CLUSTER VALUE PROBLEMS IN INFINITE-DIMENSIONAL SPACES

A Dissertation

by

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ABSTRACT

In this dissertation we study cluster value problems for Banach algebras \( H(B) \) of analytic functions on the open unit ball \( B \) of a Banach space \( X \) that contain \( X^* \) and 1. Solving cluster value problems requires understanding the cluster set of a function \( f \in H(B) \). For the Banach spaces \( X \) we focus on, such as those with a shrinking reverse monotone Finite Dimensional Decomposition and \( C(K) \), we prove cluster value theorems for a Banach algebra \( H(B) \) and a point \( x^{**} \in \bar{B}^{**} \). In doing so, we apply standard methods and results in functional analysis; in particular we use the facts that projections from \( X \) onto a finite-codimensional subspace equal \( I_X \) minus a finite rank operator and that \( C(K)^* = \ell_1(K) \) when \( K \) is compact, Hausdorff and dispersed.

We also prove that for any separable Banach space \( Y \), a cluster value problem for \( H(B_Y) \) \( (H = H^\infty \) or \( H = A_u) \) can be reduced to a cluster value problem for \( H(B_X) \) for some Banach space \( X \) that is an \( \ell_1 \)-sum of a sequence of finite-dimensional spaces. The proof relies on the construction of an isometric quotient map from a suitable \( X \) to \( Y \) that induces an isometric algebra homomorphism from \( H(B_Y) \) to \( H(B_X) \) with norm one left inverse. The left inverse is built using ultrafilter techniques. Other tools include the infinite-dimensional version of the Schwarz lemma and familiar one complex variable results such as Cauchy’s inequality and Montel’s theorem.

We conclude this work by describing the related \( \bar{\partial} \) problem and defining strong pseudoconvexity as well as uniform strong pseudoconvexity in the context of Banach spaces. Our last result is that 2-uniformly PL-convex Banach spaces have a uniformly strictly pseudoconvex unit ball. In future research we will study the \( \bar{\partial} \) problem in uniformly strictly pseudoconvex unit balls and in the open unit ball of finite-
dimensional Banach spaces such as the ball of $\ell_1^n$. 
DEDICATION

To my family and friends, for seasoning my life during these years that I have been a Ph.D. student at Texas A&M University.
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I would like to start by thanking my advisor and committee members for their valuable time put into helping me see beyond my ideas. I also want to thank the many instructors I have had during my Ph.D. for providing me with tools to be a sharp mathematician. Finally, a special thank you to all the colleagues I got to meet during these years as a graduate student at Texas A&M University and for being friends that have really understood my daily struggles as well as joys. I hope to keep learning from and with all of you.
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1. INTRODUCTION: MAIN CONCEPTS

1.1 Some particular Banach algebras

The research I present in this thesis is focused on some problems in Functional Analysis combined with Complex Analysis. To be specific, I study certain Banach algebras of bounded analytic functions on the open unit ball $B$ of a complex Banach space $X$ that contain $X^*$ (the continuous linear functionals on $X$) and 1. Indeed, every element of $X^*$ acts linearly and continuously on $B$, thus each element of $X^*$ is a bounded analytic function on $B$ (the definition of an analytic function is discussed in detail later on).

From here on, $H(B)$ will denote any such algebra. One example is the algebra $H^\infty(B)$ of all bounded analytic functions on $B$. The study of $H^\infty(B)$ is widely spread in mathematics, as in $H^\infty$ control theory, $H^\infty$ functional calculus, etc. Another well known example is the disk algebra, the analytic functions on the ball $B$ of $\mathbb{C}^n$ that extend continuously to $\bar{B}$. There are two generalizations of the disk algebra to the infinite-dimensional case: $A_u(B)$, the uniformly continuous, bounded and analytic functions on $B$, and $A(B)$, the uniform limits on $B$ of polynomials in the functions in $X^*$. These algebras satisfy the inclusions $A(B) \subset A_u(B) \subset H^\infty(B)$. In particular $A(B) = A_u(B)$ when $X$ is finite-dimensional (because each $f \in A_u(B)$ is the uniform limit of polynomials and every polynomial on a finite-dimensional space is in $A(B)$), while $A_u(B) \subset H^\infty(B)$ for every Banach space $X$ (See [6, 90-92], or check that $B \circ x^* \in H^\infty(B) \setminus A_u(B)$ for $x^* \in X^*$ of norm one with $x^*(x_1) = \|x_1\| = 1$ and $B$ a Blaschke product with zeros dense in $\partial\Delta$). Examples of infinite-dimensional spaces with $A(B) = A_u(B)$ include the $C(K)$ spaces for $K$ compact, Hausdorff and dispersed (see section 3 in [27]), while examples of spaces with $A(B) \subset A_u(B)$
include $\ell_1$, $L_1$, $C(K)$ spaces for $K$ compact, Hausdorff and not dispersed like $\ell_\infty$, $L_\infty$ (see Proposition 2.36 in [13] and the Main Theorem in [35]), and $\ell_p$, $L_p$ for $1 < p < \infty$ (because for a fixed integer $n \geq p$ and $(e_k)_k$ the canonical basis of $\ell_p$, $f(\sum a_k e_k) = \sum a_k^p$ is uniformly continuous on $B_{\ell_p}$ and analytic on complex lines, so $f \in A_u(B_{\ell_p})$; moreover $f(e_k) = 1$ for each $k \in \mathbb{N}$, while $g(e_k) \to 0$ when $g \in A(B)$, implying that $f \notin A(B)$. Then $f \circ P \in A_u(B_{L_p}) \setminus A(B_{L_p})$, for $P$ a norm one projection of $L_p$ onto a subspace that is isometric to $\ell_p$).

One of the most important topics in the study of Banach algebras $H(B)$ is the study of its set of characters, the nonzero homomorphisms from $H(B)$ to $\mathbb{C}$, called the spectrum of $H(B)$, and denoted by $M_{H(B)}$. Since our Banach algebras $H(B)$ are commutative and have an identity, the spectrum $M_{H(B)}$ is a compact Hausdorff space ([21, Theorem 2.5]). The study of the spectrum is simplified by fibering it over $\bar{B}^{**}$ (the closed unit ball of $X^{**}$) via the surjective mapping $\pi : M_{H(B)} \to \bar{B}^{**}$ given by $\pi(\tau) = \tau|_{X^{**}}$. Indeed, $\pi$ is well-defined because every $\tau \in M_{H(B)}$ is linear, continuous and of norm one; and $\pi$ is surjective because $\pi(\delta_x) = x$ for every $x \in B$ (where $\delta_x : H(B) \to \mathbb{C}$ is defined by $f \mapsto f(x)$), and since $M_{H(B)}$ is compact, $\bar{B}^{**} = \bar{B}^{w^*} \subset \pi(M_{H(B)})$.

Another related topic of interest is the Gelfand Transform: Given $f \in H(B)$, the Gelfand Transform of $f$ is the continuous map $\hat{f} : M_{H(B)} \to \mathbb{C}$ given by $\tau \mapsto \tau(f)$. The Gelfand Transform is a generalization of the Fourier Transform for $L_1(\mathbb{R})$ under convolution.

1.2 The Corona problem

One of the big open problems in the study of algebras $H(B)$ is the Corona problem on the ball, which asks whether the open unit ball of a Banach space $X$ is dense (in the weak-star topology) in $M_{H(B)}$. Note that $B$ can be seen as a subset of $M_{H(B)}$ via the
mapping $\delta : B \rightarrow M_{H(B)}$ such that $x \mapsto \delta_x$. An example of a character in $M_{H(B)} \setminus B$ can be found in Example 2.1 later on. An equivalent formulation of the Corona problem (see [24, p. 163]) is that whenever $f_1, \cdots, f_n \in H(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \epsilon > 0$ on $B$, there exist $g_1, \cdots, g_n \in H(B)$ such that $f_1g_1 + \cdots + f_ng_n = 1$.

Carleson solved the Corona problem positively for the unit disk in $\mathbb{C}$ in 1962 [9]. In 1970 Gamelin [17] discussed the corona problem for other planar domains, solving it in cases that include finitely connected planar domains, while in 1985 Garnett and Jones [23] solved positively the Corona problem for connected open subsets of the extended complex plane whose complement is a subset of the real axis. Garnett poses corona problems for other interesting planar domains in [22]. In higher dimensions, around 1970 Cole constructed an open Riemann surface which is a counterexample to a Corona theorem [20]. Even more, Sibony produced counterexamples in 1987 in [37] and in 1993 in [16] to Corona theorems in $\mathbb{C}^3$, and then in $\mathbb{C}^2$, of domains that are pseudoconvex, and strictly pseudoconvex except at one point (for example, convex sets are pseudoconvex).

As Krantz summarizes in [31], there is no domain known in the plane $\mathbb{C}$ for which the corona problem is known to fail, and there is no domain known in $\mathbb{C}^n$, for $n \geq 2$, on which the corona problem is known to hold true. In particular, it is a challenging open problem to determine whether the Corona problem holds true for the unit ball or polydisk in $\mathbb{C}^n$ for $n \geq 2$.

1.3 Cluster value problems

This thesis work deals mainly with another set of big open problems: cluster value problems. Cluster value problems are related to the Corona problem (a positive answer to a Corona problem gives a positive answer to the corresponding cluster value problem, or equivalently, a negative answer to a cluster value problem would yield
a counterexample to a Corona theorem). Roughly speaking, cluster value problems involve the understanding of the cluster sets of a function \( f \in H(B) \), that is, the limits of \( f \) over weak* convergent nets in \( B \), when \( B \) is seen as a subset of \( X^{**} \) with the weak-star topology. They also involve comparing such limits with the spectrum of \( H(B) \) evaluated at \( f \).

Kakutani [29] was among the first ones to consider cluster value problems in 1955 for domains in the complex plane, followed by I. J. Schark [26] in 1961 for the unit disk \( \Delta \) of the complex plane. I. J. Schark gave an explicit identification of the cluster values of a function \( f \) at a point \( x \) in the boundary of \( \Delta \) with the fiber over \( x \) of the spectrum evaluated at \( f \):

**Theorem 1.1.** Let \( f \in H^\infty(\Delta) \) and \( x \in \partial \Delta \). If \( M_x = \{ \tau \in M_{H^\infty(\Delta)} : \tau(id) = x \} \), then the range of \( \hat{f} \) on \( M_x \) consists of those complex numbers \( \zeta \) for which there is a sequence \( \{ \lambda_n \} \) in \( \Delta \) with \( \lambda_n \to x \) and \( f(\lambda_n) \to \zeta \).

In 1973 Gamelin [18] proved a cluster value theorem for the polydisk in \( \mathbb{C}^n \). Moreover, he proved it for finite products of open sets in \( \mathbb{C} \). Then McDonald proved in 1979 in [33] a cluster value theorem for the Euclidean unit ball in \( \mathbb{C}^n \), and actually for any strongly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary. His proof relied on a solution by Kerzman from 1971 in [30, 342-345] to a \( \bar{\partial} \) problem in a strongly pseudoconvex domain.

Being precise, we say a finite-dimensional cluster value theorem for \( B \subset \mathbb{C}^n \) holds when the following occurs: Suppose \( f \in H^\infty(B) \), \( x \in \partial B \) and \( \alpha \in \mathbb{C} \). Let \( M_x = \{ \tau \in M_{H^\infty(B)} : \tau|_{\Lambda(B)} = \delta_x \} \). There exists \( m \in M_x \) such that \( m(f) = \alpha \) if and only if there is a sequence \( \{ x_k \}_k \) in \( B \) converging to \( x \) such that \( f(x_k) \) tends to \( \alpha \).

Before discussing cluster value theorems for the unit ball of an infinite-dimensional Banach space, let us overview the basic theory of holomorphic functions on arbitrary
Banach spaces.

1.4 Complex analysis on Banach spaces

Given $U$ an open subset of a complex Banach space $X$, following Mujica [34], we say that $f : U \subset X \to \mathbb{C}$ is an analytic (or holomorphic) function on $U$ if for every $x \in U$ there exists $r > 0$ and continuous polynomials on $X$, $(P^m f(x))_{m=0}^\infty$, where $P^m f(x)$ is $m$-homogeneous, such that, if $\|y - x\| < r$ then $f(y) = \sum_{m=0}^\infty P^m f(x)(y - x)$, and the convergence is uniform on the ball of radius $r$ around $x$. The radius of convergence of $f$ at $x$, $r_c f(x)$, is the supremum of the radius of balls for which the power series around $x$ converges uniformly. Similarly, the radius of boundedness of $f$ at $x$, $r_b f(x)$, is the supremum of the radius of balls centered at $x$ and contained in $U$ on which $f$ is bounded.

An $m$-homogeneous polynomial $\hat{L}$ on $X$, for $m \in \mathbb{N}$, is the restriction to the diagonal of an $m$-linear mapping $L : X^m \to \mathbb{C}$, i.e. $\hat{L}(x) = L(x, x, \ldots, x)$ (and it is a constant function for $m = 0$).

The following two formulas are very useful (Theorems 4.3 and 7.13 in [34]).

**Proposition 1.1.** [Cauchy-Hadamard Formula] Let $U$ be an open subset of a complex Banach space $X$. If $f : U \subset X \to \mathbb{C}$ is analytic then for each $x \in U$, $1/r_c f(x) = \limsup_{m\to\infty} \|P^m f(x)\|^{1/m}$.

**Proof.** Let $x \in U$. We will first show that $1/r_c f(x) \geq \limsup_{m\to\infty} \|P^m f(x)\|^{1/m}$.

Let $r \in (0, r_c f(x))$. Then $f(y) = \sum_{m=0}^\infty P^m f(x)(y - x)$, and the convergence is uniform for $y \in B(x, r)$. Hence we can choose $m_0 \in \mathbb{N}$ such that

$$\|\sum_{j=0}^m P^j f(x)(y - x) - f(y)\| \leq 1, \quad \forall m \geq m_0 \text{ and } y \in B(x, r).$$
Thus \( \|P^m f(x)(t)\| \leq 2 \) for all \( m > m_0 \) and \( t \in B(0,r) \). Consequently \( \|P^m f(x)\| \leq 2r^{-m} \) for all \( m > m_0 \) and therefore \( \limsup_{m \to \infty} \|P^m f(x)\|^{1/m} \leq 1/r \).

Letting \( r \to r_c f(x) \) we get that \( 1/r_c f(x) \geq \limsup_{m \to \infty} \|P^m f(x)\|^{1/m} \).

Let us now show that \( 1/r_c f(x) \leq \limsup_{m \to \infty} \|P^m f(x)\|^{1/m} \).

Let \( R = \limsup_{m \to \infty} \|P^m f(x)\|^{1/m} \) and assume that \( R < \infty \). Choose \( r \in (0,1/R) \), \( s \in (r,1/R) \) and \( m_0 \in \mathbb{N} \) such that \( \|P^m f(x)\|^{1/m} < 1/s \) for all \( m \geq m_0 \). Then \( \|P^m f(x)(y-x)\| \leq (r/s)^m \) for all \( m \geq m_0 \) and \( y \in B(x,r) \), so the series \( \sum_{m=0}^{\infty} P^m f(x)(y-x) \) converges uniformly for \( y \in B(x,r) \). Thus \( r_c f(x) \geq r \). Letting \( r \to 1/R \) we obtain that \( r_c f(x) \geq 1/R \), i.e. \( 1/r_c f(x) \leq R = \limsup_{m \to \infty} \|P^m f(x)\|^{1/m} \).

**Proposition 1.2.** Let \( U \) be an open subset of a complex Banach space \( X \). If \( f : U \subset X \to \mathbb{C} \) is analytic then for each \( x \in U \), \( r_b f(x) = \min \{ r_c f(x), d_U(x) \} \) (where \( d_U(x) \) denotes the distance from \( x \) to the boundary of \( U \)).

**Proof.** Let \( x \in U \). Clearly \( d_U(x) = \sup \{ r > 0 : B(x,r) \subset U \} \), so \( r_b f(x) \leq d_U(x) \).

Thus to show \( r_b f(x) \leq \min \{ r_c f(x), d_U(x) \} \) it is enough to show \( r_b f(x) \leq r_c f(x) \).

Let \( r \in (0,r_b f(x)) \). Then \( \bar{B}(x,r) \subset U \) and \( f \) is bounded on \( \bar{B}(x,r) \), say by \( C \). It follows from Proposition 1.8 below (Cauchy Inequality) that \( \|P^m f(x)\| \leq C r^{-m} \) for all \( m \in \mathbb{N}_0 \), so by the Cauchy-Hadamard Formula we get that \( r_c f(x) = 1/\limsup_{m \to \infty} \|P^m f(x)\|^{1/m} \geq r \). Letting \( r \to r_b f(x) \) we obtain \( r_c f(x) \geq r_b f(x) \).

Let us now show that \( r_b f(x) \geq \min \{ r_c f(x), d_U(x) \} \). Choose \( 0 < r < s < \min \{ r_c f(x), d_U(x) \} \). Then \( B(x,s) \subset U \) and \( \sum_{m=0}^{\infty} P^m f(x)(y-x) \) converges for every \( y \in B(x,s) \). Thus, by Proposition 1.3 below (Identity Principle) we get that \( f(y) = \sum_{m=0}^{\infty} P^m f(x)(y-x) \), for all \( y \in B(x,s) \). Moreover, from the Cauchy-Hadamard Formula, \( \limsup_{m \to \infty} \|P^m f(x)\|^{1/m} = \frac{1}{r_c f(x)} < \frac{1}{s} \), so there exists \( C > 1 \)}
such that \( \|P_m f(x)\| \leq \frac{C}{m} \) for all \( m \in \mathbb{N}_0 \). Thus \( \|f(y)\| \leq C \sum_{m=0}^{\infty} \left( \frac{r}{s} \right)^m \) for all \( y \in B(x,r) \). Hence \( r_b f(x) \geq r \), and letting \( r \to \min\{r_c f(x), d_U(x)\} \) we get that \( r_b f(x) \geq \min\{r_c f(x), d_U(x)\} \).

Gamelin defines an analytic function in [19] differently. However, Proposition 8.6 and Theorem 8.7 in [34] prove that these definitions are equivalent. Moreover, Theorem 13.16 in [34] proves that analyticity is equivalent to \( \mathbb{C} \)-differentiability. Let us summarize these results in the following theorem.

**Theorem 1.2.** Given an open subset \( U \) of a Banach space \( X \) and \( f : U \subset X \to \mathbb{C} \), the following are equivalent:

(i) \( f \) is analytic,

(ii) \( f \) is continuous and its restriction to every complex one-dimensional affine subspace of \( X \) is analytic, i.e. for every \( x_0 \in U \) and direction \( x \in X \), the function \( \lambda \to f(x_0 + \lambda x) \) depends analytically on \( \lambda \) for \( \lambda \in \{ \zeta : x_0 + \zeta x \in U \} \),

(iii) \( f \) is locally bounded and its restriction to every complex one-dimensional affine subspace of \( X \) is analytic,

(iv) \( f \) is continuous and \( f|_{U \cap M} \) is analytic for each finite-dimensional subspace \( M \) of \( X \),

(v) \( f \) is Fréchet \( \mathbb{C} \)-differentiable, i.e. for each point \( x \in U \) there exists a \( \mathbb{C} \)-linear mapping \( L \in X^* \) such that \( \lim_{y \to x} \frac{\|f(y)-f(x)-L(y-x)\|}{\|y-x\|} = 0 \).

**Proof.** (ii) \( \Rightarrow \) (iii) Continuity clearly implies local boundedness.

(iii) \( \Rightarrow \) (ii) Let us first note that Proposition 1.9 below (Schwarz’ Lemma) still holds under our condition that the restriction of \( f \) to every complex one-dimensional affine subspace of \( X \) is analytic.
Now let \( x \in U \). Since \( f \) is locally bounded, there exists \( r > 0 \) such that \( f \) is bounded in \( B(x, r) \), say by \( C \). Then, by Schwarz’ Lemma,

\[
\|f(y) - f(x)\| \leq 2C\|y - x\|, \quad \forall y \in B(x, r).
\]

This proves that \( f \) is continuous at the point \( x \).

\((i) \Rightarrow (v)\) Let \( x \in U \) and choose \( r \in (0, r_b f(x)) \). Then \( f \) is bounded on \( B(x, r) \), say by \( C \). Moreover, since \( r_b f(x) \leq r_c f(x) \), we have that for all \( y \in B(x, r) \),

\[
f(y) = f(x) + P^1 f(x)(y - x) + \sum_{m=2}^{\infty} P^m f(x)(y - x),
\]

and so

\[
\limsup_{m \to \infty} \frac{\|f(y) - f(x) - P^1 f(x)(y - x)\|}{\|y - x\|} \leq \limsup_{m \to \infty} \sum_{m=2}^{\infty} \|P^m f(x)(\frac{y - x}{\|y - x\|})\||y - x|^{m-1}.
\]

And by Proposition 1.8 below (Cauchy Inequality), we have that for all \( m \geq 2 \) and \( y \in B(x, r) \setminus \{x\} \),

\[
\|P^m f(x)(\frac{y - x}{\|y - x\|})\| \leq r^{-m} \sup_{\|\zeta\| = r} \|f(x + \zeta(\frac{y - x}{\|y - x\|}))\| \leq Cr^{-m},
\]

so

\[
\limsup_{m \to \infty} \sum_{m=2}^{\infty} \|P^m f(x)(\frac{y - x}{\|y - x\|})\||y - x|^{m-1} \leq \limsup_{m \to \infty} \sum_{m=2}^{\infty} C\|y - x\|(\frac{\|y - x\|}{r})^m
\]

\[
= \limsup_{m \to \infty} \frac{C}{r^2}\|y - x\|(\frac{1}{1 - \|y - x\|}) = 0.
\]

Hence \( f \) is Fréchet \( C \)-differentiable.

\((v) \Rightarrow (ii)\) Let \( x_0 \in U \) and \( x \neq 0 \in X \). Since \( f \) is Fréchet \( C \)-differentiable, by
the Chain Rule also \( g(\lambda) = f(x_0 + \lambda x) \) is \( \mathbb{C} \)-differentiable on \( \Omega = \{ \zeta : x_0 + \zeta x \in U \} \).

Thus \( g \) is analytic, by the corresponding one complex variable result.

Hence the restriction of \( f \) to every complex one-dimensional affine subspace of \( X \) is analytic. And the Fréchet \( \mathbb{C} \)-differentiability of \( f \) clearly implies that \( f \) is continuous.

\((ii) \Rightarrow (iv)\) Let \( M \) be a finite-dimensional subspace of \( X \). Let \( x_0 \in U \cap M \) and let \( \{x_1, \ldots, x_n\} \) be a basis for \( M \). Then, following the proof of Proposition 1.7 (applied \( n \) times as in Corollary 7.8 in [34]) under our condition that \( f \) is continuous and its restriction to each complex line is analytic, we obtain that

\[
f(x_0 + \lambda_1 x_1 + \cdots + \lambda_n x_n) = \sum_{\alpha} c_{\alpha} \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n},
\]

and the series converges absolutely and uniformly on some polydisk \( \Delta^n(0, r) \).

Then, for each \( m \in \mathbb{N}_0 \) define \( P_m : M \to \mathbb{C} \) by

\[
P_m(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \sum_{|\alpha|=m} c_{\alpha} \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}
\]

so clearly \( P_m \) is an \( m \)-homogeneous polynomial and

\[
f(x_0 + \lambda_1 x_1 + \cdots + \lambda_n x_n) = \sum_{m=0}^{\infty} P_m(\lambda_1 x_1 + \cdots + \lambda_n x_n),
\]

where the series converges uniformly on \( \Delta^n(0, r) \).

Thus \( f|_{U \cap M} \) is analytic.

\((iv) \Rightarrow (i)\) Let \( x_0 \in U \) and \( r > 0 \) such that \( B(x_0, r) \subset U \). Then, if \( M \) is a
finite-dimensional subspace of $X$ containing $x_0$, by Proposition 1.2 applied to $f|_{U \cap M}$,

$$f(x) = \sum_{m=0}^{\infty} P_M^m(x - x_0), \quad \forall x \in M \cap B(x_0, r)$$

where each $P_M^m$ is an $m$-homogeneous polynomial.

Hence, if $M$ and $N$ are two finite-dimensional subspaces of $X$ containing $x_0$ then $P_M^m(t) = P_N^m(t) \forall t \in M \cap N, m \in \mathbb{N}_0$ (by an induction argument similar to the one given after Lemma 4.5 in [34]).

Thus, defining $P_m : X \to \mathbb{C}$ by $P_m(t) = P_M^m(t)$, where $M$ is any finite-dimensional subspace of $X$ containing $x_0$ and $t$, we get that $P_m$ is an $m$-homogeneous polynomial (after using a Polarization Formula as the one in Theorem 1.10 in [34]) and

$$f(x) = \sum_{m=0}^{\infty} P_m(x - x_0), \quad \forall x \in B(x_0, r).$$

Now since $f$ is continuous at $x_0$ there exists $s < r$ such that $f$ is bounded on $\bar{B}(x_0, s)$, say by $C$. Thus, by Proposition 1.8 below (Cauchy Inequality) applied to $f|_{U \cap M}$ for some finite-dimensional subspace $M$ of $X$ containing $x_0$ and $t \in B_X$,

$$\|P_m(t)\| \leq C s^{-m}, \quad \forall m \in \mathbb{N}_0.$$

Hence each $P_m$ is continuous and the power series $\sum_{m=0}^{\infty} P_m(x - x_0)$ converges uniformly for $x \in B(x_0, s)$, for all $s_0 < s$. \qed

Some examples [34, Section 5] of analytic functions are the following:

(i) Polynomials (finite sums of $n$-homogeneous polynomials),

(ii) Power series $\sum_{m=0}^{\infty} P_m(x)$ with infinite radius of convergence.
(iii) Power series of the form $\sum_{m=0}^{\infty} (x_m^*)^m$, where $\{x_m^*\} \subset X^*$ and $x_m^* \xrightarrow{w^*} 0$.

Also, [34, Sections 5,7] extends the following classical properties to the infinite-dimensional setting.

**Proposition 1.3.** [Identity Principle] If $f$ is analytic on a connected open set $U$ and identically zero on a nonvoid open set $V \subset U$, then $f$ is identically zero on all of $U$.

**Proof.** (a) First assume $U$ is an open ball in $X$.

Let $x \in U$ and $y \in V$, and let $\Omega = \{\lambda \in \mathbb{C} : y + \lambda(x - y) \in U\}$. Clearly $\Omega$ is a convex open set that contains 0 and 1, and in particular it is connected. Hence $g : \Omega \to \mathbb{C}$ given by $g(\lambda) = f(y + \lambda(x - y))$ is holomorphic and identically zero on a neighborhood of zero. Thus $g$ is identically zero on all of $\Omega$ by the one complex variable Identity Principle. In particular $f(x) = g(1) = 0$. Thus $f$ is identically zero on all of $U$.

(b) In the general case consider $A = \{a \in U : \exists r > 0 \text{ s.t. } f|_{B(a,r)} \equiv 0\}$. $A$ is clearly an open set, and it turns out to be closed in $U$ as well, because if $(a_n) \subset A$ satisfies $a_n \to a \in U$ then, after choosing $r > 0$ such that $B(a, r) \subset U$, we obtain $n \in \mathbb{N}$ such that $a_n \in B(a, r)$, and since there further exists $r_n$ small enough such that $B(a_n, r_n) \subset B(a, r)$ and $f|_{B(a_n, r_n)} \equiv 0$, then part (a) implies that $f|_{B(a,r)} \equiv 0$, i.e. $a \in A$. Thus $A = U$, i.e. $f$ is identically zero on all of $U$.

\[\square\]

**Proposition 1.4.** [Open Mapping Principle] If $f$ is analytic and non-constant on a connected open set $U$, then $f$ is an open mapping.

**Proof.** It is enough to show that $f(B(a,r))$ contains a neighborhood of $f(a)$ for each $B(a,r) \subset U$. So let $a \in U$ and $r > 0$ such that $B(a,r) \subset U$. By the Identity Principle
we have that $f|_{B(a,r)}$ is non-constant, so there exists $b \in B(a,r)$ such that $f(b) \neq f(a)$.

Now, $\Omega = \{ \lambda \in \mathbb{C} : a + \lambda(b - a) \in B(a,r) \}$ is a connected open set that contains 0 and 1. Hence $g : \Omega \to \mathbb{C}$ given by $g(\lambda) = f(a + \lambda(b - a))$ is an holomorphic function that satisfies $g(0) = f(a) \neq f(b) = g(1)$, so by the one complex variable Open Mapping Principle we get that $g(\Omega)$ is open, where $f(a) = g(0) \in g(\Omega) \subset f(B(a,r))$. Thus $f$ maps open sets into open sets.

\begin{proposition} [Maximum Principle] If $f$ is analytic on a connected open set $U$ and $|f|$ attains its supremum there, then $f$ is constant.
\end{proposition}
\begin{proof}
Assume $f$ is not constant. Then, by the Open Mapping Principle, for each $a \in U$ there exists $r > 0$ such that $B(f(a), r) \subset f(U)$, so $|f(a)|$ is not the supremum of $|f|$. Hence $|f|$ does not attain its supremum on $U$.
\end{proof}

\begin{proposition} [Liouville’s theorem] If $f$ is analytic and bounded on all $X$, then $f$ is constant.
\end{proposition}
\begin{proof}
Let $x \in X$. Since the function $g : \mathbb{C} \to \mathbb{C}$ given by $g(\lambda) = f(\lambda x)$ is holomorphic and bounded, then the one complex variable Liouville’s theorem implies that $g$ is constant. In particular $f(x) = g(1) = g(0) = f(0)$. So $f$ is constant.
\end{proof}

\begin{proposition} [Cauchy Integral Formula] Suppose that $f : U \to \mathbb{C}$ is holomorphic. Let $a \in U$, $t \in X$ and $r > 0$ such that $a + \zeta t \in U \forall \zeta \in \Delta(0,r)$. Then $\forall m \in \mathbb{N}_0$:

\[ P^m f(a)(t) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(a + \zeta t)}{\zeta^{m+1}} d\zeta. \]

\end{proposition}
\begin{proof}
Let $\Omega = \{ \zeta \in \mathbb{C} : a + \zeta t \in U \}$. Then $g : \Omega \to \Omega$ given by $g(\zeta) = f(a + \zeta t)$ is holomorphic on a neighborhood of $\Delta(0,r)$, so by the one complex variable Cauchy Integral Formula, \( \forall \lambda \in \Delta(0,r) \),

\[ f(a + \lambda t) = g(\lambda) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{g(\zeta)}{\zeta - \lambda} d\zeta = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(a + \zeta t)}{\zeta - \lambda} d\zeta. \]

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Now, whenever $|\zeta| = r$ and $|\lambda| < r$ we have

$$
\frac{f(a + \zeta t)}{\zeta - \lambda} = \frac{f(a + \zeta t)/\zeta}{1 - (\lambda/\zeta)} = \sum_{m=0}^{\infty} \lambda^m f(a + \zeta t) \frac{1}{\zeta^{m+1}},
$$

and the series converges absolutely and uniformly for $|\zeta| = r$ and $|\lambda| \leq s < r$, for all $s < r$, because $f$ is bounded on $\{a + \zeta t : |\zeta| = r\}$. Then, from the Dominated Convergence Theorem,

$$
f(a + \lambda t) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(a + \zeta t)}{\zeta - \lambda} d\zeta = \sum_{m=0}^{\infty} \lambda^m \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(a + \zeta t)}{\zeta^{m+1}} d\zeta,
$$

and the convergence is uniform for $|\lambda| \leq s$, for all $s < r$.

On the other hand, since $f$ is holomorphic we have a power series expansion

$$
f(a + \lambda t) = \sum_{m=0}^{\infty} P^m f(a)(\lambda t) = \sum_{m=0}^{\infty} \lambda^m P^m f(a)(t),
$$

which converges uniformly for $|\lambda| \leq s_0$, for some $s_0 > 0$. Hence, after an induction argument (such as the one given in Lemma 4.5 in [34]), we get that $\forall m \in \mathbb{N}_0$,

$$
P^m f(a)(t) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(a + \zeta t)}{\zeta^{m+1}} d\zeta.
$$

\[\square\]

**Proposition 1.8.** (Cauchy Inequality) Suppose that $f : U \rightarrow \mathbb{C}$ is holomorphic. Let $a \in U$, $t \in X$ and $r > 0$ such that $a + \zeta t \in U \forall \zeta \in \Delta(0,r)$. Then $\forall m \in \mathbb{N}_0$:

$$
\|P^m f(a)(t)\| \leq r^{-m} \sup_{|\zeta| = r} \|f(a + \zeta t)\|.
$$
Proof. From the Cauchy Integral Formula we have that \( \forall m \in \mathbb{N}_0 \),

\[
P^m f(a)(t) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(a + \zeta t)}{\zeta^{m+1}} d\zeta.
\]

Hence,

\[
||P^m f(a)(t)|| \leq \frac{1}{2\pi} \int_{|\zeta|=r} \frac{|f(a + \zeta t)|}{r^{m+1}} \leq r^{-m} \sup_{|\zeta|=r} ||f(a + \zeta t)||.
\]

\[
\Box
\]

**Proposition 1.9.** [The Schwarz Lemma] Suppose that \( f : U \to \mathbb{C} \) is holomorphic. Let \( a \in U \) and \( r > 0 \) such that \( B(a, r) \subset U \), and suppose that \( f \) is bounded on \( B(a, r) \) by \( C \). Then \( \forall x \in B(a, r) \):

\[
||f(x) - f(a)|| \leq 2C\frac{||x-a||}{r}.
\]

**Proof.** Fix \( x \in B(a, r) \setminus \{a\} \). Since \( r_b f(a) = \min\{r_c f(a), d_U(a)\} \) (proven above), and we have that \( r_b f(a) \geq r \), then \( r_c f(a) \geq r \) too, so

\[
f(a + \lambda(x - a)) = \sum_{m=0}^{\infty} P^m f(a)(x - a) \lambda^m, \quad \forall \lambda \in \Delta(0, r/||x - a||).
\]

Thus, if we define \( g : \Delta(0, r/||x - a||) \to \mathbb{C} \) by \( g(\lambda) = \frac{f(a + \lambda(x - a)) - f(a)}{\lambda} \) for \( \lambda \neq 0 \) and \( g(0) = P^1 f(a)(x - a) \), we get that actually

\[
g(\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} P^m f(a)(x - a), \quad \forall \lambda \in \Delta(0, r/||x - a||),
\]

so \( g \) is in particular holomorphic.
Now, if \( s \in (1, r/\|x - a\|) \) and \( |\lambda| = s \), then

\[
|g(\lambda)| \leq \frac{|f(a + \lambda(x - a))| + |f(a)|}{|\lambda|} \leq \frac{2C}{s},
\]

so by the Maximum Principle, if \( |\lambda| \leq s \) also \( |g(\lambda)| \leq \frac{2C}{s} \). Since \( \lambda = 1 \) satisfies \( |\lambda| \leq s \) then

\[
||f(x) - f(a)|| = |g(1)| \leq \frac{2C}{s}.
\]

Since this is true for all \( s < r/\|x - a\| \) then also

\[
||f(x) - f(a)|| \leq 2C \frac{\|x - a\|}{r}.
\]

\( \square \)

Since \( r_b f(a) = \min\{r_c f(a), d_U(a)\} \) whenever \( a \in U \) and \( f : U \to \mathbb{C} \) is holomorphic, there is a classical property that does not extend to the infinite dimensional setting: The following example [34, p. 54] exhibits that the radius of convergence of the power series of a holomorphic function around a point is not at least the distance of the point to the boundary of \( U \).

**Proposition 1.10.** Suppose that \( (x^*_m)_m \subset X^* \) is a sequence such that \( \|x^*_m\| = 1 \forall m \) and \( x^*_m \xrightarrow{w^*} 0 \). Then \( f = \sum_{m=0}^{\infty} (x^*_m)^m \) is holomorphic on all of \( X \) but has radius of convergence at 0 equal to 1.

When \( X \) is infinite-dimensional, such a sequence always exists due to the Josefson-Nissenzweig Theorem [12, 219-225].

**Remark 1.1.** If \( X \) and \( Y \) are complex Banach spaces, and \( U \subset X \) is open, we say that \( f : U \to Y \) is holomorphic if \( y^* \circ f \) is holomorphic for all \( y^* \in Y^* \). One can check that Theorem 1.2 can be extended to this setting (see [34] for the details).
We now have enough tools to discuss the cluster value problem in arbitrary Banach spaces.

1.5 Cluster value problems for infinite-dimensional Banach spaces

For arbitrary Banach spaces $X$, the cluster value theorem for $H(B)$ asserts that, for a given $x^{**} \in \overline{B}^{**}$, the sets of cluster values

$$ Cl_B(f, x^{**}) := \{ \lambda : f(x_\alpha) \to \lambda, \ x_\alpha \overset{w^*}{\to} x^{**} \} $$

coincides with the evaluation of the fiber $M_{x^{**}}(B) := \pi^{-1}(x^{**})$ at $f$,

$$ \hat{f}(M_{x^{**}}(B)) = \{ \tau(f) : \tau \in M_{x^{**}} \}, $$

for all $f \in H(B)$.

Aron, Carando, Gamelin, Lasalle and Maestre observed in [5] that for every $x^{**} \in \overline{B}^{**}$ we have the inclusion

$$ Cl_B(f, x^{**}) \subset \hat{f}(M_{x^{**}}(B)), \ \forall f \in H(B), $$

thus to prove the cluster value theorem for $H(B)$ it is enough to show $\hat{f}(M_{x^{**}}(B)) \subset Cl_B(f, x^{**}), \ \forall f \in H(B)$.

Aron, Carando, Gamelin, Lasalle and Maestre also showed in [5] that the cluster value theorem holds at every $x^{**} \in \overline{B}^{**}$ if and only if whenever $f_1, \cdots, f_{n-1} \in A(B)$ and $f_n \in H(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \epsilon > 0$ on $B$, there exist $g_1, \cdots, g_n \in H(B)$ such that $f_1g_1 + \cdots f_ng_n = 1$.

When $B$ is the unit ball of an infinite-dimensional Banach space, there are no known solutions to the Corona problem. However, Aron, Carando, Gamelin, Lasalle
and Maestre proved in 2012 in [5] a cluster value theorem at the origin for the algebra $A_u(B)$ when $X$ has a shrinking 1-unconditional basis. Examples of such spaces $X$ include $\ell_p$ for $1 < p < \infty$ and $c_0$, but not $\ell_1$, $\ell_\infty$, $L_p(0,1)$ for $1 \leq p \neq 2 \leq \infty$.

**Theorem 1.3.** If $X$ is a Banach space with a shrinking 1-unconditional basis, then the cluster value theorem holds for $A_u(B)$ at $x = 0$,

$$\text{Cl}_B(f,0) = \hat{f}(M_0), \ f \in A_u(B).$$

Aron, Carando, Gamelin, Lasalle and Maestre [5] also proved a cluster value theorem at all points of the closed unit ball of $X$ for the algebra $A_u(B)$ when $X$ is a Hilbert space. The main idea is to translate the problem to 0 via certain automorphisms of $B$, then to use the cluster value theorem at 0 for Banach spaces with a shrinking 1-unconditional basis, and to apply certain peak functions for points in $\partial B$.

**Theorem 1.4.** If $X$ is a Hilbert space, then the cluster value theorem holds for $A_u(B)$ at every $x \in \overline{B}$,

$$\text{Cl}_B(f,x) = \hat{f}(M_x), \ f \in A_u(B_X), x \in \overline{B}.$$  

**Corollary 1.1.** Let $B$ be the open unit ball of a Hilbert space. If $f_1, \cdots, f_{n-1} \in A(B)$ and $f_n \in A_u(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \epsilon > 0$ on $B$, then there exist $g_1, \cdots, g_n \in A_u(B)$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.

The same authors [5] proved a cluster value theorem at all points of the closed unit ball of $X^{**}$ for the algebra of bounded analytic functions, denoted by $H^\infty(B)$, when $X$ is $c_0$ (the space of null sequences). The proof repeatedly uses a lemma based on a solution to a $\bar{\partial}$ equation.
Theorem 1.5. If $X$ is the Banach space $c_0$ of null sequences, then the cluster value theorem holds for $H^\infty(B)$ at every $x \in \bar{B}^{**}$,

$$Cl_B(f,x) = \hat{f}(M_x), \quad f \in H^\infty(B), x \in \bar{B}^{**}.$$ 

Corollary 1.2. Let $B$ be the open unit ball of the Banach space $c_0$ of null sequences. If $f_1, \cdots, f_{n-1} \in A(B)$ and $f_n \in H^\infty(B)$ satisfy $|f_1| + \cdots + |f_n| \geq \epsilon > 0$ on $B$, then there exist $g_1, \cdots, g_n \in H^\infty(B)$ such that $f_1g_1 + \cdots + f_ng_n = 1$.

Note that Hilbert space and $c_0$ are infinite-dimensional analogues of the unit ball and the polydisk of Euclidean space, respectively.

It is open whether $L_p$ satisfies the cluster value problem, for $A_u(B_{L_p})$ or $H^\infty(B_{L_p})$, at any point of $B_{L_p}$, $1 \leq p \neq 2 \leq \infty$. Nevertheless, Lemma 4.4 in [15] implies a cluster value theorem for each point in $\partial B$ and the algebra $A_u(B)$, when $B$ is the unit ball of a uniformly convex Banach space, like $\ell_p$ and $L_p$, for $1 < p < \infty$. Also, by [1, Theorem 2.6] and [5, Corollary 2.5], there is a cluster value theorem for the algebra $A_u(B_{\ell_1})$ and each point in $\partial B_{\ell_1}$, because for each boundary point there is a function in $A(B_{\ell_1})$ peaking at it. I would like to investigate what happens at interior points.
2. OUR CLUSTER VALUE THEOREMS*

2.1 A look at finite-dimensional subspaces of a Banach space

Generalizing the ideas and techniques in [5], W. B. Johnson and I proved the following cluster value theorem in [27]:

**Theorem 2.1.** Suppose that for each finite-dimensional subspace $E$ of $X^*$ and $\epsilon > 0$ there exists a finite rank operator $S$ on $X$ so that $\| (I - S^*) |_E \| < \epsilon$ and $\| I - S \| = 1$. Then the cluster value theorem holds for $A_u(B)$ at $0$.

**Proof.** Suppose that $0 \notin Cl_B(f,0)$. We must show that $0 \notin \hat{f}(M_0)$. Since $0 \notin Cl_B(f,0)$, there exists $\delta > 0$ and a weak neighborhood $U$ of $0$ in $X$ such that $|f| \geq \delta$ on $U \cap B$. Without loss of generality we may assume $U = \cap_{i=1}^n \{ x \in X : |x_i^*| < \epsilon_0 \}$ for some $x_1^*, \cdots, x_n^* \in B_{X^*}$ and $\epsilon_0 > 0$. Let $E = \text{span}\{x_1^*, \cdots, x_n^*\}$ and let $S$ be as in the statement for $\epsilon = \epsilon_0$. Then $|f \circ (I - S)| \geq \delta$ on $B$, because for every $x \in B$ we have that $(I - S)x \in U$, indeed:

$$| < x_i^*, (I - S)x > | = | < (I - S^*)x_i^*, x > | < \epsilon_0, \text{ for } i = 1, \cdots, n.$$

Consequently $f \circ (I - S)$ is invertible in $A_u(B)$. Hence $f \circ (I - S) \neq 0$ on the fiber of the spectrum of $A_u(B)$ over $0$. From Proposition 2.1 below, we then obtain $\hat{f} \neq 0$ on $M_0$, that is $0 \notin \hat{f}(M_0)$.

For example, Banach spaces with a shrinking reverse monotone Finite Dimensional Decomposition (FDD) satisfy the conditions of the previous theorem. Recall that such Banach spaces $X$ are the union of finite-dimensional spaces $(E_n)_n$ such that...

---

∀x ∈ X, ∃!(x_n)_n with x_n ∈ E_n and x = \sum x_n; also the functions Q_n(x) = \sum_{i=n}^{\infty} x_i satisfy ||Q_n|| = 1 \forall n, and ∀x^* ∈ X^*, lim_{n \to \infty} ||x^*|_{\text{span}(\cup_{j=n}^{\infty} E_j)}|| = 0.

The proof of Theorem 2.1 relies on the following proposition in [27] (a generalization of Lemma 3.4 in [5]):

**Proposition 2.1.** Let S be a finite rank operator on X such that P = I − S has norm one. If φ ∈ M_0(B) = {τ ∈ A_u(B) : τ|_{A(B)} = δ_0}, then \( \hat{f}(\phi) = \hat{f} \circ \hat{P}(\phi) \), for all f ∈ A_u(B).

We note that the previous proposition, however, does not hold for H^\infty(B), as exhibited in the following example by Aron [27, Example 1]:

**Example 2.1.** There exists a finite rank operator S on \( \ell_2 \) so that P = I − S has norm one, and there exist φ ∈ M_0(B_{\ell_2}) = {τ ∈ H^\infty(B) : τ|_{A(B)} = δ_0} as well as f ∈ H^\infty(B_{\ell_2}) so that \( \hat{f}(\phi) \neq \hat{f} \circ \hat{P}(\phi) \).

**Proof.** Let S : \( \ell_2 \to \ell_2 \) be given by S(x) = (x_1, 0, 0, •••).

Clearly S is a finite rank operator and P = I − S has norm one.

Let (r_j) and (s_j) be sequences of positive real numbers, such that (r_j) ↓ 0 and (s_j) ↑ 1 in such a way that each \( r_j^2 + s_j^2 \) < 1 and \( r_j^2 + s_j^2 \) → 1−. For each j = 1, 2, 3, •••, let \( \delta_{r_j e_1 + s_j e_j} \) be the usual point evaluation homomorphism from H^\infty(B_{\ell_2}) → \C.

Let φ : H^\infty(B_{\ell_2}) → \C be an accumulation point of \{\delta_{r_j e_1 + s_j e_j}\} in the spectrum of H^\infty(B_{\ell_2}). Let f : B_{\ell_2} → \C be the H^\infty function given by

\[
\hat{f}(x) = \frac{x_1}{\sqrt{1 - \sum_{j=2}^{\infty} x_j^2}},
\]

where the square root is taken with respect to the branch of logarithm determined by the bounded and simply connected set \( U = \{1 - \sum_{j=2}^{\infty} x_j^2 : x \in B_{\ell_2}\} \) that does not contain 0.

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Since $f$ is clearly continuous, it is analytic because it is analytic on complex lines:

Given $x^0 = (x_n^0)_n \in B_\ell_2$ and $y^0 = (y_n^0)_n \in \ell_2$, if $\lambda \in \Omega = \{ \zeta \in \mathbb{C} : x^0 + \zeta y^0 \in B_\ell_2 \}$,

$$f(x^0 + \lambda y^0) = \frac{x^0_1 + \lambda y^0_1}{\sqrt{1 - \sum_{n=2}^{\infty} (x^0_n + \lambda y^0_n)^2}} \cdot \frac{x^0_1 + \lambda y^0_1}{\sqrt{1 - \sum_{n=2}^{\infty} (x^0_n)^2 - 2\lambda \sum_{n=2}^{\infty} x^0_n y^0_n + \lambda^2 \sum_{n=2}^{\infty} (y^0_n)^2}} = \frac{a + \lambda b}{\sqrt{c - 2\lambda d + \lambda^2 e}},$$

which is a holomorphic expression because $\Omega$ is simply connected and bounded, and $0 \notin \{c - 2\lambda d + \lambda^2 e : \lambda \in \Omega \}$.

$f$ is bounded because for all $x \in B_\ell_2$,

$$\left| \frac{x_1}{\sqrt{1 - \sum_{j=2}^{\infty} x_j^2}} \right|^2 = \frac{|x_1|^2}{1 - \sum_{j=2}^{\infty} x_j^2} \leq \frac{|x_1|^2}{1 - \sum_{n=2}^{\infty} |x_n|^2} \leq 1.$$

Finally, $\phi(f) = 1$, however $\phi(f \circ P) = 0$ because $f \circ P \equiv 0$.

We observe that Theorem 2.1 and Proposition 2.1 suggest a relationship between the cluster value problem in a Banach space and its finite-codimensional subspaces. Johnson and I [27] established the following relationship with the help of Aron and Maestre:

**Proposition 2.2.** If $Y$ is a closed finite-codimensional subspace of $X$ and $f \in A_u(B)$, then $\text{Cl}_B(f, 0) = \text{Cl}_{B_Y}(f|_Y, 0)$, where $B_Y$ is the unit ball of $Y$.

**Proof.** $A_u(B)$ coincides with the uniform limits on $\bar{B}$ of continuous polynomials on $X$ (see Theorem 7.13 in [34] and p. 56 in [6]), where polynomials are finite linear combinations of symmetric $m$-linear mappings restricted to the diagonal. Thus, by
passing to the uniform limit on $\bar{B}$, we may assume $f$ is an $m$-homogeneous polynomial, with associated symmetric $m$-linear functional $F$. Let $(x_\alpha)$ be a weakly null net in $B$ such that $f(x_\alpha) \to \lambda$.

Each $x_\alpha$ can be written uniquely as $y_\alpha + u_\alpha$, where $y_\alpha \in Y$ and $u_\alpha$ is from a fixed finite dimensional complement of $Y$ in $X$. Then

$$f(x_\alpha) = F(x_\alpha, \cdots, x_\alpha)$$

$$= f(y_\alpha) + mF(y_\alpha, \cdots, y_\alpha, u_\alpha) + \frac{m(m-1)}{2}F(y_\alpha, \cdots, y_\alpha, u_\alpha, u_\alpha) + \cdots + f(u_\alpha).$$

Now, since $(x_\alpha)$ is weakly null, the same holds for $(y_\alpha)$ and $(u_\alpha)$. However, since $(u_\alpha)$ belongs to a finite dimensional space, it follows that $||u_\alpha|| \to 0$. Thus $F(y_\alpha, \cdots, y_\alpha, u_\alpha)$, $F(y_\alpha, \cdots, y_\alpha, u_\alpha, u_\alpha)$, $\cdots$, $f(u_\alpha)$ all go to 0. Thus $f(y_\alpha) \to \lambda$. Finally, since each $||y_\alpha|| \leq ||x_\alpha|| + ||u_\alpha|| < 1 + ||u_\alpha||$, then by defining $t_\alpha = \frac{1}{1+||u_\alpha||}$ we get that $||t_\alpha y_\alpha|| < 1$ for all $\alpha$ and $t_\alpha \to 1$, and consequently, $\lim f(t_\alpha y_\alpha) = \lim t_\alpha^m f(y_\alpha) = \lambda$. Hence $\lambda \in Cl_{B_Y}(f|_Y, 0)$.

As a consequence we obtain that the cluster sets of an element $f$ of $A_u(B)$ at 0 can be described in terms of the Gelfand transforms of $f|_{B_Y}$ as $Y$ ranges over finite-codimensional subspaces of $X$:

**Proposition 2.3.** For every Banach space $X$,

$$Cl_B(f, 0) = \bigcap_{Y \subset X, \dim(X/Y) < \infty} \widehat{f|_{B_Y}}(M_0(B_Y)), \forall f \in A_u(B).$$

**Proof.** From Proposition 2.2, for every finite-codimensional subspace $Y$ of $X$,

$$Cl_B(f, 0) = Cl_{B_Y}(f|_{B_Y}, 0) \subset \widehat{f|_{B_Y}}(M_0(B_Y)).$$
For the reverse inclusion, suppose $0 \not\in \text{Cl}_B(f,0)$. Then there are $\epsilon > 0$ and a weak neighborhood $U$ of 0 such that $|f| > \epsilon$ on $U \cap B$. $U$ contains a closed finite-codimensional subspace $Y_0$ of $X$, so $|f|_{Y_0} > \epsilon$. Hence $\hat{f}|_{Y_0}$ is invertible, which implies that $0 \not\in \hat{f}|_{Y_0}(M_0(B_{Y_0}))$. □

Since $c_0$ satisfies a cluster value theorem for $A_u(B)$ and $c_0$ is 1-codimensional in $c$ then Proposition 2.2 suggests that $c$ satisfies a cluster value theorem, but [27, Example 2] shows that $c$ does not satisfy the hypothesis of Theorem 2.1:

**Example 2.2.** Let $L \in B_{c^*}$ be given by

$$L((c_n)_n) = \lim_{n \to \infty} c_n.$$  

If $S : c \to c$ is a finite rank operator with $||(S^* - I_{c^*})L|| < \epsilon$, then $||S - I_c|| \geq 2 - \epsilon$.

**Proof.** For each $k \in \mathbb{N}$, consider $L_k \in B_{c^*}$ given by

$$L_k((c_n)_n) = \frac{(\lim_{n \to \infty} c_n - c_k)}{2}.$$  

Let us show that $||S^*(L_k)|| \to 0$ as $k \to \infty$. For every $x \in B_c$, $S^*(L_k)x = L_k(Sx) \to 0$ as $k \to \infty$. Moreover, since $S$ has finite rank, $\{Sx : x \in B_c\}$ is pre-compact. Thus $S^*L_k = L_k \circ S$ converges to zero uniformly on $B_c$, i.e. $||S^*L_k|| \to 0$ as $k \to \infty$.

Now note that $||L - 2L_k|| = 1$ for each $k$, so

$$||S^* - I_{c^*}|| \geq ||(S^* - I_{c^*})(L - 2L_k)|| \geq ||2L_k - 2 \cdot S^*(L_k)|| - \epsilon \geq 2 - \epsilon - 2||S^*(L_k)||.$$  

Since $S^*(L_k) \to 0$, then $||S - I_c|| = ||S^* - I_{c^*}|| \geq 2 - \epsilon$.

□

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As we mentioned before, since $c_0$ is one-codimensional in $c$, Proposition 2.2 implies that for all $f \in A_u(B_c)$,

$$\text{Cl}_{B}(f, 0) = \text{Cl}_{B_{c_0}}(f|_{B_{c_0}}, 0).$$

Also, Propositions 1.59 and 2.8 of [13] imply that all functions in $A_u(B_{c_0})$ can be uniformly approximated on $B$ by polynomials in the functions in $X^*$, which in turn implies that each fiber at $x^{**} \in \bar{B}^{**}$ consists only of $x^{**}$, so the cluster value theorem for $A_u(B_{c_0})$ holds, and in particular

$$\text{Cl}_{B_{c_0}}(f|_{B_{c_0}}, 0) = \hat{f}|_{B_{c_0}}(M_0(B_{c_0})), \quad \forall f \in A_u(B_c).$$

Hence we are left to compare $\hat{f}|_{B_{c_0}}(M_0(B_{c_0}))$ with $\hat{f}(M_0(B_c))$ for $f \in A_u(B_c)$. Note that an inclusion is evident:

**Proposition 2.4.** For a Banach space $X$ and $Y$ a subspace of $X$,

$$\hat{f}|_{B_Y}(M_0(B_Y)) \subset \hat{f}(M_0(B)), \quad \forall f \in A_u(B).$$

*Proof.* Let $f \in A_u(B)$ and $\tau \in M_0(B_Y)$. Since $\phi_1 : A_u(B) \rightarrow A_u(B_Y)$ given by $\phi(g) = g|_Y$ for all $g \in A_u(B)$ is a continuous homomorphism that maps $A(B)$ into $A(B_Y)$, the mapping $\bar{\tau} : A_u(B) \rightarrow \mathbb{C}$ given by $\bar{\tau}(g) = \tau(g|_Y)$ for all $g \in A_u(B)$ is in the fiber $M_0(B)$. Moreover,

$$\hat{f}|_Y(\tau) = \hat{f}(\bar{\tau}).$$

The reverse inclusion is unclear. Nonetheless, $c$ is isomorphic to $c_0$ and $A_u(B_{c_0}) = A(B_{c_0})$, so by [27, Lemma 1], $A_u(B_c) = A(B_c)$, so $c$ satisfies a cluster value theorem.
for $A_u(B_c)$:

**Lemma 2.1.** Let $X$ be a Banach space so that $A_u(B_X) = A(B_X)$. If the Banach space $Y$ is isomorphic to $X$, then also $A_u(B_Y) = A(B_Y)$.

**Proof.** Let $T : Y \to X$ be the Banach space isomorphism between $Y$ and $X$.

Let $P(X)$ and $P(Y)$ denote the continuous polynomials on $X$ and $Y$ respectively, and denote by $P_f(X)$ and $P_f(Y)$ the polynomials in the functions of $X^*$ and $Y^*$ respectively (known as finite type polynomials).

Let $f \in A_u(B_Y)$. Then there exists a sequence of polynomials $P_n \in P(Y)$ such that $||P_n - f||_{B_Y} \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $P_n \circ T^{-1} \in P(X)$, so there exists a polynomial $Q_n \in P_f(X)$ such that $||P_n \circ T^{-1} - Q_n||_{B_X} \leq \frac{1}{n ||T||}$, and consequently $||P_n - Q_n \circ T||_{B_Y} \leq \frac{1}{n}$, where $Q_n \circ T \in P_f(Y)$.

Consequently, the sequence of polynomials $Q_n \circ T \in P_f(Y)$ converges to $f$ uniformly on $B_Y$, so $f \in A(B_Y)$.

**Corollary 2.1.** The Banach space $c$ satisfies the cluster value theorem for $A_u(B_c)$ at all points in $\overline{B_c}^{**}$.

**Proof.** Since every $f \in A_u(B_c)$ can be uniformly approximated on $B_c$ by polynomials in the functions in $X^*$, then each fiber at $x^{**} \in \overline{B}^{**}$ consists only of $x^{**}$, and since $Cl_B(f,x^{**}) \subset \widehat{f}(M_{x^{**}}(B)) = \{\widehat{f}(x^{**})\} \forall f \in A_u(B_c)$, where every $Cl_B(f,x^{**}) \neq \emptyset$ (see [19, p. 200] or [5, p. 2]), then the cluster value theorem for $A_u(B_c)$ holds.

Bessaga and Pełczyński proved in [7] that, when $\alpha \geq \omega_\omega$ is a countable ordinal, $C(\alpha)$ is not isomorphic to $c = C(\omega)$. Therefore we no longer can use Lemma 2.1 to obtain a cluster value theorem on such spaces of continuous functions.
Nevertheless, for $\alpha$ a countable ordinal, the intervals $[1, \alpha]$ are always compact, Hausdorff and dispersed (they contain no perfect non-void subset). The compact, Hausdorff and dispersed sets $K$ satisfy, from [35, Main Theorem], that $X = C(K)$ contains no isomorphic copy of $l_1$. Moreover, from [3, Theorem 5.4.5], $X = C(K)$ has the Dunford-Pettis property. Therefore, for dispersed $K$, the continuous polynomials on $X = C(K)$ are weakly (uniformly) continuous on bounded sets by [13, Corollary 2.37].

Moreover, since $X^* = l_1(K)$ has the approximation property, [13, Proposition 2.8] now yields that all continuous polynomials on $X$ can be uniformly approximated, on bounded sets, by polynomials of finite type. Thus the elements of $A_u(B)$ can be approximated, uniformly on $B$, by polynomials of finite type. Hence $A_u(B) = A(B)$, so each fiber at $x^{**} \in \bar{B}^{**}$ is the singleton $\{x^{**}\}$, and then $X$ satisfies the cluster value theorem for the algebra $A_u(B)$.

We now consider the cluster value problem on $X$ for the algebra of all bounded analytic functions $H^\infty(B)$.

### 2.2 A cluster value theorem for spaces of continuous functions

Following the line of proof of Theorem 5.1 in [5], and using that $C(K)^* = l_1(K)$ when $K$ is compact, Hausdorff and dispersed, we obtain a cluster value theorem for $H^\infty(B)$ when $X = C(K)$, and $K$ is compact, Hausdorff and dispersed.

**Theorem 2.2.** If $X$ is the Banach space $C(K)$, for $K$ compact, Hausdorff and dispersed, then the cluster value theorem holds for $H^\infty(B)$ at every $x \in \bar{B}^{**}$.

**Proof.** Fix $f \in H^\infty(B)$ and $w = (w_t)_{t \in K} \in \bar{B}^{**}$ (where $C(K)^{*} = l_\infty(K)$). Suppose $0 \not\in Cl_B(f, w)$. It suffices to show that $0 \not\in \hat{f}(M_w)$. 

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Since 0 is not a cluster value of \( f \) at \( w \), there exists a weak-star neighborhood \( U \) of \( w \) such that \( 0 \notin \overline{f(U \cap B)} \), where

\[
U \cap B \supset \bigcap_{i=1}^n \{ z \in B : | < (z - w), x_i^* > | < \epsilon \},
\]

for some \( \epsilon > 0 \) and \( x_1^*, \ldots, x_n^* \in X^* = l_1(K) \).

We have that \( x_i^* = (x_i^*(t))_{t \in K} \) has countably many nonzero coordinates \( \{ x_i^*(t) \}_{t \in F_i} \) for \( i = 1, \ldots, n \). Thus,

\[
U \cap B \supset \bigcap_{i=1}^n \{ z \in B : | \sum_{t \in K} (z_t - w_t) x_i^*(t) | < \epsilon \},
\]

and there is a finite set \( F \subset \bigcup_{i=1}^n F_i \) so that \( \sum_{t \notin F} | x_i^*(t) | < \epsilon / 4 \), for \( i = 1, \ldots, n \). Then,

\[
U \cap B \supset \bigcap_{t \in F} \{ z \in B : | z_t - w_t | < \delta \},
\]

where

\[
\delta = \min_{1 \leq i \leq n, t \in F} \left( \frac{\epsilon}{2 |F| |x_i^*(t)|} \right).
\]

In summary, there exist \( c > 0, \delta > 0 \) and a finite set \( F \subset K \) such that if \( z \in B \) satisfies \( | z_t - w_t | < \delta \) for \( t \in F \) then \( |f(z)| \geq c \). Relabel the indices in \( F \) as \( t_1, \ldots, t_m \), where \( m = |F| \). Then proceed as in the proof of Theorem 5.1 in [5]:

For \( 0 \leq k \leq m - 1 \), define \( U_k = \{ z \in B : |z_{t_j} - w_{t_j}| < \delta, k + 1 \leq j \leq m \} \), and set \( U_m = B \). Note that \( 1/f \) is bounded and analytic on \( U_0 \).

We claim that for each \( k, 1 \leq k \leq m \), there are functions \( g_k \) and \( h_{k,j}, 1 \leq j \leq k \), in \( H^\infty(U_k) \) that satisfy

\[
f(z)g_k(z) = 1 + (z_{t_1} - w_{t_1})h_{k1}(z) + \cdots + (z_{t_k} - w_{t_k})h_{kk}(z), \quad z \in U_k.
\] (2.1)
Once this claim is established, the proof is easily completed as follows. The functions $g_m$ and $h_{mj}$ belong to $H^\infty(B)$ and satisfy

$$\hat{f}\hat{g}_m = \hat{1} + \sum_{j=1}^{m} (\hat{z}_{t_j} - \hat{w}_{t_j})\hat{h}_{m_j}.$$ 

Since each $\hat{z}_{t_j} - \hat{w}_{t_j}$ vanishes on $M_w$ (by the definition of $M_w$), we obtain $\hat{f}\hat{g}_m = 1$ on $M_w$, and consequently $\hat{f}$ does not vanish on $M_w$, as required.

Just as in [5], the claim is established by induction on $k$. The first step, the construction of $g_1$ and $h_{11}$, is as follows. We regard $1/ f((z_t)_{t \in K})$ as a bounded analytic function of $z_t$, for $|z_t| < 1$ and $|z_t - w_{t_1}| < \delta$, with $z_t$, $t \in K - \{t_1\}$, as analytic parameters in the range $|z_t| < 1$ for $t \in K - \{t_1\}$, and $|z_{t_j} - w_{t_j}| < \delta$ for $2 \leq j \leq m$. According to lemma 5.3 in [5], we can express

$$\frac{1}{f(z)} = g_1(z) + (z_{t_1} - w_{t_1})h(z), \quad z \in U_0,$$

where $g_1 \in H^\infty(U_1)$ and $h \in H^\infty(U_0)$. We set

$$h_{11}(z) = [f(z)g_1(z) - 1]/(z_{t_1} - w_{t_1}), \quad z \in U_1,$$

so that (2.1) is valid for $k = 1$. Note that $h_{11} = -hf$ on $U_0$. Consequently $h_{11}$ is bounded and analytic on $U_0$. The defining formula then shows that $h_{11}$ is analytic on all of $U_1$, and since $|z_{t_1} - w_{t_1}| \geq \delta$ on $U_1 - U_0$, $h_{11}$ is bounded on $U_1$.

Now suppose that $2 \leq k \leq m$, and that there are functions $g_{k-1}$ and $h_{k-1,j}$ ($1 \leq j \leq k-1$) that satisfy (2.1) and are appropriately analytic. We apply lemma 5.3 in [5] to these as functions of $z_{t_k}$, with the other variables regarded as analytic parameters,
to obtain decompositions

\[ g_{k-1}(z) = g_k(z) + (z_{t_k} - w_{t_k})G_k(z) \]

and

\[ h_{k-1,j}(z) = h_{k,j}(z) + (z_{t_k} - w_{t_k})H_{k,j}(z), \quad 1 \leq j \leq m - 1, \]

where \( g_k \) and the \( h_{kj} \)'s are in \( H^\infty(U_k) \), and \( G_k \) and the \( H_{kj} \)'s are in \( H^\infty(U_{k-1}) \). From the identity (2.1), with \( k \) replaced with \( k - 1 \), we obtain

\[ fg_k = 1 + \sum_{j=1}^{k-1} (z_{t_j} - w_{t_j})h_{kj} + (z_{t_k} - w_{t_k})[-fG_k + \sum_{j=1}^{k-1} (z_{t_j} - w_{t_j})H_{kj}] \]

on \( U_{k-1} \). We define

\[ h_{kk} = \frac{[f g_k - 1 - \sum_{j=1}^{k-1} (z_{t_j} - w_{t_j})h_{kj}](z_{t_k} - w_{t_k})}{(z_{t_k} - w_{t_k})}, \quad z \in U_k. \]

Then (2.1) is valid. On \( U_{k-1} \) we have

\[ h_{kk} = -fG_k + \sum_{j=1}^{k-1} (z_{t_j} - w_{t_j})H_{kj}, \]

so that \( h_{kk} \) is bounded and analytic on \( U_{k-1} \). Since \( |z_{t_k} - w_{t_k}| \geq \delta \) on \( U_k - U_{k-1} \), we see from the defining formula that \( h_{kk} \in H^\infty(U_k) \). This establishes the induction step, and the proof is complete.

\[ \Box \]

We do not know the answer to the cluster value problem for other spaces \( C(K) \), however, we can give a partial answer to the following modification of the cluster value problem.
2.3 The cluster value problem for $H^\infty(B)$ over $A_u(B)$

In [27] we consider the following cluster value problem: Given $f_0^{**} \in \mathcal{B}^{**}$, the cluster value problem for $H^\infty(B)$ over $A_u(B)$ at $f_0^{**}$ asks whether for all $\psi \in H^\infty(B)$ and $\tau \in \mathcal{M}_{f_0^{**}}(B)$ ($\mathcal{M}_{f_0^{**}}$ is $\pi^{-1}(\delta_{f_0^{**}})$ for the restriction map $\pi : M_{H^\infty(B)} \to M_{A_u(B)}$), can we find a net $(f_\alpha) \subset \mathcal{B}$ such that $\psi(f_\alpha) \to \tau(\psi)$ and $f_\alpha$ converges to $f_0^{**}$ in the polynomial-star topology, i.e. the smallest topology that makes every extension of a polynomial on $X$ to $X^{**}$ continuous (that we denote by $\tau(\psi) \in Cl_{\mathcal{B}}(\psi,f_0^{**})$)? As before, clearly $Cl_{\mathcal{B}}(\psi,f_0^{**}) \subset \widehat{\psi}(\mathcal{M}_{f_0^{**}}(B))$, $\forall \psi \in H^\infty(B)$.

The previous problem seems to be highly nontrivial. For example, for every infinite compact Hausdorff space $K$, $C(K)$ contains a subspace $Y$ isometric to $c_0$ (Proposition 4.3.11 in [3]), so the fiber $\mathcal{M}_0(B_{C(K)})$ is huge (and from Proposition 2.6 below, so is each fiber $\mathcal{M}_{f_0}(B_{C(K)})$ for $f_0 \in B_{C(K)}$). Indeed, according to Theorem 6.6 in [10], there is a family of distinct characters $\{\tau_\alpha\}_{\alpha \in B_{\ell_\infty}}$, such that each $\tau_\alpha : H^\infty(B_Y) \to \mathbb{C}$ satisfies $\delta_0 = \tau_\alpha|_{A(B_Y)} = \tau_\alpha|_{A_u(B_Y)}$ (because $Y$ is isometric to $c_0$, so $A(B_Y) = A_u(B_Y)$). Hence $\{\tau_\alpha\}_{\alpha \in B_{\ell_\infty}} \subset \mathcal{M}_0(B_Y)$ and therefore $\{\tau_\alpha \circ R\}_{\alpha \in B_{\ell_\infty}} \subset \mathcal{M}_0(B_{C(K)})$, where $R$ is the restriction mapping $R : H^\infty(B_{C(K)}) \to H^\infty(B_Y)$, which is clearly a homomorphism. Note that the characters $\{\tau_\alpha \circ R\}_{\alpha \in B_{\ell_\infty}}$ are all distinct due to Theorem 1.1 in [4] (also [19, Theorem 2.1.3]), because $\ell_\infty$ is an isometrically injective space (Proposition 2.5.2 in [3]), so there exists a norm-one linear map $S : C(K) \to \ell_\infty$ such that $S|_{c_0} = I_{c_0}$.

The cluster value problem for $H^\infty(B)$ over $A_u(B)$ coincides with the cluster value problem for $H^\infty(B)$ when $A_u(B) = A(B)$. Thus when $K$ is compact, Hausdorff and dispersed, we have a positive answer to the previous cluster value problem for the $C(K)$ spaces.

To study the cluster value problem for $H^\infty(B)$ over $A_u(B)$ when $B = B_{C(K)}$ of
an arbitrary $C(K)$ space, in [27] we describe the following family of automorphisms $(T_{f_0})_{f_0 \in B}$:

**Proposition 2.5.** Let $f_0 \in B = B_{C(K)}$. $T_{f_0} : B \to B$ given by

$$T_{f_0}(f) = \frac{f - f_0}{1 - \overline{f_0} \cdot f} \quad \forall f \in B,$$

is biholomorphic.

This is a folklore result mentioned e.g. in [39] and [8], but inasmuch there seems to be no proof in the literature we sketch the proof.

**Proof.** Set $\delta_0 = ||f_0||$.

Let us start by showing that $T := T_{f_0}$ is well defined, i.e. $||Tf|| < 1$ when $||f|| < 1$.

Let $f \in B$. We can find $\delta \in (\delta_0, 1)$ such that $||f|| \leq \delta$.

For every $t_0 \in K$, let $z = f(t_0)$ and $c = f_0(t_0)$, so that $T(f)(t_0) = \frac{z - c}{1 - \overline{c} \cdot z}$.

Let $\Delta$ denote the open unit disk in the complex plane $\mathbb{C}$.

Since $\sigma : (\delta \cdot \overline{\Delta}) \times (\delta_0 \cdot \overline{\Delta}) \to \Delta$ given by $\sigma(z, c) = \frac{z - c}{1 - \overline{c} \cdot z}$ is continuous, then $\sigma((\delta \cdot \overline{\Delta}) \times (\delta_0 \cdot \overline{\Delta}))$ is a compact subset of $\Delta$, so there exists $\delta_1 < 1$ so that $\sigma((\delta \cdot \overline{\Delta}) \times (\delta_0 \cdot \overline{\Delta})) \subset \delta_1 \Delta$.

Thus $||Tf|| \leq \delta_1 < 1$.

Let us now show that $T$ is also holomorphic, or equivalently, Fréchet $C$-differentiable. For $f \in B$ fixed, the linear mapping $L : C(K) \to C(K)$ given by $L(h) = \frac{1 - |f_0|^2}{(1 - \overline{f_0} f)^2} h$ satisfies that, for $h \neq 0$ small enough,
\[
\frac{T(f + h) - T(f) - L(h)}{||h||} = (f + h - f_0) - \frac{f - f_0}{1 - \overline{f_0}f} - \frac{1 - |f_0|^2}{(1 - \overline{f_0}f)^2} h / ||h|| \\
= \frac{1 - |f_0|^2}{1 - \overline{f_0}f} \cdot \frac{h}{1 - \overline{f_0}(f + h)} - \frac{1 - |f_0|^2}{(1 - \overline{f_0}f)^2} h / ||h|| \\
= \frac{\overline{f_0}h}{(1 - \overline{f_0}f)^2(1 - \overline{f_0}(f + h))} (1 - |f_0|^2) h / ||h|| ,
\]

which goes to zero as \( h \to 0 \). Thus \( T \) is holomorphic.

Since \( T \) clearly has a necessarily holomorphic inverse \( (S(f) = \frac{f + f_0}{1 + \overline{f_0}f}) \), we have that \( T \) is a biholomorphic function on \( B \) that sends \( f_0 \) to the function identically zero.

**Alternative proof of analyticity.** It suffices to check that \( T \) is continuous and its restriction to each complex line is holomorphic.

To prove \( T \) is continuous, let \( f_1 \in B \) and \( \epsilon > 0 \).

For any \( f \in B \),

\[
||T(f) - T(f_1)|| = ||\frac{f - f_0}{1 - \overline{f_0}f} - \frac{f_1 - f_0}{1 - \overline{f_0}f_1}|| = ||\frac{(1 - |f_0|^2)(f - f_1)}{(1 - \overline{f_0}f)(1 - \overline{f_0}f_1)}|| ,
\]

where \( ||1 - \overline{f_0}f||, ||1 - \overline{f_0}f_1|| \geq 1 - ||f_0|| = \alpha > 0 \). Thus, when \( ||f - f_1|| < \epsilon \cdot \alpha^2 \),

\[
||T(f) - T(f_1)|| < \frac{1 \cdot \epsilon \cdot \alpha^2}{\alpha^2} = \epsilon ,
\]

so \( T \) is continuous.
To prove that $T$ is holomorphic on each line, let $f_1 \in B$, and $g_1 \neq 0 \in B$. Then, for $\lambda \in A = \{a \in \mathbb{C} : f_1 + ag_1 \in B\},$

$$\psi(\lambda) = T(f_1 + \lambda g_1) = \frac{f_1 + \lambda g_1 - f_0}{1 - \overline{f_0}(f_1 + \lambda g_1)} = \frac{\lambda g_1 + (f_1 - f_0)}{-\lambda \overline{f_0} g_1 + (1 - \overline{f_0} f_1)}.$$

Let $\lambda_0 \in A$. Set $F = \lambda_0 g_1 + (f_1 - f_0)$ and $G = 1 - \overline{f_0}(f_1 + \lambda_0 g_1)$. Clearly $G$ has modulus uniformly bounded below by $1 - ||f_0|| = \beta > 0$, so $1/G \in C(K)$. Whenever $|\lambda - \lambda_0| < \beta$, we have

$$\psi(\lambda) = \frac{(\lambda - \lambda_0)g_1 + \lambda_0 g_1 + (f_1 - f_0)}{-\lambda g_1 f_0 + \lambda_0 g_1 f_0 + (1 - \overline{f_0} f_1)} = \frac{(\lambda - \lambda_0) g_1 + F}{G} \sum_{n=0}^{\infty} \frac{(g_1 f_0)^n}{(G^n)} = (\frac{\lambda - \lambda_0}{G}) \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n (g_1 f_0)^n,$$

which converges uniformly in $\lambda$ when $|\lambda - \lambda_0| \leq \beta_0 < \beta$.

Hence $\psi$ is holomorphic, and consequently so is $T$.

$$\square$$

Let us note as a side remark that we can extend the previous conclusion to the open unit ball of the second dual of $C(K)$: Rewrite $\frac{f - f_0}{1 - \overline{f_0} f}$ as $(f - f_0) \sum_{n=0}^{\infty} (\overline{f_0} f)^n$. Since it is known that $C(K)^{**}$ is a commutative $C^*$–algebra that extends the $C^*$ structure of $C(K)$ (see [14, 310-311] and [36, p. 43]), then Proposition 2.5 extends in the following manner.
Proposition 2.6. Given \( f_0^{**} \in B_{C(K)^{**}} \), let \( T_{f_0^{**}} : B_{C(K)^{**}} \to B_{C(K)^{**}} \) be given by

\[
T_{f_0^{**}}(f^{**}) = (f^{**} - f_0^{**}) \sum_{n=0}^{\infty} (f_0^{**} f^{**})^n , \forall f^{**} \in B_{C(K)^{**}}.
\]

Then \( T_{f_0^{**}} \) is biholomorphic.

Imitating Lemmas 4.3 and 4.4 in [5], we prove in [27] that for an arbitrary \( C(K) \) space, the family of automorphisms \( (T_{f_0})_{f_0 \in B} \) of \( B \), given by \( T_{f_0} : f \to \frac{f - f_0}{1 - |f_0|} \) satisfies the following proposition:

Proposition 2.7. For each \( f_0 \in B \), the biholomorphic function \( T_{f_0} \) induces a homeomorphism \( \hat{T}_{f_0} \) on the spectrum \( M_{H(B)} \), where \( H \) denotes either the algebra \( A_u \) or the algebra \( H^\infty \), that maps \( M_{f_0}(B) \) homeomorphically onto \( M_0(B) \) (where \( M_{f_0}(B) = \{ \tau \in M_{H^\infty(B)} : \tau|_{A_u(B)} = f_0 \} \)).

Proof. Note that \( T := T_{f_0} \) is a Lipschitz function. Indeed, if \( f, g \in B \),

\[
||T(f) - T(g)|| = ||\frac{(1 - |f_0|^2)(f - g)}{(1 - f_0 f)(1 - f_0 g)}|| \leq \frac{1}{(1 - ||f_0||)^2}||f - g||.
\]

Thus for every \( \psi \in H(B), \psi \circ T \in H(B). \) So \( \hat{T} : M_{H(B)} \to M_{H(B)} \), given by

\[
\hat{T} \tau(\psi) = \tau(\psi \circ T), \forall \tau \in M_{H(B)}, \psi \in H(B),
\]

is well defined. Moreover, given \( \tau \in M_{f_0}(B) \) and \( \psi \in A_u(B) \),

\[
\hat{T} \tau(\psi) = \tau(\psi \circ T) = (\psi \circ T)(f_0) = \psi(0),
\]

i.e. \( \hat{T} \tau \in M_0(B) \), for every \( \tau \in M_{f_0}(B) \).
Now, given $\tau \in \mathcal{M}_0(B)$ it is clear that $\hat{\tau} : H(B) \to \mathbb{C}$ given by

$$\hat{\tau}(\psi) = \tau(\psi \circ T^{-1}), \quad \forall \psi \in H(B),$$

is in $M_{H(B)}$, actually in $M_{f_0}(B)$, and $\forall \psi \in H(B)$,

$$\hat{T}(\hat{\tau})(\psi) = \hat{\tau}(\psi \circ T) = \tau(\psi),$$

i.e. $\hat{T}(\hat{\tau}) = \tau$. \hfill \Box

The reader can easily check that the previous mapping $\hat{T}$ is actually a homeomorphism.

As a consequence of Proposition 2.7, we obtain that the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at 0 is equivalent to the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at every $f_0 \in B$, when $X = C(K)$.

**Corollary 2.2.** If $X$ is a Banach space $C(K)$, then the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at 0 is equivalent to the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at every $f_0 \in B$.

**Proof.** Let $f_0 \in B$ and set $T(f) = \frac{f - f_0}{1 - f_0 f}$ for $f \in B$. Then, $\forall \psi \in H^\infty(B)$,

$$\hat{\psi}(\mathcal{M}_0(B)) = \hat{\psi} \circ \hat{T}(\mathcal{M}_{f_0}(B)) = \hat{\psi} \circ \hat{T}(\mathcal{M}_{f_0}(B)),
\text{Cl}_B(\psi, 0) = \text{Cl}_B(\psi \circ T, f_0),$$

because $\psi \circ T \in H^\infty(B)$ too, and $T^{-1}(f) = (f + f_0) \sum_{n=0}^{\infty} (-\overline{f_0} f)^n$ for $f \in B_{C(K)}$ is polynomially-star continuous, because sums and norm limits of polynomially-star continuous maps are polynomially-star continuous, as well as multiplication by a fixed element of $C(K)$. \hfill \Box
Let us note that, from the Gelfand Representation Theorem, this corollary holds for any nonzero unital commutative $C^*$-algebra.

The previous result is a reduction of a cluster value problem at any point in the ball of $C(K)$ to the origin. In the next section we explain a reduction of a cluster value problem for any separable Banach space to a space with a more specific structure.

2.4 The cluster value problem for separable spaces

In [28] we prove that for any separable Banach space $Y$, a cluster value problem for $H(B_Y)$ ($H = H^\infty$ or $H = A_u$) can be reduced to a cluster value problem for $H(B_X)$ for some Banach space $X$ that is an $\ell_1$-sum of a sequence of finite-dimensional spaces. The proof relies on the construction of an isometric quotient map from a suitable $X$ to $Y$ that induces an isometric algebra homomorphism from $H(B_Y)$ to $H(B_X)$ with 1-complemented range, where the projection mapping is built using ultrafilter techniques. Other tools include the infinite-dimensional version of Schwarz' Lemma, as well as such familiar one complex variable results as Cauchy's inequality and Montel's theorem. This is done in the following two lemmas:

Lemma 2.2. Let $Y$ be a separable Banach space and $Y_1 \subset Y_2 \subset Y_3 \subset \ldots$ an increasing sequence of finite dimensional subspaces whose union is dense in $Y$. Set $X = (\sum Y_n)_1$. Then the isometric quotient map $Q : X \to Y$ defined by

$$Q(z_n)_n := \sum_{n=1}^{\infty} z_n$$

induces an isometric algebra homomorphism $Q^\# : H(B_Y) \to H(B_X)$, where $H$ denotes either the algebra $A_u$ or the algebra $H^\infty$.

The idea behind is to use the little open mapping theorem to get $Q(B_X) = B_Y$. 36
Proof. Note that for all \((z_n)_n \in X\),

\[
\|Q(z_n)_n\| = \left\| \sum_{n=1}^{\infty} z_n \right\| \leq \sum_{n=1}^{\infty} \|z_n\| = \|(z_n)_n\|_1. \tag{2.2}
\]

Let \(\tilde{Y}_n = \{(z_n)_n \in X : z_k = 0 \, \forall \, k \neq n\}\). Since \(Q(B_{\tilde{Y}_n}) = B_{Y_n}\) for all \(n \in \mathbb{N}\), we now have that \(Q(B_X)\) is dense in \(B_Y\) and hence \(Q\) is an isometric quotient map.

Then the function \(Q^\#: H(B_Y) \to H(B_X)\) given by \(Q^\#(f) = f \circ Q\) is an isometric homomorphism because \(Q^\#\) is clearly linear and for all \(f, g \in H(B_Y)\),

\[
\|Q^\#(f)\| = \sup_{x \in B_X} |f \circ Q(x)| = \sup_{y \in B_Y} |f(y)| = \|f\|.
\]

Moreover,

\[
Q^\#(f \cdot g) = (f \cdot g) \circ Q = (f \circ Q) \cdot (g \circ Q) = Q^\#(f)Q^\#(g),
\]

so \(Q^\#\) is an algebra homomorphism.

Now we find a left inverse to \(Q^\#\), that allows us to go back to \(H(B_Y)\).

Lemma 2.3. Under the assumptions of the previous lemma, there is a norm one algebra homomorphism \(T : H(B_X) \to H(B_Y)\) so that \(T(X^*) \subset Y^*\) and \(T \circ Q^\# = I_{H(B_Y)}\).

Proof. The first part of the proof consists of constructing \(T\) and verifying that \(TH(B_X) \subset H(B_Y)\).

For every \(y \in (\bigcup Y_n)\) and \(n \in \mathbb{N}\), let \(S_n(y) = (z_i)_i \in X\) be given by
\[ z_i = \begin{cases} 
  y & \text{if } i = n \text{ and } y \in Y_n \\
  0 & \text{otherwise} 
\end{cases} \]

Let \( \mathcal{U} \) be a free ultrafilter on \( \mathbb{N} \). For each \( g \in H(B_X) \) set
\[ S_g(y) = \lim_{n \in \mathcal{U}} g(S_n y) \text{ for every } y \in B(\bigcup Y_n), \]
which is well defined because \( g \) is bounded. Next we prove that \( S_g \) is continuous:

Let \( 0 < r < 1 \). Since \( g \in H^\infty(B_X) \) then Schwarz' Lemma (Thm. 7.19, [34]) and the convexity of \( B_X \) imply that \( g \in A_u(\mathbb{R}B_X) \). Let \( \epsilon > 0 \). Since \( g \) is uniformly continuous on \( \mathbb{R}B_X \) there exists \( \delta > 0 \) such that, if \( a, b \in \mathbb{R}B_X \) and \( \|a - b\| < \delta \), then \( \|g(a) - g(b)\| < \epsilon \). Thus, given \( y_1, y_2 \in \mathbb{R}B(\bigcup Y_n) \) such that \( \|y_1 - y_2\| < \delta \), we can find \( N \in \mathbb{N} \) such that \( y_1, y_2 \in Y_n \) for \( n \geq N \), and then \( \|S_n(y_1) - S_n(y_2)\| = \|y_1 - y_2\| < \delta \) eventually for \( n \), so
\[ \|S_g(y_1) - S_g(y_2)\| = \lim_{n \in \mathcal{U}} \|g(S_n(y_1)) - g(S_n(y_2))\| \leq \epsilon. \]

Each \( S_g : B(\bigcup Y_n) \to \mathbb{C} \) is uniformly continuous on \( rB(\bigcup Y_n) \) for \( 0 < r < 1 \), thus we can continuously extend each \( S_g \) to \( T_g : B_Y \to \mathbb{C} \). Moreover, it is evident that \( T_g \) is uniformly continuous on \( B_X \) when \( g \in A_u(B_X) \). It is left to show that each \( T_g \) is analytic by checking that every \( T_g \) is analytic in each complex line (Thm. 8.7, [34]).

We do this in two parts.

**Step 1** Let us check that each \( S_g \) is analytic on complex lines:

Let \( y^1 \in B(\bigcup Y_n) \) and \( y^2 \neq 0 \in (\bigcup Y_n) \). Since \( B(\bigcup Y_n) \) is open we can find \( R > 0 \) such that, if \( \|y - y^1\| < R \), then \( y \in B(\bigcup Y_n) \). Choose \( r > 0 \) such that \( r\|y^2\| < R \). Thus, if \( |\lambda| \leq r \) we have that \( \lambda \in \Lambda = \{ \zeta \in \mathbb{C} : y^1 + \zeta y^2 \in B(\bigcup Y_n) \} \).
For each \( n \in \mathbb{N} \), let \( u_n = S_n(y_1) \) and \( w_n = S_n(y_2) \).

Using the notation at the beginning of Section 1.4, we claim that

\[
Sg(y_1 + \lambda y_2) = \sum_{m=0}^{\infty} (\lim_{n \in \mathcal{U}} P^m g(u_n)(w_n))\lambda^m,
\]

uniformly on \( \lambda \), for \( |\lambda| \leq r \). Let us start by showing that for each \( m \in \mathbb{N} \) and \( \lambda \in \bar{\Delta}(0, r) \), \( \lim_{n \in \mathcal{U}} P^m g(u_n) (\lambda w_n) \) exists.

We can find \( s > 1 \) such that also \( sr\|y_2\| < R \). Then, when \( |t| \leq s \) and \( |\lambda| \leq r \), we have that \( y_1 + t\lambda y_2 \in B_{\cup Y_n} \), because

\[
\|(y_1 + t\lambda y_2) - y_1\| = |t|\|\lambda\|\|y_2\| \leq sr\|y_2\| < R,
\]

so \( u_n + t\lambda w_n \in B_X \) eventually for \( n \), and from Cauchy’s Inequality (Cor. 7.4, [34]), for each \( m \in \mathbb{N} \)

\[
\|P^m g(u_n) (\lambda w_n)\| \leq \frac{1}{s^m} \|g\|,
\]

eventually for \( n \), and then \( \lim_{n \in \mathcal{U}} P^m g(u_n) (\lambda w_n) \) exists.

Moreover, given \( M \in \mathbb{N} \) and \( \lambda \) such that \( |\lambda| \leq r \),

\[
\|Sg(y_1 + \lambda y_2) - \sum_{m=0}^{M} (\lim_{n \in \mathcal{U}} P^m g(u_n)(w_n))\lambda^m\|
\]

\[
= \| \lim_{n \in \mathcal{U}} (g(u_n + \lambda w_n) - \sum_{m=0}^{M} P^m g(u_n)(\lambda w_n)) \|
\]

\[
= \| \lim_{n \in \mathcal{U}} \sum_{m=M+1}^{\infty} P^m g(u_n)(\lambda w_n) \|
\]

\[
\leq \sum_{m=M+1}^{\infty} \frac{1}{s^m} \|g\| = \frac{\|g\|}{s^{M+1}} \frac{1}{s^r},
\]

which goes to zero as \( M \to \infty \).
Thus $Sg$ is analytic on complex lines.

Alternative proof of Step 1. Let $y^1 \in B_{(\cup Y_n)}$ and $y^2 \neq 0 \in (\cup Y_n)$. We want to show that $Sg(y^1 + \lambda y^2)$ is an analytic function of $\lambda$. It is enough to find a power series expansion.

Let $d = d_{B_{(\cup Y_n)}}(y^1)$ and let $r \in (0, d/\|y^2\|)$. Then $y^1 + \lambda y^2 \in B_{(\cup Y_n)}$ if $|\lambda| \leq r$.

For each $n \in \mathbb{N}$, let $u_n = S_n(y^1)$ and $w_n = S_n(y^2)$. Note that

$$Sg(y^1 + \lambda y^2) = \lim_{n \to \infty} g(u_n + \lambda w_n)$$

for $|\lambda| \leq r$.

Let $g_n(\lambda) = g(u_n + \lambda w_n)$ for $|\lambda| \leq r$. Since $g$ is analytic on complex lines then

$$g_n(\lambda) = \sum_{m=0}^{\infty} \frac{g_n^{(m)}(0)\lambda^m}{m!}$$

uniformly on $\lambda$, for $|\lambda| \leq r$,

because $\|(u_n + \lambda w_n) - u_n\| \leq r\|y^2\| < d_{B_{(\cup Y_n)}}(y^1) \leq d_{B_X}(u_n)$.

Now we claim that

$$Sg(y^1 + \lambda y^2) = \sum_{m=0}^{\infty} \left( \lim_{n \to \infty} g_n^{(m)}(0)/m! \right) \lambda^m$$

uniformly on $\lambda$, for $|\lambda| \leq r$. Let us first show that $\lim_{n \to \infty} g_n^{(m)}(0)/m!$ exists $\forall m \in \mathbb{N}_0$.

Let $r_2 \in (0, r)$. Since

$$g_n^{(m)}(0) = \frac{m!}{2\pi i} \int_{|\zeta| = r_2} \frac{g_n(\zeta)}{\zeta^{m+1}} d\zeta$$

then $\|g_n^{(m)}(0)/m!\| \leq \frac{1}{r_2^m} \|g\|$, so $\lim_{n \to \infty} g_n^{(m)}(0)/m!$ exists.
Moreover, for all $M \in \mathbb{N}_0$ and $|\lambda| \leq r_1 < r_2$,

$$\|Sg(y^1 + \lambda y^2) - \sum_{m=0}^{M} (\lim_{n \in \mathcal{U}} g_n^{(m)}(0)/m!\lambda^m)\|$$

$$= \| \lim_{n \in \mathcal{U}} (g_n(\lambda) - \sum_{m=0}^{M} g_n^{(m)}(0)/m!\lambda^m)\|$$

$$= \| \lim_{n \in \mathcal{U}} \sum_{m=M+1}^{\infty} g_n^{(m)}(0)/m!\lambda^m\|$$

$$\leq \sum_{m=M+1}^{\infty} \left( \frac{r_1}{r_2} \right)^m \|g\| = \left[ \frac{\|g\|}{1-r_{1}/r_2} \right] (r_1/r_2)^{M+1},$$

which goes to zero as $M \to \infty$.

**Step 2** The following general lemma should be known, but we could not find a reference.

**Lemma 2.4.** If $\phi : B_Y \to \mathbb{C}$ is bounded and uniformly continuous on $sB_Y$ for each $0 < s < 1$, and there is a dense subspace $Z$ of $Y$ such that $\phi|_Z$ is analytic on complex lines, then $\phi$ is analytic on complex lines in $B_Y$ (and hence analytic).

**Proof.** Let $y^1 \in B_Y$ and $y^2 \neq 0 \in Y$. Let $s \in (\|y^1\|, 1)$. Since $sB_Y$ is open and contains $y^1$, we can find $R > 0$ such that, if $\|y - y^1\| < R$ then $y \in sB_Y$. Choose $r > 0$ such that $r\|y^2\| \leq R$. Thus, if $|\lambda| < r$ we have that $\lambda \in \Lambda = \{ \zeta \in \mathbb{C} : y^1 + \zeta y^2 \in sB_Y \}$.

Let $f : \lambda \to \phi(y^1 + \lambda y^2)$, a function defined for $|\lambda| < r$. We want to show that $f$ is analytic.

Let $\{y^1_k\}_k \subset B_Z$ and $\{y^2_k\}_k \subset Z$ be sequences such that $\|y^1_k - y^1\| \leq \frac{1}{2^k}$ and $\|y^2_k - y^2\| \leq \frac{1}{2^k}$. Choose $K_1 \in \mathbb{N}$ such that $\frac{1+r}{2^{K_1}} \leq R - r\|y^2\|$. Then for $k \geq K_1$ and
$|\lambda| < r$ we have that
\[
\|y_k^1 + \lambda y_k^2 - y^1\| \leq \|y_k^1 - y^1\| + |\lambda|\|y_k^2 - y^2\| + |\lambda|\|y^2\| \\
< \frac{1+r}{2|\lambda|} + r\|y^2\| \\
< R,
\]
so $y_k^1 + \lambda y_k^2 \in sB_Z$.

For each $k \geq K_1$, let $f_k : \lambda \rightarrow \phi(y_k^1 + \lambda y_k^2)$, which is an analytic function for $|\lambda| < r$ by assumption.

Since $\phi$ is bounded, clearly \{${f_k}$\}$_{k \geq K_1}$ is uniformly bounded. Let us now show that \{${f_k}$\}$_{k \geq K_1}$ converges uniformly to $f$. Let $\epsilon > 0$. Since $\phi$ is uniformly continuous on $sB_Z$, we can find $\delta > 0$ such that,
\[
\|a, b \in sB_Z, \quad ||a - b|| < \delta \implies ||\phi(a) - \phi(b)|| < \epsilon.
\]

Choose $K \geq K_1$ such that $\frac{1+r}{2K} < \delta$. Then $\forall k \geq K$ and $\lambda$ with $|\lambda| < r$,
\[
\|y_k^1 + \lambda y_k^2 - (y^1 + \lambda y^2)\| \leq \|y_k^1 - y^1\| + |\lambda|\|y_k^2 - y^2\| \\
< \frac{1+r}{2K} \\
< \delta,
\]
so $\|f_k(\lambda) - f(\lambda)\| = ||\phi(y_k^1 + \lambda y_k^2) - \phi(y^1 + \lambda y^2)|| < \epsilon$.

Then, by the lemma on p. 226 in [2], $f$ is analytic.

**Alternative proof.** Let $y^1 \in B_Y$ and $y^2 \neq 0 \in Y$. Let $s \in (||y^1||, 1)$. Since $sB_Y$ is open
and contains $y^1$, we can find $R > 0$ such that, if $\|y - y^1\| < R$ then $y \in sB_Y$.

Choose $r > 0$ such that $r\|y^2\| < R$. Thus, if $|\lambda| \leq r$ we have that

$$\lambda \in \Lambda = \{\zeta \in C : y^1 + \zeta y^2 \in sB_Y\}. $$

Let $\{y^1_k\} \subset B_Z$ and $\{y^2_k\} \subset Z$ be sequences such that $\|y^1_k - y^1\| \leq \frac{1}{2^1}$ and $\|y^2_k - y^2\| \leq \frac{1}{2^r}$. Choose $K_1 \in \mathbb{N}$ such that $\frac{1+r}{2^K_1} < R - r\|y^2\|$. Then for $k \geq K_1$ and $|\lambda| \leq r$ we have that

$$\|(y^1_k + \lambda y^2_k) - y^1\| \leq \|y^1_k - y^1\| + |\lambda|\|y^2_k - y^2\| + |\lambda|\|y^2\|$$

$$\leq \frac{1+r}{2^K_1} + r\|y^2\|$$

$$< R,$$

so $y^1_k + \lambda y^2_k \in sB_Z$.

For $k \geq K_1$, Let $\varphi_k$ be the restriction of $\phi$ to $\{y^1_k + \lambda y^2_k : |\lambda| \leq r\}$. Then for $k \geq K_1$,

$$\varphi_k(y^1_k + \lambda y^2_k) = \sum_{m=0}^{\infty} (P^m \varphi_k(y^1_k)(y^2_k))\lambda^m,$$

uniformly on $\lambda$, whenever $|\lambda| \leq r$.

We claim that

$$\phi(y^1 + \lambda y^2) = \sum_{m=0}^{\infty} \lim_{k \to \infty} (P^m \varphi_k(y^1_k)(y^2_k))\lambda^m,$$

uniformly on $\lambda$, whenever $|\lambda| \leq r$. Let us start by showing that we can take the limit when $k$ goes to $\infty$.

Let $\epsilon > 0$ and choose $\delta > 0$ such that, if $a, b \in sB_Y$ and $\|a - b\| < \delta$, then $\|\phi(a) - \phi(b)\| < \epsilon$. Let $K \geq K_1$ be such that $\frac{1+r}{2^K} < \delta$. If $k_1, k_2 \geq K$, $|\lambda| \leq r$ and
$|\zeta| = 1$ then

$$\| (y_{k_1}^1 + \zeta \lambda y_{k_1}^2) - (y_{k_2}^1 + \zeta \lambda y_{k_2}^2) \| \leq \| y_{k_1}^1 - y_{k_2}^1 \| + |\zeta| \| \lambda \| \| y_{k_1}^2 - y_{k_2}^2 \| \leq \frac{1+r}{2K} < \delta,$$

so

$$\| \phi(y_{k_1}^1 + \zeta \lambda y_{k_1}^2) - \phi(y_{k_2}^1 + \zeta \lambda y_{k_2}^2) \| < \epsilon,$$

i.e.

$$\| \varphi_{k_1}(y_{k_1}^1 + \zeta \lambda y_{k_1}^2) - \varphi_{k_2}(y_{k_2}^1 + \zeta \lambda y_{k_2}^2) \| < \epsilon.$$

Hence, from the Cauchy Integral Formula, for every $m \in \mathbb{N}$

$$\| P^m \varphi_{k_1}(y_{k_1}^1)(\lambda y_{k_1}^2) - P^m \varphi_{k_2}(y_{k_2}^1)(\lambda y_{k_2}^2) \| = \| \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\varphi_{k_1}(y_{k_1}^1 + \zeta \lambda y_{k_1}^2) - \varphi_{k_2}(y_{k_2}^1 + \zeta \lambda y_{k_2}^2)}{\zeta^{m+1}} d\zeta \| < \epsilon.$$

Now, since $\frac{1+r}{2K} < R - r \| y^2 \|$ we can find $s_1 > 1$ such that $s_1 r (\frac{1}{2K} + \| y^2 \|) < R - \frac{1}{2K_1}$. Then, when $|t| \leq s_1$, $k \geq K_1$ and $|\lambda| \leq r$ we have that $y_k^1 + t \lambda y_k^2 \in B_Z$, because

$$\| (y_k^1 + t \lambda y_k^2) - y^1 \| \leq \| y_k^1 - y^1 \| + |t| \| \lambda \| \| y_k^2 \| \leq \frac{1}{2K_1} + s_1 r (\frac{1}{2K_1} + \| y^2 \|) < R,$$

so from Cauchy’s Inequality, for each $m \in \mathbb{N}$

$$\| P^m \varphi_k(y_k^1)(\lambda y_k^2) \| \leq \frac{1}{s_1^m} \| \phi \|.$$
Thus, given $M \in \mathbb{N}$ and $\lambda$ such that $|\lambda| \leq r$,

\[
\|\phi(y^1 + \lambda y^2) - \sum_{m=0}^{M} \lim_{k \to \infty} (P^m \varphi_k(y^1_k)(y^2_k))\lambda^m\|
\]

\[
= \| \lim_{k \to \infty} (\varphi_k(y^1_k + \lambda y^2_k) - \sum_{m=0}^{M} (P^m \varphi_k(y^1_k)(y^2_k))\lambda^m)\|
\]

\[
= \| \lim_{k \to \infty} \sum_{m=M+1}^{\infty} (P^m \varphi_k(y^1_k)(y^2_k))\lambda^m\|
\]

\[
\leq \sum_{m=M+1}^{\infty} \frac{1}{s_1^m} \|\phi\| = \frac{\|\phi\|}{s_1-1} \frac{1}{s_1^M},
\]

which goes to zero as $M \to \infty$.

Thus $\phi$ is analytic on complex lines in $B_Y$.

\[\square\]

From the previous two steps, we obtain that $T$ is a well defined mapping from $H(B_X)$ into $H(B_Y)$. Now, given $x^* \in X^*$, $y^1, y^2 \in B_Y$ and $\lambda \in \mathbb{C}$ such that $y^1 + \lambda y^2 \in B_Y$, we can find $\{y^1_k\}_k \subset B(\cup Y_n)$ converging to $y^1$ and $\{y^2_k\}_k \subset B(\cup Y_n)$ converging to $y^2$, and then

\[
Tx^*(y^1 + \lambda y^2) = \lim_{k \to \infty} \lim_{n \in \mathcal{U}} x^*(S_n(y^1_k + \lambda y^2_k))
\]

\[
= \lim_{k \to \infty} \lim_{n \in \mathcal{U}} (x^*(S_n(y^1_k)) + \lambda x^*(S_n(y^2_k)))
\]

\[
= Tx^*(y^1) + \lambda Tx^*(y^2),
\]

i.e. $Tx^* \in Y^*$. This shows that $TB_X = B_Y^\ast$.

Moreover, for every $f \in H(B_Y)$ and $y \in B_Y$, we can find $\{y_k\}_k \subset B(\cup Y_n)$ converg-
ing to \( y \), and thus

\[
T \circ Q^\#(f)(y) = T(f \circ Q)(y) \\
= \lim_{k \to \infty} \lim_{n \in U} f \circ Q(S_n(y_k)) \\
= \lim_{k \to \infty} f(y_k) \\
= f(y),
\]

so \( T \circ Q^\# = I_{H(B_Y)} \).

Also, \( T \) is a homomorphism because \( T \) is clearly linear and for all \( f, g \in H(B_X) \), \( y \in B_Y \), we can find \( \{y_k\}_k \subset B_{(\cup Y_n)} \) converging to \( y \), so

\[
T(f \cdot g)(y) = \lim_{k \to \infty} \lim_{n \in U} f \cdot g(S_n(y_k)) \\
= \lim_{k \to \infty} f(S_n(y_k)) \cdot g(S_n(y_k)) \\
= Tf(y) \cdot Tg(y) \\
= (Tf \cdot Tg)(y).
\]

Finally, for every \( f \in H(B_X) \),

\[
\|Tf\| = \sup_{y \in B_Y} |Tf(y)| = \sup_{y \in B_{(\cup Y_n)}} |\lim_{n \in U} f(S_n(y))| \leq \sup_{x \in B_X} |f(x)| = \|f\|,
\]

and \( \|T\| = \|T\|\|Q^\#\| \geq \|T \circ Q^\#\| = 1 \). So \( \|T\| = 1 \). 

\[\square\]

**Theorem 2.3.** Let \( Y \) be a separable Banach space and \( Y_1 \subset Y_2 \subset Y_3 \subset \ldots \) an increasing sequence of finite dimensional subspaces whose union is dense in \( Y \). Set \( X = (\sum Y_n)_1 \). Let \( H \) denote either the algebra \( A_u \) or the algebra \( H^\infty \). If \( H(B_X) \)
satisfies the cluster value theorem at every \(x^{**} \in \overline{B_X}^{**}\) then \(H(B_Y)\) satisfies the cluster value theorem at every \(y^{**} \in \overline{B_Y}^{**}\).

**Proof.** We know that \(\hat{f}(M_{x^{**}}(B_X)) \subset Cl_{B_X}(f, x^{**})\), for all \(f \in H(B_X)\) and \(x^{**} \in \overline{B_X}^{**}\). Let us show that \(\hat{g}(M_{y^{**}}(B_Y)) \subset Cl_{B_Y}(g, y^{**})\) for all \(g \in H(B_Y)\) and \(y^{**} \in \overline{B_Y}^{**}\).

Let \(y^{**} \in \overline{B_Y}^{**}\), \(\tau \in M_{y^{**}}(B_Y)\) and \(g \in H(B_Y)\). Let \(T\) be the algebra homomorphism from \(H(B_X)\) to \(H(B_Y)\) constructed in Lemma 2.3. Then \(x^{**} = y^{**} \circ T \in \overline{B_X}^{**}\), \(Q^\#(g) \in H(B_X)\) and defining \(\tilde{\tau} = \tau \circ T\), we see that \(\tilde{\tau} \in M_{x^{**}}(B_X)\) because for all \(x^* \in X^*\),

\[
\tilde{\tau}(x^*) = \tau(T x^*) = < y^{**}, T x^* > = < x^{**}, x^* >.
\]

Moreover,

\[
\overline{Q^\#(g)}(\tilde{\tau}) = \tilde{\tau}(Q^\#(g)) = \tau(T \circ Q^\#(g)) = \tau(g) = \hat{g}(\tau)
\]

and

\[
Cl_{B_X}(Q^\#(g), x^{**}) = Cl_{B_X}(g \circ Q, x^{**}) \subset Cl_{B_Y}(g, Q^{**}x^{**}) = Cl_{B_Y}(g, y^{**}),
\]

so the theorem is established. \(\square\)

**Remark 2.1.** A very special case of Theorem 2.3 is that if \(\ell_1\) satisfies the cluster value theorem, then so does \(L_1\). We do not know for \(1 < p \neq 2 < \infty\) whether the cluster value theorem for \(\ell_p\) implies the cluster value theorem for \(L_p\). Incidentally, in [5] it was proved that \(\ell_p\) for \(p\) in this range satisfies the cluster value theorem at 0, but it is open whether \(L_p\) satisfies the cluster value theorem at any point of \(B_{L_p}\).
Remark 2.2. The analogue of Theorem 2.3 for non separable spaces, which is proved by a non essential modification of the proof of Theorem 2.3, can be stated as follows. Let $Y$ be a Banach space and $(Y_\alpha)_{\alpha \in A}$ a family of finite dimensional subspaces of $Y$ that is directed by inclusion and whose union is dense in $Y$. If $(\sum_{\alpha \in A} Y_\alpha)_1$ satisfies the cluster value theorem, then so does $Y$.

Remark 2.3. There is a slight strengthening of Theorem 2.3. Let $Y$, $(Y_n)_n$, and $H$ be as in the statement of Theorem 2.3 and suppose that $(X_n)_n$ is a sequence so that $X_n$ is $1 + \epsilon_n$-isomorphic to $Y_n$ and $\epsilon_n \to 0$. If $(\sum_n X_n)_1$ satisfies the cluster value theorem for the algebra $H$, then so does $Y$. Now let $(Z_n)$ be a sequence of finite dimensional spaces so that for every finite dimensional space $Z$ and every $\epsilon > 0$, the space $Z$ is $1 + \epsilon$-isomorphic to one (and hence infinitely many) of the spaces $Z_n$. Set $C_1 = (\sum_n Z_n)_1$. As an immediate consequence of this slight improvement of Theorem 2.3 we get If $C_1$ satisfies the cluster value theorem for $H$, then so does every separable Banach space.

The proof of the improved Theorem 2.3 is essentially the same as the proof of the theorem itself. One just needs to define in Lemma 2.2 the mapping $Q$ so that the conclusion of Lemma 2.2 remains true: For each $n$ take an isomorphism $J_n : X_n \to Y_n$ so that for $x \in X_n$ the inequality $(1 + \epsilon_n)^{-1} \|x\| \leq \|J_n x\| \leq \|x\|$ is valid, and define $Q(x_n)_n = \sum_n J_n x_n$ for $(x_n)_n$ in $(\sum_n X_n)_1$. 

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3. CONCLUSION: OPEN CLUSTER VALUE PROBLEMS

3.1 Cluster value problems for $\ell_1$ and uniformly convex and uniformly smooth Banach spaces

There are plenty of open cluster value problems. The ideas of McDonald in [33] suggest that a solution to a $\bar{\partial}$ problem for the unit ball $B$ of a uniformly convex space or $\ell_1$ may help us solve a cluster value problem if the solution is weakly continuous. The $\bar{\partial}$ problem for the ball of a uniformly convex Banach space has not been solved yet, while the $\bar{\partial}$ problem for the unit ball of $\ell_1$ has been solved positively by Lempert in [32] under certain conditions. Let us describe the $\bar{\partial}$ problem in open subsets of Banach spaces:

Let $X$ and $Y$ denote complex Banach spaces, and $X_\mathbb{R}$ and $Y_\mathbb{R}$ denote the respective previous spaces seen as real Banach spaces. For every $m \in \mathbb{N}$, $L^m(X_\mathbb{R}, Y_\mathbb{R})$ denotes the continuous $m$-linear mappings $A : X_\mathbb{R}^m \to Y_\mathbb{R}$, while $L^a(m X_\mathbb{R}, Y_\mathbb{R})$ denotes the continuous $m$-linear mappings $A : X_\mathbb{R}^m \to Y_\mathbb{R}$ that are alternating, i.e.

$$A(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) = (-1)^{\sigma}A(x_1, \ldots, x_m), \ \forall \sigma \in S_m \text{ and } x_1, \ldots, x_m \in X.$$ 

Also, given $m \in \mathbb{N}$ and $p, q \in \mathbb{N}_0$ such that $p + q = m$, $L^{a(p,q)}(X_\mathbb{R}, Y_\mathbb{R})$ is the subspace of $A \in L^a(m X_\mathbb{R}, Y_\mathbb{R})$ such that

$$A(\lambda x_1, \ldots, \lambda x_m) = \lambda^p \bar{\lambda}^q A(x_1, \ldots, x_m), \ \forall \lambda \in \mathbb{C} \text{ and } x_1, \ldots, x_m \in X;$$

while $L^{apq}(m X_\mathbb{R}, Y_\mathbb{R})$ denotes the subspace of all $A \in L(m X_\mathbb{R}, Y_\mathbb{R})$ which are alternating in the first $p$ variables and are alternating in the last $q$ variables.

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The following definition can be found in [34, p.107].

**Definition 3.1.** Let \( U \) be an open subset of the complex Banach space \( X \) and let \( f : U \to Y \) be an \( \mathbb{R} \)-differentiable mapping. Let \( Df(a) \) denote the real differential of \( f \) at \( a \). Define \( Df(a) \) and \( D''f(a) \) by

\[
Df(a)(t) = 1/2[(Df(a)(t) - iDf(a)(it)),
\]

\[
D''f(a)(t) = 1/2[(Df(a)(t) + iDf(a)(it)],
\]

for every \( t \in X \). Note that \( Df(a) \) is \( \mathbb{C} \)-linear while \( D''f(a) \) is \( \mathbb{C} \)-antilinear.

Given \( A \in L^m(\mathbb{R}^X,\mathbb{R}^Y) \), define \( A^a \in L^a(\mathbb{R}^X,\mathbb{R}^Y) \) by

\[
A^a(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma A(x_{\sigma(1)}, \ldots, x_{\sigma(m)}), \quad \forall x_1, \ldots, x_m \in X.
\]

Given \( U \) an open subset of \( X \) and \( p, q \in \mathbb{N}_0 \), let \( C^\infty_{p,q}(U,Y) := C^\infty(U, L^a(p^mX, Y)) \). Then for each \( f \in C^\infty_{p,q}(U,Y) \), \( \partial f \in C^\infty_{p,q+1}(U,Y) \) is given by

\[
\partial f(x) = (m + 1)[D''f(x)]^a, \quad \forall x \in U.
\]

**Remark 3.1.** Since \( D''f(x) \in L(X, L^a(\mathbb{R}^X, \mathbb{R}^Y)) = L^{a1m}(\mathbb{R}^X, \mathbb{R}^Y) \), then Proposition 18.6 in [34] implies that \( \forall t_1, \ldots, t_{m+1} \in X \),

\[
\partial f(x)(t_1, \ldots, t_{m+1}) = (m + 1)[D''f(x)]^{a1m}(t_1, \ldots, t_{m+1})
\]

\[
= \frac{m + 1}{m + 1} \sum_{\sigma \in S_{1m}} (-1)^\sigma D''f(x)(t_{\sigma(1)}, \ldots, t_{\sigma(m+1)})
\]

where \( S_{1m} \) denotes the set of all permutations \( \sigma \in S_{m+1} \) such that \( \sigma(1) < \cdots < \sigma(m) \).
\[ \bar{\partial} f(x) (t_1, \ldots, t_{m+1}) = \sum_{j=1}^{m+1} (-1)^{j-1} D'' f(x) (t_j) (t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{m+1}). \]

The \( \bar{\partial} \) problem for \( g \in C^\infty_{p,q+1}(U,Y) \) asks whether the equation \( \bar{\partial} g = 0 \) implies the existence of \( f \in C^\infty_{p,q}(U,Y) \) such that \( \bar{\partial} f = g \).

Following the ideas of Lempert in [32] and of Kerzman in [30, pp.342-345], I aim to solve a cluster value problem for Banach spaces that are uniformly convex and uniformly smooth, and in general for those whose unit ball is strongly pseudoconvex. I will start with giving a definition of a strongly pseudoconvex domain in an infinite-dimensional Banach space.

3.2 Strong pseudoconvexity in infinite-dimensional Banach spaces

There are certain notions of pseudoconvexity in the literature ([34, p.274], [25, Theorem 2.6.12]) that hint towards a plausible extension of the definition of a strongly pseudoconvex domain to an infinite-dimensional Banach space (see [30] and [38]). As we will see, the following definitions build on the ideas exposed in Chapter VIII of [34].

**Definition 3.2.** Let \( U \) be an open and bounded subset of a complex Banach space \( X \). A function \( f : U \rightarrow [-\infty, \infty) \) is said to be strictly plurisubharmonic if \( f \) is upper semicontinuous and for each \( a \in U \) there exists \( C(a) > 0 \) such that

\[ C(a)\|b\|^2/4 \leq \frac{1}{2\pi} \int_0^{2\pi} (f(a + e^{i\theta}b) - f(a))d\theta \]

for each \( b \neq 0 \in X \) such that \( a + \Delta b \subset U \). If moreover \( C = \inf_{a \in U} C(a) > 0 \), we say that \( f \) is uniformly strictly plurisubharmonic.
Definition 3.3. An open and bounded subset $U$ of a complex Banach space $X$ is said to be strictly pseudoconvex if the function $-\log d_U$ is strictly plurisubharmonic on $U$ (where $d_U$ denotes the distance to the boundary of $U$); and $U$ is said to be uniformly strictly pseudoconvex if $-\log d_U$ is uniformly strictly plurisubharmonic on $U$.

Proposition 3.1. Let $U$ be an open and bounded subset of a complex Banach space $X$, and let $f \in C^2(U, \mathbb{R})$. Then $f$ is strictly plurisubharmonic if and only if for each $a \in U$ there exists $C(a) > 0$ such that $D'D'' f(a)(b, b) \geq C(a)\|b\|^2$, for each $b \neq 0 \in X$ such that $a + \Delta b \subset U$. And $f$ is uniformly strictly plurisubharmonic if and only if there exists $C > 0$ such that $D'D'' f(a)(b, b) \geq C\|b\|^2$, for each $a \in U$ and $b \neq 0 \in X$ such that $a + \Delta b \subset U$.

Proof. We will prove the first statement only (the second is proved similarly).

Suppose that $f$ is strictly plurisubharmonic. Then for each $a \in U$ there exists $C(a) > 0$ such that $C(a)\|b\|^2/4 \leq \frac{1}{2\pi} \int_0^{2\pi} (f(a + e^{i\theta}b) - f(a))d\theta$ for each $b \neq 0 \in X$ such that $a + \Delta b \subset U$. Fix $a \in U$ and $b \neq 0 \in X$ such that $a + \Delta b \subset U$, and consider the function $u(\zeta) = f(a + \zeta b)$, which is defined on a disk $\Delta(0, R) \supset \bar{\Delta}$.

Then, by Taylor’s Formula, for all $r \in (0, R)$,

$$u(re^{i\theta}) - u(0) = r \cos \theta \frac{\partial u}{\partial x}(0) + r \sin \theta \frac{\partial u}{\partial y}(0)$$

$$+ \frac{r^2}{2} \cos^2 \theta \frac{\partial^2 u}{\partial x^2}(s_{r, \theta} e^{i\theta}) + \frac{r^2}{2} \sin^2 \theta \frac{\partial^2 u}{\partial y^2}(s_{r, \theta} e^{i\theta})$$

$$+ r^2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y}(s_{r, \theta} e^{i\theta}), \text{ for some } s_{r, \theta} \in [0, r].$$
Thus, for all \( r \in (0, R) \),

\[
C(a) \frac{r^2 \|b\|^2}{4} \leq \frac{1}{2\pi} \int_0^{2\pi} (f(a + e^{i\theta}rb) - f(a))d\theta
= \frac{1}{2\pi} \int_0^{2\pi} (u(re^{i\theta}) - u(0))d\theta
= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{r^2}{2} \cos^2 \theta \frac{\partial^2 u}{\partial x^2}(sr, e^{i\theta}) + \frac{r^2}{2} \sin^2 \theta \frac{\partial^2 u}{\partial y^2}(sr, e^{i\theta}) + r^2 \cos \theta \sin \theta \frac{\partial^2 u}{\partial x \partial y}(sr, e^{i\theta}) \right) d\theta
\]

After dividing by \( r^2/4 \), by the Dominated Convergence Theorem we get

\[
C(a)\|b\|^2 \leq \frac{1}{\pi} \left( \frac{\partial^2 u}{\partial x^2}(0) + \frac{\partial^2 u}{\partial y^2}(0) \right) = \frac{\partial^2 u}{\partial \zeta \partial \bar{\zeta}}(0) = D'D''f(a)(b, b),
\]

where the last equality comes from exercises 35.B and 35.D in [34].

Now suppose that for each \( a \in U \) there exists \( C(a) > 0 \) such that \( D'D''f(a)(b, b) \geq C(a)\|b\|^2 \), for each \( b \neq 0 \in X \) such that \( a + \Delta b \subset U \).

Fix \( a \) and \( b \) as before, and define \( M(r) = \frac{1}{2\pi} \int_0^{2\pi} [f(a + re^{i\theta}b) - f(a)]d\theta \), for all \( r \in (0, 1] \). Consider also the function \( u(\zeta) = f(a + \zeta b) \) defined on a disk \( \Delta(0, R) \supset \bar{\Delta} \).

Then, for all \( \zeta \in \Delta(0, R) \),

\[
\frac{\partial^2 u}{\partial x^2}(\zeta) + \frac{\partial^2 u}{\partial y^2}(\zeta) = \frac{\partial^2 u}{\partial \zeta \partial \bar{\zeta}}(\zeta) = D'D''f(a + \zeta b)(b, b) \geq C(a)\|b\|^2.
\]

Since \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \), then

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) u(re^{i\theta})d\theta \geq C(a)\|b\|^2,
\]

i.e. \( M''(r) + \frac{1}{r} M'(r) \geq C(a)\|b\|^2 \), \( \forall r \in (0, 1) \).
Thus \((rM'(r) - C(a)\|b\|^2r^2)^r = rM''(r) + M'(r) - C(a)\|b\|^2r \geq 0 \forall r \in (0,1)\), so
\[ r(M'(r) - \frac{C(a)\|b\|^2}{2}r) \text{ is an increasing function of } r. \]
Since clearly \(r(M'(r) - \frac{C(a)\|b\|^2}{2}r) \to 0 \) as \(r \to 0\) (because \(M'\) is a bounded function on \((0, \epsilon)\) for some \(\epsilon > 0\)), we conclude
\[ r(M'(r) - \frac{C(a)\|b\|^2}{2}r) \geq 0 \forall r \in (0,1). \]
Hence \((M(r) - \frac{C(a)\|b\|^2}{4}r^2)^r \geq 0\) for every \(r > 0\), so \(M(r) - \frac{C(a)\|b\|^2}{4}r^2\) is an increasing function of \(r\). Since clearly \(M(r) - \frac{C(a)\|b\|^2}{4}r^2 \to 0 \) as \(r \to 0\) then \(M(r) \geq \frac{C(a)\|b\|^2}{4}r^2\) for each \(r \in (0,1)\).

Since \(M\) is continuous on \((0,1]\), we conclude that \(M(1) \geq \frac{C(a)\|b\|^2}{4}\), i.e. \(\frac{1}{2\pi} \int_0^{2\pi} [f(a + e^{i\theta}b) - f(a)] d\theta \geq \frac{C(a)\|b\|^2}{4}\), i.e. \(f\) is strictly plurisubharmonic.

Let us now exhibit some Banach spaces whose unit ball is uniformly strictly pseudoconvex. The following definition can be found in [11].

**Definition 3.4.** If \(0 < q < \infty\) and \(2 \leq r < \infty\), a continuously quasi-normed space \((X, \|\|)\) is \(r\)-uniformly PL-convex if and only if there exists \(\lambda > 0\) such that
\[
\left(\frac{1}{2\pi} \int_0^{2\pi} \|a + e^{i\theta}b\|^q d\theta\right)^{1/q} \geq \left(\|a\|^r + \lambda\|b\|^r\right)^{1/r}
\]
for all \(a\) and \(b\) in \(X\); we shall denote the largest possible value of \(\lambda\) by \(I_{r,q}(X)\).

Davis, Garling and Tomczak-Jaegermann proved in [11, Proposition 3.1] that \(I_{2,1}(\mathbb{C}) = 1/2\). Moreover, a simple modification of [11, Theorem 4.1] gives that \(L_p(\Sigma, \Omega, \mu)\) is 2-uniformly PL-convex (for \(q=1\)) when \(p \in [1,2]\), and actually \(I_{2,1}(L_p) = I_{2,1}(\mathbb{C}) = 1/2\).

The following theorem gives us that \(L_p(\Sigma, \Omega, \mu)\), for \(p \in [1,2]\), has a uniformly strictly pseudoconvex unit ball.

**Theorem 3.1.** If \(X\) is a 2-uniformly PL-convex Banach space for \(q = 1\) then \(B_X\) is uniformly strictly pseudoconvex.
Proof. Let \( a \in B_X \) and \( b \neq 0 \in X \) such that \( a + \Delta b \subset B_X \). Since \( a + \Delta b \) is compact, there exists \( r \in (0, 1) \) such that \( a + \Delta b \subset r B_X \). Hence,

\[
(\|a\|^2 + I_{2,1}(X)\|b\|^2)^{1/2} \leq \frac{1}{2\pi} \int_0^{2\pi} \|a + e^{i\theta} b\| d\theta \leq r.
\]

Let \( \lambda_0 = \sqrt{I_{2,1}(X) + 1 - 1} > 0 \). Observe that

\[
\|a\| + \lambda_0 \|b\|^2 \leq (\|a\|^2 + I_{2,1}(X)\|b\|^2)^{1/2}
\]

because

\[
(\|a\| + \lambda_0 \|b\|^2)^2 = \|a\|^2 + 2\lambda_0 \|a\| \|b\|^2 + \lambda_0^2 \|b\|^4 \leq \|a\|^2 + (2\lambda_0 + \lambda_0^2) \|b\|^2 = \|a\|^2 + ((\lambda_0 + 1)^2 - 1) \|b\|^2 = \|a\|^2 + I_{2,1}(X)\|b\|^2.
\]

Thus,

\[
\|a\| + \lambda_0 \|b\|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} \|a + e^{i\theta} b\| d\theta \leq r.
\]

Since \( x \mapsto -\log(1 - x) \) is a convex and increasing function on \([0, 1)\), an application

of Jensen’s inequality gives us that

\[
-\log(1 - (\|a\| + \lambda_0 \|b\|^2)) \leq \frac{1}{2\pi} \int_0^{2\pi} -\log(1 - \|a + e^{i\theta} b\|))d\theta.
\]

We will finish the proof by showing that \( C = 2\lambda_0 > 0 \) satisfies

\[
C\|b\|^2/4 - \log(1 - \|a\|) \leq -\log(1 - (\|a\| + \lambda_0 \|b\|^2))
\]

given that \( \|a\| + \lambda_0 \|b\|^2 \leq r < 1 \).

Letting \( t = \|a\| \) and \( s = \|b\|^2 \), we can see that (3.1) holds in case

\[
\beta := \inf\{\log\left(\frac{1-t}{1-t-\lambda_0 s}\right)/s : t \geq 0, s > 0, t + \lambda_0 s \leq r < 1\} \geq \lambda_0/2.
\]
Indeed, first note that for $1 > r > t \geq 0$ and $s > 0$,

$$\frac{1 - t}{1 - t - \lambda_0 s} \geq \frac{1 - t + \lambda_0 s}{1 - t} = 1 + \frac{\lambda_0 s}{1 - t}. $$

Hence $\beta \geq \inf_{t \in [0, r]} \inf_{s \in (0, \frac{r - t}{\lambda_0})} \log(1 + \frac{\lambda_0 s}{1 - t})/s$.

Now, for $t \in [0, r)$ fixed, $\log(1 + \frac{\lambda_0 s}{1 - t})/s$ is decreasing as a function of $s \in (0, \frac{r - t}{\lambda_0})$ because

$$\frac{d}{ds}\{\log(1 + \frac{\lambda_0 s}{1 - t})/s\} = -\frac{\log(1 + \frac{\lambda_0 s}{1 - t})}{s^2} + \frac{1}{s(1 + \frac{\lambda_0 s}{1 - t})} \cdot \frac{\lambda_0}{1 - t}$$

where the inequality $\frac{1}{x + 1} < \log(1 + \frac{1}{x})$ for $x > 0$ gives us

$$s^2 \cdot \frac{1}{s(1 + \frac{\lambda_0 s}{1 - t})} \cdot \frac{\lambda_0}{1 - t} = \frac{1}{\frac{1 - t}{\lambda_0} + 1} < \log(1 + \frac{\lambda_0 s}{1 - t})$$

i.e. $\frac{d}{ds}\{\log(1 + \frac{\lambda_0 s}{1 - t})/s\} < 0$ for every $s \in (0, \frac{r - t}{\lambda_0})$.

Consequently $\beta \geq \inf_{t \in [0, r]} \lambda_0 \log(1 + \frac{r - t}{(r - t) - r})$.

Finally, $\log(1 + \frac{r - t}{r - t})$ is increasing as a function of $t \in [0, r)$ because

$$\frac{d}{dt}\{\log(1 + \frac{r - t}{1 - t})\} = \frac{\log(2 - \frac{1 - r}{1 - t})}{(r - t)^2} + \frac{1}{r - t} \cdot \frac{1}{2 - \frac{1 - r}{1 - t}} \cdot \frac{- (1 - r)}{(1 - t)^2}$$

$$= \frac{1}{(r - t)^2} \cdot (\log(2 - \frac{1 - r}{1 - t}) - \frac{(1 - r)(r - t)}{(2 - \frac{1 - r}{1 - t})(1 - t)^2})$$

$$= \frac{1}{(r - t)^2} \cdot (\log(2 - \frac{1 - r}{1 - t}) - \frac{(1 - r)(r - t)}{((1 - t) + (r - t))(1 - t)})$$

$$= \frac{1}{(r - t)^2} \cdot (\log(1 + \frac{1 - r}{1 - t}) - \frac{(1 - r)}{(1 - t)^2} \cdot \frac{1}{(\frac{1 - r}{1 - t} + 1)})$$

$$> \frac{1}{(r - t)^2} \cdot (\log(1 + \frac{1}{\frac{1 - r}{1 - t}}) - \frac{1}{(\frac{1 - r}{1 - t} + 1)})$$

$$> 0,$$
because \( \log(1 + \frac{1}{x}) > \frac{1}{x + 1} \) whenever \( x > 0 \).

Therefore \( \beta \geq \lambda_0 \frac{\log(1 + r)}{r} \geq \frac{\lambda_0}{1 + r} \geq \frac{\lambda_0}{2} \), as we wanted to show.

3.3 Cluster value problem for \( H^\infty(B) \) and \( \bar{\partial} \) problem for \( B \) for \( \dim(X) < \infty \)

I am also interested in the cluster value problem for \( H^\infty(B) \) when \( B \) is the ball of a finite-dimensional Banach space, and in particular for the ball of \( \ell_1^n \). I anticipate that a solution to the \( \bar{\partial} \) problem for the ball of a finite-dimensional space, such as \( B_{\ell_1^n} \), interesting in its own light, may hint to a solution of the respective cluster value problem for \( \ell_1 \) and even \( \ell_1 \)-sums of finite-dimensional spaces. I intend to investigate whether Lemma 3.1 in [32] can be strengthened to a continuous solution in \( B \), as that would guarantee a solution to the respective cluster value problem for \( H^\infty(B) \).
REFERENCES


