APPLICATIONS OF POTENTIAL THEORY TO THE ANALYSIS OF PROPERTY (P_q)

A Dissertation

by

YUE ZHANG

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Emil J. Straube
Committee Members,	Harold P. Boas
	Alexei Poltoratski
	Michael Longnecker
Head of Department,	Emil J. Straube

August 2014

Major Subject: Mathematics

ABSTRACT

In the dissertation, we apply classical potential theory to study Property (P_q) and its relation with the compactness of the $\overline{\partial}$-Neumann operator N_q.

The main results in the dissertation consist of four parts. In the first part, we discuss the invariance property of Property (P_q) under holomorphic maps on any compact subset K in $\mathbb{C}^n$.

In the second part, we show that if a compact subset $K \subset \mathbb{C}^n$ has Property (P_q) $(q \geq 1)$, then for any q-dimensional affine subspace E in $\mathbb{C}^n$, $K \cap E$ has empty interior with respect to the fine topology in $\mathbb{C}^q$. We also discuss a special case of the converse statement on a smooth pseudoconvex domain when $q = 1$.

In the third part, we give two concrete examples of smooth complete Hartogs domains in $\mathbb{C}^3$ regarding the smallness of the set of weakly pseudoconvex points on the boundary. Both examples conclude that if the Hausdorff 4-dimensional measure of the set of weakly pseudoconvex points is zero then the boundary has Property (P_2).

In the fourth part, we introduce a variant of Property (P_{n-1}) on smooth pseudoconvex domains in $\mathbb{C}^n$ $(n > 2)$ which implies the compactness of the $\overline{\partial}$-Neumann operator N_{n-1}.

ACKNOWLEDGEMENTS

I would like to thank my advisor Emil Straube for his patience and guidance through my graduate study. In particular I appreciate his monograph [45] which significantly eased my study in the $\overline{\partial}$-Neumann theory and related topics in several complex variables. I also wish to thank him for providing suggestions and sharing his ideas with me through my research. In addition, I wish to thank him for his generous support for various conferences which kept me updated on the frontier of the current research field in several complex variables.

I would like to take this opportunity as well to thank Harold Boas for his help in my study. I appreciate his discussion with me at the beginning of my study on several complex variables and his generous help through my research. In addition, I wish to thank him for carefully reading the draft version of this dissertation and pointing out many inaccuracies and mistakes.

I would also like to thank Alexei Poltoratski who kindly introduced several monographs concerning harmonic analysis during my first year of graduate study, which gave me a solid background in classical analysis of one complex variable.

I owe a special thanks to the committee member Michael Longnecker for his help during my pursuing Ph.D. degree.

I would like to sincerely thank my family members in China for their patience and encouragement during my graduate studies.

Last but not least, I would like to thank graduate students Mustafa Ayyuru, Timothy Rainone, Kun Gou and postdoctoral fellow Yunus Zeytuncu for their discussion and help in my graduate studies.

TABLE OF CONTENTS

Page

ABSTRACT ii

ACKNOWLEDGEMENTS iii

TABLE OF CONTENTS iv

1. INTRODUCTION 1

2. BACKGROUND AND PRELIMINARY RESULTS 6

2.1 Levi pseudoconvexity and special boundary chart 6
2.2 The $\overline{\partial}$-Neumann Problem 7
2.3 Compactness and global regularity of the $\overline{\partial}$-Neumann operator N_q . . 9
2.4 Property (P_q) and its analysis property 11
2.5 Fine topology, basic potential theory and Property (P_q) 14

3. INVARIANCE PROPERTY OF PROPERTY (P_q) 18

3.1 Invariance property of Property (P_1) in $\mathbb{C}$ 18
3.2 Invariance property of Property (P_q) $(q \geq 2)$ 19

4. OBSTRUCTIONS TO PROPERTY (P_q) 26

4.1 Main Theorem 12 26
4.2 A partial result on the converse of Theorem 12 29

5. THE SMALLNESS OF THE WEAKLY PSEUDOCONVEX POINTS ON SMOOTH HARTOGS DOMAINS 32

5.1 First example 32
5.2 Second example 37

6. A VARIANT OF PROPERTY (P_{n-1}) ON SMOOTH PSEUDOCONVEX DOMAINS 41

6.1 Main Theorem 14 45

7. SUMMARY 48

REFERENCES 50

1. INTRODUCTION

The dissertation concerns the analysis of Property (P_q) in the $\overline{\partial}$-Neumann problem and the compactness of the $\overline{\partial}$-Neumann operator N_q on L^2-integrable forms.

Given a bounded pseudoconvex domain Ω in $\mathbb{C}^n$, the central problem in the $\overline{\partial}$-Neumann theory is to study whether there exists a bounded inverse of the complex Laplacian $\Box_q$ on the L^2-integrable forms of the domain Ω and if there exists such a bounded inverse operator, what the regularity property it has. We call the (bounded) inverse of $\Box_q$ as the $\overline{\partial}$-Neumann operator and denote it as N_q.

Hörmander ([27, 28]) showed that $\Box_q$ has a bounded inverse N_q on $L^2_{(0,q)}(\Omega)$ for bounded pseudoconvex domains. Kohn([31, 32, 33]) showed that N_q gains one derivative when the domain Ω is strictly pseudoconvex, hence has global regularity under above case. Catlin([7, 8, 10]) and D'Angelo([13, 14, 15]) introduced the finite type notion and connected the boundary geometry properties with the regularity property of the $\overline{\partial}$-Neumann operator: Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, then there exists a subelliptic estimate at a boundary point P if and only if P is a point of finite type.

We are interested in the case when N_q is compact but does not gain derivatives (the absence of subelliptic estimates). By [4] and [18], on any smooth convex domains, N_q is globally regular but can fail the compactness if the boundary contains a q-dimensional analytic variety. The compactness of N_q is also concerned in a number of useful consequences, which include the compactness of the commutators between the Bergman projection and multiplication operators ([11, 19]) regarding the Fredholm theory of Toeplitz operators ([26, 47]), existence or non-existence of Henkin-Ramirez type kernels ([24]) and certain C^* algebra results ([40]).

Based on Catlin's work ([9]), Property (P_q) implies compactness of N_q on smooth pseudoconvex domains. Within the framework of Choquet theory, Sibony ([41]) characterizes Property (P_q) by using potential theoretic tools. However, the gap between Property (P_q) of the boundary and the compactness of N_q is not clear on general pseudoconvex domains. Christ and Fu ([12]) showed that on a smooth complete pseudoconvex domain in $\mathbb{C}^2$, N_1 is compact if and only if $b\Omega$ has Property (P_1). Fu and Straube ([18]) showed that on any smooth convex domains, N_q is compact if and only if $b\Omega$ has Property (P_q).

In the first part of the dissertation, we study the invariance property of Property (P_q) on any compact subset K in $\mathbb{C}^n$. It is well known that biholomorphic mappings (smooth up to boundaries) preserve the compactness of N_q between smooth pseudoconvex domains on all levels of L^2-integrable forms (see for example in [39]). However, biholomorphic mappings are not known to preserve Property (P_q) of a compact subset in $\mathbb{C}^n$ when $q > 1$. We hope to stimulate further research in the same type of problems by our study in the first part of the dissertation, which would partially demonstrate the gap between Property (P_q) of the boundary and the compactness of N_q on pseudoconvex domains. By introducing a "twisted" type of Property (P_q) in the $\mathbb{C}^q$ subspace induced by a certain holomorphic map $\pi : \mathbb{C}^n \to \mathbb{C}^q$, we utilize the idea in [41] and obtain the following results:

Theorem 1. *Let K be a compact subset in $\mathbb{C}^n$ and $X = \pi(K) \subset \mathbb{C}^q$. Suppose $J_q^{new}(X) = X$ and for any point $x \in X$, each fiber $K \cap \pi^{-1}(x)$ has Property (P_q), then K has Property (P_q) in $\mathbb{C}^n$.*

We also obtain the following special result regarding the invariance property of Property (P_1) on the complex plane:

Proposition 1. *Let K be a compact subset in $\mathbb{C}$, and assume K has Property (P_1),* *then given any holomorphic mapping $F : \mathbb{C} \to \mathbb{C}$, $F(K)$ has Property (P_1).*

It is well known that containing a q-dimensional analytic polydisc in the boundary is an obstruction to Property (P_q). Based on Sibony's work ([41]), it is known that picking up P_q-hull is an obstruction to Property (P_q) of the boundary. In the second part of the dissertation, we apply classical potential theory results and associate the obstruction to Property (P_q) with the fine topology on the boundary, which generalizes Sibony's result ([41]) in the case of complex plane.

Theorem 2. *Let K be a compact subset of $\mathbb{C}^n$, let $1 \leq q \leq n$, and assume K has Property (P_q). Then for any q-dimensional affine subspace E in $\mathbb{C}^n$, $K \cap E$ has empty fine interior with respect to the fine topology in $\mathbb{C}^q$.*

We naturally ask whether the converse is true. Denote $\pi_P : \mathbb{C}^n \to \mathbb{C}^{n-1}$ the projection map from $\mathbb{C}^n$ onto the complex tangent space defined locally at a boundary point P on Ω. We have the following partial result regarding the case of $q = 1$ on smooth pseudoconvex domains:

Theorem 3. *Let Ω be a smooth pseudoconvex domain in $\mathbb{C}^n$ and K be the weakly pseudoconvex points in the boundary $b\Omega$. Assume that for any boundary point P and any complex line E in the complex tangent space at P, $E \cap \pi_P(K)$ has empty fine interior with respect to the fine topology in $\mathbb{C}$. Then K has Property (P_1) and hence the boundary $b\Omega$ has Property (P_1).*

In [41], Sibony showed that given a smooth pseudoconvex domain Ω in $\mathbb{C}^n$, if the set of the weakly pseudoconvex points on the boundary $b\Omega$ has Hausdorff 2-dimensional measure zero in $\mathbb{C}^n$, then the boundary $b\Omega$ has Property (P_1). In the third part of the dissertation, we explore two examples of smooth complete Hartogs domains in $\mathbb{C}^3$ regarding the smallness of weakly pseudoconvex points on the boundary.

Proposition 2. *Define a smooth complete Hartogs domain* $\Omega \subset \mathbb{C}^3$ *by:*

$$\Omega = \{(z_1, z_2, z_3) \mid |z_2|^2 + |z_3|^2 < e^{-\varphi(z_1)}, \quad z_1 \in \mathbb{D}(0,1)\}.$$

Assume $\varphi \in C^\infty(\mathbb{D}(0,1))$*,* φ *is subharmonic on* $\mathbb{D}(0,1)$ *and* φ *has extra regularity property such that boundary points* (z_1, z_2, z_3) *are strictly pseudoconvex when* $|z_1|$ *is close to 1. If the Hausdorff* 4*-dimensional measure of the weakly pseudoconvex points of* $b\Omega$ *is zero, then* $b\Omega$ *has Property* (P_1) *and the* $\overline{\partial}$*-Neumann operator* N_1 *is compact.*

Proposition 3. *Define a smooth complete Hartogs domain* $\Omega \subset \mathbb{C}^3$ *by:*

$$\Omega = \{(z_1, z_2, z_3) |\ |z_3|^2 < e^{-\varphi(z_1)-\psi(z_2)}, z_1 \in \mathbb{D}(0,1), z_2 \in \mathbb{D}(0,1)\}.$$

Assume that $\varphi, \psi \in C^\infty(\mathbb{D}(0,1))$ *and subharmonic on* $\mathbb{D}(0,1)$ *in the respective complex plane. Assume further that the boundary points* (z_1, z_2, z_3) *are strictly pseudoconvex when* (z_1, z_2) *is close to* $b(\mathbb{D}(0,1) \times \mathbb{D}(0,1))$*. If the Hausdorff* 4*-dimensional measure of the weakly pseudoconvex points of* $b\Omega$ *is zero, then* $b\Omega$ *has Property* (P_2) *and the* $\overline{\partial}$*-Neumann operator* N_2 *is compact.*

In the first example, the result is unexpected in the sense that the set of weakly pseudoconvex points would be expected to only have Property (P_2) when we assume its Hausdorff 4-dimensional measure is zero. This result indicates that Hausdorff measure is a crude tool to completely capture the information of Property (P_q) on higher levels of forms. In second example, we develop an approach to control second derivatives of the function λ occurring in the definition of Property (P_q) and we also utilize Boas's ([3]) idea of summing functions in the proof. The general case is still open as Sibony's approach ([41]) in the case of $q = 1$ cannot be carried over to the cases of higher level forms.

Besides Property (P_q), McNeal's ([37]) Property $(\widetilde{P_q})$ implies the compactness of N_q on smooth pseudoconvex domains in $\mathbb{C}^n$ and Straube's ([44]) "short time flow" condition implies the compactness of N_1 on smooth pseudoconvex domains in $\mathbb{C}^2$. In the fourth part of the dissertation, we develop a variant of Property (P_{n-1}) (denoted as Property $(P^{\#}_{n-1})$) on any smooth pseudoconvex domain in $\mathbb{C}^n$ $(n > 2)$ which implies the compactness of N_{n-1} on the domain. Our work is based on a Hörmander-Kohn-Morrey type formula developed by Ahn ([1]) and Zampieri ([46]). Our main theorem is the following:

Theorem 4. *Let $\Omega \subset \mathbb{C}^n$ $(n > 2)$ be a smooth bounded pseudoconvex domain. If $b\Omega$ has Property $(P^{\#}_{n-1})$, then the $\overline{\partial}$-Neumann operator N_{n-1} is compact on $L^2_{(0,n-1)}(\Omega)$.*

It is clear that Property (P_{n-2}) implies Property $(P^{\#}_{n-1})$, but it is still unclear what the relation is between Property (P_{n-1}) and Property $(P^{\#}_{n-1})$. Our definition of Property $(P^{\#}_{n-1})$ does not depend on the eigenvalues of the complex Hessian of λ in the definition of the original Property (P_{n-1}), indeed only the diagonal entries in the complex Hessian of λ are involved in our definition of Property $(P^{\#}_{n-1})$.

2. BACKGROUND AND PRELIMINARY RESULTS

Let Ω be a bounded domain in $\mathbb{C}^n$. Ω has a C^k smooth boundary ($1 \leq k \leq \infty$) if there exist an open neighborhood U of $\overline{\Omega}$ and a C^k smooth function ρ defined on U such that: $\Omega = \{z \in U | \rho(z) < 0\}$, the boundary $b\Omega = \{z \in U | \rho(z) = 0\}$, $U \backslash \overline{\Omega} = \{z \in U | \rho(z) > 0\}$ and $|\nabla \rho| > 0$ on $b\Omega$. ρ is called the defining function of Ω.

2.1 Levi pseudoconvexity and special boundary chart

We briefly discuss the Levi pseudoconvexity for a C^2 smooth domain Ω. Let Ω be a domain with C^2 smooth boundary, Ω is called pseudoconvex if:

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k \geq 0, \quad z \in b\Omega, \; w \in \mathbb{C}^n, \text{ with} \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j = 0. \tag{2.1}$$

We define the complex tangent space to $b\Omega$ at z by:

$$T_z^{\mathbb{C}}(b\Omega) = \{w \in \mathbb{C}^n | \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j = 0\}.$$

Given a C^2 smooth pseudoconvex domain, if the inequality (2.1) is strict at the boundary point P (for all $w \neq 0$), we say the boundary point P is a strictly pseudoconvex boundary point of $b\Omega$. We say the boundary point P is a weakly pseudoconvex point of $b\Omega$ if the left side of the inequality (2.1) equal to 0 at P for some $w \neq 0 \in T_P^{\mathbb{C}}(b\Omega)$. If each boundary point is a strictly pseudoconvex point, then we say the domain Ω is a strictly pseudoconvex domain. The quadratic form in (2.1), i.e., the restriction to $T_z^{\mathbb{C}}(b\Omega)$ of the complex Hessian of ρ, is called the Levi form of $b\Omega$ at z.

In general, we define a bounded domain Ω to be pseudoconvex if it can be exhausted by an increasing sequence of C^2 smooth strictly pseudoconvex subdomains. This definition agrees with our definition when the boundary is C^2 smooth. For equivalence of various definitions of pseudoconvexity, we refer the reader to [38].

For a C^2 smooth domain Ω, near a boundary point P, we choose vector fields $L_1, \cdots, L_{n-1}$ of type $(1,0)$ which are orthonormal and span $T_z^{\mathbb{C}}(b\Omega_\epsilon)$ for z near P, where $\Omega_\epsilon = \{z \in \Omega | \rho(z) < -\epsilon\}$. L_n is defined to be the complex normal and we can normalize the length of L_n to be 1. Note that $\{L_j\}_{j=1}^n$ locally induces an orthonormal coordinate system near the boundary point P.

Define $(1,0)$-forms $\{\omega_j\}_{j=1}^n$ to be the dual basis of $\{L_j\}_{j=1}^n$ near P. By taking wedge products of $\overline{\omega}_j$'s, we have a local orthonormal bases for $(0,q)$-forms ($q \geq 1$) near P. We say $\{\omega_j\}_{j=1}^n$, $\{L_j\}_{j=1}^n$ and their induced coordinates form a special boundary chart near P. The definition of special boundary chart is introduced in [16], we also refer the reader there for further details.

2.2 The $\overline{\partial}$-Neumann Problem

Spencer ([42]) and Garabedian ([22]) formulated the $\overline{\partial}$-Neumann problem to generalize Hodge theory to non-compact complex manifolds. One of the most important application of $\overline{\partial}$-Neumann Problem is to study the $\overline{\partial}$-problem. We refer the reader to [29] and [34] for a history of the $\overline{\partial}$-Neumann Problem.

We briefly discuss the set up of the $\overline{\partial}$-Neumann problem in this section. Let Ω be a bounded domain in $\mathbb{C}^n$ $(n \geq 2)$. Let $L^2_{(0,q)}(\Omega)$ be the space of $(0,q)$-forms $(1 \leq q \leq n)$ with L^2-integrable coefficients. Given any $(0,q)$-form u, we can write $u = \sum_J' u_J d\bar{z}_J$, and the L^2-norm is defined as $\| \sum_J' u_J d\bar{z}_J \|^2 = \sum_J' \int_\Omega |u_J|^2 dV(z)$, where the summation is over an increasing multi-index $J = (j_1, \ldots, j_q)$. $L^2_{(0,q)}(\Omega)$ is a Hilbert space with above norm and induced inner product. Define $\overline{\partial} : L^2_{(0,q)}(\Omega) \to$

$L^2_{(0,q+1)}(\Omega)$ by:

$$\overline{\partial}(\sum_J{}' u_J d\overline{z}_J) = \sum_{j=1}^n \sum_J{}' \frac{\partial u_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J,$$

where the derivatives are viewed as distributions. We denote the domain of $\overline{\partial}$ by $\text{dom}(\overline{\partial}) = \{u \in L^2_{(0,q)}(\Omega) | \overline{\partial} u \in L^2_{(0,q+1)}(\Omega)\}$. By functional analysis results, $\overline{\partial}$ is a linear, closed, densely defined operator on $L^2_{(0,q)}(\Omega)$ and hence has a Hilbert adjoint $\overline{\partial}^*$. We denote the domain of $\overline{\partial}^*$ by $\text{dom}(\overline{\partial}^*) = \{v \in L^2_{(0,q+1)}(\Omega) | \exists C > 0,\ |(v, \overline{\partial} u)| \leq C||u||, \forall u \in \text{dom}(\overline{\partial})\}$. When Ω has a C^2 smooth boundary, by using integration by parts, we know that given any $u \in C^1_{(0,q+1)}(\overline{\Omega})$, $u \in \text{dom}(\overline{\partial}^*)$ if and only if $\sum_{j=1}^n u_{jK} \frac{\partial \rho}{\partial z_j} = 0$ on $b\Omega$ for all multi-indices K of length q.

In a special boundary chart near any boundary point P of a C^2 smooth domain Ω, we have a simple expression for $\text{dom}(\overline{\partial}^*)$: given any $u \in C^1_{(0,q)}(\overline{\Omega})$ and u is supported in a special boundary chart, $u \in \text{dom}(\overline{\partial}^*)$ if and only if $u_J = 0$ on $b\Omega$ when $n \in J$. If $u = \sum_J' u_J \overline{\omega}_J$ in a special boundary chart, the tangential part of u is defined as $u_{\text{Tan}} = \sum_{n \notin J}' u_J \overline{\omega}_J$ and the normal part of u is defined as $u_{\text{Norm}} = \sum_{n \in J}' u_J \overline{\omega}_J$.

Now we define the complex Laplacian as $\Box_q u := \overline{\partial}^* \overline{\partial} u + \overline{\partial}\overline{\partial}^* u$ on $L^2_{(0,q)}$ forms. For any $u \in L^2_{(0,q)}(\Omega)$, we denote the domain of $\Box_q$ as $\text{dom}(\Box_q) := \{u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) |\ \overline{\partial} u \in \text{dom}(\overline{\partial}^*),\ \overline{\partial}^* u \in \text{dom}(\overline{\partial})\}$. Here we suppress the subscript of the level of the form in $\overline{\partial}$ and $\overline{\partial}^*$ for simplicity.

$\Box_q$ is a densely defined, closed and unbounded linear operator on $L^2_{(0,q)}(\Omega)$. The $\overline{\partial}$-Neumann problem is to find a solution to $\Box_q u = f$ on Ω for $u \in \text{dom}(\Box_q)$. Whether $\Box_q$ has an bounded inverse on $L^2_{(0,q)}(\Omega)$ is the central question in the $\overline{\partial}$-Neumann problem. We call the (bounded) inverse operator of $\Box_q$ as the $\overline{\partial}$-Neumann operator, and denote it as N_q.

In early 1960s, Kohn ([30]) proved that $\Box_q$ does have a bounded inverse N_q on $L^2_{(0,q)}(\Omega)$ for strictly pseudoconvex domains. Hörmander ([27, 28]) went further and showed that $\Box_q$ has a bounded inverse N_q on $L^2_{(0,q)}(\Omega)$ for bounded pseudoconvex domains.

2.3 Compactness and global regularity of the $\overline{\partial}$-Neumann operator N_q

From a partial differential equations perspective, it is natural to study the global regularity property of N_q. We say the $\overline{\partial}$-Neumann operator N_q is globally regular on Ω if for any $u \in C^\infty_{(0,q)}(\overline{\Omega})$, $N_q u \in C^\infty_{(0,q)}(\overline{\Omega})$. Starting from the early 1960s, plenty of important results have been obtained regarding the global regularity of the $\overline{\partial}$-Neumann operator N_q in the perspective of geometric analysis and partial differential equations. Kohn([31, 32, 33]) showed that N_q gains one derivative when the domain Ω is strictly pseudoconvex, hence has global regularity under above case. Catlin([7, 8, 10]) and D'Angelo([13, 14, 15]) introduced the finite type notion and characterized the existence of a subelliptic estimate (with a fractional gain of less than one derivative) at a boundary point P: let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, then there exists a subelliptic estimate at a boundary point P if and only if P is a point of finite type. In particular for any strictly pseudoconvex point P on a C^2 smooth pseudoconvex domain, P is a point of finite type. It is then clear that if a smooth pseudoconvex domain is a domain of finite type (all boundary points are of finite type), the $\overline{\partial}$-Neumann operator N_1 is global regular on Ω. We refer the reader to [5] for a more comprehensive survey of the global regularity of the $\overline{\partial}$-Neumann problem.

We are interested in the case when N_q is compact but does not gain derivatives (the absence of subelliptic estimates). In the perspective of functional analysis, N_q is said to be compact on $L^2_{(0,q)}(\Omega)$ if the image of the unit ball in $L^2_{(0,q)}(\Omega)$ under N_q is relatively compact in $L^2_{(0,q)}(\Omega)$. In the perspective of partial differential equations, it is known that the compactness of N_q can be characterized by compactness estimates:

Proposition 4 ([35]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \leq q \leq n$. Then the following are equivalent:*

(i) N_q is compact as an operator on $L^2_{(0,q)}(\Omega)$.

(ii) For every $\epsilon > 0$, there exists a constant C_ϵ such that we have the compactness estimate:

$$||u||^2 \leq \epsilon(||\overline{\partial}u||^2 + ||\overline{\partial}^* u||^2) + C_\epsilon ||u||^2_{-1} \text{ for } u \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*).$$

$|| \cdot ||_{-1}$ is the Sobolev W^{-1}-norm defined coefficientwise for any $(0,q)$-form u, i.e., a form $u = \sum'_J u_J d\overline{z}_J$ is in $W^{-1}(\Omega)$ if and only if $u_J \in W^{-1}(\Omega)$ for all J. In general, we define the Sobolev W^s-norm ($s \in \mathbb{R}$) for any $(0,q)$-form u in the same way as above: a form $u = \sum'_J u_J d\overline{z}_J$ is in $W^s(\Omega)$ if and only if $u_J \in W^s(\Omega)$ for all J. Proposition 4 is essentially folklore but see for example [35].

Kohn and Nirenberg ([35]) showed that on smooth bounded pseudoconvex domains, the compactness of N_q on $L^2_{(0,q)}(\Omega)$ implies global regularity of N_q. It has become clear in recent years that that global regularity of N_q is subtle, while the compactness of N_q is stronger, for these results we refer to [4, 18, 19, 45].

It is well known that compactness of N_q is a local property, the following proposition is taken from [45].

Proposition 5 ([45]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \leq q \leq n$. If for every boundary point P there exists a pseudoconvex domain U such that $P \in U$ and $U \cap \Omega$ is a domain and N_q on $U \cap \Omega$ is compact, then N_q on Ω is compact.*

Catlin ([9]) introduced the notion of Property (P) which implies the compactness of N_1 on smooth pseudoconvex domains and the assumption of smoothness can be removed by Straube's result ([43]). A natural generalization of the notion above to

Property (P_q) regarding the $\overline{\partial}$-Neumann operator N_q $(1 \leq q \leq n)$ can be carried out and Catlin's work still shows that Property (P_q) of the boundary $b\Omega$ implies the compactness of N_q.

Theorem 5 ([9])**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let $1 \leq q \leq n$. If $b\Omega$ satisfies Property (P_q), then N_q is compact.*

We postpone the definition of Property (P_q) to the next section and list the following two propositions which have been frequently used in the literature and in our dissertation when proving the compactness estimate of N_q. The first proposition below appears for example in [9] and [35] or more recently in [45].

Proposition 6 (Sobolev Interpolation)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Given any real number $s_1 > s > s_2$, for any $\epsilon > 0$, we have the following interpolation estimates on any $(0, q)$-form $u \in W^{s_1}(\Omega)$:*

$$||u||_s^2 \leq \epsilon ||u||_{s_1}^2 + C_\epsilon ||u||_{s_2}^2.$$

Here C_ϵ is independent of u.

The following density lemma in [27] is useful when passing estimates from smooth forms in $\text{dom}(\overline{\partial}^*)$ to general forms in $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*)$.

Proposition 7 ([27])**.** *Let Ω be a C^{k+1} $(1 \leq k \leq \infty)$ smooth domain, then $C^k_{(0,q)}(\overline{\Omega}) \cap$ dom$(\overline{\partial}^*)$ is dense in dom$(\overline{\partial}) \cap$ dom$(\overline{\partial}^*)$ in the graph norm $u \mapsto (||u||^2 + ||\overline{\partial} u||^2 + ||\overline{\partial}^* u||^2)^{\frac{1}{2}}$.*

2.4 Property (P_q) and its analysis property

By Sibony's ([41]) work, Property (P_q) can be studied by classical Choquet theory with respect to the function family $P_q(K)$. The analysis property of Property (P_q)

is closely related with various results in classical potential theory. We introduce the definition of Property (P_q) and present the results in [41] which we need in the dissertation.

Definition 1. *A compact set $K \subset \mathbb{C}^n$ has Property (P_q) $(1 \leq q \leq n)$ if for any $M > 0$, there exists an open neighborhood U of K and a C^2 smooth function λ on U such that $0 \leq \lambda \leq 1$ on U and $\forall z \in U$, the sum of any q eigenvalues of the complex Hessian $\left(\frac{\partial^2 \lambda}{\partial z_j \partial \bar{z}_k}\right)_{j,k}$ is at least M.*

The following linear algebra result is useful when proving Property (P_q). See for example in [9] for its application in proving Property (P_q), here we follow [45].

Lemma 1 ([45])**.** *Let λ be a C^2 smooth function in $\mathbb{C}^n$. Fix any $z \in \mathbb{C}^n$, $1 \leq q \leq n$ and let u be any $(0,q)$-form at z. The following are equivalent:*

(i) The sum of any q eigenvalues of $\left(\frac{\partial^2 \lambda}{\partial z_j \partial \bar{z}_k}\right)_{j,k}$ is at least M.

(ii) $\displaystyle\sum_{|K|=q-1}' \sum_{j,k=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}} \geq M|u|^2.$

(iii) $\displaystyle\sum_{s=1}^{q} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k} (\mathbf{e}^s)_j \overline{(\mathbf{e}^s)_k} \geq M$, *whenever $\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^q$ are orthonormal vectors in $\mathbb{C}^n$.*

By Lemma 1, it is clear that Property (P_q) is preserved under unitary coordinates change.

Definition 2. *Let K be a compact subset in $\mathbb{C}^n$.*

(1) Let U be an open neighborhood containing K. Define the function family $P_q(U)$ on U by: $P_q(U) = \{f \in C(U) |\ f$ is subharmonic on $E \cap U$ for any complex q-dimensional affine subspace $E\}$.

(2) Define the function family $P_q(K)$ on K to be the closure in $C(K)$ of the functions that belong to $P_q(U)$ for some open neighborhood U of K (U is allowed to depend on the function).

(3) Define a probability measure μ on K to be a q-Jensen measure for $z \in K$ with respect to $P_q(K)$ if $h(z) \leq \int_K h\, d\mu$ for $\forall h \in P_q(K)$ and define the associated Choquet boundary $J_q(K)$ as $J_q(K) = \{z \in K |\ \mu$ is the unit point mass at z if μ is a q-Jensen measure at z with respect to $P_q(K)\}$.

The following theorem in [41] shows that if a compact subset K has Property (P_q), the function family $P_q(K)$ has a good approximation property in $C(K)$.

Theorem 6 ([41]). *Let K be a compact subset of $\mathbb{C}^n$. The following are equivalent:*

(i) K satisfies property (P_q).

(ii) $P_q(K) = C(K)$.

(iii) $J_q(K) = K$.

(iv) The function $-|z|^2$ belongs to $P_q(K)$.

By the maximum principle for subharmonic functions and the equivalence between (i) and (ii) in above theorem, analytic discs in the boundary of a domain Ω in $\mathbb{C}^n$ are obstructions to Property (P_1) of the boundary.

The following two propositions in [41] show that Property (P_q) is a local property and is preserved by countable unions.

Proposition 8 ([41]). *Let K be a compact subset of $\mathbb{C}^n$. Assume that for every $z \in K$, there exists $r > 0$ such that $K \cap \overline{B(z,r)}$ satisfies Property (P_q). Then K has Property (P_q).*

Proposition 9 ([41]). *Let $K = \bigcup_{m=1}^{\infty} K_m$ with K_m compact for all m. Assume that K is compact, if all K_m satisfy Property (P_q), then so does K.*

2.5 Fine topology, basic potential theory and Property (P_q)

We will introduce some basic notions and results in potential theory, and then relate them with the analysis property of Property (P_q).

The fine topology in $\mathbb{C}^n$ is the weakest topology in which all subharmonic functions are continuous. The fine topology is strictly stronger than the usual Euclidean topology in $\mathbb{R}^n$ (see examples in [41]) and any Euclidean open set in $\mathbb{R}^n$ is finely open. Finely open sets are still massive near a point in the sense of Lebesgue measure. We have the following result (see Corollary 7.2.4 in [2], or Corollary 10.5 in [25]):

Lemma 2. *Let M be a compact subset in $\mathbb{R}^n$ and if z is a fine interior point of M, then we have:*

$$\lim_{r \to 0} \frac{\sigma(M \cap \partial B(z,r))}{\sigma(\partial B(z,r))} = 1, \tag{2.2}$$

where σ is the surface measure on the sphere $\partial B(z,r)$ and $B(z,r) = \{|z| < r\} \subset \mathbb{R}^n$.

The following theorem from [6] connects the fine boundary of a compact set K and the Choquet boundary K with respect to a special class of harmonic functions.

Theorem 7 ([6]). *In a Green space Ω_0, consider a compact set K and the set $\mathcal{F}$ of functions on K such that each one is the restriction on K of a harmonic function on an open neighborhood of K. Then the Choquet boundary of K with respect to $\mathcal{F}$ is the fine boundary of K.*

For more basic background of fine topology and its application in classical potential theory, we refer the reader to [2] and [25].

For a Euclidean open subset U in $\mathbb{R}^n$, denote $\lambda_1(U)$ the smallest eigenvalue of the Dirichlet problem for the (real) Laplacian on U. It is well known that $\lambda_1(U) =$

$\inf\{\int |\nabla u|^2|\ u \in C_0^\infty(U), \int |u|^2 = 1\}$ (see for example in [23]). Li and Yau ([36]) showed that the eigenvalues of the Dirichlet problem has lower bounds regarding the volume measure of the open set U:

Theorem 8 ([36]). *Let U be a Euclidean open set in $\mathbb{R}^n$, and assume the eigenvalues of the Dirichlet problem for the (real) Laplacian on U are monotonically ordered by $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$. Then we have the following estimate for the lower bound of each λ_k ($k \geq 1$):*

$$\lambda_k \geq \frac{nC_n}{n+2}\left(\frac{k}{vol(U)}\right)^{\frac{2}{n}}.$$

C_n is a constant only depend on n and $vol(U)$ is the volume measure of U.

On the other hand, the Dirichlet problem can be formulated regarding the Dirichlet Laplacian on a bounded finely open set V in $\mathbb{R}^n$. Most of the results thereafter are similar as the classical Dirichlet problem case (in particular the eigenvalues coincide when considering both problems on a Euclidean open set in $\mathbb{R}^n$), but have some extra stability of eigenvalues (with respect to the bounded finely open set case) under unions or intersections of sequences of domains. We refer the reader to [17] for the set up of the Dirichlet problem on a bounded finely open set and we only list the following proposition from [17] which we will use in the dissertation. Assume the eigenvalues of the Dirichlet problem on a finely open set V are monotonically ordered by $\lambda_1(V) < \lambda_2(V) \leq \cdots \leq \lambda_k(V) \leq \cdots$.

Proposition 10 ([17]). *Let D_i ($i \geq 1$) be a monotonically decreasing sequence of uniformly bounded finely open subsets of $\mathbb{R}^n$ such that D_i converges to a finely open set D in $\mathbb{R}^n$: $int_f \bigcap_i D_i = D$. Here int_f denotes the fine interior. Then for all $k \geq 1$, $\lambda_k(D_i) \to \lambda_k(D)$ as $i \to \infty$.*

The following proposition is formulated in [20], but the equivalence of (i) and (ii) is credited to Sibony ([41]).

Proposition 11 ([20, 41])**.** *Let K be a compact subset of $\mathbb{C}$. The following are equivalent:*

(i) K satisfies Property (P_1).

(ii) K has empty fine interior in $\mathbb{C}$.

(iii) K supports no nonzero function in $W_0^1(\mathbb{C})$.

(iv) For any sequence of Euclidean open sets $\{U_j\}_{j=1}^{\infty}$ such that $K \subset U_{j+1} \subset U_j$ and $\bigcap_{j=1}^{\infty} U_j = K$, the smallest eigenvalue $\lambda_1(U_j)$ of the Dirichlet problem on U_j satisfies: $\lambda_1(U_j) \to \infty$ as $j \to \infty$.

Corollary 1 ([41])**.** *Let K be a compact subset in $\mathbb{C}$ and K has Lebesgue measure zero in $\mathbb{C}$. Then K has Property (P_1).*

We point out that since most of the classical potential results which were used in Proposition 11 and Corollary 1 are formulated in $\mathbb{R}^n$, hence a part of Proposition 11 can be generalized verbatim to the case of $\mathbb{C}^n$ regarding Property (P_n). We list the following generalization of Corollary 1 and prove it due to the importance of its application in the dissertation.

Proposition 12. *Let K be a compact subset in $\mathbb{C}^n$ and K has Lebesgue measure zero in $\mathbb{C}^n$. Then K has Property (P_n) in $\mathbb{C}^n$.*

Proof. Assume we have any sequence of Euclidean open sets $\{U_j\}_{j=1}^{\infty}$ such that $K \subset U_{j+1} \subset U_j$ and $\bigcap_{j=1}^{\infty} U_j = K$. Apply Theorem 8, we have $\lambda_1(U_j) \to \infty$ as $j \to \infty$. Here $\lambda_1(U_j)$ is the smallest eigenvalue of the Dirichlet problem for the (real) Laplacian on (Euclidean open set) U_j.

Now apply Proposition 10, we have:

$$\lim_{j\to\infty} \lambda_1(U_j) = \lambda_1(int_f \bigcap_j U_j) = \lambda_1(int_f K). \tag{2.3}$$

Here, λ_1 in equation (2.3) is the smallest eigenvalue of the Dirichlet problem on a finely open set. But in the left hand side of (2.3), each $\lambda_1(U_j)$ is equal to the smallest eigenvalue of the Dirichlet problem on the Euclidean open set U_j. Now using our conclusion in the first paragraph of our proof and by equation (2.3) we conclude that $\lambda_1(int_f K) = \infty$. This forces $int_f K = \emptyset$, K has empty fine interior in $\mathbb{C}^n$.

Now we will show that given K has empty fine interior in $\mathbb{C}^n$, K has Property (P_n). For any $z_0 \in K$, since K has empty fine interior, z_0 is a fine boundary point of K. Assume μ is an n-Jensen measure for z_0 with respect to $P_n(K)$, then in particular for any f which is harmonic in a neighborhood of K, we have: $f|_K \in P_n(K)$ and $f|_K(z_0) \leq \int_K f|_K \, d\mu$. Now apply Theorem 7, we conclude that z_0 is in the Choquet boundary of K with respect to the function family $\mathcal{F}$ in Theorem 7. Note that Theorem 7 requires K be a compact subset of a Green space, but in our case we know that the fine interior of K with respect to any bounded open subset $V \supset K$ in $\mathbb{C}^n$ is the same as the fine interior of K with respect to $\mathbb{C}^n$, and hence Theorem 7 still applies in our case. We conclude that μ is a point mass measure at z_0. Therefore, $z_0 \in J_n(K)$. Since z_0 is arbitrary, $K = J_n(K)$ and K has Property (P_n) by Theorem 6. □

One of the most interesting applications of above results is that we can construct a smooth pseudoconvex complete Hartogs domain Ω in $\mathbb{C}^2$ such that the set of weakly pseudoconvex boundary points has positive surface measure on $b\Omega$, but the boundary $b\Omega$ still has Property (P_1) and hence the $\overline{\partial}$-Neumann operator N_1 is compact. We refer the reader to [41] for the construction.

3. INVARIANCE PROPERTY OF PROPERTY (P_q)

In this section, we will study the invariance property of Property (P_q) based on Sibony's ([41]) work. Our motivation is originated from studying the gap between Property (P_q) on the boundary of a given smooth pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and the compactness of N_q on $L^2_{(0,q)}(\Omega)$. It is well known that biholomorphic mappings (smooth up to boundaries) preserve the compactness of N_q between smooth pseudoconvex domains on all levels of L^2-integrable forms (see for example in [39]). However, biholomorphic mappings are not known to preserve Property (P_q) of a compact subset in $\mathbb{C}^n$ when $q > 1$. Therefore the difference in the invariance property would partially demonstrate the gap between Property (P_q) and the compactness of N_q. By studying the invariance property of Property (P_q) $(q > 1)$, we hope to stimulate further research in the same types of problems.

3.1 Invariance property of Property (P_1) in $\mathbb{C}$

We start with a special invariance property of Property (P_1) on the complex plane.

Proposition 13. *Let K be a compact subset in $\mathbb{C}$, and assume K has Property (P_1), then given any holomorphic mapping $F : \mathbb{C} \to \mathbb{C}$, $F(K)$ has Property (P_1).*

Proof. Our first observation is the following fact: If F is biholomorphic on an open set $U \subset \mathbb{C}$, then for any subharmonic function f on U, $f \circ F^{-1}$ is also subharmonic on $F(U)$. Hence for any compact subset $M \subset U$ which has Property (P_1), $F(M)$ has Property (P_1).

Define $K_m = \{z \in K \,\big|\, |F'(z)| \geq \frac{1}{m}\}$, and $K_0 = \{z \in K \mid F'(z) = 0\}$, where $F'(z)$

is the first derivatives of F at z. We have:

$$F(K) = F(K_0) \cup \bigcup_{m \geq 1} F(K_m). \tag{3.1}$$

For each $K_m \subset K$, K_m has Property (P_1). Since F is biholomorphic in a neighborhood of each point in K_m, and Property (P_1) is a local property, by using the fact at the beginning of our proof, we conclude that $F(K_m)$ has Property (P_1).

By Sard's theorem, $F(K_0)$ has Lebesgue measure 0 in $\mathbb{C}$ and hence by Corollary 1, $F(K_0)$ has Property (P_1).

Since by Proposition 9 Property (P_1) is preserved by countable unions of compact subset, by (3.1), $F(K)$ has Property (P_1). □

Remark 1. *(1) One of the difficulties in proving such type of invariance property is that the pullback F^{-1} is not known to be holomorphic. Hence given any $f \in P_1(K)$ (or in general $P_q(K)$ for $1 \leq q \leq n$), it is not known whether $f \circ F^{-1}$ is in $P_1(K)$ (or $P_q(K)$ for $1 \leq q \leq n$).*

(2) It is not known whether we can generalize Proposition 13 to higher dimension case regarding Property (P_q) in $\mathbb{C}^n$ for $1 \leq q \leq n$. When $q > 1$, $f \circ F^{-1}$ is not necessarily in $P_q(K)$ even F is biholomorphic. When $q = 1$ and by Proposition 12, we only know that $F(K_0)$ has Property (P_n) which is insufficient for the case of $n > 1$.

3.2 Invariance property of Property (P_q) $(q \geq 2)$

Now we study the invariance property of Property (P_q) $(q \geq 2)$ of a compact subset $K \subset \mathbb{C}^m$ under holomorphic mappings $\pi : \mathbb{C}^m \to \mathbb{C}^n$. Our main question is the following: Given a holomorphic mapping $\pi : \mathbb{C}^m \to \mathbb{C}^n$ $(n \leq m)$ and a compact subset $K \subset \mathbb{C}^m$, assume the image set $\pi(K) \subset \mathbb{C}^n$ has Property (P_q), does the set K have Property (P_q) in $\mathbb{C}^m$?

When $q = 1$, Sibony ([41]) proved the following result:

Theorem 9 ([41])**.** *Let $K \subset \mathbb{C}^m$ and $\pi : \mathbb{C}^m \to \mathbb{C}^n$ be a holomorphic mapping. Denote $X \subset \mathbb{C}^n$ as the image set $\pi(K)$. If X has Property (P_1) and for all $x \in X$ the fiber $\pi^{-1}(x) \cap K$ has Property (P_1), then K has Property (P_1).*

When $q \geq 2$, we can find some nontrivial holomorphic mapping π, which the image set $\pi(K)$ has a "twisted" Property (P_q), and Sibony's result can be generalized to the case $q > 1$.

Let K be a compact subset in $\mathbb{C}^n$ and define the holomorphic mapping $\pi : \mathbb{C}^n \to \mathbb{C}^q$ as $\pi(z_1, \cdots, z_n) = (\sum_{j=1}^{n-1} z_j + g_1(z_n), \cdots, \sum_{j=1}^{n-1} z_j + g_k(z_n), \cdots, \sum_{j=1}^{n-1} z_j + g_q(z_n))$, where g_k $(k = 1, \cdots, q)$ is holomorphic on z_n-plane and $\frac{\partial g_1}{\partial z_n} = \cdots = \frac{\partial g_k}{\partial z_n} = \cdots = \frac{\partial g_q}{\partial z_n}$ on the projection set of K onto z_n-plane. By our definition, if the projection set of K onto z_n-plane has an accumulation point, then g_k is different from each other by a constant.

Definition 3. *(a) Let U be an open subset in $\mathbb{C}^q$, define the function family $P_q^{new}(U)$ on U as:*

$$P_q^{new}(U) = \left\{ f \in C^2(U) \middle| \sum_{j,k=1}^{q} \frac{\partial^2 f}{\partial \xi_j \partial \bar{\xi}_k} \geq 0 \text{ on } U \right\}.$$

Note that the function family $P_q^{new}(U)$ is a convex cone in $C^2(U)$.

(b) For any compact subset X in $\mathbb{C}^q$, define the function family $P_q^{new}(X)$ on X by: $P_q^{new}(X)$ is the closure in $C(X)$ of the functions that belong to $P_q^{new}(U)$ for some open set $U \subset \mathbb{C}^q$ containing X, where U is allowed to depend on the function. Note that the function family $P_q^{new}(X)$ is still a convex cone in $C(X)$.

(c) Define a probability measure μ on X to be a q-Jensen measure for $z \in X$ with respect to $P_q^{new}(X)$ if $h(z) \leq \int_X h \, d\mu$ for $\forall h \in P_q^{new}(X)$ and define the associated

Choquet boundary $J_q^{new}(X)$ *as* $J_q^{new}(X) = \{z \in X | \ \mu$ *is the unit point mass at* z *if* μ *is a q-Jensen measure at* z *with respect to* $P_q^{new}(X)\}$.

Proposition 14. *Let* π *be defined as above and assume* K *is a compact set in* $\mathbb{C}^n$. *Denote* $X \subset \mathbb{C}^q$ *as the image set* $\pi(K)$. *If* $f \in P_q^{new}(X)$, *then* $f \circ \pi \in P_1(K)$ *and hence* $f \circ \pi \in P_q(K)$.

Proof. We first show that if $f \in P_q^{\text{new}}(U)$, where U is an open neighborhood of X, then $f \circ \pi$ defined on V has nonnegative complex Hessian on K, where $V = \pi^{-1}(U)$ is an open neighborhood of K. This is done by calculating the eigenvalues of the complex Hessian of $f \circ \pi$ on $V \subset \mathbb{C}^n$ and then restricting to K. We have:

$$\begin{aligned}
\frac{\partial^2 (f \circ \pi)}{\partial z_j \partial \overline{z}_k}(z_1, \cdots, z_n) &= \sum_{s,t=1}^{q} \frac{\partial^2 f}{\partial \xi_s \partial \overline{\xi}_t}(\pi(z_1, \cdots, z_n)) \ \text{ on } V, \ j, k = 1, \cdots, n-1; \\
\frac{\partial^2 (f \circ \pi)}{\partial z_n \partial \overline{z}_k}(z_1, \cdots, z_n) &= \sum_{s,t=1}^{q} \overline{\frac{\partial g_t}{\partial z_n}} \frac{\partial^2 f}{\partial \xi_s \partial \overline{\xi}_t}(\pi(z_1, \cdots, z_n)) \ \text{ on } V, \ k = 1, \cdots, n-1; \\
\frac{\partial^2 (f \circ \pi)}{\partial z_n \partial \overline{z}_n}(z_1, \cdots, z_n) &= \sum_{s,t=1}^{q} \frac{\partial g_s}{\partial z_n} \overline{\frac{\partial g_t}{\partial z_n}} \frac{\partial^2 f}{\partial \xi_s \partial \overline{\xi}_t}(\pi(z_1, \cdots, z_n)) \ \text{ on } V.
\end{aligned}$$

Consider $z \in K$ and put in the condition $\frac{\partial g_1}{\partial z_n} = \cdots = \frac{\partial g_k}{\partial z_n} = \cdots = \frac{\partial g_q}{\partial z_n}$, then the complex Hessian of $f \circ \pi$ is:

$$\begin{aligned}
\frac{\partial^2 (f \circ \pi)}{\partial z_j \partial \overline{z}_k}(z_1, \cdots, z_n) &= \sum_{s,t=1}^{q} \frac{\partial^2 f}{\partial \xi_s \partial \overline{\xi}_t}(\pi(z_1, \cdots, z_n)) \ \text{ on } K, \ j, k = 1, \cdots, n-1; \\
\frac{\partial^2 (f \circ \pi)}{\partial z_n \partial \overline{z}_k}(z_1, \cdots, z_n) &= \overline{\frac{\partial g_1}{\partial z_n}} \cdot \sum_{s,t=1}^{q} \frac{\partial^2 f}{\partial \xi_s \partial \overline{\xi}_t}(\pi(z_1, \cdots, z_n)) \ \text{ on } K, \ k = 1, \cdots, n-1; \\
\frac{\partial^2 (f \circ \pi)}{\partial z_n \partial \overline{z}_n}(z_1, \cdots, z_n) &= \left| \frac{\partial g_1}{\partial z_n} \right|^2 \sum_{s,t=1}^{q} \frac{\partial^2 f}{\partial \xi_s \partial \overline{\xi}_t}(\pi(z_1, \cdots, z_n)) \ \text{ on } K.
\end{aligned}$$

The rank of the complex Hessian of $f \circ \pi$ is at most 1 on K, therefore there are $n-1$ eigenvalues of 0. By calculating the coefficient of $(n-1)$-th power term in the characteristic polynomial of the complex Hessian, the nonzero eigenvalue is $\sum_{j,k=1}^{q} \frac{\partial^2 f}{\partial \xi_j \partial \bar{\xi}_k} \left(|\frac{\partial g_1}{\partial z_n}|^2 + n - 1 \right) \geq 0$ on K. Therefore, the complex Hessian of $f \circ \pi$ is nonnegative on K.

On V, we take $\epsilon > 0$ such that the complex Hessian of $f \circ \pi + \epsilon |z|^2$ is nonnegative on V, i.e., $f \circ \pi + \epsilon |z|^2 \in P_1(V)$. The choice of ϵ depends on f and V, and when V shrinks down to K, $\epsilon \to 0$.

Now take any $f \in P_q^{\text{new}}(X)$. By definition, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ such that $f_n \in P_q^{\text{new}}(U_n)$ and $\lim_{n \to \infty} ||f_n - f||_{\infty, X} = 0$, where U_n is an open neighborhood of X. We can arrange U_n such that $U_{n+1} \subset U_n$ and $\bigcap_{n=1}^{\infty} \overline{U_n} = X$, hence U_n shrinks down to X. Since $||f_n \circ \pi - f \circ \pi||_{\infty,K} \leq ||f_n - f||_{\infty,X}$, $\lim_{n \to \infty} ||f_n \circ \pi + \epsilon_n |z|^2 - f \circ \pi||_{\infty,K} = 0$. Each $f_n \circ \pi + \epsilon_n |z|^2 \in P_1(V_n)$ by above argument, where $V_n = \pi^{-1}(U_n) \supset K$ and ϵ_n is defined in the last paragraph, therefore $f \circ \pi \in P_1(K)$ and hence $f \circ \pi \in P_q(K)$. □

Remark 2. *(1) We proved that the pullback $f \circ \pi \in P_1(K)$ in the above proposition, however in the main theorem below, we only need $f \circ \pi \in P_q(K)$. Hence it is interesting to see whether one can construct a holomorphic mapping π such that $f \circ \pi \in P_q(K)$ but $f \circ \pi \notin P_{q-1}(K)$ under our context.*

(2) When $q = 1$, consider a projection map $\pi : (z_1, \cdots, z_n) \mapsto z_1$. It is trivially true that if $f \in P_1(X)$ then $f \circ \pi \in P_1(K)$. However this is no longer true when we consider a projection map $\pi : (z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_q)$ for $q > 1$: let $f = |z_1|^2 - |z_2|^2 \in P_2(U)$ for an open set $U \subset \mathbb{C}^2$, but $f \circ \pi \notin P_2(U \times \mathbb{C})$, where $\pi : (z_1, z_2, z_3) \mapsto (z_1, z_2)$. Hence Proposition 14 partially overcomes the difficulty which is not detected in the case of $q = 1$.

To prove the main theorem, we need one density lemma and the idea of such density lemma is implicit in [41].

Lemma 3. *Let π, K, X be defined as above. For any $x \in K$, $P_q(K)$ is dense in $P_q(\pi^{-1}(\pi(x)) \cap K)$ with respect to the $||\cdot||_\infty$ topology.*

Proof. Let φ be a strictly positive function on an open neighborhood V of $\pi^{-1}(\pi(x)) \cap K$ such that $\varphi \in P_q(V)$. We will find a function θ such that $\theta \in P_q(K)$ and $\theta = \varphi$ on $\pi^{-1}(\pi(x)) \cap K$. Then the lemma follows by taking approximation as we did in the proof of Proposition 14.

We claim that there exists a function ψ defined on a neighborhood of K such that $\psi \in P_q(K)$, $\psi < 0$ on $\pi^{-1}(\pi(x)) \cap K$ and $\psi \geq \delta > 0$ on $\partial V \cap K$, where δ is a positive constant.

Assume our claim first, then we can take a large constant $C > 0$ such that $C \cdot \psi > \varphi$ in an neighborhood of $\partial V \cap K$ and $C \cdot \psi < 0$ on $\pi^{-1}(\pi(x)) \cap K$. Define $\theta = \max(\varphi, C\psi)$ on $V \cap K$ and $C\psi$ on $K \backslash V$. By our construction, $\theta \in P_q(K)$ and $\theta = \varphi$ on $\pi^{-1}(\pi(x)) \cap K$. Therefore the lemma follows.

Finally, to prove our claim above, consider $h(\xi) = |\xi - \pi(x)|^2$ on X. By calculating the complex Hessian of h, $\sum_{j,k=1}^{q} \frac{\partial^2 h}{\partial \xi_j \partial \bar{\xi}_k} = q > 0$. Hence $h \in P_q^{\text{new}}(X)$, and by observation, $h(\pi(x)) = 0$ and $h\,|_{\partial V' \cap X} \geq \delta' > 0$, where V' is some neighborhood of $\pi(x)$. By taking h to be $h - \frac{\delta'}{2}$, we can assume $h < 0$ on $\pi(x)$ and $h \geq \delta' > 0$ on $\partial V' \cap X$.

Now take $\psi = h \circ \pi$. Since $h \in P_q^{\text{new}}(X)$, by Proposition 14, $\psi \in P_q(K)$. By the construction of h, $\psi < 0$ on $\pi^{-1}(\pi(x)) \cap K$ and $\psi \geq \delta > 0$ on $\partial V \cap K$, where V is a neighborhood of $\pi^{-1}(\pi(x)) \cap K$. Therefore our claim is proved. □

Theorem 10. *Let π, K, X be defined as above. Suppose $J_q^{new}(X) = X$ and for any point $x \in X$, each fiber $K \cap \pi^{-1}(x)$ has Property (P_q), then K has Property (P_q) in $\mathbb{C}^n$.*

Proof. Our proof relies on Sibony's ([41]) argument in the case of $q = 1$. Suppose μ is a q-Jensen measure for $z \in K$ with respect to $P_q(K)$. The pushforward measure $\pi_*\mu$ on X is defined as $\pi_*\mu(B) = \mu(\pi^{-1}(B))$ for $B \subseteq X$ and $\int_X f \, d(\pi_*\mu) = \int_K f \circ \pi \, d\mu$ for any measurable function f on X.

If $f \in P_q^{\text{new}}(X)$, by Proposition 14, $f \circ \pi \in P_q(K)$ and we have the following:

$$f(\pi(z)) = f \circ \pi(z) \leq \int_K f \circ \pi \, d\mu = \int_X f \, d(\pi_*\mu). \tag{3.2}$$

Hence $\pi_*(\mu)$ is a q-Jensen measure for $\pi(z)$ with respect to $P_q^{\text{new}}(X)$. Since $J_q^{\text{new}}(X) = X$, $\pi_*(\mu) = \delta_{\pi(z)}$, where $\delta_{\pi(z)}$ is the point mass measure at $\pi(z)$. Therefore, μ has support only on $\pi^{-1}(\pi(z)) \cap K$.

Claim: $\mu\big|_{\pi^{-1}(\pi(z))\cap K}$ is a q-Jensen measure for z with respect to $P_q(\pi^{-1}(\pi(z)) \cap K)$.

Assume our claim true first. Since each fiber $K \cap \pi^{-1}(x)$ has Property (P_q) for any $x \in X$, $J_q(\pi^{-1}(\pi(z)) \cap K) = \pi^{-1}(\pi(z)) \cap K$ with respect to $P_q(\pi^{-1}(\pi(z)) \cap K)$. Now apply our claim above, $\mu\big|_{\pi^{-1}(\pi(z))\cap K}$ is a point mass measure at z and hence $\mu = \delta_z$.

We showed that for any q-Jensen measure μ for $z \in K$ with respect to $P_q(K)$, μ is a point mass measure. Therefore $J_q(K) = K$ with respect to $P_q(K)$, and hence the theorem follows.

To prove our claim, since the q-Jensen measure μ has support only on $\pi^{-1}(\pi(z)) \cap K$, we have the following:

$$h(z) \leq \int_K h \, d\mu = \int_{\pi^{-1}(\pi(z))\cap K} h \, d\mu, \quad \forall h \in P_q(K). \tag{3.3}$$

By Lemma 3, $P_q(K)$ is dense in $P_q(\pi^{-1}(\pi(z)) \cap K)$ with respect to the $|| \cdot ||_\infty$

topology, hence by approximation we have:

$$h(z) \leq \int_{\pi^{-1}(\pi(z))\cap K} h \, d\mu, \quad \forall h \in P_q(\pi^{-1}(\pi(z)) \cap K). \tag{3.4}$$

The claim is proved. $\square$

Remark 3. *(1) The condition $J_q^{new}(X) = X$ can be interpreted as a twisted version of Property (P_q) in $\mathbb{C}^q$. It is interesting to see whether this twisted Property (P_q) still has a similar analysis property as Property (P_q) has. We briefly discuss the relation between $P_q^{new}(X) = C(X)$ and $J_q^{new}(X) = X$ here.*

On the one hand, it is straightforward to see that given any compact subset $X \subset \mathbb{C}^q$, if $P_q^{new}(X) = C(X)$, then $J_q^{new}(X) = X$. Because $P_q^{new}(X) = C(X)$ implies that the subaveraging inequality in the definition of q-Jensen measures holds for all continuous functions, we have the desired conclusion.

On the other hand, the opposite implication is not necessarily true. In [41], the opposite implication relies on a result of Edwards ([21], Theorem 1.2), which is not necessarily true in our case. Although the function family $P_q^{new}(X)$ is a convex cone, it is not necessarily preserved by the maximum function. More precisely, the differential operator $\sum_{j,k=1}^{q} \frac{\partial^2}{\partial \xi_j \partial \bar{\xi}_k}$ in the definition of $P_q^{new}(X)$ does not have maximum principle.

(2) The intersection of $P_q^{new}(X)$ and $P_q(X)$ is nonempty: it contains C^2 smooth subharmonic functions defined on some neighborhood of X whose complex Hessians are diagonal. In particular, $|z|^2$ belongs to the intersection.

(3) Take $q = 1$ in our main theorem, our result reduces to a special case of Theorem 9.

4. OBSTRUCTIONS TO PROPERTY (P_q)

In [41], Sibony characterized Property (P_1) of the boundary of a hyperconvex domain in $\mathbb{C}^n$ (in particular true for any smooth pseudoconvex domain) with the existence of a peak function $\psi \in P_1(\overline{\Omega})$. Generalizing Sibony's result to Property (P_q), it is known that picking up P_q-hull is an obstruction to Property (P_q) of the boundary. (Compare the remarks in [45], section 4.8.)

We are interested in the study of obstructions to Property (P_q) in terms of the geometry or topology of complex q-dimensional varieties in the boundary of the domain. In particular, we want to generalize the following theorem of Sibony ([41]) (which is contained in Proposition 11) to the case of $q > 1$ in higher dimensions:

Theorem 11 ([41])**.** *Let K be a compact subset of $\mathbb{C}$. Then K has Property (P_1) if and only if K has empty fine interior.*

4.1 Main Theorem 12

We have the following main theorem:

Theorem 12. *Let K be a compact subset of $\mathbb{C}^n$, let $1 \leq q \leq n$, and assume K has Property (P_q). Then for any q-dimensional affine subspace E in $\mathbb{C}^n$, $K \cap E$ has empty fine interior with respect to the fine topology in $\mathbb{C}^q$.*

Proof. **Claim 1**: $K \cap E$ has Property (P_q) in E.

Denote $\{\mathbf{e^1}, \cdots, \mathbf{e^q}\}$ the q orthonormal vectors in $\mathbb{C}^n$ which span E. For any $\xi \in E$, we write $\xi = \sum_{i=1}^{q} \xi_i \mathbf{e^i}$.

Fix any $z_0 \in K$, given a function f on $\mathbb{C}^n$, we defined the following function $\widetilde{f}$ on E by:

$$\widetilde{f}(\xi_1, \cdots, \xi_q) = f(z_0 + \xi_1 \mathbf{e^1} + \cdots + \xi_q \mathbf{e^q}).$$

For any C^2 smooth function f on $\mathbb{C}^n$, we have the following:

$$\begin{aligned}
\Delta\widetilde{f} = \sum_{i=1}^{n} \frac{\partial^2 \widetilde{f}}{\partial \xi_i \partial \overline{\xi}_i} \quad &= \quad \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} (\mathbf{e^1})_j \overline{(\mathbf{e^1})_k} \\
&\quad + \cdots + \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} (\mathbf{e^s})_j \overline{(\mathbf{e^s})_k} \\
&\quad + \cdots + \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} (\mathbf{e^n})_j \overline{(\mathbf{e^n})_k} \\
&= \quad \sum_{s=1}^{q} \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} (\mathbf{e^s})_j \overline{(\mathbf{e^s})_k}. \qquad (4.1)
\end{aligned}$$

Since K has Property (P_q) in $\mathbb{C}^n$, for any $M > 0$, there exists an open neighborhood U of K in $\mathbb{C}^n$ and a C^2 smooth function $0 \leq \lambda_M \leq 1$ on U such that the sum of any q eigenvalues of the complex Hessian of λ_M is at least M on U. By Lemma 1 and (4.1), we conclude that $\Delta\widetilde{\lambda_M} \geq M$ and $0 \leq \widetilde{\lambda_M} \leq 1$ on a neighborhood of $K \cap E$ in E (as a copy of $\mathbb{C}^q$). Claim 1 follows.

Claim 2: Let Q be a compact subset in $\mathbb{C}^m$ $(m \geq 1)$, if Q has Property (P_m) then Q has empty fine interior in $\mathbb{C}^m$.

We denote $\delta(V)$ be the smallest eigenvalue of Dirichlet problem on a Euclidean open set V.

Since Q has Property (P_m), for any $j > 0$, there exists an open neighborhood V_j of Q and a C^2 smooth λ_j on V_j such that $0 \leq \lambda_j \leq 1$, $\Delta\lambda_j \geq j$ on V_j. We can shrink each V_j such that we further assume V_j has smooth boundary, $V_{j+1} \subset V_j$ and $\lambda_j \in C^\infty(\overline{V_j})$. Fix j now.

On V_j, consider the solution h to $\Delta h = 0$ on V_j and $h|_{\partial V_j} = \lambda_j|_{\partial V_j}$. Replacing λ_j by $\lambda_j - h$, we can assume that $\lambda_j \in W_0^1(V_j) \cap C^\infty(\overline{V_j})$, $-1 \leq \lambda_j \leq 1$ on V_j and $\Delta\lambda_j \geq j$ on V_j.

On V_j, let $w_j \in W_0^1(V_j)$ be the eigenfunction of $-\Delta$ to the eigenvalue $\delta(V_j)$, i.e.,

$\Delta w_j = -\delta(V_j)w_j$ on V_j and $w_j|_{\partial V_j} = 0$. By Theorem 8.38 in [23], we can assume that w_j is nonnegative on V_j.

We have the following inequality:

$$
\begin{aligned}
j\int_{V_j} w_j &\le \int_{V_j} (\Delta\lambda_j)w_j \\
&= \int_{V_j} \lambda_j \Delta w_j \\
&= -\delta(V_j)\int_{V_j} \lambda_j w_j \\
&\le \delta(V_j)\int_{V_j} w_j.
\end{aligned} \tag{4.2}
$$

Note that since $\lambda_j \in W_0^1(V_j) \cap C^\infty(\overline{V_j})$ and $w_j \in W_0^1(V_j)$, we can use integration by parts to switch the Laplacian in the second equation of (4.2). And the last estimate in (4.2) follows by taking absolute values and using the fact that $-1 \le \lambda_j \le 1$ on V_j.

Now since w_j is an eigenfunction, $\int_{V_j} w_j > 0$, so by (4.2) $\delta(V_j) \ge j$. Hence we have:

$$
\begin{aligned}
+\infty = \lim_{j\to\infty} \delta(V_j) &= \delta(\text{fine interior of } \bigcap_{j=1}^{\infty} V_j) \\
&= \delta(\text{fine interior of } Q).
\end{aligned} \tag{4.3}
$$

The second equality in (4.3) follows from Proposition 10. Note that we abuse the notation in (4.3), the three δ in (4.3) are defined as the smallest eigenvalue of the Dirichlet problem on a finely open set. δ agrees with the usual definition on Euclidean open sets ([17]), so the left hand side of (4.3) is valid.

By (4.3), Q has empty fine interior in $\mathbb{C}^m$. Claim 2 is proved.

Now apply claim 2 to $K \cap E$ and take $m = q$, the theorem follows. □

Remark 4. *It is well known (see [45] for example) that if $b\Omega$ contains a q-dimensional* *affine (or even analytic) polydisc where Ω is a bounded pseudoconvex domain, then $b\Omega$ does not satisfy Property (P_q). Now notice that a Euclidean open set in $\mathbb{C}^n$ is finely open in $\mathbb{C}^n$, so the condition in Theorem 12 is more general than the absence of complex q-dimensional affine varieties in the boundary.*

4.2 A partial result on the converse of Theorem 12

Now let Ω be a smooth bounded pseudoconvex domain and K be the weakly pseudoconvex points of $b\Omega$, we also wish to address that whether the converse of Theorem 12 is true. To be precise, Given Ω and K as above, if for any q-dimensional affine subspace E in $\mathbb{C}^n$, $K \cap E$ has empty fine interior with respect to the fine topology in $\mathbb{C}^q$, is it true that K has Property (P_q)? We obtained the following partial result when $q = 1$:

Denote $\pi_P : \mathbb{C}^n \to \mathbb{C}^{n-1}$ the projection map from $\mathbb{C}^n$ onto the complex tangent space defined locally at a boundary point P on Ω.

Theorem 13. *Given Ω and K above, assume that for any boundary point P and any complex line E in the complex tangent space at P, $E \cap \pi_P(K)$ has empty fine interior with respect to the fine topology in $\mathbb{C}$. Then K has Property (P_1) and hence the boundary $b\Omega$ has Property (P_1).*

Proof. Take a neighborhood $U \subset \mathbb{C}^n$ of P such that the local complex tangent system is well-defined. We will prove $K \cap \overline{V}$ has Property (P_1) for any open set $V \subset U$.

Denote $\{\xi_j\}_{j=1}^{n-1}$ the orthonormal coordinates which span the complex tangent space at P and ξ_n the complex normal at P. Denote E_j $(j = 1, \cdots, n-1)$ the complex line spanned by each ξ_j $(j = 1, \cdots, n-1)$ passing through P.

By assumption, $E_j \cap \pi_P(K)$ has empty fine interior with respect to the fine topology in $\mathbb{C}$, therefore for any $M > 0$, there exists an open neighborhood $U_j \subset E_j$

of $E_j \cap \pi_P(K)$, and a C^2 smooth function (of one variable) $\lambda_j(\xi_j)$ on U_j such that $0 \le \lambda_j \le 1$ and $\frac{\partial^2 \lambda_j}{\partial \xi_j \partial \bar{\xi}_j} \ge M$ on U_j.

Define the linear projection map η_j from $U \subset \mathbb{C}^n$ to E_j by $\eta_j(\xi_1, \cdots, \xi_n) = \xi_j$, $j = 1, \cdots, n-1$. We can now define a function λ on a neighborhood of $K \cap \overline{V}$ by: $\lambda = \sum_{j=1}^{n-1} \lambda_j \circ \eta_j + M\rho^2$, where ρ is the defining function of Ω.

Notice that $\lambda_j \circ \eta_j(\xi_1, \cdots, \xi_n) = \lambda_j(\xi_j)$, we can calculate the complex Hessian A of $\sum_{j=1}^{n-1} \lambda_j \circ \eta_j$ on a neighborhood of $K \cap \overline{V}$ with respect to the coordinates $\{\xi_j\}_{j=1}^n$:

$$A = \begin{pmatrix} \frac{\partial^2 \lambda_1}{\partial \xi_1 \partial \bar{\xi}_1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\partial^2 \lambda_2}{\partial \xi_2 \partial \bar{\xi}_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2 \lambda_{n-1}}{\partial \xi_{n-1} \partial \bar{\xi}_{n-1}} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We calculate the complex Hessian B of ρ^2 on $b\Omega$ with respect to the coordinates $\{\xi_j\}_{j=1}^n$:

$$B = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence for any $\epsilon > 0$ (ϵ is independent of M and λ), we can choose a neighborhood W of $b\Omega$ such that the eigenvalues of the complex Hessian B_ϵ of ρ^2 on W are $\eta_1, \cdots, \eta_n$ with $|\eta_j| < \varepsilon$ for $j = 1, \cdots, n-1$ and $|\eta_n - 1| < \varepsilon$. Denote $B_0 = B$ when $\epsilon = 0$.

Summing A and B_ϵ on a neighborhood of $K \cap \overline{V}$. When $\epsilon = 0$, the smallest eigenvalue of the complex Hessian of λ is at least M on $K \cap \overline{V}$. Use the continuity of the eigenvalues of the complex Hessian of ρ^2 and notice that ϵ is independent of

M, therefore the smallest eigenvalue of the complex Hessian of λ is at least $\frac{M}{2}$ on a neighborhood of $K \cap \overline{V}$. Since $\rho = 0$ on $b\Omega$, by the construction of λ, we also conclude that $0 \leq \lambda \leq n$ on a neighborhood of $K \cap \overline{V}$. Since Property (P_1) (or generally Property (P_q)) is preserved by unitary change of coordinate systems, we conclude that $K \cap \overline{V}$ has Property (P_1). Since Property (P_1) is a local property by Proposition 8, our theorem follows. □

Remark 5. *In [3], the idea of summing functions in each z_j-plane $(j = 1, \cdots, n)$ is used to create the function in the definition of Property (P_1). We adapted this idea in our proof of the theorem, however, we only need to sum $n-1$ functions in our case and the last function $M\rho^2$ comes for free. The key observation is that the function $M\rho^2$ only has positive eigenvalue in the complex normal direction on the boundary, and such property can be used to produce an arbitrarily big eigenvalue in the complex normal direction.*

5. THE SMALLNESS OF THE WEAKLY PSEUDOCONVEX POINTS ON SMOOTH HARTOGS DOMAINS

Our study in the smallness of the weakly pseudoconvex points on the boundary of a smooth bounded pseudoconvex Hartogs domain Ω in terms of Hausdorff measure is motivated by the results of Sibony ([41]) and Boas ([3]) on general pseudoconvex domains: Let $q = 1$ and assume that the set K of the weakly pseudoconvex points on the boundary $b\Omega$ has Hausdorff 2-dimensional measure zero in $\mathbb{C}^n$, then the boundary $b\Omega$ has Property (P_1) and hence the $\overline{\partial}$-Neumann operator N_1 is compact on $L^2_{(0,1)}(\Omega)$. (Boas ([3]) has an explicit construction of the function λ involved in the definition of Property (P_1).)

The general case is the following: Given a smooth bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$, assume the set K of the weakly pseudoconvex points on the boundary $b\Omega$ has Hausdorff $2q$-dimensional measure zero in $\mathbb{C}^n$, then is it true that the boundary $b\Omega$ has Property (P_q) in $\mathbb{C}^n$ and the $\overline{\partial}$-Neumann operator N_q is compact on $L^2_{(0,q)}(\Omega)$?

Sibony's approach can not be generalized to the case $q > 1$ (see remarks after Proposition 14). Therefore it is not clear (or unknown) that whether K(or $b\Omega$) always has Property (P_q). In this section, we give two examples of smooth complete pseudoconvex Hartogs domains in $\mathbb{C}^3$ which have the desired property.

5.1 First example

Let $\Omega = \{(z_1, z_2, z_3) \,\big|\, |z_2|^2 + |z_3|^2 < e^{-\varphi(z_1)},\ z_1 \in \mathbb{D}(0,1)\}$ and we assume $\varphi \in C^\infty(\mathbb{D}(0,1))$ and φ is subharmonic on $\mathbb{D}(0,1)$. We assume further that φ has extra regularity property such that $b\Omega$ is C^∞ smooth and boundary points (z_1, z_2, z_3) are strictly pseudoconvex when $|z_1|$ is close to 1.

Denote the defining function $\rho(z_1, z_2, z_3) = |z_2|^2 + |z_3|^2 - e^{-\varphi(z_1)}$, the complex

Hessian of ρ is:

$$\left(\frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}\right)_{j,k=1,2,3} = \begin{pmatrix} -e^{\varphi}|\frac{\partial\varphi}{\partial z_1}|^2 + e^{-\varphi}\Delta\varphi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.1}$$

The complex tangent space $T^{\mathbb{C}}_{(z_1,z_2,z_3)}(b\Omega) = \{(\xi_1,\xi_2,\xi_3)|\ e^{-\varphi(z_1)}\frac{\partial\varphi}{\partial z_1}\xi_1 + \overline{z}_2\xi_2 + \overline{z}_3\xi_3 = 0\}$. We discuss three types of boundary points as follows:

Type I: On the boundary points of $\{(z_1,z_2,z_3) \in b\Omega|\ z_2 = 0\}$.

The complex tangent space $T^{\mathbb{C}}_{(z_1,z_2,z_3)}(b\Omega)$ can be expressed as: $T^{\mathbb{C}}_{(z_1,z_2,z_3)}(b\Omega) = (a, b, -a\frac{e^{-\varphi(z_1)}}{\overline{z}_3}\frac{\partial\varphi}{\partial z_1})$, where $a, b \in \mathbb{C}$. Calculate the Levi form on Type I points (and put in the boundary condition $|z_2|^2 + |z_3|^2 = e^{-\varphi(z_1)}$):

$$\text{Levi form} = |b|^2 + e^{-\varphi(z_1)}\frac{\partial^2\varphi}{\partial z_1 \partial \overline{z}_1}|a|^2 \geq 0. \tag{5.2}$$

We refer the reader to section 2.1 for the definition of Levi form. By (5.2), the Levi form on Type I points is nonnegative. By taking $b = 0$, the weakly pseudoconvex points of Type I points are precisely $\{(z_1,z_2,z_3) \in b\Omega|\ \Delta\varphi(z_1) = 0, z_2 = 0\}$.

Type II: On the boundary points of $\{(z_1,z_2,z_3) \in b\Omega|\ z_3 = 0\}$.

Similar as Type I points, we conclude that the Levi form is nonnegative on Type II points and the weakly pseudoconvex points of Type II points are precisely $\{(z_1,z_2,z_3) \in b\Omega|\ \Delta\varphi(z_1) = 0, z_3 = 0\}$.

Type III: On the boundary points of $\{(z_1,z_2,z_3) \in b\Omega|\ z_2 \neq 0, z_3 \neq 0, \frac{\partial\varphi}{\partial z_1} = 0\}$.

The complex tangent space $T^{\mathbb{C}}_{(z_1,z_2,z_3)}(b\Omega)$ can be expressed as: $T^{\mathbb{C}}_{(z_1,z_2,z_3)}(b\Omega) = (a, b, -\frac{\overline{z}_2}{\overline{z}_3}b)$, where $a, b \in \mathbb{C}$.

Calculate the Levi form on Type III points (and put in the boundary condition $|z_2|^2 + |z_3|^2 = e^{-\varphi(z_1)}$):

$$\text{Levi form} = e^{-\varphi}(|a|^2\Delta\varphi + \frac{|b|^2}{|z_3|^2}) \geq 0. \tag{5.3}$$

We conclude that the Levi form is nonnegative on Type III points and the weakly pseudoconvex points of Type III points are precisely $\{(z_1, z_2, z_3) \in b\Omega|\ \Delta\varphi(z_1) = 0, \frac{\partial\varphi}{\partial z_1} = 0, z_2 \neq 0, z_3 \neq 0\}$.

Type IV: On the boundary points $\{(z_1, z_2, z_3) \in b\Omega|\ z_2 \neq 0, z_3 \neq 0, \frac{\partial\varphi}{\partial z_1} \neq 0\}$.

The complex tangent space is spanned by $(1, -\frac{e^{-\varphi}}{\bar{z}_2}\frac{\partial\varphi}{\partial z_1}, 0)$ and $(1, 0, -\frac{e^{-\varphi}}{\bar{z}_3}\frac{\partial\varphi}{\partial z_1})$. We express any complex tangent at Type IV points as $(a+b, -a\frac{e^{-\varphi}}{\bar{z}_2}\frac{\partial\varphi}{\partial z_1}, -b\frac{e^{-\varphi}}{\bar{z}_3}\frac{\partial\varphi}{\partial z_1})$, where $a, b \in \mathbb{C}$. We calculate the Levi form at Type IV points (and put in the boundary condition $|z_2|^2 + |z_3|^2 = e^{-\varphi(z_1)}$):

$$\begin{aligned}
&\text{Levi form} \\
= \;& e^{-\varphi(z_1)}\left[\Delta\varphi \cdot |a+b|^2 + \left|\frac{\partial\varphi}{\partial z_1}\right|^2 \left(|a|^2 + |b|^2 - |a+b|^2 + |a|^2\frac{|z_3|^2}{|z_2|^2} + |b|^2\frac{|z_2|^2}{|z_3|^2}\right)\right] \\
\geq \;& e^{-\varphi(z_1)}\left[\Delta\varphi \cdot |a+b|^2 + \left|\frac{\partial\varphi}{\partial z_1}\right|^2 \left(|a|\frac{|z_3|}{|z_2|} - |b|\frac{|z_2|}{|z_3|}\right)^2\right] \\
\geq \;& 0.
\end{aligned}$$

Therefore the Levi form is nonnegative on Type IV points. By setting the first equality to zero in above calculation, the Levi form vanishes exactly when $a = \frac{|z_2|^2}{|z_3|^2}b$ on points $\{(z_1, z_2, z_3) \in b\Omega|\ \Delta\varphi(z_1) = 0, \frac{\partial\varphi}{\partial z_1} \neq 0, z_2 \neq 0, z_3 \neq 0\}$. Note that taking $a = -b$ in the calculation above does not produce any weakly pseudoconvex points. Therefore, the weakly pseudoconvex points of Type IV points are precisely $\{(z_1, z_2, z_3) \in b\Omega|\ \Delta\varphi(z_1) = 0, \frac{\partial\varphi}{\partial z_1} \neq 0, z_2 \neq 0, z_3 \neq 0\}$.

Now take the union of all weakly pseudoconvex points of each type points, we conclude that the weakly pseudoconvex points of $b\Omega$ are exactly $\{(z_1, z_2, z_3) \in b\Omega|\ \Delta\varphi(z_1) = 0, z_1 \in \mathbb{D}(0, 1)\}$.

We have proved the following lemma:

Lemma 4. *Define a smooth complete Hartogs domain* $\Omega \subset \mathbb{C}^3$ *by:*

$$\Omega = \{(z_1, z_2, z_3) \,\big|\, |z_2|^2 + |z_3|^2 < e^{-\varphi(z_1)}, \quad z_1 \in \mathbb{D}(0,1)\}.$$

Assume $\varphi \in C^\infty(\mathbb{D}(0,1))$*,* φ *is subharmonic on* $\mathbb{D}(0,1)$ *and* φ *has extra regularity property such that boundary points* (z_1, z_2, z_3) *are strictly pseudoconvex when* $|z_1|$ *is close to 1. Then* Ω *is pseudoconvex and the weakly pseudoconvex set of* $b\Omega$ *is precisely* $W = \{(z_1, z_2, z_3) \in b\Omega |\ \Delta\varphi(z_1) = 0, z_1 \in \mathbb{D}(0,1)\}$.

Lemma 5. *Let* Ω *be as above in Lemma 4. If the weakly pseudoconvex points of* $b\Omega$ *has Hausdorff* 4*-dimensional measure zero in* $\mathbb{C}^3$*, then* $\{z_1 |\ \Delta\varphi(z_1) = 0\}$ *has Lebesgue measure zero in* $\mathbb{C}$*.*

Proof. Denote the projection set of W (defined in Lemma 4) onto $z_1 - z_2$ plane as A. Since W has Hausdorff 4-dim measure zero in $\mathbb{C}^3$, the Hausdorff 4-dim measure of A in $z_1 - z_2$ plane is zero. Since Hausdorff $2q$-dim measure is equal to a constant multiplying Lebesgue measure in $\mathbb{C}^q$ ($q \geq 1$), we conclude that the set A has Lebesgue measure zero in $\mathbb{C}^2$.

Since $W = \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\ \Delta\varphi(z_1) = 0, z_1 \in \mathbb{D}(0,1), |z_2|^2 + |z_3|^2 = e^{-\varphi(z_1)}\}$, we have: $A = \{(z_1, z_2) |\ \Delta\varphi(z_1) = 0, z_1 \in \mathbb{D}(0,1), |z_2|^2 \leq e^{-\varphi(z_1)}\}$.

For each fixed $\xi_1 \in \mathbb{C}$, $A_{\xi_1} := \{z_2 |\ (\xi_1, z_2) \in A\}$ contains a disk $\mathbb{D}(0,r)$ with $0 < r < e^{-\varphi(\xi_1)}$. Define $\delta = \min_{\xi_1 \in \{\Delta\varphi = 0\}} e^{-\varphi(\xi_1)} > 0$.

Since A has Lebesgue measure zero in $\mathbb{C}^2$ and by Fubini Theorem we have:

$$\begin{aligned} 0 = m(A) &= \int_{\{(z_1,z_2)\in A\}} dm(z_1, z_2) \\ &= \int_{\{\Delta\varphi = 0\}} dm(z_1) \int_{z_2 \in A_{z_1}} dm(z_2), \end{aligned}$$

m is the Lebesgue measure on each space respectively. By above discussion, for each $z_1 \in \{\Delta\varphi = 0\}$, $\int_{z_2 \in A_{z_1}} dm(z_2) \geq \pi\delta^2 > 0$, hence $m(\{\Delta\varphi = 0\}) = 0$ and the lemma follows. $\square$

Proposition 15. *Ω is defined as in Lemma 4. If the Hausdorff 4-dimensional measure of the weakly pseudoconvex points of $b\Omega$ is zero, then $b\Omega$ has Property (P_1) and the $\overline{\partial}$-Neumann operator N_1 is compact.*

Proof. Any strictly pseudoconvex points of $b\Omega$ is of finite type and hence any compact subsets of strictly pseudoconvex points of $b\Omega$ has Property (P_1) ([9]). Therefore it suffices to prove that the weakly pseudoconvex points W of $b\Omega$ has Property (P_1) and apply Proposition 9 to conclude our claim. Define the projection map $\pi(z_1, z_2, z_3) = z_1$. By Lemma 5, $\pi(W)$ has Lebesgue measure zero in z_1-plane. By Corollary 1, $\pi(W)$ has Property (P_1).

On W, the function $-|z|^2 = -|z_1|^2 - e^{-\varphi(z_1)}$, which becomes a function of z_1 alone. Since $\pi(W)$ has Property (P_1), $-|z_1|^2 - e^{-\varphi(z_1)} \in P_1(\pi(W))$. We can find a sequence of function $\{\lambda_m(z_1)\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} ||\lambda_m(z_1) - (-|z_1|^2 - e^{-\varphi(z_1)})||_{\infty,\pi(W)} = 0$, and each $\lambda_m(z_1) \in P_1(U_m)$ for some open neighborhood $U_m \supset \pi(W)$. Since $\lambda_m \circ \pi(z_1, z_2, z_3) = \lambda_m(z_1)$, we have:

$$\begin{aligned} & \lim_{m\to\infty} ||\lambda_m \circ \pi - (-|z|^2)||_{\infty,W} \\ = & \lim_{m\to\infty} ||\lambda_m \circ \pi - (-|z_1|^2 - e^{-\varphi(z_1)})||_{\infty,W} \\ \leq & \lim_{m\to\infty} ||\lambda_m(z_1) - (-|z_1|^2 - e^{-\varphi(z_1)})||_{\infty,\pi(W)} \\ = & \ 0. \end{aligned}$$

Therefore $-|z|^2 \in P_1(W)$ by above inequality together with the fact that each $\lambda_m \circ \pi \in P_1(\pi^{-1}(U_m)) \subset P_1(V_m)$, where each $V_m \subset \mathbb{C}^3$ is an open neighborhood of W.

Therefore W has Property (P_1) and our proposition follows. $\square$

Remark 6. *Our result on Ω is unexpected in the sense that W would be expected to only have Property (P_2) when we assume the Hausdorff 4-dimensional measure of W is zero. Hence this example suggests that Hausdorff measure is a crude tool to characterize Property (P_q) of the boundary for the case of $q > 1$.*

5.2 Second example

We look at another smooth complete Hartogs domain in $\mathbb{C}^3$, although most calculation procedure remains the same, our result demonstrates an approach to control second derivatives when proving Property (P_q) for $q > 1$.

Let $\Omega = \{(z_1, z_2, z_3) | \, |z_3|^2 < e^{-\varphi(z_1)-\psi(z_2)}, z_1 \in \mathbb{D}(0,1), z_2 \in \mathbb{D}(0,1)\}$. We assume that $\varphi, \psi \in C^\infty(\mathbb{D}(0,1))$ and subharmonic on $\mathbb{D}(0,1)$ in the respective complex plane. Assume further that the boundary points (z_1, z_2, z_3) are strictly pseudoconvex when (z_1, z_2) is close to $b(\mathbb{D}(0,1) \times \mathbb{D}(0,1))$. By replacing the distinguished boundary of $\mathbb{D}(0,1) \times \mathbb{D}(0,1)$ with some smooth boundary (for example the boundary of any ball), we may assume Ω has a smooth boundary.

Denote the defining function $\rho(z_1, z_2, z_3) = |z_3|^2 - e^{-\varphi(z_1)-\psi(z_2)}$, the complex Hessian of ρ is:

$$\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}\right)_{j,k=1,2,3} = \begin{pmatrix} -e^{\varphi-\psi}(\Delta_{z_1}\varphi - \left|\frac{\partial \varphi}{\partial z_1}\right|^2) & -\frac{\partial \varphi}{\partial z_1}\overline{\frac{\partial \psi}{\partial z_2}} e^{-\varphi-\psi} & 0 \\ -\overline{\frac{\partial \varphi}{\partial z_1}}\frac{\partial \psi}{\partial z_2} e^{-\varphi-\psi} & -e^{\varphi-\psi}(\Delta_{z_2}\varphi - \left|\frac{\partial \psi}{\partial z_2}\right|^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The complex tangent space at a boundary point (z_1, z_2, z_3) is:

$$T^{\mathbb{C}}_{(z_1,z_2,z_3)}(b\Omega) = \left\{ (\xi_1, \xi_2, \xi_3) \,\middle|\, e^{-\varphi(z_1)-\psi(z_2)}\frac{\partial \varphi}{\partial z_1}\xi_1 + e^{-\varphi(z_1)-\psi(z_2)}\frac{\partial \psi}{\partial z_2}\xi_2 + \overline{z}_3\xi_3 = 0 \right\}.$$

The complex tangent space is spanned by $(0,1,-\frac{\frac{\partial\psi}{\partial z_2}}{\overline{z}_3}e^{-\varphi-\psi})$ and $(1,0,-\frac{\frac{\partial\varphi}{\partial z_1}}{\overline{z}_3}e^{-\varphi-\psi})$, hence we have:

$$T^{\mathbb{C}}_{(z_1,z_2,z_3)}(b\Omega) = \left(b, a, -\frac{e^{-\varphi-\psi}}{\overline{z}_3}(a\frac{\partial\psi}{\partial z_2} + b\frac{\partial\varphi}{\partial z_1})\right),$$

where $a, b \in \mathbb{C}$. Now calculate the Levi form (and put in the boundary condition $|z_3|^2 = e^{-\varphi(z_1)-\psi(z_2)}$):

$$\text{Levi form} = e^{-\varphi-\psi}(\Delta_{z_1}\varphi \cdot |b|^2 + \Delta_{z_2}\psi \cdot |a|^2) \geq 0.$$

Therefore Ω is pseudoconvex and by taking $a = 0$ and $b = 0$ in above equation respectively, the weakly pseudoconvex points on Ω are precisely the union of the following two sets (I) and (II):

(I): $\{(z_1, z_2, z_3) \in b\Omega | \ \frac{\partial^2\varphi}{\partial z_1 \partial \overline{z}_1} = 0;\ z_2 \in \mathbb{D}(0,1)\}$.

(II): $\{(z_1, z_2, z_3) \in b\Omega | \ \frac{\partial^2\psi}{\partial z_2 \partial \overline{z}_2} = 0;\ z_1 \in \mathbb{D}(0,1)\}$.

Notice that neither (I) nor (II) has Property (P_1) in $\mathbb{C}^3$, since each of them contains a copy of $\mathbb{D}(0,1)$.

Proposition 16. *Define the smooth complete Hartogs domain Ω as above. Ω is pseudoconvex. If the Hausdorff 4-dimensional measure of the weakly pseudoconvex points of $b\Omega$ is zero, then $b\Omega$ has Property (P_2) and the $\overline{\partial}$-Neumann operator N_2 is compact.*

Proof. We first prove that the set (I) above has Property (P_2) in $\mathbb{C}^3$. Denote A_1 as the projection set of (I) onto $z_1 - z_2$ plane. Then $A_1 = \{(z_1, z_2) | \Delta_{z_1}\varphi = 0;\ z_2 \in \mathbb{D}(0,1)\}$, and by the same argument in Proposition 15, A_1 has Lebesgue measure zero in $\mathbb{C}^2$.

Apply Fubini Theorem, we have:

$$0 = m(A_1) = \int_{\{\Delta_{z_1}\varphi=0\}} dm(z_1) \int_{\mathbb{D}(0,1)} dm(z_2),$$

where m is the Lebesgue measure in the respective complex plane. Hence $m(\{\Delta_{z_1}\varphi = 0\}) = 0$, and by Corollary 1, $\{z_1 \in \mathbb{D}(0,1) | \frac{\partial^2\varphi}{\partial z_1 \partial \overline{z}_1} = 0\}$ has Property (P_1).

Denote A_2 as the projection set of (I) onto $z_2 - z_3$ plane, again we conclude that A_2 has Lebesgue measure zero in $\mathbb{C}^2$. Hence A_2 has Property (P_2) in $\mathbb{C}^2$ by Proposition 12.

Fix any $M > 0$, there exists an open neighborhood $U \subset \mathbb{C}^2$ of A_2 and a C^2 smooth function $\eta(z_2, z_3)$ such that $0 \le \eta \le 1$ and $\frac{\partial^2\eta}{\partial z_2 \partial \overline{z}_2} + \frac{\partial^2\eta}{\partial z_3 \partial \overline{z}_3} \ge M$ on U.

Now define $M' = \sup_{(z_2,z_3)\in A_2} \left(2 \sum_{j,k=1}^{2} \left|\frac{\partial^2\eta}{\partial z_j \partial \overline{z}_k}(z_2, z_3)\right|\right) > 2M$.

Given M', there exists an open neighborhood $V \subset \mathbb{C}$ of $\{z_1 \in \mathbb{D}(0,1) | \frac{\partial^2\varphi}{\partial z_1 \partial \overline{z}_1} = 0\}$ and a C^2 smooth function $\gamma(z_1)$ such that $0 \le \gamma \le 1$ and $\frac{\partial^2\gamma}{\partial z_1 \partial \overline{z}_1} > M'$ on V.

We define $\lambda(z_1, z_2, z_3) = \frac{1}{2}(\gamma(z_1) + \eta(z_2, z_3))$, by our construction, λ is well-defined on a neighborhood S of the set (I), and $0 \le \lambda \le 1$ on S.

For all $(0,2)$-forms u at $z \in S$, we have:

$$\begin{aligned}
& 2\sum_{|K|=1}\sum_{j,k=1}^{3} \frac{\partial^2\lambda}{\partial z_j \partial \overline{z}_k}(z_1, z_2, z_3) u_{jK}\overline{u_{kK}} \\
= \ & |u_{12}|^2 \left(\frac{\partial^2\gamma}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2\eta}{\partial z_2 \partial \overline{z}_2}\right) + |u_{13}|^2 \left(\frac{\partial^2\gamma}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2\eta}{\partial z_3 \partial \overline{z}_3}\right) \\
& + |u_{23}|^2 \left(\frac{\partial^2\eta}{\partial z_2 \partial \overline{z}_2} + \frac{\partial^2\eta}{\partial z_3 \partial \overline{z}_3}\right) + 2Re\left(\frac{\partial^2\eta}{\partial z_2 \partial \overline{z}_3} u_{12}\overline{u_{13}}\right) \\
\ge \ & |u_{12}|^2 \left(\frac{\partial^2\gamma}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2\eta}{\partial z_2 \partial \overline{z}_2} - \left|\frac{\partial^2\eta}{\partial z_2 \partial \overline{z}_3}\right|\right) \\
& + |u_{13}|^2 \left(\frac{\partial^2\gamma}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2\eta}{\partial z_3 \partial \overline{z}_3} - \left|\frac{\partial^2\eta}{\partial z_2 \partial \overline{z}_3}\right|\right)
\end{aligned}$$

$$
\begin{aligned}
& +|u_{23}|^2 \left(\frac{\partial^2 \eta}{\partial z_2 \partial \overline{z}_2} + \frac{\partial^2 \eta}{\partial z_3 \partial \overline{z}_3} \right) \\
> \quad & M(|u_{12}|^2 + |u_{13}|^2 + |u_{23}|^2) \\
= \quad & M|u|^2.
\end{aligned}
$$

Hence the set (I) has Property (P_2) in $\mathbb{C}^3$ by Lemma 1. Similarly, the set (II) has Property (P_2) in $\mathbb{C}^3$. The rest of boundary points are strictly pseudoconvex points, by the same argument at the beginning of Proposition 15, our proposition follows. $\square$

6. A VARIANT OF PROPERTY (P_{n-1}) ON SMOOTH PSEUDOCONVEX DOMAINS

In this section, we study a different variant of Property (P_{n-1}) on smooth pseudoconvex domains in $\mathbb{C}^n$, which implies the compactness of N_{n-1} on $L^2_{(0,n-1)}(\Omega)$.

Besides Property (P_q), McNeal's ([37]) Property ($\widetilde{P_q}$) implies the compactness of N_q on smooth pseudoconvex domains in $\mathbb{C}^n$ and Straube's ([44]) "short time flow" condition implies the compactness of N_1 on smooth pseudoconvex domains in $\mathbb{C}^2$. However, the relation between both conditions and Property (P_q) on the respective level of forms is not fully understood.

Let U be a neighborhood of any boundary point of Ω, $\{\omega^1, \cdots, \omega^n\}$ be $(1,0)$-forms on U which form a special boundary frame and $\{L_1, \cdots, L_n\}$ be the dual basis of $\{\omega^1, \cdots, \omega^n\}$, where L_i $(i = 1, \cdots, n-1)$ are complex tangents and L_n is the complex normal. We refer the reader to section 2.1 for the definition of special boundary chart and the notations there. Given a function $f \in C^2(U)$, define $\{f_{jk}\}$ as the coefficients in the following summation: $\partial\overline{\partial} f = \sum_{j,k=1}^n f_{jk}\omega^j \wedge \overline{\omega}^k$. Define $u = \sum_J' u_J \overline{\omega}_J \in C^\infty_{(0,n-1)}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$ with $\mathrm{supp}(u) \in \overline{\Omega} \cap U$ and $\varphi \in C^2(\overline{\Omega})$. Denote ρ as the defining function of Ω. Our start point is a variant Hörmander-Kohn-Morrey type formula which is due to Ahn ([1]) and Zampieri ([46]).

Proposition 17 ([1, 46])**.** *For every integer s with $1 \le s \le n-1$:*

$$\begin{aligned} & C(||\overline{\partial} u||^2_\varphi + ||\overline{\partial}^*_\varphi u||^2_\varphi) + C||u||^2_\varphi \qquad (6.1) \\ \ge \; & \sideset{}{'}\sum_{|K|=n-2} \sum_{j,k=1}^n \int_\Omega \varphi_{jk} u_{jK} \overline{u_{kK}} e^{-\varphi} \, dV - \sideset{}{'}\sum_{|J|=n-1} \sum_{j\le s} \int_\Omega \varphi_{jj} |u_J|^2 e^{-\varphi} \, dV \\ & + \sideset{}{'}\sum_{|K|=n-2} \sum_{j,k=1}^n \int_{b\Omega} \rho_{jk} u_{jK} \overline{u_{kK}} e^{-\varphi} \, d\sigma - \sideset{}{'}\sum_{|J|=n-1} \sum_{j\le s} \int_{b\Omega} \rho_{jj} |u_J|^2 e^{-\varphi} \, d\sigma. \end{aligned}$$

Here u, φ, Ω and ρ are defined as above.

To apply above estimate in our case, notice that since we work with $(0, n-1)$ form u, the only tangential part of u is $u_{1,2,\cdots,n-1}\,\overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \cdots \wedge \overline{\omega}_{n-1}$ (see section 2.2 for the definition of the tangential part of u), therefore if we control the regularity estimate of $u_{1,2,\cdots,n-1}$, we can derive the desired compactness estimate.

Proposition 18. *Let Ω be a smooth pseudoconvex domain, $u = \sum_J u_J \overline{\omega}_J \in C^\infty_{(0,n-1)}(\overline{\Omega})$ $\cap dom(\overline{\partial}^*)$ with $supp(u) \in \overline{\Omega} \cap U$, where U and $\{\omega_j\}_{j=1}^n$ forms a special boundary chart defined as above. Let $\varphi \in C^2(\overline{\Omega})$ and denote ρ as the defining function of Ω. We have the following estimates:*

$$\begin{aligned}
&\int_\Omega (\sum_{s=1}^{n-1} \varphi_{ss} - \varphi_{tt})|u_{1,2,\cdots,n-1}|^2 e^{-\varphi}\, dV \\
&\leq C(||\overline{\partial}u||^2_\varphi + ||\overline{\partial}^*_\varphi u||^2_\varphi + ||u||^2_\varphi) + C_\varphi ||e^{-\frac{\varphi}{2}}u||^2_{-1}, \quad \forall 1 \leq t \leq n-1. \qquad (6.2)
\end{aligned}$$

Proof. We make use of the estimate in Proposition 17 in our proof. Take $s = 1$ in Proposition 17.

We start with the last two terms in the estimate (6.1) and put in the condition $u_{nK} = 0$ on $b\Omega$ (since $u \in \text{dom}(\overline{\partial}^*)$):

$$\sum_{|K|=n-2}' \sum_{j,k=1}^n \int_{b\Omega} \rho_{jk} u_{jK} \overline{u_{kK}} e^{-\varphi}\, d\sigma = \int_{b\Omega} (\sum_{j=1}^{n-1} \rho_{jj})|u_{1,2,\cdots,n-1}|^2 e^{-\varphi}\, d\sigma,$$

$$\sum_{|J|=n-1}' \sum_{j\leq s} \int_{b\Omega} \rho_{jj} |u_J|^2 e^{-\varphi}\, d\sigma = \int_{b\Omega} \rho_{11} |u_{1,2,\cdots,n-1}|^2 e^{-\varphi}\, d\sigma.$$

Therefore, the last line in the estimate (6.1) becomes:

$$\sum_{|K|=n-2}' \sum_{j,k=1}^n \int_{b\Omega} \rho_{jk} u_{jK} \overline{u_{kK}} e^{-\varphi}\, d\sigma - \sum_{|J|=n-1}' \sum_{j\leq s} \int_{b\Omega} \rho_{jj} |u_J|^2 e^{-\varphi}\, d\sigma$$

$$= \int_{b\Omega} (\sum_{j=2}^{n-1} \rho_{jj}) |u_{1,2,\cdots,n-1}|^2 e^{-\varphi}\, d\sigma \geq 0. \tag{6.3}$$

Notice that we use the fact $\rho_{jj} \geq 0$ for all $j \geq 1$ by pseudoconvexity of Ω.

To estimate the second line in the estimate (6.1), we first take the two sums running over the indices of the tangential part of u:

$$\begin{aligned} &\sum_{|\widetilde{K}|=n-2}' \sum_{j,k=1}^{n} \int_\Omega \varphi_{jk} u_{j\widetilde{K}} \overline{u_{k\widetilde{K}}} e^{-\varphi}\, dV - \sum_{|\widetilde{J}|=n-1}' \sum_{j \leq s} \int_\Omega \varphi_{jj} |u_{\widetilde{J}}|^2 e^{-\varphi}\, dV \\ &= \int_\Omega (\sum_{j=1}^{n-1} \varphi_{jj}) |u_{1,2,\cdots,n-1}|^2 e^{-\varphi}\, dV - \int_\Omega \varphi_{11} |u_{1,2,\cdots,n-1}|^2 e^{-\varphi}\, dV \\ &= \int_\Omega (\sum_{j=2}^{n-1} \varphi_{jj}) |u_{1,2,\cdots,n-1}|^2 e^{-\varphi}\, dV, \end{aligned} \tag{6.4}$$

where $\widetilde{K}$ is the set of $(n-2)$-tuples of K which do not contain n and $\widetilde{J}$ is the of $(n-1)$-tuples of J which do not contain n.

To estimate the error terms from the difference of indices, we notice that the error terms only involve (coefficients of) the normal parts of u. These terms can be estimated in a standard argument: Let I be an increasing $(n-1)$-tuple fixed. By the classical Sobolev estimates of Δ (see [23] for example), we have:

$$\begin{aligned} ||(u_{\text{Norm}})_I \cdot e^{-\frac{\varphi}{2}}||_1 &\leq ||\Delta((u_{\text{Norm}})_I \cdot e^{-\frac{\varphi}{2}})||_{-1} \\ &\leq C_\varphi(||u \cdot e^{-\frac{\varphi}{2}}|| + ||\overline{\partial} u \cdot e^{-\frac{\varphi}{2}}|| + ||\overline{\partial}^*_\varphi u \cdot e^{-\frac{\varphi}{2}}||). \end{aligned} \tag{6.5}$$

The second inequality of (6.5) follows from the fact that $\overline{\partial}\vartheta + \vartheta\overline{\partial}$ acts coefficientwise as $-\frac{1}{4}\Delta$ on domains in $\mathbb{C}^n$ (see for example in [45], lemma 2.11), where ϑ is formal adjoint of $\overline{\partial}$. Since we only need to estimate the L^2 norm of the normal parts of u, applying Proposition 6, we can use the interpolation of Sobolev norms (from

W^1-norm to W^{-1}-norm) to make the constant C_φ in (6.5) be independent of φ:

$$\begin{aligned}
& ||(u_{\text{Norm}})_I \cdot e^{-\frac{\varphi}{2}}||_0 \\
\leq\ & \epsilon||(u_{\text{Norm}})_I \cdot e^{-\frac{\varphi}{2}}||_1 + C_\epsilon||(u_{\text{Norm}})_I \cdot e^{-\frac{\varphi}{2}}||_{-1} \\
\leq\ & \epsilon C_\varphi||u||_{0,\varphi} + \epsilon C_\varphi||\overline{\partial}u||_{0,\varphi} + \epsilon C_\varphi||\overline{\partial}^*_\varphi u||_{0,\varphi} + C_{\varphi,\varepsilon}||ue^{-\frac{\varphi}{2}}||_{-1}. \qquad (6.6)
\end{aligned}$$

Take $\epsilon < \frac{1}{C_\varphi}$, and hence we have:

$$||(u_{\text{Norm}})_I||_{0,\varphi} \leq C(||u||_{0,\varphi} + ||\overline{\partial}u||_{0,\varphi} + ||\overline{\partial}^*_\varphi u||_{0,\varphi}) + C_\varphi||ue^{-\frac{\varphi}{2}}||^2_{-1}. \qquad (6.7)$$

Now first apply Cauchy inequality to all normal parts of u in the second line of the estimate (6.1), use (6.7) (the coefficients φ_{jk} can be absorbed by ϵ in (6.6)) to estimate the normal parts of u, then use (6.4) to estimate the tangential parts of u in the second line of the estimate (6.1) and apply Proposition 17, we have:

$$\begin{aligned}
& \int_\Omega (\sum_{s=2}^{n-1} \varphi_{ss})|u_{1,2,\cdots,n-1}|^2 e^{-\varphi}\, dV \\
& \leq C(||\overline{\partial}u||^2_\varphi + ||\overline{\partial}^*_\varphi u||^2_\varphi + ||u||^2_\varphi) + C_\varphi||e^{-\frac{\varphi}{2}}u||^2_{-1}. \qquad (6.8)
\end{aligned}$$

We proved the proposition for $t = 1$, for the rest cases we just need to permute the basis in the special boundary chart and by symmetry, our proposition follows. □

Now cover $b\Omega$ by finitely many special boundary charts $\{V_j\}_{j=1}^N$.

Definition 4. *For a smooth bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ $(n > 2)$, $b\Omega$ has Property $(P^\#_{n-1})$ if the following holds on each chart V_j: For any $M > 0$, there exists a neighborhood U of $b\Omega$ and a C^2 smooth function λ on $U \cap V_j$, such that $0 \leq \lambda(z) \leq 1$ and there exists t $(1 \leq t \leq n-1)$ such that $\sum_{s=1}^{n-1} \lambda_{ss} - \lambda_{tt} \geq M$ on $U \cap V_j$.*

Remark 7. *(1) Our definition of Property ($P^{\#}_{n-1}$) does not depend on the eigenvalues of the complex Hessian of λ in the definition of the original Property (P_{n-1}), indeed only the diagonal entries in the complex Hessian of λ are involved in our definition of Property ($P^{\#}_{n-1}$). However such Property ($P^{\#}_{n-1}$) can only be formulated within the special boundary charts.*

(2) By Schur majorization theorem, Property (P_{n-2}) implies Property ($P^{\#}_{n-1}$), but it is still unclear what the relation is between Property (P_{n-1}) and Property ($P^{\#}_{n-1}$).

6.1 Main Theorem 14

Now we prove the main theorem in this section:

Theorem 14. *Let $\Omega \subset \mathbb{C}^n$ ($n > 2$) be a smooth bounded pseudoconvex domain. If $b\Omega$ has Property ($P^{\#}_{n-1}$), then the $\overline{\partial}$-Neumann operator N_{n-1} is compact on $L^2_{(0,n-1)}(\Omega)$.*

Proof. Fix $M > 0$, by Proposition 4 we need to prove the following compactness estimate for $(0, n-1)$ forms $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$:

$$||u||^2 \leq \frac{C}{M}(||\overline{\partial}u||^2 + ||\overline{\partial}^* u||^2) + C_M||u||^2_{-1}. \tag{6.9}$$

It suffices to establish (6.9) for $u \in C^\infty_{(0,n-1)}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$ by using the density of these forms in $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ (See Proposition 7).

Since $b\Omega$ has Property ($P^{\#}_{n-1}$), on each special boundary chart V_j, there exists an open neighborhood U_M of $b\Omega$ and a C^2 smooth function λ_M on $U_M \cap V_j$ such that $0 \leq \lambda_M \leq 1$ and $\exists t$ ($1 \leq t \leq n-1$) such that $\sum_{s=1}^{n-1} \lambda_{M_{ss}} - \lambda_{M_{tt}} \geq M$ on $U_M \cap V_j$. By choosing a function η in $C^2(\overline{\Omega})$ which agrees near $U_M \cap V_j$ with λ_M and $0 \leq \eta \leq 1$ on $\overline{\Omega}$, we can further assume $\lambda \in C^2(\overline{\Omega})$ and $0 \leq \lambda \leq 1$.

Now assume first that u is supported near the boundary and by a partition of unity, we may assume that u is supported in $V_j \cap U_M$ for some j. We apply

Proposition 18 with $\varphi = \lambda_M$ and notice that the weighted norm is comparable to the usual unweighted L^2-norm since $0 \leq \lambda_M \leq 1$, hence we have:

$$\int_\Omega |u_{1,2,\cdots,n-1}|^2 \, dV \leq \frac{C}{M}(||\overline{\partial}u||^2 + ||\overline{\partial}^* u||^2 + ||u||^2) + C_M||u||^2_{-1}. \tag{6.10}$$

By estimate (6.10), we only need to estimate the normal part of u, but this can be done exactly the same as we did in the proof of Proposition 18. Hence estimate (6.10) holds when we replace the left side with normal components of u. Now absorbing the the $\frac{C}{M}||u||^2$ into the left side, we have:

$$||u||^2 \leq \frac{C}{M}(||\overline{\partial}u||^2 + ||\overline{\partial}^* u||^2) + C_M||u||^2_{-1}. \tag{6.11}$$

Hence the compactness estimate is established when u is supported near the boundary.

When u has compact support in Ω, the desired compactness estimate follows from the interior elliptic regularity of $\overline{\partial} \oplus \overline{\partial}^*$ with the constant C independent of the support: Let V contains the support of U and by the interior elliptic regularity:

$$||u||^2_{1,V} \leq C_V(||\overline{\partial}u||^2_{0,V} + ||\overline{\partial}^* u||^2_{0,V} + ||u||^2_{W^{-1}(\Omega)}). \tag{6.12}$$

We refer the reader to [23] for general discussions of interior elliptic regularity and see also [45] under the context of $\overline{\partial}$-Neumann problem. Note that in above estimate we also use the fact that $||\cdot||_{W^{-1}(V)} \lesssim ||\cdot||_{W^{-1}(\Omega)}$ by duality. Since we only need to estimate the L^2-norm of u, we can again use interpolation of Sobolev norms (between W^1-norm and W^{-1}-norm) in the same way as we did in Proposition 18 to make the constant C_V before the terms $||\overline{\partial}u||$ and $||\overline{\partial}^* u||$ independent of V. Hence the compactness estimate follows for u compactly supported in Ω.

Finally when $u \in C^\infty_{(0,n-1)}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$, choose a partition of unity of $\overline{\Omega}$, say χ_0 and χ_1, such that χ_0 is supported in Ω and χ_1 is supported near $b\Omega$. We have established the compactness estimates for $\chi_0 u$ and $\chi_1 u$. Notice that $\overline{\partial}$ or $\overline{\partial}^*$ produces derivatives of χ_0 and χ_1 which contain no derivatives of u. Hence these terms are compactly supported in Ω and can be estimated in the same way as in the last two paragraphs. Therefore our compactness estimate holds and the theorem follows. $\square$

7. SUMMARY

In section 1, we briefly discussed the significance of studying Property (P_q) and the compactness of the $\overline{\partial}$-Neumann operator N_q. We also discuss the motivation of our research on the analysis of Property (P_q), related with the main results in the dissertation.

In the first part of section 2, we gave the set up of the $\overline{\partial}$-Neumann problem and introduced various regularity properties of the $\overline{\partial}$-Neumann operator N_q. We also introduced the definition of Property (P_q) and its basic properties. In the second part of section 2, we discussed preliminary results from classical potential theory and their applications in the study of Property (P_q).

In section 3, we first gave a special result regarding the invariance property of Property (P_1) for a compact subset in $\mathbb{C}$. Then we introduced a twisted Property (P_q) induced by a certain holomorphic mapping $\pi : \mathbb{C}^n \to \mathbb{C}^q$ and show that if given a compact subset K in $\mathbb{C}^n$, the image set $\pi(K)$ has the twisted Property (P_q) in the $\mathbb{C}^q$ subspace and each fiber of $K \cap \pi^{-1}(x)$ has Property (P_q) in $\mathbb{C}^n$ for every $x \in \pi(K)$, then K has Property (P_q) in $\mathbb{C}^n$. This invariance property is a partial generalization of Sibony's ([41]) result. Our proof partially overcome the difficulty in the case of $q > 1$ which is not detected in the case of $q = 1$.

In section 4, we first studied the obstruction to Property (P_q) for a compact set K in $\mathbb{C}^n$. We proved that if K has Property (P_q), then for any q-dimensional affine subspace E in $\mathbb{C}^n$, $K \cap E$ has empty fine interior with respect to the fine topology in $\mathbb{C}^q$. Our proof utilized several results in the classical potential theory. Our result generalizes Sibony's ([41]) result on the complex plane. We then proved a special case regarding the converse of the previous result on a smooth pseudoconvex domain.

In the proof of this special case, we utilized Boas's ([3]) idea of summing functions.

In section 5, we gave two concrete examples of smooth complete Hartogs domains in $\mathbb{C}^3$ concerning the smallness of weakly pseudoconvex points on the boundary. While both examples conclude that if the Hausdorff 4-dimensional measure of the set of weakly pseudoconvex points is zero then the boundary has Property (P_2), the first example suggested that Hausdorff measure is a crude tool to completely capture the information of Property (P_q) $(q > 1)$ on higher levels of forms, which was not detected in the case of $q = 1$ in Sibony's ([41]) results. In the second example we developed an approach to control second derivatives of the function λ occurring in the definition of Property (P_q) and we also utilized Boas's ([3]) idea of summing functions in the proof.

In section 6, we introduced a variant of Property (P_{n-1}) on smooth pseudoconvex domains in $\mathbb{C}^n$ $(n > 2)$ which implies the compactness of the $\overline{\partial}$-Neumann operator N_{n-1}. Our new Property $(P^{\#}_{n-1})$ does not depend on the eigenvalues of the complex Hessian of λ in the definition of the original Property (P_{n-1}), indeed only the diagonal entries in the complex Hessian of λ are involved in our definition of Property $(P^{\#}_{n-1})$. However, whether such definition can be generalized to the other level of forms is still unclear.

REFERENCES

[1] H. Ahn, Global boundary regularity for the $\overline{\partial}$-equation on q-pseudoconvex domains. *Math. Nachr.* 280 (2007), 343-350.

[2] D. H. Armitage and S. J. Gardiner, *Classical potential theory.* Springer Monogr. Math., Springer-Verlag, London 2001.

[3] H. P. Boas, Small sets of infinite type are benign for the $\overline{\partial}$-Neumann problem. *Proc. Amer. Math. Soc.* 103 (1988), 569-578.

[4] H. P. Boas and E. J. Straube, Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in $\mathbb{C}^n$ admitting a defining function that is plurisubharmonic on the boundary. *Math. Z.* 206 (1991), 81-88.

[5] H. P. Boas and E. J. Straube, Global regularity of the $\overline{\partial}$-Neumann problem: a survey of the L^2-Sobolev theory. In *Several complex variables* (MSRI 1995-96). Cambridge University Press, Cambridge 1999, 79-111.

[6] M. Brelot, *On topologies and boundaries in potential theory.* Enlarged edition of a course of lectures delivered in 1966, Lecture Notes in Math. 175, Springer-Verlag, Berlin 1971.

[7] D. Catlin, Necessary conditions for subellipticity of the $\overline{\partial}$-Neumann problem. *Ann. of Math.* (2) 117 (1983), 147-171.

[8] D. Catlin, Boundary invariants of pseudoconvex domains. *Ann. of Math.*(2) 120 (1984), 529-586.

[9] D. Catlin, Global regularity of the $\overline{\partial}$-Neumann problem. In *Complex analysis of several variables* (Madison, 1982). Proc. Sympos. Pure Math. 41, Amer. Math. Soc., Providence 1984, 39-49.

[10] D. Catlin, Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains. *Ann. of Math.* (2) 126 (1987), 131-191.

[11] D. W. Catlin and J. P. D'Angelo, Positivity conditions for bihomogeneous polynomials. *Math. Res. Lett.* 4 (1997), 555-567.

[12] M. Christ and S. Fu, Compactness in the $\overline{\partial}$-Neumann problem, magnetic Schrödinger operators, and the Aharonov-Bohm effect. *Adv. Math.* 197 (2005),1-40.

[13] J. P. D'Angelo, Finite type conditions for real hypersurfaces. *J. Differential Geom.* 14 (1979), 59-66 (1980).

[14] J. P. D'Angelo, Subelliptic estimates and failure of semicontinuity for orders of contact. *Duke Math. J.* 47 (1980), 955-957.

[15] J. P. D'Angelo, Real hypersurfaces, orders of contact, and applications. *Ann. of Math.* (2) 115 (1982), 615-637.

[16] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex.* Ann. of Math. Stud. 75, Princeton University Press, Princeton, N.J., 1972.

[17] B. Fuglede, The Dirichlet Laplacian on finely open sets. *Potential Anal.* 10 (1999), 91-101.

[18] S. Fu and E. J. Straube, Compactness in the $\overline{\partial}$-Neumann problem on convex domains. *J. Funct. Anal.* 159 (1998), 629-641.

[19] S. Fu and E. J. Straube, Compactness in the $\overline{\partial}$-Neumann problem. In *Complex analysis and geometry* (Columbus 1999). Ohio State Univ. Math. Res. Inst. Publ. 9, Walter de Gruyter, Berlin 2001, 141-160.

[20] S. Fu and E. J. Straube, Semi-Classical analysis of Schrödinger operators and compactness in the $\overline{\partial}$-Neumann problem, *J. Math. Anal. Appl.* 271 (2002), 267-282; Correction in *ibid* 280 (2003), 195-196.

[21] T. W. Gamelin, *Uniform algebras and Jensen measures.* London Math. Soc. Lecture Note Ser. 32, Cambridge University Press, Cambridge 1978.

[22] P. R. Garabedian and D. C. Spencer, Complex boundary value problems. *Trans. Amer. Math. Soc.* 73 (1952), 223-242.

[23] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd edition, Grundlehren Math. Wiss. 224, Springer-Verlag, Berlin 1998.

[24] T. Hefer and I. Lieb, On the compactness of the $\overline{\partial}$-Neumann operator. *Ann. Fac. Sci. Toulouse Math. (6)* 9 (2000), 415-432.

[25] L. L. Helms, *Introduction to potential theory.* Pure Appl. Math. 22, Wiley-Interscience, New York, London, Sydney 1969.

[26] G. M. Henkin and A. Iordan, Compactness of the Neumann operator for hyperconvex domains with non-smooth B-regular boundary. *Math. Ann.* 307 (1997), 151-168.

[27] L. Hörmander, L^2 estimates and existence theorem for the $\overline{\partial}$ operator. *Acta Math.* 113 (1965), 89-152.

[28] L. Hörmander, *An introduction to complex analysis in several variables.* 3rd ed., North-Holland Math. Library 7, North-Holland Publishing Co., Amsterdam 1990.

[29] L. Hörmander, A history of existence theorems for the Cauchy-Riemann complex in L^2 spaces. *J. Geom. Anal.* 13 (2003), 329-357.

[30] J. J. Kohn, Solution of the $\overline{\partial}$-Neumann problem on strongly pseudo-convex manifolds. *Proc. Nat. Acad. Sci. U.S.A.* 47 (1961), 1198-1202.

[31] J. J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds. I. *Ann. of Math.* (2) 78 (1963), 112-148.

[32] J. J. Kohn, Regularity at the boundary of the $\overline{\partial}$-Neumann problem. *Proc. Nat. Acad. Sci. U.S.A* 49 (1963), 206-213.

[33] J. J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds. II. *Ann. of Math.* (2) 79 (1964), 450-472.

[34] J. J. Kohn, A survey of the $\overline{\partial}$-Neumann problem. In *Complex analysis of several variables* (Madison, Wis., 1982), Proc. Sympos. Pure Math 41, Amer. Math. Soc., Providence, R.I., 1984, 137-145.

[35] J. J. Kohn and L. Nirenberg, Non-coercive boundary value problems. *Comm. Pure Appl. Math.* 18 (1965), 443-492.

[36] P. Li and S. Yau, On the Schrödinger equation and the eigenvalue problem. *Comm. Math. Phys.* 88 (1983), 309-318.

[37] J. McNeal, A sufficient condition for compactness of the $\overline{\partial}$-Neumann operator. *J. Funct. Anal.* 195 (2002), 190-205.

[38] R. M. Range, *Holomorphic functions and integral representations in several complex variables.* Grad. Texts in Math. 108, Springer-Verlag, New York 1986.

[39] S. Şahutoğlu, Compactness of the $\overline{\partial}$-Neumann problem and Stein neighborhood bases. Dissertation, Texas A&M University, 2006.

[40] N. Salinas, Noncompactness of the $\overline{\partial}$-Neumann problem and Toeplitz C^*-algebras. In *Several complex variables and complex geometry*, Part 3 (Santa

Cruz, CA, 1989), Proc. Sympos. Pure Math. 52, Amer. Math. Soc., Providence, R.I., 1991, 329-334.

[41] N. Sibony, Une classe de domaines pseudoconvexes. *Duke Math. J.* 55 (1987), 299-319.

[42] D. C. Spencer, Les opérateurs de Green et Neumann sur les variétés ouvertes riemanniennes et hermitiennes. Lecures given at the Collège de France, copy in the library of the Institut Henri Poincaré, 1955.

[43] E. J. Straube, Plurisubharmonic functions and subellipticity of the $\overline{\partial}$-Neumann problem on non-smooth domains. *Math. Res. Lett.* 4 (1997), 459-467.

[44] E. J. Straube, Geometric conditions which imply compactness of the $\overline{\partial}$-Neumann operator. *Ann. Inst. Fourier (Grenoble)* 54 (2004), 699-710.

[45] E. J. Straube, *Lectures on the L^2-Sobolev Theory of the $\overline{\partial}$-Neumann Problem.* ESI Lectures in Mathematics and Physics, European Math. Society Publishing House, Zürich, 2010.

[46] G. Zampieri, *Complex analysis and CR geometry.* University Lecture Ser. 43, Amer. Math. Soc., Providence, R.I., 2008.

[47] U. Venugopalkrishna, Fredholm operators associated with strongly pseudoconvex domains in $\mathbb{C}^n$. *J. Funct. Anal.* 9 (1972), 349-373.