 will become necessary. As mentioned before, Van Vyve [105] proved Theorem 4 *indirectly* by defining the extended formulation Q^{δ} and showing that every extreme point (ray) of the set Q has a counterpart in Q^{δ} (see [105] for details). We have developed a *direct* proof for Theorem 4, which only uses the original inequalities and the cycle structure. We believe this proof can be insightful in further pursuit of research in this area. Here, we present an alternative form of Theorem 4 and provide our proof:

Lemma 1. *Let $C = (V_C, A_C)$ be a directed Hamiltonian cycle over q nodes, where $V_C = \{1, \ldots, q\}$, $A_C := \{(1, i_2), (i_2, i_3), \ldots, (i_q, 1)\}$, and $i_2, \ldots, i_q \in \{2, \ldots, q\}$ are distinct. Let $\sigma \in \mathbb{R}, \alpha \in \mathbb{R}_+$, and to each node $i \in \{1, \ldots, q\}$ assign the values $\omega_i \in \mathbb{R}_+$, $\kappa_i \in \mathbb{Z}$, and $\gamma_i \in \mathbb{R}_+$ such that $\gamma_i < \alpha, i = 1, \ldots, q$, $\gamma_{i-1} < \gamma_i, i = 2, \ldots, q$. If*

$$\sigma + \omega_i + \alpha\kappa_i \geq \gamma_i \quad i = 1, \ldots, q, \tag{20}$$

then the cycle inequality

$$\sum_{(i,j)\in F} (\sigma + \omega_i - \gamma_j + (\gamma_i - \gamma_j + \alpha)\,\kappa_i) + \sum_{(i,j)\in B} (\omega_i + (\gamma_i - \gamma_j)\,\kappa_i) \geq 0, \tag{21}$$

is valid, where F and B are the sets of forward and backward arcs in A_C, respectively (i.e. $F = \{(i,j) \in A_C : i < j\}$ and $B = \{(i,j) \in A_C : i > j\}$).

Proof. For $p \in \{1,\ldots,q\}$, let A_p be the arcs in the path from 1 to i_{p+1} in C, i.e. $A_p := \{(1,i_2),(i_2,i_3),\ldots,\ (i_p,i_{p+1})\}$ (we define $i_{q+1} := 1$). Denote the set of forward and backward arcs in A_p by F_p and B_p, respectively (note that if $p' < p$, then $A_{p'} \subset A_p$, $F_{p'} \subseteq F_p$, and $B_{p'} \subseteq B_p$). Also, let $T(.)$ be an operator that, when applied on an arc set, returns the set of tail nodes of the arcs in that arc set. Define the index $g_p \in \{i_1,\ldots,i_p\}$ recursively as follows: $g_1 := 1$, and

$$g_p := \begin{cases} g_{p-1} & \text{if } i_p \in T(F_p), g_{p-1} \in T(F_{p-1}), \kappa_{g_{p-1}} \geq \kappa_{i_p}, \\ i_p & \text{if } i_p \in T(F_p), g_{p-1} \in T(F_{p-1}), \kappa_{g_{p-1}} < \kappa_{i_p}, \\ g_{p-1} & \text{if } i_p \in T(F_p), g_{p-1} \in T(B_{p-1}), \kappa_{g_{p-1}} > \kappa_{i_p}, \\ i_p & \text{if } i_p \in T(F_p), g_{p-1} \in T(B_{p-1}), \kappa_{g_{p-1}} \leq \kappa_{i_p}, \\ g_{p-1} & \text{if } i_p \in T(B_p), g_{p-1} \in T(B_{p-1}), \kappa_{g_{p-1}} \leq \kappa_{i_p}, \\ i_p & \text{if } i_p \in T(B_p), g_{p-1} \in T(B_{p-1}), \kappa_{g_{p-1}} > \kappa_{i_p}, \\ g_{p-1} & \text{if } i_p \in T(B_p), g_{p-1} \in T(F_{p-1}), \kappa_{g_{p-1}} < \kappa_{i_p}, \\ i_p & \text{if } i_p \in T(B_p), g_{p-1} \in T(F_{p-1}), \kappa_{g_{p-1}} \geq \kappa_{i_p}, \end{cases}$$

for $p = 2,\ldots,q$ and for $p = 1,\ldots,q$, define $\Delta_p = \gamma_{g_p} - \gamma_{i_{p+1}}$, if $g_p \in T(B_p)$, and 0 if

$g_p \in T(F_p)$. In order to prove the theorem, we first show that the inequality

$$\sum_{(i,j)\in F_p} (\gamma_i - (\gamma_i - \gamma_j + \alpha)\,\kappa_i) + \sum_{(i,j)\in B_p} (\gamma_i - \gamma_j)\,(1-\kappa_i)$$
$$\le (|F_p|-1)\sigma + \sum_{i\in T(A_p)\setminus\{g_p\}} \omega_i - \left(\gamma_1 - \gamma_{i_{p+1}} + \alpha\right)\kappa_{g_p} + \gamma_1 + \Delta_p, \tag{22}$$

is valid for $p = 1,\ldots,q$. We prove this by induction on p. For $p = 1$, the inequality (22) is trivial because $A_1 = \{(1, i_2)\}$, $F_1 = A_1$, $B_1 = \emptyset$, and $\Delta_1 = 0$, and therefore, both sides of the inequality reduce to $\gamma_1 - (\gamma_1 - \gamma_{i_2} + \alpha)\,\kappa_1$.

For simplicity, we denote the left-hand and right-hand sides of inequality (22) for p by L_p and R_p, respectively. Now as the induction hypothesis we assume $L_{p-1} \le R_{p-1}$. We then prove $L_p \le R_p$. Consider the following cases (which correspond to the cases in the g_p definition):

I. $i_p \in T(F_p)$. This means $\gamma_{i_p} < \gamma_{i_{p+1}}$, $F_p = F_{p-1} \cup \{(i_p, i_{p+1})\}$, and $B_p = B_{p-1}$. Therefore we can write

$$\begin{aligned} L_p &= L_{p-1} + \gamma_{i_p} - \left(\gamma_{i_p} - \gamma_{i_{p+1}} + \alpha\right)\kappa_{i_p} \\ &\le (|F_{p-1}|-1)\sigma + \sum_{i\in T(A_{p-1})\setminus\{g_{p-1}\}} \omega_i \\ &\quad - \left(\gamma_1 - \gamma_{i_p} + \alpha\right)\kappa_{g_{p-1}} + \gamma_1 + \Delta_{p-1} + \gamma_{i_p} - \left(\gamma_{i_p} - \gamma_{i_{p+1}} + \alpha\right)\kappa_{i_p} \end{aligned} \tag{23}$$

where the last inequality is based on (22) for $p-1$. Now, consider the following subcases:

I.1. $g_{p-1} \in T(F_{p-1})$, $\kappa_{g_{p-1}} \ge \kappa_{i_p}$. This implies $g_p = g_{p-1}$, and hence $\Delta_p = \Delta_{p-1} = 0$. Now notice that $0 \le \left(\gamma_{i_p} - \gamma_{i_{p+1}}\right)\left(\kappa_{i_p} - \kappa_{g_{p-1}}\right)$, and by inequality (20) for i_p, $0 \le \sigma + \omega_{i_p} + \alpha\kappa_{i_p} - \gamma_{i_p}$. Adding these two inequalities to inequality

(23), we get

$$\begin{aligned} L_p &\le |F_{p-1}|\sigma + \sum_{i\in T(A_{p-1})\setminus\{g_{p-1}\}} \omega_i + \omega_{i_p} \\ &- \left(\gamma_1 - \gamma_{i_{p+1}} + \alpha\right)\kappa_{g_{p-1}} + \gamma_1 + \Delta_{p-1} \\ &= (|F_p|-1)\sigma + \sum_{i\in T(A_p)\setminus\{g_p\}} \omega_i \\ &- \left(\gamma_1 - \gamma_{i_{p+1}} + \alpha\right)\kappa_{g_p} + \gamma_1 + \Delta_p = R_p. \end{aligned} \tag{24}$$

The first identity is true because $|F_{p-1}| = |F_p| - 1$, $T(A_{p-1}) \cup \{i_p\} = T(A_p)$, $g_{p-1} = g_p$, and $\Delta_{p-1} = \Delta_p (= 0)$.

I.2. $g_{p-1} \in T(F_{p-1})$, $\kappa_{g_{p-1}} < \kappa_{i_p}$. This implies $g_p = i_p$, and hence $g_p \in T(F_p)$. Therefore, $\Delta_{p-1} = \Delta_p = 0$. Notice that $0 \le \left(\gamma_1 - \gamma_{i_p}\right)\left(\kappa_{g_{p-1}} + 1 - \kappa_{i_p}\right)$, $0 \le \gamma_{g_{p-1}} - \gamma_1$, and by inequality (20) for g_{p-1}, $0 \le \sigma + \omega_{g_{p-1}} + \alpha\kappa_{g_{p-1}} - \gamma_{g_{p-1}}$. By adding these three inequalities to inequality (23), we get

$$\begin{aligned} L_p &\le |F_{p-1}|\sigma + \sum_{i\in T(A_{p-1})\setminus\{g_{p-1}\}} \omega_i + \omega_{g_{p-1}} \\ &- \left(\gamma_1 - \gamma_{i_{p+1}} + \alpha\right)\kappa_{i_p} + \gamma_1 + \Delta_{p-1} = R_p. \end{aligned}$$

The final identity is true because $|F_{p-1}| = |F_p| - 1$, $T(A_{p-1}) = T(A_p)\setminus\{i_p\}$, $i_p = g_p$, and $\Delta_{p-1} = \Delta_p (= 0)$.

I.3. $g_{p-1} \in T(B_{p-1})$, $\kappa_{g_{p-1}} > \kappa_{i_p}$. This means $g_p = g_{p-1}$, $\Delta_{p-1} = \gamma_{g_{p-1}} - \gamma_{i_p}$, and $\Delta_p = \gamma_{g_p} - \gamma_{i_{p+1}} = \gamma_{g_{p-1}} - \gamma_{i_{p+1}}$. Adding valid inequalities $0 \le \left(\gamma_{i_p} - \gamma_{i_{p+1}}\right)\left(\kappa_{i_p} + 1 - \kappa_{g_{p-1}}\right)$ and $0 \le \sigma + \omega_{i_p} + \alpha\kappa_{i_p} - \gamma_{i_p}$ to inequality (23) gives

$$L_p \le |F_{p-1}|\sigma + \sum_{i \in T(A_{p-1})\setminus\{g_{p-1}\}} \omega_i + \omega_{i_p} \\ - \left(\gamma_1 - \gamma_{i_{p+1}} + \alpha\right)\kappa_{g_{p-1}} + \gamma_1 + \Delta_{p-1} + \gamma_{i_p} - \gamma_{i_{p+1}} = R_p. \tag{25}$$

The final identity holds because $|F_{p-1}| = |F_p| - 1$, $T(A_{p-1}) \cup \{i_p\} = T(A_p)$, $g_{p-1} = g_p$, and $\Delta_{p-1} + \gamma_{i_p} - \gamma_{i_{p+1}} = \gamma_{g_{p-1}} - \gamma_{i_{p+1}} = \Delta_p$.

I.4. $g_{p-1} \in T(B_{p-1})$, $\kappa_{g_{p-1}} \le \kappa_{i_p}$. This means $g_p = i_p$, and hence $g_p \in T(F_p)$. Therefore, $\Delta_p = 0$. Also, $\Delta_{p-1} = \gamma_{g_{p-1}} - \gamma_{i_p}$. Now adding valid inequalities $0 \le \left(\gamma_1 - \gamma_{i_p}\right)\left(\kappa_{g_{p-1}} - \kappa_{i_p}\right)$ and $0 \le \sigma + \omega_{g_{p-1}} + \alpha\kappa_{g_{p-1}} - \gamma_{g_{p-1}}$ to inequality (23) gives

$$L_p \le |F_{p-1}|\sigma + \sum_{i \in T(A_{p-1})\setminus\{g_{p-1}\}} \omega_i + \omega_{g_{p-1}} \\ - \left(\gamma_1 - \gamma_{i_{p+1}} + \alpha\right)\kappa_{i_p} + \gamma_1 + \Delta_{p-1} + \gamma_{i_p} - \gamma_{g_{p-1}} = R_p.$$

The final identity is true because $|F_{p-1}| = |F_p| - 1$, $T(A_{p-1}) = T(A_p)\setminus\{i_p\}$, $i_p = g_p$, $\Delta_{p-1} + \gamma_{i_p} - \gamma_{g_{p-1}} = 0$, and $\Delta_p = 0$.

II. $i_p \in T(B_p)$. This means $\gamma_{i_p} > \gamma_{i_{p+1}}$, $F_p := F_{p-1}$, and $B_p := B_{p-1} \cup \{(i_p, i_{p+1})\}$. Therefore we can write

$$\begin{aligned} L_p &= L_{p-1} + \left(\gamma_{i_p} - \gamma_{i_{p+1}}\right)\left(1 - \kappa_{i_p}\right) \\ &\le (|F_{p-1}| - 1)\sigma + \sum_{i \in T(A_{p-1})\setminus\{g_{p-1}\}} \omega_i \\ &\quad - \left(\gamma_1 - \gamma_{i_p} + \alpha\right)\kappa_{g_{p-1}} + \gamma_1 + \Delta_{p-1} + \left(\gamma_{i_p} - \gamma_{i_{p+1}}\right)\left(1 - \kappa_{i_p}\right) \end{aligned} \tag{26}$$

where the last inequality is based on (22) for $p - 1$. Now, consider the following subcases:

II.1. $g_{p-1} \in T(B_{p-1})$, $\kappa_{g_{p-1}} \leq \kappa_{i_p}$. This means $g_p = g_{p-1}$, $\Delta_{p-1} = \gamma_{g_{p-1}} - \gamma_{i_p}$, and $\Delta_p = \gamma_{g_p} - \gamma_{i_{p+1}} = \gamma_{g_{p-1}} - \gamma_{i_{p+1}}$. Adding valid inequalities $0 \leq \left(\gamma_{i_p} - \gamma_{i_{p+1}}\right)\left(\kappa_{i_p} - \kappa_{g_{p-1}}\right)$ and $0 \leq \omega_{i_p}$ to inequality (26), we get the same inequality as (25) except for the coefficient of σ which will be $|F_{p-1}| - 1$. This inequality is true for the same reasons stated in case I.3 and the fact that $|F_{p-1}| = |F_p|$ in this case.

II.2. $g_{p-1} \in T(B_{p-1})$, $\kappa_{g_{p-1}} > \kappa_{i_p}$. This means $\Delta_{p-1} = \gamma_{g_{p-1}} - \gamma_{i_p}$. Also, $g_p = i_p$, and hence $g_p \in T(B_p)$. Therefore, $\Delta_p = \gamma_{g_p} - \gamma_{i_{p+1}} = \gamma_{i_p} - \gamma_{i_{p+1}}$. Adding valid inequalities $0 \leq \left(\gamma_1 - \gamma_{i_p} + \alpha\right)\left(\kappa_{g_{p-1}} - \kappa_{i_p} - 1\right)$, $0 \leq \gamma_1 - \gamma_{g_{p-1}} + \alpha$, and $0 \leq \omega_{g_{p-1}}$ to inequality (26) gives

$$
\begin{aligned}
L_p \leq \left(|F_{p-1}| - 1\right)\sigma + &\sum_{i \in T(A_{p-1}) \setminus \{g_{p-1}\}} \omega_i + \omega_{g_{p-1}} \\
&- \left(\gamma_1 - \gamma_{i_{p+1}} + \alpha\right)\kappa_{i_p} + \gamma_1 + \Delta_{p-1} + \gamma_{i_p} - \gamma_{g_{p-1}} + \gamma_{i_p} - \gamma_{i_{p+1}} = R_p.
\end{aligned}
$$

The final identity is true because $|F_{p-1}| = |F_p|$, $T(A_{p-1}) = T(A_p) \setminus \{i_p\}$, $i_p = g_p$, $\Delta_{p-1} = \gamma_{g_{p-1}} - \gamma_{i_p}$, and $\gamma_{i_p} - \gamma_{i_{p+1}} = \Delta_p$.

II.3. $g_{p-1} \in T(F_{p-1})$, $\kappa_{g_{p-1}} < \kappa_{i_p}$. This implies $g_p = g_{p-1}$, and hence $\Delta_p = \Delta_{p-1} = 0$. Adding valid inequalities $0 \leq \left(\gamma_{i_p} - \gamma_{i_{p+1}}\right)\left(\kappa_{i_p} - 1 - \kappa_{g_{p-1}}\right)$ and $0 \leq \omega_{i_p}$ to inequality (26), we get the same inequality as (24) except for the coefficient of σ which will be $|F_{p-1}| - 1$. This inequality is true for the same reasons stated in case I.1 and the fact that $|F_{p-1}| = |F_p|$ in this case.

II.4. $g_{p-1} \in T(F_{p-1})$, $\kappa_{g_{p-1}} \geq \kappa_{i_p}$. This means $\Delta_{p-1} = 0$. Also, $g_p = i_p$, and hence $g_p \in T(B_p)$. Therefore $\Delta_p = \gamma_{g_p} - \gamma_{i_{p+1}} = \gamma_{i_p} - \gamma_{i_{p+1}}$. Adding valid inequalities $0 \leq \left(\gamma_1 - \gamma_{i_p} + \alpha\right)\left(\kappa_{g_{p-1}} - \kappa_{i_p}\right)$ and $0 \leq \omega_{g_{p-1}}$ to inequality (26) gives

$$
\begin{aligned}
L_p \le |F_{p-1}|\sigma + &\sum_{i\in T(A_{p-1})\setminus\{g_{p-1}\}} \omega_i + \omega_{g_{p-1}} \\
&- \left(\gamma_1 - \gamma_{i_{p+1}} + \alpha\right)\kappa_{i_p} + \gamma_1 + \Delta_{p-1} + \gamma_{i_p} - \gamma_{i_{p+1}} = R_p.
\end{aligned}
$$

The final identity is true because $|F_{p-1}| = |F_p|$, $T(A_{p-1}) = T(A_p)\setminus\{i_p\}$, $i_p = g_p$, $\Delta_{p-1} = 0$, and $\gamma_{i_p} - \gamma_{i_{p+1}} = \Delta_p$.

All cases are exhausted, and therefore, inequality (22) is valid for any $p = 1, \ldots, q$. Now recall that $i_{q+1} = 1$. This implies $A_q = A_C$, and therefore,

$$
\begin{aligned}
L_q &= \sum_{(i,j)\in F} \left(\gamma_i - \left(\gamma_i - \gamma_j + \alpha\right)\kappa_i\right) + \sum_{(i,j)\in B} \left(\gamma_i - \gamma_j\right)\left(1 - \kappa_i\right) \\
&= \sum_{(i,j)\in F} \left(\gamma_j - \left(\gamma_i - \gamma_j + \alpha\right)\kappa_i\right) - \sum_{(i,j)\in B} \left(\gamma_i - \gamma_j\right)\kappa_i
\end{aligned}
\tag{27}
$$

The second identity is true because $\sum_{(i,j)\in F}\gamma_i + \sum_{(i,j)\in B}\left(\gamma_i - \gamma_j\right) = \sum_{(i,j)\in F}\left(\gamma_i - \gamma_j + \gamma_j\right) + \sum_{(i,j)\in B}\left(\gamma_i - \gamma_j\right) = \sum_{(i,j)\in F}\gamma_j + \sum_{(i,j)\in A_C}\left(\gamma_i - \gamma_j\right) = \sum_{(i,j)\in F}\gamma_j$. Note that $\sum_{(i,j)\in A_C}\left(\gamma_i - \gamma_j\right) = 0$ because the arcs in A_C form a cycle. Now based on inequality (22) for $p = q$ and inequality (27), we have

$$
\begin{aligned}
&\sum_{(i,j)\in F} \left(\gamma_j - \left(\gamma_i - \gamma_j + \alpha\right)\kappa_i\right) - \sum_{(i,j)\in B} \left(\gamma_i - \gamma_j\right)\kappa_i \\
&\le (|F| - 1)\sigma + \sum_{i\in T(A_C)\setminus\{g_q\}} \omega_i - \alpha\kappa_{g_q} + \gamma_1 + \Delta_q, \\
&\le |F|\sigma + \sum_{i\in T(A_C)} \omega_i + \Delta_q + \gamma_1 - \gamma_{g_q} \le |F|\sigma + \sum_{i\in T(A_C)} \omega_i,
\end{aligned}
\tag{28}
$$

where the second inequality is true by adding the valid inequality $0 \le \sigma + \omega_{g_q} + \alpha\kappa_{g_q} - \gamma_{g_q}$ to the first inequality, and the third inequality is true because we have either $\Delta_q = 0$ or $\Delta_q = \gamma_{g_q} - \gamma_{i_{q+1}} = \gamma_{g_q} - \gamma_1$, and hence, $\Delta_q + \gamma_1 - \gamma_{g_q} \le \min\{\gamma_1 - \gamma_{g_q}, 0\} =$

$\gamma_1 - \gamma_{g_q} \leq 0$. By rearranging the terms in inequality (28), we get inequality (21). This completes the proof. $\square$

Now given $n' \in \{1, \ldots, n\}$, we develop the n'-step cycle inequalities for $Q^{m,n}$ as follows: Without loss of generality, we assume $\beta_{i-1}^{(n')} \leq \beta_i^{(n')}$, $i = 2, \ldots, m$, where $\beta_i^{(n')}$ is defined as (11). Also define $\beta_0 := 0$. Now similar to the graph defined for the cycle inequalities (see Section II.2.2), here we define a directed graph $G_{n'} = (V, A)$, where $V := \{0, 1, \ldots, m\}$ and $A := \{(i,j) : i, j \in V, \beta_i^{(n')} \neq \beta_j^{(n')}\}$. $G_{n'}$ is a complete graph except for the arcs (i,j) where $\beta_i^{(n')} = \beta_j^{(n')}$. Here to each arc $(i,j) \in A$, we associate the linear function $\psi_{ij}^{n'}(y, v, s)$ defined as

$$\psi_{ij}^{n'}(y,v,s) := \begin{cases} s + v_i + \sum\limits_{t=n'+1}^{n} \alpha_t y_t^i + \beta_{ij}^{(n')} \left(1 - \phi_i^{n'}(y^i)\right) - \beta_j^{(n')} & \text{if } i < j, \\ v_i + \sum\limits_{t=n'+1}^{n} \alpha_t y_t^i + \left(\beta_i^{(n')} - \beta_j^{(n')}\right) \left(1 - \phi_i^{n'}(y^i)\right) & \text{if } i > j, \end{cases} \tag{29}$$

where $\beta_{ij}^{(n')} := \beta_i^{(n')} - \beta_j^{(n')} + \alpha_{n'}$ for all $(i,j) \in A$, $i < j$, the functions $\phi_i^{n'}(y^i)$, $i = 1, \ldots, m$, are defined as (17) and by definition, $v_0 := 0$, $y^0 := 0$, and $\phi_0^{n'}(y^0) := 1$.

We show that each elementary cycle of graph $G_{n'}$ corresponds to a valid inequality for the set $Q^{m,n}$, which we refer to as the *n'-step cycle inequality.* To do this in addition to Lemma 1, we need the following lemma:

Lemma 2. *For $i \in \{1, \ldots, m\}$ and $n' \in \{1, \ldots, n\}$, the inequality*

$$s + v_i + \sum_{t=n'+1}^{n} \alpha_t y_t^i + \alpha_{n'} \left(1 - \phi_i^{n'}(y^i)\right) \geq \beta_i^{(n')} \tag{30}$$

is valid for $Q^{m,n}$ if the n'-step MIR conditions (9) hold for constraint i of $Q^{m,n}$, i.e. $\alpha_t \left\lceil \beta_i^{(t-1)}/\alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n'$.

Proof. Kianfar and Fathi [62] proved that the following inequality

$$
\begin{aligned}
&s + v_i + \sum_{t=n'+1}^{n} \alpha_t y_t^i \\
&+ \alpha_{n'} \left(\sum_{t=1}^{n'} \prod_{l=t+1}^{n'} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil y_t^i - \prod_{l=1}^{n'} \left\lceil \frac{\beta_i^{(l-1)}}{\alpha_l} \right\rceil + \left\lceil \frac{\beta_i^{(n'-1)}}{\alpha_{n'}} \right\rceil \right) \geq \beta_i^{(n'-1)}
\end{aligned} \tag{31}
$$

is valid for the relaxation of $Q^{m,n}$ defined by its i'th constraint, i.e. $\{(y^i, v_i, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1}) \times \mathbb{R}_+ \times \mathbb{R}_+ : \sum_{t=1}^{n} \alpha_t y_t^i + v_i + s \geq \beta_i\}$, if the n'-step MIR conditions for constraint i hold. Therefore, it is also valid for $Q^{m,n}$. Subtracting $\alpha_{n'} \left\lceil \frac{\beta_i^{(n'-1)}}{\alpha_{n'}} \right\rceil$ from both sides and rearranging the terms in (31) gives (30). □

Theorem 5. *Given $n' \in \{1, \ldots, n\}$ and an elementary cycle $C = (V_C, A_C)$ of graph $G_{n'}$, the n'-step cycle inequality*

$$
\sum_{(i,j) \in A_C} \psi_{ij}^{n'}(y, v, s) \geq 0 \tag{32}
$$

is valid for $Q^{m,n}$ if the n'-step MIR conditions for $i \in V_C$, i.e.

$$
\alpha_t \left\lceil \beta_i^{(t-1)} / \alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n', i \in V_C. \tag{33}
$$

Proof. Consider a point $(\hat{y}, \hat{v}, \hat{s}) \in Q^{m,n}$. Based on Lemma 2, inequality (30) is satisfied by the point $(\hat{y}, \hat{v}, \hat{s})$ for each $i \in V_C \backslash \{0\}$ because of (33). But notice that inequality (30) for this point is the same as inequality (20) if we define $\sigma := \hat{s}$, $\alpha := \alpha_{n'}$, and $\omega_i := \hat{v}_i + \sum_{t=n'+1}^{n} \alpha_t \hat{y}_t^i$, $\kappa_i := 1 - \phi_{n'}^i(\hat{y}^i)$, $\gamma_i := \beta_i^{(n')}$, $i \in V_C \backslash \{0\}$. Also, in case $0 \in V_C$, if we define ω_0, κ_0, and γ_0 in a similar way, inequality (20) for $i = 0$ reduces to the valid inequality $\hat{s} \geq 0$ because as we defined before $y^0 := 0$, $v_0 := 0$, $\phi_0^{n'}(y^0) := 1$, and $\beta_0 := 0$. With these definitions, we have $\omega_i \geq 0$, $\kappa_i \in \mathbb{Z}$, $i \in V_C$ and $0 = \gamma_0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_{|V_C|} < \alpha_{n'}$. Therefore, according to Lemma 1, inequality (21) in which σ, α and $\omega_i, \kappa_i, \gamma_i, i \in V_C$ are replaced with the values defined here is valid. It is easy to see that this inequality is exactly the same as the n'-step cycle inequality (32) for the point $(\hat{y}, \hat{v}, \hat{s})$. This completes the proof. $\square$

Special Cases: The n-step MIR inequalities [62] and the mixed n-step MIR inequalities [96] are special cases of the n-step cycle inequalities.

I. The n-step cycle inequality (32) written for cycle $C = (V_C, A_C)$ such that $A_C = \{(0,i),(i,0)\}$ and $v_i = 0$ gives the n-step MIR inequality (10) written for constraint i in $Q_0^{m,n}$.

II. The n-step cycle inequality (32) written for cycle $C = (V_C, A_C)$ such that $A_C = \{(i_1,i_2),\dots,\ (i_q,i_1)\}$ with only one forward arc (i_1,i_2), followed by backward arcs $(i_1,i_2),\dots,(i_q,i_1)$ and $v_i = 0$ for all $i \in K$, gives the following inequalities for $Q_0^{m,n}$: the type I mixed n-step MIR inequality (15) where $K = \{i_q,\dots,i_2\}$, if $i_1 = 0$, and the type II mixed n-step MIR inequality (16) where $K = \{i_q,\dots,i_1\}$, if $i_1 \neq 0$.

Remark: For the special case where the parameters $\alpha_1,\dots,\alpha_{n'}$ are divisible, i.e. $\alpha_t|\alpha_{t-1},\ t = 2,\dots,n'$, the n'-step MIR conditions are automatically satisfied no matter what the value of β_i is.

Example 1. *Consider the following continuous multi-mixing set with 6 rows:*

$$
\begin{aligned}
Q^{6,2} = \{(y, v, s) \in (\mathbb{Z} \times \mathbb{Z}_+)^6 \times \mathbb{R}^7_+ :\\
50y^1_1 + 12y^1_2 + v_1 + s \geq 87,\\
50y^2_1 + 12y^2_2 + v_2 + s \geq 39,\\
50y^3_1 + 12y^3_2 + v_3 + s \geq 141,\\
50y^4_1 + 12y^4_2 + v_4 + s \geq 93,\\
50y^5_1 + 12y^5_2 + v_5 + s \geq 45,\\
50y^6_1 + 12y^6_2 + v_6 + s \geq 71\}.
\end{aligned}
$$

So we have $\alpha = (\alpha_1, \alpha_2) = (50, 12)$, $\beta_1 = 87$, $\beta_2 = 39$, $\beta_3 = 141$, $\beta_4 = 93$, $\beta_5 = 45$, $\beta_6 = 71$. *Note that* $\beta^{(1)}_6 = 21 < \beta^{(1)}_1 = 37 < \beta^{(1)}_2 = 39 < \beta^{(1)}_3 = 41 < \beta^{(1)}_4 = 43 < \beta^{(1)}_5 = 45$ *and* $\beta^{(2)}_1 = 1 < \beta^{(2)}_2 = 3 < \beta^{(2)}_3 = 5 < \beta^{(2)}_4 = 7 < \beta^{(2)}_5 = \beta^{(2)}_6 = 9$. *Note that* $\left\lceil \beta^{(1)}_i/\alpha_2 \right\rceil = 4$ *for* $i = 1, \ldots, 5$, $\left\lceil \beta^{(1)}_6/\alpha_2 \right\rceil = 3$ *and clearly the 2-step MIR conditions* (33), *i.e.* $\alpha_1 \geq \alpha_2 \left\lceil \beta^{(1)}_i/\alpha_2 \right\rceil$, *are satisfied for* $i = 1, \ldots, 6$.

2-step cycle inequalities for $Q^{6,2}$: *Setting* $n' = 2$, *the set of nodes and arcs of the graph* G_2 *will be* $V_2 = \{0, \ldots, 6\}$ *and* $A_2 = \{(i, j) : i, j \in V_2\} \backslash \{(5, 6), (6, 5)\}$ *because* $\beta^{(2)}_5 = \beta^{(2)}_6$. *The linear function* $\psi^2_{ij}(y, v, s)$ *associated with each arc* $(i, j) \in A_2$ *is defined by* (1) *where* $n' = 2$, *i.e.*

$$
\psi^2_{ij}(y, v, s) := \begin{cases} s + v_i + \left(\beta^{(2)}_i - \beta^{(2)}_j + \alpha_2\right)(1 - \phi^2_i(y^i)) - \beta^{(2)}_j & \text{if } \beta^{(2)}_i < \beta^{(2)}_j, \\ v_i + \left(\beta^{(2)}_i - \beta^{(2)}_j\right)(1 - \phi^2_i(y^i)) & \text{if } \beta^{(2)}_i > \beta^{(2)}_j, \end{cases}
$$

where $\phi^2_i(y^i) = \left\lceil \beta^{(1)}_i/\alpha_2 \right\rceil \lceil \beta_i/\alpha_1 \rceil - \left\lceil \beta^{(1)}_i/\alpha_2 \right\rceil y^i_1 - y^i_2$, *for* $i = 1, \ldots, 6$, *and* $v_0 := 0$, $y^0 := 0$, *and* $\phi^2_0(y^0) := 1$. *Based on Theorem 5, the 2-step cycle inequali-*

ties corresponding to the cycles in G_2 are valid for $Q^{6,2}$. For example, the 2-step cycle inequality corresponding to a cycle $C_1^2 = (V_{C_1^2}, A_{C_1^2})$ in G_2 where $A_{C_1^2} = \{(1,3),(3,6),(6,4),(4,5),(5,2),(2,1)\}$ is $\psi_{13}^2 + \psi_{36}^2 + \psi_{64}^2 + \psi_{45}^2 + \psi_{52}^2 + \psi_{21}^2 \geq 0$, i.e.

$$
\begin{aligned}
&(s + v_1 + 32y_1^1 + 8y_2^1 - 61) + (s + v_3 + 32y_1^3 + 8y_2^3 - 97) \\
&+ (v_6 + 6y_1^6 + 2y_2^6 - 10) + (s + v_4 + 40y_1^4 + 10y_2^4 - 79) \\
&+ (v_5 + 24y_1^5 + 6y_2^5 - 18) + (v_2 + 8y_1^2 + 2y_2^2 - 6) \geq 0.
\end{aligned} \tag{34}
$$

Likewise, for a cycle C_2^2 in G_2 with $A_{C_2^2} = \{(2,4),(4,3),(3,5),(5,2)\}$, the 2-step cycle inequality is $\psi_{24}^2 + \psi_{43}^2 + \psi_{35}^2 + \psi_{52}^2 \geq 0$, i.e.

$$
\begin{aligned}
&(s + v_2 + 32y_1^2 + 8y_2^2 - 31) + (v_4 + 8y_1^4 + 2y_2^4 - 14) \\
&+ (s + v_3 + 32y_1^3 + 8y_2^3 - 33) + (v_5 + 24y_1^5 + 6y_2^5 - 18) \geq 0,
\end{aligned} \tag{35}
$$

and for a cycle C_3^2 in G_2 with $A_{C_3^2} = \{(0,6),(6,4),(4,1),(1,0)\}$, the 2-step cycle inequality is $\psi_{06}^2 + \psi_{64}^2 + \psi_{41}^2 + \psi_{10}^2 \geq 0$, i.e.

$$
\begin{aligned}
&(s - 9) + (v_6 + 6y_1^6 + 2y_2^6 - 10) + (v_4 + 24y_1^4 + 6y_2^4 - 56) \\
&+ (v_1 + 4y_1^1 + y_2^1 - 7) \geq 0.
\end{aligned} \tag{36}
$$

1-step Cycle Inequalities for $Q^{6,2}$: *Setting $n' = 1$, the set of nodes and arcs of the graph G_1 will be $V_1 = \{0, 6, 1, \ldots, 5\}$ and $A_1 = \{(i,j) : i, j \in V_1\}$ because $\beta_6^{(1)} < \beta_1^{(1)}$. The linear function $\psi_{ij}^1(y, v, s)$ associated with each arc $(i,j) \in A_1$ is defined by (1)* *where* $n' = 1$, *i.e.*

$$\psi^1_{ij}(y,v,s) := \begin{cases} s + v_i + \alpha_2 y^i_2 + \left(\beta^{(1)}_i - \beta^{(1)}_j + \alpha_1\right)(1 - \phi^1_i(y^i)) - \beta^{(1)}_j & \text{if } \beta^{(1)}_i < \beta^{(1)}_j, \\ v_i + \alpha_2 y^i_2 + \left(\beta^{(1)}_i - \beta^{(1)}_j\right)(1 - \phi^1_i(y^i)) & \text{if } \beta^{(1)}_i > \beta^{(1)}_j, \end{cases}$$

where $\phi^1_i(y^i) = \lceil \beta_i/\alpha_1 \rceil - y^i_1$, *for* $i = 1, \ldots, 5$, *and* $v_0 := 0$, $y^0 := 0$, *and* $\phi^1_0(y^0) := 1$. *Based on Theorem 5, the 1-step cycle inequalities corresponding to the cycles in* G_1 *are valid for* $Q^{6,2}$. *For example, the 1-step cycle inequality corresponding to a cycle* $C^1_1 = (V_{C^1_1}, A_{C^1_1})$ *in* G_1 *where* $A_{C^1_1} = \{(6,1),(1,2),(2,3),(3,4),(4,5),(5,6)\}$ *is* $\psi^1_{61} + \psi^1_{12} + \psi^1_{23} + \psi^1_{34} + \psi^1_{45} + \psi^1_{56} \geq 0$, *i.e.*

$$\begin{aligned} &(s + v_6 + 34y^6_1 + 12y^6_2 - 71) + (s + v_1 + 48y^1_1 + 12y^1_2 - 87) \\ &+ (s + v_2 + 48y^2_1 + 12y^2_2 - 41) + (s + v_3 + 48y^3_1 + 12y^3_2 - 139) \\ &+ (s + v_4 + 48y^4_1 + 12y^4_2 - 93) + (v_5 + 24y^5_1 + 12y^5_2) \geq 0. \end{aligned} \tag{37}$$

Likewise, for a cycle C^1_2 *in* G_1 *with* $A_{C^1_2} = \{(6,2),(2,5),(5,6)\}$, *the 1-step cycle inequality is* $\psi^1_{62} + \psi^1_{25} + \psi^1_{56} \geq 0$, *i.e.*

$$\begin{aligned} &(s + v_6 + 32y^6_1 + 12y^6_2 - 71) + (s + v_2 + 44y^2_1 + 12y^2_2 - 45) \\ &+ (v_5 + 24y^5_1 + 12y^5_2) \geq 0, \end{aligned} \tag{38}$$

and for a cycle C^1_3 *in* G_1 *with* $A_{C^1_3} = \{(0,4),(4,6),(6,0)\}$, *the 1-step cycle inequality is* $\psi^1_{04} + \psi^1_{46} + \psi^1_{60} \geq 0$, *i.e.*

$$(s - 43) + (v_4 + 22y^4_1 + 12y^5_2 - 22) + (v_6 + 21y^6_1 + 12y^6_2 - 21) \geq 0. \tag{39}$$

Theorem 6. *The following linear program is a compact extended formulation for*

$Q^{m,n}$, *if conditions* (33) *hold.*

$$\psi_{ij}^{n'}(y,v,s) \geq \delta_i^{n'} - \delta_j^{n'} \text{ for all } (i,j) \in A, n' \in \{1,\ldots,n\} \tag{40}$$

$$\textstyle\sum_{t=1}^{n} \alpha_t y_t^i + v_i + s \geq \beta_i, i = 1,\ldots,m \tag{41}$$

$$y \in (\mathbb{R} \times \mathbb{R}_+^{n-1})^m, v \in \mathbb{R}_+^m, s \in \mathbb{R}_+, \delta \in \mathbb{R}^{n(m+1)}. \tag{42}$$

Proof. Let $Q^{m,n,\delta} := \{(y,v,s,\delta)$ satisfying (40)-(42)$\}$. Clearly $Proj_{y,v,s}(Q^{m,n,\delta})$ is defined by the set of all n'-step cycle inequalities (32), for $n' = 1,\ldots,n$, and bound constraints $s, v \geq 0$. This means all the inequalities which define $Proj_{y,v,s}(Q^{m,n,\delta})$ are valid for $Q^{m,n}$ if conditions (33) hold which implies $Q^{m,n} \subseteq Proj_{y,v,s}(Q^{m,n,\delta})$ under the same conditions. This proves that $Q^{m,n,\delta}$ is an extended formulation for $Q^{m,n}$. □

III.2 Facet-Defining n-step Cycle Inequalities

In this section, we show that for any $n' \in \{1,\ldots,n\}$, the n'-step cycle inequalities define facets for $conv(Q^{m,n})$ under certain conditions. In order to prove this, we first define some points and provide some properties for them.

Definition 13. *For* $i \in \{1,\ldots,m\}$*, define the points* $\mathcal{P}^{i,d}, \mathcal{Q}^{i,d} \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$*,* $d = 1,\ldots,n$*, as follows:*

$$\mathcal{P}_t^{i,d} := \begin{cases} \left\lfloor \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rfloor & t = 1,\ldots,d-1, \\ \left\lceil \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rceil & t = d, \\ 0 & t = d+1,\ldots,n, \end{cases} \qquad \mathcal{Q}_t^{i,d} := \begin{cases} \left\lfloor \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rfloor & t = 1,\ldots,d, \\ 0 & t = d+1,\ldots,n, \end{cases}$$

and the point $\mathcal{R}^i \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$ *(assuming* $\left\lfloor \beta_i^{(n'-1)}/\alpha_{n'} \right\rfloor \geq 1$*) as* $\mathcal{R}^i = \mathcal{Q}^{i,n'} - e_{n'}$*,* *where $e_{n'}$ is the n'th unit vector in $\mathbb{R}^n$. Also, define the points $\mathcal{S}^{i,d} \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$, $d = 2, \ldots, n'$, (assuming $\left\lfloor \beta_i^{(d-1)}/\alpha_d \right\rfloor \geq 1$, $d = 2, \ldots, n'$) as follows:*

$$\mathcal{S}_t^{i,d} := \begin{cases} \mathcal{Q}_t^{i,n'} & t = 1, \ldots, d-2, d+1, \ldots, n \\ \left\lfloor \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rfloor - 1 & t = d-1, \\ 2\left\lfloor \frac{\beta_i^{(t-1)}}{\alpha_t} \right\rfloor + 1 & t = d. \end{cases}$$

Moreover, for $i, j \in \{1, \ldots, m\}$ such that $\beta_i^{(n')} > \beta_j^{(n')}$, define the points $\mathcal{T}^{i,j,d} \in \mathbb{Z} \times \mathbb{Z}_+^{n-1}$, $d = n', \ldots, n$, as $\mathcal{T}_t^{i,j,d} := \mathcal{Q}_t^{i,n'}$ for $t = 1, \ldots, n', d+1, \ldots, n$ and $\left\lfloor \frac{\beta_{ij}^{(n',t-1)}}{\alpha_t} \right\rfloor$ for $t = n'+1, \ldots, d$, where $\beta_{ij}^{(n',n')} := \beta_i^{(n')} - \beta_j^{(n')}$ and $\beta_{ij}^{(n',t)} := \beta_{ij}^{(n',t-1)} - \alpha_t \left\lfloor \beta_{ij}^{(n',t-1)}/\alpha_t \right\rfloor$, $t = n'+1, \ldots, n$.

Lemma 3. *The point $(\hat{y}, \hat{v}, \hat{s}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ satisfies constraint $i \in \{1, \ldots, m\}$ of $Q^{m,n}$ if any of the following is true*

(a). $\hat{y}^i = \mathcal{P}^{i,d}$ for some $d \in \{1, \ldots, n\}$

(b). $\hat{y}^i = \mathcal{Q}^{i,d}$ for some $d \in \{1, \ldots, n\}$ and $\hat{v}_i + \hat{s} \geq \beta_i^{(d)}$,

(c). $\hat{y}^i = \mathcal{R}^i$ and $\hat{v}_i + \hat{s} \geq \alpha_{n'} + \beta_i^{(n')}$,

(d). $\hat{y}^i = \mathcal{S}^{i,d}$ for some $d \in \{2, \ldots, n'\}$ and $\hat{v}_i + \hat{s} \geq \beta_i^{(n')} + \alpha_{d-1} - \alpha_d \left\lceil \beta_i^{(d-1)}/\alpha_d \right\rceil$,

(e). $\hat{y}^i = \mathcal{T}^{i,j,d}$ for some $d \in \{n', \ldots, n\}$ and $j \in \{1, \ldots, m\}$ and $\hat{v}_i + \hat{s} \geq \beta_j^{(n')} + \beta_{ij}^{(n',d)}$.

Proof. Cases (a) and (b) can be easily proved similar to the proof of Lemma 5 in [96]. Cases (c) and (d) can also be easily proved similar to the proof of Lemma 9 in [96]. For (e), notice that by substituting the point $(\hat{y}, \hat{v}, \hat{s})$ in constraint i of $Q^{m,n}$, we get

$\sum_{t=1}^{n'} \alpha_t \left\lfloor \beta_i^{(t-1)}/\alpha_t \right\rfloor + \sum_{t=n'+1}^{d} \alpha_t \left\lfloor \beta_{ij}^{(n',t-1)}/\alpha_t \right\rfloor + \hat{v}_i + \hat{s} \geq \beta_i$, or $\hat{v}_i + \hat{s} \geq \beta_j^{(n')} + \beta_{ij}^{(n',d)}$, which is true by the assumption of case (e). □

Lemma 4. *For $i \in \{1, \ldots, m\}$ and $n' \in \{1, \ldots, n\}$,*

(a). $\phi_i^{n'}(\mathcal{P}^{i,d}) = 0,\ d = 1, \ldots, n'$,

(b). $\phi_i^{n'}(\mathcal{Q}^{i,d}) = 1,\ d = n', \ldots, n$,

(c). $\phi_i^{n'}(\mathcal{R}^{i}) = 2$,

(d). $\phi_i^{n'}(\mathcal{S}^{i,d}) = 1,\ d = 2, \ldots, n'$,

(e). $\phi_i^{n'}(\mathcal{T}^{i,j,d}) = 1,\ d = n', \ldots, n$, for $j \in \{1, \ldots, m\}$ such that $\beta_i^{(n')} > \beta_j^{(n')}$.

Proof. Cases (a), (b) and (e) can be proved similar to Lemma 6 of [96] and cases (c) and (d) can be proved similar to Lemma 10 of [96]. □

As before, given a cycle $C = (V_C, A_C)$ of $G_{n'}$, let F and B be the set of forward arcs and backward arcs of the cycle C, respectively, i.e. $F := \{(i,j) \in A_C : i < j\}$ and $B := \{(i,j) \in A_C : j < i\}$.

Theorem 7. *For $n' \in \{1, \ldots, n\}$, the n'-step cycle inequality (32) for an elementary cycle $C = (V_C, A_C)$ of graph G is facet-defining for conv$(Q^{m,n})$ if (in addition to the n'-step MIR conditions (33)) the following conditions hold*

(a) $\left\lfloor \beta_k^{(d-1)}/\alpha_d \right\rfloor \geq 1,\ d = 2, \ldots, n$, for all $(k,l) \in F$,

(b) $\beta_l^{(n')} - \beta_k^{(n')} \geq \max\left\{\alpha_{d-1} - \alpha_d \left\lceil \frac{\beta_k^{(d-1)}}{\alpha_d} \right\rceil, d = 2, \ldots, n'\right\}$ for all $(k,l) \in F$,

(c) $\left\lfloor \beta_{kl}^{(n',d-1)}/\alpha_d \right\rfloor \geq 1,\ d = n'+1, \ldots, n$, for all $(k,l) \in B$.

Proof. Consider the supporting hyperplane of inequality (32) for the cycle C. Note that this hyperplane can be written as

$$\sum_{(i,j)\in F}\left(s+v_i+\sum_{t=n'+1}^{n}\alpha_t y_t^i-\beta_i^{(n')}+\left(\beta_i^{(n')}-\beta_j^{(n')}+\alpha_{n'}\right)\left(1-\phi_i^{n'}(y^i)\right)\right)$$
$$=\sum_{(i,j)\in B}\left(\left(\beta_i^{(n')}-\beta_j^{(n')}\right)\phi_i^{n'}(y^i)-\sum_{t=n'+1}^{n}\alpha_t y_t^i-v_i\right) \tag{43}$$

because $-\sum_{(i,j)\in F}\beta_j^{(n')}+\sum_{(i,j)\in B}\left(\beta_i^{(n')}-\beta_j^{(n')}\right)=-\sum_{(i,j)\in F}\beta_i^{(n')}$. Let $\Gamma=\{(y,v,s)\in conv(Q^{m,n}):(43)\}$ be the face of $conv(Q^{m,n})$ defined by hyperplane (43).

First, we prove that Γ is a facet of $Q^{m,n}$ under conditions (a) (note that under conditions (a), $0\notin V_C$ because $\beta_0=0$ and does not satisfy conditions (a)). Let

$$\sum_{i=1}^{m}\sum_{t=1}^{n}\lambda_t^i y_t^i+\sum_{i=1}^{m}\rho_i v_i+\rho_0 s=\theta \tag{44}$$

be a hyperplane passing through Γ. We prove that (44) must be a multiple of (43).

Notice that for each $k\in\{1,\dots,m\}\backslash V_C$ and $d\in\{1,\dots,n\}$, the unit vector $\mathcal{E}_1^{k,d}=(y^1,\dots,y^m,v_1,\dots,v_m,s)\in(\mathbb{Z}\times\mathbb{Z}_+^{n-1})^m\times\mathbb{R}_+^{m+1}$, in which $y_d^k=1$ and all other coordinates are zero, is a direction for both the set $Q^{m,n}$ and the hyperplane defined by (43), and hence a direction for the face Γ. This implies that $\lambda_d^k=0$ for all $k\in\{1,\dots,m\}\backslash V_C$ and $d=1,\dots,n$. By similar reasoning, for each $k\in\{1,\dots,m\}\backslash V_C$, the unit vector $\mathcal{E}_2^k=(y^1,\dots,y^m,v_1,\dots,v_m,s)\in(\mathbb{Z}\times\mathbb{Z}_+^{n-1})^m\times\mathbb{R}_+^{m+1}$, in which $v_k=1$ and all other coordinates are zero, is a direction for the face Γ, implying that $\rho_k=0$, $k\in\{1,\dots,m\}\backslash V_C$. These reduce the hyperplane (44) to

$$\sum_{i\in V_C}\sum_{t=1}^{n}\lambda_t^i y_t^i+\sum_{i\in V_C}\rho_i v_i+\rho_0 s=\theta \tag{45}$$

Next, consider the point $\mathcal{A} = (y, v, s) = (y^1, \ldots, y^m, v_1, \ldots, v_m, 0) \in (\mathbb{Z}\times\mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ such that, for $i = 1, \ldots, m$, $(y^i, v_i) = (\mathcal{Q}^{i,n'}, \beta_i^{(n')})$ if $i \in T(F)$, and $(y^i, v_i) = (\mathcal{P}^{i,1}, 0)$ if $i \notin T(F)$. Based on Lemma 3(a,b), $\mathcal{A} \in Q^{m,n}$ and using Lemma 4(a,b), it can be easily verified that $\mathcal{A}$ satisfies (43). So, $\mathcal{A} \in \Gamma$ and hence must satisfy (45). Substituting $\mathcal{A}$ into (45) gives

$$\sum_{i\in T(F)} \left(\rho_i \beta_i^{(n')} + \sum_{t=1}^{n'} \lambda_t^i \left\lfloor \beta_i^{(t-1)}/\alpha_t \right\rfloor \right) + \sum_{i\in T(B)} \lambda_1^i \left\lceil \beta_i/\alpha_1 \right\rceil = \theta. \tag{46}$$

Using (46), hyperplane (45) reduces to

$$\begin{aligned} &\sum_{i\in T(F)} \left(\rho_i \left(v_i - \beta_i^{(n')} \right) + \sum_{t=1}^{n'} \lambda_t^i \left(y_t^i - \left\lfloor \beta_i^{(t-1)}/\alpha_t \right\rfloor \right) + \sum_{t=n'+1}^{n} \lambda_t^i y_t^i \right) \\ &+ \rho_0 s = \sum_{i\in T(B)} \left(\lambda_1^i \left(\left\lceil \beta_i/\alpha_1 \right\rceil - y_1^i \right) - \sum_{t=2}^{n} \lambda_t^i y_t^i - \rho_i v_i \right). \end{aligned} \tag{47}$$

Now, consider the points $\mathcal{B}^{k,d} = (y, v, s) = (y^1, \ldots, y^m, v_1, \ldots, v_m, 0) \in (\mathbb{Z}\times\mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ for $k \in T(F)$ and $d = n' + 1, \ldots, n$ such that

$$(y^i, v_i) = \begin{cases} (\mathcal{Q}^{i,n'}, \beta_i^{(n')}) & \text{if } i \in T(F)\backslash\{k\}, \\ (\mathcal{Q}^{i,d}, \beta_i^{(d)}) & \text{if } i = k, \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin T(F), \end{cases}$$

for $i = 1, \ldots, m$. By Lemma 3(a,b), $\mathcal{B}^{k,d} \in Q^{m,n}$, for all $k \in T(F)$ and $d = n' + 1, \ldots, n$. Using Lemma 4(a,b), one can easily verify that all these points also satisfy (43). So for all $k \in T(F)$ and $d = n' + 1, \ldots, n$, $\mathcal{B}^{k,d} \in \Gamma$, and hence must satisfy (47). Now if for each $k \in T(F)$, we substitute the points $\mathcal{B}^{k,n'+1}, \ldots, \mathcal{B}^{k,n}$ one after the other into (47), (since conditions (a) holds) we get

$$\lambda_d^k = \alpha_d \rho_k,\ d = n'+1,\dots,n,\ k \in T(F) \tag{48}$$

Next, consider the points $\mathcal{C}^{k,d} = (y, v, s) = (y^1, \dots, y^m, v_1, \dots, v_m, 0) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ for $k \in T(B)$, $d = 2, \dots, n'$ such that

$$(y^i, v_i) = \begin{cases} (\mathcal{Q}^{i,n'}, \beta_i^{(n')}) & \text{if } i \in T(F), \\ (\mathcal{P}^{i,d}, 0) & \text{if } i = k, \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin T(F) \cup \{k\}, \end{cases}$$

for $i = 1, \dots, m$. By Lemma 3(a,b), $\mathcal{C}^{k,d} \in Q^{m,n}$, for all $k \in T(B)$ and $d = 2, \dots, n'$. Using Lemma 4(a,b), one can easily verify that all these points also satisfy (43). So for all $k \in T(B)$ and $d = 2, \dots, n'$, $\mathcal{C}^{k,d} \in \Gamma$, and hence must satisfy (47). For each $k \in T(B)$, substituting the points $\mathcal{C}^{k,2}, \dots, \mathcal{C}^{k,n'}$ one after the other into (47) gives $\lambda_{d-1}^k = \lambda_d^k \left\lceil \beta_k^{(d-1)}/\alpha_d \right\rceil$, $d = 2, \dots, n'$, $k \in T(B)$, which implies $\lambda_d^k = \lambda_{n'}^k \prod_{l=d+1}^{n'} \left\lceil \beta_k^{(l-1)}/\alpha_l \right\rceil$, $d = 1, \dots, n'$, $k \in T(B)$. This, along with (48), reduces hyperplane (47) to

$$\begin{aligned} &\sum_{i \in T(F)} \left(\rho_i \left(v_i + \sum_{t=n'+1}^{n} \alpha_t y_t^i - \beta_i^{(n')} \right) + \sum_{t=1}^{n'} \lambda_t^i \left(y_t^i - \left\lfloor \beta_i^{(t-1)}/\alpha_t \right\rfloor \right) \right) \\ &+ \rho_0 s = \sum_{i \in T(B)} \left(\lambda_{n'}^i \phi_i^{n'}(y^i) - \sum_{t=n'+1}^{n} \lambda_t^i y_t^i - \rho_i v_i \right). \end{aligned} \tag{49}$$

Now, consider the point $\mathcal{D} = (y, v, s) = (y^1, \dots, y^m, v_1, \dots, v_m, \eta) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$, where $\eta = \min\{\beta_i^{(n)} : i \in T(F)\}$, such that for $i = 1, \dots, m$, $(y^i, v_i) = (\mathcal{Q}^{i,n'}, \beta_i^{(n')} - \eta)$ if $i \in T(F)$, and $(y^i, v_i) = (\mathcal{P}^{i,1}, 0)$ if $i \notin T(F)$. By Lemma 3(a,b), it is clear that $\mathcal{D} \in Q^{m,n}$ and using Lemma 4(a,b), one can easily verify that it also satisfies (43). So $\mathcal{D} \in \Gamma$, and hence must satisfy (49). Substituting $\mathcal{D}$ into (49) gives

$$\rho_0 = \sum_{i \in T(F)} \rho_i. \tag{50}$$

Now for $i \in V_C$, let $N(i)$ be the node in V_C such that $(i, N(i)) \in A_C$. For each $(k,l) \in A_C$, since conditions (a) holds, consider the points $\mathcal{F}^{k,l} = (y, v, s) = (y^1, \ldots, y^m, v_1, \ldots, v_m, \beta_l^{(n')}) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ such that

$$(y^i, v_i) = \begin{cases} (\mathcal{R}^i, \beta_i^{(n')} - \beta_l^{(n')} + \alpha_{n'}) & \text{if } i \in T(F), N(i) < l \\ (\mathcal{Q}^{i,n'}, 0) & \text{if } i \in T(F), i < l \leq N(i) \\ (\mathcal{Q}^{i,n'}, \beta_i^{(n')} - \beta_l^{(n')}) & \text{if } i \in T(F), i \geq l \\ (\mathcal{Q}^{i,n'}, 0) & \text{if } i \in T(B), i < l \\ (\mathcal{Q}^{i,n'}, \beta_i^{(n')} - \beta_l^{(n')}) & \text{if } i \in T(B), N(i) < l \leq i \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \in T(B), N(i) \geq l \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin V_C, \end{cases}$$

for $i = 1, \ldots, m$. By Lemma 3(a,b,c), it is clear that $\mathcal{F}^{k,l} \in Q^{m,n}$ for all $(k,l) \in A_C$.

Using Lemma 4(a,b,c), if we substitute $\mathcal{F}^{k,l}$ into (43), we get

$$\begin{aligned}
&\sum_{(i,j)\in F; i,j<l} \left(\beta_i^{(n')} - \beta_j^{(n')}\right) + \sum_{(i,j)\in B; i,j<l} \left(\beta_i^{(n')} - \beta_j^{(n')}\right) \\
&+ \sum_{(i,j)\in F; i<l\le j} \left(\beta_i^{(n')} - \beta_l^{(n')}\right) + \sum_{(i,j)\in B; j<l\le i} \left(\beta_l^{(n')} - \beta_j^{(n')}\right) \\
&= - \sum_{(i,j)\in F; i<l\le j} \beta_i^{(n')} + \sum_{(i,j)\in B; j<l\le i} \beta_j^{(n')} \\
&+ \sum_{(i,j)\in F; i<l\le j} \beta_i^{(n')} - \sum_{(i,j)\in B; j<l\le i} \beta_j^{(n')} = 0,
\end{aligned} \tag{51}$$

which is obviously true. Therefore, the points $\mathcal{F}^{k,l}$, for all $(k,l) \in A_C$, also satisfy (43). Hence, they belong to Γ, and must satisfy (49). Now, note that in the point $\mathcal{F}^{k,l}$, $(k,l) \in F$, by definition we have $(y^k, v_k) = (\mathcal{Q}^{k,n'}, 0)$. For each $(k,l) \in F$, define another point $\mathcal{F}_1^{k,l} = (y, v, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{F}^{k,l}$ except that $(y^k, v_k) = (\mathcal{R}^k, \beta_k^{(n')} - \beta_l^{(n')} + \alpha_{n'})$. For precisely the same reasons stated for $\mathcal{F}^{k,l}$, the points $\mathcal{F}_1^{k,l}$, $(k,l) \in F$, must also satisfy (49) (note that substituting $\mathcal{F}_1^{k,l}$ in (43) gives identity (51) again). Now if for each $(k,l) \in F$, we substitute $\mathcal{F}^{k,l}$ and $\mathcal{F}_1^{k,l}$ into (49) and subtract one equality from the other, we get

$$\lambda_{n'}^k = \rho_k \left(\beta_k^{(n')} - \beta_l^{(n')} + \alpha_{n'}\right), \text{ for all } (k,l) \in F. \tag{52}$$

Next, for each $(k,l) \in F$ and $d = 2, \ldots, n'$, since conditions (a) hold, define the point $\mathcal{F}_2^{k,l,d} = (y, v, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{F}^{k,l}$ except that $(y^k, v_k) = (\mathcal{S}^{k,d}, 0)$. By Lemma 3(a,b,c,d) and because of conditions (b), it is clear that $\mathcal{F}_2^{k,l,d} \in Q^{m,n}$ for all $(k,l) \in F$ and $d = 2, \ldots, n'$. Using Lemma 4(a,b,c,d), one can easily verify that they also satisfy (43) (note that substituting

$\mathcal{F}_2^{k,l,d}$ in (43) gives identity (51) again), and hence belong to Γ and must satisfy (49). Now if for each $(k,l) \in F$ and $d = 2, \ldots, n'$, we substitute the points $\mathcal{F}^{k,l}$ and $\mathcal{F}_2^{k,l,d}$ into (49) and subtract one equality from the other, we get

$$\lambda_{d-1}^k = \lambda_d^k \left\lceil \beta_k^{(d-1)}/\alpha_d \right\rceil, d \in \{2, \ldots, n'\}, k \in T(F). \tag{53}$$

This implies

$$\lambda_d^k = \lambda_{n'}^k \prod_{p=d+1}^{n'} \left\lceil \beta_k^{(p-1)}/\alpha_p \right\rceil, d = 1, \ldots, n', k \in T(F). \tag{54}$$

Next, note that in the point $\mathcal{F}^{k,l}$, $(k,l) \in B$, by definition we have $(y^k, v_k) = (\mathcal{P}^{k,1}, 0)$. For each $(k,l) \in B$ and $d = n', \ldots, n$, define the point $\mathcal{F}_3^{k,l,d} = (y, v, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{F}^{k,l}$ except that $(y^k, v_k) = (\mathcal{T}^{k,l,d}, \beta_{kl}^{(n',d)})$. By Lemma 3(a,b,c,e), it is clear that $\mathcal{F}_3^{k,l,d} \in Q^{m,n}$ for all $(k,l) \in B$ and $d = n', \ldots, n$. Using Lemma 4(a,b,c,e), we can easily verify that they also satisfy (43) (note that substituting $\mathcal{F}_3^{k,l,d}$ in (43) gives identity (51) again), and hence belong to Γ and must satisfy (49). Now if for each $(k,l) \in B$, we substitute $\mathcal{F}^{k,l}$ and $\mathcal{F}_3^{k,l,n'}$ into (49) and subtract one equality from the other, we get

$$\lambda_{n'}^k = \rho_k \left(\beta_k^{(n')} - \beta_l^{(n')} \right), \text{ for all } (k,l) \in B, \tag{55}$$

and if we continue to do the same with $\mathcal{F}_3^{k,l,n'+1}, \ldots, \mathcal{F}_3^{k,l,n}$ one after the other, in light of condition (c), we get

$$\lambda_d^k = \alpha_d \rho_k, d = n' + 1, \ldots, n, \text{ for all } (k,l) \in B. \tag{56}$$

Based on (50), (52), (54), (55), (56), and using (18), hyperplane (49) reduces to

$$\begin{aligned}&\sum_{(i,j)\in F}\rho_i\left(s+v_i+\sum_{t=n'+1}^{n}\alpha_t y_t^i-\beta_i^{(n')}+\left(\beta_i^{(n')}-\beta_j^{(n')}+\alpha_{n'}\right)\left(1-\phi_i^{n'}(y^i)\right)\right)\\&=\sum_{(i,j)\in B}\rho_i\left(\left(\beta_i^{(n')}-\beta_j^{(n')}\right)\phi_i^{n'}(y^i)-\sum_{t=n'+1}^{n}\alpha_t y_t^i-v_i\right).\end{aligned}\tag{57}$$

Now, for $i \in V_C$, let $P(i)$ be the node in V_C such that $(P(i), i) \in A_C$, and define $i_a := \min\{j \in V_C : i < j\}$ and $i_b := \max\{j \in V_C : j < i\}$. Also let $i_{max} = \max\{i : i \in V_C\}$ and $i_{min} = \min\{i : i \in V_C\}$. For $l \in V_C\backslash\{i_{max}\}$, if we substitute the point $\mathcal{F}^{P(l),l}$ and $\mathcal{F}^{P(l_a),l_a}$ into (57) (note that both points must satisfy (57) as argued for all points $\mathcal{F}^{k,l}$) and subtract the two equalities, we get $\sum_{\substack{(i,j)\in F\\ i<l_a\le j}}\rho_i\left(\beta_l^{(n')}-\beta_{l_a}^{(n')}\right)+\sum_{\substack{(i,j)\in B\\ j<l_a\le i}}\rho_i\left(\beta_{l_a}^{(n')}-\beta_l^{(n')}\right)=0$. Since $\beta_l^{(n')}\neq\beta_{l_a}^{(n')}$, we get

$$\sum_{(i,j)\in F; i<l_a\le j}\rho_i-\sum_{(i,j)\in B; j<l_a\le i}\rho_i=0.\tag{58}$$

Likewise, for $l \in V_C\backslash\{i_{min}\}$, if we substitute the point $\mathcal{F}^{P(l_b),l_b}$ and $\mathcal{F}^{P(l),l}$ into equality (57) and subtract the two equalities, we get

$$\sum_{(i,j)\in F; i<l\le j}\rho_i-\sum_{(i,j)\in B; j<l\le i}\rho_i=0\tag{59}$$

because $\beta_{l_b}^{(n')}\neq\beta_l^{(n')}$. Notice that if $l = P(i_{max})$, then $l_a = i_{max}$, and identity (58) reduces to

$$\rho_{P(i_{max})}=\rho_{i_{max}}\tag{60}$$

Also if for each $l \in V_C \backslash \{i_{min}, i_{max}\}$, we subtract (58) from (59), we get

$$\rho_{P(l)} = \rho_l, \quad l \in V_C \backslash \{i_{min}, i_{max}\}. \tag{61}$$

Identities (60) and (61) imply that $\rho_{P(l)} = \rho_l$ for all $l \in V_C$ (because $P(i) = i_{min}$ for some $i \in V_C \backslash \{i_{min}\}$). Therefore,

$$\rho_i = \rho_j \text{ for all } i, j \in V_C \tag{62}$$

as C is a cycle. This reduces hyperplane (57) to a constant multiple (by (50) this multiple is $\rho_0/|F|$) of (43), which completes the proof . □

Example 1 (continued). *Notice that for $n' = 1$, each cycle $C = (V_C, A_C)$ in graph G_1 with a set of backward arcs $B = \{(i, 6)\}$, for $i \in \{1, \ldots, 5\}$, satisfy the additional conditions required for Theorem 7, i.e. (a) $\left\lfloor \beta_k^{(1)}/\alpha_2 \right\rfloor = 3 \geq 1$, for $k \in \{1, \ldots, 5\}$, $\left\lfloor \beta_6^{(1)}/\alpha_2 \right\rfloor = 2 \geq 1$, (b) this condition is automatically satisfied for $n' = 1$, and (c) $\left\lfloor \beta_{k,6}^{(1,1)}/\alpha_2 \right\rfloor = \left\lfloor \left(\beta_k^{(1)} - \beta_6^{(1)}\right)/\alpha_2 \right\rfloor \geq 1$, for $k = 1, \ldots, 5$. Therefore, the 1-step cycle inequality* (32) *corresponding to a cycle C in G_1, where $B = \{(i, 6)\}$ for $i \in \{1, \ldots, 5\}$, defines facet for conv$(Q^{6,2})$. In particular, the 1-step cycle inequalities corresponding to the cycles C_1^1 and C_2^1 are facet-defining for conv$(Q^{6,2})$.*

Now, for $n' = 2$, the coefficients of $Q^{6,2}$ also satisfy the additional conditions required in Theorem 7, i.e. (a) $\left\lfloor \beta_k^{(1)}/\alpha_2 \right\rfloor = 3 \geq 1$, for $k \in \{1, \ldots, 5\}$, $\left\lfloor \beta_6^{(1)}/\alpha_2 \right\rfloor = 2 \geq 1$, (b) $\beta_l^{(2)} - \beta_k^{(2)} \geq 2 = \alpha_1 - \alpha_2 \left\lceil \beta_k^{(1)}/\alpha_2 \right\rceil$ for all $(k, l) \in A_2$ such that $1 \leq k < l \leq 6$, and there is no condition (c) for $n' = n = 2$. Therefore, the 2-step cycle inequality (32) *corresponding to each cycle $C = (V_C, A_C)$ in graph G_2, where $V_C \subseteq \{1, \ldots, 6\}$, defines a facet for conv$(Q^{6,2})$. In particular, 2-step cycle inequalities corresponding to the cycles C_1^2 and C_2^2 are facet-defining for conv$(Q^{6,2})$.*

Theorem 8. *For $n' \in \{1, \ldots, n\}$, the n'-step cycle inequality* (32) *for an elementary cycle $C = (V_C, A_C)$ of graph G is facet-defining for conv$(Q^{m,n})$ if (in addition to the n'-step MIR conditions* (33)*) the following condition hold*

(a) $T(F) = \{0\}$,

(b) $\left\lfloor \beta_{kl}^{(n',d-1)} / \alpha_d \right\rfloor \geq 1$, $d = n'+1, \ldots, n$, *for all* $(k,l) \in B$.

Proof. As shown before, the supporting hyperplane of inequality (32) can be written as (43), which for the C considered in this theorem reduces to

$$s = \sum_{(i,j) \in B} \left(\left(\beta_i^{(n)} - \beta_j^{(n)} \right) \phi^i(y^i) - \sum_{t=n'+1}^{n} \alpha_t y_t^i - v_i \right) \tag{63}$$

because by condition (a), the cycle C has only one forward arc, which goes out of node 0, and we have $v_0 = 0$, $y^0 = 0$ and $\phi_0^{n'}(y^0) := 1$ by definition. Let Γ be the face of $Q^{m,n}$ defined by hyperplane (63). We prove that any generic hyperplane

$$\rho_0 s + \sum_{i=1}^{m} \rho_i v_i + \sum_{i=1}^{m} \sum_{t=1}^{n} \lambda_j^i y_j^i = \theta \tag{64}$$

that passes through Γ is a scalar multiple of (63). By the same reasoning we reduced hyperplane (44) to (45) in Theorem 7, we can reduce hyperplane (64) to

$$\sum_{i \in V_C \setminus \{0\}} \sum_{t=1}^{n} \lambda_t^i y_t^i + \sum_{i \in V_C \setminus \{0\}} \rho_i v_i + \rho_0 s = \theta. \tag{65}$$

Now consider the following points (correspondig to the points with the same name in the proof of Theorem 7): The point $\mathcal{A} = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$ such that $(y^i, v_i) = (\mathcal{P}^{i,1}, 0)$, $i = 1, \ldots, m$, and $s = 0$; the points $\mathcal{C}^{k,d} = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^m \times \mathbb{R}_+$, for $k \in T(B)$, $d = 2, \ldots, n'$, such that $(y^k, v_k) = (\mathcal{P}^{k,d}, 0)$ and $(y^i, v_i) = (\mathcal{P}^{i,1}, 0)$ for $i \in \{1, \ldots, m\} \backslash (T(F) \cup \{k\})$, and $s = 0$; the points $\mathcal{F}^{k,l} = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^m \times \mathbb{R}_+$, for $(k, l) \in B$, such that

$$(y^i, v_i) = \begin{cases} (\mathcal{Q}^{i,n'}, 0) & \text{if } i \in T(B), i \leq l \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \in T(B), N(i) \geq l \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin V_C, \end{cases}$$

for $i = 1, \ldots, m$, and $s = \beta_l{}^{(n')}$; and the points $\mathcal{F}_3^{k,l,d} \in (\mathbb{Z} \times \mathbb{Z}_+^{n-1})^m \times \mathbb{R}_+^{m+1}$, for $(k, l) \in B$, $d = n', \ldots, n$, whose coordinates are all exactly the same as $\mathcal{F}^{k,l}$ except that $(y^k, v_k) = (\mathcal{T}^{k,l,d}, \beta_{kl}^{(n',d)})$.

By Lemma 3(a,b,e), all the aforementioned points belong to $Q^{m,n}$, and by Lemma 4(a,b,e), it is easy to verify that they also satisfy (63). So, they belong to Γ, and hence must satisfy (65). Therefore, given conditions (b), all these points can be used in the same fashion the points with similar names were used in the proof of Theorem 7 to reduce the hyperplane (65) to an equality which is ρ_0 times the hyperplane (63). This completes the proof. □

Example 1 (continued)**.** *Notice that for $n' = 1$, each cycle $C = (V_C, A_C)$ in graph G_1 with $A_C = \{(0, i), (i, 0)\}$ for $i \in \{1, \ldots, 6\}$ or $A_C = \{(0, i), (i, 6), (6, 0)\}$ for $i \in \{1, \ldots, 5\}$ satisfies the conditions required for Theorem 8, i.e. (a) $T(F) = \{0\}$, and (b) $\left\lfloor \beta_{k,l}^{(1,1)} / \alpha_2 \right\rfloor = \left\lfloor \left(\beta_k^{(1)} - \beta_l^{(1)} \right) / \alpha_2 \right\rfloor \geq 1$ for all $(k, l) \in \left\{ (i, j) \in A_C : \beta_i^{(1)} > \beta_j^{(1)} \right\}$. Therefore, the 1-step cycle inequality* (32) *corresponding to each cycle C defines a facet for conv($Q^{6,2}$). In particular, 1-step cycle inequality corresponding to the cycle C_3^1 is facet-defining for conv($Q^{6,2}$). Moreover, the 2-step cycle inequality* (32) *corresponding to each cycle $C = (V_C, A_C)$ in $G_2 = (V_2, A_2)$, where $T(F) = \{0\}$, also* *defines facet for conv*$(Q^{6,2})$ *because there is no condition (b) for* $n' = n = 2$. *In particular, 2-step cycle inequality corresponding to the cycle* C_3^2 *is facet-defining for conv*$(Q^{6,2})$.

III.3 Separation Algorithm

Given a point $(\hat{y}, \hat{v}, \hat{s})$ and $n' \in \{1, \ldots, n\}$, it is possible to solve the exact separation problem over all the n'-step cycle inequalities for the set $Q^{m,n}$. The goal is to find an n'-step cycle inequality (32) that is violated by $(\hat{y}, \hat{v}, \hat{s})$, if any. This can be done by detecting a negative weight cycle (if any) in the directed graph $G_{n'} = (V, A)$ with weights $\psi_{ij}^{n'}(\hat{y}, \hat{v}, \hat{s})$ for each arc $(i, j) \in A$. This means that the most negative cycle in $G_{n'}$ (if it exists) corresponds to the n'-step cycle inequality that is most violated by $(\hat{y}, \hat{v}, \hat{s})$. However, the problem of finding the most negative cycle in a graph is strongly NP-hard [99]. A method proposed by Cherkassy and Goldberg [25] (which is a combination of the cycle detection strategy of Tarjan [100] and the Bellman-Ford-Moore's labeling algorithm [33]), denoted by BFCT, is one of the fastest known algorithms to detect a negative cycle. BFCT terminates when it finds the first negative cycle; however, there may be cycles with smaller weight in the graph which would lead to stronger inequalities. Therefore, we devised a modified version of BFCT, denoted by MBFCT. The pseudocode of MBFCT is presented in Algorithm 1 and it works as follows:

For each node $i \in V$, we maintain $distance(i)$, $parent(i)$, and $status(i) \in \{unreached, labeled,\ scanned\}$ (refer Lines 2-4 of Algorithm 1). Initially for every node $i \in V, distance(i) = \infty, parent(i) = null$, and $status(i) =$ "*unreached*." The algorithm starts by setting $status(0) =$ "*labeled*" and $distance(0) = 0$ in Line 5. It also maintains a set of labeled nodes, denoted by $label := \{i \in V : status(i) = \text{"}labeled\text{"}\}$, in a first-in, first-out queue. This means a newly labeled node is added at the tail

Algorithm 1 Separation Algorithm for n'-step Cycle Inequalities

1: **function** MBFCT($G_{n'}, (\hat{y}, \hat{v}, \hat{s}), n'$)
2:     **for** $i \in V$ **do**
3:         $d(i) \leftarrow \infty$; $parent(i) \leftarrow Null$; $status(i) \leftarrow$ "unreached";
4:     **end for**
5:     $NC \leftarrow \emptyset, label \leftarrow \{0\}$; $status(0) \leftarrow$ "labeled"; $d(0) \leftarrow 0$; $Count \leftarrow 0$;
6:     **for** $i \in label$ and $Count \leq 3|V|$ **do** ▷ `FIFO selection rule`
7:         **for** $(i, j) \in A$ **do**
8:             **if** $d(i) + \psi_{ij}^{n'}(\hat{y}, \hat{v}, \hat{s}) < d(j)$ **then**
9:                 $d(j) \leftarrow d(i) + \psi_{ij}^{n'}(\hat{y}, \hat{v}, \hat{s})$; $status(j) \leftarrow$ "labeled"; $parent(j) \leftarrow i$;
10:                 $\bar{A}_p \leftarrow \{(parent(j), j) : j \in V, parent(j) \neq Null\}$;
11:                 Construct graph $\bar{G}^p = (V, \bar{A}_p)$
12:                 **if** the subtree of $\bar{G}^p$ rooted at j contains i **then**
13:                     $NC \leftarrow NC \cup \{(j \sim i - j)\}$
14:                     ▷ $j \sim i$ `denotes the path from node` j `to node` i `in` $\bar{G}^p$
15:                 **else**
16:                     remove all the nodes of subtree except j from $\bar{G}^p$
17:                     and change their *status* to unreached
18:                 **end if**
19:             **end if**
20:         **end for**
21:         $label \leftarrow label \backslash \{i\}; status(i) \leftarrow$ "scanned"; $Count \leftarrow Count + 1$;
22:     **end for**
23:     **return** the most negative cycle in NC (if exist)
24: **end function**

of the queue if they are not already on it. Therefore, at the start we set $label = \{0\}$ in Line 5. For each step, we remove the head node i from the queue *label* such that $status(i) =$ "*labeled*," and scan node i. The scanning of a labeled node i is performed as follows. For each arc $(i, j) \in A$ where $distance(i) + \psi_{ij}^{n'}(\hat{y}, \hat{v}, \hat{s}) < distance(j)$ (Line 8), we set $distance(j) = distance(i) + \psi_{ij}^{n'}(\hat{y}, \hat{v}, \hat{s}), parent(j) = i,$

$status(j)$ = "*labeled*", and add j at the tail of the queue *label* if $j \notin label$ (Line 9). This is called the labeling operation. Now, let $\bar{G}^p = (V, \bar{A}_p)$ be a subgraph of $G_{n'}$ such that $\bar{A}_p := \{(parent(j), j) : j \in V, parent(j) \neq null\}$. When the labeling operation is applied to an arc (i, j), the subtree of $\bar{G}^p$ rooted at j is traversed to find if it contains i (which implies that a negative cycle in $G_{n'}$ exists). On the other hand, if the node i is not in the subtree, all the nodes except j are removed from the current tree and their status is changed to "unreached." After scanning, the status of node i is updated to "scanned."

Unlike the BFCT [25], MBFCT does not stop after finding the first negative cycle and continues the search for other negative cycles (if any) until a certain termination condition is satisfied (see Line 6 in Algorithm 1). Out of all the cycles found by MBFCT, the one with the most negative weight is used to generate the n'-step cycle inequality (32) that separates $(\hat{y}, \hat{v}, \hat{s})$ with the largest violation among all generated cycles. Clearly, if MBFCT does not return any negative cycle, the point cannot be separated using the n'-step cycle inequalities.

We also note that as presented in [105] for the case of $n = 1$, for a general $n' \in \{1, \ldots, n\}$ we can also formulate the separation problem associated with the n'-step cycle inequalities as follows:

$$\min \left\{ \sum_{(i,j) \in E} \psi_{ij}^{n'}(\hat{y}, \hat{v}, \hat{s}) z_{ij} : \mathbf{M} z = 0, z \geq 0 \right\}. \tag{66}$$

where z_{ij} is a variable representing the flow along arc (i, j), $\mathbf{M}$ is the node-arc incidence matrix of G, and the goal is to test whether linear program (66) has a strictly negative solution value.

CHAPTER IV

CONTINUOUS MULTI-MIXING SET WITH GENERAL COEFFICIENTS

In this chapter, we relax the n-step MIR conditions imposed on the coefficients of continuous multi-mixing set (discussed in previous chapter) and consider the continuous multi-mixing set with general coefficients, denoted by

$$Y^m := \left\{ (y, v, s) \in \mathbb{Z}_+^{m \times N} \times \mathbb{R}_+^m \times \mathbb{R}_+ : \sum_{t \in \mathcal{N}} a_{it} y_t^i + v_i + s \geq b_i, i = 1, \ldots, m \right\}$$

where $\mathcal{N} := \{1, \ldots, N\}$, $a \in \mathbb{R}^{m \times N}$, and $b \in \mathbb{R}^m$ (no conditions are imposed on the coefficients). Note that the mixed integer knapsack set Y_0^1 is a special case of Y^m where $N = 1$. It is the projection of $Y^1 \cap \{v = 0\}$ on (y, s). In Section IV.1, we generalize n-step cycle inequalities, $n \in \mathbb{N}$, for Y^m, and discuss how the n-step MIR inequalities [62] are special cases of the n-step cycle inequalities. We also introduce a compact extended formulation for Y^m and observe that the separation over the set of all n-step cycle inequalities, $n \in \mathbb{N}$, for set Y^m can be performed using the separation algorithm (discussed in Chapter III) with slight modifications. In Section IV.2, we obtain conditions under which n-step cycle inequalities are facet-defining for $conv(Y^m)$.

IV.1 Valid Inequalities and Extended Formulation

In this section, given $n \in \mathbb{N}$, first we develop the n-step cycle inequalities for Y^m as follows: We choose a parameter vector $\alpha = (\alpha_1, \ldots, \alpha_n) > 0$ and without loss of generality, we assume $b_{i-1}^{(n)} \leq b_i^{(n)}$, $i = 2, \ldots, m$, where $b_i^{(n)}$ is defined as (11). Also define $b_0 := 0$. Now similar to the graph defined for the cycle inequalities (see Section

II.2.2), here we define a directed graph $G_n = (V, A)$, where $V := \{0, 1, \ldots, m\}$ and $A := \{(i, j) : i, j \in V, b_i^{(n)} \neq b_j^{(n)}\}$. G_n is a complete graph except for the arcs (i, j) where $b_i^{(n)} = b_j^{(n)}$. Here to each arc $(i, j) \in A$, we associate the linear function $\Psi_{ij}^n(y, v, s)$ defined as

$$\Psi_{ij}^n(y, v, s) := \begin{cases} s + v_i + \displaystyle\sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}_n^{i,n}}} a_{it}^{(n)} y_t^i + b_{ij}^{(n)} \left(1 - \Phi_i^n(y^i)\right) - b_j^{(n)} & \text{if } i < j, \\ v_i + \displaystyle\sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}_n^{i,n}}} a_{it}^{(n)} y_t^i + \left(b_i^{(n)} - b_j^{(n)}\right) \left(1 - \Phi_i^n(y^i)\right) & \text{if } i > j, \end{cases} \tag{67}$$

where $b_{ij}^{(n)} := b_i^{(n)} - b_j^{(n)} + \alpha_n$ for all $(i, j) \in A$, $i < j$,

$$\mathcal{I}_g^{i,n} := \{x \in \mathbb{R} : x^{(q)} < b_i^{(q)}, q = 1, \ldots, g, x^{(g+1)} \geq b_i^{(g+1)}\},$$
$$\mathcal{I}_n^{i,n} := \{x \in \mathbb{R} : x^{(q)} < b_i^{(q)}, q = 1, \ldots, n\},$$

for $g = 0, \ldots, n-1$, $i = 1, \ldots, m$, and the functions $\Phi_i^n(y^i)$, $i = 1, \ldots, m$, in its open form can be defined as

$$\begin{aligned} \Phi_i^n(y^i) := \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{g=0}^{n-1} \sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}_g^{i,n}}} \Bigg(\sum_{q=1}^{g} \prod_{l=q+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{it}^{(q-1)}}{\alpha_q} \right\rfloor \\ + \prod_{l=g+2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_{it}^{(g)}}{\alpha_{g+1}} \right\rceil \Bigg) y_t^i - \sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}_n^{i,n}}} \sum_{q=1}^{n} \prod_{l=q+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{it}^{(q-1)}}{\alpha_q} \right\rfloor y_t^i \end{aligned} \tag{68}$$

and by definition, $v_0 := 0$, $y^0 := 0$, and $\Phi_0^n(y^0) := 1$.

We show that each elementary cycle of graph G_n corresponds to a valid inequality for the set Y^m, which we also refer to as the *n-step cycle inequality*. To do this in addition to Lemma 1, we need the following lemma:

Lemma 5. *For $i \in \{1, \dots, m\}$ and $n \in \mathbb{N}$, the inequality*

$$s + v_i + \sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}_n^{i,n}}} a_{it}^{(n)} y_t^i + \alpha_n \left(1 - \Phi_i^n(y^i)\right) \geq b_i^{(n)} \tag{69}$$

is valid for Y^m if $\alpha_d \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{d-1}, d = 2, \dots, n$.

Proof. Kianfar and Fathi [62] proved that the following inequality

$$\begin{aligned} s + v_i + \sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}_n^{i,n}}} a_{it}^{(n)} y_t^i + \alpha_n \Bigg\{ \sum_{g=0}^{n-1} \sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}_g^{i,n}}} \Bigg(\sum_{q=1}^{g} \prod_{l=q+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{it}^{(q-1)}}{\alpha_q} \right\rfloor \\ + \prod_{l=g+2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_{it}^{(g)}}{\alpha_{g+1}} \right\rceil \Bigg) y_t^i + \sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}_n^{i,n}}} \sum_{q=1}^{n} \prod_{l=q+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{it}^{(q-1)}}{\alpha_q} \right\rfloor y_t^i \Bigg\} \\ - \alpha_n \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil + \alpha_n \geq b_i^{(n)} \end{aligned} \tag{70}$$

is valid for the relaxation of Y^m defined by its i'th constraint, i.e. $\{(y^i, v_i, s) \in \mathbb{Z}_+^N \times \mathbb{R}_+ \times \mathbb{R}_+ : \sum_{t \in \mathcal{N}} a_{it} y_t^i + v_i + s \geq b_i\}$, for $\alpha := (\alpha_1, \dots, \alpha_n)$ satisfying $\alpha_d \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{d-1}, d = 2, \dots, n$. Therefore, it is also valid for Y^m. Note that rearranging the terms in (70) and using (68) gives (69). □

Theorem 9. *Given $n \in \mathbb{N}$ and an elementary cycle $C = (V_C, A_C)$ of graph G_n, the n-step cycle inequality*

$$\sum_{(i,j) \in A_C} \Psi_{ij}^n(y, v, s) \geq 0 \tag{71}$$

is valid for Y^m if the parameters $(\alpha_1, \dots, \alpha_n)$ satisfy

$$\alpha_d \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{d-1}, d = 2, \dots, n, i \in V_C. \tag{72}$$

Proof. Consider a point $(\hat{y}, \hat{v}, \hat{s}) \in Y^m$. Based on Lemma 5, inequality (69) is satisfied by the point $(\hat{y}, \hat{v}, \hat{s})$ for each $i \in V_C \setminus \{0\}$ because of (72). But notice that inequality (69) for this point is the same as inequality (20) if we define $\sigma := \hat{s}$, $\alpha := \alpha_n$, and $\omega_i := \hat{v}_i + \sum_{t \in \mathcal{N}, a_{it} \in \mathcal{I}_n^{i,n}} a_{it}^{(n)} \hat{y}_t^i$, $\kappa_i := 1 - \Phi_n^i(\hat{y}^i)$, $\gamma_i := b_i^{(n)}$, $i \in V_C \setminus \{0\}$. Also, in case $0 \in V_C$, if we define ω_0, κ_0, and γ_0 in a similar way, inequality (20) for $i = 0$ reduces to the valid inequality $\hat{s} \geq 0$ because as we defined before $y^0 := 0$, $v_0 := 0$, $\Phi_0^n(y^0) := 1$, and $b_0 := 0$. With these definitions, we have $\omega_i \geq 0$, $\kappa_i \in \mathbb{Z}$, $i \in V_C$ and $0 = \gamma_0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_{|V_C|} < \alpha_n$. Therefore, according to Lemma 1, inequality (21) in which σ, α and $\omega_i, \kappa_i, \gamma_i, i \in V_C$ are replaced with the values defined here is valid. It is easy to see that this inequality is exactly the same as the n-step cycle inequality (71) for the point $(\hat{y}, \hat{v}, \hat{s})$. This completes the proof. $\square$

Special Cases: For each $n \in \mathbb{N}$, the n-step cycle inequality (71) written for cycle $C = (V_C, A_C)$ such that $A_C = \{(0, i), (i, 0)\}$ gives the n-step MIR inequality (13) written for constraint i in Y^m.

Separation Algorithm. Given a point $(\hat{y}, \hat{v}, \hat{s})$ and $n \in \mathbb{N}$, we can also formulate the separation problem associated with the n-step cycle inequalities (71) as follows:

$$\min \left\{ \sum_{(i,j) \in A} \Psi_{ij}^n(\hat{y}, \hat{v}, \hat{s}) z_{ij} : \mathbf{M}z = 0, z \geq 0 \right\}. \tag{73}$$

where z_{ij} is a variable representing the flow along arc (i, j), $\mathbf{M}$ is the node-arc incidence matrix of G_n, and the goal is to test whether linear program (113) has a strictly negative solution value. Therefore, for the point $(\hat{y}, \hat{v}, \hat{s})$, we find an n-step cycle inequality (71) that is violated by $(\hat{y}, \hat{v}, \hat{s})$, if any, by detecting a negative weight cycle (if any) in the directed graph G_n with weights $\Psi_{ij}^n(\hat{y}, \hat{v}, \hat{s})$ for each arc $(i, j) \in A$ (refer to Section III.3 for details).

Example 2. *Consider the following continuous multi-mixing set with 5 rows and general coefficients:*

$$
\begin{aligned}
Y^5 = \{(y, v, s) \in \mathbb{Z}_+^{6\times 5} \times \mathbb{R}_+^6 :& \\
& 52y_1^1 + 35y_2^1 - 125y_3^1 + 17y_4^1 - 19y_5^1 - 57y_6^1 + v_1 + s \geq 88, \\
& 33y_1^2 + 35y_2^2 + 84y_3^2 + 17y_4^2 - 53y_5^2 - 125y_6^2 + v_2 + s \geq 163, \\
& 16y_1^3 + 35y_2^3 - 3y_3^3 + 17y_4^3 + 34y_5^3 + 48y_6^3 + v_3 + s \geq 61, \\
& -21y_1^4 + 35y_2^4 + 87y_3^4 + 17y_4^4 + 122y_5^4 - 36y_6^4 + v_4 + s \geq 135, \\
& 56y_1^5 + 35y_2^5 + 64y_3^5 + 17y_4^5 + 19y_5^5 + 52y_6^5 + v_5 + s \geq 86\}.
\end{aligned}
$$

We have $\mathcal{N} = \{1, \ldots, 6\}$*,* $b_1 = 88$*,* $b_2 = 163$*,* $b_3 = 61$*,* $b_4 = 135$*, and* $b_5 = 86$*. Assuming* $(\alpha_1, \alpha_2) = (35, 17)$*, we have* $b_5^{(1)} = 16 < b_1^{(1)} = 18 < b_2^{(1)} = 23 < b_3^{(1)} = 26 < b_4^{(1)} = 30$*, and* $b_1^{(2)} = 1 < b_2^{(2)} = 5 < b_3^{(2)} = 9 < b_4^{(2)} = 13 < b_5^{(2)} = 16$*. Note that* $\left\lceil b_i^{(1)}/\alpha_2 \right\rceil = 2$ *for* $i = 1, \ldots, 4$*,* $\left\lceil b_5^{(1)}/\alpha_2 \right\rceil = 1$*, and clearly the conditions (72), i.e.* $\alpha_1 \geq \alpha_2 \left\lceil b_i^{(1)}/\alpha_2 \right\rceil$*, are satisfied for* $i = 1, \ldots, 5$*. Note that* $a_{13}, a_{15}, a_{16} \in \mathcal{I}_1^{1,2}$*,* $a_{11}, a_{12}, a_{14} \in \mathcal{I}_2^{1,2}$*,* $a_{21} \in \mathcal{I}_0^{2,2}$*,* $a_{23}, a_{25} \in \mathcal{I}_1^{2,2}$*,* $a_{22}, a_{24}, a_{26} \in \mathcal{I}_2^{2,2}$*,* $a_{33}, a_{35} \in \mathcal{I}_0^{3,2}$*,* $a_{31}, a_{36} \in \mathcal{I}_1^{3,2}$*,* $a_{32}, a_{34} \in \mathcal{I}_2^{3,2}$*,* $a_{46} \in \mathcal{I}_0^{4,2}$*,* $a_{41} \in \mathcal{I}_1^{4,2}$*,* $a_{42}, a_{43}, a_{44}, a_{45} \in \mathcal{I}_2^{4,2}$*,* $a_{51}, a_{53}, a_{55}, a_{56} \in \mathcal{I}_0^{5,2}$*, and* $a_{52}, a_{54} \in \mathcal{I}_2^{5,2}$*. Observe that for* $i = 1, \ldots, 5$*,* $a_{i2} = \alpha_1$*,* $a_{i4} = \alpha_2$*,* $a_{i2}, a_{i4} \in \mathcal{I}_2^{i,2}$ *and* $a_{i2}^{(2)} = a_{i4}^{(2)} = 0$*. Therefore, we define* $\mathcal{N}_\alpha = \{2, 4\}$*. We also have* $a_{ir}^{(2)} = 0$*, for* $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ *and* $i = 1, \ldots, 5$*, where* $a_{ir} \in \mathcal{I}_2^{i,2}$*.*

2-step cycle inequalities for $\boldsymbol{Y^5}$: *Setting* $n = 2$*, the set of nodes and arcs of the graph* G_2 *will be* $V_2 = \{0, \ldots, 5\}$ *and* $A_2 = \{(i, j) : i, j \in V_2\}$*. The linear function*

$\Psi^2_{ij}(y,v,s)$ *associated with each arc* $(i,j) \in A_2$ *is defined by* (67) *where* $n = 2$*, i.e.*

$$\Psi^2_{ij}(y,v,s) := \begin{cases} s + v_i + \displaystyle\sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}^{i,2}_2}} a^{(2)}_{it} y^i_t + b^{(2)}_{ij}\left(1 - \Phi^2_i(y^i)\right) - b^{(2)}_j & \text{if } i < j, \\ v_i + \displaystyle\sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}^{i,2}_2}} a^{(2)}_{it} y^i_t + \left(b^{(2)}_i - b^{(2)}_j\right)\left(1 - \Phi^2_i(y^i)\right) & \text{if } i > j, \end{cases}$$

where $b^{(2)}_{ij} := b^{(2)}_i - b^{(2)}_j + \alpha_2$ *for all* $(i,j) \in A$, $i < j$,

$$\Phi^2_i(y^i) := \prod_{l=1}^{2} \left\lceil \frac{b^{(l-1)}_i}{\alpha_l} \right\rceil - \sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}^{i,2}_0}} \left\lceil \frac{b^{(1)}_i}{\alpha_2} \right\rceil \left\lceil \frac{a_{it}}{\alpha_1} \right\rceil y^i_t - \sum_{\substack{t \in \mathcal{N} \\ a_{it} \in \mathcal{I}^{i,2}_1}} \left(\left\lceil \frac{b^{(1)}_i}{\alpha_2} \right\rceil \left\lfloor \frac{a_{it}}{\alpha_1} \right\rfloor + \left\lceil \frac{a^{(1)}_{it}}{\alpha_2} \right\rceil \right) y^i_t$$
$$- \sum_{\substack{t \in \mathcal{N} \setminus \mathcal{N}_\alpha \\ a_{it} \in \mathcal{I}^{i,2}_2}} \sum_{q=1}^{2} \prod_{l=q+1}^{2} \left\lceil \frac{b^{(l-1)}_i}{\alpha_l} \right\rceil \left\lfloor \frac{a^{(q-1)}_{it}}{\alpha_q} \right\rfloor y^i_t - \left\lceil \frac{b^{(1)}_i}{\alpha_2} \right\rceil y^i_2 - y^i_4,$$

and $v_0 := 0$, $y^0 := 0$, $a_{0t} = 0$ *for* $t \in \mathcal{N}$, *and* $\Phi^2_0(y^0) := 1$. *Based on Theorem 9, the 2-step cycle inequalities corresponding to the cycles in* G_2 *are valid for* Y^5. *For example, the 2-step cycle inequality corresponding to a cycle* $C = (V_C, A_C)$ *in* G_2 *where* $A_C = \{(1,3),(3,5),(5,4),(4,2)\}$ *is*

$$\Psi^2_{13} + \Psi^2_{35} + \Psi^2_{54} + \Psi^2_{42} \geq 0. \tag{74}$$

Likewise, for a cycle C *in* G_2 *with* $A_C = \{(1,4),(4,2),(2,5),(5,1)\}$, *the 2-step cycle inequality is*

$$\Psi^2_{14} + \Psi^2_{42} + \Psi^2_{25} + \Psi^2_{51} \geq 0, \tag{75}$$

and for a cycle C *in* G_2 *with* $A_C = \{(0,5),(5,4),(4,1),(1,0)\}$, *the 2-step cycle inequality is*

$$\Psi^2_{05} + \Psi^2_{54} + \Psi^2_{41} + \Psi^2_{10} \geq 0. \tag{76}$$

Theorem 10. *The following linear program is a compact extended formulation for Y^m, if conditions* (72) *hold.*

$$\Psi_{ij}^n(y, v, s) \geq \delta_i^n - \delta_j^n \text{ for all } (i, j) \in A, n \in \{1, \dots, n\} \tag{77}$$

$$\textstyle\sum_{t=1}^n a_{it} y_t^i + v_i + s \geq b_i, i = 1, \dots, m \tag{78}$$

$$y \in \mathbb{R}_+^{mn}, v \in \mathbb{R}_+^m, s \in \mathbb{R}_+, \delta \in \mathbb{R}^{n(m+1)}. \tag{79}$$

Proof. Let $Y^{m,\delta} := \{(y, v, s, \delta)$ satisfying (77)-(79)$\}$. Clearly $Proj_{y,v,s}(Y^{m,\delta})$ is defined by the set of all n-step cycle inequalities (71), for $n = 1, \dots, n$, and bound constraints $s, v \geq 0$. This means all the inequalities which define $Proj_{y,v,s}(Y^{m,\delta})$ are valid for Y^m if the parameters $(\alpha_1, \dots, \alpha_n)$ satisfy conditions (72) which implies $Y^m \subseteq Proj_{y,v,s}(Y^{m,\delta})$ under the same conditions. This proves that $Y^{m,\delta}$ is an extended formulation for Y^m. □

IV.2 Facet-Defining n-step Cycle Inequalities

In this section, we show that for any $n \in \mathbb{N}$, the n-step cycle inequalities (71) define facets for $conv(Y^m)$ under certain conditions. In order to prove this, we first define $\mathcal{N}_\alpha := \{t_1, \dots, t_n\} \subseteq \mathcal{N}$ such that for $t \in \mathcal{N}_\alpha$, $a_{it} = a_{jt} (> 0)$, $i, j \in V_C$. Then we assign parameter $\alpha_d = a_{it_d}$ for $i \in V_C$ and $d = 1, \dots, n$ and re-write (68) as follows:

$$\Phi_i^n(y^i) := \prod_{l=1}^{n}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil - \sum_{g=0}^{n-1}\sum_{\substack{t\in\mathcal{N}\setminus\mathcal{N}_\alpha\\ a_{it}\in\mathcal{I}_g^{i,n}}}\left(\sum_{q=1}^{g}\prod_{l=q+1}^{n}\left\lceil\frac{b_i^{(l-1)}}{\alpha_l}\right\rceil\left\lfloor\frac{a_{it}^{(q-1)}}{\alpha_q}\right\rfloor\right.$$
$$\left. + \prod_{l=g+2}^{n}\left\lceil\frac{b_i^{(l-1)}}{\alpha_l}\right\rceil\left\lceil\frac{a_{it}^{(g)}}{\alpha_{g+1}}\right\rceil\right)y_t^i - \sum_{\substack{t\in\mathcal{N}\setminus\mathcal{N}_\alpha\\ a_{it}\in\mathcal{I}_n^{i,n}}}\sum_{q=1}^{n}\prod_{l=q+1}^{n}\left\lceil\frac{b_i^{(l-1)}}{\alpha_l}\right\rceil\left\lfloor\frac{a_{it}^{(q-1)}}{\alpha_q}\right\rfloor y_t^i \quad (80)$$
$$-\sum_{d=1}^{n}\prod_{l=d+1}^{n}\left\lceil\frac{b_i^{(l-1)}}{\alpha_l}\right\rceil y_{t_d}^i.$$

Next, we redefine some points (introduced in Chapter III), introduce some new points, and provide some properties for them. Note that in the following definitions we only describe nonzero components for each point.

Definition 14. *For $i \in \{1,\ldots,m\}$, define the points $\mathcal{P}^{i,r}, \mathcal{Q}^{i,r} \in \mathbb{Z}_+^N$, $r = 1,\ldots,n$, as follows:*

$$\mathcal{P}_{t_d}^{i,r} := \begin{cases}\left\lfloor\frac{b_i^{(d-1)}}{\alpha_d}\right\rfloor & d = 1,\ldots,r-1,\\ \left\lceil\frac{b_i^{(d-1)}}{\alpha_d}\right\rceil & d = r\end{cases} \qquad \mathcal{Q}_{t_d}^{i,r} := \left\{\left\lfloor\frac{b_i^{(d-1)}}{\alpha_d}\right\rfloor \quad d = 1,\ldots,r,\right.$$

and the point $\mathcal{R}^i \in \mathbb{Z}_+^N$ (assuming $\left\lfloor b_i^{(n-1)}/\alpha_n\right\rfloor \geq 1$) as $\mathcal{R}^i = \mathcal{Q}^{i,n} - e_{t_n}$, where e_{t_n} is the t_nth unit vector in $\mathbb{R}^n$. Also, define the points $\mathcal{S}^{i,r} \in \mathbb{Z}_+^N$, $r = 2,\ldots,n$, (assuming $\left\lfloor b_i^{(r-1)}/\alpha_r\right\rfloor \geq 1$, $r = 1,\ldots,n$) as follows:

$$\mathcal{S}_{t_d}^{i,r} := \begin{cases}\mathcal{Q}_{t_d}^{i,n} & d = 1,\ldots,r-2,r+1,\ldots,n\\ \left\lfloor\frac{b_i^{(d-1)}}{\alpha_d}\right\rfloor - 1 & d = r-1,\\ 2\left\lfloor\frac{b_i^{(d-1)}}{\alpha_d}\right\rfloor + 1 & d = r,\end{cases}$$

the points $\mathcal{T}^{i,g,r}, \mathcal{U}^{i,g,r} \in \mathbb{Z}_+^N$, $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ *where* $a_{ir} \in \mathcal{I}_g^{i,n}$ *and* $g \in \{0, \ldots, n-1\}$, *as follows:*

$$\mathcal{T}_t^{i,g,r} := \begin{cases} \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lceil \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = 1, \ldots, g+1, \\ 1 & t = r, \end{cases}$$

$$\mathcal{U}_t^{i,g,r} := \begin{cases} \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lceil \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = 1, \ldots, g, \\ \left\lfloor \frac{b_i^{(d-1)}}{\alpha_d} \right\rfloor - \left\lceil \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = g+1, \\ \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = g+2, \ldots, n, \\ 1 & t = r, \end{cases}$$

(note that by definition $a_{ir}^{(g)} < b_i^{(g)}$ *and* $a_{ir}^{(g+1)} = a_{ir}^{(g)} - \alpha_{g+1} \left\lfloor a_{ir}^{(g)}/\alpha_{g+1} \right\rfloor \geq b_i^{(g+1)} = b_i^{(g)} - \alpha_{g+1} \left\lfloor b_i^{(g)}/\alpha_{g+1} \right\rfloor > a_{ir}^{(g)} - \alpha_{g+1} \left\lfloor b_i^{(g)}/\alpha_{g+1} \right\rfloor$ *which implies* $\left\lfloor b_i^{(g)}/\alpha_{g+1} \right\rfloor > \left\lfloor a_{ir}^{(g)}/\alpha_{g+1} \right\rfloor$ *or* $\left\lfloor b_i^{(g)}/\alpha_{g+1} \right\rfloor \geq \left\lceil a_{ir}^{(g)}/\alpha_{g+1} \right\rceil$*), and the points* $\mathcal{V}^{i,r}, \mathcal{W}^{i,r} \in \mathbb{Z}_+^N$, $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ *where* $a_{ir} \in \mathcal{I}_n^{i,n}$, *as follows:*

$$\mathcal{V}_t^{i,r} := \begin{cases} \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lceil \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = 1, \ldots, n, \\ 1 & t = r, \end{cases}$$

$$\mathcal{W}_t^{i,r} := \begin{cases} \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lceil \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = 1, \dots, n-1, \\ \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor & t = t_d, d = n, \\ 1 & t = r. \end{cases}$$

Lemma 6. *The point* $(\hat{y}, \hat{v}, \hat{s}) \in \mathbb{Z}_+^{m \times N} \times \mathbb{R}_+^{m+1}$ *satisfies constraint* $i \in \{1, \dots, m\}$ *of* Y^m *if any of the following is true*

(a). $\hat{y}^i = \mathcal{P}^{i,r}$ *for some* $r \in \{1, \dots, n\}$

(b). $\hat{y}^i = \mathcal{Q}^{i,r}$ *for some* $r \in \{1, \dots, n\}$ *and* $\hat{v}_i + \hat{s} \geq b_i^{(r)}$,

(c). $\hat{y}^i = \mathcal{R}^i$ *and* $\hat{v}_i + \hat{s} \geq \alpha_n + b_i^{(n)}$,

(d). $\hat{y}^i = \mathcal{S}^{i,r}$ *for some* $r \in \{2, \dots, n\}$ *and* $\hat{v}_i + \hat{s} \geq b_i^{(n)} + \alpha_{r-1} - \alpha_r \left\lceil b_i^{(r-1)}/\alpha_r \right\rceil$,

(e). $\hat{y}^i = \mathcal{T}^{i,g,r}$ *for some* $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ *where* $a_{ir} \in \mathcal{I}_g^{i,n}$ *and* $g \in \{0, \dots, n-1\}$,

(f). $\hat{y}^i = \mathcal{U}^{i,g,r}$ *for some* $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ *where* $a_{ir} \in \mathcal{I}_g^{i,n}$ *and* $g \in \{0, \dots, n-1\}$, *and* $\hat{v}_i + \hat{s} \geq b_i^{(n)} + \alpha_{g+1} - a_{ir}^{(g+1)}$,

(g). $\hat{y}^i = \mathcal{V}^{i,r}$ *for some* $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ *where* $a_{ir} \in \mathcal{I}_n^{i,n}$, *and* $\hat{v}_i + \hat{s} \geq b_i^{(n)} - a_{ir}^{(n)}$,

(h). $\hat{y}^i = \mathcal{W}^{i,r}$ *for some* $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ *where* $a_{ir} \in \mathcal{I}_n^{i,n}$.

Proof. Cases (a) and (b) can be easily proved similar to the proof of Lemma 5 in [96]. Cases (c) and (d) can also be easily proved similar to the proof of Lemma 9 in [96]. For (e), notice that by substituting the point $(\hat{y}, \hat{v}, \hat{s})$ in constraint i of Y^m, we get $\sum_{d=1}^{g+1} \alpha_d \left(\left\lfloor b_i^{(d-1)}/\alpha_d \right\rfloor - \left\lfloor a_{ir}^{(d-1)}/\alpha_d \right\rfloor \right) + a_{ir} + \hat{v}_i + \hat{s} \geq b_i$, or $\hat{v}_i + \hat{s} \geq 0$, which is true by the definition of a_{ir}, i.e. $a_{ir}^{(g+1)} \geq b_i^{(g+1)}$. For (f), notice that by substituting the point $(\hat{y}, \hat{v}, \hat{s})$ in constraint i of Y^m, we get $\sum_{d=1}^{g+1} \alpha_d \left(\left\lfloor b_i^{(d-1)}/\alpha_d \right\rfloor - \left\lfloor a_{ir}^{(d-1)}/\alpha_d \right\rfloor \right) -$

$\alpha_{g+1} + \sum_{d=g+2}^{n} \alpha_d \left\lfloor b_i^{(d-1)}/\alpha_d \right\rfloor + a_{ir} + \hat{v}_i + \hat{s} \geq b_i$, or $\hat{v}_i + \hat{s} \geq b_i^{(n)} + \alpha_{g+1} - a_{ir}^{(g+1)}$, which is true by the assumption of (f). For (g), notice that by substituting the point $(\hat{y}, \hat{v}, \hat{s})$ in constraint i of Y^m, we get $\sum_{d=1}^{n} \alpha_d \left(\left\lfloor b_i^{(d-1)}/\alpha_d \right\rfloor - \left\lfloor a_{ir}^{(d-1)}/\alpha_d \right\rfloor \right) + a_{ir} + \hat{v}_i + \hat{s} \geq b_i$, or $\hat{v}_i + \hat{s} \geq b_i^{(n)} - a_{ir}^{(n)}$, which is true by the assumption of (g). For (h), notice that by substituting the point $(\hat{y}, \hat{v}, \hat{s})$ in constraint i of Y^m, we get $\sum_{d=1}^{n} \alpha_d \left(\left\lfloor b_i^{(d-1)}/\alpha_d \right\rfloor - \left\lfloor a_{ir}^{(d-1)}/\alpha_d \right\rfloor \right) + \alpha_n + a_{ir} + \hat{v}_i + \hat{s} \geq b_i$, or $\hat{v}_i + \hat{s} \geq 0$, which is true because $\alpha_n + a_{ir}^{(n)} \geq b_r^{(n)}$. □

Lemma 7. *For $i \in \{1, \ldots, m\}$ and $n \in \mathbb{N}$,*

(a). $\Phi_i^n(\mathcal{P}^{i,r}) = 0$, $r = 1, \ldots, n$,

(b). $\Phi_i^n(\mathcal{Q}^{i,r}) = 1$, $r = 1, \ldots, n$,

(c). $\Phi_i^n(\mathcal{R}^{i}) = 2$,

(d). $\Phi_i^n(\mathcal{S}^{i,r}) = 1$, $r = 2, \ldots, n$,

(e). $\Phi_i^n(\mathcal{T}^{i,g,r}) = 0$, for each $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{ir} \in \mathcal{I}_g^{i,n}$ and $g \in \{0, \ldots, n-1\}$,

(f). $\Phi_i^n(\mathcal{U}^{i,g,r}) = 1$, for each $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{ir} \in \mathcal{I}_g^{i,n}$ and $g \in \{0, \ldots, n-1\}$,

(g). $\Phi_i^n(\mathcal{V}^{i,r}) = 1$, for each $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{ir} \in \mathcal{I}_n^{i,n}$,

(h). $\Phi_i^n(\mathcal{W}^{i,r}) = 0$, for each $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{ir} \in \mathcal{I}_n^{i,n}$.

Proof. Cases (a) and (b) can be proved similar to Lemma 6 of [96] and cases (c) and (d) can be proved similar to Lemma 10 of [96]. The remaining cases are proved as follows: For $i \in \{1, \ldots, m\}$, $n \in \mathbb{N}$, and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{ir} \in \mathcal{I}_g^{i,n}$ and $g \in$

$\{0, \ldots, n-1\}$, we have

$$
\begin{aligned}
\Phi_i^n(\mathcal{T}^{i,g,r}) =& \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{g} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor - \prod_{l=g+2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_{ir}^{(g)}}{\alpha_{g+1}} \right\rceil \\
&- \sum_{d=1}^{g+1} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor - 1 \right) \\
=& \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \left(\prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) - \ldots \\
&- \left(\prod_{l=g}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=g+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) - \prod_{l=g+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil = 0,
\end{aligned}
$$

$$
\begin{aligned}
\Phi_i^n(\mathcal{U}^{i,g,r}) =& \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{g} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor - \prod_{l=g+2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_{ir}^{(g)}}{\alpha_{g+1}} \right\rceil \\
&- \sum_{d=1}^{g} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor - 1 \right) \\
&- \prod_{l=g+2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(g)}}{\alpha_{g+1}} \right\rceil - 1 - \left\lceil \frac{a_{ir}^{(g)}}{\alpha_{g+1}} \right\rceil \right) \\
&- \sum_{d=g+2}^{n} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - 1 \right) \\
=& \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \left(\prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) - \ldots \\
&- \left(\prod_{l=n}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=n+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) = 1.
\end{aligned}
$$

Finally, for $i \in \{1, \dots, m\}$, $n \in \mathbb{N}$, and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{ir} \in \mathcal{I}_n^{i,n}$, we have

$$\begin{aligned}
\Phi_i^n(\mathcal{V}^{i,r}) =& \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{n} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor \\
& - \sum_{d=1}^{n} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor - 1 \right) \\
=& \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \left(\prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) - \dots \\
& - \left(\prod_{l=n}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=n+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) = 1,
\end{aligned}$$

$$\begin{aligned}
\Phi_i^n(\mathcal{W}^{i,r}) =& \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{n} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor \\
& - \sum_{d=1}^{n-1} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_{ir}^{(d-1)}}{\alpha_d} \right\rfloor - 1 \right) \\
& - \prod_{l=n+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(n-1)}}{\alpha_n} \right\rceil - \left\lfloor \frac{a_{ir}^{(n-1)}}{\alpha_n} \right\rfloor \right) = 0.
\end{aligned}$$

This completes the proof. □

As before, given a cycle $C = (V_C, A_C)$ of G_n, let F and B be the set of forward arcs and backward arcs of the cycle C, respectively, i.e. $F := \{(i,j) \in A_C : i < j\}$ and $B := \{(i,j) \in A_C : j < i\}$.

Theorem 11. *For $n \in \mathbb{N}$, the n-step cycle inequality (71) for an elementary cycle $C = (V_C, A_C)$ of graph G is facet-defining for conv(Y^m) if the following conditions hold:*

(a) For $i \in V_C$, $\alpha_d = a_{it_d}$ where $t_d \in \mathcal{N}_\alpha$ for $d = 1, \dots, n$ such that $\alpha_{t_d} \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{t_{d-1}}, d = 2, \dots, n$;

(b) $\left\lfloor b_k^{(d-1)}/\alpha_d \right\rfloor \geq 1,\ d = 1, \ldots, n,$ *for all* $(k,l) \in F$;

(c) $a_{ir}^{(n)} = 0,\ r \in \mathcal{N}\backslash\mathcal{N}_\alpha$ *where* $a_{ir} \in \mathcal{I}_n^{i,n}$ *and* $i \in V_C$;

(d) $b_l^{(n)} - b_k^{(n)} \geq \max\left\{\alpha_{d-1} - \alpha_d \left\lceil \frac{b_k^{(d-1)}}{\alpha_d} \right\rceil, d = 2, \ldots, n\right\}$ *for all* $(k,l) \in F$;

(e) $b_l^{(n)} - b_k^{(n)} \geq \max\left\{\alpha_{g+1} - a_{kr}^{(g+1)}, r \in \mathcal{N}\backslash\mathcal{N}_\alpha, a_{kr} \in \mathcal{I}_g^{k,n}, g \in \{0. \ldots, n-1\}\right\}$ *for all* $(k,l) \in F$.

Proof. Consider the supporting hyperplane of inequality (71) for the cycle C. Note that this hyperplane can be written as

$$\sum_{(i,j)\in F}\left(s + v_i + \sum_{\substack{t\in\mathcal{N}\\ a_{it}\in\mathcal{I}_n^{i,n}}} a_{it}^{(n)} y_t^i - b_i^{(n)} + \left(b_i^{(n)} - b_j^{(n)} + \alpha_n\right)\left(1 - \Phi_i^n(y^i)\right)\right)$$
$$= \sum_{(i,j)\in B}\left(\left(b_i^{(n)} - b_j^{(n)}\right)\Phi_i^n(y^i) - \sum_{\substack{t\in\mathcal{N}\\ a_{it}\in\mathcal{I}_n^{i,n}}} a_{it}^{(n)} y_t^i - v_i\right) \tag{81}$$

because $-\sum_{(i,j)\in F} b_j^{(n)} + \sum_{(i,j)\in B}\left(b_i^{(n)} - b_j^{(n)}\right) = -\sum_{(i,j)\in F} b_i^{(n)}$. Note that in the light of conditions (a), $\Phi_i^n(y^i), i \in V_C$, in (81) is defined by (80). Let $\Gamma = \{(y, v, s) \in conv(Y^m) : (81)\}$ be the face of $conv(Y^m)$ defined by hyperplane (81).

First, we prove that Γ is a facet of Y^m under conditions (b) (note that under conditions (b), $0 \notin V_C$ because $b_0 = 0$ and does not satisfy conditions (a)). Let

$$\sum_{i=1}^{m}\sum_{t=1}^{n} \lambda_t^i y_t^i + \sum_{i=1}^{m} \rho_i v_i + \rho_0 s = \theta \tag{82}$$

be a hyperplane passing through Γ. We prove that (82) must be a multiple of (81).

Notice that for each $k \in \{1, \ldots, m\}\backslash V_C$ and $d \in \{1, \ldots, n\}$ where $a_{kd} \geq 0$, the unit vector $\mathcal{A}_1^{k,d} = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$, in which $y_d^k = 1$ and all other coordinates are zero, is a direction for both the set Y^m and the hyperplane defined by (81), and hence a direction for the face Γ. This implies that $\lambda_d^k = 0$ for all $k \in \{1, \ldots, m\} \backslash V_C$ and $d \in \{1, \ldots, n\}$ where $a_{kd} \geq 0$. Furthermore, for each $k \in \{1, \ldots, m\} \backslash V_C$ and $d \in \{1, \ldots, n\}$ where $a_{kd} < 0$, the unit vector $\mathcal{A}_2^{k,d} = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$, in which $y_d^k = 1$, $y_{t_1}^k = \lceil -a_{kd}/\alpha_1 \rceil$, and all other coordinates are zero, is a direction for both the set Y^m and the hyperplane defined by (81), and hence a direction for the face Γ. This implies that $\lambda_d^k = 0$ for all $k \in \{1, \ldots, m\} \backslash V_C$ and $d \in \{1, \ldots, n\}$ where $a_{kd} \geq 0$. By similar reasoning, for each $k \in \{1, \ldots, m\} \backslash V_C$, the unit vector $\mathcal{A}_3^k = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$, in which $v_k = 1$ and all other coordinates are zero, is a direction for the face Γ, implying that $\rho_k = 0$, $k \in \{1, \ldots, m\} \backslash V_C$. These reduce the hyperplane (82) to

$$\sum_{i \in V_C} \sum_{t=1}^{n} \lambda_t^i y_t^i + \sum_{i \in V_C} \rho_i v_i + \rho_0 s = \theta \tag{83}$$

Next, consider the point $\mathcal{B} = (y, v, s) = (y^1, \ldots, y^m, v_1, \ldots, v_m, 0) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ such that

$$(y^i, v_i) = \begin{cases} (\mathcal{Q}^{i,n}, b_i^{(n)}) & \text{if } i \in T(F), \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin T(F), \end{cases}$$

for $i = 1, \ldots, m$. Based on Lemma 6(a,b), $\mathcal{B} \in Y^m$ and using Lemma 7(a,b), it can be easily verified that $\mathcal{B}$ satisfies (81). So, $\mathcal{B} \in \Gamma$ and hence must satisfy (83). Substituting $\mathcal{B}$ into (83) gives

$$\sum_{i \in T(F)} \left(\rho_i b_i^{(n)} + \sum_{d=1}^{n} \lambda_{t_d}^i \left\lfloor b_i^{(d-1)}/\alpha_d \right\rfloor \right) + \sum_{i \in T(B)} \lambda_{t_1}^i \lceil b_i/\alpha_1 \rceil = \theta. \tag{84}$$

Using (84), hyperplane (83) reduces to

$$\sum_{i\in T(F)}\left(\rho_i\left(v_i-b_i^{(n)}\right)+\sum_{d=1}^{n}\lambda_{t_d}^i\left(y_{t_d}^i-\left\lfloor b_i^{(d-1)}/\alpha_d\right\rfloor\right)+\sum_{t\in\mathcal{N}\setminus\mathcal{N}_\alpha}\lambda_t^i y_t^i\right)$$
$$+\rho_0 s=\sum_{i\in T(B)}\left(\lambda_{t_1}^i\left(\lceil b_i/\alpha_1\rceil-y_{t_1}^i\right)-\sum_{t\in\mathcal{N}\setminus\{t_1\}}\lambda_t^i y_t^i-\rho_i v_i\right). \tag{85}$$

Now, consider the points $\mathcal{C}^{k,d}=(y,v,s)=(y^1,\ldots,y^m,v_1,\ldots,v_m,0)\in\mathbb{Z}_+^{mN}\times\mathbb{R}_+^{m+1}$ for $k\in T(B)$, $d=2,\ldots,n$ such that

$$(y^i,v_i)=\begin{cases}(\mathcal{Q}^{i,n},b_i^{(n)}) & \text{if } i\in T(F),\\ (\mathcal{P}^{i,d},0) & \text{if } i=k,\\ (\mathcal{P}^{i,1},0) & \text{if } i\notin T(F)\cup\{k\},\end{cases}$$

for $i=1,\ldots,m$. By Lemma 6(a,b), $\mathcal{C}^{k,d}\in Y^m$, for all $k\in T(B)$ and $d=2,\ldots,n$. Using Lemma 7(a,b), one can easily verify that all these points also satisfy (81). So for all $k\in T(B)$ and $d=2,\ldots,n$, $\mathcal{C}^{k,d}\in\Gamma$, and hence must satisfy (85). For each $k\in T(B)$, substituting the points $\mathcal{C}^{k,2},\ldots,\mathcal{C}^{k,n}$ one after the other into (85) gives

$$\lambda_{t_{d-1}}^k=\lambda_{t_d}^k\left\lceil b_k^{(d-1)}/\alpha_d\right\rceil, d=2,\ldots,n,k\in T(B),$$

which implies

$$\lambda_{t_d}^k=\lambda_{t_n}^k\prod_{l=d+1}^{n}\left\lceil b_k^{(l-1)}/\alpha_l\right\rceil, d=1,\ldots,n,k\in T(B). \tag{86}$$

Now, note that in the point $\mathcal{C}^{k,d}$, $k\in T(B)$, $d\in\{2,\ldots,n\}$, by definition we have $(y^k,v_k)=(\mathcal{P}^{k,d},0)$. For each $k\in T(B)$ and $r\in\mathcal{N}\setminus\mathcal{N}_\alpha$ where $a_{kr}\in\mathcal{I}_g^{k,n}$,

$g \in \{0, \ldots, n-1\}$, we define another point $\mathcal{C}_1^{k,g,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are exactly the same as $\mathcal{C}^{k,d}$ except that $(y^k, v_k) = (\mathcal{T}^{k,g,r}, 0)$. By Lemma 6(a,b,e), $\mathcal{C}_1^{k,g,r} \in Y^m$, for all $k \in T(B)$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_g^{k,n}$, $g \in \{0, \ldots, n-1\}$. Using Lemma 7(a,b,e), one can easily verify that all these points also satisfy (81). So for all $k \in T(B)$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_g^{k,n}$, $g \in \{0, \ldots, n-1\}$, $\mathcal{C}_1^{k,g,r} \in \Gamma$, and hence must satisfy (85). Now for each $k \in T(B)$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_g^{k,n}$, $g \in \{0, \ldots, n-1\}$, $\mathcal{C}_1^{k,g,r} \in \Gamma$, substituting the point $\mathcal{C}_1^{k,g,r}$ in (85) and using (86) gives

$$\begin{aligned} \lambda_r^k &= \lambda_{t_n}^k \left(\prod_{l=1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{g+1} \prod_{l=d+1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_k^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_{kr}^{(d-1)}}{\alpha_d} \right\rfloor - 1 \right) \right) \\ &= \lambda_{t_n}^k \left(\sum_{d=1}^{g} \prod_{l=d+1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{kr}^{(d-1)}}{\alpha_d} \right\rfloor + \prod_{l=g+2}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_{kr}^{(g)}}{\alpha_{g+1}} \right\rceil \right). \end{aligned} \tag{87}$$

The last equality holds because

$$\begin{aligned} &\prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{g} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - 1 \right) - \prod_{l=g+2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{b_i^{(g)}}{\alpha_{g+1}} \right\rceil \\ &= \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \left(\prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) - \ldots - \left(\prod_{l=g}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right. \\ &\quad \left. - \prod_{l=g+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) - \prod_{l=g+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil = 0. \end{aligned}$$

Next, for each $k \in T(B)$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$, we define another point $\mathcal{C}_2^{k,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are exactly the same as $\mathcal{C}^{k,d}$ except that $(y^k, v_k) = (\mathcal{W}^{k,r}, 0)$. By Lemma 6(a,b,h), $\mathcal{C}_2^{k,r} \in Y^m$, for all $k \in T(B)$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$. Using Lemma 7(a,b,h) and condition (c), one can easily verify that all these points also satisfy (81). So for all $k \in T(B)$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$, $\mathcal{C}_2^{k,r} \in \Gamma$, and hence must satisfy (85). Now for each $k \in T(B)$ and

$r \in \mathcal{N}\backslash\mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$, $\mathcal{C}_2^{k,r} \in \Gamma$, substituting the point $\mathcal{C}_2^{k,r}$ in (85) and using (86) gives

$$\begin{aligned}
\lambda_r^k = \lambda_{t_n}^k \Bigg(\prod_{l=1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{n-1} \prod_{l=d+1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_k^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_{kr}^{(d-1)}}{\alpha_d} \right\rfloor - 1 \right) \\
- \prod_{l=n+1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_k^{(n-1)}}{\alpha_n} \right\rceil - \left\lfloor \frac{a_{kr}^{(n-1)}}{\alpha_n} \right\rfloor \right) \Bigg) \\
= \lambda_{t_n}^k \left(\sum_{d=1}^{n} \prod_{l=d+1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{kr}^{(d-1)}}{\alpha_d} \right\rfloor \right). \qquad (88)
\end{aligned}$$

The last equality holds because

$$\begin{aligned}
\prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{n-1} \prod_{l=d+1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - 1 \right) - \left\lceil \frac{b_i^{(n-1)}}{\alpha_n} \right\rceil \\
= \prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \left(\prod_{l=1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=2}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) - \dots \\
- \left(\prod_{l=n-1}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \prod_{l=n}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \right) - \prod_{l=n}^{n} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil = 0.
\end{aligned}$$

Based on (86), (87), and (88), hyperplane (85) reduces to

$$\begin{aligned}
\sum_{i \in T(F)} \left(\rho_i \left(v_i - b_i^{(n)} \right) + \sum_{d=1}^{n} \lambda_{t_d}^i \left(y_{t_d}^i - \left\lfloor b_i^{(d-1)}/\alpha_d \right\rfloor \right) + \sum_{t \in \mathcal{N}\backslash\mathcal{N}_\alpha} \lambda_t^i y_t^i \right) \\
+ \rho_0 s = \sum_{i \in T(B)} \left(\lambda_{t_n}^i \Phi_i^n(y^i) - \rho_i v_i \right). \qquad (89)
\end{aligned}$$

Now, consider the point $\mathcal{D} = (y, v, s) = (y^1, \dots, y^m, v_1, \dots, v_m, \eta) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$, where $\eta = \min\{b_i^{(n)} : i \in T(F)\}$, such that

$$(y^i, v_i) = \begin{cases} (\mathcal{Q}^{i,n}, b_i^{(n)} - \eta) & \text{if } i \in T(F), \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin T(F), \end{cases}$$

for $i = 1, \ldots, m$. By Lemma 6(a,b), it is clear that $\mathcal{D} \in Y^m$ and using Lemma 7(a,b), one can easily verify that it also satisfies (81). So $\mathcal{D} \in \Gamma$, and hence must satisfy (89). Substituting $\mathcal{D}$ into (89) gives

$$\rho_0 = \sum_{i \in T(F)} \rho_i. \tag{90}$$

Now for $i \in V_C$, let $N(i)$ be the node in V_C such that $(i, N(i)) \in A_C$. For each $(k,l) \in A_C$, since conditions (a) holds, consider the points $\mathcal{E}^{k,l} = (y, v, s) = (y^1, \ldots, y^m, v_1, \ldots, v_m, b_l^{(n)}) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ such that

$$(y^i, v_i) = \begin{cases} (\mathcal{R}^i, b_i^{(n)} - b_l^{(n)} + \alpha_n) & \text{if } i \in T(F), N(i) < l \\ (\mathcal{Q}^{i,n}, 0) & \text{if } i \in T(F), i < l \leq N(i) \\ (\mathcal{Q}^{i,n}, b_i^{(n)} - b_l^{(n)}) & \text{if } i \in T(F), i \geq l \\ (\mathcal{Q}^{i,n}, 0) & \text{if } i \in T(B), i < l \\ (\mathcal{Q}^{i,n}, b_i^{(n)} - b_l^{(n)}) & \text{if } i \in T(B), N(i) < l \leq i \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \in T(B), N(i) \geq l \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin V_C, \end{cases}$$

for $i = 1, \ldots, m$. By Lemma 6(a,b,c), it is clear that $\mathcal{E}^{k,l} \in Y^m$ for all $(k,l) \in A_C$.

Using Lemma 7(a,b,c), if we substitute $\mathcal{E}^{k,l}$ into (81), we get

$$\begin{aligned} &\sum_{(i,j)\in F;i,j<l}\left(b_i^{(n)}-b_j^{(n)}\right)+\sum_{(i,j)\in B;i,j<l}\left(b_i^{(n)}-b_j^{(n)}\right)\\ &+\sum_{(i,j)\in F;i<l\leq j}\left(b_i^{(n)}-b_l^{(n)}\right)+\sum_{(i,j)\in B;j<l\leq i}\left(b_l^{(n)}-b_j^{(n)}\right)\\ &=-\sum_{(i,j)\in F;i<l\leq j}b_i^{(n)}+\sum_{(i,j)\in B;j<l\leq i}b_j^{(n)}\\ &+\sum_{(i,j)\in F;i<l\leq j}b_i^{(n)}-\sum_{(i,j)\in B;j<l\leq i}b_j^{(n)}=0, \end{aligned}\tag{91}$$

which is obviously true. Therefore, the points $\mathcal{E}^{k,l}$, for all $(k,l)\in A_C$, also satisfy (81). Hence, they belong to Γ, and must satisfy (89). Now, note that in the point $\mathcal{E}^{k,l}$, $(k,l)\in F$, by definition we have $(y^k,v_k)=(\mathcal{Q}^{k,n},0)$. For each $(k,l)\in F$, define another point $\mathcal{E}_1^{k,l}=(y,v,s)\in\mathbb{Z}_+^{mN}\times\mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{F}^{k,l}$ except that $(y^k,v_k)=(\mathcal{R}^k,b_k^{(n)}-b_l^{(n)}+\alpha_n)$. For precisely the same reasons stated for $\mathcal{E}^{k,l}$, the points $\mathcal{E}_1^{k,l}$, $(k,l)\in F$, must also satisfy (89) (note that substituting $\mathcal{E}_1^{k,l}$ in (81) gives identity (91) again). Now if for each $(k,l)\in F$, we substitute $\mathcal{E}^{k,l}$ and $\mathcal{E}_1^{k,l}$ into (89) and subtract one equality from the other, we get

$$\lambda_{t_n}^k=\rho_k\left(b_k^{(n)}-b_l^{(n)}+\alpha_n\right),\ \text{for all } (k,l)\in F.\tag{92}$$

Next, for each $(k,l)\in F$ and $d=2,\ldots,n$, since conditions (b) hold, define the point $\mathcal{E}_2^{k,l,d}=(y,v,s)\in\mathbb{Z}_+^{mN}\times\mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{E}^{k,l}$ except that $(y^k,v_k)=(\mathcal{S}^{k,d},0)$. By Lemma 6(a,b,c,d) and because of conditions (d), it is clear that $\mathcal{E}_2^{k,l,d}\in Y^m$ for all $(k,l)\in F$ and $d=2,\ldots,n$. Using Lemma 7(a,b,c,d), one can easily verify that they also satisfy (81) (note that substituting $\mathcal{E}_2^{k,l,d}$ in (81) gives identity (91) again), and hence belong to Γ and must satisfy (89).

Now if for each $(k,l) \in F$ and $d = 2, \ldots, n$, we substitute the points $\mathcal{E}^{k,l}$ and $\mathcal{E}_2^{k,l,d}$ into (89) and subtract one equality from the other, we get

$$\lambda_{t_{d-1}}^k = \lambda_{t_d}^k \left\lceil b_k^{(d-1)}/\alpha_d \right\rceil, d \in \{2, \ldots, n\}, k \in T(F).$$

This implies

$$\lambda_{t_d}^k = \lambda_{t_n}^k \prod_{p=d+1}^{n} \left\lceil b_k^{(p-1)}/\alpha_p \right\rceil, d = 1, \ldots, n, k \in T(F). \tag{93}$$

For each $(k,l) \in F$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$, since conditions (e) hold, define the point $\mathcal{E}_3^{k,l,g,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{E}^{k,l}$ except that $(y^k, v_k) = (\mathcal{U}^{k,g,r}, 0)$. By Lemma 6(a,b,c,f) and because of conditions (f), it is clear that $\mathcal{E}_3^{k,l,g,r} \in Y^m$ for all $(k,l) \in F$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_g^{k,n}$, $g \in \{0, \ldots, n-1\}$. Using Lemma 7(a,b,c,f), one can easily verify that they also satisfy (81) (note that substituting $\mathcal{E}_3^{k,l,g,r}$ in (81) gives identity (91) again), and hence belong to Γ and must satisfy (89). Now if for each $(k,l) \in F$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_g^{k,n}$, $g \in \{0, \ldots, n-1\}$, we substitute the points $\mathcal{E}^{k,l}$ and $\mathcal{E}_3^{k,l,g,r}$ into (89), subtract one equality from the other, and use equalities (93), we get

$$\lambda_r^k = \lambda_{t_n}^k \left(\sum_{d=1}^{g} \prod_{l=d+1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{kr}^{(d-1)}}{\alpha_d} \right\rfloor + \prod_{l=g+2}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_{kr}^{(g)}}{\alpha_{g+1}} \right\rceil \right). \tag{94}$$

Also, for each $(k,l) \in F$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$, define the point $\mathcal{E}_4^{k,l,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{E}^{k,l}$ except that $(y^k, v_k) = (\mathcal{V}^{k,r}, 0)$. By Lemma 6(a,b,c,g) and because $b_l^{(n)} > b_k^{(n)}$, it is clear that $\mathcal{E}_4^{k,l,g,r} \in Y^m$ for all $(k,l) \in F$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$. Using Lemma 7(a,b,c,g), one can easily verify that they also satisfy (81) (note that substituting

$\mathcal{E}_4^{k,l,r}$ in (81) gives identity (91) again), and hence belong to Γ and must satisfy (89). Now if for each $(k,l) \in F$ and $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$, we substitute the points $\mathcal{E}^{k,l}$ and $\mathcal{E}_4^{k,l,r}$ into (89), subtract one equality from the other, and use equalities (93), we get

$$\lambda_r^k = \lambda_{t_n}^k \left(\sum_{d=1}^{n} \prod_{l=d+1}^{n} \left\lceil \frac{b_k^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_{kr}^{(d-1)}}{\alpha_d} \right\rfloor \right). \tag{95}$$

Next, note that in the point $\mathcal{E}^{k,l}$, $(k,l) \in B$, by definition we have $(y^k, v_k) = (\mathcal{P}^{k,1}, 0)$. For each $(k,l) \in B$, define the point $\mathcal{E}_5^{k,l} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{E}^{k,l}$ except that $(y^k, v_k) = (\mathcal{Q}^{k,n}, b_k^{(n)} - b_l^{(n)})$. By Lemma 6(a,b,c), it is clear that $\mathcal{E}_5^{k,l} \in Y^m$ for all $(k,l) \in B$. Using Lemma 7(a,b,c), we can easily verify that they also satisfy (81) (note that substituting $\mathcal{E}_5^{k,l}$ in (81) gives identity (91) again), and hence belong to Γ and must satisfy (89). Now if for each $(k,l) \in B$, we substitute $\mathcal{E}^{k,l}$ and $\mathcal{E}_5^{k,l}$ into (89) and subtract one equality from the other, we get

$$\lambda_{t_n}^k = \rho_k \left(b_k^{(n)} - b_l^{(n)} \right), \text{ for all } (k,l) \in B. \tag{96}$$

Based on (90), (92), (93), (94), (95), (96), and assumption (c), hyperplane (49) reduces to

$$\begin{aligned} &\sum_{(i,j) \in F} \rho_i \Bigg(s + v_i + \sum_{\substack{t \in \mathcal{N} \backslash \mathcal{N}_\alpha \\ a_{it} \in \mathcal{I}_n^n}} a_{it}^{(n)} y_t^i - b_i^{(n)} + \left(b_i^{(n)} - b_j^{(n)} + \alpha_n \right) \left(1 - \Phi_i^n(y^i) \right) \Bigg) \\ &= \sum_{(i,j) \in B} \rho_i \left(\left(b_i^{(n)} - b_j^{(n)} \right) \Phi_i^n(y^i) - \sum_{\substack{t \in \mathcal{N} \backslash \mathcal{N}_\alpha \\ a_{it} \in \mathcal{I}_n^n}} a_{it}^{(n)} y_t^i - v_i \right). \end{aligned} \tag{97}$$

Now, for $i \in V_C$, let $P(i)$ be the node in V_C such that $(P(i), i) \in A_C$, and define $i_a :=$

$\min\{j \in V_C : i < j\}$ and $i_b := \max\{j \in V_C : j < i\}$. Also let $i_{max} = \max\{i : i \in V_C\}$ and $i_{min} = \min\{i : i \in V_C\}$. For $l \in V_C \backslash \{i_{max}\}$, if we substitute the point $\mathcal{E}^{P(l),l}$ and $\mathcal{E}^{P(l_a),l_a}$ into (97) (note that both points must satisfy (97) as argued for all points $\mathcal{E}^{k,l}$) and subtract the two equalities, we get $\sum_{\substack{(i,j)\in F \\ i<l_a\leq j}} \rho_i \left(b_l^{(n)} - b_{l_a}^{(n)}\right) + \sum_{\substack{(i,j)\in B \\ j<l_a\leq i}} \rho_i \left(b_{l_a}^{(n)} - b_l^{(n)}\right) = 0$. Since $b_l^{(n)} \neq b_{l_a}^{(n)}$, we get

$$\sum_{(i,j)\in F; i<l_a\leq j} \rho_i - \sum_{(i,j)\in B; j<l_a\leq i} \rho_i = 0. \tag{98}$$

Likewise, for $l \in V_C \backslash \{i_{min}\}$, if we substitute the point $\mathcal{E}^{P(l_b),l_b}$ and $\mathcal{E}^{P(l),l}$ into equality (97) and subtract the two equalities, we get

$$\sum_{(i,j)\in F; i<l\leq j} \rho_i - \sum_{(i,j)\in B; j<l\leq i} \rho_i = 0 \tag{99}$$

because $b_{l_b}^{(n)} \neq b_l^{(n)}$. Notice that if $l = P(i_{max})$, then $l_a = i_{max}$, and identity (98) reduces to

$$\rho_{P(i_{max})} = \rho_{i_{max}} \tag{100}$$

Also if for each $l \in V_C \backslash \{i_{min}, i_{max}\}$, we subtract (98) from (99), we get

$$\rho_{P(l)} = \rho_l, \quad l \in V_C \backslash \{i_{min}, i_{max}\}. \tag{101}$$

Identities (100) and (101) imply that $\rho_{P(l)} = \rho_l$ for all $l \in V_C$ (because $P(i) = i_{min}$ for some $i \in V_C \backslash \{i_{min}\}$). Therefore,

$$\rho_i = \rho_j \text{ for all } i, j \in V_C \tag{102}$$

as C is a cycle. This reduces hyperplane (97) to a constant multiple (by (90) this multiple is $\rho_0/|F|$) of (81), which completes the proof . $\square$

Example 2 (continued)**.** *Notice that for* $n = 2$*, the coefficients of* Y^5 *also satisfy the additional conditions required in Theorem 11, i.e. (b)* $\left\lfloor b_k^{(0)}/\alpha_1 \right\rfloor \geq 1$*,* $\left\lfloor b_k^{(1)}/\alpha_2 \right\rfloor = 1$*, for* $k \in T(F) \subseteq \{1, \ldots, 4\}$*, (c)* $a_{kr}^{(2)} = 0$ *for* $k = 1, \ldots, 5$ *and* $r \in \mathcal{N}$ *such that* $a_{kr} \in \mathcal{I}_2^{k,2}$*, (d)* $b_l^{(2)} - b_k^{(2)} \geq 1 = \alpha_1 - \alpha_2 \left\lceil b_k^{(1)}/\alpha_2 \right\rceil$ *for all* $(k, l) \in A_2$ *such that* $1 \leq k < l \leq 5$*, and (e)* $b_l^{(2)} - b_k^{(2)} \geq 3 = \max\left\{\alpha_1 - a_{kr}^{(1)}, r \in \mathcal{N} \backslash \mathcal{N}_\alpha, a_{kr} \in \mathcal{I}_0^{k,2}\right\}$ *and* $b_l^{(2)} - b_k^{(2)} \geq \max\left\{\alpha_2 - a_{kr}^{(2)}, r \in \mathcal{N} \backslash \mathcal{N}_\alpha, a_{kr} \in \mathcal{I}_1^{k,2}\right\}$ *for all* $(k, l) \in A_2$ *such that* $1 \leq k < l \leq 5$*. Therefore, the 2-step cycle inequality* (71) *corresponding to each cycle* $C = (V_C, A_C)$ *in graph* G_2*, where* $V_C \subseteq \{1, \ldots, 5\}$*, defines a facet for conv*(Y^5)*. In particular, 2-step cycle inequalities* (74) *and* (75) *are facet-defining for conv*(Y^5)*.*

Theorem 12. *For* $n \in \mathbb{N}$*, the n-step cycle inequality* (71) *for an elementary cycle* $C = (V_C, A_C)$ *of graph* G *is facet-defining for conv*(Y^m) *if the following conditions hold:*

(a) $T(F) = \{0\}$*;*

(b) For $i \in T(B)$*,* $\alpha_d = a_{it_d}$ *where* $t_d \in \mathcal{N}_\alpha$ *for* $d = 1, \ldots, n$ *such that* $\alpha_{t_d} \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{t_{d-1}}, d = 2, \ldots, n$*;*

(c) For $i \in T(B)$*,* $a_{ir}^{(n)} = 0$*,* $r \in \mathcal{N} \backslash \mathcal{N}_\alpha$ *where* $a_{ir} \in \mathcal{I}_n^{i,n}$*.*

Proof. As shown before, the supporting hyperplane of inequality (71) can be written as (81), which for the C considered in this theorem reduces to

$$s = \sum_{(i,j) \in B} \left(\left(b_i^{(n)} - b_j^{(n)}\right) \Phi^i(y^i) - \sum_{\substack{t \in \mathcal{N} \backslash \mathcal{N}_\alpha \\ a_{it} \in \mathcal{I}_n^{i,n}}} a_{it}^{(n)} y_t^i - v_i \right) \tag{103}$$

because by condition (a), the cycle C has only one forward arc, which goes out of node 0, and we have $v_0 = 0$, $y^0 = 0$ and $\Phi_0^n(y^0) := 1$ by definition. Let Γ be the face of Y^m defined by hyperplane (103). We prove that any generic hyperplane

$$\rho_0 s + \sum_{i=1}^{m} \rho_i v_i + \sum_{i=1}^{m} \sum_{t=1}^{n} \lambda_j^i y_j^i = \theta \tag{104}$$

that passes through Γ is a scalar multiple of (103). By the same reasoning we reduced hyperplane (82) to (83) in Theorem 11, we can reduce hyperplane (104) to

$$\sum_{i \in V_C \setminus \{0\}} \sum_{t=1}^{n} \lambda_t^i y_t^i + \sum_{i \in V_C \setminus \{0\}} \rho_i v_i + \rho_0 s = \theta. \tag{105}$$

Now consider the following points (correspondig to the points with the same name in the proof of Theorem 11): The point $\mathcal{B} = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ such that $(y^i, v_i) = (\mathcal{P}^{i,1}, 0)$, $i = 1, \ldots, m$, and $s = 0$; the points $\mathcal{C}^{k,d} = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^m \times \mathbb{R}_+$, for $k \in T(B)$, $d = 2, \ldots, n$, such that $(y^k, v_k) = (\mathcal{P}^{k,d}, 0)$ and $(y^i, v_i) = (\mathcal{P}^{i,1}, 0)$ for $i \in \{1, \ldots, m\} \setminus (T(F) \cup \{k\})$, and $s = 0$; the points $\mathcal{C}_1^{k,g,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$, for $k \in T(B)$ and $r \in \mathcal{N} \setminus \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_g^{k,n}$, $g \in \{0, \ldots, n-1\}$, whose coordinates are exactly the same as $\mathcal{C}^{k,d}$ except that $(y^k, v_k) = (\mathcal{T}^{k,g,r}, 0)$; the points $\mathcal{C}_2^{k,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$, for $k \in T(B)$ and $r \in \mathcal{N} \setminus \mathcal{N}_\alpha$ where $a_{kr} \in \mathcal{I}_n^{k,n}$, whose coordinates are exactly the same as $\mathcal{C}^{k,d}$ except that $(y^k, v_k) = (\mathcal{W}^{k,r}, 0)$; the points $\mathcal{E}^{k,l} = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^m \times \mathbb{R}_+$, for $(k, l) \in B$, such that

$$(y^i, v_i) = \begin{cases} (\mathcal{Q}^{i,n}, 0) & \text{if } i \in T(B), i \leq l \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \in T(B), N(i) \geq l \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin V_C, \end{cases}$$

for $i = 1, \ldots, m$, and $s = b_l{}^{(n)}$; and the points $\mathcal{E}_5^{k,l} \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$, for $(k,l) \in B$, whose coordinates are all exactly the same as $\mathcal{E}^{k,l}$ except that $(y^k, v_k) = (\mathcal{Q}^{k,n}, b_k^{(n)} - b_l^{(n)})$.

By Lemma 6(a,b,e,h), all the aforementioned points belong to Y^m, and by Lemma 7(a,b,e,h), it is easy to verify that they also satisfy (103). So, they belong to Γ, and hence must satisfy (105). Therefore, given conditions (c), all these points can be used in the same fashion the points with similar names were used in the proof of Theorem 11 to reduce the hyperplane (105) to an equality which is ρ_0 times the hyperplane (103). This completes the proof. $\square$

Example 2 (continued)**.** *Moreover, the 2-step cycle inequality* (71) *corresponding to each cycle* $C = (V_C, A_C)$ *in* $G_2 = (V_2, A_2)$*, where* $T(F) = \{0\}$*, also defines facet for conv*(Y^5) *because condition (c) holds for* $n = 2$*, i.e.* $a_{kr}^{(2)} = 0$ *for* $k = 1, \ldots, 5$ *and* $r \in \mathcal{N}$ *such that* $a_{kr} \in \mathcal{I}_2^{k,2}$*. In particular, 2-step cycle inequality* (36) *is facet-defining for conv*(Y^5)*.*

CHAPTER V

CONTINUOUS MULTI-MIXING SET WITH GENERAL COEFFICIENTS AND BOUNDED INTEGER VARIABLES

In this chapter, we unify the concepts of continuous multi-mixing and n-step mingling by incorporating upper bounds on the integer variables of the continuous multi-mixing set (where no conditions are imposed on the coefficients) and by developing new families of valid inequalities for this set (which we refer to as the mingled n-step cycle inequalities, $n \in \mathbb{N}$). We denote this new generalization of continuous multi-mixing set by

$$Z^m := \Bigg\{(y, v, s) \in \mathbb{Z}_+^{m\times N} \times \mathbb{R}_+^m \times \mathbb{R}_+ : \\ \sum_{t\in T} a_t y_t^i + \sum_{k\in K} a_k y_k^i + v_i + s \geq b_i, y^i \leq u^i, i = 1, \dots, m\Bigg\}$$

where (T, K) is a partitioning of $\mathcal{N} := \{1, \dots, N\}$ with $a_t > 0$ for $t \in T$, $a_k < 0$ for $k \in K$, and $u^i \in \mathbb{Z}_+^N$ for $i \in \{1, \dots, m\}$. Observe that the mixed integer knapsack set with bounded integer variables Z_0^1 (studied in [6, 7, 10, 74]) is a special case of Z^m where $n = 1$. It is the projection of $Z^1 \cap \{v = 0\}$ on (y, s). In Section V.1, we assume that $b_i \geq 0$, $i = 1, \dots, m$, and for each $n \in \mathbb{N}$, we develop a new class of valid inequalities for Z^m which we refer to as mingled n-step cycle inequalities. We observe how the n-step mingling [6, 7], n-step MIR inequalities [62], and n-step cycle inequalities (introduced in Chapter IV) are special cases of the mingled n-step cycle inequalities. We also introduce a compact extended formulation for Z^m and an exact separation algorithm to separate over the set of all mingled n-step cycle inequalities for a given $n \in \mathbb{N}$. In Section V.2, we obtain conditions under which a special case of mingled n-step cycle inequalities (which we refer to as the mingled n-step mixing inequalities) are facet-defining for $conv(Z^m)$.

V.1 Valid Inequalities and Extended Formulation

In this section, for each $n \in \mathbb{N}$, we develop a new class of valid inequalities for Z^m. First, for each $i \in \{1, \ldots, m\}$, we introduce the following notations (assuming $b_i \geq 0$): Let $T_i^+ := \{1, \ldots, n_i^+\} \subseteq \{t \in T : a_t > b_i\}$ and $\bar{K}_i := \{k \in K : a_k + \sum_{t \in T_i^+} a_t u_t^i < 0\}$. We index T_i^+ in non-increasing order of a_t's. For $k \in K \setminus \bar{K}_i$, we define a set T_{ik}, an integer l_{ik}, and the numbers $\bar{u}_{tk}^i$ such that $\bar{u}_{tk}^i \leq u_t^i$ for $t \in T_k$ as follows:

$$
\begin{aligned}
T_{ik} &:= \{1, \ldots, q(i,k)\}, \text{ where } q(i,k) := \min\left\{q \in T_i^+ : a_k + \sum_{t=1}^{q} a_t u_t^i \geq 0\right\}; \\
l_{ik} &:= \min\left\{l \in \mathbb{Z}_+ : a_k + \sum_{t=1}^{q(i,k)-1} a_t u_t^i + a_{q(i,k)} l \geq 0\right\}; \text{ and} \\
\bar{u}_{tk}^i &:= \begin{cases} u_t^i, & \text{if } t < q(i,k), \\ l_{ik}, & \text{if } t = q(i,k). \end{cases}
\end{aligned}
$$

Now for $i \in \{1, \ldots, m\}$ and $k \in \bar{K}_i$, let $T_{ik} := T_i^+, q(i,k) := n_i^+, l_{ik} := u_{n_i^+}^i$, and $\bar{u}_{tk}^i := u_t^i$ for $t \in T_{ik}$. We also define $K_{it} := \{k \in K : k \in T_{ik}\}$ (as a result, for $t \in T \setminus T_i^+$, $K_{it} = \emptyset$),

$$
\tau_{ik} := \min\left\{b_i, a_k + \sum_{t \in T_{ik}} a_t \bar{u}_{tk}^i\right\} \text{ for } i \in \{1, \ldots, m\}, k \in K \tag{106}
$$

(therefore, $0 \leq \tau_{ik} \leq b_i$ for $k \in K \setminus \bar{K}$ and $\tau_{ik} < 0$ for $k \in \bar{K}$).

Next, we choose a parameter vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) > 0$ and without loss of generality, we assume $b_{i-1}^{(n-1)} \leq b_i^{(n-1)}$, $i = 2, \ldots, m$, where $b_i^{(n-1)}$ is defined as (11).

Also define $b_0 := 0$ and for $g = 0, \dots, n-2$, $i = 1, \dots, m$,

$$\mathcal{I}_g^{i,n-1} := \{x \in \mathbb{R} : x^{(q)} < b_i^{(q)}, q = 1, \dots, g, x^{(g+1)} \geq b_i^{(g+1)}\},$$
$$\mathcal{I}_{n-1}^{i,n-1} := \{x \in \mathbb{R} : x^{(q)} < b_i^{(q)}, q = 1, \dots, n-1\}.$$

Now similar to the graph defined for the cycle inequalities (see Section II.2.2), here we define a directed graph $\bar{G}_n = (V, A)$, where $V := \{0, 1, \dots, m\}$ and $A := \{(i,j) : i, j \in V, b_i^{(n-1)} \neq b_j^{(n-1)}\}$. $\bar{G}_n$ is a complete graph except for the arcs (i,j) where $b_i^{(n-1)} = b_j^{(n-1)}$. Here to each arc $(i,j) \in A$, we associate the linear function $\pi_{ij}^n(y, v, s)$ defined as (note that some of the notations used in this chapter have already been introduced in Subsection II.2.4)

$$\pi_{ij}^n(y,v,s) := \begin{cases} s + v_i + \sum\limits_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} a_t^{(n-1)} y_t^i + \sum\limits_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_{n-1}^{i,n-1}}} \tau_{ik}^{(n-1)} y_k^i + b_{ij}^{(n-1)} \left(1 - \xi_i^n(y^i)\right) - b_j^{(n-1)} & \text{if } i < j, \\ v_i + \sum\limits_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} a_t^{(n-1)} y_t^i + \sum\limits_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_{n-1}^{i,n-1}}} \tau_{ik}^{(n-1)} y_k^i + \left(b_i^{(n-1)} - b_j^{(n-1)}\right) \left(1 - \xi_i^n(y^i)\right) & \text{if } i > j, \end{cases} \tag{107}$$

where $b_{ij}^{(n-1)} := b_i^{(n-1)} - b_j^{(n-1)} + \alpha_{n-1}$ for all $(i,j) \in A$, $i < j$, and the functions

$\xi_i^n(y^i)$, $i = 1, \ldots, m$, in its open form can be defined as

$$\begin{aligned}
\xi_i^n(y^i) :=& \prod_{l=1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{t \in T_i^+} \prod_{l=1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(y_t^i - \sum_{k \in K_t} \bar{u}_{tk}^i y_k^i \right) \\
&- \sum_{g=0}^{n-2} \sum_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_g^{i,n-1}}} \left(\sum_{q=1}^{g} \prod_{l=q+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_t^{(q-1)}}{\alpha_q} \right\rfloor + \prod_{l=g+2}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_t^{(g)}}{\alpha_{g+1}} \right\rceil \right) y_t^i \\
&- \sum_{g=0}^{n-2} \sum_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_g^{i,n-1}}} \left(\sum_{q=1}^{g} \prod_{l=q+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\tau_{ik}^{(q-1)}}{\alpha_q} \right\rfloor + \prod_{l=g+2}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{\tau_{ik}^{(g)}}{\alpha_{g+1}} \right\rceil \right) y_k^i \\
&- \sum_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} \sum_{q=1}^{n-1} \prod_{l=q+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_t^{(q-1)}}{\alpha_q} \right\rfloor y_t^i - \sum_{\substack{k \in K \\ \tau_k \in \mathcal{I}_{n-1}^{i,n-1}}} \sum_{q=1}^{n-1} \prod_{l=q+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\tau_{ik}^{(q-1)}}{\alpha_q} \right\rfloor y_k^i
\end{aligned} \tag{108}$$

and by definition, $v_0 := 0$, $y^0 := 0$, and $\xi_0^n(y^0) := 1$.

We show that each elementary cycle of graph $\bar{G}_n$ corresponds to a valid inequality for the set Z^m, which we also refer to as the *mingled n-step cycle inequality.* To do this in addition to Lemma 1, we need the following lemma:

Lemma 8. *For $i \in \{1, \ldots, m\}$ and $n \in \mathbb{N}$, the inequality*

$$s + v_i + \sum_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} a_t^{(n-1)} y_t^i + \sum_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_{n-1}^{i,n-1}}} \tau_{ik}^{(n-1)} y_k^i + \alpha_{n-1} \left(1 - \xi_i^n(y^i)\right) \geq b_i^{(n-1)} \tag{109}$$

is valid for Z^m if $\alpha_d \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{d-1}, d = 2, \ldots, n-1.$

Proof. Atamtürk and Kianfar [7] proved that the following inequality

$$
\begin{aligned}
& s + v_i + \alpha_{n-1}\Bigg[1 - \prod_{l=1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil + \sum_{t \in T_i^+} \prod_{l=1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \Bigg(y_t^i - \sum_{k \in K_t} \bar{u}_{tk}^i y_k^i\Bigg) \\
& + \sum_{g=0}^{n-2} \sum_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_g^{i,n-1}}} \left(\sum_{q=1}^{g} \prod_{l=q+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_t^{(q-1)}}{\alpha_q} \right\rfloor + \prod_{l=g+2}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_t^{(g)}}{\alpha_{g+1}} \right\rceil \right) y_t^i \\
& + \sum_{g=0}^{n-2} \sum_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_g^{i,n-1}}} \left(\sum_{q=1}^{g} \prod_{l=q+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\tau_{ik}^{(q-1)}}{\alpha_q} \right\rfloor + \prod_{l=g+2}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{\tau_{ik}^{(g)}}{\alpha_{g+1}} \right\rceil \right) y_k^i \Bigg] \qquad (110) \\
& + \alpha_{n-1} \sum_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} \sum_{q=1}^{n-1} \prod_{l=q+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_t^{(q-1)}}{\alpha_q} \right\rfloor y_t^i + \sum_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} a_t^{(n-1)} y_t^i \\
& + \alpha_{n-1} \sum_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_{n-1}^{i,n-1}}} \sum_{q=1}^{n-1} \prod_{l=q+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\tau_{ik}^{(q-1)}}{\alpha_q} \right\rfloor y_k^i + \sum_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_{n-1}^{i,n-1}}} \tau_{ik}^{(n-1)} y_k^i \geq b_i^{(n-1)}
\end{aligned}
$$

is valid for a relaxation of Z^m defined by its i'th constraint, i.e. $\{(y^i, v_i, s) \in \mathbb{Z}_+^N \times \mathbb{R}_+ \times \mathbb{R}_+ : \sum_{t \in T} a_t y_t^i + \sum_{k \in K} a_k y_k^i + v_i + s \geq b_i, y^i \leq u\}$, for $\alpha := (\alpha_1, \ldots, \alpha_{n-1})$ satisfying $\alpha_d \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{d-1}, d = 2, \ldots, n-1$. Therefore, it is also valid for Z^m. Note that rearranging the terms in (110) and using (108) gives (109). $\square$

Theorem 13. *Given $n \in \mathbb{N}$ and an elementary cycle $C = (V_C, A_C)$ of graph $\bar{G}_n$, the mingled n-step cycle inequality*

$$
\sum_{(i,j) \in A_C} \pi_{ij}^n(y, v, s) \geq 0 \qquad (111)
$$

is valid for Z^m if the parameters $(\alpha_1, \ldots, \alpha_{n-1})$ satisfy

$$
\alpha_d \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{d-1}, d = 2, \ldots, n-1, i \in V_C. \qquad (112)
$$

Proof. Consider a point $(\hat{y}, \hat{v}, \hat{s}) \in Z^m$. Based on Lemma 8, inequality (109) is satisfied by the point $(\hat{y}, \hat{v}, \hat{s})$ for each $i \in V_C \setminus \{0\}$ because of (112). But notice that inequality (109) for this point is the same as inequality (20) if we define $\sigma := \hat{s}$, $\alpha := \alpha_{n-1}$, and

$$\omega_i := \hat{v}_i + \sum_{\substack{t \in T \setminus T_i^+ \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} a_t^{(n-1)} \hat{y}_t^i + \sum_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_{n-1}^{i,n-1}}} \tau_{ik}^{(n-1)} \hat{y}_k^i,$$

$\kappa_i := 1 - \xi_i^n(\hat{y}^i)$, $\gamma_i := b_i^{(n-1)}$, $i \in V_C \backslash \{0\}$. Also, in case $0 \in V_C$, if we define ω_0, κ_0, and γ_0 in a similar way, inequality (20) for $i = 0$ reduces to the valid inequality $\hat{s} \geq 0$ because as we defined before $y^0 := 0$, $v_0 := 0$, $\xi_0^n(y^0) := 1$, and $b_0 := 0$. With these definitions, we have $\omega_i \geq 0$, $\kappa_i \in \mathbb{Z}$, $i \in V_C$ and $0 = \gamma_0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_{|V_C|} < \alpha_{n-1}$. Therefore, according to Lemma 1, inequality (21) in which σ, α and $\omega_i, \kappa_i, \gamma_i, i \in V_C$ are replaced with the values defined here is valid. It is easy to see that this inequality is exactly the same as the mingled n-step cycle inequality (111) for the point $(\hat{y}, \hat{v}, \hat{s})$. This completes the proof. □

Special Cases: The following are few special cases of the mingled n-step cycle inequalities:

- The mingled n-step cycle inequality (111) written for cycle $C = (V_C, A_C)$ such that $A_C = \{(0, i), (i, 0)\}$ gives the n-step mingling inequality (14) written for constraint i in Z^m;
- The mingled n-step cycle inequality (111) reduces to $(n-1)$-step cycle inequalities (71) in case $T_i^+ = \emptyset$ for all $i \in V_C$;
- For $\bar{K} = \emptyset$ and $\alpha_{n-1} = \alpha_{n-2}$, the mingled n-step cycle inequality (111) becomes mingled $(n-1)$-step cycle inequalities.

Separation Algorithm. Given a point $(\hat{y}, \hat{v}, \hat{s})$ and $n \in \mathbb{N}$, we can also formulate the separation problem associated with the mingled n-step cycle inequalities (111) as follows:

$$\min\left\{\sum_{(i,j)\in A}\pi^n_{ij}(\hat{y},\hat{v},\hat{s})z_{ij} : \overline{\mathbf{M}}z=0, z\geq 0\right\}. \tag{113}$$

where z_{ij} is a variable representing the flow along arc (i,j), $\overline{\mathbf{M}}$ is the node-arc incidence matrix of $\bar{G}_n$, and the goal is to test whether linear program (113) has a strictly negative solution value. Therefore, for the point $(\hat{y},\hat{v},\hat{s})$, we can find a mingled n-step cycle inequality (111) that is violated by $(\hat{y},\hat{v},\hat{s})$, if any, by detecting a negative weight cycle (if any) in the directed graph $\bar{G}_n$ with weights $\pi^n_{ij}(\hat{y},\hat{v},\hat{s})$ for each arc $(i,j)\in A$ (refer to Section III.3 for details).

Example 3. *Consider the following continuous multi-mixing set with general coefficients, bounded integer variables, and 4 rows:*

$$\begin{aligned} Z^4 =\{&(y,v,s)\in\mathbb{Z}^{9\times 4}_+\times\mathbb{R}^5_+ : y^i_1\leq 1, y^i_2\leq 1, y^i_3\leq 1, y^i_4\leq 2, y^i_6\leq 2, i=1,\ldots,4,\\ &37y^1_1+33y^1_2+31y^1_3+15y^1_4+5y^1_5+6y^1_6-64y^1_7-81y^1_8-106y^1_9+v_1+s\geq 16,\\ &37y^2_1+33y^2_2+31y^2_3+15y^2_4+5y^2_5+6y^2_6-64y^2_7-81y^2_8-106y^2_9+v_2+s\geq 29,\\ &37y^3_1+33y^3_2+31y^3_3+15y^3_4+5y^3_5+6y^3_6-64y^3_7-81y^3_8-106y^3_9+v_3+s\geq 24,\\ &37y^4_1+33y^4_2+31y^4_3+15y^4_4+5y^4_5+6y^4_6-64y^4_7-81y^4_8-106y^4_9+v_4+s\geq 25\}.\end{aligned}$$

We have $\mathcal{N}=\{1,\ldots,6\}$, $T=\{1,\ldots,6\}$, $K=\{7,8,9\}$, *for* $i=1,\ldots,4$, *upper bound array* $u^i=\{1,1,1,2,u^i_5,2,u^i_7,u^i_8,u^i_9\}$ *where* $u^i_5,u^i_7,u^i_8,u^i_9\in\mathbb{Z}_+$, $b_1=16$, $b_2=29$, $b_3=24$, *and* $b_4=25$. *For* $T^+_i=\{t\in T: a_t>b_i\}=\{1,2,3\}$, $i=1,\ldots,4$, *we have* $\bar{K}_i=\{9\}$, $T_{i7}=\{1,2\}$, $T_{i8}=T_{i9}=\{1,2,3\}$, *and so* $K_{i1}=K_{i2}=\{7,8,9\}$ *and* $K_{i3}=\{8,9\}$ *for all* $i=1,\ldots,4$. *Also,* $\tau_{i7}=6$, $\tau_{i8}=20$, $\tau_{i9}=-5$ *for* $i=1,\ldots,4$. *Assuming* $(\alpha_1,\alpha_2)=(15,6)$, *we have* $b^{(1)}_1=1<b^{(1)}_3=9<b^{(1)}_4=10<b^{(1)}_2=14$, *and*

$b_1^{(2)} = 1 < b_2^{(2)} = 2 < b_3^{(2)} = 3 < b_4^{(2)} = 4$. *Note that* $\left\lceil b_1^{(1)}/\alpha_2 \right\rceil = 1$, $\left\lceil b_i^{(1)}/\alpha_2 \right\rceil = 2$ *for* $i = 2, 3, 4$, *and clearly the conditions* (112), *i.e.* $\alpha_1 \geq \alpha_2 \left\lceil b_i^{(1)}/\alpha_2 \right\rceil$, *are satisfied for* $i = 1, \ldots, 4$. *Note that* $a_4, a_6 \in \mathcal{I}_2^{i,2}, i = 1, \ldots, 4$, $a_5, \tau_{17}, \tau_{18}, \tau_{19} \in \mathcal{I}_0^{1,2}$, $a_5, \tau_{i8} \in \mathcal{I}_1^{i,2}, i = 2, 3, 4$, $\tau_{i7} \in \mathcal{I}_2^{i,2}, i = 2, 3, 4$, $\tau_{29} \in \mathcal{I}_1^{2,2}$, *and* $\tau_{i9} \in \mathcal{I}_0^{i,2}, i = 3, 4$. *Observe that* $a_2 = \alpha_1$, $a_4 = \alpha_2$, *and* $a_2^{(2)} = a_4^{(2)} = 0$. *Therefore, we define* $T_\alpha = \{2, 4\}$. *We also have* $\tau_{ir}^{(2)} = 0$, *where* $\tau_{ir} \in \mathcal{I}_2^{i,2}$, *for* $r \in K$ *and* $i = 1, \ldots, 4$.

Mingled 3-step cycle inequalities for Z^4: *Setting* $n = 2$, *the set of nodes and arcs of the graph* $\bar{G}_2$ *will be* $V_2 = \{0, \ldots, 4\}$ *and* $A_2 = \{(i, j) : i, j \in V_2\}$. *The linear function* $\pi_{ij}^2(y, v, s)$ *associated with each arc* $(i, j) \in A_2$ *is defined by* (107) *where* $n = 2$. *Based on Theorem 13, the mingled 3-step cycle inequalities corresponding to the cycles in* G_2 *are valid for* Z^4. *For example, the mingled 3-step cycle inequality corresponding to a cycle* $C = (V_C, A_C)$ *in* G_2 *where* $A_C = \{(0, 4), (4, 3), (3, 1), (1, 0)\}$ *is*

$$\pi_{04}^2 + \pi_{43}^2 + \pi_{31}^2 + \pi_{10}^2 \geq 0. \tag{114}$$

Theorem 14. *The following linear program is a compact extended formulation for* Z^m, *if conditions* (112) *hold.*

$$\pi_{ij}^n(y, v, s) \geq \delta_i^n - \delta_j^n \text{ for all } (i, j) \in A, n \in \{1, \ldots, N\} \tag{115}$$

$$\textstyle\sum_{t \in T} a_t y_t^i + \sum_{k \in K} a_k y_k^i + v_i + s \geq b_i, i = 1, \ldots, m \tag{116}$$

$$y_t^i \leq u_t, t \in \mathcal{N}, i = 1, \ldots, m \tag{117}$$

$$y \in \mathbb{R}_+^{mn}, v \in \mathbb{R}_+^m, s \in \mathbb{R}_+, \delta \in \mathbb{R}^{N(m+1)}. \tag{118}$$

Proof. Let $Z^{m,\delta} := \{(y, v, s, \delta)$ satisfying (115)-(118)$\}$. Clearly $Proj_{y,v,s}(Z^{m,\delta})$ is defined by the set of all mingled n-step cycle inequalities (111), for $n = 1, \ldots, N$, and bound constraints $s, v \geq 0$. This means all the inequalities which define $Proj_{y,v,s}(Z^{m,\delta})$

are valid for Z^m if the parameters $(\alpha_1, \ldots, \alpha_{n-1})$ satisfy conditions (112) which implies $Z^m \subseteq Proj_{y,v,s}(Z^{m,\delta})$ under the same conditions. This proves that $Z^{m,\delta}$ is an extended formulation for Z^m. □

V.2 Facet-Defining Mingled n-step Cycle Inequalities

In this section, we introduce a special case of the mingled n-step cycle inequalities which we refer to as the *mingled n-step mixing inequalities*. The mingled n-step cycle inequality (111) written for cycle $C = (V_C, A_C)$ such that $A_C = \{(0, i_1), (i_1, i_2), \ldots, (i_{q-1}, i_q), (i_q, 0)\}$ with only one forward arc $(0, i_1)$, followed by backward arcs (i_1, i_2), $\ldots, (i_q, 0)$ gives the mingled n-step mixing inequalities, i.e.

$$s \geq \sum_{(i,j) \in B} \left(\left(b_i^{(n-1)} - b_j^{(n-1)}\right) \Phi_i^n(y^i) - \sum_{\substack{t \in T \setminus (T_i^+ \cup T_\alpha) \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} a_t^{(n-1)} y_t^i - \sum_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_{n-1}^{i,n-1}}} \tau_{ik}^{(n-1)} y_k^i - v_i \right) \tag{119}$$

where $B = \{(i_1, i_2), \ldots, (i_{q-1}, i_q), (i_q, 0)\}$. We show that for any $n \in \mathbb{N}$, the mingled n-step mixing inequalities define facets for $conv(Z^m)$ under certain conditions. In order to prove this, we first define $T_\alpha := \{t_1, \ldots, t_{n-1}\} \subseteq T \setminus \left(\cup_{i \in V_C} T_i^+\right)$, assign $\alpha_d = a_{t_d}$ for $d = 1, \ldots, n-1$, and re-write (108) as follows:

$$
\begin{aligned}
\xi_i^n(y^i) := &\prod_{l=1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil - \sum_{t\in T_i^+}\prod_{l=1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil \left(y_t^i - \sum_{k\in K_t}\bar{u}_{tk}^i y_k^i\right) \\
&-\sum_{g=0}^{n-2}\sum_{\substack{t\in T\setminus(T_i^+\cup T_\alpha)\\ a_t\in\mathcal{I}_g^{i,n-1}}}\left(\sum_{q=1}^{g}\prod_{l=q+1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil \left\lfloor \frac{a_t^{(q-1)}}{\alpha_q}\right\rfloor + \prod_{l=g+2}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil \left\lceil \frac{a_t^{(g)}}{\alpha_{g+1}}\right\rceil\right) y_t^i \\
&-\sum_{g=0}^{n-2}\sum_{\substack{k\in K\\ \tau_{ik}\in\mathcal{I}_g^{i,n-1}}}\left(\sum_{q=1}^{g}\prod_{l=q+1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil \left\lfloor \frac{\tau_{ik}^{(q-1)}}{\alpha_q}\right\rfloor + \prod_{l=g+2}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil \left\lceil \frac{\tau_{ik}^{(g)}}{\alpha_{g+1}}\right\rceil\right) y_k^i \\
&-\sum_{\substack{t\in T\setminus(T_i^+\cup T_\alpha)\\ a_t\in\mathcal{I}_{n-1}^{i,n-1}}}\sum_{q=1}^{n-1}\prod_{l=q+1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil \left\lfloor \frac{a_t^{(q-1)}}{\alpha_q}\right\rfloor y_t^i \\
&-\sum_{\substack{k\in K\\ \tau_{ik}\in\mathcal{I}_{n-1}^{i,n-1}}}\sum_{q=1}^{n-1}\prod_{l=q+1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil \left\lfloor \frac{\tau_{ik}^{(q-1)}}{\alpha_q}\right\rfloor y_k^i - \sum_{\substack{d=1\\ t_d\in T_\alpha}}^{n-1}\prod_{l=d+1}^{n-1}\left\lceil \frac{b_i^{(l-1)}}{\alpha_l}\right\rceil y_{t_d}^i. \qquad (120)
\end{aligned}
$$

Next, we redefine some points (introduced in Chapters III and IV), introduce some new points, and provide some properties for them. Note that in the following definitions we only describe nonzero components for each point.

Definition 15. *For $i \in \{1,\dots,m\}$, define the points $\mathcal{P}^{i,r}, \mathcal{Q}^{i,r} \in \mathbb{Z}_+^N$, $r = 1,\dots,n-1$, as follows:*

$$
\mathcal{P}_{t_d}^{i,r} := \begin{cases} \left\lfloor \frac{b_i^{(d-1)}}{\alpha_d} \right\rfloor & d = 1,\dots,r-1, \\ \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil & d = r \end{cases} \qquad \mathcal{Q}_{t_d}^{i,r} := \left\{ \left\lfloor \frac{b_i^{(d-1)}}{\alpha_d} \right\rfloor \quad d = 1,\dots,r, \right.
$$

the points $\mathcal{R}^{1,i,g,r} \in \mathbb{Z}_+^N$, $r \in T\setminus(T_i^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_g^{i,n-1}$ and $g \in \{0,\dots,n-2\}$, *as follows:*

$$\mathcal{R}_t^{1,i,g,r} := \begin{cases} \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lceil \frac{a_r^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = 1, \ldots, g+1, \\ 1 & t = r, \end{cases}$$

and the points $\mathcal{R}^{2,i,g,r} \in \mathbb{Z}_+^N$, $r \in K$ *where* $\tau_{ir} \in \mathcal{I}_g^{i,n-1}$ *and* $g \in \{0, \ldots, n-2\}$, *as follows:*

$$\mathcal{R}_t^{2,i,g,r} := \begin{cases} \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lceil \frac{\tau_{ir}^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = 1, \ldots, g+1, \\ 1 & t = r, \\ \bar{u}_{tr}^i & \textit{for all } t \in T_{ir}. \end{cases}$$

Furthermore, we introduce the points $\mathcal{S}^{1,i,r} \in \mathbb{Z}_+^N$, $r \in T \backslash (T_i^+ \cup T_\alpha)$ *where* $a_r \in \mathcal{I}_{n-1}^{i,n-1}$, *as follows:*

$$\mathcal{S}_t^{1,i,r} := \begin{cases} \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lceil \frac{a_r^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = 1, \ldots, n-2, \\ \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_r^{(d-1)}}{\alpha_d} \right\rfloor & t = t_d, d = n-1, \\ 1 & t = r. \end{cases}$$

and the points $\mathcal{S}^{2,i,r} \in \mathbb{Z}_+^N$, $r \in K$ *where* $\tau_{ir} \in \mathcal{I}_{n-1}^{i,n-1}$, *as follows:*

$$\mathcal{S}_t^{2,i,r} := \begin{cases} \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lceil \frac{\tau_{ir}^{(d-1)}}{\alpha_d} \right\rceil & t = t_d, d = 1, \ldots, n-2, \\ \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{\tau_{ir}^{(d-1)}}{\alpha_d} \right\rfloor & t = t_d, d = n-1, \\ 1 & t = r, \\ \bar{u}_{tr}^i & \textit{for all } t \in T_{ir}. \end{cases}$$

Lemma 9. *For* $i \in \{1, \ldots, m\}$, *assuming* $u_{t_1}^i \geq \left\lceil \frac{b_i}{\alpha_1} \right\rceil - \left\lceil \frac{\min\{\tau_{ik} : k \in \bar{K}_i\}}{\alpha_1} \right\rceil$ *and* $u_{t_d}^i \geq \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil$, $d = 2, \ldots, n-1$, *the point* $(\hat{y}, \hat{v}, \hat{s}) \in \mathbb{Z}_+^{m \times N} \times \mathbb{R}_+^{m+1}$ *satisfies constraint* i *of* Z^m *if any of the following is true*

(a). $\hat{y}^i = \mathcal{P}^{i,r}$ *for some* $r \in \{1, \ldots, n-1\}$

(b). $\hat{y}^i = \mathcal{Q}^{i,r}$ *for some* $r \in \{1, \ldots, n-1\}$ *and* $\hat{v}_i + \hat{s} \geq b_i^{(r-1)}$,

(c). $\hat{y}^i = \mathcal{R}^{1,i,g,r}$ *for some* $r \in T \backslash (T_i^+ \cup T_\alpha)$ *where* $a_r \in \mathcal{I}_g^{i,n-1}$ *and* $g \in \{0, \ldots, n-2\}$,

(d). $\hat{y}^i = \mathcal{R}^{2,i,g,r}$ *for some* $r \in K$ *where* $\tau_{ir} \in \mathcal{I}_g^{i,n-1}$ *and* $g \in \{0, \ldots, n-2\}$,

(e). $\hat{y}^i = \mathcal{S}^{1,i,r}$ *for some* $r \in T \backslash (T_i^+ \cup T_\alpha)$ *where* $a_r \in \mathcal{I}_{n-1}^{i,n-1}$,

(f). $\hat{y}^i = \mathcal{S}^{2,i,r}$ *for some* $r \in K$ *where* $\tau_{ir} \in \mathcal{I}_{n-1}^{i,n-1}$.

Proof. Cases (a) and (b) can be easily proved similar to the proof of Lemma 5 in [96]. Cases (c)-(f) can be easily proved similar to the proof of Lemma 6 in previous chapter. □

Lemma 10. *For* $i \in \{1, \ldots, m\}$ *and* $n \in \mathbb{N}$,

(a). $\xi_i^n(\mathcal{P}^{i,r}) = 0$, $r = 1, \ldots, n-1$,

(b). $\xi_i^n(\mathcal{Q}^{i,r}) = 1,\ r = 1, \ldots, n-1,$

(c). $\xi_i^n(\mathcal{R}^{1,i,g,r}) = 0,$ *for each* $r \in T \backslash (T_i^+ \cup T_\alpha)$ *where* $a_r \in \mathcal{I}_g^{i,n-1}, g \in \{0, \ldots, n-2\},$

(d). $\xi_i^n(\mathcal{R}^{2,i,g,r}) = 0,$ *for each* $r \in K$ *where* $\tau_{ir} \in \mathcal{I}_g^{i,n-1}$ *and* $g \in \{0, \ldots, n-2\},$

(e). $\xi_i^n(\mathcal{S}^{1,i,r}) = 0,$ *for each* $r \in T \backslash (T_i^+ \cup T_\alpha)$ *where* $a_r \in \mathcal{I}_{n-1}^{i,n-1},$

(f). $\xi_i^n(\mathcal{S}^{2,i,r}) = 0,$ *for each* $r \in K$ *where* $\tau_{ir} \in \mathcal{I}_{n-1}^{i,n-1}.$

Proof. Cases (a) and (b) can be proved similar to Lemma 6 of [96]. The remaining cases, i.e. Cases (c)-(f), can be proved similar to Lemma 7 in previous chapter. □

Theorem 15. *For* $n \in \mathbb{N}$*, the mingled* n*-step cycle inequality* (111) *for an elementary cycle* $C = (V_C, A_C)$ *of graph* $\bar{G}_n$ *is facet-defining for* $conv(Z^m)$ *if (in addition to* $T(F) = \{0\}$*) the following conditions hold:*

(a) $\alpha_d = a_{t_d}$ *where* $t_d \in T \backslash \left(\cup_{i \in V_C} T_i^+\right)$ *for* $d = 1, \ldots, n-1;$

(b) $T_i^+ = \{t \in T : a_t \geq \alpha_1 \lceil b_i/\alpha_1 \rceil\}$ *and* $\alpha_{t_d} \left\lceil b_i^{(d-1)}/\alpha_d \right\rceil \leq \alpha_{t_{d-1}}$ *for* $d = 2, \ldots, n-1, i \in T(B);$

(c) $u_{t_1}^i \geq \left\lceil \frac{b_i}{\alpha_1} \right\rceil - \left\lceil \frac{\min\{\tau_{ik} : k \in \bar{K}_i\}}{\alpha_1} \right\rceil$ *and* $u_{t_d}^i \geq \left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil$*,* $d = 2, \ldots, n-1$ *for* $i \in T(B);$

(d) $a_r^{(n-1)} = 0,\ r \in T \backslash (T_i^+ \cup T_\alpha)$ *where* $a_r \in \mathcal{I}_{n-1}^{i,n-1},\ i \in T(B);$

(e) $\tau_{ir}^{(n-1)} = 0,\ r \in K$ *where* $\tau_{ir} \in \mathcal{I}_{n-1}^{i,n-1},\ i \in T(B);$

Proof. Consider the supporting hyperplane of inequality (111) for the cycle C with $T(F) = \{0\}$. Note that this hyperplane can be written as

$$s \geq \sum_{(i,j) \in B} \left(\left(b_i^{(n-1)} - b_j^{(n-1)}\right) \xi_i^n(y^i) - \sum_{\substack{t \in T \backslash (T_i^+ \cup T_\alpha) \\ a_t \in \mathcal{I}_{n-1}^{i,n-1}}} a_t^{(n-1)} y_t^i - \sum_{\substack{k \in K \\ \tau_{ik} \in \mathcal{I}_{n-1}^{i,n-1}}} \tau_{ik}^{(n-1)} y_k^i - v_i \right) \tag{121}$$

because the cycle C has only one forward arc, which goes out of node 0, and we have $v_0 = 0$, $y^0 = 0$, $\xi_0^n(y^0) := 1$, and B is the set of backward arcs of the cycle C i.e. $B := \{(i,j) \in A_C : j < i\}$ by definition. Let $\Gamma = \{(y,v,s) \in conv(Z^m) : (121)\}$ be the face of $conv(Z^m)$ defined by hyperplane (121) and

$$\sum_{i=1}^{m}\sum_{t=1}^{n} \lambda_t^i y_t^i + \sum_{i=1}^{m} \rho_i v_i + \rho_0 s = \theta \tag{122}$$

be a hyperplane passing through Γ. We prove that (122) must be a multiple of (121). Now, consider the point $\mathcal{A} = (y,v,s) = (y^1,\ldots,y^m,v_1,\ldots,v_m,0) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ such that

$$(y^i, v_i) = \begin{cases} (\mathcal{P}^{i,1}, 0) & \text{if } i \in T(B), \\ (0, b_i) & \text{if } i \notin T(B), \end{cases}$$

Based on Lemma 9(a), $\mathcal{A} \in Z^m$ and using Lemma 10(a), it can be easily verified that $\mathcal{A}$ satisfies (121). So, $\mathcal{A} \in \Gamma$ and hence must satisfy (122). Substituting $\mathcal{A}$ into (122) gives

$$\sum_{i \in T(B)} \lambda_{t_1}^i \lceil b_i/\alpha_1 \rceil + \sum_{\substack{i=1 \\ i \notin T(B)}}^{m} \rho_i b_i = \theta. \tag{123}$$

Using (123), hyperplane (122) reduces to

$$\begin{aligned} \rho_0 s = &\sum_{i \in T(B)} \left(\lambda_{t_1}^i \left(\lceil b_i/\alpha_1 \rceil - y_{t_1}^i \right) - \sum_{t \in \mathcal{N}\setminus\{t_1\}} \lambda_t^i y_t^i - \rho_i v_i \right) \\ &+ \sum_{\substack{i=1 \\ i \notin T(B)}}^{m} \left(\rho_i (b_i - v_i) - \sum_{t \in \mathcal{N}} \lambda_t^i y_t^i \right). \end{aligned} \tag{124}$$

Next, for $p = 1,\ldots,m$ and $r \in T_p^+$, consider the points $\mathcal{A}_1^{p,r} = (y,v,s) = (y^1,\ldots,y^m,$ $v_1,\ldots,v_m,0) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ such that $(y^i, v_i) = (\mathcal{P}^{i,1}, 0)$ for all $i \in \{1,\ldots,m\}\setminus(T(B) \cup$

$\{p\})$, $(y^i, v_i) = (0, b_i)$ for $i \in T(B)\backslash\{p\}$, and

$$y_t^p = \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{if } t \neq r, \end{cases}$$

for $t \in \mathcal{N}$, and $v_p = 0$. Based on Lemma 9(a) and the definition of T_p^+ (i.e. $a_r > b_p$ for $r \in T_p^+$), $\mathcal{A}_1^{p,r} \in Z^m$ and using Lemma 10(a), it can be easily verified that $\mathcal{A}_1^{p,r}$ satisfies (121). So, $\mathcal{A}_1^{p,r} \in \Gamma$ and hence must satisfy (124). Substituting $\mathcal{A}_1^{p,r}$ into (124) gives

$$\lambda_r^p = \lambda_{t_1}^p \lceil b_p/\alpha_1 \rceil \text{ for } p = 1, \ldots, m. \tag{125}$$

Notice that for each $p \in \{1, \ldots, m\}\backslash T(B)$, the unit vector $\mathcal{B}_1^p = (y^1, \ldots, y^m, v_1, \ldots, v_m, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$, in which $v_p = 1$ and all other coordinates are zero, is a direction for both the set Z^m and the hyperplane defined by (121), and hence a direction for the face Γ. This implies that

$$\rho_p = 0 \text{ for all } p \in \{1, \ldots, m\}\backslash T(B). \tag{126}$$

For each $p \in \{1, \ldots, m\}\backslash T(B)$ and $d \in \mathcal{N}$, consider the point $\mathcal{B}_2^{p,d} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are exactly same as $\mathcal{A}$ except that $y_d^p = 1$ and $v_p = \min\{0, 1 - a_d\}$. Based on Lemma 9(a), $\mathcal{B}_2^{p,d} \in Z^m$ and using Lemma 10(a), it can be easily verified that $\mathcal{B}_2^{p,d}$ satisfies (121). So, $\mathcal{B}_2^{p,d} \in \Gamma$ and hence must satisfy (124). Substituting $\mathcal{B}_2^{p,d}$ into (124) and using (126) gives

$$\lambda_d^p = 0 \text{ for } p \in \{1, \ldots, m\}\backslash T(B), d \in \mathcal{N}. \tag{127}$$

These reduce the hyperplane (124) to

$$\rho_0 s = \sum_{i\in T(B)} \left(\lambda^i_{t_1} \left(\lceil b_i/\alpha_1 \rceil - y^i_{t_1} \right) - \sum_{t\in \mathcal{N}\setminus\{t_1\}} \lambda^i_t y^i_t - \rho_i v_i \right). \tag{128}$$

Now, consider the points $\mathcal{C}^{p,d} = (y,v,s) = (y^1,\ldots,y^m,v_1,\ldots,v_m,0) \in \mathbb{Z}^{mN}_+ \times \mathbb{R}^{m+1}_+$ for $k \in T(B)$, $d = 2,\ldots,n-1$ such that

$$(y^i, v_i) = \begin{cases} (\mathcal{P}^{i,d}, 0) & \text{if } i = p, \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \neq p, \end{cases}$$

for $i = 1,\ldots,m$. By Lemma 9(a), $\mathcal{C}^{p,d} \in Z^m$, for all $p \in T(B)$ and $d = 2,\ldots,n-1$. Using Lemma 10(a), one can easily verify that all these points also satisfy (121). So for all $p \in T(B)$ and $d = 2,\ldots,n-1$, $\mathcal{C}^{p,d} \in \Gamma$, and hence must satisfy (128). For each $p \in T(B)$, substituting the points $\mathcal{C}^{p,2},\ldots,\mathcal{C}^{p,n-1}$ one after the other into (128) gives

$$\lambda^p_{t_{d-1}} = \lambda^p_{t_d} \left\lceil b^{(d-1)}_p/\alpha_d \right\rceil, d = 2,\ldots,n-1, p \in T(B),$$

which implies

$$\lambda^p_{t_d} = \lambda^p_{t_{n-1}} \prod_{l=d+1}^{n-1} \left\lceil b^{(l-1)}_p/\alpha_l \right\rceil, d = 1,\ldots,n-2, p \in T(B). \tag{129}$$

Also, using (125) and (129), we get

$$\lambda^p_r = \lambda^p_{t_{n-1}} \prod_{l=1}^{n-1} \left\lceil b^{(l-1)}_p/\alpha_l \right\rceil \text{ for all } r \in T^+_p, p \in T(B). \tag{130}$$

Note that in the point $\mathcal{C}^{p,d}$, $p \in T(B)$, $d \in \{2,\ldots,n-1\}$, by definition we have

$(y^p, v_p) = (\mathcal{P}^{p,d}, 0)$. For each $p \in T(B)$ and $r \in T\backslash(T_p^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_g^{p,n-1}$, $g \in \{0, \ldots, n-2\}$, we define another point $\mathcal{C}_1^{p,g,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are exactly the same as $\mathcal{C}^{p,d}$ except that $(y^p, v_p) = (\mathcal{R}^{1,p,g,r}, 0)$. By Lemma 9(a,c), $\mathcal{C}_1^{p,g,r} \in Z^m$, for all $p \in T(B)$ and $r \in T\backslash(T_p^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_g^{p,n}$, $g \in \{0, \ldots, n-2\}$. Using Lemma 10(a,c), one can easily verify that all these points also satisfy (121). So for all $p \in T(B)$ and $r \in T\backslash(T_p^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_g^{p,n}$, $g \in \{0, \ldots, n-2\}$, $\mathcal{C}_1^{p,g,r} \in \Gamma$, and hence must satisfy (128). Now for each $p \in T(B)$ and $r \in T\backslash(T_p^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_g^{p,n}$, $g \in \{0, \ldots, n-2\}$, $\mathcal{C}_1^{p,g,r} \in \Gamma$, substituting the point $\mathcal{C}_1^{p,g,r}$ in (128) and using (129) gives

$$\begin{aligned} \lambda_r^p &= \lambda_{t_{n-1}}^p \left(\prod_{l=1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{g+1} \prod_{l=d+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{a_r^{(d-1)}}{\alpha_d} \right\rfloor - 1 \right) \right) \\ &= \lambda_{t_{n-1}}^p \left(\sum_{d=1}^{g} \prod_{l=d+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_r^{(d-1)}}{\alpha_d} \right\rfloor + \prod_{l=g+2}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{a_r^{(g)}}{\alpha_{g+1}} \right\rceil \right). \end{aligned} \tag{131}$$

Next, for each $p \in T(B)$ and $r \in T\backslash(T_p^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_{n-1}^{p,n-1}$, we define another point $\mathcal{C}_2^{p,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are exactly the same as $\mathcal{C}^{p,d}$ except that $(y^p, v_p) = (\mathcal{S}^{1,p,r}, 0)$. By Lemma 9(a,e), $\mathcal{C}_2^{p,r} \in Z^m$, for all $p \in T(B)$ and $r \in T\backslash(T_p^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_{n-1}^{p,n-1}$. Using Lemma 10(a,e) and condition (d), one can easily verify that all these points also satisfy (121). So for all $p \in T(B)$ and $r \in T\backslash(T_p^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_{n-1}^{p,n-1}$, $\mathcal{C}_2^{p,r} \in \Gamma$, and hence must satisfy (128). Now for each $p \in T(B)$ and $r \in T\backslash(T_p^+ \cup T_\alpha)$ where $a_r \in \mathcal{I}_{n-1}^{p,n-1}$, $\mathcal{C}_2^{p,r} \in \Gamma$, substituting the point $\mathcal{C}_2^{p,r}$ in (128) and using (129) gives

$$\lambda_r^p = \lambda_{t_{n-1}}^p \left(\sum_{d=1}^{n-1} \prod_{l=d+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_r^{(d-1)}}{\alpha_d} \right\rfloor \right). \tag{132}$$

For each $p \in T(B)$ and $r \in K$ where $\tau_{pr} \in \mathcal{I}_g^{p,n-1}$, $g \in \{0, \ldots, n-2\}$, we define another point $\mathcal{C}_3^{p,g,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are exactly the same as $\mathcal{C}^{p,d}$ except that $(y^p, v_p) = (\mathcal{R}^{2,p,g,r}, 0)$. By Lemma 9(a,d), $\mathcal{C}_3^{p,g,r} \in Z^m$, for all $p \in T(B)$ and $r \in K$ where $o_{pr} \in \mathcal{I}_g^{p,n-1}$, $g \in \{0, \ldots, n-2\}$. Using Lemma 10(a,d), one can easily verify that all these points also satisfy (121). So for all $p \in T(B)$ and $r \in K$ where $\tau_{pr} \in \mathcal{I}_g^{p,n-1}$, $g \in \{0, \ldots, n-2\}$, $\mathcal{C}_3^{p,g,r} \in \Gamma$, and hence must satisfy (128). Now for each $p \in T(B)$ and $r \in K$ where $\tau_{pr} \in \mathcal{I}_g^{p,n-1}$, $g \in \{0, \ldots, n-2\}$, $\mathcal{C}_3^{p,g,r} \in \Gamma$, substituting the point $\mathcal{C}_3^{p,g,r}$ in (128) and using (129) and (130) gives

$$\begin{aligned}
\lambda_r^p &= \lambda_{t_{n-1}}^p \Bigg(\prod_{l=1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil - \sum_{d=1}^{g+1} \prod_{l=d+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left(\left\lceil \frac{b_i^{(d-1)}}{\alpha_d} \right\rceil - \left\lfloor \frac{\tau_{pr}^{(d-1)}}{\alpha_d} \right\rfloor - 1 \right) \\
&\qquad - \sum_{t \in T_i^+} \prod_{l=1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \bar{u}_{tr}^p \Bigg) \\
&= \lambda_{t_{n-1}}^p \Bigg(\sum_{d=1}^{g} \prod_{l=d+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{a_r^{(d-1)}}{\alpha_d} \right\rfloor + \prod_{l=g+2}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{\tau_{pr}^{(g)}}{\alpha_{g+1}} \right\rceil \\
&\qquad - \sum_{t \in T_i^+} \prod_{l=1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \bar{u}_{tr}^p \Bigg). \qquad (133)
\end{aligned}$$

Next, for each $p \in T(B)$ and $r \in K$ where $\tau_{pr} \in \mathcal{I}_{n-1}^{p,n-1}$, we define another point $\mathcal{C}_4^{p,r} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are exactly the same as $\mathcal{C}^{p,d}$ except that $(y^p, v_p) = (\mathcal{S}^{2,p,r}, 0)$. By Lemma 9(a,f), $\mathcal{C}_4^{p,r} \in Z^m$, for all $p \in T(B)$ and $r \in K$ where $\tau_{pr} \in \mathcal{I}_{n-1}^{p,n-1}$. Using Lemma 10(a,f) and condition (e), one can easily verify that all these points also satisfy (121). So for all $p \in T(B)$ and $r \in K$ where $\tau_{pr} \in \mathcal{I}_{n-1}^{p,n-1}$, $\mathcal{C}_4^{p,r} \in \Gamma$, and hence must satisfy (128). Now for each $p \in T(B)$ and $r \in K$ where $\tau_{pr} \in \mathcal{I}_{n-1}^{p,n-1}$, $\mathcal{C}_4^{p,r} \in \Gamma$, substituting the point $\mathcal{C}_4^{p,r}$ in (128) and using (129) and (130) gives

$$\lambda_r^p = \lambda_{t_{n-1}}^p \Bigg(\sum_{d=1}^{n-1} \prod_{l=d+1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{\tau_{pr}^{(d-1)}}{\alpha_d} \right\rfloor - \sum_{t \in T_i^+} \prod_{l=1}^{n-1} \left\lceil \frac{b_i^{(l-1)}}{\alpha_l} \right\rceil \bar{u}_{tr}^p \Bigg). \qquad (134)$$

Based on (129), (130), (131), (132), (133), and (134), hyperplane (128) reduces to

$$\rho_0 s = \sum_{i \in T(B)} \left(\lambda^i_{t_{n-1}} \xi^n_i(y^i) - \rho_i v_i \right). \tag{135}$$

Now for $i \in V_C$, let $N(i)$ be the node in V_C such that $(i, N(i)) \in A_C$. For each $(p, q) \in B$, consider the points $\mathcal{D}^{p,q} = (y, v, s) = (y^1, \ldots, y^m, v_1, \ldots, v_m, b_q^{(n-1)}) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ such that

$$(y^i, v_i) = \begin{cases} (\mathcal{Q}^{i,n-1}, 0) & \text{if } i \in T(B), i < q \\ (\mathcal{Q}^{i,n-1}, b_i^{(n-1)} - b_q^{(n-1)}) & \text{if } i \in T(B), N(i) < q \leq i \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \in T(B), N(i) \geq q \\ (\mathcal{P}^{i,1}, 0) & \text{if } i \notin V_C, \end{cases}$$

for $i = 1, \ldots, m$. By Lemma 9(a,b), it is clear that $\mathcal{D}^{p,q} \in Z^m$ for all $(p, q) \in B$. Using Lemma 10(a,b), it is easy to show that points $\mathcal{D}^{p,q}$, for all $(p, q) \in B$, also satisfy (121). Hence, they belong to Γ, and must satisfy (135). Now, note that in the point $\mathcal{D}^{p,q}$, $(p, q) \in B$, by definition we have $(y^p, v_p) = (\mathcal{Q}^{p,n-1}, b_p^{(n-1)} - b_q^{(n-1)})$. For each $(p, q) \in B$, define another point $\mathcal{D}_1^{p,q} = (y, v, s) \in \mathbb{Z}_+^{mN} \times \mathbb{R}_+^{m+1}$ whose coordinates are all exactly the same as $\mathcal{D}^{p,q}$ except that $(y^p, v_p) = (\mathcal{Q}^{p,n-1}, 0)$. For precisely the same reasons stated for $\mathcal{D}^{p,q}$, the points $\mathcal{D}_1^{p,q}$, $(p, q) \in B$, must also satisfy (135). Now if for each $(p, q) \in B$, we substitute $\mathcal{D}^{p,q}$ and $\mathcal{D}_1^{p,q}$ into (135) and subtract one equality from the other, we get

$$\lambda^p_{t_{n-1}} = \rho_p \left(b_p^{(n-1)} - b_q^{(n-1)} \right), \text{ for all } (p, q) \in B. \tag{136}$$

Based on (136), and assumptions $(c), (d)$, hyperplane (135) reduces to

$$\rho_0 s = \sum_{(i,j)\in B} \rho_i \left(\left(b_i^{(n)} - b_j^{(n)}\right) \xi_i^n(y^i) - \sum_{\substack{t\in T\setminus(T_i^+\cup T_\alpha) \\ a_t\in\mathcal{I}_{n-1}^{i,n-1}}} a_t^{(n-1)} y_t^i - \sum_{\substack{k\in K \\ \tau_{ik}\in\mathcal{I}_{n-1}^{i,n-1}}} \tau_{ik}^{(n-1)} y_k^i - v_i \right). \tag{137}$$

Assuming $B := \{(i_1, i_2), \ldots, (i_q, 0)\}$ where $i_1 > i_2 > \ldots > i_q$, we substitute points $\mathcal{D}_1^{i_q,0}$, $\ldots, \mathcal{D}_1^{i_1,i_2}$ one after another in (137) and get

$$\rho_i = \rho_0 \text{ for all } i \in T(B). \tag{138}$$

This reduces hyperplane (137) to a constant multiple of (121), which completes the proof.

□

Example 3 (continued)**.** *Notice that for $n = 2$, the coefficients of Z^4 also satisfy the additional conditions required in Theorem 15, i.e. (c) $u_4^i = u_6^i = 2$ for $i = 1, \ldots, 4$, (d) $a_r^{(2)} = 0$ for $r \in T\setminus T_k^+$, $k = 1, \ldots, 4$, where $a_r \in \mathcal{I}_2^{k,2}$, (e) $a_r^{(2)} = 0$ for $r \in T\setminus T_k^+$, $k = 1, \ldots, 4$, where $a_r \in \mathcal{I}_2^{k,2}$. Therefore, the mingled 3-step cycle inequality* (111) *corresponding to each cycle $C = (V_C, A_C)$ in graph $\bar{G}_2$, where $T(F) = \{0\}$, defines a facet for conv(Z^4). In particular, mingled 3-step cycle inequalities* (114) *is facet-defining for conv(Z^4).*

CHAPTER VI

CUTS FOR MMLS, MMFL, AND MMND PROBLEMS

In this chapter, we introduce new classes of multi-row cuts for MIPs involving "*multi-modularity capacity constraints*". More specifically, in Sections VI.1, VI.2, VI.3, we utilize the facets of continuous multi-mixing set (discussed in Chapter III) to develop valid inequalities for multi-module capacitated lot-sizing (MMLS) problem with(out) backlogging (MML-(W)B), multi-module capacitated facility location (MMFL), and multi-module capacitated network design (MMND) problems, respectively, which subsume various well-known classes of inequalities earlier developed for these problems. Furthermore, in Section VI.4, we computationally evaluate the effectiveness of the developed cuts (applied using our separation algorithm) in solving the MML-(W)B problem.

VI.1 Cuts for Multi-Module Capacitated Lot-Sizing Problem

In this section, we use n-step cycle inequalities to develop cutting planes for MML-(W)B problem. We define MML-B as follows. Let $P := \{1, \ldots, m\}$ be the set of time periods and $\{\alpha_1, \ldots, \alpha_n\}$ be the set of sizes of the n available capacity modules. The setup cost per module of size $\alpha_t, t = 1, \ldots, n$ in period p is denoted by f_p^t. Given the demand, the production per unit cost, the inventory per unit cost, and the per unit shortage (backlog) cost in period p, denoted by d_p, c_p, h_p, and b_p, respectively, the MML-B problem can be formulated as:

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$$\min \sum_{p\in P} c_p x_p + \sum_{p\in P} h_p s_p + \sum_{p\in P} b_p r_p + \sum_{p\in P}\sum_{t=1}^{n} f_p^t z_p^t \tag{139}$$

$$s_{p-1} - r_{p-1} + x_p = d_p + s_p - r_p, p \in P \tag{140}$$

$$x_p \leq \sum_{t=1}^{n} \alpha_t z_p^t, p \in P \tag{141}$$

$$(z,x,r,s) \in \mathbb{Z}_+^{m\times n} \times \mathbb{R}_+^m \times \mathbb{R}_+^{m+1} \times \mathbb{R}_+^{m+1} \tag{142}$$

where x_p is the production in period p, s_p and r_p are the inventory and backlog, respectively, at the end of period $p, s_0 = r_m = 0$, and z_p^t is the number of capacity modules of size $\alpha_t, t = 1,\ldots,n$, used in period p. Let X^{MML-B} denote the set of feasible solutions to constraints (140)-(142). Note that every valid inequality for X^{MML-B} also gives a valid inequality for the set of feasible solutions to the MML-WB problem which is the projection of $X^{MML-B} \cap \{r = 0\}$ on (z,x,s).

In order to generate valid inequalities for X^{MML-B}, we consider periods $k,\ldots,l$, for any $k,l \in P$ where $k < l$. Let $S \subseteq \{k,\ldots,l\}$ such that $k \in S$. For $i \in S$, let $S_i := S \cap \{k,\ldots,i\}$, $m_i = \min\{p : p \in S\backslash S_i\}$ with $m_i = l+1$ if $S\backslash S_i = \emptyset$, and $b_i = \sum_{p=k}^{m_i-1} d_p$. Now, by adding equalities (140) from period k to period $m_i - 1$, we get

$$s_{k-1} + r_{m_i-1} + \sum_{p=k}^{m_i-1} x_p = b_i + s_{m_i-1} + r_{k-1}. \tag{143}$$

Note that $S_i \subseteq \{k,\ldots,m_i-1\}$ by definition. If we relax $x_p, p \in S_i$, in (143) to its upper bound based on (141) and drop $r_{k-1}, s_{m_i-1}(\geq 0)$, we get the following valid inequality:

$$s_{k-1} + r_{m_i-1} + \sum_{p\in\{k,\ldots,m_i-1\}\backslash S_i} x_p + \sum_{t=1}^{n} \alpha_t \sum_{p\in S_i} z_p^t \geq b_i. \tag{144}$$

Setting

$$s := s_{k-1}, v_i := r_{m_i-1} + \sum_{p\in\{k,\ldots,m_i-1\}\setminus S_i} x_p, \text{ and } y_t^i := \sum_{p\in S_i} z_p^t, \tag{145}$$

inequality (144) becomes

$$s + v_i + \sum_{t=1}^{n} \alpha_t y_t^i \geq b_i, \tag{146}$$

which is of the same form as the defining inequalities of continuous multi-mixing set (notice that $s, v_i \in \mathbb{R}_+, y_t^i \in \mathbb{Z}_+, t = 1, \ldots, n$). Therefore we can form a set of base inequalities consisting of inequalities (144) for all $i \in S$ such that the n-step MIR conditions, i.e. $\alpha_t \left\lceil b_i^{(t-1)}/\alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n$, hold. We construct a directed graph for these base inequalities in the same fashion as we did for the continuous multi-mixing set $Q^{m,n}$ in Chapter III. The n-step cycle inequalities corresponding to each elementary cycle C in this graph is valid for X^{MML-B}. We refer to these inequalities as the n-step (k, l, S, C) cycle inequalities. The same procedure also provides a new class of valid inequalities for MML-WB which subsume the valid inequalities generated using the mixed n-step MIR inequalities [96] for MML-WB.

Note that a procedure similar to what was presented above for n can also be used to develop n'-step (k, l, S, C) cycle inequalities for MML-(W)B problem for any $n' \in \{1, \ldots, n\}$ in general.

VI.2 Cuts for Multi-Module Capacitated Facility Location Problem

In this section, we use n-step cycle inequalities to develop cutting planes for MMFL problem. We define MMFL (first introduced in [96]) as follows. Let $P := \{1, \ldots, m\}$ be a set of potential facilities, $P' := \{1, \ldots, m'\}$ be a set of clients, and $\{\alpha_1, \ldots, \alpha_n\}$ be the set of sizes of the n available capacity modules. The setup cost per module of size $\alpha_t, t = 1, \ldots, n$ at facility p is denoted by f_p^t. Given the demand of client p' and the distribution cost per unit between facility p and client p', denoted by $d_{p'}$ and $c_{pp'}$, respectively, the MMFL problem can be formulated as:

$$\min \sum_{p \in P} \sum_{p' \in P'} c_{pp'} x_{pp'} + \sum_{p \in P} \sum_{t=1}^{n} f_p^t z_p^t \tag{147}$$

$$\sum_{p \in P} x_{pp'} = d_{p'}, p' \in P' \tag{148}$$

$$\sum_{p' \in P'} x_{pp'} \leq \sum_{t=1}^{n} \alpha_t z_p^t, p \in P \tag{149}$$

$$(z, x) \in \mathbb{Z}_+^{m \times n} \times \mathbb{R}_+^{m \times m'} \tag{150}$$

where $x_{pp'}$ is the portion of demand of client p' satisfied by facility p, and z_p^t is the number of capacity modules of size $\alpha_t, t = 1, \dots, n$, used at facility p. Let X^{MMFL} denote the set of feasible solutions to constraints (148)-(150).

In order to generate valid inequalities for X^{MMFL}, we consider facilities $k, \dots, l$, for any $k, l \in P$ where $k < l$. Let $S \subseteq \{k, \dots, l\}$ such that $k \in S$. For $i \in S$, let $S_i := S \cap \{k, \dots, i\}$, $S_i' \subseteq P'$, and $b_i = \sum_{p' \in S_i'} d_{p'}$. Now, by adding equalities (148) for clients $p' \in S_i'$, we get

$$\sum_{p \in P} \sum_{p' \in S_i'} x_{pp'} = b_i. \tag{151}$$

If we relax $\sum_{p' \in S_i'} x_{pp'}, p \in S_i$, in (151) to its upper bound based on (149), we get the following valid inequality:

$$\sum_{p \in P \setminus S_i} \sum_{p' \in S_i'} x_{pp'} + \sum_{t=1}^{n} \alpha_t \sum_{p \in S_i} z_p^t \geq b_i. \tag{152}$$

Assuming $S_i' \subset S_{i+1}'$, for all i and setting

$$s := \sum_{p \in P \setminus S} \sum_{p' \in S_1'} x_{pp'}, v_i := \sum_{p \in P \setminus S_i} \sum_{p' \in S_i'} x_{pp'} - \sum_{p \in P \setminus S} \sum_{p' \in S_1'} x_{pp'}, \text{ and } y_t^i := \sum_{p \in S_i} z_p^t, \tag{153}$$

inequality (152) becomes of the same form as the defining inequalities of continuous multi-mixing set (notice that $s, v_i \in \mathbb{R}_+, y_t^i \in \mathbb{Z}_+, t = 1, \ldots, n$ because $\{(p,p') : p \in P/S, p' \in S_1'\} \subseteq \{(p,p') : p \in P/S_i, p' \in S_i'\}$ for all $i \in S$). Therefore we can form a set of base inequalities consisting of inequalities (152) for all $i \in S$ such that the n-step MIR conditions, i.e. $\alpha_t \left\lceil b_i^{(t-1)}/\alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n$, hold. We construct a directed graph for these base inequalities in the same fashion as we did for the continuous multi-mixing set $Q^{m,n}$ in Chapter III. The n-step cycle inequalities corresponding to each elementary cycle C in this graph is valid for X^{MMFL}. These inequalities subsume the valid inequalities generated using the mixed n-step MIR inequalities [96] for MMFL. Note that a procedure similar to what was presented above for n can also be used to develop a new family of valid inequalities for MMFL problem for any $n' \in \{1, \ldots, n\}$ in general.

VI.3 Cuts for Multi-Module Capacitated Network Design Problem

We next develop a new class of valid inequalities for multi-module capacitated network design (MMND) problem by utilizing the n-step cycle inequalities for $Q^{m,n}$. The MMND is the problem of finding the optimal flow and combination of capacity modularities over the arcs of a (directed) graph to satisfy the net demand at each node, where there are costs associated with the flow and the installed arc capacity modules. Interestingly, the MMLS and MMFL problems can be viewed as special cases of the MMND problem. We define it as follows. Let $G = (V, A)$ be a (directed) graph where $V := \{1, \ldots, m\}$ and $\{\alpha_1, \ldots, \alpha_n\}$ be the set of sizes of the n available capacity modules. The setup cost per module of size $\alpha_t, t = 1, \ldots, n$ and flow cost at arc $(p,p') \in A$ are denoted by $f_{pp'}^t$ and $c_{pp'}$, respectively. Given the net demand d_p (negative demand is supply) at each node $p \in V$, the MMND problem can be formulated as:

$$\min \sum_{(p,p')\in A} \left(c_{pp'} x_{pp'} + \sum_{t=1}^{n} f^t_{pp'} z^t_{pp'} \right) \tag{154}$$

$$\sum_{(p,p')\in A} x_{p'p} - \sum_{(p,p')\in A} x_{pp'} = d_p, p \in V \tag{155}$$

$$x_{pp'} \le \sum_{t=1}^{n} \alpha_t z^t_{pp'}, (p,p') \in A \tag{156}$$

$$(z,x) \in \mathbb{Z}_+^{n|A|} \times \mathbb{R}_+^{|A|} \tag{157}$$

where $x_{pp'}$ corresponds to the flow on the directed arc (p,p'), and $z^t_{pp'}$ is the number of capacity modules of size $\alpha_t, t = 1, \ldots, n$, used at arc (p,p'). Let X^{MMND} denote the set of feasible solutions to constraints (155)-(157).

In order to generate valid inequalities for X^{MMND}, we consider nodes $k, \ldots, l$, for any $k, l \in P$ where $k < l$. Let $S \subseteq \{k, \ldots, l\}$ such that $k \in S$. For $i \in S$, let $S_i = S \cap \{k, \ldots, i\}$, $b_i = \sum_{p \in V \setminus S_i} d_p$, $a(S_i) := \{(p,p'), (p',p) \in A : p \in S_i, p' \in V \setminus S_i\}$, $A_i \subseteq a(S_i)$, and $A'_i = \{(p,p') \in a(S) \setminus A_i : p \in S_i, p' \in V \setminus S_i\}$. Now, by adding equalities (155) for nodes $p \in V \setminus S_i$ and relaxing $x_{pp'}, (p,p') \in A_i$, to its upper bound based on (156), we get the following valid inequality:

$$\sum_{(p,p')\in A'_i} x_{pp'} + \sum_{t=1}^{n} \alpha_t \sum_{(p,p')\in A_i} z^t_{pp'} \ge b_i. \tag{158}$$

Assuming $A'_i \subset A'_{i+1}$, for all i and setting

$$s := \sum_{(p,p')\in A'_k} x_{pp'}, v_i := \sum_{(p,p')\in A'_i \setminus A'_k} x_{pp'}, \text{ and } y^i_t := \sum_{(p,p')\in A_i} z^t_{pp'}, \tag{159}$$

inequality (158) becomes of the same form as the defining inequalities of continuous multi-mixing set (notice that $s, v_i \in \mathbb{R}_+, y^i_t \in \mathbb{Z}_+, t = 1, \ldots, n$). Therefore we can form a set of base inequalities consisting of inequalities (152) for all $i \in S$ such that the n-step MIR conditions, i.e. $\alpha_t \left\lceil b_i^{(t-1)}/\alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n$, hold. Hence, a procedure similar to what was presented above for MML-B (Section VI.1) and MMFL (Section VI.2) can also be used to develop a new family of valid inequalities for MMND problem for any $n' \in \{1, \ldots, n\}$ in general. Interestingly, the cuts developed in [19, 70, 72] for two-modularity ND with divisible capacities (2MND-DC) and in [9] for MMND can be derived just using 1-step MIR procedure. Furthermore, two-modularity cut-set inequalities for 3MND-DC [70] and the partition inequalities for the single-arc MMND-DC [89] can be derived using the 2-step MIR [36, 62] and the n-step MIR, respectively. Our inequalities derived in this section for MMND subsume all these existing valid inequalities developed for this problem and its special cases.

VI.4 Computational Results

In this section, we computationally evaluate the effectiveness of the n'-step cycle inequalities, $n' \in \{1, \ldots, n\}$, for the MML-(W)B problem using our separation algorithm (discussed in Section III.3). We chose $n = 2$ for our experiments in this paper and refer to the MML-WB and MML-B problem with two capacity modules ($n = 2$) as 2ML-WB and 2ML-B, respectively. We created random 2ML-WB and 2ML-B instances with 60 time periods, i.e. $P = \{1, \ldots, 60\}$, and varying cost and capacity characteristics. The demand d_p, production cost c_p, and holding cost h_p in each period were drawn from integer *uniform*$[10, 190]$, integer *uniform*$[81, 119]$, and real *uniform*$[1, 19]$, respectively. For each instance of 2ML-B, the backlog cost b_p in each period equals h_p plus a real number drawn from *uniform*$[1, 10]$. We used three sets of capacity modules $\alpha = (\alpha_1, \alpha_2)$: (70, 34), (100, 35), and (180, 80), denoted by M_a, M_b, and M_c respectively, and four sets of setup costs $(f_p^1, f_p^2), p \in P$: (1000, 600), (5000, 2600), (10500, 6600), and (13000, 10600), denoted by F_I, F_{II}, F_{III}, and F_{IV} respectively. This leads to 12 instance categories where the first set of setup costs (i.e. F_I) leads to easy instances and the remaining three lead to hard instances. Note that some of the instance generation ideas we used here are inspired by the ideas used in [96] for 2ML-WB.

For each 2ML-(W)B instance, we first solved the problem (defined in Section VI.1), for $n = 2$, without adding any of our own cuts using CPLEX 11.0 with its default settings (2ML-(W)B-DEF). In a separate run, for each $n' \in \{1, 2\}$, we used our cut generation algorithm, denoted by CutGen(n'), to add n'-step (k, l, S, C) cycle inequalities to the problem at the root node. The pseudocode of CutGen is presented in Algorithm 2. This algorithm calls our separation algorithm in Line 14 for several choices of (k, l, S) (see Lines 3-11) to generate n'-step (k, l, S, C) cycle inequalities (Lines 12-14) that are violated by the LP relaxation optimal solution, which is updated after adding each cut (see Lines 15-19). Note that each choice of (k, l, S) provides one set of base inequalities (144) (where $n = 2$) and we solve an exact separation problem over the set of all 2-step (k, l, S, C) cycle inequalities corresponding to the base inequalities which satisfy the n-step MIR conditions (discussed in Section VI.1). We then removed the inactive cuts and used CPLEX 11.0 with its default settings to solve the problem (2ML-(W)B-1CUTS for $n' = 1$, and 2ML-(W)B-2CUTS for $n' = 2$). We implemented our codes in Microsoft Visual C++ 2010 and all the experiments were run on a PC which has two Intel Xeon E5620 2.40GHz processors and 12 GB of RAM.

The results of our computational experiments are shown in Tables 1 and 2. Each row of these tables reports the average results for 10 instances of the corresponding instance category. Note that an instance category corresponding to a set of setup costs (say F_I) and a set of capacity module (say M_a) is denoted by I-a. We report the percentage of the integrality gap closed by our cuts, i.e. $G\% = 100 \times (zcut - zlp)/(zmip - zlp)$, where $zlp, zcut$, and $zmip$ are the opti-

Algorithm 2 Generating n'-step (k, l, S, C) Cycle Inequalities for MML-(W)B

function CutGen(n')
  $(\hat{z}, \hat{x}, \hat{r}, \hat{s}) \leftarrow$ optimal solution of the LP relaxation ▷ $\hat{r} = 0$ `for 2ML-WB instance`
  for $k = 1$ to m **do**
    for $l = k + 1$ to m **do**
      for $SS = 1$ to 3 **do**
        if $SS = 1$ **then** $S = \{k, \ldots, l\}$
        else if $SS = 2$ **then**
          $S = \{k\} \cup \{p \in \{k+1, \ldots, l\} : \hat{z}_p^1 > 0 \text{ or } \hat{z}_p^2 > 0\}$
        else if $SS = 3$ **then**
          $S = \{k\} \cup \{p \in \{k+1, \ldots, l\} : \hat{z}_p^1 \notin \mathbb{Z} \text{ or } \hat{z}_p^2 \notin \mathbb{Z}\}$
        end if
        Each choice of (k, l, S) provides directed graph G
        Obtain $(\hat{y}, \hat{v}, \hat{s})$ from $(\hat{z}, \hat{x}, \hat{r}, \hat{s})$ ▷ `see Section VI.1`
        $C := MBFCT(G, (\hat{y}, \hat{v}, \hat{s}), n')$
        if n'-step (k, l, S, C) cycle inequality is violated by $(\hat{z}, \hat{x}, \hat{r}, \hat{s})$ **then**
          Add the n'-step (k, l, S, C) cycle inequality as a cut
          Re-optimize the LP relaxation
          $(\hat{z}, \hat{x}, \hat{r}, \hat{s}) \leftarrow$ optimal solution of the LP relaxation
        end if
      end for
    end for
  end for
end function

mal objective values of the LP relaxation without our cuts, LP relaxation with our cuts, and MIP, respectively. We also report the number of branch-and-bound nodes (*Nodes*), and the time (in seconds) to solve 2ML-(W)B-DEF (T_{Def}), 2ML-(W)B-1CUTS (T^1_{Opt}), and 2ML-(W)B-2CUTS (T^2_{Opt}) to optimality. Note that T^1_{Opt} and T^2_{Opt} exclude the cut generation time. For each $n' \in \{1, 2\}$, the number of active n'-step (k, l, S, C) cycle cuts added at the root node (*Cuts*), the time (in seconds) to generate n'-step (k, l, S, C) cycle cuts (denoted by T^1_{Cut} for $n' = 1$ and T^2_{Cut} for $n' = 2$), and the total time (including the cut generation time) to solve 2ML-(W)B-1CUTS and 2ML-(W)B-2CUTS, denoted by $T^1 = T^1_{Cut} + T^1_{Opt}$ and $T^2 = T^2_{Cut} + T^2_{Opt}$

respectively, are also reported.

Table 2: Results of computational experiments on 2ML-WB instances

Inst.	2ML-WB-DEF		2ML-WB-1CUTS						2ML-WB-2CUTS					
	T_{Def}	*Node*	*Cut*	T^1_{Cut}	T^1_{Opt}	T^1	*Node*	*G%*	*Cut*	T^2_{Cut}	T^2_{Opt}	T^2	*Node*	*G%*
I-a	0.46	811	102	2.0	6.10	8.10	24289	54	114	3.3	0.27	3.6	314	90
I-b	0.73	1296	114	14	0.66	14.7	1099	77	93	8.6	0.23	8.8	212	90
I-c	0.31	347	90	14	0.23	14.2	304	56	123	12	0.07	12.1	32	91
II-a	1128	6.0×10^6	104	2.1	1636	1638	6.8×10^6	42	76	3.3	48.4	51.7	123665	86
II-b	152	356302	82	14	56	70	167450	75	70	9.0	6.42	15.4	14003	81
II-c	700	1.3×10^6	112	15	719	734	1.1×10^6	50	98	12	4.87	16.9	11027	87
III-a	1699	1.0×10^7	64	2.4	1417	1419	5.8×10^6	63	60	3.2	194	197	616257	81
III-b	2448	8.4×10^6	68	15	993	1008	1.3×10^6	75	56	9.1	16.0	25.1	43513	80
III-c	313	663551	76	15	325	340	1.0×10^6	70	76	12	20.0	32.0	38633	86
IV-a	1852	1.1×10^7	64	2.7	434	437	2.1×10^6	76	57	3.0	3.87	6.9	11343	88
IV-b	1972	7.1×10^6	67	14	400	414	605580	82	58	7.7	36.1	43.8	95252	84
IV-c	266	319360	72	16	16	32	40533	77	62	12	2.47	14.5	4234	87

In Table 2, comparing the time to optimize the 2ML-WB problem before and after adding the 2-step (k, l, S, C) cycle cuts (i.e. T^2_{Opt} vs. T_{Def}), we see significant improvement obtained by adding these cuts in both easy instances (on average 3 times) and hard instances (on average 112 times). There is also a substantial reduction in the number of branch-and-bound nodes (on average 6.5 times for easy instances and 174 times for hard instances). The percentage of integrality gap closed by the 2-step (k, l, S, C) cycle cuts is between 80.32% and 91.15% (the average is 85.90%). These results show the strength of 2-step (k, l, S, C) cycle inequalities. Interestingly, in these instances adding 2-step (k, l, S, C) cycle inequalities over 1-step

(k, l, S, C) cycle inequalities has improved the closed integrality gap by 19.48% (in average), the number of nodes by 43 times (in average), and the solution time (i.e. T^2_{Opt} vs. T^1_{Opt}) by 36 times (in average).

Table 3: Results of computational experiments on 2ML-B instances

Inst	2ML-B-DEF		2ML-B-1CUTS						2ML-B-2CUTS					
	T_{Def}	*Node*	*Cut*	T^1_{Cut}	T^1_{Opt}	T^1	*Node*	*G%*	*Cut*	T^2_{Cut}	T^2_{Opt}	T^2	*Node*	*G%*
I-a	0.34	582	105	1.9	3.35	5.25	11150	58	109	4.0	0.18	4.2	197	91
I-b	0.31	691	113	2.1	0.25	2.4	446	80	91	2.9	0.10	3.0	116	88
I-c	0.13	277	98	2.0	0.10	2.1	169	55	125	3.7	0.03	3.7	23	93
II-a	1133	5.2×10^6	113	2.2	2085	2087	8.4×10^6	50	86	4.9	135	140	274503	83
II-b	7.8	31909	93	2.4	10.8	13.2	29551	83	81	3.7	6.1	9.8	12065	83
II-c	28.6	117942	121	2.2	96.4	98.6	300361	57	101	4.6	3.8	8.4	6986	87
III-a	854	4.6×10^6	72	2.5	244	246	1.1×10^6	78	80	5.7	28.0	33.7	81743	83
III-b	122	660454	79	2.8	5.5	8.3	12906	91	79	4.2	3.9	8.1	4601	87
III-c	28.2	130383	88	2.5	56	59	146378	79	86	6.2	13.4	19.6	31979	84
IV-a	1211	6.8×10^6	104	2.9	323	326	753257	80	93	4.2	38	42	82211	83
IV-b	527	3.0×10^6	138	3.3	335	338	1.1×10^6	94	84	3.2	198	201	921530	88
IV-c	37	213644	89	2.8	8.8	11.6	21719	85	88	5.8	4.4	10.2	7151	86

Moreover, going to Table 3 we observe that in all the instance categories of 2ML-B, adding the 2-step (k, l, S, C) cycle cuts cuts to 2ML-B-DEF has reduced the solution time (on average 3 times for easy instances and 13.8 times for hard instances) and the number of branch-and-bound node (on average 6.9 times for easy instances and 39.9 times for hard instances). The percentage of integrality gap closed by these cuts is between 82.94% and 92.52% (the average is 86.75%) for 2ML-B instances. Notice that in these instances adding 2-step (k, l, S, C) cycle inequalities over 1-step

(k, l, S, C) inequalities has improved the closed gap by 16% (in average), the number of nodes by 14 times (in average), and the solution time (i.e. T^2_{Opt} vs. T^1_{Opt}) by 7.7 times (in average).

Also, observe that for the hard instances in Tables 1 and 2, the cut generation time for 2-step (k, l, S, C) cycle cuts (T^2_{Cut}) in negligible compared to T_{Def}. This combined with the highly improved optimization time after adding these cuts has resulted in a total solution time (T^2_{Total}) which is on average 58 times and 9.9 times smaller than the total time to solve 2ML-WB-DEF and 2ML-B-DEF, respectively, (T_{Def}). The collection of these observations show that the 2-step (k, l, S, C) cycle inequalities are very effective in solving the 2ML-WB and 2ML-B problems.

CHAPTER VII

CONCLUSION AND FUTURE RESEARCH

VII.1 Conclusion

In this dissertation, we developed facet-defining valid inequalities for the following new generalizations of the well-studied continuous mixing set: 1) Continuous multi-mixing set with the so-called n-step MIR conditions on the coefficients, (2) Continuous multi-mixing set with *general coefficients*, and (3) Continuous multi-mixing set with general coefficients and *bounded integer variables*. This resulted in new cut-generating procedures for the mixed integer programs and generalizations of MIR, mixed MIR, continuous mixing, n-step MIR, mixed n-step MIR, mingling, and n-step mingling. We provided a knowledge base for developing new families of cutting planes for MIP problems involving "*multi-modularity capacity constraints*" (MMCCs), in particular multi-module capacitated lot-sizing (MMLS), multi-module capacitated facility location (MMFL), and multi-module capacitated network design (MMND). These cutting planes generalize various well-known families of cuts for MMLS, MMFL, and MMND problems, and significantly improve the efficiency of algorithms for solving them.

In the first step, we unified the concepts of the continuous mixing and the n-step MIR by developing a class of valid inequalities (n-step cycle inequalities) for continuous multi-mixing set (a generalization of the continuous mixing set and the n-mixing set) where the coefficients satisfy the so-called "n-step MIR conditions." We provided the facet-defining properties of the n'-step cycle inequalities, $n' \in \{1, \ldots, n\}$, for the continuous multi-mixing set, and showed that the 1-step cycle inequalities

[105], n-step MIR inequalities [62], and mixed n-step MIR inequalities [96] form special cases of the n-step cycle inequalities. Note that the n-step MIR conditions are automatically satisfied if the parameters $\alpha_1, \ldots, \alpha_n$ are divisible. We also presented a compact extended formulation for the continuous multi-mixing set and an exact separation algorithm to separate over the set of all n-step cycle inequalities.

In the next step, we extended the results of the first step to the case where no conditions are imposed on the coefficients of the continuous multi-mixing set. We relaxed the n-step MIR conditions and considered the continuous multi-mixing set with general coefficients. This lead to an extended formulation and generalization of the n-step cycle inequalities. We identified the conditions under which they are facet-defining.

In the third step, we unified the concepts of continuous multi-mixing and n-step mingling by incorporating upper bounds on the integer variables of the continuous multi-mixing set with general coefficients. For each $n \in \mathbb{N}$, we developed new families of valid inequalities for this set, referred to as the mingled n-step cycle inequalities. We derived the facet-defining conditions of these inequalities and provide an exact separation algorithm to separate over a set of all mingled n-step cycle inequalities for a given $n \in \mathbb{N}$. Note that these inequalities generalize n-step cycle inequalities [16, 15] and n-step mingling inequalities [7] (which subsume continuous cover and reverse continuous cover inequalities of Marchand and Wolsey [73] as well as the continuous integer knapsack cover and pack inequalities of Atamtürk [10, 11] derived earlier by superadditive lifting techniques).

Finally, we utilized the results of first step to develop new families of valid inequalities for MIPs involving MMCCs. In particular, we focused on the multi-modularity generalizations of three following high-impact classes of capacitated MIPs: lot-sizing, facility location, and network design problems. We showed that the n-step cycle inequalities can be used to generate cuts for the MMLS with(out) backlogging (MML-(W)B), MMFL, and MMND problems which subsume valid inequalities introduced in [51, 87, 96] for LS problems, [2, 51, 96] for FL problems, and [9, 19, 51, 52, 61, 70, 72, 89] for ND problems, respectively. We also computationally evaluate the effectiveness of the n-step cycle inequalities (applied using our separation algorithm) for the MML-(W)B problem. Our computational results show that our cuts are very effective in solving the MML instances with(out) backlogging, resulting in substantial reduction in the integrality gap, number of nodes, and total solution time.

VII.2 Future Plans

The methodological developments in this dissertation creates pathways to several new research problems. Some research directions originating from the results in this dissertation are as follows:

(i). **Multi-Module Capacitated Lot-Sizing Problem.** On the first path, in the light of the computational results in this dissertation, we intend to investigate the facet-defining properties of the valid inequalities (developed using 2-step cycle inequalities) for two-module capacitated lot-sizing problem with(out) backlogging. Furthermore, we are examining the computational complexity of MML-(W)B. If the number of modularities (n) is part of the input data, these problems are clearly NP-hard (mixed integer knapsack problem can be reduced to single-period versions of these problems). However, the complexity for a fixed n is an open question, which we are already investigating. In addition, we are exploring the solution structure for these problems to develop strong extended formulations and optimization algorithms for them.

(ii). **Superincreasing Continuous Multi-Mixing Set.** On the second path, we intend to develop facets for continuous multi-mixing set with bounded integer variables where coefficients of integer variables and their upper bounds together form a *superincreasing* sequence of tuples. We also plan to describe the convex hull of this set. If successful, this research will generalize the results for superincreasing (0/1) knapsack polyhedron.

(iii). **(New) Facets for New/Existing Base Sets.** In this task, we intend to investigate the polyhedral structure and develop facet-defining valid inequalities for new base sets which we will later use to develop cuts for general and special structure MIPs. We also plan to investigate the possibility of developing new families of facets for continuous multi-mixing set and its generalizations.

(iv). **Separation Algorithms.** In applying the cuts (developed using the facets of (new) base sets) while solving MIPs with MMCCs, the separation problem must be solved many times. As a result, developing efficient separation methods to use these cuts is crucial. We will pursue the following directions in this regard: We will study developing exact separation algorithms for such cuts if that is achievable within reasonable effort. However if the effort proves to be prohibitive due to the complexity of the separation problem, we will develop intelligent and fast separation heuristics. In order to develop the fastest and most effective separation methods, we will theoretically and computationally investigate how the choices of constraint selection strategy and other input parameters to the separation algorithm affect the cut generation time and the amount by which the LP relaxation solution violates the generated cut.

(v). **Computational Research.** On this path, we plan to investigate the very important issue of using the above mentioned valid inequalities in practice. What we need are intelligent methods to evaluate these valid inequalities and use them most effectively in general algorithms for solving MIP like branch-and-cut. We plan to perform theoretical and experimental research in this area to address questions like how to find the strongest cuts among infinite possibilities, which constraints to use for this purpose, and in what order to use cuts in the branch-and-cut tree in different problem contexts.

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