PHASE RETRIEVAL OF SPARSE SIGNALS FROM MAGNITUDE INFORMATION

A Thesis

by

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ABSTRACT

The ability to recover the phase information of a signal of interest from a measurement process plays an important role in many practical applications. When only the Fourier transform magnitude of the signal is recorded, recovering the complete signal from these nonlinear measurements turns into a problem of phase retrieval.

Many practical algorithms exist to handle the phase retrieval problem. However, they present the drawback of convergence to a local minimum because of the non-convex Fourier magnitude constraints. Recent approaches formulating the problem in a higher dimensional space overcome this drawback but require a sufficiently large number of measurements. By using compressive sensing (CS) techniques, the number of measurements required for phase retrieval can be reduced with the additional information pertaining to the signal structure.

With the aim of reducing the number of measurements, this dissertation focuses on the problem of signal recovery by exploiting the sparsity information present in the signal samples. In this thesis, two approaches are proposed to accomplish sparse signal recovery from fewer magnitude measurements, modified Phase Cut and improved Phase Lift. In these approaches, we combine the phase retrieval methods, both *Phase Cut* and *Phase Lift*, which formulate the problem in a higher dimensional space, with l_1-norm minimization idea in CS by exploiting the sparse structure of the signals. The minimum number of measurements required for signal recovery by the proposed approaches is less than the number that Phase Cut and Phase Lift methods require. Both the modified Phase

Cut and the improved Phase Lift approaches outperform another variation of the Phase Lift method, Compressive Phase Retrieval via Lifting; namely, better signal reconstruction rate is obtained for different sparsity degrees. However, in terms of computation time, Phase Lift based methods are faster than the Phase Cut based methods.

Ultimately, combining the phase retrieval methods with the l_1-norm minimization enables the usage of the sparse structure of the signal for the exact recovery up to a sparsity degree from fewer magnitude measurements. However, challenges remain, particularly those related with computation time of methods and the sparsity degree of the signal which the methods could recover up to by fewer measurements.

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NOMENCLATURE

CPRL	Compressive Phase Retrieval via Lifting
CS	Compressive Sensing
IID	Independent and Identically Distributed
MSE	Mean Square Error
Rel. MSE	Relative Mean Square Error
RIP	Restricted Isometry Property
SDP	Semidefinite Programming
SNR	Signal-to-Noise Ratio

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1. INTRODUCTION AND LITERATURE REVIEW

Reconstruction of the phase information of a signal from its magnitude measurement plays a significant role in many applications in physics and engineering. In many cases the magnitude information can be recorded during the measurement process while the phase information is lost. In order to recover the signal exactly, the phase information is crucial and this problem is referred to as phase retrieval.

The phase retrieval problem was motivated by applications in fields such as X-ray crystallography, optics, astronomy, blind channel estimation, and radar. One example of phase retrieval is in optics, the shape of the object is included both in the amplitude and phase of the propagating electromagnetic wave [1]. However, only the amplitude may be measured and the phase information is lost. Therefore, phase retrieval algorithms are employed to recover the lost phase information. Another example is in X-ray crystallography, which is a method for determining the atomic and molecular structure of a crystal. This method is based on the fact that X-rays are diffracted by crystals. When incident X-ray beams pass through the crystal, they diffract into many specific directions. The diffraction pattern consists of reflections of different intensities. By using the intensities and angles of these diffracted beams, the electron density map within the crystal can be pictured. In these applications, the phase retrieval methods aim to recover the crystal shape from the magnitude of its Fourier transform. Since an X-ray detector can only record intensities, i.e., the square of the magnitude of the Fourier transform, the phase information is lost during the measurement process.

Mathematically, if **s** stands for the diffraction vector and **r** denotes the relative position of the electron to the one at the origin, the Fourier transform of the electron density function $\rho(\mathbf{r})$ is given by

$$F(\mathbf{s}) = \int_{crystal} \rho(\mathbf{r}) e^{-2\pi j \langle \mathbf{r},\mathbf{s} \rangle} d\mathbf{r}. \quad (1)$$

The integral in (1) is complex-valued, and it presents an amplitude and angle when it is expressed in polar form as $F(\mathbf{s}) = |F(\mathbf{s})| e^{j\angle F(\mathbf{s})}$. Only the magnitude $|F(\mathbf{s})|$ is obtained by taking the square root of the intensities measured in the diffraction experiments. The phase information plays an important role in determining the electron density function. Other examples come from the field of image processing since most of the information about images is stored in their phase. The images in Figure 1.1 clearly demonstrate the importance of the phase of the Fourier transform in preservation of the original image.

Figure 1.1. Illustration of the importance of phase in a two dimensional Fourier transform. **(**Top Left) Original *Lena* image; (Top Right) Original *Barbara* image; (Bottom Left) Image is obtained using the magnitude of the Fourier transform of *Lena* and the phase of the Fourier transform of *Barbara*; (Bottom Right) Using the magnitude of the Fourier transform of *Barbara* and the phase of the Fourier transform of *Lena*.

The methods proposed in the literature to solve the phase retrieval problem seek a solution based on some measurements in the object domain and in the transformed domain. Many methods for this problem rely on Gerchberg-Saxton [2] and “input-output” [3] algorithms and their variations. The iterative Gerchberg-Saxton algorithm consists of Fourier and inverse Fourier transformation steps imposing object and Fourier domain constraints. This algorithm is based on alternating projections between two non-convex sets. The greedy Gerchberg-Saxton algorithm has lead up to many researches to improve the phase retrieval process. Fienup’s input-output method [3], which is based on Gerchberg-Saxton algorithm, uses a negative feedback idea in the object domain operation and it is faster than the Gerchberg-Saxton method. While these classical methods which are based on error reduction algorithms work practically well, their convergence to the global optimum cannot be guaranteed, and they require prior information. The convergence of the methods based on the iterative alternating projections can be achieved for Gaussian measurements [4].

For the general case, the phase retrieval problem reduces to finding the signal $\mathbf{x} \in \mathbb{C}^n$, from m measurements expressed as $|\mathbf{Ax}| = \mathbf{b}$, where $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$. In this problem, the signal can be recovered up to a global phase factor such that if $\mathbf{x}$ is a solution to this problem, $c\mathbf{x}$ is also a solution for any c with $|c| = 1$. The unique recovery up to a global phase factor can be achieved with an oversampling ratio at least greater than 2, independent of the dimension [5]. The number of measurements required for unique recovery is $4n - 2$ generic measurements in the absence of noise [6]. The injectivity of the mapping over n dimensional complex signals is achieved by using a larger set of intensity measurements.

Interpreting the phase retrieval problem via convex optimization helps to better understand the phase retrieval algorithms based on alternating projections such as the "input-output" algorithm [7], and also to understand how they can be implemented with fewer measurements [8-10]. The non-convex constraints in the phase retrieval problem may be relaxed into a set of convex constraints by formulating the problem in a higher dimensional space. The authors of the [8] formulate the phase retrieval problem as recovering a rank-one matrix in which the quadratic measurements of the signal are interpreted as linear measurements of the rank-one matrix. The problem is then cast as a rank minimization problem with affine constraints. Since the rank minimization problem is an NP-hard problem, a trace-norm relaxation approach instead of minimizing the rank has been proposed, and the resulting algorithm, which is a convex program, is referred to as *Phase Lift*. In [9], the *Phase Lift* method is combined with multiple structured illuminations. Several diffraction patterns are collected in order to yield uniqueness. In [10], what the authors call *Phase Cut* is a semidefinite program (SDP relaxation) similar to the one in the max-cut problem. In this algorithm, the authors separate the phase and amplitude variables and optimize the phase variables. These convex optimization based methods show similar performances, and achieve exact recovery when the measurements are normally distributed, and the number of measurements is on the order of $n \log n$, where n is the dimension of the signal of interest. However, treating the phase retrieval problem as a matrix recovery problem increases the computational cost.

Reducing the number of measurements required for phase retrieval would be beneficial to reduce the cost caused by collecting additional measurements. When the signal of interest is sparse, employing compressed sensing techniques may help to reduce the required number of measurements. Compressed sensing [11, 12] is a technique for recovering a sparse signal efficiently from an underdetermined system of linear equations, and it finds the sparsest solution. When the signal $\mathbf{x}$ is a linear combination of K basis vectors, and it is sparse in some domain, it is referred to as a K-sparse signal. If the matrix containing the K basis vectors as columns is $\mathbf{\Psi}$ and the vector of the coefficients in this domain is $\mathbf{s}$, then $\mathbf{x} = \mathbf{\Psi s}$. Let $\mathbf{y} = \mathbf{\Phi x} = \mathbf{\Phi \Psi s} = \mathbf{\Theta s}$ denote the linear measurements of the signal $\mathbf{x}$ with length n, and let $\mathbf{\Phi}$ represent the measurements matrix. Compressed sensing techniques may reconstruct $\mathbf{x}$ with a small error using only m measurements, in which m is on the order of $O(K \log n/K)$ [12, 13]. In order to recover the sparse signal from $m < n$ measurements, the measurement matrix must satisfy some properties such as the restricted isometry property. Random Gaussian measurement matrices hold this property with a high probability, and capture the information in the structured signal [14, 15]. Given such a measurement matrix, exact recovery will be attained via l_1-norm minimization which is a convex optimization problem:

$$\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad s.t.\, \mathbf{y} = \mathbf{\Phi x}\,. \tag{2}$$

Since the number of measurements is less than the number of unknowns, there are infinitely many $\mathbf{s}'$ that satisfy $\mathbf{y} = \mathbf{\Theta s}'$. The sparse solution of this underdetermined linear system of equations lies in the $(n - m)$-dimensional translated null space of $\mathbf{\Theta}$, which is $H = \mathcal{N}(\boldsymbol{\Theta}) + \mathbf{s}$. The geometry of the l_1-norm minimization is given in Figure 1.2.

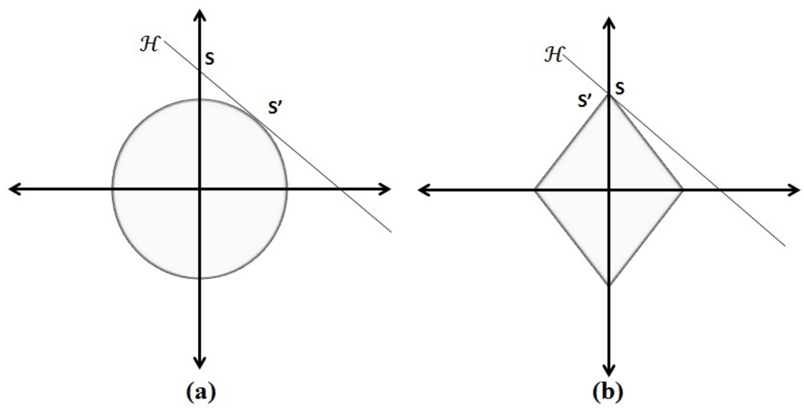


Figure 1.2. (a) Visualization of l_2-norm minimization. It finds a non-sparse point at the intersection of l_2-ball and H. **(b)** Visualization of l_1-norm minimization. It finds the sparse point with high probability at the intersection of l_1-ball and H.

Combining compressive sensing techniques and phase retrieval methods may enable the recovery of the unknown sparse signal from fewer magnitude measurements, so algorithms that exploit the signal structure might be possible. A convex formulation of the sparse phase retrieval problem is proposed in [16]. Similar to *Phase Lift* method, the authors of [16] lift up the problem to the space of $\boldsymbol{n} \times \boldsymbol{n}$ matrices, where $\mathbf{X} = \mathbf{x}\mathbf{x}^*$ is a rank-one matrix. Additionally, they introduce the l_1-norm regularization by adding a design parameter that is multiplied with $\|\mathbf{X}\|_1$, entry-wise l_1-norm for the matrix $\mathbf{X}$, to the objective function of the semidefinite programming (SDP) problem. Other approaches based on SDP for the sparse phase retrieval problem are used in [17-19]. In [17], the support of the signal is found prior to the signal reconstruction. Given the autocorrelation of the signal and the number of locations of the nonzero values of the signal, the authors of [17] aim to find the support of the signal by promoting the sparsity property. Following the support of the signal, a convex optimization based framework, which is actually the *Phase Lift* [8, 9] method, is employed to recover the signal. The procedure proposed in [17] finds the support of the signal from the support of its autocorrelation sequence if the sparsity of the signal is up to $O\left(\sqrt{n}\right)$.

In this research, we first present the mathematical formulation of the phase retrieval problem as a matrix recovery problem as in the *Phase Lift* [8] and *Phase Cut* [10] methods. The matrix recovery problem induced by the phase retrieval enables the usage of tools from convex optimization, which are stable in the presence of noise. For exact recovery, these methods require at least on the order of $n \log n$ measurements. The similarities and differences exhibited by these algorithms are described in mathematical terms. Also, these algorithms are compared in terms of computational complexity, performance and stability in the presence of noisy measurements at different signal-to-noise ratio (SNR) values.

When the number of measurements is smaller than the number of data samples necessary for exact reconstruction, some additional information is needed. This additional information pertains to the signal structure. With the aim of reducing the number of measurements, our research focuses on the exact signal recovery by imposing sparsity in the signal samples. To take advantage of the sparsity of the signal, an l_1-norm minimization approach was proposed in the field of compressed sensing (CS) [11, 12]. Therefore, we mainly concentrate on the usage of the *Phase Lift* and *Phase Cut* methods for the reconstruction of sparse signals. Combining these methods with the l_1- norm minimization problem in compressive sampling, the number of measurements can be further reduced for unique signal recovery.

While the recently proposed CPRL (*Compressive Phase Retrieval via Lifting*) method [16] which is motivated both by the l_1-norm minimization approach and *Phase Lift* technique, works for sparse signal recovery from fewer measurements, a similar approach called the *Phase Cut* technique fails. Therefore, in this thesis we propose an approach by modifying the *Phase Cut* method by adding constraints to make it converge to the true solution in order to recover sparse signals from fewer magnitude measurements, and we refer to this approach as modified Phase Cut. In this thesis, we conduct simulations to compare the modified Phase Cut and CPRL [16]. In addition, we also propose an approach which is inspired by a different version of the *Phase Lift* method in [20] and the l_1-norm minimization in order to enable the usage of the signal sparse structure for signal recovery from magnitude measurements. Finally, we conduct the performance analyses of these approaches in terms of reconstruction rate, complexity and robustness in the presence of noise, and perform a comparison of the proposed approaches with the CPRL method.

In Section 2, the general phase retrieval problem is expressed and the methods formulating this problem as a semidefinite program, which are *Phase Lift* [8] and *Phase Cut* [10] methods, are discussed. In Section 3, the proposed approaches based on *Phase Lift* and *Phase Cut* methods are defined for sparse signal recovery from fewer magnitude measurements. In Section 4.1 simulation results are given for signal recovery from magnitude information without assuming any constraint about the sparse structure of the signal. Finally, several results are presented in Section 4.2 that describe the performance of the proposed approaches in comparison with CPRL in terms of recovering sparse signals from fewer magnitude measurements.

2. PHASE RETRIEVAL METHODS

Phase retrieval methods focus on seeking a solution $\mathbf{x}$, a n-dimensional signal, given the magnitudes of linear measurements of $\mathbf{x}$, i.e., the magnitudes of inner products $|\langle \mathbf{x}, \mathbf{a}_i \rangle|$ for $i = 1, \ldots, m$. In short, this problem can be expressed as

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{such that} && \mathbf{b} = |\mathbf{A}\mathbf{x}|. \end{aligned} \tag{3}$$

Matrix $\mathbf{A}$ stands for the $m \times n$ measurement matrix and its $i-th$ row is represented by the vector $\mathbf{a}_i$. The problem of phase retrieval is generally studied in two set-ups. In the first one, the measurements are obtained through a Fourier transformation process. Therefore, the number of measurements is equal to n and there is some prior information about the unknown signal. Without any additional information or constraints, knowledge of only the Fourier magnitude is, in general, insufficient to uniquely determine the signal. This is due to the fact that the convolution of the true signal $\mathbf{x}$ with an all-pass signal will produce another signal with the same Fourier magnitude [21]. The additional information about the signal can be positivity, magnitude information about the signal, sparsity and so on. In the second setting, the phase retrieval problem is cast as a signal recovery problem from oversampled data, i.e., $m > n$, while there may or may not be any prior information about the signal.

Classical approaches for solving this problem are the error reduction algorithms based on alternating projections [2, 3]. For example, in phase retrieval problems, prior information in the signal domain may refer to the support of the signal $\mathbf{x}$ which might be contained in some set. These iterative methods alternate between projecting $\mathbf{Ax}_k$ onto the magnitude constraint, yielding $\mathbf{y}_k$ and then projecting $\mathbf{A}^\dagger\mathbf{y}_k$ onto a known support constraint, yielding $\mathbf{x}_{k+1}$. Therefore, these approaches aim to find a common point in the intersection of two constraint sets. For any two closed convex sets, the alternating projection method is guaranteed to find some points in the intersection of these sets. However, due to the non-convexity of the magnitude constraints set, convergence is not guaranteed and the method can be trapped in local minima.

A recent approach for solving this problem formulates the non-convex magnitude constraint as a system of linear matrix equations. Then, the phase retrieval problem is defined as finding a rank-one matrix satisfying the linear constraints. In this formulation, rank minimization is relaxed by trace minimization. Therefore, the resulting algorithm becomes a convex program. This approach is called "*Phase Lift* [8]". Another approach as a convex optimization framework is based on SDP relaxation and the resulting algorithm is referred to as "*Phase Cut* [10]". In Sections 2.3 and 2.4 these methods will be described in detail.

Before giving detailed explanations about these methods, the inherent ambiguity of phase retrieval problem is explained in Section 2.1.

2.1. Ambiguity in Phase Retrieval Problem for One-Dimensional Signals

The phase retrieval problem is considered as the signal recovery from the magnitude of Fourier transform. Before discussing about the uniqueness of the solution for the phase retrieval problem, it should be noted that the following transformations on the input signal which are $-\mathrm{x}, \mathrm{x}[n-m]$ and $x[-n]$ will not change the magnitude of the

Fourier transform. In the problem of recovery of the signal from the magnitude of its Fourier transform, a sign change, unknown shift in time and time reversal will be considered acceptable ambiguities for unique recovery.

Since the autocorrelation function and power spectrum are Fourier transform pairs, the phase retrieval problem can be interpreted as the signal recovery from the autocorrelation sequence. The autocorrelation sequence of a real-valued signal can be expressed as the convolution of the signal itself with its time reversed version:

$$r_{xx}(\tau) = x(\tau) * x(-\tau). \tag{4}$$

The Fourier transform of the autocorrelation function is then given by $\mathsf{X}(\omega)\mathsf{X}(-\omega)$, where $\mathsf{X}(\omega)$ is the Fourier transform of the signal x. If the signal of interest is real-valued signal, then the Fourier transform of the autocorrelation function is given by $\mathsf{X}(\omega)\mathsf{X}^*(\omega)$. Therefore, as the Wiener-Khinchin theorem states, the power spectrum $|\mathsf{X}(\omega)|^2 = \mathsf{X}(\omega)\mathsf{X}^*(\omega)$ is simply the Fourier transform of the autocorrelation function.

Since we are dealing with finite length sequences, the finite autocorrelation sequence of the finite length sequence **x** for length n is given by

$$a_k = \sum_{i=0}^{n-k} x_i x_{i+k}. \tag{5}$$

The z-transform of the autocorrelation sequence **a** is given by $\mathsf{A}(z) = \mathsf{X}(z)\mathsf{X}\left(z^{-1}\right)$, where $\mathsf{X}(z)$ is the z-transform of the signal **x**. If **x** is the real-valued signal, then the polynomial $\mathsf{X}(z)$ has real-valued coefficients. Therefore, the roots of $\mathsf{X}(z)$ occur in conjugate pairs. In addition, because $\mathsf{A}(z) = \mathsf{A}\left(z^{-1}\right)$, the roots of $\mathsf{A}(z)$ appear in quadruples of the form $(z_0, z_0^*, z_0^{-1}, z_0^{-*})$. The root pairs (z_0, z_0^*) and (z_0^{-1}, z_0^{-*}) can be assigned to $\mathrm{X}(z)$ or $\mathrm{X}(z^{-1})$, or vice versa. These different assignments will lead to different polynomials $\mathrm{X}(z)$.

The ambiguities due to the different assignments of the roots can be avoided by restricting the set of admissible solutions to the minimum phase sequences which will enable the sequence to be uniquely defined by the magnitude of its Fourier transform [22]. Another example is given by a real-valued finite length signal that has an irreducible z-transform; such a signal is uniquely defined by the magnitude of its Fourier transform. However, such constraints would not cover a very large class of one-dimensional signals.

In Figure 2.1(top), a real-valued sparse signal $\mathbf{x_1}$ having a length of 16 samples is given. Its z-transform has three real-valued roots and twelve complex-valued roots. The non-sparse signal $\mathbf{x_2}$ is obtained by reflecting the two complex-valued roots of the z-transform of $\mathbf{x_1}$ about the unit circle. Both signals $\mathbf{x_1}$ and $\mathbf{x_2}$ present the same magnitude for the Fourier Transform, while these two signals are different in the time domain. A different assignment of the roots will give rise to a different signal in the time domain while both of them present the same autocorrelation sequence or power spectrum density. Thus, in order to guarantee a unique solution, it is essential to impose some constraints to solve the ambiguity caused by the changes in the locations of zeros and poles.

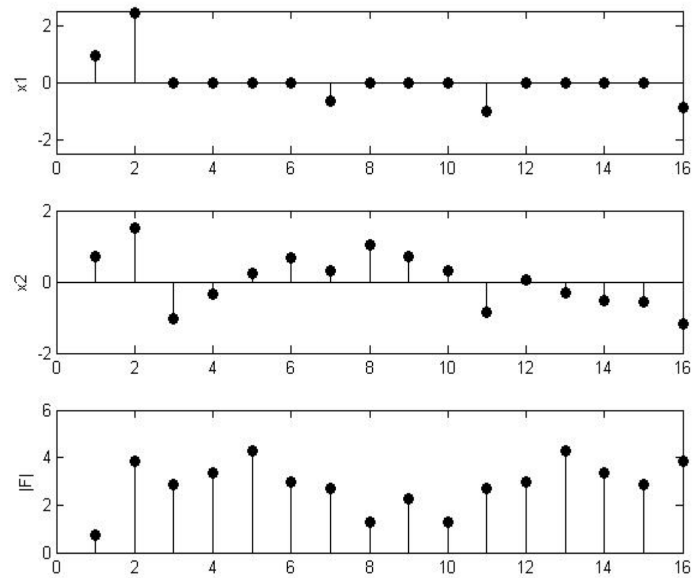


Figure 2.1. (Top) A sparse signal contains 5 nonzero values. (Middle) A different signal having the same magnitude of Fourier transform as the top one. (Bottom) The Fourier transform of both signals.

2.2. General Phase Retrieval Problem via Convex Optimization

In the general phase retrieval problem, which we implemented in our simulations, there are m measurements which are given in the form of the inner product magnitude $|\langle \mathbf{x}, \mathbf{a}_i \rangle|$ or $|\langle \mathbf{x}, \mathbf{a}_i \rangle|^2$ for $i = 1, \dots, m$. The sensing vectors $\mathbf{a}_i$ are independent and identically distributed (IID) standard normal random vectors. The magnitude measurement data are invariant under a global phase factor. Therefore, both signals $\mathbf{x}$ and $e^{i\theta}\mathbf{x}$, where $\theta \in [0,2\pi]$, are accepted as solutions of the problem.

Let $A\colon \mathbb{C}^{n\times n} \to \mathbb{R}^m$ be the linear transformation defined via the mapping $A(\mathbf{X}) = \{\mathbf{a}_i^*\mathbf{X}\mathbf{a}_i\}_{1\le i\le \mathrm{m}}$. Quadratic measurements in the form of $|\langle \mathbf{x}, \mathbf{a}_i \rangle|^2$ can be interpreted as linear measurements of rank-one matrix $\mathbf{X} = \mathbf{x}\mathbf{x}^*$, in which the measurements $|\langle \mathbf{x}, \mathbf{a}_i \rangle|^2$

are given in the form of $A(\mathbf{xx}^*)$. Therefore, the phase retrieval problem is equivalent to finding a rank-one matrix $\mathbf{X}$ from linear measurements of it. The non-convex constraints of the phase retrieval problem are now linear measurements of the positive semidefinite matrix $\mathbf{X}$; however, the rank constraint is non-convex and the problem of rank minimization is in general NP-hard. Therefore, rank minimization is computationally intractable.

The well-known heuristic that is used to solve the general rank minimization problem is the nuclear norm. The nuclear norm of a matrix $\mathbf{X}$, represented as $\|\mathbf{X}\|_*$, is the sum of the singular values of the matrix $\mathbf{X}$, i.e., $\|\mathbf{X}\|_* \coloneqq \sum_{i=1}^{r} \sigma_i$. The nuclear norm heuristic is employed in order to solve the rank minimization problems because the convex envelope (convex hull) of $rank(\mathbf{X})$ on the set $S = \{\mathbf{X} \in \mathbb{R}^{m\times n} \mid \|\mathbf{X}\| \leq 1\}$ is the nuclear norm $\|\mathbf{X}\|_*$ [23, 24]. Therefore, the nuclear norm can be interpreted as the convex approximation of the rank. The convex envelope of a function $f(x): C \to \mathbb{R}$, where C is a convex set, is defined as the largest convex function g such that $g(x) \leq f(x)$ for all $x \in C$ [25].

When the matrix $\mathbf{X}$ is a positive semidefinite matrix, $\mathbf{X} \succcurlyeq 0$ and $\mathbf{X} \in \mathbb{R}^{n\times n}$, minimization of the nuclear norm is equivalent to the trace minimization approach to solve the rank minimization problem. The trace of a matrix is given by the sum of its eigenvalues $\mathrm{Tr}(\mathbf{X}) = \sum_{i=1}^{n} \lambda_i(\mathbf{X})$. Since a positive semidefinite matrix has non-negative eigenvalues, the l_1-norm of the vector containing these eigenvalues, $\|\lambda(\mathbf{X})\|_{l1} = \sum_{i=1}^{n} |\lambda_i(\mathbf{X})|$ will be equal to the trace of $\mathbf{X}$. From compressive sensing, it is known that the l_1-norm minimization of a vector results in a sparse vector. Therefore, minimizing the l_1-norm of the vector of eigenvalues will produce many zero-valued eigenvalues which yields a low-rank matrix. This procedure intuitively explains why trace minimization heuristic is used instead of rank minimization of a positive semidefinite matrix. In the phase retrieval problem, since the optimization variable $\mathbf{X} = \mathbf{x}\mathbf{x}^*$ is a positive semidefinite matrix, the trace minimization heuristic is employed to solve this problem. Trace minimization from convex constraints in the phase retrieval problem is a convex program, which can be solved efficiently.

In the following two sections, the methods applied to solve the phase retrieval problem via a convex optimization framework will be described.

2.2.1. *Phase Lift Method*

Given m quadratic intensity measurements of the form $|\langle \mathbf{x}, \mathbf{a}_i \rangle|^2, i = 1, \dots, m$, the goal is to recover the unknown signal which can be a complex-valued discrete-time signal, $\mathbf{x} \in \mathbb{C}^n$. In many applications both phase and magnitude information are necessary, and such information comes through knowledge of the inner product $\langle \mathbf{x}, \mathbf{a}_i \rangle$. However, in many applications, only the magnitude information is recorded, and the phase information is lost. The *Phase Lift* method was recently proposed as a semidefinite programming framework for the phase retrieval problem [8]. The signal $\mathbf{x}$ can be recovered, up to a global phase factor, exactly from the magnitude square measurements if the measurement vectors $\mathbf{a}_i$ are independent and uniformly sampled on unit sphere.

Let $\mathbf{a}_i \in \mathbb{C}^n$ be independent and identically distributed (IID) standard normal random vectors, and define b_i, $i = 1, \dots, m$, as follows:

$$b_i = |\langle \mathbf{x}, \mathbf{a}_i \rangle|^2, i = 1, \dots, m. \tag{6}$$

When the number of measurements is at least on the order of $n \log n$, phase retrieval can be accomplished with high probability [8]. The problem of phase recovery from quadratic constraints is lifted up into the problem of recovering a rank-one matrix from affine constraints by casting it as a trace-minimization problem via semidefinite programming. The quadratic measurements can be expressed as

$$b_i = \left(\mathbf{a}_i^{\mathsf{H}}\mathbf{x}\right)^{\mathsf{H}}\left(\mathbf{a}_i^{\mathsf{H}}\mathbf{x}\right) = \mathbf{x}^{\mathsf{H}}\mathbf{a}_i\mathbf{a}_i^{\mathsf{H}}\mathbf{x} = \mathrm{Tr}(\mathbf{\Theta}_i\mathbf{X}), i = 1, \dots, m. \quad (7)$$

The matrices $\mathbf{\Theta}_i$ and $\mathbf{X}$ are given by $\mathbf{\Theta}_i = \mathbf{a}_i\mathbf{a}_i^{\mathsf{H}}$ and $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathsf{H}}$, respectively. Therefore, the phase retrieval problem is expressed as

$$\begin{array}{ll} \text{minimize} & \mathrm{rank}(\mathbf{X}) \\ \text{subject to} & b_i = \mathrm{Tr}(\mathbf{\Theta}_i\mathbf{X}), \qquad i = 1, \dots, m \\ & \mathbf{X} \succcurlyeq 0. \end{array} \quad (8)$$

Since the rank minimization problem is NP-hard, instead of minimizing it, a trace-norm relaxation is adopted as shown below:

$$\begin{array}{ll} \text{minimize} & \mathrm{Tr}(\mathbf{X}) \\ \text{subject to} & b_i = \mathrm{Tr}(\mathbf{\Theta}_i\mathbf{X}), \qquad i = 1, \dots, m \\ & \mathbf{X} \succcurlyeq 0. \end{array} \quad (9)$$

When the number of measurements is on the order of $n \log n$, the trace minimization solution is exact, and the matrix $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathsf{H}}$ is the unique solution to the above optimization problem. This holds with probability at least $1 - 3e^{-\gamma\frac{m}{n}}$, where γ is a positive absolute constant, for the models in which the $\mathbf{a}_i$'s are independently and identically distributed [8].

In [20], exact signal recovery from the quadratic measurements is accomplished from the independent and identically distributed standard normal random measurement vectors $\mathbf{a}_i$ when the number of measurements is at least $m \geq c_0 n$, where c_0 is a sufficiently large constant. The phase retrieval problem is solved by finding the positive semidefinite matrix $\mathbf{X}$ that best fits the observed measurements in the l_1-norm sense [20] as follows:

$$\begin{array}{ll} \text{minimize} & \sum_{1\leq i\leq m} |\mathrm{Tr}(\mathbf{\Theta}_i \mathbf{X}) - b_i| \\ \text{subject to} & \mathbf{X} \succcurlyeq 0. \end{array} \tag{10}$$

The approach proposed by the authors of [20] improves the *Phase Lift* method in the sense that the exact signal recovery holds with probability at least $1 - e^{-\gamma m}$.

In Section 2.4.1, we will propose a method using the formulation in (10) in order to recover sparse signals.

2.2.2. *Phase Cut Method*

Another method developed for phase recovery only from the magnitude of linear measurements is based on formulating the problem as a quadratic optimization problem over unit complex phase vectors [10]. Similar to *Phase Lift*, complex measurements of a signal $\mathbf{x} \in \mathbb{C}^n$ are obtained from a linear injective operator, and the measurements are random allowing the uniqueness to be guaranteed.

The problem of recovering the signal $\mathbf{x} \in \mathbb{C}^n$ from the amplitude vector $\mathbf{b} = |\mathbf{Ax}|, \mathbf{b} \in \mathbb{R}^m$, i.e., given m measurements is solved by separating the amplitude and phase variables, and only the phase variables are optimized. The equation $|\mathbf{Ax}| = \mathbf{b}$ is then expressed as $\mathbf{Ax} = diag(\mathbf{b})\mathbf{u}$, where $\mathbf{u} \in \mathbb{C}^m$ satisfies the property $|u_i| = 1$ and is a phase vector. Matrix $diag(\mathbf{b})$ is an $m \times m$ diagonal matrix and all its main diagonal entries are the elements of the vector $\mathbf{b}$. The phase retrieval problem is expressed as

$$\min_{\mathbf{u}\in \mathbb{C}^m, |u_i|=1, \mathbf{x}\in\mathbb{C}^n} \|\mathbf{A}\mathbf{x} - diag(\mathbf{b})\mathbf{u}\|_2^2 \,. \tag{11}$$

The least squares solution of the objective function in (11) is given by $\mathbf{x}_{\mathrm{ls}} = \mathbf{A}^{\dagger}\, diag(\mathbf{b})\mathbf{u}$. The matrix $\mathbf{A}^{\dagger}$ denotes the pseudo-inverse of the measurement matrix $\mathbf{A}$. Plugging $\mathbf{x}_{\mathrm{ls}}$ into equation (11), we express the objective function in the following form:

$$\begin{aligned}
&\left\|\mathbf{A}\mathbf{A}^{\dagger}\, diag(\mathbf{b})\mathbf{u} - diag(\mathbf{b})\mathbf{u}\right\|_2^2 \\
&= \left(\mathbf{A}\mathbf{A}^{\dagger}\, diag(\mathbf{b})\mathbf{u} - diag(\mathbf{b})\mathbf{u}\right)^{\mathrm{H}}\left(\mathbf{A}\mathbf{A}^{\dagger}\, diag(\mathbf{b})\mathbf{u} - diag(\mathbf{b})\mathbf{u}\right) \\
&= \mathbf{u}^{\mathrm{H}} diag(\mathbf{b})\left(\mathbf{A}\mathbf{A}^{\dagger} - \mathbf{I}\right)^{H}\left(\mathbf{A}\mathbf{A}^{\dagger} - \mathbf{I}\right) diag(\mathbf{b})\mathbf{u} = \mathbf{u}^{\mathrm{H}} diag(\mathbf{b})\left(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}\right) diag(\mathbf{b})\mathbf{u}\,.
\end{aligned} \tag{12}$$

The last statement in equation (12) is obtained because of the orthogonal projector onto the left null space of matrix $\mathbf{A}$, $\mathscr{P}_{\mathscr{N}(\mathbf{A}^{\mathrm{H}})} = \mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}$, which is also the projector onto the orthogonal complement of image of $\mathbf{A}$. Finally, the phase retrieval problem becomes a quadratic optimization problem when it is expressed as follows:

$$\begin{aligned}
&\text{minimize} && \mathbf{u}^{\mathrm{H}}\mathbf{M}\mathbf{u} \\
&\text{subject to} && |u_i| = 1,\, i = 1, \dots, m,
\end{aligned} \tag{13}$$

where the matrix $\mathbf{M}$ is given by $\mathbf{M} = diag(\mathbf{b})\left(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}\right) diag(\mathbf{b})$. Using the equation $\mathbf{u}^{\mathrm{H}}\mathbf{M}\mathbf{u} = \mathrm{Tr}(\mathbf{M}\mathbf{U})$, this problem becomes a semidefinite program by adopting a convex relaxation which can be achieved by eliminating the rank constraint $\mathrm{rank}(\mathbf{U}) = 1$. The phase retrieval problem expressed as a semidefinite program is formulated below, and it is referred to as the *Phase Cut* method [10]:

$$\begin{aligned}
&\text{minimize} && \mathrm{Tr}(\mathbf{M}\mathbf{U}) \\
&\text{subject to} && diag(\mathbf{U}) = \mathbf{1},\, \mathbf{U} \succcurlyeq 0.
\end{aligned} \tag{14}$$

Matrix $\mathbf{U}$ is a Hermitian matrix, $\mathbf{U} \in \mathbf{H}_m$. When the solution of this optimization problem has rank one, then $\mathbf{U} = \mathbf{u}\mathbf{u}^{\mathrm{H}}$ is the optimal solution. If the resulting matrix $\mathbf{U}$

has a larger rank, then the normalized eigenvector of the matrix $\mathbf{U}$ corresponding to the largest eigenvalue is used as solution.

When the signal of interest is real-valued, the phase retrieval problem can be expressed as

$$\min_{u_{*i}^2+u_{*(i+m)}^2=1, i=1,\ldots,m} \left\| \left(\mathbf{A}_* \mathbf{A}_*^{\dagger} \mathbf{B}_* - \mathbf{B}_* \right) \mathbf{u}_* \right\|_2^2. \tag{15}$$

The matrices $\mathbf{A}_* \in \mathbb{R}^{2m\times n}$ and $\mathbf{B}_* \in \mathbb{R}^{2m\times 2m}$, and the vector $\mathbf{u}_* \in \mathbb{R}^{2m}$ are expressed as $\mathbf{A}_* = \begin{pmatrix} \mathrm{Re}(\mathbf{A}) \\ \mathrm{Im}(\mathbf{A}) \end{pmatrix}$, $\mathbf{B}_* = diag\begin{pmatrix} \mathbf{b} \\ \mathbf{b} \end{pmatrix}$ and $\mathbf{u}_* = \begin{pmatrix} \mathrm{Re}(\mathbf{u}) \\ \mathrm{Im}(\mathbf{u}) \end{pmatrix}$ [10]. The minimization problem is then expressed similarly as shown below:

$$\begin{array}{ll} \text{minimize} & \mathbf{u}_*^{\mathsf{T}} \mathbf{B}_*^{\mathsf{T}} (\mathbf{I} - \mathbf{A}_* \mathbf{A}_*{}^{\dagger}) \mathbf{B}_* \mathbf{u}_* \\ \text{subject to} & u_{*i}^2 + u_{*(i+m)}^2 = 1, i = 1, \ldots, m, \end{array} \tag{16}$$

where u_{*i} denotes the $i-th$ element of the phase vector $\mathbf{u}_*$. The objective function of the problem in (16) is equal to $\mathrm{Tr}(\mathbf{M}_* \mathbf{U}_*)$ in which the matrix $\mathbf{U}_* \in \mathbf{S}_{2m}$ is a rank-one matrix. The matrices $\mathbf{U}_*$ and $\mathbf{M}_*$ are defined as $\mathbf{U}_* = \mathbf{u}_* \mathbf{u}_*^{\mathsf{T}}$ and $\mathbf{M}_* = \mathbf{B}_*^{\mathsf{T}} (\mathbf{I} - \mathbf{A}_* \mathbf{A}_*{}^{\dagger}) \mathbf{B}_*$, respectively. As a result, the semidefinite program of phase retrieval problem is expressed as

$$\begin{array}{ll} \text{minimize} & \mathrm{Tr}(\mathbf{M}_* \mathbf{U}_*) \\ \text{subject to} & \mathbf{U}_{*(i,i)} + \mathbf{U}_{*(m+i,m+i)} = 1, \ \mathbf{U}_* \succcurlyeq 0, \end{array} \tag{17}$$

where $\mathbf{U}_{*(i,j)}$ denotes the entry of matrix $\mathbf{U}_*$ corresponding to the $i-th$ row and $j-th$ column. Finally, after obtaining the optimization variable $\mathbf{U}_*$, the reconstructed signal is then computed as $\mathbf{x}_{\mathrm{rec}} = \mathbf{A}_*{}^{\dagger} \mathbf{B}_* \mathbf{u}_*$.

2.3. Comparison of Phase Lift and Phase Cut Methods

In order to compare these two methods, we will use a similar procedure to that employed in [26]. Two cases will be addressed. In the first case, the number of measurements (m) is greater than the number of unknowns (n), while in the second case the number of measurements is less than the number of unknowns, which is an underdetermined system of equations.

Briefly, the *Phase Lift* technique is expressed as

$$\begin{array}{ll} \text{minimize} & \mathrm{Tr}(\mathbf{X}) \\ \text{subject to} & diag(\mathbf{AXA}^*) = \mathbf{b} \odot \mathbf{b} \\ & \mathbf{X} \succcurlyeq 0, \end{array} \tag{18}$$

where the measurement vector $\mathbf{b}$ is given by $\mathbf{b} = |\mathbf{Ax}|$ and $\mathbf{b} \in \mathbb{R}^m$. The term $\mathbf{b} \odot \mathbf{b}$ denotes the Hadamard product of the vector $\mathbf{b}$ with itself [27]. The affine constraints in the form of $b_i^2 = \mathrm{Tr}(\mathbf{\Theta}_i \mathbf{X})$, for $i = 1, . . , m$, correspond to the entries of the vector $\mathbf{b} \odot \mathbf{b} = diag(\mathbf{AXA}^*)$.

The Phase Cut method for the problem in (18) is defined by the following semidefinite program:

$$\begin{array}{ll} \text{minimize} & \mathrm{Tr}(\mathbf{MU}) \\ \text{subject to} & diag(\mathbf{U}) = \mathbf{1}, \mathbf{U} \succcurlyeq 0, \end{array} \tag{19}$$

where the matrix $\mathbf{M}$ is given by $\mathbf{M} = diag(\mathbf{b})\left(\mathbf{I} - \mathbf{AA}^{\dagger}\right)diag(\mathbf{b})$.

When there are $m \geq n$ measurements and the matrix $\mathbf{A}$ has full column rank, $\mathbf{A}^{\dagger}$, the pseudo-inverse of the measurement matrix $\mathbf{A}$, is a left inverse and the product $\mathbf{A}^{\dagger}\mathbf{A}$ is the identity matrix. Let us assume that the matrix $\hat{\mathbf{X}}$ is the optimal solution of the semidefinite program in equation (18). Therefore, the objective function in (18) obtained by $\hat{\mathbf{X}}$ is less than any other value obtained by other positive semidefinite matrix in the feasible region of the problem, $\mathrm{Tr}(\hat{\mathbf{X}}) \leq \mathrm{Tr}(\mathbf{X}) = \mathrm{Tr}(\mathbf{x}\mathbf{x}^*)$. The optimal solution $\hat{\mathbf{X}} = \hat{\mathbf{x}}\hat{\mathbf{x}}^*$ can be expressed as follows because of $\mathbf{A}\hat{\mathbf{x}} = diag(\mathbf{b})\hat{\mathbf{u}}$ for some $\hat{\mathbf{u}}$:

$$\hat{\mathbf{X}} = \mathbf{A}^{\dagger} diag(\mathbf{b}) \hat{\mathbf{U}} diag(\mathbf{b}) (\mathbf{A}^{\dagger})^{\mathsf{H}}. \tag{20}$$

By using the fact $\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{I}$, matrix $\hat{\mathbf{U}}$ will take the expression in (21) for the equality in (20) to hold

$$\hat{\mathbf{U}} = diag(\mathbf{b})^{-1} \mathbf{A}\hat{\mathbf{X}}\mathbf{A}^{\mathsf{H}} diag(\mathbf{b})^{-1}. \tag{21}$$

Therefore, it is observed that $diag(\hat{\mathbf{U}}) = \mathbf{1}$ (i.e., an all ones vector) and $\hat{\mathbf{U}} \succcurlyeq 0$. The matrix $\hat{\mathbf{U}}$ is a positive semidefinite matrix because we already know $\hat{\mathbf{X}} \succcurlyeq 0$. With this knowledge and defining the vector $\mathbf{t} = \mathbf{A}^{\mathsf{H}} diag(\mathbf{b})^{-1}\mathbf{w}$ for some vector $\mathbf{w}$, positive definiteness of the matrix $\hat{\mathbf{U}}$ will be clear from $\mathbf{w}^{\mathsf{H}}\hat{\mathbf{U}}\mathbf{w} = \mathbf{t}^{\mathsf{H}}\hat{\mathbf{X}}\mathbf{t} \geq 0$. Therefore, the matrix $\hat{\mathbf{U}}$ ensures the constraints of the *Phase Cut* problem. The remaining step is to show that $\hat{\mathbf{U}}$ gives the minimum solution of the objective function among the matrices of the feasible set. In this regard, the objective function is expressed in terms of $\hat{\mathbf{U}}$ as shown below:

$$\begin{aligned} \mathrm{Tr}(\mathbf{M}\hat{\mathbf{U}}) &= \mathrm{Tr}\left(diag(\mathbf{b})(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}) diag(\mathbf{b}) diag(\mathbf{b})^{-1} \mathbf{A}\hat{\mathbf{X}}\mathbf{A}^{\mathsf{H}} diag(\mathbf{b})^{-1}\right) \\ &= \mathrm{Tr}\left(diag(\mathbf{b})(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}) \mathbf{A}\hat{\mathbf{X}}\mathbf{A}^{\mathsf{H}} diag(\mathbf{b})^{-1}\right) = \mathrm{Tr}(\mathbf{0}) = 0. \end{aligned} \tag{22}$$

Therefore, the matrix $\hat{\mathbf{U}} = \hat{\mathbf{u}}\hat{\mathbf{u}}^{\mathsf{H}}$ is the optimal value of the *Phase Cut* problem.

The *Phase Cut* algorithm developed by [10] does not achieve exact recovery when the signal of interest is sparse, and the number of measurements is less than the number of unknowns. When the system is underdetermined ($m < n$), there are infinitely many solutions $\mathbf{x}$ such that $\mathbf{Ax} = diag(\mathbf{b})\mathbf{u}$ for some given phase vector $\mathbf{u}$.

When the measurement matrix $\mathbf{A}$ has full row rank $\mathrm{r} = m < n$, for each $diag(\mathbf{b})\mathbf{u} \in \mathbb{R}^m$ there is a solution set of all solutions $\{\mathbf{x} \mid \mathbf{Ax} = diag(\mathbf{b})\mathbf{u}\} = \{\mathbf{x}_\mathrm{p} + \mathbf{z} \mid \mathbf{z} \in \mathcal{N}(\mathbf{A})\}$, where $\mathbf{x}_\mathrm{p}$ is any particular solution. For the least squares solution of the problem $\mathbf{x}_\mathrm{ls} = \mathbf{A}^\dagger\, diag(\mathbf{b})\mathbf{u}$, we obtain the following relations:

$$\begin{aligned}
&\mathbf{A}(\mathbf{x} - \mathbf{x}_\mathrm{ls}) = 0\\
&(\mathbf{x} - \mathbf{x}_\mathrm{ls})^\mathrm{H}\,\mathbf{A}^\mathrm{H} = 0\\
&(\mathbf{x} - \mathbf{x}_\mathrm{ls})^\mathrm{H}\,\mathbf{A}^\mathrm{H}\left(\mathbf{AA}^\mathrm{T}\right)^{-1} diag(\mathbf{b})\mathbf{u} = 0\\
&(\mathbf{x} - \mathbf{x}_\mathrm{ls})^\mathrm{H}\mathbf{A}^\dagger\, diag(\mathbf{b})\mathbf{u} = 0\\
&(\mathbf{x} - \mathbf{x}_\mathrm{ls})^\mathrm{H}\mathbf{x}_\mathrm{ls} = 0.
\end{aligned} \tag{23}$$

From equation (23), it is concluded that $(\mathbf{x} - \mathbf{x}_\mathrm{ls})$ is orthogonal to $\mathbf{x}_\mathrm{ls}$, $(\mathbf{x} - \mathbf{x}_\mathrm{ls}) \perp \mathbf{x}_\mathrm{ls}$, and it leads to $\|\mathbf{x}\|^2 = \|\mathbf{x}_\mathrm{ls}\|^2 + \|\mathbf{x} - \mathbf{x}_\mathrm{ls}\|^2 > \|\mathbf{x}_\mathrm{ls}\|^2$. As a result, the least-squares solution, $\mathbf{x}_\mathrm{ls} = \mathbf{A}^\dagger\, diag(\mathbf{b})\mathbf{u}$ is the minimum $\boldsymbol{l}_2$-norm solution.

For the phase vector $\mathbf{u}$, the rank-one positive semidefinite matrix $\mathbf{X}$ is expressed as $\mathbf{X} = \mathbf{A}^\dagger\, diag(\mathbf{b})\mathbf{U}diag(\mathbf{b})\left(\mathbf{A}^\dagger\right)^\mathrm{H}$ by using the solution $\mathbf{x} = \mathbf{A}^\dagger\, diag(\mathbf{b})\mathbf{u}$. Therefore, the *Phase Cut* method can be expressed as in (24) for sparse signal recovery [10]:

$$\begin{aligned}
\text{minimize} \quad & \mathrm{Tr}(\mathbf{MU}) + \left\|\mathbf{A}^\dagger\, diag(\mathbf{b})\mathbf{U}diag(\mathbf{b})\left(\mathbf{A}^\dagger\right)^\mathrm{H}\right\|_1\\
\text{subject to} \quad & diag(\mathbf{U}) = \mathbf{1},\ \mathbf{U} \succcurlyeq 0.
\end{aligned} \tag{24}$$

However, in the underdetermined system of equations, the matrix $\mathbf{M}$ given in the *Phase Cut* problem in (24) is equal to zero, $\mathbf{M} = diag(\mathbf{b})\left(\mathbf{I} - \mathbf{AA}^\dagger\right)diag(\mathbf{b}) = 0$, since $\mathbf{AA}^\dagger = \mathbf{I}$. In this case the first term of the objective function in this problem, $\mathrm{Tr}(\mathbf{MU})$

(where $\mathbf{U} = \mathbf{u}\mathbf{u}^{H}$) is constant and equal to 0. Therefore, the *Phase Cut* problem is reduced to the problem in (25) by exploiting the sparsity information as shown below:

$$\begin{aligned} &\text{minimize} && \left\| \mathbf{A}^{\dagger}\, diag(\mathbf{b}) \mathbf{U} diag(\mathbf{b}) \left(\mathbf{A}^{\dagger}\right)^{H} \right\|_1 \\ &\text{subject to} && diag(\mathbf{U}) = \mathbf{1}, \mathbf{U} \succcurlyeq 0. \end{aligned} \tag{25}$$

Any solution $\mathbf{U} = \mathbf{u}\mathbf{u}^{H}$ of the problem in (25) leads to the recovered signal $\mathbf{x} = \mathbf{A}^{\dagger}\, diag(\mathbf{b})\mathbf{u}$ which may or may not be the same as the true unknown sparse signal. In other words, the *Phase Cut* algorithm chooses the minimum $\boldsymbol{l}_1$-norm solution among the least squares solutions [26].

3. SPARSE SIGNAL RECOVERY

In the signal recovery problem when a priori knowledge exists about the sparsity of the signal, the recovery is achieved from a system of underdetermined linear equations by employing l_1-norm minimization, a convex optimization problem, which is a common technique employed in compressed sensing. The sparsity constraint does not need to be in the signal domain. The signal can be sparse in other bases. In compressed sensing applications, the system of interest in general consists of linear measurements of some unknown signal. However, the phase retrieval problem is nonlinear because of the magnitude measurements.

The semidefinite programming based methods, which are mentioned in Section 2, do not assume any structure about the signal. These methods, *Phase Lift* and *Phase Cut*, require the measurement vectors to be random and sufficiently large in order to uniquely determine the signal. Sparsity can be used to restrict the number of solutions of the phase retrieval problem and it can also be used as a feature to reconstruct the signals from fewer measurements. Thus, by using the sparsity constraint of the signal, methods could be developed to converge to the exact solution from an underdetermined system of magnitude measurements.

In the following subsections, the extended versions of the semidefinite programming based phase retrieval methods for sparse signal recovery from the quadratic measurements are described. First, in Section 3.1, a recent method which is called compressive phase retrieval via lifting (CPRL) [16] is discussed. In the following

Section 3.1.1, we propose the improved Phase Lift approach by combining the phase retrieval problem with l_1-norm minimization to impose the sparse structure of the signal. In Section 3.2, the modified Phase Cut approach is proposed to recover sparse signals via the *Phase Cut* based method.

3.1. Phase Lift Method for Sparse Signal Recovery

In [16], compressive sensing is applied to the problem of reconstructing a signal only from the magnitude information, and the compressive phase retrieval problem is expressed as a convex optimization problem. The problem is considered as a signal recovery from magnitude information $b_i = |\langle \mathbf{x}, \mathbf{a}_i \rangle|^2, i = 1, \dots, m$ as in [8]. Additionally, it is assumed that the signal of interest is sparse. Similar to the *Phase Lift* method, the authors of [16] use a trace-norm relaxation instead of rank minimization, and exploit the l_1-norm minimization criterion to cope with sparsity. The formulation in (26) is then referred as compressive phase retrieval via lifting (*CPRL* [16]):

$$\begin{array}{ll} \text{minimize} & \mathrm{Tr}(\mathbf{X}) + \lambda \|\mathbf{X}\|_1 \\ \text{subject to} & b_i = \mathrm{Tr}(\mathbf{\Theta}_i \mathbf{X}) \ i = 1, \dots, m \\ & \mathbf{X} \succcurlyeq 0, \end{array} \tag{26}$$

where notation $\|\mathbf{X}\|_1$ for matrix $\mathbf{X}$ denotes the entry-wise l_1-norm. The matrices $\mathbf{\Theta}_i$ and $\mathbf{X}$ are given by $\mathbf{\Theta}_i = \mathbf{a}_i \mathbf{a}_i^{\mathrm{H}}$ and $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{H}}$, respectively, and $\lambda > 0$ is a design parameter, forcing the signal to be sparse. Ultimately, the problem is solved by choosing the estimated signal as the normalized leading eigenvector of $\mathbf{X}$.

In compressed sensing, the signal is reconstructed from underdetermined linear measurements exactly provided that the signal is sufficiently sparse and the linear operator obeys the restricted isometry property (RIP) [28]. Intuitively, the restricted isometry property claims that the matrix corresponding to the linear operator preserves the distance between two vectors that present the same sparsity.

For a linear operator $A(\mathbf{X})$ and all $\mathbf{X} \neq 0$ such that $\|\mathbf{X}\|_0 \leq k$, the isometry constant is defined as the smallest number ϵ satisfying

$$1 - \epsilon \leq \frac{\|A(\mathbf{X})\|_2^2}{\|\mathbf{X}\|_2^2} \leq 1 + \epsilon. \tag{27}$$

For the linear operator used in the phase retrieval problem, the authors of [8] claim that the restricted isometry property in l_2-norm is not valid since $\|A(\mathbf{X})\|_2^2$ involves fourth order moments of Gaussian variables. Instead of RIP for the l_2-norm, RIP-1 is proposed as follows [16]. A linear operator $A(\mathbf{X})$ is $(\epsilon, 2k)$-RIP-1 if $1 - \epsilon \leq \frac{\|A(\mathbf{X})\|_1}{\|\mathbf{X}\|_1} \leq 1 + \epsilon$ for all matrices $\mathbf{X} \neq 0$ such that $\|\mathbf{X}\|_0 \leq k$. The solution of compressive phase retrieval via lifting algorithm $\widetilde{\mathbf{X}}$ gives the sparsest solution $\tilde{\mathbf{x}}$ if it has rank-one and $(\epsilon, 2\|\widetilde{\mathbf{X}}\|_0)$-RIP-1 with $\epsilon < 1$ [16]. However, the restricted isometry property and the given theoretical bounds may be hard to check for a matrix. Therefore, random Gaussian matrices, which are known to satisfy the RIP property, are used in our simulations.

3.1.1. *Improved Phase Lift Approach for Sparse Signal Recovery*

In [20], the *Phase Lift* method is improved in terms of the number of measurements necessary for exact signal recovery such that at least $m \geq c_0 n$ measurements are necessary, where c_0 is a sufficiently large constant. In order to further decrease the number of measurements, we proposed the following approach in (28) by exploiting the sparsity information present in the signal:

$$\begin{array}{ll} \text{minimize} & \sum_{1\le i\le m} |\mathrm{Tr}(\boldsymbol{\Theta}_i \mathbf{X}) - b_i| + \lambda \|\mathbf{X}\|_1 \\ \text{subject to} & \mathbf{X} \succcurlyeq 0, \end{array} \tag{28}$$

where $\lambda > 0$ is a design parameter enforcing the sparsity. The matrix $\mathbf{X}$ is given by $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{H}}$. The recovered signal is estimated by extracting the largest rank-one component of the matrix $\widehat{\mathbf{X}} = \sum_{k=1}^{n} \hat{\lambda}_k \hat{\boldsymbol{u}}_k \hat{\boldsymbol{u}}_k^H$, where $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n$ The estimated signal is then computed as $\hat{\mathbf{x}} = \sqrt{\hat{\lambda}_1} \hat{\boldsymbol{u}}_1 \hat{\boldsymbol{u}}_1^H$. In Section 4.2, it will be illustrated that the approach in (28) slightly improves the CPRL approach for sparse signal recovery. However, it is observed empirically that the reconstruction performance of this approach highly depends on the value of design parameter λ.

3.2. Modified Phase Cut Approach for Sparse Signal Recovery

When the signal is sparse and fewer measurements are recorded, the *Phase Cut* algorithm fails to recover the true signal as mentioned in Section 2.3.

Phase recovery via the *Phase Cut* method can be interpreted as a projection problem [10]. This interpretation enables us to modify the *Phase Cut* method to achieve sparse signal recovery from fewer measurements.

First, the objective function in (14) of *Phase Cut* method is expressed as $\mathrm{Tr}(\mathbf{MU})$, where $\mathbf{M} = diag(\mathbf{b})(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger})diag(\mathbf{b})$. Since $\mathrm{Tr}(\mathbf{MU}) = \mathrm{Tr}(\mathbf{UM})$, it follows that:

$$\mathrm{Tr}(\mathbf{UM}) = \mathrm{Tr}\Big((\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger})diag(\mathbf{b})\mathbf{U}diag(\mathbf{b})\Big). \tag{29}$$

Define the matrix $\mathbf{V}$ via $\mathbf{V} = diag(\mathbf{b})\mathbf{U}diag(\mathbf{b})$. Therefore, $\mathrm{Tr}(\mathbf{UM}) = \mathrm{Tr}\Big((\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger})\mathbf{V}\Big)$, and it is equal to the matrix inner product given by $\langle \mathbf{V}, (\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}) \rangle = \langle \mathbf{V}, \mathscr{P}_{\mathscr{R}(\mathbf{A})^{\perp}} \rangle$. This matrix inner product is then expressed as a distance associated to the trace norm for all $\mathbf{V} \in \mathbf{H}_m$ such that $\mathbf{V} \succcurlyeq 0$,

$$\mathrm{Tr}\left(\mathbf{V}\left(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}\right)\right) = d_1(\mathbf{V}, \mathscr{F}), \tag{30}$$

where the set F is defined as $F = \left\{\mathbf{V} \in \mathbf{H}_m : \ \left(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}\right)\mathbf{V}\left(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}\right) = \mathbf{0}\right\}$ [10].

Therefore, the signal recovery problem is interpreted as minimizing a distance function for the constraints $\mathbf{V} \succcurlyeq 0$ and $\mathbf{V}_{i,i} = b_i^2, i = 1, \dots, m$. However, when the signal of interest is sparse and the number of measurements is less than the number of unknowns, the interpretation of the signal recovery problem as a distance minimization is not valid because $\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{I}$ makes the distance constant (and 0 which is defined in (30)).

In this thesis, based on interpreting the phase retrieval problem as a projection problem as in [10], the *Phase Cut* method is modified to accommodate sparse signal recovery using fewer measurements. When we have fewer measurements, the *Phase Cut* method tries to find the sparsest solution in the class of least squares solutions corresponding to different phase vectors $\mathbf{u}$. Therefore, additional constraints limiting the feasible region are included in order to converge to the exact solution in (31) by limiting the error to 0 between the observed magnitude data and the magnitude measured for the solution of the problem.

The *Phase Cut* method for recovering a sparse signal from underdetermined measurements is proposed as follows:

$$\begin{array}{ll} \text{minimize} & \lambda\left\|\mathbf{PUP}^{\mathrm{H}}\right\|_1 \\ \text{subject to} & \left\|\mathbf{b}\odot\mathbf{b} - \mathrm{diag}(\mathbf{APUP}^{\mathbf{H}}\mathbf{A}^{\mathrm{H}})\right\|_2 < \epsilon \\ & \mathbf{U} \succcurlyeq 0, \mathrm{diag}(\mathbf{U}) = 1, \end{array} \tag{31}$$

where $\mathbf{U} = \mathbf{u}\mathbf{u}^{\mathrm{H}}$ and $\mathbf{P} = \mathbf{A}^{\dagger}\mathrm{diag}(\mathbf{b})$. In the presence of noisy measurements ϵ is adapted according to the noise level. The modified *Phase Cut* method in equation (31) recovers the sparse signal from fewer measurements.

In Figure 3.1 an example is given to demonstrate that the modified Phase Cut approach recovers the true sparse signal while the *Phase Cut* method fails to converge to true solution and it results in a dense reconstructed signal. The true signal has 3 nonzero samples out of 32 samples, and we have 20 magnitude measurements. As seen in the figure, *Phase cut* fails to identify the signal with a degree of sparsity equal to 3. When we apply the modified Phase Cut approach, the true sparse signal is correctly identified.

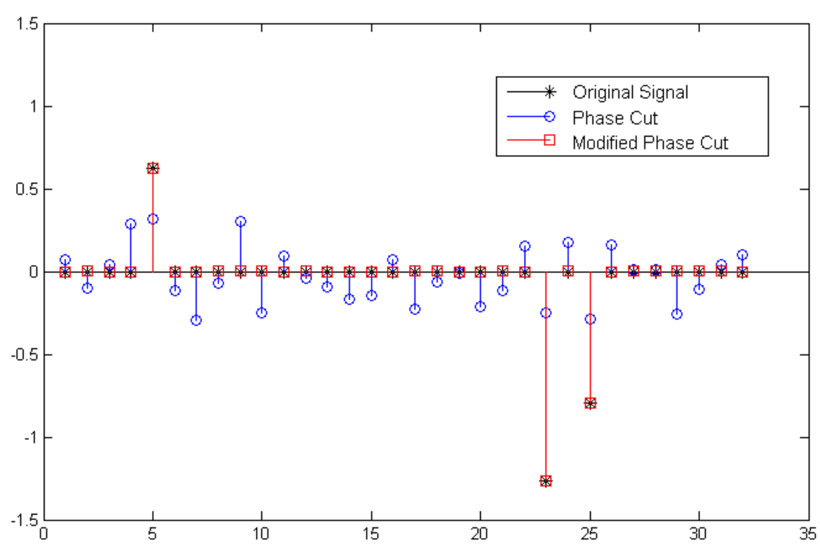


Figure 3.1. The original real-valued signal which has a length of 32 samples and the reconstructed signals by *Phase Cut* and the modified Phase Cut methods.

4. NUMERICAL SIMULATIONS

In this section of the thesis, a number of experiments are given to analyze and compare the phase retrieval methods based on semidefinite programming. Mainly *Phase Lift* [8] and *Phase Cut* [10] algorithms and their variations for the sparse signal recovery will be considered. First, in Section 4.1 we present the simulation results for the comparison of *Phase Lift* and *Phase Cut* methods in which no sparsity assumption on signal structure is made. In Section 4.2 the main results of the thesis in terms of computer simulations are presented and focus on sparse signal recovery by our proposed approaches modified Phase Cut and improved Phase Lift in comparison with CPRL [16].

Analyses of these methods are performed in terms of complexity, stability and performance for both non-sparse and sparse signal recovery. The simulations are implemented in Matlab using CVX, a package for solving convex programs [29, 30], and to solve low dimensional SDP.

The conducted computer simulations focus on establishing the relation between the computational complexity of the algorithm and reconstruction of the signal from a small number of measurements by exploiting the sparse structure of the signal, and on evaluating the performance of these algorithms in the presence of noisy measurements.

We conducted the experiments with randomly generated test signals. The magnitude measurements are obtained by transforming the signal into the Fourier domain followed by random projections.

4.1. Simulations for the Recovery of Non-Sparse Signals

To recover the signal $\mathbf{x}$ uniquely from $\mathbf{b} = |\mathbf{Ax}|$, $m \geq 4n - 2$ or $m \geq 2n - 2$ generic measurements are sufficient for complex-valued and real-valued signals, respectively [6]. If the number of identically and independently distributed Gaussian measurements is on the order of $n \log n$, the *Phase Lift* method recovers the signal $\mathbf{x}$ exactly with a high probability. When the phase recovery problem is formulated as a convex relaxation of a quadratic optimization problem, such as in the *Phase Cut* method, the number of measurements required for the exact recovery is at least as large as the number of measurements required for the *Phase Lift* technique.

Figure 4.1 shows the minimum number of measurements required to have 100% success rate both in the *Phase Lift* and the *Phase Cut* methods. In this experiment, we used real-valued test signals of length 16 samples. The magnitude measurements $\mathbf{b} = |\mathbf{RFx}|$, which are real-valued $\mathbf{b} \in \mathbb{R}^m$, are obtained by transforming the signal into the Fourier domain by $\mathbf{F} \in \mathbb{C}^{16\times16}$ followed by random projections $\mathbf{R} \in \mathbb{C}^{m\times16}$, where m is the number of measurements generated from a standard normal distribution. Therefore, the measurement matrix $\mathbf{A}$ can be expressed as the product of a random matrix $\mathbf{R}$ with a Fourier matrix $\mathbf{F}$ such that $\mathbf{A} = \mathbf{RF}$. This experiment is repeated 10 times for 10 different random non-sparse signals. When the number of measurements reaches 38, exact recovery occurs for both *Phase Lift* and *Phase Cut* methods, and they show very similar reconstruction rates. In this case, unique recovery is achieved by using a larger number of measurements instead of taking the advantage of the signal structure such as having a few nonzero elements.

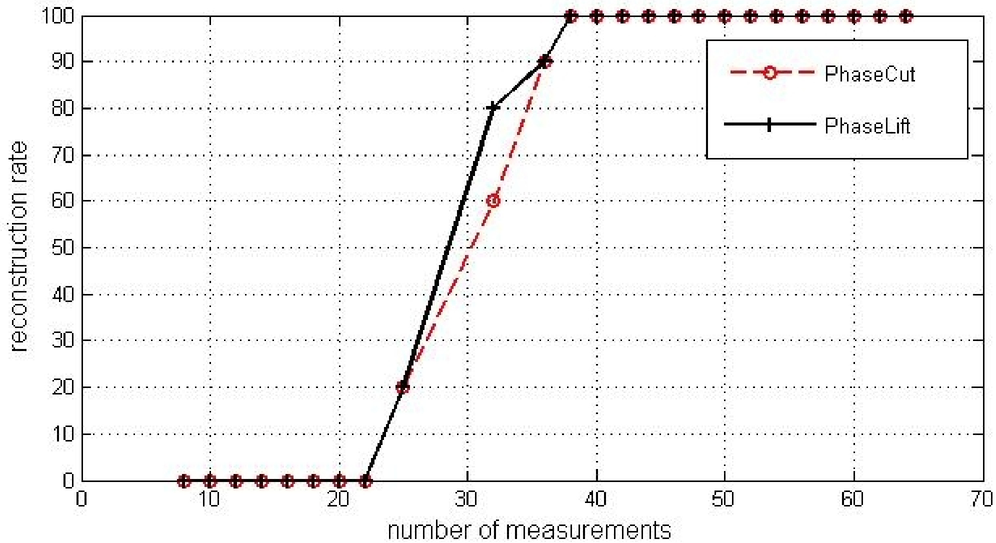


Figure 4.1. Reconstruction rate of a real-valued dense signal which has a length of 16 samples is shown for the given number of measurements. The straight line represents the *Phase Lift* method, and the dotted line represents the *Phase Cut* method.

In order to compare the computational complexities of these algorithms, the computation time (expressed in seconds) and number of iterations necessary to solve the semidefinite program in Matlab via the CVX package are given in Table 4.1. The different lengths of the unknown signals are chosen as 8, 16 and 32. The number of measurements for the given signals with specified dimension n is $m = 4n$ in order to guarantee exact signal recovery. This experiment is repeated ten times with ten different random signals, and then the average number of iterations and the average computation time are recorded. The number of iterations does not seem to depend significantly on the signal size, while the computation time increases with the signal dimension. The computation time for the *Phase Cut* method is longer than that of the *Phase Lift* method to recover real-valued signals from magnitude measurements. The reason for this behavior is the fact that the complex matrices are expressed as larger real-valued matrices with a sub-block capturing their real part and another sub-block for their imaginary part, as is the case in the *Phase Cut* algorithm. Therefore, this modification would increase the computational time of the algorithm whereas it would not have an effect on the number of iterations required to converge to the optimal solution.

Table 4.1 Computation time and the number of iterations in SDP for *Phase Lift* and *Phase Cut* methods

Method	**Size (n)**	**Iteration**	**Time (sec)**
Phase Lift	8	20	0.74
	16	14	1.21
	32	17	13.16
Phase Cut	8	18	1.11
	16	21	1.78
	32	22	21.99

As discussed in Section 2.3 about the comparison of *Phase Lift* and *Phase Cut* methods, both algorithms find the same exact solution, up to a global phase factor, when the number of measurements is larger than the number of unknowns, i.e., when the linear mapping is injective. Figure 4.2 illustrates an example of reconstructed signals via *Phase Lift* and *Phase Cut* methods. It can be seen that both methods result in the same exact solution for the signal recovery problem.

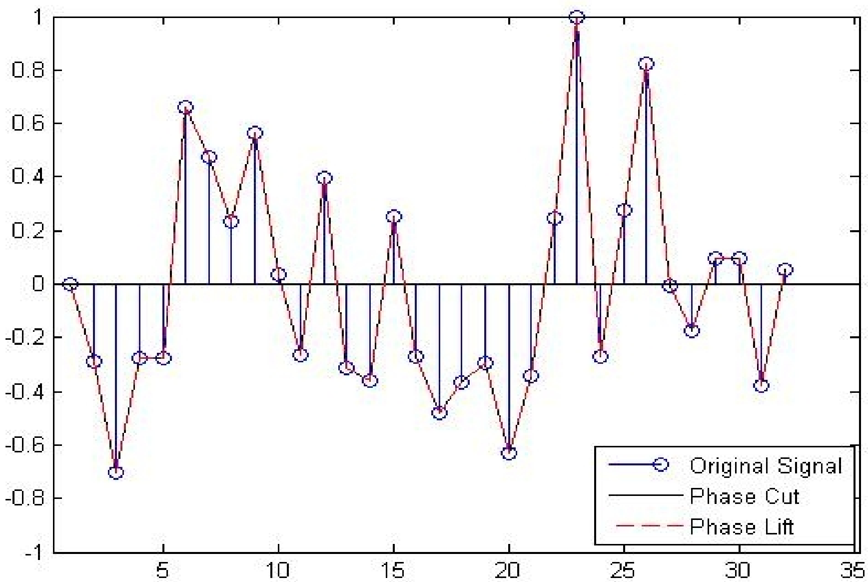


Figure 4.2. The original real-valued signal which has a length of 32 samples (plotted by stem function) and the reconstructed signals by *Phase Lift* and *Phase Cut* methods.

Stability analysis of algorithms plays a significant role for practical applications. Therefore, the methods proposed for the phase retrieval problem need to be robust when the measurements are contaminated by additive noise. Because squared magnitudes are measured in many practical applications, i.e., in Fraunhofer diffraction in optics, we use the following model to test the *Phase Lift* method:

$$b_i = |\langle \mathbf{x}, \mathbf{a}_i \rangle|^2 + \upsilon_i, \;\; i = 1, \dots, m, \tag{32}$$

where υ_i is the noise term bounded by $\|\upsilon_i\|_2^2 \leq \|\epsilon\|_2^2$. Using this model, the *Phase Lift* method is reformulated as shown below in the presence of noisy measurements.

$$\begin{array}{ll} \text{minimize} & \mathrm{Tr}(\mathbf{X}) \\ \text{subject to} & \left\|\mathbf{b} - diag(\mathbf{A}\mathbf{X}\mathbf{A}^{\mathrm{H}})\right\|_2 \leq \epsilon, i = 1, \dots, m \\ & \mathbf{X} \succcurlyeq 0. \end{array} \tag{33}$$

The recovered signal is then computed by using the normalized eigenvector of the solution matrix **X** corresponding to the largest eigenvalue. When the noisy measurements of the signal are recorded, the solution of the *Phase Lift* method obeys the following inequalities $\|\tilde{\mathbf{X}} - \mathbf{x}\mathbf{x}^{\mathsf{H}}\|_2 \leq C_0\epsilon$ an$\|\tilde{\mathbf{x}} - \mathbf{x}\|_2 \leq C_0 \min\left(\|\boldsymbol{x}\|_2, \epsilon/\|\boldsymbol{x}\|_2\right)$d, for some positive constant C_0 with high probability [8]. Existence of such bounds shows that the *Phase Lift* method is stable in the presence of noisy measurements.

In order to test the performance of the *Phase Cut* method in the presence of noisy measurements, we adopt the model of measurements in the form of $\mathbf{b} = |\mathbf{A}\mathbf{x}| + \mathbf{b}_{\mathsf{noise}}$ because the measurements are obtained from the magnitude of a linear system, not the square of the magnitude, in the *Phase Cut* approach.

In order to compare both the *Phase Lift* and the *Phase Cut* methods in the presence of noisy measurements, their robustness is assessed empirically. First, noisy measurement models are adopted as follows: $b_i = |\langle \mathbf{x}, \mathbf{a}_i \rangle|^2 + v_i, i = 1, \dots, m,$ and $\mathbf{b} = |\mathbf{A}\mathbf{x}| + \mathbf{b}_{\mathsf{noise}}$ for the *Phase Lift* and the *Phase Cut* methods, respectively. For each of the models, the additive noise terms are from a Gaussian distribution. In the test, the Gaussian noise is added at different SNR levels: 5, 10, 25, 50 and 100 dB. For each of the SNR levels, different real-valued test signals of size 16, measurements of size 64 and the noise signals of the same size as the measurements are generated, and the experiment is repeated 100 times. The average relative mean square error (MSE) is calculated by using the results of these 100 experiments. Since the reconstructed signals present a phase ambiguity factor, the relative MSE is then calculated by the following formula [9]:

$$\text{rel. MSE} = \min_{c:|c|=1} \frac{\|c\mathbf{x} - \tilde{\mathbf{x}}\|_2^2}{\|\mathbf{x}\|_2^2}. \tag{34}$$

The vector $\tilde{\mathbf{x}}$ is the reconstructed signal and the vector $\mathbf{x}$ is the true signal. The relative MSE in the dB scale is calculated by $10\log_{10}(\text{rel. MSE})$. The relative MSE and the relative MSE in a dB scale for different SNR levels are plotted in Figures 4.3 and 4.4, respectively. As shown in Figures 4.3 and 4.4, both methods are stable in the presence of noisy measurements.

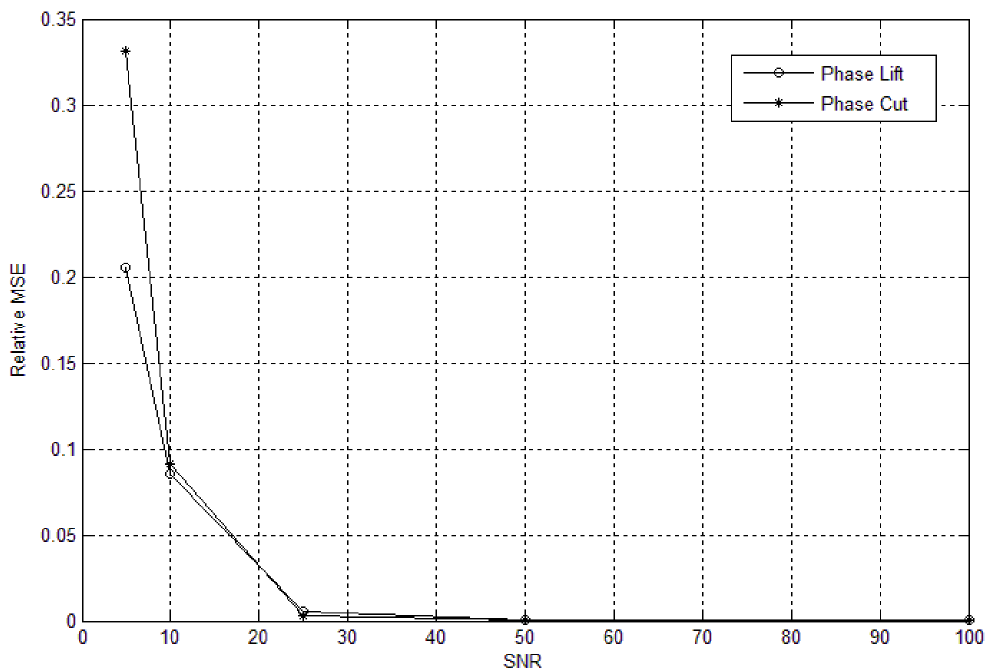


Figure 4.3. Performance of *Phase Lift* and *Phase Cut* method in the presence of Gaussian noise (relative MSE versus SNR).

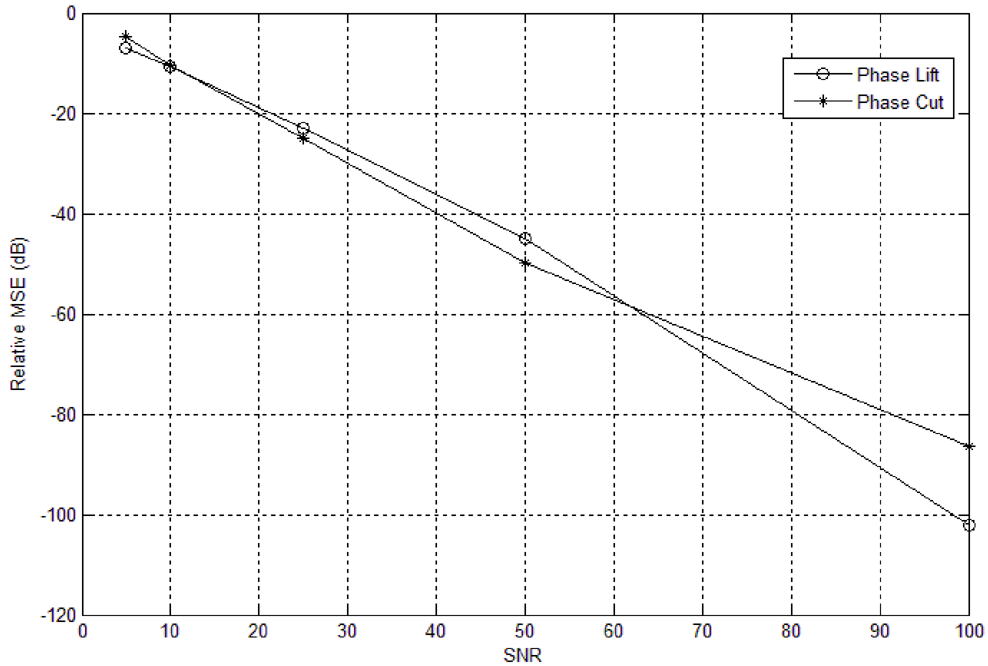
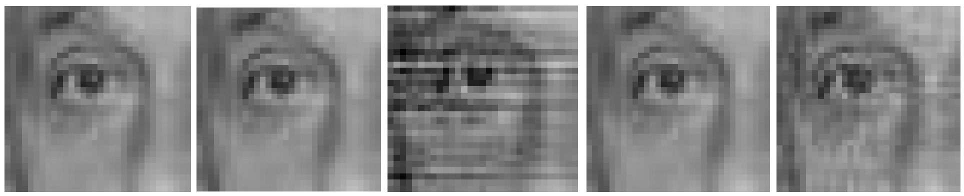


Figure 4.4. Performance of *Phase Lift* and *Phase Cut* methods in the presence of Gaussian noise (relative MSE on log scale versus SNR).

In the last example, we used a cropped image of dimension 32×32 in order to demonstrate the performance of the signal reconstruction methods on 2D-signals. The measurements are obtained by transforming the image into the Fourier domain, and multiplying the 2D-Fourier transform of the image by a random matrix of dimension 128×32. Finally, the absolute values of the entries of the resulting 2D signal are recorded as the magnitude measurements. We implemented both algorithms on a column by column basis, and then the recovery is achieved by combining these columns in order to generate an image because CVX package cannot handle this problem in the scale of vectorized images. In Figures 4.5(c)-(e), the images are recovered from the magnitude measurements which are contaminated by additive Gaussian noise, the signal-to-noise ratio being equal to 25 dB.



(a) **(b)** **(c)** **(d)** **(e)**

Figure 4.5. **(a)** The true image of dimension 32×32. **(b)** Reconstructed image by *Phase Lift* from noise free measurements **(c)** Reconstructed image by *Phase Lift* from noisy measurements. **(d)** Reconstructed image by *Phase Cut* from noise free measurements. **(e)** Reconstructed image by *Phase Cut* from noisy measurements.

4.2. Simulations for the Recovery of Sparse Signals

In the phase retrieval problem, the exact solution is recovered when the number of measurements is $m > n$. Motivated by compressed sensing, the phase retrieval approaches are combined with l_1-norm minimization techniques to recover the sufficiently sparse unknown signal from fewer measurements. Unique signal recovery, up to a global phase factor, is targeted by using the signal structure, which is sparse in this case, instead of recording more measurements.

In this section, the numerical results exhibited by sparse versions of the *Phase Lift* and *Phase Cut* methods for recovering the sparse signals from underdetermined measurements are described. First, the simulations are performed to compare CPRL [16] and the proposed modified Phase Cut, which were mentioned in the Sections 3.1 and 3.2, respectively. Later, the simulation results are presented by the proposed improved Phase Lift approach in comparison with CPRL.

The magnitude measurements are obtained from $\mathbf{b} = |\mathbf{RFx}|$ and $\mathbf{b} = |\mathbf{RFx}|^2$, $\mathbf{b} \in \mathbb{R}^m$ for modified Phase Cut and CPRL, respectively. The sparse signal is transformed into the Fourier domain by multiplying the signal with the Fourier matrix $\mathbf{F} \in \mathbb{C}^{n \times n}$ which is then further transformed via the random projection matrix $\mathbf{R} \in \mathbb{C}^{m \times n}$ generated using a standard normal distribution. For sparse signal recovery, the number of measurements is less than the number of unknowns, $m < n$.

We have mentioned in Section 2.3 that the *Phase Cut* method is not tractable when the number of measurements is less than number of unknowns. In order to illustrate that the CPRL approach works, which is based on the *Phase Lift* method, while the similar approach applied to the *Phase Cut* method fails, the reconstruction performance of these methods is shown in Figure 4.6. The signal of interest that we want to recover is real-valued and its length is 32 samples. It has 3 nonzero values, in other words the sparsity degree of the signal is 3. For sparse signal recovery, 20 magnitude measurements of the signal are recorded. While CPRL recovers the true signal exactly, the l_1-norm based *Phase Cut* method fails to find the true sparse signal.

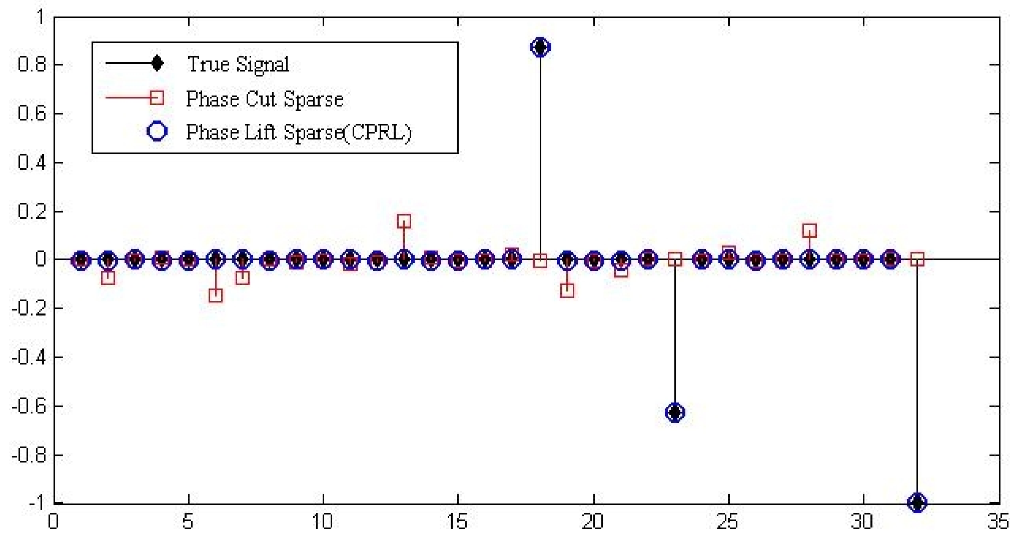


Figure 4.6. True sparse signal in diamond shape, and reconstructed signals by *Phase Cut* and *Phase Lift* methods promoted with l_1-norm minimization.

Phase Cut algorithm fails for sparse signal recovery; however, the proposed modified Phase Cut method with additional constraints, which are described in Section 3.2, works in recovering the sparse signals from fewer measurements. In Figure 4.7, it can be observed that the modified Phase Cut method recovers the real-valued signal, which has a length of 32 samples and 3 non-zero values, from 20 magnitude measurements.

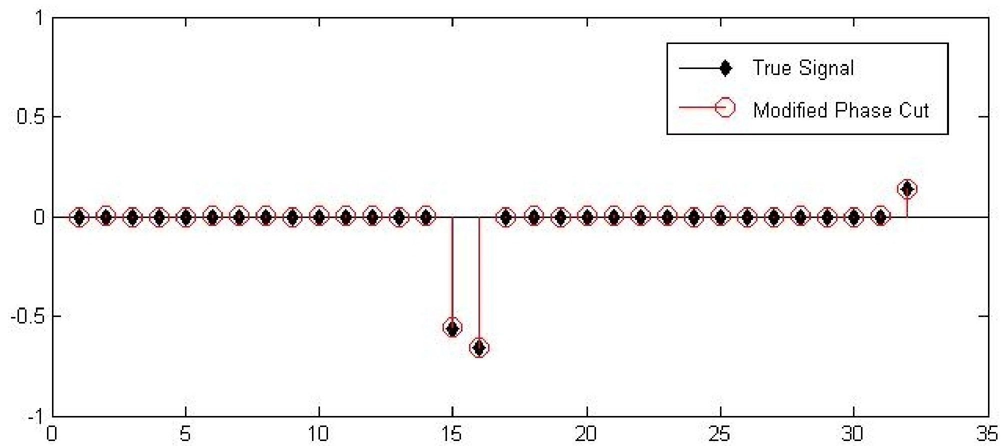


Figure 4.7. True sparse signal in diamond shape, and reconstructed signal by modified Phase Cut in circle (n = 32, sparsity degree = 3, m = 20).

In order to demonstrate the performance of the modified Phase Cut method, first the success rate of the proposed approach in 100 trials is recorded for a fixed length signal with fixed number of measurements while the degree of sparsity is varying. In Figure 4.8, the success rates of the CPRL and modified Phase Cut approaches for different values of signal sparsity are given. While both modified Phase Cut approach and CPRL work for signals which present degree of sparsity up to 2 at a success rate 90%-100%, their performance decrease sharply beyond the sparsity degree 2. It is observed that the modified Phase Cut method improves the performance of CPRL especially when the sparsity degree of the signal is 3.

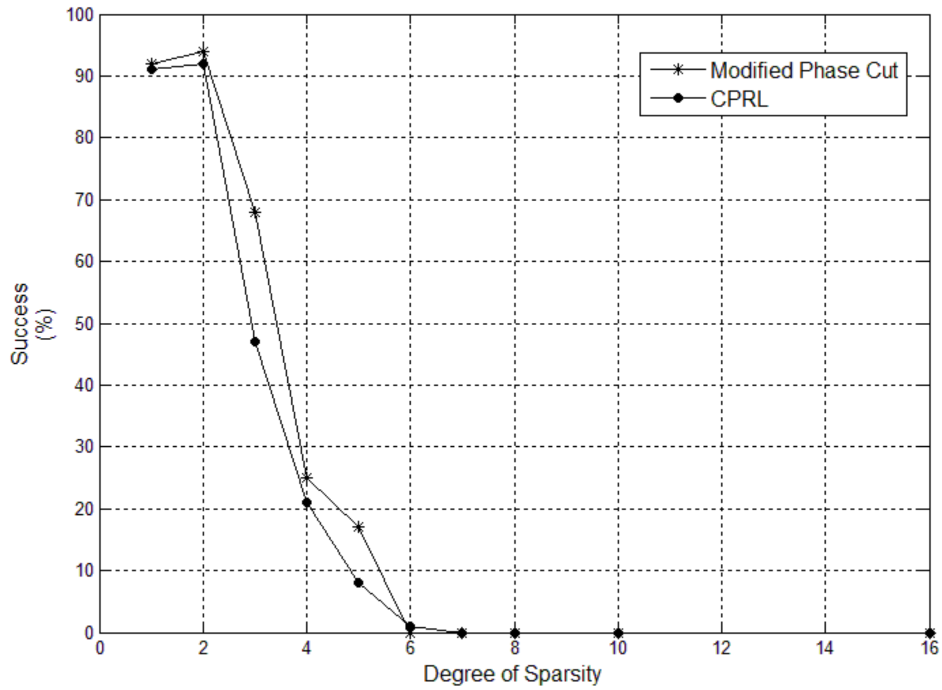


Figure 4.8. For the signal which has a length of 32 samples, the number of successes in 100 experiments versus the degree of sparsity is given when we have 20 magnitude measurements (Modified Phase Cut and CPRL).

In Figure 4.9, the relative MSE error of both methods is given for different values in the degree of signal sparsity. 100 computer simulations are conducted, and in each simulation, the test signals are randomly generated. The relative MSE values are then calculated by taking the average of the error in each test. Similarly, relative MSE of the modified Phase Cut approach is slightly less than the relative MSE of the CPRL method.

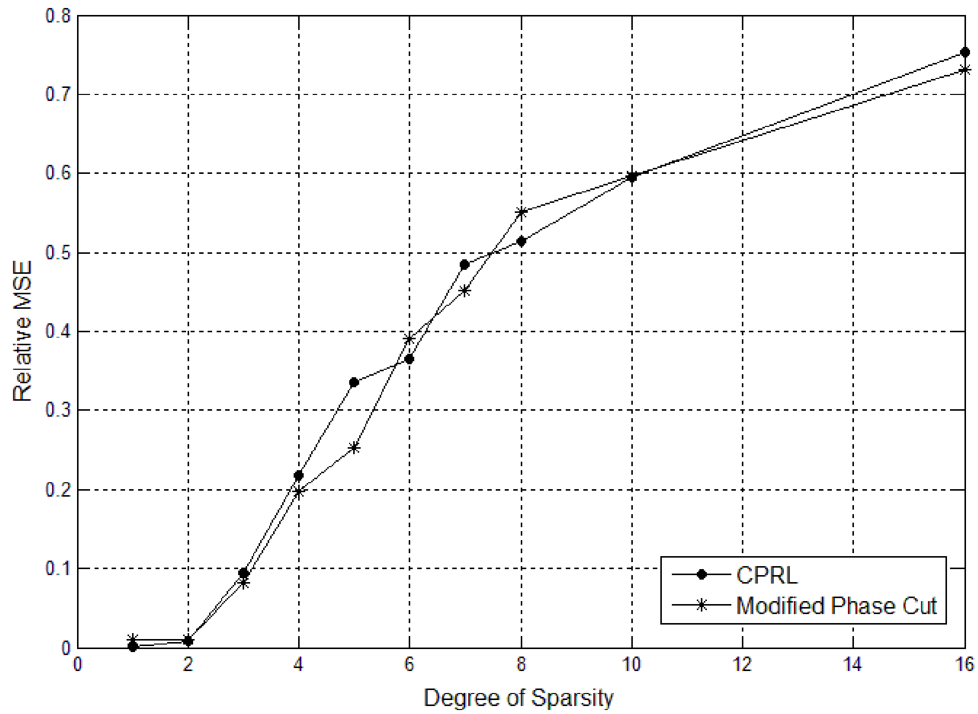


Figure 4.9. Relative MSE versus different sparsity degrees when the number of measurements is fixed to 20 for a signal having a length of 32 samples.

In the second experiment, the minimum number of measurements required to have exact signal recovery is illustrated for different values of the sparsity degree for a real-valued signal which has a length of 16 samples. This experiment is repeated 100 times and then the numbers of measurements required for unique signal recovery are averaged. In order to compare their performances, the results exhibited by CPRL and the modified Phase Cut methods are described. Both methods show similar performances and the number of measurements required to have exact signal recovery is fewer than $4n$ measurements when the sparse structure of the signal is promoted with an l_1-norm minimization as shown in Figure 4.10.

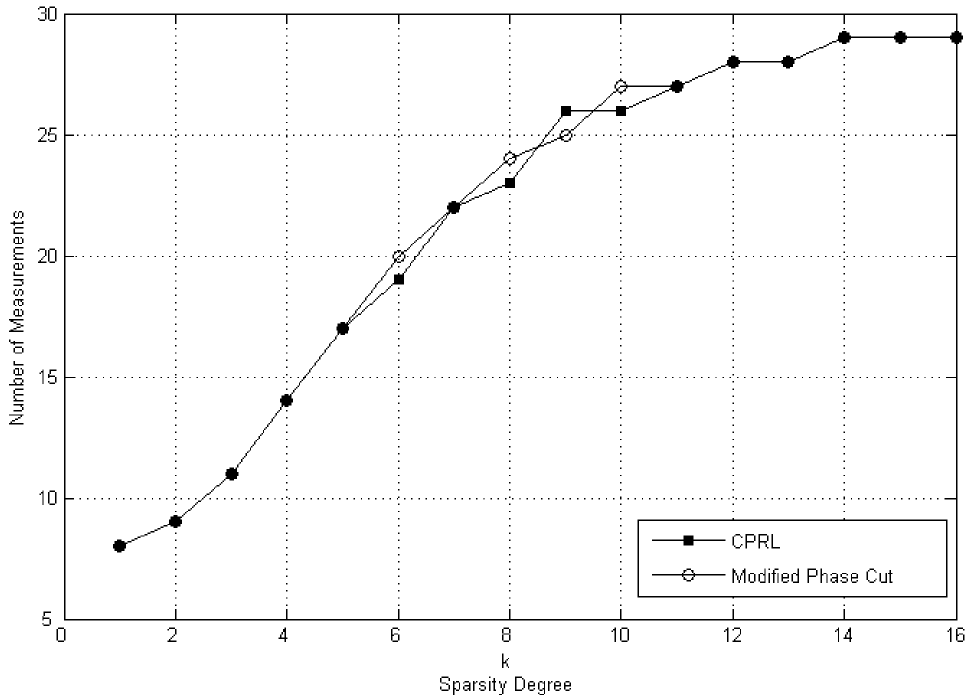


Figure 4.10. Number of measurements required to have exact signal recovery when the original signal (of size 16) is sparse. This experiment is repeated 100 times and then the results are averaged.

In addition to determining the number of measurements required for unique signal recovery for different values of sparsity degree, we also computed the number of measurements required for unique signal recovery for signals with different lengths. This experiment is repeated 10 times for 10 different random signals, and then the average number of required measurements is calculated. The lengths of the signals are chosen as 8, 16 and 32 samples, respectively, while the sparsity degree of these signals is fixed to 3. The results of this experiment are displayed in Table 4.2. When the degree of sparsity of the signals is fixed and the lengths of the signals vary, the sparse signals are exactly recovered from fewer measurements for both CPRL and modified Phase Cut methods.

Table 4.2. The number of measurements required for exact signal recovery for CPRL and the modified Phase Cut approaches

Method	Size (n)	Number of Measurements Required for Exact Signal Recovery
CPRL	8	10
	16	12
	32	15
Modified *Phase Cut*	8	10
	16	12
	32	18

In order to compare the computational complexities of the CPRL and the modified *Phase Cut* approaches, the computation time expressed in seconds and the number of iterations required to solve the semidefinite program in Matlab via the CVX package are evaluated. In Table 4.3, the lengths of the signals change while the sparsity degree of the signals is fixed to 3. In this case, it is observed that the computational complexities of both approaches depend on the signal length, especially for the modified Phase Cut method. When we compare the computation time for *Phase Lift* method in Table 4.1 with the computation time of CPRL in Table 4.3, CPRL achieves exact recovery in a shorter time. However, this improvement in the computational time for the modified Phase Cut method is not observed while exact signal recovery is achieved from fewer measurements.

Table 4.3. Computation time and the number of iterations in SDP for the CPRL and modified Phase Cut approaches (sparsity degree of the signals is fixed to 3)

Method	**Size (n)**	**Iteration**	**Time (sec)**
CPRL	8	18	0.64
	16	22	0.86
	32	27	3.17
Modified Phase Cut	8	19	1.16
	16	20	3.08
	32	23	42.03

Table 4.4. Computation time and the number of iterations in SDP for the CPRL and modified Phase Cut approaches (length of the signals is fixed to 16 samples)

Method	**Sparsity Degree of the signal (k)**	**Iteration**	**Time (sec)**
CPRL	3	22	0.87
	6	25	1.01
	10	28	1.25
Modified Phase Cut	3	23	4.32
	6	26	8.03
	10	24	15.31

In Table 4.4, the runtime of both approaches is given for different sparsity degrees of the signal. For each of the sparsity values, the number of measurements is chosen such that the approaches will achieve the exact signal recovery. First, when the signal of interest has more non-zero samples, the computation time for CPRL and modified Phase Cut increases. However, this increase is sharper in the computation time of modified Phase Cut approach. If Tables 4.1 and 4.4 are compared, it turns out that there is a trade-off between the computational complexity of the algorithms and the undersampling factor. While exploiting the sparse structure of the signals decreases the necessary number of measurements for exact recovery, the runtime of the modified Phase Cut method increases slightly. However, in terms of recovering a signal from fewer measurements, exploiting the structure of the signals enables the convergence of the CPRL approach to the exact solution slightly faster as expected.

To evaluate the performance of these approaches in the presence of noisy measurements, we employed the same procedure in the non-sparse signal recovery case described in Section 4.1. In the test, the Gaussian noise is added to the magnitude measurements at different SNR levels such as 5, 10, 25, 50 and 100 dB. For each of the SNR levels, different real-valued test signals which present 4 nonzero samples out of the 16 samples are generated. In order to guarantee the exact recovery, the number of measurements is fixed to 16. Each experiment is repeated 100 times. In Figure 4.11, the relative MSE values for different SNR levels are illustrated. When the SNR level is low, the relative MSE of the CPRL approach is slightly lower than the relative MSE of the modified Phase Cut approach. For higher SNR levels, they present similar performances.

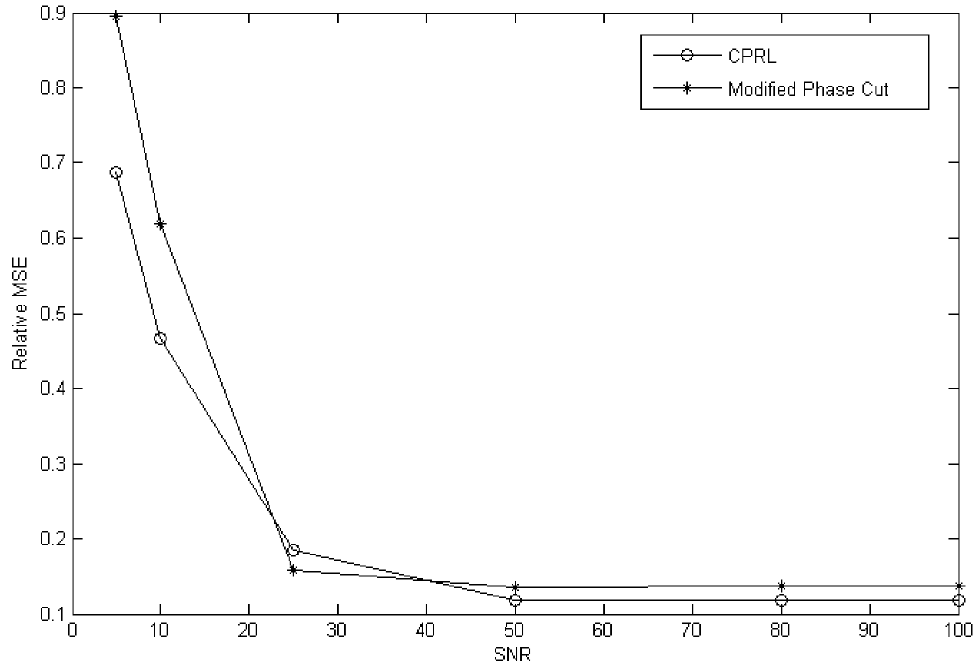


Figure 4.11. Performance of CPRL and modified Phase Cut in the presence of Gaussian noise (relative MSE versus SNR).

Finally, in order to improve the *Phase Lift* method for sparse signal recovery, we proposed the approach in (35) to exploit the sparsity information present in the signal by combining the method proposed in [20] with a l_1-norm minimization:

$$\begin{array}{ll} \text{minimize} & \sum_{1\leq i\leq m} |\mathrm{Tr}(\boldsymbol{\Theta}_i \mathbf{X}) - b_i| + \lambda \|\mathbf{X}\|_1 \\ \text{subject to} & \mathbf{X} \succcurlyeq 0, \end{array} \tag{35}$$

where $\lambda > 0$ is a design parameter enforcing the sparsity, and $\|\mathbf{X}\|_1$ denotes the entry-wise l_1-norm for the matrix $\mathbf{X}$.

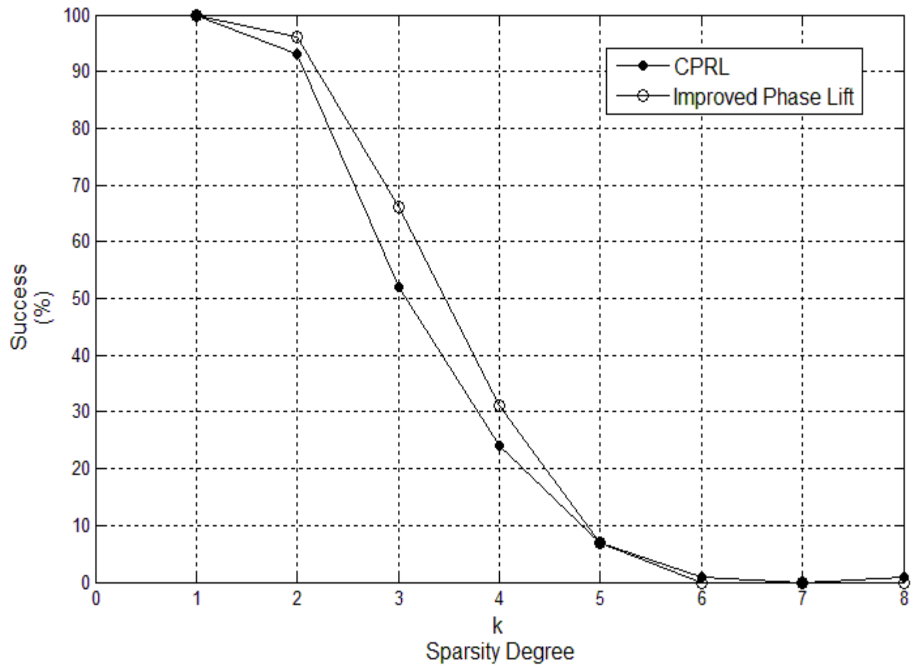


Figure 4.12. Success rate of the signal recovery using CPRL and the proposed approach.

Figure 4.12 demonstrates the performance of the approach in (35) and CPRL as a function of the sparsity of the signal. While the performance of the proposed approach highly depends on the value of design parameter λ, CPRL method does not exhibit such a high dependence. For this reason, in order to compare their reconstruction rates, λ is fixed to 1.

In Figure 4.12, it is illustrated that the proposed approach slightly improves the performance of CPRL approach for sparse signal recovery. For larger degrees of sparsity of the signal, the reconstruction rate of the proposed approach may be further increased by adjusting the design parameter. This experiment consists of 100 trials, and each of the simulations is tested with randomly generated signals with length equal to 32 samples.

The number of measurements is chosen equal to 20. In Figure 4.13 the relative MSE error of both methods is given for different values of the degree of signal sparsity.

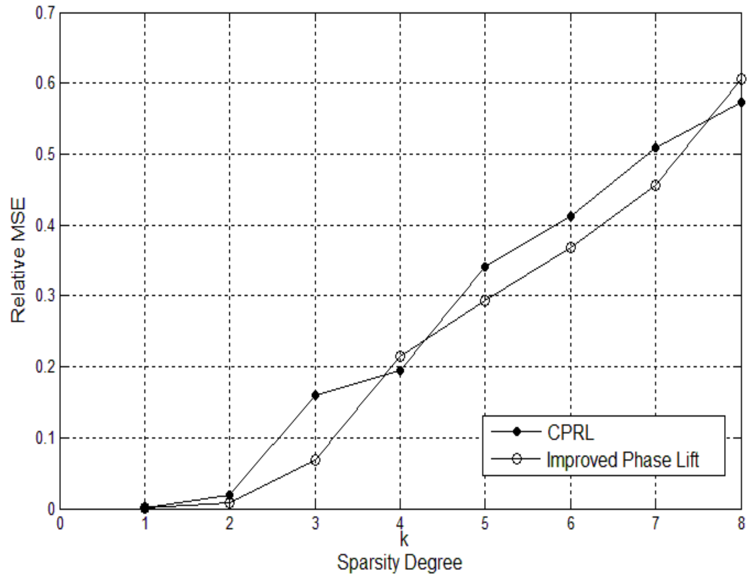


Figure 4.13. Relative MSE for different values of sparsity degree is given when the number of measurements is fixed to 20 for a signal having a length of 32 samples.

5. SUMMARY AND CONCLUSION

In the presented research, the phase retrieval problem from nonlinear magnitude measurements is studied. The classical phase retrieval algorithms mainly depend on the alternating projection method. These methods work practically well; however, exact recovery of these methods is not guaranteed because of the nonlinear magnitude measurement constraint. Therefore, this study concentrates on the *Phase Lift* [8] and *Phase Cut* [10] methods which interpret the phase retrieval problem in a higher dimensional space by formulating the signal recovery problem as a rank-one matrix recovery problem. The *Phase Lift* and the *Phase Cut* methods which are based on semidefinite programming require redundant number of measurements in order to guarantee the exact signal recovery. However, in real-word scenarios, it can be expensive and time consuming to collect a large number of measurements. Therefore, in this study, the sparse structure of the signals is employed in order to reduce the number of measurements. With this goal, SDP based *Phase Lift* and *Phase Cut* methods, which requires sufficiently large amount of measurements, are combined with l_1-norm minimization in compressive sensing.

In this thesis, we proposed the modified Phase Cut and improved Phase Lift approaches in order to recover sparse signals from fewer measurements and they are analyzed in comparison to a recent method, called compressive phase retrieval via lifting (CPRL) [16]. The proposed approaches are developed by exploiting the sparsity information present in the signal by employing l_1-norm minimization.

The *Phase Lift* and the *Phase Cut* methods recover the signals exactly with high probability when the number of measurements is sufficiently large in the presence of IID random vectors. Their performances are very similar when the number of measurements is greater than the number of unknowns. If both methods have optimal solutions for the semidefinite programs defined for each of them, these solutions are the same and exact. Despite their similar performance in the presence of overdetermined systems, in an underdetermined system of equations with sparsity constraints, the *Phase Cut* method fails because of the least squares criterion.

The proposed modified Phase Cut method handles this problem by adding extra affine constraints in the range space of the linear mapping. The reconstruction performance of the modified Phase Cut approach is then better than the reconstruction rate of CPRL approach for different sparsities. The number of measurements required for exact signal recovery exhibits a similar trend in both approaches for different degrees of sparsity. However, these approaches, i.e., the *Phase Lift* and *Phase Cut* methods and their versions for the recovery of sparse signals, differ in terms of computational complexity. The *Phase Lift* and CPRL (the version of *Phase Lift* method for sparse signal recovery) method present less computational complexity than the *Phase Cut* and modified Phase Cut approach, respectively. Because the configuring of the *Phase Cut* based approaches in terms of dealing with complex matrices increases the matrix dimension according to the configuration of *Phase Lift* based approaches, and this configuration increases the runtime of these methods. In addition, decreasing the number of measurements by using sparsity information of the signal improves the computational complexity, as well as the runtime, of the CPRL approach with respect to the *Phase Lift* method. However, surprisingly this improvement is not observed in *Phase Cut* based methods. Reducing the number of measurements by using the sparse structure of the signal causes the runtime of the modified Phase Cut method to increase, although it is achieved in sparse signal recovery, and it improves the performance of CPRL. Finally, both CPRL and modified Phase Cut approaches are stable in the presence of noisy measurements. Their performances improve with an increase in the signal-to-noise ratio.

The other approach that we proposed in order to improve the *Phase Lift* method for sparse signal recovery gives better reconstruction performance than the CPRL approach. However, it is observed empirically that the performance of this approach highly depends on the value of design parameter λ. Better reconstruction performances of the improved Phase Lift approach can be achieved by carefully adjusting the design parameter.

In conclusion, combining the phase retrieval methods based on semidefinite programming with the l_1-norm minimization idea in compressive sensing enables the usage of the sparse structure of the signal for the exact recovery from fewer magnitude measurements. The proposed approaches which are modified Phase Cut and improved Phase Lift perform slightly better than CPRL in terms of reconstruction rate. However, the methods that are based on *Phase Cut* are computationally slower than the developed version of the *Phase Lift* method. For both methods, the exact signal recovery could be accomplished up to a lower degree of sparsity. Thus, these techniques could be improved in the future in order to recover signals at higher sparsity levels from fewer measurements. Finally, formulating the phase retrieval in a higher dimensional space and solving the semidefinite programs by interior-point methods in the CVX package [30] is challenging in terms of computational time in large scale problems. Therefore, for the future directions, some improvements could be done in terms of computation time by using some SDP acceleration techniques.

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