CLUSTER VALUE PROBLEMS IN INFINITE-DIMENSIONAL SPACES

A Dissertation

by

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ABSTRACT

In this dissertation we study cluster value problems for Banach algebras $H(B)$ of analytic functions on the open unit ball B of a Banach space X that contain X^* and 1. Solving cluster value problems requires understanding the cluster set of a function $f \in H(B)$. For the Banach spaces X we focus on, such as those with a shrinking reverse monotone Finite Dimensional Decomposition and $C(K)$, we prove cluster value theorems for a Banach algebra $H(B)$ and a point $x^{**} \in \bar{B}^{**}$. In doing so, we apply standard methods and results in functional analysis; in particular we use the facts that projections from X onto a finite-codimensional subspace equal I_X minus a finite rank operator and that $C(K)^* = \ell_1(K)$ when K is compact, Hausdorff and dispersed.

We also prove that for any separable Banach space Y, a cluster value problem for $H(B_Y)$ ($H = H^\infty$ or $H = A_u$) can be reduced to a cluster value problem for $H(B_X)$ for some Banach space X that is an ℓ_1-sum of a sequence of finite-dimensional spaces. The proof relies on the construction of an isometric quotient map from a suitable X to Y that induces an isometric algebra homomorphism from $H(B_Y)$ to $H(B_X)$ with norm one left inverse. The left inverse is built using ultrafilter techniques. Other tools include the infinite-dimensional version of the Schwarz lemma and familiar one complex variable results such as Cauchy's inequality and Montel's theorem.

We conclude this work by describing the related $\bar{\partial}$ problem and defining strong pseudoconvexity as well as uniform strong pseudoconvexity in the context of Banach spaces. Our last result is that 2-uniformly PL-convex Banach spaces have a uniformly strictly pseudoconvex unit ball. In future research we will study the $\bar{\partial}$ problem in uniformly strictly pseudoconvex unit balls and in the open unit ball of finite-

dimensional Banach spaces such as the ball of ℓ_1^n.

DEDICATION

To my family and friends, for seasoning my life during these years that I have been a Ph.D. student at Texas A&M University.

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I would like to start by thanking my advisor and committee members for their valuable time put into helping me see beyond my ideas. I also want to thank the many instructors I have had during my Ph.D. for providing me with tools to be a sharp mathematician. Finally, a special thank you to all the colleagues I got to meet during these years as a graduate student at Texas A&M University and for being friends that have really understood my daily struggles as well as joys. I hope to keep learning from and with all of you.

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1. INTRODUCTION: MAIN CONCEPTS

1.1 Some particular Banach algebras

The research I present in this thesis is focused on some problems in Functional Analysis combined with Complex Analysis. To be specific, I study certain Banach algebras of bounded analytic functions on the open unit ball B of a complex Banach space X that contain X^* (the continuous linear functionals on X) and 1. Indeed, every element of X^* acts linearly and continuously on B, thus each element of X^* is a bounded analytic function on B (the definition of an analytic function is discussed in detail later on).

From here on, $H(B)$ will denote any such algebra. One example is the algebra $H^\infty(B)$ of all bounded analytic functions on B. The study of $H^\infty(B)$ is widely spread in mathematics, as in H^∞ control theory, H^∞ functional calculus, etc. Another well known example is the disk algebra, the analytic functions on the ball B of $\mathbb{C}^n$ that extend continuously to $\bar{B}$. There are two generalizations of the disk algebra to the infinite-dimensional case: $A_u(B)$, the uniformly continuous, bounded and analytic functions on B, and $A(B)$, the uniform limits on B of polynomials in the functions in X^*. These algebras satisfy the inclusions $A(B) \subset A_u(B) \subset H^\infty(B)$. In particular $A(B) = A_u(B)$ when X is finite-dimensional (because each $f \in A_u(B)$ is the uniform limit of polynomials and every polynomial on a finite-dimensional space is in A(B)), while $A_u(B) \subsetneq H^\infty(B)$ for every Banach space X (See [6, 90-92], or check that $B \circ x^* \in H^\infty(B) \setminus A_u(B)$ for $x^* \in X^*$ of norm one with $x^*(x_1) = \|x_1\| = 1$ and B a Blaschke product with zeros dense in $\partial\Delta$). Examples of infinite-dimensional spaces with $A(B) = A_u(B)$ include the $C(K)$ spaces for K compact, Hausdorff and dispersed (see section 3 in [27]), while examples of spaces with $A(B) \subsetneq A_u(B)$

include ℓ_1, L_1, $C(K)$ spaces for K compact, Hausdorff and not dispersed like ℓ_∞, L_∞ (see Proposition 2.36 in [13] and the Main Theorem in [35]), and ℓ_p, L_p for $1 < p < \infty$ (because for a fixed integer $n \geq p$ and $(e_k)_k$ the canonical basis of ℓ_p, $f(\sum a_k e_k) = \sum a_k^n$ is uniformly continuous on B_{ℓ_p} and analytic on complex lines, so $f \in A_u(B_{\ell_p})$; moreover $f(e_k) = 1$ for each $k \in \mathbb{N}$, while $g(e_k) \to 0$ when $g \in A(B)$, implying that $f \notin A(B)$. Then $f \circ P \in A_u(B_{L_p}) \setminus A(B_{L_p})$, for P a norm one projection of L_p onto a subspace that is isometric to ℓ_p).

One of the most important topics in the study of Banach algebras $H(B)$ is the study of its set of characters, the nonzero homomorphisms from $H(B)$ to $\mathbb{C}$, called the *spectrum of* $H(B)$, and denoted by $M_{H(B)}$. Since our Banach algebras $H(B)$ are commutative and have an identity, the spectrum $M_{H(B)}$ is a compact Hausdorff space ([21, Theorem 2.5]). The study of the spectrum is simplified by fibering it over $\bar{B}^{**}$ (the closed unit ball of X^{**}) via the surjective mapping $\pi : M_{H(B)} \to \bar{B}^{**}$ given by $\pi(\tau) = \tau|_{X^*}$. Indeed, π is well-defined because every $\tau \in M_{H(B)}$ is linear, continuous and of norm one; and π is surjective because $\pi(\delta_x) = x$ for every $x \in B$ (where $\delta_x : H(B) \to \mathbb{C}$ is defined by $f \mapsto f(x)$), and since $M_{H(B)}$ is compact, $\bar{B}^{**} = \bar{B}^{w^*} \subset \pi(M_{H(B)})$.

Another related topic of interest is the Gelfand Transform: Given $f \in H(B)$, the Gelfand Transform of f is the continuous map $\hat{f} : M_{H(B)} \to \mathbb{C}$ given by $\tau \mapsto \tau(f)$. The Gefand Transform is a generalization of the Fourier Transform for $L_1(\mathbb{R})$ under convolution.

1.2 The Corona problem

One of the big open problems in the study of algebras $H(B)$ is the *Corona problem* on the ball, which asks whether the open unit ball of a Banach space X is dense (in the weak-star topology) in $M_{H(B)}$. Note that B can be seen as a subset of $M_{H(B)}$ via the mapping $\delta : B \to M_{H(B)}$ such that $x \mapsto \delta_x$. An example of a character in $M_{H(B)} \setminus B$ can be found in Example 2.1 later on. An equivalent formulation of the Corona problem (see [24, p. 163]) is that whenever $f_1, \cdots, f_n \in H(B)$ satisfy $|f_1|+\cdots+|f_n| \geq \epsilon > 0$ on B, there exist $g_1, \cdots, g_n \in H(B)$ such that $f_1g_1 + \cdots + f_ng_n = 1$.

Carleson solved the Corona problem positively for the unit disk in $\mathbb{C}$ in 1962 [9]. In 1970 Gamelin [17] discussed the corona problem for other planar domains, solving it in cases that include finitely connected planar domains, while in 1985 Garnett and Jones [23] solved positively the Corona problem for connected open subsets of the extended complex plane whose complement is a subset of the real axis. Garnett poses corona problems for other interesting planar domains in [22]. In higher dimensions, around 1970 Cole constructed an open Riemann surface which is a counterexample to a Corona theorem [20]. Even more, Sibony produced counterexamples in 1987 in [37] and in 1993 in [16] to Corona theorems in $\mathbb{C}^3$, and then in $\mathbb{C}^2$, of domains that are pseudoconvex, and strictly pseudoconvex except at one point (for example, convex sets are pseudoconvex).

As Krantz summarizes in [31], there is no domain known in the plane $\mathbb{C}$ for which the corona problem is known to fail, and there is no domain known in $\mathbb{C}^n$, for $n \geq 2$, on which the corona problem is known to hold true. In particular, it is a challenging open problem to determine whether the Corona problem holds true for the unit ball or polydisk in $\mathbb{C}^n$ for $n \geq 2$.

1.3 Cluster value problems

This thesis work deals mainly with another set of big open problems: cluster value problems. Cluster value problems are related to the Corona problem (a positive answer to a Corona problem gives a positive answer to the corresponding cluster value problem, or equivalently, a negative answer to a cluster value problem would yield a counterexample to a Corona theorem). Roughly speaking, cluster value problems involve the understanding of the cluster sets of a function $f \in H(B)$, that is, the limits of f over weak* convergent nets in B, when B is seen as a subset of X^{**} with the weak-star topology. They also involve comparing such limits with the spectrum of $H(B)$ evaluated at f.

Kakutani [29] was among the first ones to consider cluster value problems in 1955 for domains in the complex plane, followed by I. J. Schark [26] in 1961 for the unit disk Δ of the complex plane. I. J. Schark gave an explicit identification of the cluster values of a function f at a point x in the boundary of Δ with the fiber over x of the spectrum evaluated at f:

Theorem 1.1. *Let $f \in H^\infty(\Delta)$ and $x \in \partial\Delta$. If $M_x = \{\tau \in M_{H^\infty(\Delta)} : \tau(id) = x\}$, then the range of $\hat{f}$ on M_x consists of those complex numbers ζ for which there is a sequence $\{\lambda_n\}$ in Δ with $\lambda_n \to x$ and $f(\lambda_n) \to \zeta$.*

In 1973 Gamelin [18] proved a cluster value theorem for the polydisk in $\mathbb{C}^n$. Moreover, he proved it for finite products of open sets in $\mathbb{C}$. Then McDonald proved in 1979 in [33] a cluster value theorem for the Euclidean unit ball in $\mathbb{C}^n$, and actually for any strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. His proof relied on a solution by Kerzman from 1971 in [30, 342-345] to a $\bar{\partial}$ problem in a strongly pseudoconvex domain.

Being precise, we say a finite-dimensional *cluster value theorem for $B \subset \mathbb{C}^n$* holds when the following occurs: Suppose $f \in H^\infty(B)$, $x \in \partial B$ and $\alpha \in \mathbb{C}$. Let $M_x = \{\tau \in M_{H^\infty(B)} : \tau|_{A(B)} = \delta_x\}$. There exists $m \in M_x$ such that $m(f) = \alpha$ if and only if there is a sequence $\{x_k\}_k$ in B converging to x such that $f(x_k)$ tends to α.

Before discussing cluster value theorems for the unit ball of an infinite-dimensional Banach space, let us overview the basic theory of holomorphic functions on arbitrary

Banach spaces.

1.4 Complex analysis on Banach spaces

Given U an open subset of a complex Banach space X, following Mujica [34], we say that $f : U \subset X \to \mathbb{C}$ is an analytic (or holomorphic) function on U if for every $x \in U$ there exists $r > 0$ and continuous polynomials on X, $(P^m f(x))_{m=0}^{\infty}$, where $P^m f(x)$ is m-homogeneous, such that, if $\|y - x\| < r$ then $f(y) = \sum_{m=0}^{\infty} P^m f(x)(y - x)$, and the convergence is uniform on the ball of radius r around x. The radius of convergence of f at x, $r_c f(x)$, is the supremum of the radius of balls for which the power series around x converges uniformly. Similarly, the radius of boundedness of f at x, $r_b f(x)$, is the supremum of the radius of balls centered at x and contained in U on which f is bounded.

An m-homogeneous polynomial $\hat{L}$ on X, for $m \in \mathbb{N}$, is the restriction to the diagonal of a m-linear mapping $L : X^m \to \mathbb{C}$, i.e. $\hat{L}(x) = L(x, x, \cdots, x)$ (and it is a constant function for $m = 0$).

The following two formulas are very useful (Theorems 4.3 and 7.13 in [34]).

Proposition 1.1. *[Cauchy-Hadamard Formula] Let U be an open subset of a complex Banach space X. If $f : U \subset X \to \mathbb{C}$ is analytic then for each $x \in U$, $1/r_c f(x) = \limsup_{m\to\infty} \|P^m f(x)\|^{1/m}$.*

Proof. Let $x \in U$. We will first show that $1/r_c f(x) \geq \limsup_{m\to\infty} \|P^m f(x)\|^{1/m}$.

Let $r \in (0, r_c f(x))$. Then $f(y) = \sum_{m=0}^{\infty} P^m f(x)(y - x)$, and the convergence is uniform for $y \in B(x, r)$. Hence we can choose $m_0 \in \mathbb{N}$ such that

$$\|\sum_{j=0}^{m} P^j f(x)(y - x) - f(y)\| \leq 1, \quad \forall m \geq m_0 \text{ and } y \in B(x, r).$$

Thus $\|P^m f(x)(t)\| \leq 2$ for all $m > m_0$ and $t \in B(0,r)$. Consequently $\|P^m f(x)\| \leq 2r^{-m}$ for all $m > m_0$ and therefore $\limsup_{m\to\infty} \|P^m f(x)\|^{1/m} \leq 1/r$.

Letting $r \to r_c f(x)$ we get that $1/r_c f(x) \geq \limsup_{m\to\infty} \|P^m f(x)\|^{1/m}$.

Let us now show that $1/r_c f(x) \leq \limsup_{m\to\infty} \|P^m f(x)\|^{1/m}$.

Let $R = \limsup_{m\to\infty} \|P^m f(x)\|^{1/m}$ and assume that $R < \infty$. Choose $r \in (0, 1/R)$, $s \in (r, 1/R)$ and $m_0 \in \mathbb{N}$ such that $\|P^m f(x)\|^{1/m} < 1/s$ for all $m \geq m_0$. Then $\|P^m f(x)(y - x)\| \leq (r/s)^m$ for all $m \geq m_0$ and $y \in B(x,r)$, so the series $\sum_{m=0}^{\infty} P^m f(x)(y-x)$ converges uniformly for $y \in B(x,r)$. Thus $r_c f(x) \geq r$. Letting $r \to 1/R$ we obtain that $r_c f(x) \geq 1/R$, i.e. $1/r_c f(x) \leq R = \limsup_{m\to\infty} \|P^m f(x)\|^{1/m}$.

$\square$

Proposition 1.2. *Let U be an open subset of a complex Banach space X. If $f : U \subset X \to \mathbb{C}$ is analytic then for each $x \in U$, $r_b f(x) = \min\{r_c f(x), d_U(x)\}$ (where $d_U(x)$ denotes the distance from x to the boundary of U).*

Proof. Let $x \in U$. Clearly $d_U(x) = \sup\{r > 0 : B(x,r) \subset U\}$, so $r_b f(x) \leq d_U(x)$. Thus to show $r_b f(x) \leq \min\{r_c f(x), d_U(x)\}$ it is enough to show $r_b f(x) \leq r_c f(x)$. Let $r \in (0, r_b f(x))$. Then $\bar{B}(x,r) \subset U$ and f is bounded on $\bar{B}(x,r)$, say by C. It follows from Proposition 1.8 below (Cauchy Inequality) that $\|P^m f(x)\| \leq Cr^{-m}$ for all $m \in \mathbb{N}_0$, so by the Cauchy-Hadamard Formula we get that $r_c f(x) = 1/\limsup_{m\to\infty} \|P^m f(x)\|^{1/m} \geq r$. Letting $r \to r_b f(x)$ we obtain $r_c f(x) \geq r_b f(x)$.

Let us now show that $r_b f(x) \geq \min\{r_c f(x), d_U(x)\}$. Choose $0 < r < s < \min\{r_c f(x), d_U(x)\}$. Then $B(x,s) \subset U$ and $\sum_{m=0}^{\infty} P^m f(x)(y-x)$ converges for every $y \in B(x,s)$. Thus, by Proposition 1.3 below (Identity Principle) we get that $f(y) = \sum_{m=0}^{\infty} P^m f(x)(y-x)$, for all $y \in B(x,s)$. Moreover, from the Cauchy-Hadamard Formula, $\limsup_{m\to\infty} \|P^m f(x)\|^{1/m} = \frac{1}{r_c f(x)} < \frac{1}{s}$, so there exists $C > 1$

such that $\|P^m f(x)\| \leq \frac{C}{s^m}$ for all $m \in \mathbb{N}_0$. Thus $\|f(y)\| \leq C \sum_{m=0}^{\infty} (\frac{r}{s})^m$ for all $y \in B(x, r)$. Hence $r_b f(x) \geq r$, and letting $r \to \min\{r_c f(x), d_U(x)\}$ we get that $r_b f(x) \geq \min\{r_c f(x), d_U(x)\}$. $\square$

Gamelin defines an analytic function in [19] differently. However, Proposition 8.6 and Theorem 8.7 in [34] prove that these definitions are equivalent. Moreover, Theorem 13.16 in [34] proves that analyticity is equivalent to $\mathbb{C}$-differentiability. Let us summarize these results in the following theorem.

Theorem 1.2. *Given an open subset U of a Banach space X and $f : U \subset X \to \mathbb{C}$, the following are equivalent:*

(i) f is analytic,

(ii) f is continuous and its restriction to every complex one-dimensional affine subspace of X is analytic, i.e. for every $x_0 \in U$ and direction $x \in X$, the function $\lambda \to f(x_0 + \lambda x)$ depends analytically on λ for $\lambda \in \{\zeta : x_0 + \zeta x \in U\}$,

(iii) f is locally bounded and its restriction to every complex one-dimensional affine subspace of X is analytic,

(iv) f is continuous and $f|_{U \cap M}$ is analytic for each finite-dimensional subspace M of X,

(v) f is Fréchet $\mathbb{C}$-differentiable, i.e. for each point $x \in U$ there exists a $\mathbb{C}$-linear mapping $L \in X^*$ such that $\lim_{y \to x} \frac{\|f(y) - f(x) - L(y-x)\|}{\|y - x\|} = 0$.

Proof. $(ii) \Rightarrow (iii)$ Continuity clearly implies local boundedness.

$(iii) \Rightarrow (ii)$ Let us first note that Proposition 1.9 below (Schwarz' Lemma) still holds under our condition that the restriction of f to every complex one-dimensional affine subspace of X is analytic.

Now let $x \in U$. Since f is locally bounded, there exists $r > 0$ such that f is bounded in $B(x, r)$, say by C. Then, by Schwarz' Lemma,

$$\|f(y) - f(x)\| \leq 2C\|y - x\|, \quad \forall y \in B(x, r).$$

This proves that f is continuous at the point x.

$(i) \Rightarrow (v)$ Let $x \in U$ and choose $r \in (0, r_b f(x))$. Then f is bounded on $\bar{B}(x, r)$, say by C. Moreover, since $r_b f(x) \leq r_c f(x)$, we have that for all $y \in B(x, r)$,

$$f(y) = f(x) + P^1 f(x)(y - x) + \sum_{m=2}^{\infty} P^m f(x)(y - x),$$

and so

$$\limsup_{m\to\infty} \frac{\|f(y) - f(x) - P^1 f(x)(y - x)\|}{\|y - x\|} \leq \limsup_{m\to\infty} \sum_{m=2}^{\infty} \|P^m f(x)(\frac{y - x}{\|y - x\|})\|\|y-x\|^{m-1}.$$

And by Proposition 1.8 below (Cauchy Inequality), we have that for all $m \geq 2$ and $y \in B(x, r) \setminus \{x\}$,

$$\|P^m f(x)(\frac{y - x}{\|y - x\|})\| \leq r^{-m} \sup_{|\zeta|=r} \|f(x + \zeta(\frac{y - x}{\|y - x\|})\| \leq Cr^{-m},$$

so

$$\begin{aligned}\limsup_{m\to\infty} \sum_{m=2}^{\infty} \|P^m f(x)(\frac{y - x}{\|y - x\|})\|\|y - x\|^{m-1} &\leq \limsup_{m\to\infty} \sum_{m=2}^{\infty} C/\|y - x\|(\frac{\|y - x\|}{r})^m \\ &= \limsup_{m\to\infty} \frac{C}{r^2}\|y - x\|(\frac{1}{1 - \frac{\|y-x\|}{r}}) = 0.\end{aligned}$$

Hence f is Fréchet $\mathbb{C}$-differentiable.

$(v) \Rightarrow (ii)$ Let $x_0 \in U$ and $x \neq 0 \in X$. Since f is Fréchet $\mathbb{C}$-differentiable, by the Chain Rule also $g(\lambda) = f(x_0 + \lambda x)$ is $\mathbb{C}$-differentiable on $\Omega = \{\zeta : x_0 + \zeta x \in U\}$. Thus g is analytic, by the corresponding one complex variable result.

Hence the restriction of f to every complex one-dimensional affine subspace of X is analytic. And the Fréchet $\mathbb{C}$-differentiability of f clearly implies that f is continuous.

$(ii) \Rightarrow (iv)$ Let M be a finite-dimensional subspace of X. Let $x_0 \in U \cap M$ and let $\{x_1, \cdots, x_n\}$ be a basis for M. Then, following the proof of Proposition 1.7 (applied n times as in Corollary 7.8 in [34]) under our condition that f is continuous and its restriction to each complex line is analytic, we obtain that

$$f(x_0 + \lambda_1 x_1 + \cdots \lambda_n x_n) = \sum_{\alpha} c_\alpha \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n},$$

and the series converges absolutely and uniformly on some polydisk $\Delta^n(0, r)$.

Then, for each $m \in \mathbb{N}_0$ define $P_m : M \to \mathbb{C}$ by

$$P_m(\lambda_1 x_1 + \cdots \lambda_n x_n) = \sum_{|\alpha|=m} c_\alpha \lambda_1^{\alpha_1} \cdots \lambda_n,^{\alpha_n}$$

so clearly P_m is an m-homogeneous polynomial and

$$f(x_0 + \lambda_1 x_1 + \cdots \lambda_n x_n) = \sum_{m=0}^{\infty} P_m(\lambda_1 x_1 + \cdots \lambda_n x_n),$$

where the series converges uniformly on $\Delta^n(0, r)$.

Thus $f|_{U \cap M}$ is analytic.

$(iv) \Rightarrow (i)$ Let $x_0 \in U$ and $r > 0$ such that $B(x_0, r) \subset U$. Then, if M is a finite-dimensional subspace of X containing x_0, by Proposition 1.2 applied to $f|_{U\cap M}$,

$$f(x) = \sum_{m=0}^{\infty} P_m^M(x - x_0), \quad \forall x \in M \cap B(x_0, r)$$

where each P_m^M is an m-homogeneous polynomial.

Hence, if M and N are two finite-dimensional subspaces of X containing x_0 then $P_m^M(t) = P_m^N(t)$ $\forall t \in M \cap N, m \in \mathbb{N}_0$ (by an induction argument similar to the one given after Lemma 4.5 in [34]).

Thus, defining $P_m : X \to \mathbb{C}$ by $P_m(t) = P_m^M(t)$, where M is any finite-dimensional subspace of X containing x_0 and t, we get that P_m is an m-homogeneous polynomial (after using a Polarization Formula as the one in Theorem 1.10 in [34]) and

$$f(x) = \sum_{m=0}^{\infty} P_m(x - x_0), \quad \forall x \in B(x_0, r).$$

Now since f is continuous at x_0 there exists $s < r$ such that f is bounded on $\bar{B}(x_0, s)$, say by C. Thus, by Proposition 1.8 below (Cauchy Inequality) applied to $f|_{U\cap M}$ for some finite-dimensional subspace M of X containing x_0 and $t \in B_X$,

$$\|P_m(t)\| \leq Cs^{-m}, \quad \forall m \in \mathbb{N}_0.$$

Hence each P_m is continuous and the power series $\sum_{m=0}^{\infty} P_m(x - x_0)$ converges uniformly for $x \in B(x_0, s)$, for all $s_0 < s$. □

Some examples [34, Section 5] of analytic functions are the following:

(i) Polynomials (finite sums of n-homogeneous polynomials),

(ii) Power series $\sum_{m=0}^{\infty} P_m(x)$ with infinite radius of convergence.

(iii) Power series of the form $\sum_{m=0}^{\infty}(x_m^*)^m$, where $\{x_m^*\} \subset X^*$ and $x_m^* \xrightarrow{w^*} 0$.

Also, [34, Sections 5,7] extends the following classical properties to the infinite-dimensional setting.

Proposition 1.3. *[Identity Principle] If f is analytic on a connected open set U and identically zero on a nonvoid open set $V \subset U$, then f is identically zero on all of U.*

Proof. (a) First assume U is an open ball in X.

Let $x \in U$ and $y \in V$, and let $\Omega = \{\lambda \in \mathbb{C} : y + \lambda(x - y) \in U\}$. Clearly Ω is a convex open set that contains 0 and 1, and in particular it is connected. Hence $g : \Omega \to \mathbb{C}$ given by $g(\lambda) = f(y + \lambda(x - y))$ is holomorphic and identically zero on a neighborhood of zero. Thus g is identically zero on all of Ω by the one complex variable Identity Principle. In particular $f(x) = g(1) = 0$. Thus f is identically zero on all of U.

(b) In the general case consider $A = \{a \in U : \exists r > 0 \text{ s.t. } f|_{B(a,r)} \equiv 0\}$. A is clearly an open set, and it turns out to be closed in U as well, because if $(a_n)_n \subset A$ satisfies $a_n \to a \in U$ then, after choosing $r > 0$ such that $B(a,r) \subset U$, we obtain $n \in \mathbb{N}$ such that $a_n \in B(a,r)$, and since there further exists r_n small enough such that $B(a_n, r_n) \subset B(a,r)$ and $f|_{B(a_n,r_n)} \equiv 0$, then part (a) implies that $f|_{B(a,r)} \equiv 0$, i.e. $a \in A$. Thus $A = U$, i.e. f is identically zero on all of U.

□

Proposition 1.4. *[Open Mapping Principle] If f is analytic and non-constant on a connected open set U, then f is an open mapping.*

Proof. It is enough to show that $f(B(a,r))$ contains a neighborhood of $f(a)$ for each $B(a,r) \subset U$. So let $a \in U$ and $r > 0$ such that $B(a,r) \subset U$. By the Identity Principle we have that $f|_{B(a,r)}$ is non-constant, so there exists $b \in B(a,r)$ such that $f(b) \neq f(a)$. Now, $\Omega = \{\lambda \in \mathbb{C} : a+\lambda(b-a) \in B(a,r)\}$ is a connected open set that contains 0 and 1. Hence $g : \Omega \to \mathbb{C}$ given by $g(\lambda) = f(a+\lambda(b-a))$ is an holomorphic function that satisfies $g(0) = f(a) \neq f(b) = g(1)$, so by the one complex variable Open Mapping Principle we get that $g(\Omega)$ is open, where $f(a) = g(0) \in g(\Omega) \subset f(B(a,r))$. Thus f maps open sets into open sets. $\square$

Proposition 1.5. *[Maximum Principle] If f is analytic on a connected open set U and $|f|$ attains its supremum there, then f is constant.*

Proof. Assume f is not constant. Then, by the Open Mapping Principle, for each $a \in U$ there exists $r > 0$ such that $B(f(a), r) \subset f(U)$, so $|f(a)|$ is not the supremum of $|f|$. Hence $|f|$ does not attain its supremum on U. $\square$

Proposition 1.6. *[Liouville's theorem] If f is analytic and bounded on all X, then f is constant.*

Proof. Let $x \in X$. Since the function $g : \mathbb{C} \to \mathbb{C}$ given by $g(\lambda) = f(\lambda x)$ is holomorphic and bounded, then the one complex variable Liouville's theorem implies that g is constant. In particular $f(x) = g(1) = g(0) = f(0)$. So f is constant. $\square$

Proposition 1.7. *[Cauchy Integral Formula] Suppose that $f : U \to \mathbb{C}$ is holomorphic. Let $a \in U$, $t \in X$ and $r > 0$ such that $a + \zeta t \in U$ $\forall \zeta \in \bar{\Delta}(0,r)$. Then $\forall m \in \mathbb{N}_0$: $P^m f(a)(t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a+\zeta t)}{\zeta^{m+1}} d\zeta$.*

Proof. Let $\Omega = \{\zeta \in \mathbb{C} : a + \zeta t \in U\}$. Then $g : \Omega \to \Omega$ given by $g(\zeta) = f(a + \zeta t)$ is holomorphic on a neighborhood of $\bar{\Delta}(0,r)$, so by the one complex variable Cauchy Integral Formula, $\forall \lambda \in \Delta(0,r)$,

$$f(a + \lambda t) = g(\lambda) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{g(\zeta)}{\zeta - \lambda} d\zeta = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a+\zeta t)}{\zeta - \lambda} d\zeta.$$

Now, whenever $|\zeta| = r$ and $|\lambda| < r$ we have

$$\frac{f(a+\zeta t)}{\zeta - \lambda} = \frac{f(a+\zeta t)/\zeta}{1-(\lambda/\zeta)} = \sum_{m=0}^{\infty} \lambda^m \frac{f(a+\zeta t)}{\zeta^{m+1}},$$

and the series converges absolutely and uniformly for $|\zeta| = r$ and $|\lambda| \leq s < r$, for all $s < r$, because f is bounded on $\{a + \zeta t : |\zeta| = r\}$. Then, from the Dominated Convergence Theorem,

$$f(a+\lambda t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a+\zeta t)}{\zeta - \lambda} d\zeta = \sum_{m=0}^{\infty} \lambda^m \left(\frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a+\zeta t)}{\zeta^{m+1}} d\zeta\right),$$

and the convergence is uniform for $|\lambda| \leq s$, for all $s < r$.

On the other hand, since f is holomorphic we have a power series expansion

$$f(a+\lambda t) = \sum_{m=0}^{\infty} P^m f(a)(\lambda t) = \sum_{m=0}^{\infty} \lambda^m P^m f(a)(t),$$

which converges uniformly for $|\lambda| \leq s_0$, for some $s_0 > 0$. Hence, after an induction argument (such as the one given in Lemma 4.5 in [34]), we get that $\forall m \in \mathbb{N}_0$,

$$P^m f(a)(t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a+\zeta t)}{\zeta^{m+1}} d\zeta.$$

$\square$

Proposition 1.8. *[Cauchy Inequality] Suppose that $f : U \to \mathbb{C}$ is holomorphic. Let $a \in U$, $t \in X$ and $r > 0$ such that $a + \zeta t \in U$ $\forall \zeta \in \bar{\Delta}(0, r)$. Then $\forall m \in \mathbb{N}_0$: $\|P^m f(a)(t)\| \leq r^{-m} \sup_{|\zeta|=r} \|f(a+\zeta t)\|$.*

Proof. From the Cauchy Integral Formula we have that $\forall m \in \mathbb{N}_0$,

$$P^m f(a)(t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a+\zeta t)}{\zeta^{m+1}} d\zeta.$$

Hence,

$$||P^m f(a)(t)|| \leq \frac{1}{2\pi} \int_{|\zeta|=r} \frac{|f(a+\zeta t)|}{r^{m+1}} \leq r^{-m} \sup_{|\zeta|=r} ||f(a+\zeta t)||.$$

□

Proposition 1.9. *[The Schwarz Lemma] Suppose that $f : U \to \mathbb{C}$ is holomorphic. Let $a \in U$ and $r > 0$ such that $B(a,r) \subset U$, and suppose that f is bounded on $B(a,r)$ by C. Then $\forall x \in B(a,r)$:*

$\|f(x) - f(a)\| \leq 2C\frac{\|x-a\|}{r}$.

Proof. Fix $x \in B(a,r) \setminus \{a\}$. Since $r_b f(a) = \min\{r_c f(a), d_U(a)\}$ (proven above), and we have that $r_b f(a) \geq r$, then $r_c f(a) \geq r$ too, so

$$f(a+\lambda(x-a)) = \sum_{m=0}^{\infty} P^m f(a)(x-a)\lambda^m, \quad \forall \lambda \in \Delta(0, r/\|x-a\|).$$

Thus, if we define $g : \Delta(0, r/\|x-a\|) \to \mathbb{C}$ by $g(\lambda) = \frac{f(a+\lambda(x-a))-f(a)}{\lambda}$ for $\lambda \neq 0$ and $g(0) = P^1 f(a)(x-a)$, we get that actually

$$g(\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} P^m f(a)(x-a), \quad \forall \lambda \in \Delta(0, r/\|x-a\|),$$

so g is in particular holomorphic.

Now, if $s \in (1, r/\|x-a\|)$ and $|\lambda| = s$, then

$$|g(\lambda)| \leq \frac{|f(a+\lambda(x-a))| + |f(a)|}{|\lambda|} \leq \frac{2C}{s},$$

so by the Maximum Principle, if $|\lambda| \leq s$ also $|g(\lambda)| \leq \frac{2C}{s}$. Since $\lambda = 1$ satisfies $|\lambda| \leq s$ then

$$||f(x) - f(a)|| = |g(1)| \leq \frac{2C}{s}.$$

Since this is true for all $s < r/\|x-a\|$ then also

$$||f(x) - f(a)|| \leq 2C\frac{\|x-a\|}{r}.$$

□

Since $r_b f(a) = \min\{r_c f(a), d_U(a)\}$ whenever $a \in U$ and $f : U \to \mathbb{C}$ is holomorphic, there is a classical property that does not extend to the infinite dimensional setting: The following example [34, p. 54] exhibits that the radius of convergence of the power series of a holomorphic function around a point is not at least the distance of the point to the boundary of U.

Proposition 1.10. *Suppose that $(x_m^*)_m \subset X^*$ is a sequence such that $\|x_m^*\| = 1\ \forall m$ and $x_m^* \xrightarrow{w^*} 0$. Then $f = \sum_{m=0}^{\infty}(x_m^*)^m$ is holomorphic on all of X but has radius of convergence at 0 equal to 1.*

When X is infinite-dimensional, such a sequence always exists due to the Josefson-Nissenzweig Theorem [12, 219-225].

Remark 1.1. If X and Y are complex Banach spaces, and $U \subset X$ is open, we say that $f : U \to Y$ is holomorphic if $y^* \circ f$ is holomorphic for all $y^* \in Y^*$. One can check that Theorem 1.2 can be extended to this setting (see [34] for the details).

We now have enough tools to discuss the cluster value problem in arbitrary Banach spaces.

1.5 Cluster value problems for infinite-dimensional Banach spaces

For arbitrary Banach spaces X, the *cluster value theorem for* $H(B)$ asserts that, for a given $x^{**} \in \bar{B}^{**}$, the sets of cluster values

$$Cl_B(f, x^{**}) := \{\lambda : f(x_\alpha) \to \lambda,\ x_\alpha \xrightarrow{w^*} x^{**}\}$$

coincides with the evaluation of the fiber $M_{x^{**}}(B) := \pi^{-1}(x^{**})$ at f,

$$\hat{f}(M_{x^{**}}(B)) = \{\tau(f) : \tau \in M_{x^{**}}\},$$

for all $f \in H(B)$.

Aron, Carando, Gamelin, Lasalle and Maestre observed in [5] that for every $x^{**} \in \bar{B}^{**}$ we have the inclusion

$$Cl_B(f, x^{**}) \subset \widehat{f}(M_{x^{**}}(B)),\ \forall f \in H(B), \tag{1.1}$$

thus to prove the cluster value theorem for $H(B)$ it is enough to show $\widehat{f}(M_{x^{**}}(B)) \subset Cl_B(f, x^{**}),\ \forall f \in H(B)$.

Aron, Carando, Gamelin, Lasalle and Maestre also showed in [5] that the cluster value theorem holds at every $x^{**} \in \bar{B}^{**}$ if and only if whenever $f_1, \cdots, f_{n-1} \in A(B)$ and $f_n \in H(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \epsilon > 0$ on B, there exist $g_1, \cdots, g_n \in H(B)$ such that $f_1 g_1 + \cdots f_n g_n = 1$.

When B is the unit ball of an infinite-dimensional Banach space, there are no known solutions to the Corona problem. However, Aron, Carando, Gamelin, Lasalle and Maestre proved in 2012 in [5] a cluster value theorem at the origin for the algebra $A_u(B)$ when X has a shrinking 1-unconditional basis. Examples of such spaces X include ℓ_p for $1 < p < \infty$ and c_0, but not ℓ_1, ℓ_∞, $L_p(0,1)$ for $1 \leq p \neq 2 \leq \infty$.

Theorem 1.3. *If X is a Banach space with a shrinking 1-unconditional basis, then the cluster value theorem holds for $A_u(B)$ at $x = 0$,*

$$Cl_B(f, 0) = \hat{f}(M_0), \quad f \in A_u(B).$$

Aron, Carando, Gamelin, Lasalle and Maestre [5] also proved a cluster value theorem at all points of the closed unit ball of X for the algebra $A_u(B)$ when X is a Hilbert space. The main idea is to translate the problem to 0 via certain automorphisms of B, then to use the cluster value theorem at 0 for Banach spaces with a shrinking 1-unconditional basis, and to apply certain peak functions for points in ∂B.

Theorem 1.4. *If X is a Hilbert space, then the cluster value theorem holds for $A_u(B)$ at every $x \in \bar{B}$,*

$$Cl_B(f, x) = \hat{f}(M_x), \quad f \in A_u(B_X), x \in \bar{B}.$$

Corollary 1.1. *Let B be the open unit ball of a Hilbert space. If $f_1, \cdots, f_{n-1} \in A(B)$ and $f_n \in A_u(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \epsilon > 0$ on B, then there exist $g_1, \cdots, g_n \in A_u(B)$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.*

The same authors [5] proved a cluster value theorem at all points of the closed unit ball of X^{**} for the algebra of bounded analytic functions, denoted by $H^\infty(B)$, when X is c_0 (the space of null sequences). The proof repeatedly uses a lemma based on a solution to a $\bar{\partial}$ equation.

Theorem 1.5. *If X is the Banach space c_0 of null sequences, then the cluster value theorem holds for $H^\infty(B)$ at every $x \in \bar{B}^{**}$,*

$$Cl_B(f,x) = \hat{f}(M_x), \quad f \in H^\infty(B), x \in \bar{B}^{**}.$$

Corollary 1.2. *Let B be the open unit ball of the Banach space c_0 of null sequences. If $f_1, \cdots f_{n-1} \in A(B)$ and $f_n \in H^\infty(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \epsilon > 0$ on B, then there exist $g_1, \cdots, g_n \in H^\infty(B)$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.*

Note that Hilbert space and c_0 are infinite-dimensional analogues of the unit ball and the polydisk of Euclidean space, respectively.

It is open whether L_p satisfies the cluster value problem, for $A_u(B_{L_p})$ or $H^\infty(B_{L_p})$, at any point of B_{L_p}, $1 \leq p \neq 2 \leq \infty$. Nevertheless, Lemma 4.4 in [15] implies a cluster value theorem for each point in ∂B and the algebra $A_u(B)$, when B is the unit ball of a uniformly convex Banach space, like ℓ_p and L_p, for $1 < p < \infty$. Also, by [1, Theorem 2.6] and [5, Corollary 2.5], there is a cluster value theorem for the algebra $A_u(B_{\ell_1})$ and each point in ∂B_{ℓ_1}, because for each boundary point there is a function in $A(B_{\ell_1})$ peaking at it. I would like to investigate what happens at interior points.

2. OUR CLUSTER VALUE THEOREMS*

2.1 A look at finite-dimensional subspaces of a Banach space

Generalizing the ideas and techniques in [5], W. B. Johnson and I proved the following cluster value theorem in [27]:

Theorem 2.1. *Suppose that for each finite-dimensional subspace E of X^* and $\epsilon > 0$ there exists a finite rank operator S on X so that $\|(I - S^*)|_E\| < \epsilon$ and $\|I - S\| = 1$. Then the cluster value theorem holds for $A_u(B)$ at 0.*

Proof. Suppose that $0 \notin Cl_B(f, 0)$. We must show that $0 \notin \hat{f}(M_0)$. Since $0 \notin Cl_B(f, 0)$, there exists $\delta > 0$ and a weak neighborhood U of 0 in X such that $|f| \geq \delta$ on $U \cap B$. Without loss of generality we may assume $U = \cap_{i=1}^{n} \{x \in X : \ |x_i^*| < \epsilon_0\}$ for some $x_1^*, \cdots, x_n^* \in B_{X^*}$ and $\epsilon_0 > 0$. Let $E = \text{span}\{x_1^*, \cdots, x_n^*\}$ and let S be as in the statement for $\epsilon = \epsilon_0$. Then $|f \circ (I - S)| \geq \delta$ on B, because for every $x \in B$ we have that $(I - S)x \in U$, indeed:

$$| < x_i^*, (I - S)x > | = | < (I - S^*)x_i^*, x > | < \epsilon_0, \text{ for } i = 1, \cdots, n.$$

Consequently $f \circ (I - S)$ is invertible in $A_u(B)$. Hence $\widehat{f \circ (I - S)} \neq 0$ on the fiber of the spectrum of $A_u(B)$ over 0. From Proposition 2.1 below, we then obtain $\hat{f} \neq 0$ on M_0, that is $0 \notin \hat{f}(M_0)$. $\square$

For example, Banach spaces with a shrinking reverse monotone Finite Dimensional Decomposition (FDD) satisfy the conditions of the previous theorem. Recall that such Banach spaces X are the union of finite-dimensional spaces $(E_n)_n$ such that

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$\forall x \in X$, $\exists!$ $(x_n)_n$ with $x_n \in E_n$ and $x = \sum x_n$; also the functions $Q_n(x) = \sum_{i=n}^{\infty} x_i$ satisfy $\|Q_n\| = 1$ $\forall n$, and $\forall x^* \in X^*$, $\lim_{n\to\infty} \|x^*|_{\mathrm{span}(\cup_{j=n}^{\infty} E_j)}\| = 0$.

The proof of Theorem 2.1 relies on the following proposition in [27] (a generalization of Lemma 3.4 in [5]):

Proposition 2.1. *Let S be a finite rank operator on X such that $P = I - S$ has norm one. If $\phi \in M_0(B) = \{\tau \in A_u(B) : \tau|_{A(B)} = \delta_0\}$, then $\hat{f}(\phi) = \widehat{f \circ P}(\phi)$, for all $f \in A_u(B)$.*

We note that the previous proposition, however, does not hold for $H^\infty(B)$, as exhibited in the following example by Aron [27, Example 1]:

Example 2.1. *There exists a finite rank operator S on ℓ_2 so that $P = I - S$ has norm one, and there exist $\phi \in M_0(B_{\ell_2}) = \{\tau \in H^\infty(B) : \tau|_{A(B)} = \delta_0\}$ as well as $f \in H^\infty(B_{\ell_2})$ so that $\hat{f}(\phi) \neq \widehat{f \circ P}(\phi)$.*

Proof. Let $S : \ell_2 \to \ell_2$ be given by $S(x) = (x_1, 0, 0, \cdots)$.

Clearly S is a finite rank operator and $P = I - S$ has norm one.

Let (r_j) and (s_j) be sequences of positive real numbers, such that $(r_j) \downarrow 0$ and $(s_j) \uparrow 1$ in such a way that each $r_j^2 + s_j^2 < 1$ and $r_j^2 + s_j^2 \to 1^-$. For each $j = 1, 2, 3, \cdots$, let $\delta_{r_j e_1 + s_j e_j}$ be the usual point evaluation homomorphism from $H^\infty(B_{\ell_2}) \to \mathbb{C}$. Let $\phi : H^\infty(B_{\ell_2}) \to \mathbb{C}$ be an accumulation point of $\{\delta_{r_j e_1 + s_j e_j}\}$ in the spectrum of $H^\infty(B_{\ell_2})$. Let $f : B_{\ell_2} \to \mathbb{C}$ be the H^∞ function given by

$$f(x) = \frac{x_1}{\sqrt{1 - \sum_{j=2}^{\infty} x_j^2}},$$

where the square root is taken with respect to the branch of logarithm determined by the bounded and simply connected set $U = \{1 - \sum_{j=2}^{\infty} x_j^2 : x \in B_{\ell_2}\}$ that does not contain 0.

Since f is clearly continuous, it is analytic because it is analytic on complex lines: Given $x^0 = (x^0_n)_n \in B_{\ell_2}$ and $y^0 = (y^0_n)_n \in \ell_2$, if $\lambda \in \Omega = \{\zeta \in \mathbb{C} : x^0 + \zeta y^0 \in B_{\ell_2}\}$,

$$\begin{aligned} f(x^0 + \lambda y^0) &= \frac{x^0_1 + \lambda y^0_1}{\sqrt{1 - \sum_{n=2}^{\infty}(x^0_n + \lambda y^0_n)^2}} \\ &= \frac{x^0_1 + \lambda y^0_1}{\sqrt{1 - \sum_{n=2}^{\infty}(x^0_n)^2 - 2\lambda \sum_{n=2}^{\infty} x^0_n y^0_n + \lambda^2 \sum_{n=2}^{\infty}(y^0_n)^2}} \\ &= \frac{a + \lambda b}{\sqrt{c - 2\lambda d + \lambda^2 e}}, \end{aligned}$$

which is a holomorphic expression because Ω is simply connected and bounded, and $0 \notin \{c - 2\lambda d + \lambda^2 e : \lambda \in \Omega\}$.

f is bounded because for all $x \in B_{\ell_2}$,

$$\left|\frac{x_1}{\sqrt{1 - \sum_{j=2}^{\infty} x_j^2}}\right|^2 = \frac{|x_1|^2}{|1 - \sum_{j=2}^{\infty} x_j^2|} \leq \frac{|x_1|^2}{1 - \sum_{n=2}^{\infty} |x_n|^2} \leq 1.$$

Finally, $\phi(f) = 1$, however $\phi(f \circ P) = 0$ because $f \circ P \equiv 0$.

□

We observe that Theorem 2.1 and Proposition 2.1 suggest a relationship between the cluster value problem in a Banach space and its finite-codimensional subspaces. Johnson and I [27] established the following relationship with the help of Aron and Maestre:

Proposition 2.2. *If Y is a closed finite-codimensional subspace of X and $f \in A_u(B)$, then $Cl_B(f, 0) = Cl_{B_Y}(f|_Y, 0)$, where B_Y is the unit ball of Y.*

Proof. $A_u(B)$ coincides with the uniform limits on $\bar{B}$ of continuous polynomials on X (see Theorem 7.13 in [34] and p. 56 in [6]), where polynomials are finite linear combinations of symmetric m-linear mappings restricted to the diagonal. Thus, by passing to the uniform limit on $\bar{B}$, we may assume f is an m-homogeneous polynomial, with associated symmetric m-linear functional F. Let (x_α) be a weakly null net in B such that $f(x_\alpha) \to \lambda$.

Each x_α can be written uniquely as $y_\alpha + u_\alpha$, where $y_\alpha \in Y$ and u_α is from a fixed finite dimensional complement of Y in X. Then

$$\begin{aligned} f(x_\alpha) &= F(x_\alpha, \cdots, x_\alpha) \\ &= f(y_\alpha) + mF(y_\alpha, \cdots, y_\alpha, u_\alpha) + \frac{m(m-1)}{2} F(y_\alpha, \cdots, y_\alpha, u_\alpha, u_\alpha) + \cdots + f(u_\alpha). \end{aligned}$$

Now, since (x_α) is weakly null, the same holds for (y_α) and (u_α). However, since (u_α) belongs to a finite dimensional space, it follows that $||u_\alpha|| \to 0$. Thus $F(y_\alpha \cdots, y_\alpha, u_\alpha)$, $F(y_\alpha, \cdots, y_\alpha, u_\alpha, u_\alpha), \cdots, f(u_\alpha)$ all go to 0. Thus $f(y_\alpha) \to \lambda$. Finally, since each $||y_\alpha|| \leq ||x_\alpha|| + || - u_\alpha|| < 1 + ||u_\alpha||$, then by defining $t_\alpha = \frac{1}{1+||u_\alpha||}$ we get that $||t_\alpha y_\alpha|| < 1$ for all α and $t_\alpha \to 1$, and consequently, $\lim f(t_\alpha y_\alpha) = \lim t_\alpha^m f(y_\alpha) = \lambda$. Hence $\lambda \in Cl_{B_Y}(f|_Y, 0)$.

□

As a consequence we obtain that the cluster sets of an element f of $A_u(B)$ at 0 can be described in terms of the Gelfand transforms of $f|_{B_Y}$ as Y ranges over finite-codimensional subspaces of X:

Proposition 2.3. *For every Banach space* X,

$$Cl_B(f, 0) = \bigcap_{Y \subset X, \dim(X/Y) < \infty} \widehat{f|_{B_Y}}(M_0(B_Y)), \ \forall f \in A_u(B).$$

Proof. From Proposition 2.2, for every finite-codimensional subspace Y of X,

$$Cl_B(f, 0) = Cl_{B_Y}(f|_{B_Y}, 0) \subset \widehat{f|_{B_Y}}(M_0(B_Y)).$$

For the reverse inclusion, suppose $0 \notin Cl_B(f,0)$. Then there are $\epsilon > 0$ and a weak neighborhood U of 0 such that $|f| > \epsilon$ on $U \cap B$. U contains a closed finite-codimensional subspace Y_0 of X, so $|f|_{B_{Y_0}}| > \epsilon$. Hence $\widehat{f|_{B_{Y_0}}}$ is invertible, which implies that $0 \notin \widehat{f|_{B_{Y_0}}}(M_0(B_{Y_0}))$. □

Since c_0 satisfies a cluster value theorem for $A_u(B)$ and c_0 is 1-codimensional in c then Proposition 2.2 suggests that c satisfies a cluster value theorem, but [27, Example 2] shows that c does not satisfy the hypothesis of Theorem 2.1:

Example 2.2. *Let $L \in B_{c^*}$ be given by*

$$L((c_n)_n) = \lim_{n\to\infty} c_n.$$

If $S : c \to c$ is a finite rank operator with $||(S^* - I_{c^*})L|| < \epsilon$, then $||S - I_c|| \geq 2 - \epsilon$.

Proof. For each $k \in \mathbb{N}$, consider $L_k \in B_{c^*}$ given by

$$L_k((c_n)_n) = (\lim_{n\to\infty} c_n - c_k)/2.$$

Let us show that $||S^*(L_k)|| \to 0$ as $k \to \infty$. For every $x \in B_c$, $S^*(L_k)x = L_k(Sx) \to 0$ as $k \to \infty$. Moreover, since S has finite rank, $\{Sx : \ x \in B_c\}$ is pre-compact. Thus $S^*L_k = L_k \circ S$ converges to zero uniformly on B_c, i.e. $||S^*L_k|| \to 0$ as $k \to \infty$.

Now note that $||L - 2L_k|| = 1$ for each k, so

$$||S^* - I_{c^*}|| \geq ||(S^* - I_{c^*})(L - 2L_k)|| \geq ||2L_k - 2 \cdot S^*(L_k)|| - \epsilon \geq 2 - \epsilon - 2||S^*(L_k)||.$$

Since $S^*(L_k) \to 0$, then $||S - I_c|| = ||S^* - I_{c^*}|| \geq 2 - \epsilon$.

□

As we mentioned before, since c_0 is one-codimensional in c, Proposition 2.2 implies that for all $f \in A_u(B_c)$,

$$Cl_{B_c}(f, 0) = Cl_{B_{c_0}}(f|_{B_{c_0}}, 0).$$

Also, Propositions 1.59 and 2.8 of [13] imply that all functions in $A_u(B_{c_0})$ can be uniformly approximated on B by polynomials in the functions in X^*, which in turn implies that each fiber at $x^{**} \in \bar{B}^{**}$ consists only of x^{**}, so the cluster value theorem for $A_u(B_{c_0})$ holds, and in particular

$$Cl_{B_{c_0}}(f|_{B_{c_0}}, 0) = \widehat{f|_{B_{c_0}}}(M_0(B_{c_0})), \;\; \forall f \in A_u(B_c).$$

Hence we are left to compare $\widehat{f|_{B_{c_0}}}(M_0(B_{c_0}))$ with $\widehat{f}(M_0(B_c))$ for $f \in A_u(B_c)$. Note that an inclusion is evident:

Proposition 2.4. *For a Banach space X and Y a subspace of X,*

$$\widehat{f|_{B_Y}}(M_0(B_Y)) \subset \widehat{f}(M_0(B)), \;\; \forall f \in A_u(B).$$

Proof. Let $f \in A_u(B)$ and $\tau \in M_0(B_Y)$. Since $\phi_1 : A_u(B) \to A_u(B_Y)$ given by $\phi(g) = g|_Y$ for all $g \in A_u(B)$ is a continuous homomorphism that maps $A(B)$ into $A(B_Y)$, the mapping $\tilde{\tau} : A_u(B) \to \mathbb{C}$ given by $\tilde{\tau}(g) = \tau(g|_Y)$ for all $g \in A_u(B)$ is in the fiber $M_0(B)$. Moreover,

$$\widehat{f|_Y}(\tau) = \hat{f}(\tilde{\tau}).$$

$\square$

The reverse inclusion is unclear. Nonetheless, c is isomorphic to c_0 and $A_u(B_{c_0}) = A(B_{c_0})$, so by [27, Lemma 1], $A_u(B_c) = A(B_c)$, so c satisfies a cluster value theorem for $A_u(B_c)$:

Lemma 2.1. *Let X be a Banach space so that $A_u(B_X) = A(B_X)$. If the Banach space Y is isomorphic to X, then also $A_u(B_Y) = A(B_Y)$.*

Proof. Let $T : Y \to X$ be the Banach space isomorphism between Y and X.

Let $P(X)$ and $P(Y)$ denote the continuous polynomials on X and Y respectively, and denote by $P_f(X)$ and $P_f(Y)$ the polynomials in the functions of X^* and Y^* respectively (known as finite type polynomials).

Let $f \in A_u(B_Y)$. Then there exists a sequence of polynomials $P_n \in \mathcal{P}(Y)$ such that $||P_n - f||_{B_Y} \leq \frac{1}{n}, \ \forall n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $P_n \circ T^{-1} \in \mathcal{P}(X)$, so there exists a polynomial $Q_n \in \mathcal{P}_f(X)$ such that $||P_n \circ T^{-1} - Q_n||_{B_X} < \frac{1}{n \cdot ||T||}$, and consequently $||P_n - Q_n \circ T||_{B_Y} < \frac{1}{n}$, where $Q_n \circ T \in \mathcal{P}_f(Y)$.

Consequently, the sequence of polynomials $Q_n \circ T \in \mathcal{P}_f(Y)$ converges to f uniformly on B_Y, so $f \in A(B_Y)$.

□

Corollary 2.1. *The Banach space c satisfies the cluster value theorem for $A_u(B_c)$ at all points in $\overline{B_c}^{**}$.*

Proof. Since every $f \in A_u(B_c)$ can be uniformly approximated on B_c by polynomials in the functions in X^*, then each fiber at $x^{**} \in \bar{B}^{**}$ consists only of x^{**}, and since $Cl_B(f, x^{**}) \subset \widehat{f}(M_{x^{**}}(B)) = \{\widehat{f}(x^{**})\} \ \forall f \in A_u(B_c)$, where every $Cl_B(f, x^{**}) \neq \emptyset$ (see [19, p. 200] or [5, p. 2]), then the cluster value theorem for $A_u(B_c)$ holds. □

Bessaga and Pełczyński proved in [7] that, when $\alpha \geq \omega^\omega$ is a countable ordinal, $C(\alpha)$ is not isomorphic to $c = C(\omega)$. Therefore we no longer can use Lemma 2.1 to obtain a cluster value theorem on such spaces of continuous functions.

Nevertheless, for α a countable ordinal, the intervals $[1, \alpha]$ are always compact, Hausdorff and dispersed (they contain no perfect non-void subset). The compact, Hausdorff and dispersed sets K satisfy, from [35, Main Theorem], that $X = C(K)$ contains no isomorphic copy of l_1. Moreover, from [3, Theorem 5.4.5], $X = C(K)$ has the Dunford-Pettis property. Therefore, for dispersed K, the continuous polynomials on $X = C(K)$ are weakly (uniformly) continuous on bounded sets by [13, Corollary 2.37].

Moreover, since $X^* = l_1(K)$ has the approximation property, [13, Proposition 2.8] now yields that all continuous polynomials on X can be uniformly approximated, on bounded sets, by polynomials of finite type. Thus the elements of $A_u(B)$ can be approximated, uniformly on B, by polynomials of finite type. Hence $A_u(B) = A(B)$, so each fiber at $x^{**} \in \bar{B}^{**}$ is the singleton $\{x^{**}\}$, and then X satisfies the cluster value theorem for the algebra $A_u(B)$.

We now consider the cluster value problem on X for the algebra of all bounded analytic functions $H^\infty(B)$.

2.2 A cluster value theorem for spaces of continuous functions

Following the line of proof of Theorem 5.1 in [5], and using that $C(K)^* = \ell_1(K)$ when K is compact, Hausdorff and dispersed, we obtain a cluster value theorem for $H^\infty(B)$ when $X = C(K)$, and K is compact, Hausdorff and dispersed.

Theorem 2.2. *If X is the Banach space $C(K)$, for K compact, Hausdorff and dispersed, then the cluster value theorem holds for $H^\infty(B)$ at every $x \in \bar{B}^{**}$.*

Proof. Fix $f \in H^\infty(B)$ and $w = (w_t)_{t \in K} \in \bar{B}^{**}$ (where $C(K)^{**} = l_\infty(K)$). Suppose $0 \notin Cl_B(f, w)$. It suffices to show that $0 \notin \hat{f}(M_w)$.

Since 0 is not a cluster value of f at w, there exists a weak-star neighborhood U of w such that $0 \notin \overline{f(U \cap B)}$, where

$$U \cap B \supset \cap_{i=1}^{n} \{z \in B : | < (z-w), x_i^* > | < \epsilon\},$$

for some $\epsilon > 0$ and $x_1^*, \cdots, x_n^* \in X^* = l_1(K)$.

We have that $x_i^* = (x_i^*(t))_{t \in K}$ has countably many nonzero coordinates $\{x_i^*(t)\}_{t \in F_i}$ for $i = 1, \cdots, n$. Thus,

$$U \cap B \supset \cap_{i=1}^{n} \{z \in B : |\sum_{t \in K} (z_t - w_t) x_i^*(t)| < \epsilon\},$$

and there is a finite set $F \subset \cup_{i=1}^{n} F_i$ so that $\sum_{t \notin F} |x_i^*(t)| < \epsilon/4$, for $i = 1, \cdots, n$. Then,

$$U \cap B \supset \cap_{t \in F} \{z \in B : |z_t - w_t| < \delta\},$$

where

$$\delta = \min_{1 \le i \le n, t \in F} \frac{\epsilon}{(2|F|)|x_i^*(t)|}.$$

In summary, there exist $c > 0$, $\delta > 0$ and a finite set $F \subset K$ such that if $z \in B$ satisfies $|z_t - w_t| < \delta$ for $t \in F$ then $|f(z)| \ge c$. Relabel the indices in F as $t_1, \cdots, t_m$, where $m = |F|$. Then proceed as in the proof of Theorem 5.1 in [5]:

For $0 \le k \le m-1$, define $U_k = \{z \in B : |z_{t_j} - w_{t_j}| < \delta, k+1 \le j \le m\}$, and set $U_m = B$. Note that $1/f$ is bounded and analytic on U_0.

We claim that for each k, $1 \le k \le m$, there are functions g_k and $h_{k,j}$, $1 \le j \le k$, in $H^\infty(U_k)$ that satisfy

$$f(z) g_k(z) = 1 + (z_{t_1} - w_{t_1}) h_{k1}(z) + \cdots + (z_{t_k} - w_{t_k}) h_{kk}(z), \quad z \in U_k. \tag{2.1}$$

Once this claim is established, the proof is easily completed as follows. The functions g_m and h_{mj} belong to $H^\infty(B)$ and satisfy

$$\widehat{f}\widehat{g_m} = \widehat{1} + \sum_{j=1}^{m} (\widehat{z_{t_j} - w_{t_j}})\widehat{h_{mj}}.$$

Since each $\widehat{z_{t_j}} - w_{t_j}$ vanishes on M_w (by the definition of M_w), we obtain $\widehat{f}\widehat{g_m} = 1$ on M_w, and consequently $\widehat{f}$ does not vanish on M_w, as required.

Just as in [5], the claim is established by induction on k. The first step, the construction of g_1 and h_{11}, is as follows. We regard $1/f((z_t)_{t\in K})$ as a bounded analytic function of z_{t_1} for $|z_{t_1}| < 1$ and $|z_{t_1} - w_{t_1}| < \delta$, with z_t, $t \in K - \{t_1\}$, as analytic parameters in the range $|z_t| < 1$ for $t \in K - \{t_1\}$, and $|z_{t_j} - w_{t_j}| < \delta$ for $2 \leq j \leq m$. According to lemma 5.3 in [5], we can express

$$\frac{1}{f(z)} = g_1(z) + (z_{t_1} - w_{t_1})h(z), \quad z \in U_0,$$

where $g_1 \in H^\infty(U_1)$ and $h \in H^\infty(U_0)$. We set

$$h_{11}(z) = [f(z)g_1(z) - 1]/(z_{t_1} - w_{t_1}), \quad z \in U_1,$$

so that (2.1) is valid for $k = 1$. Note that $h_{11} = -hf$ on U_0. Consequently h_{11} is bounded and analytic on U_0. The defining formula then shows that h_{11} is analytic on all of U_1, and since $|z_{t_1} - w_{t_1}| \geq \delta$ on $U_1 - U_0$, h_{11} is bounded on U_1.

Now suppose that $2 \leq k \leq m$, and that there are functions g_{k-1} and $h_{k-1,j}$ $(1 \leq j \leq k-1)$ that satisfy (2.1) and are appropriately analytic. We apply lemma 5.3 in [5] to these as functions of z_{t_k}, with the other variables regarded as analytic parameters, to obtain decompositions

$$g_{k-1}(z) = g_k(z) + (z_{t_k} - w_{t_k})G_k(z)$$

and

$$h_{k-1,j}(z) = h_{k,j}(z) + (z_{t_k} - w_{t_k})H_{k,j}(z), \quad 1 \leq j \leq m-1,$$

where g_k and the h_{kj}'s are in $H^\infty(U_k)$, and G_k and the H_{kj}'s are in $H^\infty(U_{k-1})$. From the identity (2.1), with k replaced with $k-1$, we obtain

$$fg_k = 1 + \sum_{j=1}^{k-1}(z_{t_j} - w_{t_j})h_{kj} + (z_{t_k} - w_{t_k})[-fG_k + \sum_{j=1}^{k-1}(z_{t_j} - w_{t_j})H_{kj}]$$

on U_{k-1}. We define

$$h_{kk} = [fg_k - 1 - \sum_{j=1}^{k-1}(z_{t_j} - w_{t_j})h_{kj}]/(z_{t_k} - w_{t_k}), \quad z \in U_k.$$

Then (2.1) is valid. On U_{k-1} we have

$$h_{kk} = -fG_k + \sum_{j=1}^{k-1}(z_{t_j} - w_{t_j})H_{kj},$$

so that h_{kk} is bounded and analytic on U_{k-1}. Since $|z_{t_k} - w_{t_k}| \geq \delta$ on $U_k - U_{k-1}$, we see from the defining formula that $h_{kk} \in H^\infty(U_k)$. This establishes the induction step, and the proof is complete.

$\square$

We do not know the answer to the cluster value problem for other spaces $C(K)$, however, we can give a partial answer to the following modification of the cluster value problem.

2.3 The cluster value problem for $H^\infty(B)$ over $A_u(B)$

In [27] we consider the following cluster value problem: Given $f_0^{**} \in \overline{B}^{**}$, the cluster value problem for $H^\infty(B)$ over $A_u(B)$ at f_0^{**} asks whether for all $\psi \in H^\infty(B)$ and $\tau \in \mathcal{M}_{f_0^{**}}(B)$ ($\mathcal{M}_{f_0^{**}}$ is $\pi^{-1}(\delta_{f_0^{**}})$ for the restriction map $\pi : M_{H^\infty(B)} \to M_{A_u(B)}$), can we find a net $(f_\alpha) \subset B$ such that $\psi(f_\alpha) \to \tau(\psi)$ and f_α converges to f_0^{**} in the polynomial-star topology, i.e. the smallest topology that makes every extension of a polynomial on X to X^{**} continuous (that we denote by $\tau(\psi) \in \mathsf{Cl}_B(\psi, f_0^{**})$)? As before, clearly $\mathsf{Cl}_B(\psi, f_0^{**}) \subset \widehat{\psi}(\mathcal{M}_{f_0^{**}}(B))$, $\forall \psi \in H^\infty(B)$.

The previous problem seems to be highly nontrivial. For example, for every infinite compact Hausdorff space K, $C(K)$ contains a subspace Y isometric to c_0 (Proposition 4.3.11 in [3]), so the fiber $\mathcal{M}_0(B_{C(K)})$ is huge (and from Proposition 2.6 below, so is each fiber $\mathcal{M}_{f_0}(B_{C(K)})$ for $f_0 \in B_{C(K)}$). Indeed, according to Theorem 6.6 in [10], there is a family of distinct characters $\{\tau_\alpha\}_{\alpha \in B_{\ell_\infty}}$, such that each $\tau_\alpha : H^\infty(B_Y) \to \mathbb{C}$ satisfies $\delta_0 = \tau_\alpha|_{A(B_Y)} = \tau_\alpha|_{A_u(B_Y)}$ (because Y is isometric to c_0, so $A(B_Y) = A_u(B_Y)$). Hence $\{\tau_\alpha\}_{\alpha \in B_{\ell_\infty}} \subset \mathcal{M}_0(B_Y)$ and therefore $\{\tau_\alpha \circ R\}_{\alpha \in B_{\ell_\infty}} \subset \mathcal{M}_0(B_{C(K)})$, where R is the restriction mapping $R : H^\infty(B_{C(K)}) \to H^\infty(B_Y)$, which is clearly a homomorphism. Note that the characters $\{\tau_\alpha \circ R\}_{\alpha \in B_{\ell_\infty}}$ are all distinct due to Theorem 1.1 in [4] (also [19, Theorem 2.1.3]), because ℓ_∞ is an isometrically injective space (Proposition 2.5.2 in [3]), so there exists a norm-one linear map $S : C(K) \to \ell_\infty$ such that $S|_{c_0} = I_{c_0}$.

The cluster value problem for $H^\infty(B)$ over $A_u(B)$ coincides with the cluster value problem for $H^\infty(B)$ when $A_u(B) = A(B)$. Thus when K is compact, Hausdorff and dispersed, we have a positive answer to the previous cluster value problem for the $C(K)$ spaces.

To study the cluster value problem for $H^\infty(B)$ over $A_u(B)$ when $B = B_{C(K)}$ of an arbitrary $C(K)$ space, in [27] we describe the following family of automorphisms $(T_{f_0})_{f_0 \in B}$:

Proposition 2.5. *Let $f_0 \in B = B_{C(K)}$. $T_{f_0} : B \to B$ given by*

$$T_{f_0}(f) = \frac{f - f_0}{1 - \overline{f_0} \cdot f} \quad \forall f \in B,$$

is biholomorphic.

This is a folklore result mentioned e.g. in [39] and [8], but inasmuch there seems to be no proof in the literature we sketch the proof.

Proof. Set $\delta_0 = ||f_0||$.

Let us start by showing that $T := T_{f_0}$ is well defined, i.e. $||Tf|| < 1$ when $||f|| < 1$.

Let $f \in B$. We can find $\delta \in (\delta_0, 1)$ such that $||f|| \leq \delta$.

For every $t_0 \in K$, let $z = f(t_0)$ and $c = f_0(t_0)$, so that $T(f)(t_0) = \frac{z-c}{1-\bar{c}z}$.

Let Δ denote the open unit disk in the complex plane $\mathbb{C}$.

Since $\sigma : (\delta \cdot \overline{\Delta}) \times (\delta_0 \cdot \overline{\Delta}) \to \Delta$ given by $\sigma(z, c) = \frac{z-c}{1-\bar{c}z}$ is continuous, then $\sigma((\delta \cdot \overline{\Delta}) \times (\delta_0 \cdot \overline{\Delta}))$ is a compact subset of Δ, so there exists $\delta_1 < 1$ so that $\sigma((\delta \cdot \overline{\Delta}) \times (\delta_0 \cdot \overline{\Delta})) \subset \delta_1 \overline{\Delta}$.

Thus $||Tf|| \leq \delta_1 < 1$.

Let us now show that T is also holomorphic, or equivalently, Fréchet $\mathbb{C}$-differentiable. For $f \in B$ fixed, the linear mapping $L : C(K) \to C(K)$ given by $L(h) = \frac{1-|f_0|^2}{(1-\overline{f_0}f)^2}h$ satisfies that, for $h \neq 0$ small enough,

$$
\begin{aligned}
\frac{T(f+h)-T(f)-L(h)}{||h||} &= \left(\frac{f+h-f_0}{1-\overline{f_0}(f+h)} - \frac{f-f_0}{1-\overline{f_0}f} - \frac{1-|f_0|^2}{(1-\overline{f_0}f)^2}h\right)/||h|| \\
&= \left(\frac{1-|f_0|^2}{1-\overline{f_0}f} \cdot \frac{h}{1-\overline{f_0}(f+h)} - \frac{1-|f_0|^2}{(1-\overline{f_0}f)^2}h\right)/||h|| \\
&= \frac{\overline{f_0}h}{(1-\overline{f_0}f)^2(1-\overline{f_0}(f+h))}(1-|f_0|^2)h/||h||,
\end{aligned}
$$

which goes to zero as $h \to 0$. Thus T is holomorphic.

Since T clearly has a necessarily holomorphic inverse ($S(f) = \frac{f+f_0}{1+\overline{f_0}\cdot f}$), we have that T is a biholomorphic function on B that sends f_0 to the function identically zero.

Alternative proof of analyticity. It suffices to check that T is continuous and its restriction to each complex line is holomorphic.

To prove T is continuous, let $f_1 \in B$ and $\epsilon > 0$.

For any $f \in B$,

$$
||T(f) - T(f_1)|| = ||\frac{f-f_0}{1-\overline{f_0}\cdot f} - \frac{f_1-f_0}{1-\overline{f_0}\cdot f_1}|| = ||\frac{(1-|f_0|^2)(f-f_1)}{(1-\overline{f_0}\cdot f)(1-\overline{f_0}\cdot f_1)}||,
$$

where $||1-\overline{f_0}\cdot f||, ||1-\overline{f_0}\cdot f_1|| \geq 1-||f_0|| = \alpha > 0$. Thus, when $||f-f_1|| < \epsilon\cdot\alpha^2$,

$$
||T(f)-T(f_1)|| < \frac{1\cdot\epsilon\cdot\alpha^2}{\alpha^2} = \epsilon,
$$

so T is continuous.

To prove that T is holomorphic on each line, let $f_1 \in B$, and $g_1 \neq 0 \in B$. Then, for $\lambda \in A = \{a \in \mathbb{C} : f_1 + ag_1 \in B\}$,

$$\psi(\lambda) = T(f_1 + \lambda g_1) = \frac{f_1 + \lambda g_1 - f_0}{1 - \overline{f_0}(f_1 + \lambda g_1)} = \frac{\lambda g_1 + (f_1 - f_0)}{-\lambda \overline{f_0} g_1 + (1 - \overline{f_0} f_1)}.$$

Let $\lambda_0 \in A$. Set $F = \lambda_0 g_1 + (f_1 - f_0)$ and $G = 1 - \overline{f_0}(f_1 + \lambda_0 g_1)$. Clearly G has modulus uniformly bounded below by $1 - ||f_0|| = \beta > 0$, so $1/G \in C(K)$. Whenever $|\lambda - \lambda_0| < \beta$, we have

$$\begin{aligned}
\psi(\lambda) &= \frac{(\lambda - \lambda_0)g_1 + \lambda_0 g_1 + (f_1 - f_0)}{-(\lambda - \lambda_0)g_1\overline{f_0} - \lambda_0 g_1 \overline{f_0} + (1 - \overline{f_0} f_1)} \\
&= \frac{(\lambda - \lambda_0)g_1 + F}{-(\lambda - \lambda_0)g_1\overline{f_0} + G} \\
&= \frac{(\lambda - \lambda_0)g_1 + F}{G} \frac{1}{1 - (\lambda - \lambda_0)(g_1\overline{f_0}/G)} \\
&= (\frac{(\lambda - \lambda_0)g_1 + F}{G}) \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n (\frac{g_1\overline{f_0}}{G})^n,
\end{aligned}$$

which converges uniformly in λ when $|\lambda - \lambda_0| \leq \beta_0 < \beta$.

Hence ψ is holomorphic, and consequently so is T.

$\square$

Let us note as a side remark that we can extend the previous conclusion to the open unit ball of the second dual of $C(K)$: Rewrite $\frac{f - f_0}{1 - \overline{f_0} \cdot f}$ as $(f - f_0) \sum_{n=0}^{\infty} (\overline{f_0} f)^n$. Since it is known that $C(K)^{**}$ is a commutative C^*-algebra that extends the C^* structure of $C(K)$ (see [14, 310-311] and [36, p. 43]), then Proposition 2.5 extends in the following manner.

Proposition 2.6. *Given $f_0^{**} \in B_{C(K)^{**}}$, let $T_{f_0^{**}} : B_{C(K)^{**}} \to B_{C(K)^{**}}$ be given by*

$$T_{f_0^{**}}(f^{**}) = (f^{**} - f_0^{**}) \sum_{n=0}^{\infty} (\overline{f_0^{**}} f^{**})^n \quad , \forall f^{**} \in B_{C(K)^{**}}.$$

Then $T_{f_0^{**}}$ is biholomorphic.

Imitating Lemmas 4.3 and 4.4 in [5], we prove in [27] that for an arbitrary $C(K)$ space, the family of automorphisms $(T_{f_0})_{f_0 \in B}$ of B, given by $T_{f_0} : f \to \frac{f - f_0}{1 - \overline{f_0} f}$ satisfies the following proposition:

Proposition 2.7. *For each $f_0 \in B$, the biholomorphic function T_{f_0} induces a homeomorphism $\hat{T}_{f_0}$ on the spectrum $M_{H(B)}$, where H denotes either the algebra A_u or the algebra H^∞, that maps $\mathcal{M}_{f_0}(B)$ homeomorphically onto $\mathcal{M}_0(B)$ (where $\mathcal{M}_{f_0}(B) = \{\tau \in M_{H^\infty(B)} : \tau|_{A_u(B)} = f_0\}$).*

Proof. Note that $T := T_{f_0}$ is a Lipschitz function. Indeed, if $f, g \in B$,

$$||T(f) - T(g)|| = ||\frac{(1 - |f_0|^2)(f - g)}{(1 - \overline{f_0} f)(1 - \overline{f_0} g)}|| \leq \frac{1}{(1 - ||f_0||)^2} ||f - g||.$$

Thus for every $\psi \in H(B)$, $\psi \circ T \in H(B)$. So $\hat{T} : M_{H(B)} \to M_{H(B)}$, given by

$$\hat{T}(\tau)(\psi) = \tau(\psi \circ T), \;\; \forall \tau \in M_{H(B)}, \; \psi \in H(B),$$

is well defined. Moreover, given $\tau \in \mathcal{M}_{f_0}(B)$ and $\psi \in A_u(B)$,

$$\hat{T}(\tau)(\psi) = \tau(\psi \circ T) = (\psi \circ T)(f_0) = \psi(0),$$

i.e. $\hat{T}(\tau) \in \mathcal{M}_0(B)$, for every $\tau \in \mathcal{M}_{f_0}(B)$.

Now, given $\tau \in \mathcal{M}_0(B)$ it is clear that $\hat{\tau} : H(B) \to \mathbb{C}$ given by

$$\hat{\tau}(\psi) = \tau(\psi \circ T^{-1}), \quad \forall \psi \in H(B),$$

is in $M_{H(B)}$, actually in $\mathcal{M}_{f_0}(B)$, and $\forall\, \psi \in H(B)$,

$$\hat{T}(\hat{\tau})(\psi) = \hat{\tau}(\psi \circ T) = \tau(\psi),$$

i.e. $\hat{T}(\hat{\tau}) = \tau$. □

The reader can easily check that the previous mapping $\hat{T}$ is actually a homeomorphism.

As a consequence of Proposition 2.7, we obtain that the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at 0 is equivalent to the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at every $f_0 \in B$, when $X = C(K)$.

Corollary 2.2. *If X is a Banach space $C(K)$, then the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at 0 is equivalent to the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at every $f_0 \in B$.*

Proof. Let $f_0 \in B$ and set $T(f) = \frac{f - f_0}{1 - \overline{f_0} f}$ for $f \in B$. Then, $\forall\, \psi \in H^\infty(B)$,

$$\begin{aligned} \hat{\psi}(\mathcal{M}_0(B)) &= \hat{\psi} \circ \hat{T}(\mathcal{M}_{f_0}(B)) = \widehat{\psi \circ T}(\mathcal{M}_{f_0}(B)), \\ \mathsf{Cl}_B(\psi, 0) &= \mathsf{Cl}_B(\psi \circ T, f_0), \end{aligned}$$

because $\psi \circ T \in H^\infty(B)$ too, and $T^{-1}(f) = (f + f_0) \sum_{n=0}^{\infty} (-\overline{f_0} f)^n$ for $f \in B_{C(K)}$ is polynomially-star continuous, because sums and norm limits of polynomially-star continuous maps are polynomially-star continuous, as well as multiplication by a fixed element of $C(K)$. □

Let us note that, from the Gelfand Representation Theorem, this corollary holds for any nonzero unital commutative C^*-algebra.

The previous result is a reduction of a cluster value problem at any point in the ball of $C(K)$ to the origin. In the next section we explain a reduction of a cluster value problem for any separable Banach space to a space with a more specific structure.

2.4 The cluster value problem for separable spaces

In [28] we prove that for any separable Banach space Y, a cluster value problem for $H(B_Y)$ ($H = H^\infty$ or $H = A_u$) can be reduced to a cluster value problem for $H(B_X)$ for some Banach space X that is an ℓ_1-sum of a sequence of finite-dimensional spaces. The proof relies on the construction of an isometric quotient map from a suitable X to Y that induces an isometric algebra homomorphism from $H(B_Y)$ to $H(B_X)$ with 1-complemented range, where the projection mapping is built using ultrafilter techniques. Other tools include the infinite-dimensional version of Schwarz' Lemma, as well as such familiar one complex variable results as Cauchy's inequality and Montel's theorem. This is done in the following two lemmas:

Lemma 2.2. *Let Y be a separable Banach space and $Y_1 \subset Y_2 \subset Y_3 \subset \dots$ an increasing sequence of finite dimensional subspaces whose union is dense in Y. Set $X = (\sum Y_n)_1$. Then the isometric quotient map $Q : X \to Y$ defined by*

$$Q(z_n)_n := \sum_{n=1}^{\infty} z_n$$

induces an isometric algebra homomorphism $Q^\# : H(B_Y) \to H(B_X)$, where H denotes either the algebra A_u or the algebra H^∞.

The idea behind is to use the little open mapping theorem to get $Q(B_X) = B_Y$.

Proof. Note that for all $(z_n)_n \in X$,

$$\|Q(z_n)_n\| = \|\sum_{n=1}^{\infty} z_n\| \leq \sum_{n=1}^{\infty} \|z_n\| = \|(z_n)_n\|_1. \tag{2.2}$$

Let $\widetilde{Y_n} = \{(z_n)_n \in X : \; z_k = 0 \; \forall \; k \neq n\}$. Since $Q(B_{\widetilde{Y_n}}) = B_{Y_n}$ for all $n \in \mathbb{N}$, we now have that $Q(B_X)$ is dense in B_Y and hence Q is an isometric quotient map.

Then the function $Q^{\#} : H(B_Y) \to H(B_X)$ given by $Q^{\#}(f) = f \circ Q$ is an isometric homomorphism because $Q^{\#}$ is clearly linear and for all $f, g \in H(B_Y)$,

$$\|Q^{\#}(f)\| = \sup_{x \in B_X} |f \circ Q(x)| = \sup_{y \in B_Y} |f(y)| = \|f\|.$$

Moreover,

$$Q^{\#}(f \cdot g) = (f \cdot g) \circ Q = (f \circ Q) \cdot (g \circ Q) = Q^{\#}(f) Q^{\#}(g),$$

so $Q^{\#}$ is an algebra homomorphism.

□

Now we find a left inverse to $Q^{\#}$, that allows us to go back to $H(B_Y)$.

Lemma 2.3. *Under the assumptions of the previous lemma, there is a norm one algebra homomorphism $T : H(B_X) \to H(B_Y)$ so that $T(X^*) \subset Y^*$ and $T \circ Q^{\#} = I_{H(B_Y)}$.*

Proof. The first part of the proof consists of constructing T and verifying that $TH(B_X) \subset H(B_Y)$.

For every $y \in (\cup Y_n)$ and $n \in \mathbb{N}$, let $S_n(y) = (z_i)_i \in X$ be given by

$$z_i = \begin{cases} y & \text{if } i = n \text{ and } y \in Y_n \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. For each $g \in H(B_X)$ set

$$Sg(y) = \lim_{n \in \mathcal{U}} g(S_n y) \text{ for every } y \in B_{(\cup Y_n)},$$

which is well defined because g is bounded. Next we prove that Sg is continuous:

Let $0 < r < 1$. Since $g \in H^\infty(B_X)$ then Schwarz' Lemma (Thm. 7.19, [34]) and the convexity of B_X imply that $g \in A_u(rB_X)$. Let $\epsilon > 0$. Since g is uniformly continuous on rB_X there exists $\delta > 0$ such that, if $a, b \in rB_X$ and $\|a - b\| < \delta$, then $\|g(a) - g(b)\| < \epsilon$. Thus, given $y_1, y_2 \in rB_{(\cup Y_n)}$ such that $\|y_1 - y_2\| < \delta$, we can find $N \in \mathbb{N}$ such that $y_1, y_2 \in Y_n \;\; \forall\, n \geq N$, and then $\|S_n(y_1) - S_n(y_2)\| = \|y_1 - y_2\| < \delta$ eventually for n, so

$$\|Sg(y_1) - Sg(y_2)\| = \lim_{n \in \mathcal{U}} \|g(S_n(y_1)) - g(S_n(y_2))\| \leq \epsilon.$$

Each $Sg : B_{(\cup Y_n)} \to \mathbb{C}$ is uniformly continuous on $rB_{(\cup Y_n)}$ for $0 < r < 1$, thus we can continuously extend each Sg to $Tg : B_Y \to \mathbb{C}$. Moreover, it is evident that Tg is uniformly continuous on B_X when $g \in A_u(B_X)$. It is left to show that each Tg is analytic by checking that every Tg is analytic in each complex line (Thm. 8.7, [34]). We do this in two parts.

Step 1 Let us check that each Sg is analytic on complex lines:

Let $y^1 \in B_{(\cup Y_n)}$ and $y^2 \neq 0 \in (\cup Y_n)$. Since $B_{(\cup Y_n)}$ is open we can find $R > 0$ such that, if $\|y - y^1\| < R$, then $y \in B_{(\cup Y_n)}$. Choose $r > 0$ such that $r\|y^2\| < R$. Thus, if $|\lambda| \leq r$ we have that $\lambda \in \Lambda = \{\zeta \in \mathbb{C} :\; y^1 + \zeta y^2 \in B_{(\cup Y_n)}\}$.

For each $n \in \mathbb{N}$, let $u_n = S_n(y^1)$ and $w_n = S_n(y^2)$.

Using the notation at the beginning of Section 1.4, we claim that

$$Sg(y^1 + \lambda y^2) = \sum_{m=0}^{\infty} (\lim_{n \in \mathcal{U}} P^m g(u_n)(w_n))\lambda^m,$$

uniformly on λ, for $|\lambda| \leq r$. Let us start by showing that for each $m \in \mathbb{N}$ and $\lambda \in \bar{\Delta}(0,r)$, $\lim_{n\in\mathcal{U}} P^m g(u_n)(\lambda w_n)$ exists.

We can find $s > 1$ such that also $sr\|y^2\| < R$. Then, when $|t| \leq s$ and $|\lambda| \leq r$, we have that $y^1 + t\lambda y^2 \in B_{(\cup Y_n)}$, because

$$\|(y^1 + t\lambda y^2) - y^1\| = |t\|\lambda\|y^2\| \leq sr\|y^2\| < R,$$

so $u_n + t\lambda w_n \in B_X$ eventually for n, and from Cauchy's Inequality (Cor. 7.4, [34]), for each $m \in \mathbb{N}$

$$\|P^m g(u_n)(\lambda w_n)\| \leq \frac{1}{s^m}\|g\|,$$

eventually for n, and then $\lim_{n\in\mathcal{U}} P^m g(u_n)(\lambda w_n)$ exists.

Moreover, given $M \in \mathbb{N}$ and λ such that $|\lambda| \leq r$,

$$\begin{aligned}
\|Sg(y^1 + \lambda y^2) - \sum_{m=0}^{M} (\lim_{n\in\mathcal{U}} P^m g(u_n)(w_n))\lambda^m\| & \\
&= \|\lim_{n\in\mathcal{U}} (g(u_n + \lambda w_n) - \sum_{m=0}^{M} P^m g(u_n)(\lambda w_n))\| \\
&= \|\lim_{n\in\mathcal{U}} \sum_{m=M+1}^{\infty} P^m g(u_n)(\lambda w_n)\| \\
&\leq \sum_{m=M+1}^{\infty} \tfrac{1}{s^m}\|g\| = \tfrac{\|g\|}{s-1}\tfrac{1}{s^M},
\end{aligned}$$

which goes to zero as $M \to \infty$.

Thus Sg is analytic on complex lines.

Alternative proof of Step 1. Let $y^1 \in B_{(\cup Y_n)}$ and $y^2 \neq 0 \in (\cup Y_n)$. We want to show that $Sg(y^1 + \lambda y^2)$ is an analytic function of λ. It is enough to find a power series expansion.

Let $d = d_{B_{(\cup Y_n)}}(y^1)$ and let $r \in (0, d/\|y^2\|)$. Then $y^1 + \lambda y^2 \in B_{(\cup Y_n)}$ if $|\lambda| \leq r$.

For each $n \in \mathbb{N}$, let $u_n = S_n(y^1)$ and $w_n = S_n(y^2)$. Note that

$$Sg(y^1 + \lambda y^2) = \lim_{n \in \mathcal{U}} g(u_n + \lambda w_n) \text{ for } |\lambda| \leq r.$$

Let $g_n(\lambda) = g(u_n + \lambda w_n)$ for $|\lambda| \leq r$. Since g is analytic on complex lines then

$$g_n(\lambda) = \sum_{m=0}^{\infty} \frac{g_n^{(m)}(0)\lambda^m}{m!} \text{ uniformly on } \lambda, \text{ for } |\lambda| \leq r,$$

because $\|(u_n + \lambda w_n) - u_n\| \leq r\|y^2\| < d_{B_{(\cup Y_n)}}(y^1) \leq d_{B_X}(u_n)$.

Now we claim that

$$Sg(y^1 + \lambda y^2) = \sum_{m=0}^{\infty} (\lim_{n \in \mathcal{U}} g_n^{(m)}(0)/m!)\lambda^m$$

uniformly on λ, for $|\lambda| \leq r$. Let us first show that $\lim_{n \in \mathcal{U}} g_n^{(m)}(0)/m!$ exists $\forall m \in \mathbb{N}_0$:

Let $r_2 \in (0, r)$. Since

$$g_n^{(m)}(0) = \frac{m!}{2\pi i} \int_{|\zeta| = r_2} \frac{g_n(\zeta)}{\zeta^{m+1}} d\zeta$$

then $\|g_n^{(m)}(0)/m!\| \leq \frac{1}{r_2^m}\|g\|$, so $\lim_{n \in \mathcal{U}} g_n^{(m)}(0)/m!$ exists.

Moreover, for all $M \in \mathbb{N}_0$ and $|\lambda| \leq r_1 < r_2$,

$$\begin{aligned}
\|Sg(y^1 + \lambda y^2) - \sum_{m=0}^{M} (\lim_{n\in\mathcal{U}} g_n^{(m)}(0)/m!)\lambda^m\| \\
&= \| \lim_{n\in\mathcal{U}} (g_n(\lambda) - \sum_{m=0}^{M} g_n^{(m)}(0)/m!\lambda^m)\| \\
&= \| \lim_{n\in\mathcal{U}} \sum_{m=M+1}^{\infty} g_n^{(m)}(0)/m!\lambda^m\| \\
&\leq \sum_{m=M+1}^{\infty} (\tfrac{r_1}{r_2})^m \|g\| = \tfrac{\|g\|}{1-r_1/r_2}(r_1/r_2)^{M+1},
\end{aligned}$$

which goes to zero as $M \to \infty$.

Step 2 The following general lemma should be known, but we could not find a reference.

Lemma 2.4. *If $\phi : B_Y \to \mathbb{C}$ is bounded and uniformly continuous on sB_Y for each $0 < s < 1$, and there is a dense subspace Z of Y such that $\phi|_{B_Z}$ is analytic on complex lines, then ϕ is analytic on complex lines in B_Y (and hence analytic).*

Proof. Let $y^1 \in B_Y$ and $y^2 \neq 0 \in Y$. Let $s \in (\|y^1\|, 1)$. Since sB_Y is open and contains y^1, we can find $R > 0$ such that, if $\|y - y^1\| < R$ then $y \in sB_Y$. Choose $r > 0$ such that $r\|y^2\| \leq R$. Thus, if $|\lambda| < r$ we have that $\lambda \in \Lambda = \{\zeta \in \mathbb{C} : \ y^1 + \zeta y^2 \in sB_Y\}$.

Let $f : \lambda \to \phi(y^1 + \lambda y^2)$, a function defined for $|\lambda| < r$. We want to show that f is analytic.

Let $\{y_k^1\}_k \subset B_Z$ and $\{y_k^2\}_k \subset Z$ be sequences such that $\|y_k^1 - y^1\| \leq \frac{1}{2^k}$ and $\|y_k^2 - y^2\| \leq \frac{1}{2^k}$. Choose $K_1 \in \mathbb{N}$ such that $\frac{1+r}{2^{K_1}} \leq R - r\|y^2\|$. Then for $k \geq K_1$ and

$|\lambda| < r$ we have that

$$\begin{aligned}\|(y_k^1 + \lambda y_k^2) - y^1\| &\leq \|y_k^1 - y^1\| + |\lambda|\|y_k^2 - y^2\| + |\lambda|\|y^2\| \\ &< \tfrac{1+r}{2^{K_1}} + r\|y^2\| \\ &\leq R,\end{aligned}$$

so $y_k^1 + \lambda y_k^2 \in sB_Z$.

For each $k \geq K_1$, let $f_k : \lambda \to \phi(y_k^1 + \lambda y_k^2)$, which is an analytic function for $|\lambda| < r$ by assumption.

Since ϕ is bounded, clearly $\{f_k\}_{k \geq K_1}$ is uniformly bounded. Let us now show that $\{f_k\}_{k \geq K_1}$ converges uniformly to f. Let $\epsilon > 0$. Since ϕ is uniformly continuous on sB_Z, we can find $\delta > 0$ such that,

$$a, b \in sB_Z, \quad ||a - b|| < \delta \Longrightarrow ||\phi(a) - \phi(b)|| < \epsilon.$$

Choose $K \geq K_1$ such that $\frac{1+r}{2^K} < \delta$. Then $\forall k \geq K$ and λ with $|\lambda| < r$,

$$\begin{aligned}\|(y_k^1 + \lambda y_k^2) - (y^1 + \lambda y^2)\| &\leq \|y_k^1 - y^1\| + |\lambda|\|y_k^2 - y^2\| \\ &\leq \frac{1+r}{2^k} \\ &< \delta,\end{aligned}$$

so $\|f_k(\lambda) - f(\lambda)\| = \|\phi(y_k^1 + \lambda y_k^2) - \phi(y^1 + \lambda y^2)\| < \epsilon$.

Then, by the lemma on p. 226 in [2], f is analytic.

Alternative proof. Let $y^1 \in B_Y$ and $y^2 \neq 0 \in Y$. Let $s \in (\|y^1\|, 1)$. Since sB_Y is open and contains y^1, we can find $R > 0$ such that, if $\|y - y^1\| < R$ then $y \in sB_Y$. Choose $r > 0$ such that $r\|y^2\| < R$. Thus, if $|\lambda| \leq r$ we have that $\lambda \in \Lambda = \{\zeta \in \mathbb{C} : \ y^1 + \zeta y^2 \in sB_Y\}$.

Let $\{y_k^1\}_k \subset B_Z$ and $\{y_k^2\}_k \subset Z$ be sequences such that $\|y_k^1 - y^1\| \leq \frac{1}{2^k}$ and $\|y_k^2 - y^2\| \leq \frac{1}{2^k}$. Choose $K_1 \in \mathbb{N}$ such that $\frac{1+r}{2^{K_1}} < R - r\|y^2\|$. Then for $k \geq K_1$ and $|\lambda| \leq r$ we have that

$$\begin{aligned}
\|(y_k^1 + \lambda y_k^2) - y^1\| &\leq \|y_k^1 - y^1\| + |\lambda|\|y_k^2 - y^2\| + |\lambda|\|y^2\| \\
&\leq \tfrac{1+r}{2^{K_1}} + r\|y^2\| \\
&< R,
\end{aligned}$$

so $y_k^1 + \lambda y_k^2 \in sB_Z$.

For $k \geq K_1$, Let φ_k be the restriction of ϕ to $\{y_k^1 + \lambda y_k^2 : \ |\lambda| \leq r\}$. Then for $k \geq K_1$,

$$\varphi_k(y_k^1 + \lambda y_k^2) = \sum_{m=0}^{\infty} (P^m \varphi_k(y_k^1)(y_k^2))\lambda^m,$$

uniformly on λ, whenever $|\lambda| \leq r$.

We claim that

$$\phi(y^1 + \lambda y^2) = \sum_{m=0}^{\infty} \lim_{k \to \infty} (P^m \varphi_k(y_k^1)(y_k^2))\lambda^m,$$

uniformly on λ, whenever $|\lambda| \leq r$. Let us start by showing that we can take the limit when k goes to ∞.

Let $\epsilon > 0$ and choose $\delta > 0$ such that, if $a, b \in sB_Y$ and $\|a - b\| < \delta$, then $\|\phi(a) - \phi(b)\| < \epsilon$. Let $K \geq K_1$ be such that $\frac{1+r}{2^K} < \delta$. If $k_1, k_2 \geq K$, $|\lambda| \leq r$ and

$|\zeta| = 1$ then

$$\|(y^1_{k_1} + \zeta\lambda y^2_{k_1}) - (y^1_{k_2} + \zeta\lambda y^2_{k_2})\| \le \|y^1_{k_1} - y^1_{k_2}\| + |\zeta||\lambda|\|y^2_{k_1} - y^2_{k_2}\| \le \tfrac{1+r}{2^K} < \delta,$$

so

$$\|\phi(y^1_{k_1} + \zeta\lambda y^2_{k_1}) - \phi(y^1_{k_2} + \zeta\lambda y^2_{k_2})\| < \epsilon,$$

i.e.

$$\|\varphi_{k_1}(y^1_{k_1} + \zeta\lambda y^2_{k_1}) - \varphi_{k_2}(y^1_{k_2} + \zeta\lambda y^2_{k_2})\| < \epsilon.$$

Hence, from the Cauchy Integral Formula, for every $m \in \mathbb{N}$

$$\begin{aligned}
\|P^m\varphi_{k_1}(y^1_{k_1})(\lambda y^2_{k_1}) &- P^m\varphi_{k_2}(y^1_{k_2})(\lambda y^2_{k_2})\| \\
&= \|\tfrac{1}{2\pi i}\int_{|\zeta|=1} \frac{\varphi_{k_1}(y^1_{k_1} + \zeta\lambda y^2_{k_1}) - \varphi_{k_2}(y^1_{k_2} + \zeta\lambda y^2_{k_2})}{\zeta^{m+1}}\, d\zeta\| \\
&< \epsilon.
\end{aligned}$$

Now, since $\frac{1+r}{2^{K_1}} < R - r\|y^2\|$ we can find $s_1 > 1$ such that $s_1 r(\frac{1}{2^{K_1}} + \|y^2\|) < R - \frac{1}{2^{K_1}}$. Then, when $|t| \le s_1$, $k \ge K_1$ and $|\lambda| \le r$ we have that $y^1_k + t\lambda y^2_k \in B_Z$, because

$$\begin{aligned}
\|(y^1_k + t\lambda y^2_k) - y^1\| &\le \|y^1_k - y^1\| + |t|\|\lambda|\|y^2_k\| \\
&\le \tfrac{1}{2^{K_1}} + s_1 r(\tfrac{1}{2^{K_1}} + \|y^2\|) \\
&< R,
\end{aligned}$$

so from Cauchy's Inequality, for each $m \in \mathbb{N}$

$$\|P^m\varphi_k(y^1_k)(\lambda y^2_k)\| \le \frac{1}{s_1^m}\|\phi\|.$$

Thus, given $M \in \mathbb{N}$ and λ such that $|\lambda| \le r$,

$$\begin{aligned}
\|\phi(y^1 + \lambda y^2) - \sum_{m=0}^{M} \lim_{k\to\infty} (P^m \varphi_k(y_k^1)(y_k^2))\lambda^m\| \\
&= \| \lim_{k\to\infty} (\varphi_k(y_k^1 + \lambda y_k^2) - \sum_{m=0}^{M} (P^m \varphi_k(y_k^1)(y_k^2))\lambda^m)\| \\
&= \| \lim_{k\to\infty} \sum_{m=M+1}^{\infty} (P^m \varphi_k(y_k^1)(y_k^2))\lambda^m\| \\
&\le \sum_{m=M+1}^{\infty} \tfrac{1}{s_1^m}\|\phi\| = \tfrac{\|\phi\|}{s_1-1}\tfrac{1}{s_1^M},
\end{aligned}$$

which goes to zero as $M \to \infty$.

Thus ϕ is analytic on complex lines in B_Y.

$\square$

From the previous two steps, we obtain that T is a well defined mapping from $H(B_X)$ into $H(B_Y)$. Now, given $x^* \in X^*$, $y^1, y^2 \in B_Y$ and $\lambda \in \mathbb{C}$ such that $y^1 + \lambda y^2 \in B_Y$, we can find $\{y_k^1\}_k \subset B_{(\cup Y_n)}$ converging to y^1 and $\{y_k^2\}_k \subset B_{(\cup Y_n)}$ converging to y^2, and then

$$\begin{aligned}
Tx^*(y^1 + \lambda y^2) &= \lim_{k\to\infty} \lim_{n\in\mathcal{U}} x^*(S_n(y_k^1 + \lambda y_k^2)) \\
&= \lim_{k\to\infty} \lim_{n\in\mathcal{U}} (x^*(S_n(y_k^1)) + \lambda x^*(S_n(y_k^2))) \\
&= Tx^*(y^1) + \lambda Tx^*(y^2),
\end{aligned}$$

i.e. $Tx^* \in Y^*$. This shows that $TB_{X^*} = B_{Y^*}$.

Moreover, for every $f \in H(B_Y)$ and $y \in B_Y$, we can find $\{y_k\}_k \subset B_{(\cup Y_n)}$ converging to y, and thus

$$\begin{aligned}
T \circ Q^{\#}(f)(y) &= T(f \circ Q)(y) \\
&= \lim_{k\to\infty} \lim_{n\in\mathcal{U}} f \circ Q(S_n(y_k)) \\
&= \lim_{k\to\infty} f(y_k) \\
&= f(y),
\end{aligned}$$

so $T \circ Q^{\#} = I_{H(B_Y)}$.

Also, T is a homomorphism because T is clearly linear and for all $f, g \in H(B_X)$, $y \in B_Y$, we can find $\{y_k\}_k \subset B_{(\cup Y_n)}$ converging to y, so

$$\begin{aligned}
T(f \cdot g)(y) &= \lim_{k\to\infty} \lim_{n\in\mathcal{U}} f \cdot g(S_n(y_k)) \\
&= \lim_{k\to\infty} \lim_{n\in\mathcal{U}} f(S_n(y_k)) \cdot g(S_n(y_k)) \\
&= Tf(y) \cdot Tg(y) \\
&= (Tf \cdot Tg)(y).
\end{aligned}$$

Finally, for every $f \in H(B_X)$,

$$\|Tf\| = \sup_{y\in B_Y} |Tf(y)| = \sup_{y\in B_{(\cup Y_n)}} |\lim_{n\in\mathcal{U}} f(S_n(y))| \le \sup_{x\in B_X} |f(x)| = \|f\|,$$

and $\|T\| = \|T\|\|Q^{\#}\| \ge \|T \circ Q^{\#}\| = 1$. So $\|T\| = 1$.

$\square$

Theorem 2.3. *Let Y be a separable Banach space and $Y_1 \subset Y_2 \subset Y_3 \subset \dots$ an increasing sequence of finite dimensional subspaces whose union is dense in Y. Set $X = (\sum Y_n)_1$. Let H denote either the algebra A_u or the algebra H^∞. If $H(B_X)$*

satisfies the cluster value theorem at every $x^{**} \in \overline{B_X}^{**}$ *then* $H(B_Y)$ *satisfies the cluster value theorem at every* $y^{**} \in \overline{B_Y}^{**}$.

Proof. We know that $\hat{f}(M_{x^{**}}(B_X)) \subset Cl_{B_X}(f, x^{**})$, for all $f \in H(B_X)$ and $x^{**} \in \overline{B_X}^{**}$. Let us show that $\hat{g}(M_{y^{**}}(B_Y)) \subset Cl_{B_Y}(g, y^{**})$ for all $g \in H(B_Y)$ and $y^{**} \in \overline{B_Y}^{**}$.

Let $y^{**} \in \overline{B_Y}^{**}$, $\tau \in M_{y^{**}}(B_Y)$ and $g \in H(B_Y)$. Let T be the algebra homomorphism from $H(B_X)$ to $H(B_Y)$ constructed in Lemma 2.3. Then $x^{**} = y^{**} \circ T \in \overline{B_X}^{**}$, $Q^{\#}(g) \in H(B_X)$ and defining $\tilde{\tau} = \tau \circ T$, we see that $\tilde{\tau} \in M_{x^{**}}(B_X)$ because for all $x^* \in X^*$,

$$\tilde{\tau}(x^*) = \tau(Tx^*) =< y^{**}, Tx^* >=< x^{**}, x^* > .$$

Moreover,

$$\widehat{Q^{\#}(g)}(\tilde{\tau}) = \tilde{\tau}(Q^{\#}(g)) = \tau(T \circ Q^{\#}(g)) = \tau(g) = \hat{g}(\tau)$$

and

$$Cl_{B_X}(Q^{\#}(g), x^{**}) = Cl_{B_X}(g \circ Q, x^{**}) \subset Cl_{B_Y}(g, Q^{**}x^{**}) = Cl_{B_Y}(g, y^{**}),$$

so the theorem is established. □

Remark 2.1. A very special case of Theorem 2.3 is that if ℓ_1 satisfies the cluster value theorem, then so does L_1. We do not know for $1 < p \neq 2 < \infty$ whether the cluster value theorem for ℓ_p implies the cluster value theorem for L_p. Incidentally, in [5] it was proved that ℓ_p for p in this range satisfies the cluster value theorem at 0, but it is open whether L_p satisfies the cluster value theorem at any point of B_{L_p}.

Remark 2.2. The analogue of Theorem 2.3 for non separable spaces, which is proved by a non essential modification of the proof of Theorem 2.3, can be stated as follows. *Let Y be a Banach space and $(Y_\alpha)_{\alpha\in A}$ a family of finite dimensional subspaces of Y that is directed by inclusion and whose union is dense in Y. If $(\sum_{\alpha\in A} Y_\alpha)_1$ satisfies the cluster value theorem, then so does Y.*

Remark 2.3. There is a slight strengthening of Theorem 2.3. *Let Y, $(Y_n)_n$, and H be as in the statement of Theorem 2.3 and suppose that $(X_n)_n$ is a sequence so that X_n is $1+\epsilon_n$-isomorphic to Y_n and $\epsilon_n \to 0$. If $(\sum_n X_n)_1$ satisfies the cluster value theorem for the algebra H, then so does Y.* Now let (Z_n) be a sequence of finite dimensional spaces so that for every finite dimensional space Z and every $\epsilon > 0$, the space Z is $1+\epsilon$-isomorphic to one (and hence infinitely many) of the spaces Z_n. Set $C_1 = (\sum_n Z_n)_1$. As an immediate consequence of this slight improvement of Theorem 2.3 we get *If C_1 satisfies the cluster value theorem for H, then so does every separable Banach space.*

The proof of the improved Theorem 2.3 is essentially the same as the proof of the theorem itself. One just needs to define in Lemma 2.2 the mapping Q so that the conclusion of Lemma 2.2 remains true: For each n take an isomorphism $J_n : X_n \to Y_n$ so that for $x \in X_n$ the inequality $(1+\epsilon_n)^{-1}\|x\| \leq \|J_n x\| \leq \|x\|$ is valid, and define $Q(x_n)_n = \sum_n J_n x_n$ for $(x_n)_n$ in $(\sum_n X_n)_1$.

3. CONCLUSION: OPEN CLUSTER VALUE PROBLEMS

3.1 Cluster value problems for ℓ_1 and uniformly convex and uniformly smooth Banach spaces

There are plenty of open cluster value problems. The ideas of McDonald in [33] suggest that a solution to a $\bar{\partial}$ problem for the unit ball B of a uniformly convex space or ℓ_1 may help us solve a cluster value problem if the solution is weakly continuous. The $\bar{\partial}$ problem for the ball of a uniformly convex Banach space has not been solved yet, while the $\bar{\partial}$ problem for the unit ball of ℓ_1 has been solved positively by Lempert in [32] under certain conditions. Let us describe the $\bar{\partial}$ problem in open subsets of Banach spaces:

Let X and Y denote complex Banach spaces, and $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ denote the respective previous spaces seen as real Banach spaces. For every $m \in \mathbb{N}$, $L({}^m X_{\mathbb{R}}, Y_{\mathbb{R}})$ denotes the continuous m-linear mappings $A : X_{\mathbb{R}}^m \to Y_{\mathbb{R}}$, while $L^a({}^m X_{\mathbb{R}}, Y_{\mathbb{R}})$ denotes the continuous m-linear mappings $A : X_{\mathbb{R}}^m \to Y_{\mathbb{R}}$ that are *alternating*, i.e.

$$A(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) = (-1)^{\sigma} A(x_1, \cdots, x_m), \ \forall \sigma \in S_m \text{ and } x_1, \cdots, x_m \in X.$$

Also, given $m \in \mathbb{N}$ and $p, q \in \mathbb{N}_0$ such that $p + q = m$, $L^a({}^{p,q} X_{\mathbb{R}}, Y_{\mathbb{R}})$ is the subspace of $A \in L^a({}^m X_{\mathbb{R}}, Y_{\mathbb{R}})$ such that

$$A(\lambda x_1, \cdots, \lambda x_m) = \lambda^p \bar{\lambda}^q A(x_1, \cdots, x_m), \ \forall \lambda \in \mathbb{C} \text{ and } x_1, \cdots, x_m \in X;$$

while $L^{apq}({}^m X_{\mathbb{R}}, Y_{\mathbb{R}})$ denotes the subspace of all $A \in L({}^m X_{\mathbb{R}}, Y_{\mathbb{R}})$ which are alternating in the first p variables and are alternating in the last q variables.

The following definition can be found in [34, p.107].

Definition 3.1. *Let U be an open subset of the complex Banach space X and let $f : U \to Y$ be an $\mathbb{R}$-differentiable mapping. Let $Df(a)$ denote the real differential of f at a. Define $D'f(a)$ and $D''f(a)$ by*

$$D'f(a)(t) = 1/2[Df(a)(t) - iDf(a)(it)],$$

$$D''f(a)(t) = 1/2[Df(a)(t) + iDf(a)(it)],$$

for every $t \in X$. Note that $D'f(a)$ is $\mathbb{C}$-linear while $D''f(a)$ is $\mathbb{C}$-antilinear.

Given $A \in L({}^m X_{\mathbb{R}}, Y_{\mathbb{R}})$, define $A^a \in L^a({}^m X_{\mathbb{R}}, Y_{\mathbb{R}})$ by

$$A^a(x_1, \cdots, x_m) = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma A(x_{\sigma(1)}, \cdots, x_{\sigma(m)}),\ \forall x_1, \cdots, x_m \in X.$$

Given U an open subset of X and $p, q \in \mathbb{N}_0$, let $C^\infty_{p,q}(U, Y) := C^\infty(U, L^a({}^{p,q} X_{\mathbb{R}}, Y_{\mathbb{R}}))$. Then for each $f \in C^\infty_{p,q}(U, Y)$, $\bar{\partial} f \in C^\infty_{p,q+1}(U, Y)$ is given by

$$\bar{\partial} f(x) = (m+1)[D''f(x)]^a,\ \forall x \in U.$$

Remark 3.1. Since $D''f(x) \in L(X_{\mathbb{R}}, L^a({}^m X_{\mathbb{R}}, Y_{\mathbb{R}})) = L^{a1m}({}^{m+1} X_{\mathbb{R}}, Y_{\mathbb{R}})$, then Proposition 18.6 in [34] implies that $\forall t_1, \cdots, t_{m+1} \in X$,

$$\begin{aligned} \bar{\partial} f(x)(t_1, \cdots, t_{m+1}) &= (m+1)[D''f(x)]^{a1m}(t_1, \cdots, t_{m+1}) \\ &= \frac{m+1}{m+1} \sum_{\sigma \in S_{1m}} (-1)^\sigma D''f(x)(t_{\sigma(1)}, \cdots, t_{\sigma(m+1)}) \end{aligned}$$

where S_{1m} denotes the set of all permutations $\sigma \in S_{m+1}$ such that $\sigma(1) < \cdots < \sigma(m)$, so

$$\bar{\partial} f(x)(t_1, \cdots, t_{m+1}) = \sum_{j=1}^{m+1} (-1)^{j-1} D'' f(x)(t_j)(t_1, \cdots, t_{j-1}, t_{j+1}, \cdots, t_{m+1}).$$

The $\bar{\partial}$ problem for $g \in C^{\infty}_{p,q+1}(U, Y)$ asks whether the equation $\bar{\partial} g = 0$ implies the existence of $f \in C^{\infty}_{p,q}(U, Y)$ such that $\bar{\partial} f = g$.

Following the ideas of Lempert in [32] and of Kerzman in [30, pp.342-345], I aim to solve a cluster value problem for Banach spaces that are uniformly convex and uniformly smooth, and in general for those whose unit ball is strongly pseudoconvex. I will start with giving a definition of a strongly pseudoconvex domain in an infinite-dimensional Banach space.

3.2 Strong pseudoconvexity in infinite-dimensional Banach spaces

There are certain notions of pseudoconvexity in the literature ([34, p.274], [25, Theorem 2.6.12]) that hint towards a plausible extension of the definition of a strongly pseudoconvex domain to an infinite-dimensional Banach space (see [30] and [38]). As we will see, the following definitions build on the ideas exposed in Chapter VIII of [34].

Definition 3.2. *Let U be an open and bounded subset of a complex Banach space X. A function $f : U \to [-\infty, \infty)$ is said to be strictly plurisubharmonic if f is upper semicontinuous and for each $a \in U$ there exists $C(a) > 0$ such that*

$$C(a)\|b\|^2/4 \leq \frac{1}{2\pi} \int_0^{2\pi} (f(a + e^{i\theta} b) - f(a)) d\theta$$

for each $b \neq 0 \in X$ such that $a + \bar{\Delta} b \subset U$. If moreover $C = \inf_{a \in U} C(a) > 0$, we say that f is uniformly strictly plurisubharmonic.

Definition 3.3. *An open and bounded subset U of a complex Banach space X is said to be strictly pseudoconvex if the function $-\log d_U$ is strictly plurisubharmonic on U (where d_U denotes the distance to the boundary of U); and U is said to be uniformly strictly pseudoconvex if $-\log d_U$ is uniformly strictly plurisubharmonic on U.*

Proposition 3.1. *Let U be an open and bounded subset of a complex Banach space X, and let $f \in C^2(U, \mathbb{R})$. Then f is strictly plurisubharmonic if and only if for each $a \in U$ there exists $C(a) > 0$ such that $D'D''f(a)(b,b) \geq C(a)\|b\|^2$, for each $b \neq 0 \in X$ such that $a + \bar{\Delta}b \subset U$. And f is uniformly strictly plurisubharmonic if and only if there exists $C > 0$ such that $D'D''f(a)(b,b) \geq C\|b\|^2$, for each $a \in U$ and $b \neq 0 \in X$ such that $a + \bar{\Delta}b \subset U$.*

Proof. We will prove the first statement only (the second is proved similarly).

Suppose that f is strictly plurisubharmonic. Then for each $a \in U$ there exists $C(a) > 0$ such that $C(a)\|b\|^2/4 \leq \frac{1}{2\pi}\int_0^{2\pi}(f(a + e^{i\theta}b) - f(a))d\theta$ for each $b \neq 0 \in X$ such that $a + \bar{\Delta}b \subset U$. Fix $a \in U$ and $b \neq 0 \in X$ such that $a + \bar{\Delta}b \subset U$, and consider the function $u(\zeta) = f(a + \zeta b)$, which is defined on a disk $\Delta(0, R) \supset \bar{\Delta}$.

Then, by Taylor's Formula, for all $r \in (0, R)$,

$$\begin{aligned} u(re^{i\theta}) - u(0) = {} & r\cos\theta\frac{\partial u}{\partial x}(0) + r\sin\theta\frac{\partial u}{\partial y}(0) \\ & + \frac{r^2}{2}\cos^2\theta\frac{\partial^2 u}{\partial x^2}(s_{r,\theta}e^{i\theta}) + \frac{r^2}{2}\sin^2\theta\frac{\partial^2 u}{\partial y^2}(s_{r,\theta}e^{i\theta}) \\ & + r^2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}(s_{r,\theta}e^{i\theta}), \text{ for some } s_{r\theta} \in [0, r]. \end{aligned}$$

Thus, for all $r \in (0, R)$,

$$
\begin{aligned}
&C(a)\frac{r^2\|b\|^2}{4} \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}(f(a+e^{i\theta}rb)-f(a))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}(u(re^{i\theta})-u(0))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}(\frac{r^2}{2}\cos^2\theta\frac{\partial^2 u}{\partial x^2}(s_{r,\theta}e^{i\theta})+\frac{r^2}{2}\sin^2\theta\frac{\partial^2 u}{\partial y^2}(s_{r,\theta}e^{i\theta})+r^2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}(s_{r,\theta}e^{i\theta}))d\theta
\end{aligned}
$$

After dividing by $r^2/4$, by the Dominated Convergence Theorem we get

$$C(a)\|b\|^2 \leq \frac{1}{\pi}\pi(\frac{\partial^2 u}{\partial x^2}(0)+\frac{\partial^2 u}{\partial y^2}(0)) = \frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}}(0) = D'D''f(a)(b,b),$$

where the last equality comes from exercises 35.B and 35.D in [34].

Now suppose that for each $a \in U$ there exists $C(a) > 0$ such that $D'D''f(a)(b,b) \geq C(a)\|b\|^2$, for each $b \neq 0 \in X$ such that $a + \bar{\Delta}b \subset U$.

Fix a and b as before, and define $M(r) = \frac{1}{2\pi}\int_0^{2\pi}[f(a+re^{i\theta}b)-f(a)]d\theta$, for all $r \in (0,1]$. Consider also the function $u(\zeta) = f(a+\zeta b)$ defined on a disk $\Delta(0,R) \supset \bar{\Delta}$. Then, for all $\zeta \in \Delta(0,R)$,

$$\frac{\partial^2 u}{\partial x^2}(\zeta)+\frac{\partial^2 u}{\partial y^2}(\zeta) = \frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}}(\zeta) = D'D''f(a+\zeta b)(b,b) \geq C(a)\|b\|^2.$$

Since $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^u}{\partial\theta^2}$, then

$$\frac{1}{2\pi}\int_0^{2\pi}(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})u(re^{i\theta})d\theta \geq C(a)\|b\|^2,$$

i.e. $M''(r)+\frac{1}{r}M'(r) \geq C(a)\|b\|^2$, $\forall r \in (0,1)$.

Thus $(rM'(r) - C(a)\|b\|^2\frac{r^2}{2})' = rM''(r) + M'(r) - C(a)\|b\|^2 r \geq 0\ \forall r \in (0,1)$, so $r(M'(r) - \frac{C(a)\|b\|^2}{2}r)$ is an increasing function of r. Since clearly $r(M'(r) - \frac{C(a)\|b\|^2}{2}r) \to 0$ as $r \to 0$ (because M' is a bounded function on $(0,\epsilon)$ for some $\epsilon > 0$), we conclude that $r(M'(r) - \frac{C(a)\|b\|^2}{2}r) \geq 0$ for every $r \in (0,1)$. Hence $(M(r) - \frac{C(a)\|b\|^2}{4}r^2)' \geq 0$ for every $r > 0$, so $M(r) - \frac{C(a)\|b\|^2}{4}r^2$ is an increasing function of r. Since clearly $M(r) - \frac{C(a)\|b\|^2}{4}r^2 \to 0$ as $r \to 0$ then $M(r) \geq \frac{C(a)\|b\|^2}{4}r^2$ for each $r \in (0,1)$.

Since M is continuous on $(0,1]$, we conclude that $M(1) \geq \frac{C(a)\|b\|^2}{4}$, i.e. $\frac{1}{2\pi}\int_0^{2\pi}[f(a+e^{i\theta}b) - f(a)]d\theta \geq \frac{C(a)\|b\|^2}{4}$, i.e. f is strictly plurisubharmonic. □

Let us now exhibit some Banach spaces whose unit ball is uniformly strictly pseudoconvex. The following definition can be found in [11].

Definition 3.4. *If $0 < q < \infty$ and $2 \leq r < \infty$, a continuously quasi-normed space $(X, \|\|)$ is r-uniformly PL-convex if and only if there exists $\lambda > 0$ such that*

$$(\frac{1}{2\pi}\int_0^{2\pi} \|a + e^{i\theta}b\|^q d\theta)^{1/q} \geq (\|a\|^r + \lambda\|b\|^r)^{1/r}$$

for all a and b in X; we shall denote the largest possible value of λ by $I_{r,q}(X)$.

Davis, Garling and Tomczak-Jaegermann proved in [11, Propositon 3.1] that $I_{2,1}(\mathbb{C}) = 1/2$. Moreover, a simple modification of [11, Theorem 4.1] gives that $L_p(\Sigma, \Omega, \mu)$ is 2-uniformly PL-convex (for q=1) when $p \in [1,2]$, and actually $I_{2,1}(L_p) = I_{2,1}(\mathbb{C}) = 1/2$.

The following theorem gives us that $L_p(\Sigma, \Omega, \mu)$, for $p \in [1,2]$, has a uniformly strictly pseudoconvex unit ball.

Theorem 3.1. *If X is a 2-uniformly PL-convex Banach space for $q = 1$ then B_X is uniformly strictly pseudoconvex.*

Proof. Let $a \in B_X$ and $b \neq 0 \in X$ such that $a + \bar{\Delta} b \subset B_X$. Since $a + \bar{\Delta} b$ is compact, there exists $r \in (0,1)$ such that $a + \bar{\Delta} b \subset rB_X$. Hence,

$$(\|a\|^2 + I_{2,1}(X)\|b\|^2)^{1/2} \leq \frac{1}{2\pi}\int_0^{2\pi} \|a + e^{i\theta}b\| d\theta \leq r.$$

Let $\lambda_0 = \sqrt{I_{2,1}(X)+1} - 1 > 0$. Observe that $\|a\| + \lambda_0\|b\|^2 \leq (\|a\|^2 + I_{2,1}(X)\|b\|^2)^{1/2}$ because $(\|a\| + \lambda_0\|b\|^2)^2 = \|a\|^2 + 2\lambda_0\|a\|\|b\|^2 + \lambda_0^2\|b\|^4 \leq \|a\|^2 + (2\lambda_0 + \lambda_0^2)\|b\|^2 = \|a\|^2 + ((\lambda_0+1)^2 - 1)\|b\|^2 = \|a\|^2 + I_{2,1}(X)\|b\|^2$. Thus,

$$\|a\| + \lambda_0\|b\|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \|a + e^{i\theta}b\| d\theta \leq r.$$

Since $x \mapsto -\log(1-x)$ is a convex and increasing function on $[0,1)$, an application of Jensen's inequality gives us that

$$-\log(1 - (\|a\| + \lambda_0\|b\|^2)) \leq \frac{1}{2\pi}\int_0^{2\pi} -\log(1 - \|a + e^{i\theta}b\|) d\theta.$$

We will finish the proof by showing that $C = 2\lambda_0 > 0$ satisfies

$$C\|b\|^2/4 - \log(1 - \|a\|) \leq -\log(1 - (\|a\| + \lambda_0\|b\|^2)) \tag{3.1}$$

given that $\|a\| + \lambda_0\|b\|^2 \leq r < 1$.

Letting $t = \|a\|$ and $s = \|b\|^2$, we can see that (3.1) holds in case

$$\beta := \inf\{\log(\frac{1-t}{1-t-\lambda_0 s})/s : \; t \geq 0, s > 0, t + \lambda_0 s \leq r < 1\} \geq \lambda_0/2.$$

Indeed, first note that for $1 > r > t \geq 0$ and $s > 0$,

$$\frac{1-t}{1-t-\lambda_0 s} \geq \frac{1-t+\lambda_0 s}{1-t} = 1 + \frac{\lambda_0 s}{1-t}.$$

Hence $\beta \geq \inf_{t\in[0,r)} \inf_{s\in(0,\frac{r-t}{\lambda_0}]} \log(1+\frac{\lambda_0 s}{1-t})/s$.

Now, for $t \in [0,r)$ fixed, $\log(1+\frac{\lambda_0 s}{1-t})/s$ is decreasing as a function of $s \in (0, \frac{r-t}{\lambda_0}]$ because

$$\frac{d}{ds}\{\log(1+\frac{\lambda_0 s}{1-t})/s\} = \frac{-\log(1+\frac{\lambda_0 s}{1-t})}{s^2} + \frac{1}{s(1+\frac{\lambda_0 s}{1-t})} \cdot \frac{\lambda_0}{1-t}$$

where the inequality $\frac{1}{x+1} < \log(1+\frac{1}{x})$ for $x > 0$ gives us

$$s^2 \cdot \frac{1}{s(1+\frac{\lambda_0 s}{1-t})} \cdot \frac{\lambda_0}{1-t} = \frac{1}{\frac{1-t}{\lambda_0 s}+1} < \log(1+\frac{\lambda_0 s}{1-t})$$

i.e. $\frac{d}{ds}\{\log(1+\frac{\lambda_0 s}{1-t})/s\} < 0$ for every $s \in (0, \frac{r-t}{\lambda_0}]$.

Consequently $\beta \geq \inf_{t\in[0,r)} \lambda_0 \frac{\log(1+\frac{r-t}{1-t})}{r-t}$.

Finally, $\frac{\log(1+\frac{r-t}{1-t})}{r-t}$ is increasing as a function of $t \in [0,r)$ because

$$\begin{aligned}
\frac{d}{dt}\{\frac{\log(2-\frac{1-r}{1-t})}{r-t}\} &= \frac{\log(2-\frac{1-r}{1-t})}{(r-t)^2} + \frac{1}{r-t} \cdot \frac{1}{2-\frac{1-r}{1-t}} \cdot \frac{-(1-r)}{(1-t)^2} \\
&= \frac{1}{(r-t)^2} \cdot (\log(2-\frac{1-r}{1-t}) - \frac{(1-r)(r-t)}{(2-\frac{1-r}{1-t})(1-t)^2}) \\
&= \frac{1}{(r-t)^2} \cdot (\log(2-\frac{1-r}{1-t}) - \frac{(1-r)(r-t)}{((1-t)+(r-t))(1-t)}) \\
&= \frac{1}{(r-t)^2} \cdot (\log(1+\frac{1-r}{1-t}) - \frac{(1-r)}{(1-t)} \cdot \frac{1}{(\frac{1-t}{r-t}+1)}) \\
&> \frac{1}{(r-t)^2} \cdot (\log(1+\frac{1}{(\frac{1-t}{1-r})}) - \frac{1}{(\frac{1-t}{r-t}+1)}) \\
&> 0,
\end{aligned}$$

because $\log(1+\frac{1}{x}) > \frac{1}{x+1}$ whenever $x > 0$.

Therefore $\beta \geq \lambda_0 \frac{\log(1+r)}{r} \geq \frac{\lambda_0}{1+r} \geq \frac{\lambda_0}{2}$, as we wanted to show. $\square$

3.3 Cluster value problem for $H^\infty(B)$ and $\bar{\partial}$ problem for B for $\dim(X) < \infty$

I am also interested in the cluster value problem for $H^\infty(B)$ when B is the ball of a finite-dimensional Banach space, and in particular for the ball of ℓ_1^n. I anticipate that a solution to the $\bar{\partial}$ problem for the ball of a finite-dimensional space, such as $B_{\ell_1^n}$, interesting in its own light, may hint to a solution of the respective cluster value problem for ℓ_1 and even ℓ_1-sums of finite-dimensional spaces. I intend to investigate whether Lemma 3.1 in [32] can be strengthened to a continuous solution in $\bar{B}$, as that would guarantee a solution to the respective cluster value problem for $H^\infty(B)$.

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