# IMPLICIT RATE-TYPE MODELS FOR ELASTIC BODIES: DEVELOPMENT, INTEGRATION, LINEARIZATION AND APPLICATION

A Dissertation

by

RONALD CRAIG BRIDGES, II

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2011

Major Subject: Mechanical Engineering

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# ABSTRACT

Implicit Rate-Type Models for Elastic Bodies:
Development, Integration, Linearization and Application. (August 2011)
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Chair of Advisory Committee: Dr. K. R. Rajagopal

In this dissertation, we use the second law of thermodynamics to find restrictions on the Gibbs potential so that there can be no dissipation associated with the deformation, in any process. We find that the use of the Gibbs potential leads to a much richer class of models than traditional elasticity wherein the Helmholtz potential is utilized, and the relationship between the stress and stretch is governed by a tensor-valued rate equation. For the special case of an isotropic body, we obtain a solution to this rate equation and after a linearization process, show that such models offer the possibility of the stress "blowing up," while the strain remains finite which is entirely consistent with the theory; such is not possible in traditional elasticity. This possibility has a wide array of applications including fracture in metals and delamination of composite bodies. A boundary value problem is proposed and solved numerically, for a particular Gibbs potential, in order to illustrate the efficacy of the framework. More applications such as boundary value problems within the context of finite elasticity and dissipative mechanisms are also discussed.

This dissertation is dedicated to all the family, friends, and professors who have encouraged me throughout these past ten years. I would like to give considerable credit to my fiancée, Lindsey Harrison. While I have only come to know her at the tail end of this adventure, she has provided me with a wealth of support and inspiration in my life.

## ACKNOWLEDGMENTS

There are very few students who get to study under someone with such a broad understanding of the world as Professor Rajagopal. I consider myself especially lucky that I met him while I was an undergraduate, thus enabling me the opportunity to begin learning from him as a very young student. I will always remember his teachings, not only on mechanics-but also philosophy, history, and english, and I thank him for his support over the past ten years.

# NOMENCLATURE

Below definitions are assigned to various symbols and acronyms. For the case of a symbol having dual meaning, the context should make the symbol clear.

| | |
|---|---|
| $\mathbb{B}$ | Set of Points Comprising the "Abstract" Body |
| $\mathbf{B}$ | Left Cauchy-Green Stretch Tensor |
| $\mathbf{b}$ | Body Force per Unit Mass |
| $C^n[a,b]$ | Continously Differentiable $n$ Times on the Interval $[a,b]$ |
| $\mathbf{D}$ | Symmetric Part of the Velocity Gradient |
| $\det(\mathbf{A})$ | Determinant of the Linear Transformation $\mathbf{A}$ |
| $\boldsymbol{\varepsilon}$ | Linearized Strain Tensor |
| $\mathbf{E}_H$ | Hencky Strain |
| $\mathcal{E}$ | Three-Dimensional Euclidean Space |
| $\zeta$ | Rate of Entropy Production |
| $\mathbf{e}_i$ | Vector Basis in an Orthogonal Coordinate System |
| $\varepsilon$ | Internal Energy per Unit Mass |
| $\mathbf{F}$ | Deformation Gradient |
| $\mathrm{I}_{\mathbf{A}}$, $\mathrm{II}_{\mathbf{A}}$, $\mathrm{III}_{\mathbf{A}}$ | First, Second, and Third Invariants of a Linear Transformation $\mathbf{A}$ |
| $\kappa$ | Placer Function Which Maps a Set of Points into Another |
| $\kappa_R(\mathbb{B})$ | Placment of the Reference Configuration |
| $\kappa_t(\mathbb{B})$ | Placment of the Current Configuration |
| $\mathbf{L}$ | Eulerian Gradient of the Velocity Vector |
| $\ln(\mathbf{A})$ | Natural Logarithm of the Symmetric Linear Transformation $\mathbf{A}$ |
| $\frac{df}{dt}$ | Material Time Derivative of the Function $f$ |
| $\boldsymbol{\Omega}$ | Skew-Symmetric Tensor |
| $\Phi$ | The Gibbs Potential |
| PDE | Partial Differential Equation(s) |

| | |
|---|---|
| $\mathbf{q}$ | Heat Flux Vector |
| $\varrho$ | Mass Density in the Current Configuration |
| $\varrho_R$ | Mass Density in the Reference Configuration |
| $r$ | Heating per Unit Mass |
| $s.t.$ | Such That |
| $\mathbf{R}$ | Rotation Tensor in the Polar Decomposition of the Deformation Gradient |
| $\mathrm{tr}(\mathbf{A})$ | Trace of the Linear Transformation $\mathbf{A}$ |
| $\theta$ | Absolute Temperature or Angular Coordinate in a Curvillinear Coordinate System |
| $\mathbf{T}$ | Cauchy Stress Tensor |
| $\mathbf{t_n}$ | Traction Vector acting on a Surface with a Normal Vector $\mathbf{n}$ |
| $t$ | Time |
| $\mathbf{U}$ | Right Stretch Tensor in the Polar Decomposition of the Deformation Gradient |
| $\mathbf{u}$ | Displacement Vector |
| $\mathcal{V}$ | Translation Space |
| $\mathbf{V}$ | Left Stretch Tensor in the Polar Decomposition of the Deformation Gradient |
| $\mathbf{v}$ | Velocity Vector |
| $\boldsymbol{\chi}_{\kappa_R}$ | Mapping of the Material Points in the Reference Configuration to the Current Configuration |
| $\xi$ | Rate of Dissipation of Energy |
| $\mathbf{X}$ | Material Point in the Reference Configuration |
| $\mathbf{x}$ | Material Point in the Current Configuration |
| $\Psi$ | Helmholtz Potential |
| $\mathbf{a} \cdot \mathbf{b}$ | Scalar Product between Two Elements $\mathbf{a}$, $\mathbf{b} \in \mathcal{V}$ |
| $\otimes$ | Tensor Product Operation |

| | |
|---|---|
| $\|\mathbf{A}\|$ | Trace Norm of the Linear Transformation $\mathbf{A}$ |
| $\mathbf{a} \times \mathbf{b}$ | Cross Product between Two Vector Fields $\mathbf{a}$, $\mathbf{b}$ over a Three-Dimensional Space |
| $\mathbf{1}$ | Identity Transformation |

# GLOSSARY

In this section keywords used in the development of the dissertation are given meaning.

| | |
|---|---|
| Big "O" | We have $h = O(g)$ as $y \to \hat{y}$ if there exists a $C$ s.t. $\|h(y)\| \leq Cg(y)$ for each $y$ sufficiently close to $\hat{y}$ ($C$-constant). |
| bvp4c | A finite difference code that uses the three-stage Lobatto IIIa formula, and the collocation polynomial provides a $C^1$[a,b]-continuous solution that is fourth-order accurate uniformly on the interval [a,b]. |
| Fortran | A general purpose, imperative programming language that is especially useful for numeric computation. |
| MATLAB | A program that integrates numerical analysis computation, signal processing, graphics in a user-friendly environment. |

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# 1. INTRODUCTION

Elasticity is perhaps one of the world's eldest topics. In fact, ancient Greeks undoubtably possessed some understanding of the subject by virtue of their elaborate columns and structures [1]. Leonardo Da Vinci tested the strength of copper wires through the (ingenious) use of a hopper/basket tensile test configuration, and Galileo, whom supposedly knew of Da Vinci's work through Cardan's notebooks [2], also executed tests on various materials and coined the phrase "absolute resistence to fracture" with regard to the strength of copper [3]. Of course, Robert Hooke was the first scientist to determine the one dimensional form of what we use today as linear elasticity [4]. In this work entitled "Ut Tensio se Vis," Hooke explains that the internal force in a spring should be proportional to the amount by which the spring is stretched. However, even in these early states of the development of linear elasticity, some persons understood that its use warranted many assumptions. For example, Antoine Parent realized that Hooke's Law can only be utilized in very special circumstances as he states (with regard to determining the tensile stresses in a beam) [4]:

*"We know that the distribution of stresses represented by two equal triangles is correct only so long as the material obeys Hooke's Law and it cannot be used for calculating the ultimate bending load L."*

The founders of what we would call the modern day field of elasticity bear the illustrious names of Navier, Green, Cauchy, St. Venant, Stokes, Airy and Rankine (amongst others). Not only did these titans derive the equations of motion, initially

This dissertation follows the style of *Computers and Mathematics with Applications*.

from a microscopic perspective, but they formulated the equations for special problems and created some of the most interesting methods for solution; such methods are still utilized even today [5]. In fact, the progression of mathematics in the $18^{th}$ and $19^{th}$ centuries is, in part, a result of great thinkers determined to find solutions to these equations. Moreover, sophisticated methods such as conformal mapping have been invaluable for solving plane stress and plane strain type problems [6]. However, even after conquering such feats, we still have such little understanding of this topic. For example, is an elastic body one which should return to its original shape on unloading or is it one in which there can be no dissipation in any process? These two notions are *not* the same. Furthermore, it is well known that Cauchy elasticity does not result in the same models as Green elasticity unless thermodynamical considerations such as the word assumption are employed. However, such a debate is not the purpose of this work as these ideas have been clearly addressed by Rajagopal [7].

Herein the aim is to elaborate on recent work presented in [7], [8] which shows that what one typically refers to as elasticity is far too narrow, and therefore so are the classes of models commonly used to describe the response of non-dissipative phenomena. Rajagopal and Srinivasa subsequently published work which showed that the state function (e.g., Helmholtz potential) depending on both the stress and strain leads not only to a larger class of models in elasticity, but also models capable of describing a wide range of responses. Furthermore, it was shown that, in keeping with classical thermodynamics, it is more convenient to use the Gibbs potential when the stress is involved in the constitutive formulation [8]. Since these interesting papers have been published, several authors have expanded this new field of elasticity. Farina et al. [9] use these types of models to capture the response of a body with a stretching threshold (see the bottom figure on page 19), and they develop a very interesting free boundary value problem. Rajagopal and Bustamante [10] showed that if one begins with

$$\boldsymbol{\varepsilon} = \mathbf{g}(\mathbf{T}), \tag{1.1}$$

then the traditional Airy's stress functions can still be utilized for solving plane stress and plane strain type problems, however, now one must contend with solving a non-linear equation. As such, the authors proceed to develop the weak formulation of the governing equation. More recently, Rajagopal and Walton [11] have used models of the form (1.1) to set up the classical anti-plane fracture model. They perform an asymptotic analysis of the solution near the crack tip to show that even though the stresses may tend to infinity, models such as (1.1), are capable of predicting small strains, which is entirely keeping with using the use of the linearized measure of strain; such is not possible in the classical theory [12].

The purpose of this work is to expand upon these ideas using the Gibbs potential and include the absolute temperature into the formulation. Such models are in fact capable of dissipation via conduction, however, we use the Clausius-Duhem form of the second law of thermodynamics to determine the restrictions on the Gibbs potential so that there is no dissipation due to deformation. It is shown that the stress and strain must satisfy a first-order, tensor-valued differential equation, which must be solved together with the balance equations, in general. For a special structure of the Gibbs potential, i.e., that associated with an isotropic body, we find that this equation can be integrated to show that the Hencky strain is derivable from this potential. On assuming that the displacement gradient is sufficently small, the Hencky strain reduces to the linearized measure of strain. Thus, on prescribing the Gibbs potential, one obtains a relationship of the form (1.1), which need not be invertible. Such a relationship is very interesting since it allows the non-linearities associated with the kinematics to be separated from those from the constitutive equation. Such a case is not possible within either Cauchy or Green elasticity as the assumption of small strain leads to Hooke's law in both cases, i.e., the linearization process causes the entire class of models to dissolve into a single model. Moreover, equations of the form (1.1) are capable of describing threshold type response as well as limiting strain behavior (see the figures at the bottom of page 19).

In order to set the stage for the development of the theory, we first review some basic kinematical ideas in Section 2. For the sake of completeness, the balance equations are derived in Section 3. Next, the (re)introduction of the Gibbs potential and the requirements we shall stipulate for it are addressed. The latter half of Section 4 shows how equations of the form (1.1) can be obtained under the assumption of small displacement gradient. Finally, in Section 5 we show how to apply these concepts to solve a meaningful boundary value problem which illustrates the efficacy of the framework. Lastly, we summarize this work and show how the current theory may be utilized to solve more problems for bodies undergoing large deformations. It is also shown that one may expand the ideas in this treatise in order to model dissipative phenomena.

# 2. KINEMATICS

Let $\mathbb{B}$ represent the abstract body and $\kappa_R(\mathbb{B})$ and $\kappa_t(\mathbb{B})$ denote the reference configuration and the current configuration at time $t \in [0, \infty)$, respectively. By a motion, we mean a one-to-one mapping $\boldsymbol{\chi}_{\kappa_R} : \kappa_R(\mathbb{B}) \times [0, \infty) \to \mathcal{E}$ such that [13]:

$$\mathbf{x} = \boldsymbol{\chi}_{\kappa_R}(\mathbf{X}, t), \quad \mathbf{X} \in \kappa_R(\mathbb{B}) \;\text{ and }\; \mathbf{x} \in \kappa_t(\mathbb{B}). \tag{2.1}$$

For the purposes of this work, we shall assume that the motion is sufficiently smooth so that all of the derivative operations have meaning. The displacement and velocity of the particle $\mathbf{X}$ are defined through the relations:

$$\mathbf{u}_{\kappa_R}(\mathbf{X}, t) = \boldsymbol{\chi}_{\kappa_R}(\mathbf{X}, t) - \mathbf{X} \quad \text{and} \quad \mathbf{v}_{\kappa_R}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}_{\kappa_R}}{\partial t}(\mathbf{X}, t) := \frac{d\boldsymbol{\chi}_{\kappa_R}}{dt}, \tag{2.2}$$

respectively, and the quantity $\dfrac{d(\cdot)}{dt}$ represents the usual material time derivative. The deformation gradient is defined in the usual manner:

$$\mathbf{F}_{\kappa_R}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}_{\kappa_R}}{\partial \mathbf{X}} = \mathbf{1} + \frac{\partial \mathbf{u}_{\kappa_R}}{\partial \mathbf{X}}. \tag{2.3}$$

Let us henceforth suppress the subscript $\kappa_R$ for the sake of convenience. Now, according to the Polar Decomposition theorem, the deformation gradient can decomposed as follows:

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.4}$$

where $\mathbf{R}$ is an orthogonal transformation and $\mathbf{U}$ and $\mathbf{V}$ are both positive definite and symmetric stretch tensors. We shall assume that $\det(\mathbf{F}_{\kappa_R}) > 0$, and therefore $\mathbf{R}$ represents a rotation and is also a unique transformation. The left Cauchy-Green stretch tensor is given by the following relationship:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 = \mathbf{1} + 2\boldsymbol{\varepsilon} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \left( \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \right)^T, \tag{2.5a}$$

respectively, where the linearized measure of strain is given by through the formula:

$$\boldsymbol{\varepsilon} := \frac{1}{2} \left[ \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left( \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \right)^T \right]. \tag{2.5b}$$

Since the motion is presumed to be one-to-one for each instant of time, the mapping (2.1) can be inverted so that:

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t), \tag{2.6}$$

and therefore any quantity of interest $h$ can be expressed in the following forms:

$$h = \hat{h}(\mathbf{X}, t) = \tilde{h}(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t) = \bar{h}(\mathbf{x}, t). \tag{2.7}$$

Note that we shall use lower case letters to denote differentiation executed with respect to $\mathbf{x}$, e.g., $\mathrm{div}(\cdot)$, and capital letters to signify that the derivative operation is carried out with respect to $\mathbf{X}$. The Eulerian gradient of the velocity is given by the formula:

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{d\mathbf{F}}{dt}\mathbf{F}^{-1}, \tag{2.8}$$

and the stretching tensor, i.e., the symmetric part of the velocity gradient, takes the form:

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) = \frac{1}{2}\left(\frac{d\mathbf{V}}{dt}\mathbf{V}^{-1} + \mathbf{V}^{-1}\frac{d\mathbf{V}}{dt} + \mathbf{V}\boldsymbol{\Omega}\mathbf{V}^{-1} - \mathbf{V}^{-1}\boldsymbol{\Omega}\mathbf{V}\right), \tag{2.9}$$

where $\boldsymbol{\Omega} := \frac{d\mathbf{R}}{dt}\mathbf{R}^T$ is a skew-symmetric transformation. We shall find in particularly convenient to write $\mathbf{D}$ in terms of the left stretch tensor as in the second part of equation (2.9). For the development of the constitutive theory we shall require the selection of a set of invariants. Thus, for any symmetric second order tensor $\mathbf{A}$, we choose the principal invariants of $\mathbf{A}$ to be given by the relations:

$$\mathrm{I}_{\mathbf{A}} := \mathrm{tr}(\mathbf{A}), \quad \mathrm{II}_{\mathbf{A}} := \mathrm{tr}(\mathbf{A}^2), \quad \mathrm{III}_{\mathbf{A}} := \det(\mathbf{A}). \tag{2.10}$$

One identity, which shall be used many times herein, involves the third invariant and shows how volume elements in the reference configuration transform when the points are mapped to the current configuration, and this relationship is given be the equation [14]:

$$dv = \mathrm{III}_{\mathbf{F}} dV. \tag{2.11}$$

We should note that while this set of invariants is entirely sound from a mathematical perspective, recent evidence shows that (systematic) error is greatly exxacerbated by the fact that this set does not form an orthonormal integrity basis [15]. Nevertheless, we shall use the proposed set of invariants for the development of the theory as the implementation of the set of invariants presented in [15] is relatively straightforward. We shall also require a Galilean invariant derivative in the work to follow, and there are many to choose from. Herein we work with the rate $\frac{d\mathbf{A}^*}{dt}$ which is given by the formula:

$$\frac{d\mathbf{A}^*}{dt} := \mathbf{R}^T \left( \frac{d\mathbf{A}}{dt} - \mathbf{\Omega}\mathbf{A} + \mathbf{A}\mathbf{\Omega} \right) \mathbf{R}. \tag{2.12}$$

These are the only kinematical quanitites which will be used in the development of the constitutive equations. In the next section we shall use these ideas to develop the balance laws that we shall assume, for the purpose of this work, must hold in any process.

# 3. BALANCE EQUATIONS

Here we develop the balance equations which we shall assume must hold for all processes. The balance of mass, linear and angular momentum, and energy are developed in integral form for an arbitrary subpart of the body and after some trivial manipulations, requiring continuity of the integrands yields the local form of the balance rules. Furthermore, the second law of thermodynamics in the Clausis-Duhem form is given but later on reintroduced in a slighly different form.

## 3.1 Balance of Mass

The statement that the total mass $m$ of a subpart $P_R$ must be the same between the reference and current configuration is given by:

$$m = \int_{P_R} \varrho_R dV = \int_{P_t} \varrho dv = \int_{P_R} \varrho \det(\mathbf{F}) dV, \tag{3.1a}$$

where $\varrho$ and $\varrho_R$ denote the densities in the current and reference configuration, respectively, and we have used the identity (2.11). On collecting like terms from $(3.1a)_1$ and $(3.1a)_3$ and requiring continuity of the integrand we find:

$$\varrho_R = \varrho \det(\mathbf{F}). \tag{3.1b}$$

Thus, if the material is incompressible, then we must satisfy $\det(\mathbf{F}) = 1$ for each instant of time, and this implies that $\varrho = \varrho_R$. On taking the material time derivative of equation (3.1b), using the identity $\frac{d}{dt}[\det(\mathbf{F})] = \mathrm{div}(\mathbf{v})\det(\mathbf{F})$, and assuming that $\det(\mathbf{F}) \neq 0$, we obtain the other commonly used form of mass balance:

$$\frac{d}{dt}\left[\varrho \det(\mathbf{F})\right] = 0 \Rightarrow \frac{d\varrho}{dt} + \varrho\,\mathrm{div}(\mathbf{v}) = 0. \tag{3.1c}$$

## 3.2 Balance of Linear and Angular Momentum

We shall use Euler's axiom to develop the balance of linear momentum which states that the rate of change of linear momentum equals the total force acting on the body. Thus, we can write the balance of linear in the following form:

$$\frac{d}{dt}\int_{P_t} \varrho \mathbf{v} dv = \int_{\partial P_t} \mathbf{t_n} da + \int_{P_t} \varrho \mathbf{b} dv, \tag{3.2a}$$

where $\mathbf{t_n}$ is the (simple) traction vector acting on a surface with unit normal vector $\mathbf{n}$, and $\mathbf{b}$ is the specific body force. We fix the configuration in order to bring the material time derivative to the integrand by transforming the integral to one over a volume measure in the reference configuration by once again utilizing the identity (2.11) and also balance of mass (3.1$c$):

$$\begin{aligned}\frac{d}{dt}\int_{P_t} \varrho \mathbf{v} dv = \frac{d}{dt}\int_{P_R} \varrho \mathbf{v} \det(\mathbf{F}) dV &= \int_{P_R} \left\{ \varrho \frac{d\mathbf{v}}{dt}\det(\mathbf{F}) + \frac{d}{dt}\left[\varrho \det(\mathbf{F})\right] \right\} dV \\ &= \int_{P_t} \varrho \frac{d\mathbf{v}}{dt} dv. \end{aligned} \tag{3.2b}$$

Furthermore, we can transform the integral over the surface to one over the volume by using the divergence theorem:

$$\int_{\partial P_t} \mathbf{t_n} da = \int_{\partial P_t} \mathbf{T}^T \mathbf{n} da = \int_{P_t} \operatorname{div}(\mathbf{T}) dv. \tag{3.2c}$$

On inserting (3.2b) and (3.2c) into equation (3.2a), and requiring continuity we find:

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div}(\mathbf{T}) + \varrho \mathbf{b}. \tag{3.2d}$$

Euler's axiom for the balance of angular momentum requires that the rate of change of angular momentum equal the total moment force acting on the body, i.e.:

$$\frac{d}{dt}\int_{P_t} \mathbf{x} \times \varrho \mathbf{v} dv = \int_{\partial P_t} \mathbf{x} \times \mathbf{t_n} da + \int_{P_t} \mathbf{x} \times \varrho \mathbf{b} dv. \tag{3.3a}$$

On making manipulations similar to those in (3.2b) and (3.2c) we find:

$$\begin{aligned}\frac{d}{dt}\int_{P_t} \mathbf{x} \times \varrho \mathbf{v} dv &= \int_{P_R} \left\{ \mathbf{v} \times \varrho \mathbf{v} \det(\mathbf{F}) + \mathbf{x} \times \varrho \frac{d\mathbf{v}}{dt}\det(\mathbf{F}) + \mathbf{x} \times \mathbf{v}\frac{d}{dt}\left[\varrho \det(\mathbf{F})\right] \right\} dV \\ &= \int_{P_t} \mathbf{x} \times \varrho \frac{d\mathbf{v}}{dt} dv, \end{aligned} \tag{3.3b}$$

and

$$\int_{\partial P_t} \mathbf{x} \times \mathbf{t_n} da = \int_{\partial P_t} \mathbf{x} \times \mathbf{T}^T \mathbf{n} da = -\int_{P_t} \mathbf{w} dv + \int_{P_t} \mathbf{x} \times \text{div}(\mathbf{T}) dv, \tag{3.3c}$$

respectively, where $\mathbf{w}$ is the axial vector of $\mathbf{T} - \mathbf{T}^T$, i.e., $\mathbf{w}$ satisfies:

$$(\mathbf{T} - \mathbf{T}^T)\mathbf{p} = \mathbf{w} \times \mathbf{p}, \tag{3.3d}$$

$\forall$ fixed $\mathbf{p} \in \mathcal{V}$. On combining the last terms in (3.3a), (3.3b), and (3.3c), and utilizing the balance of linear momentum (3.2d), we find:

$$\int_{P_t} \mathbf{x} \times \left[ \varrho \frac{d\mathbf{v}}{dt} - \text{div}(\mathbf{T}) - \varrho \mathbf{b} \right] dv = \mathbf{0}. \tag{3.3e}$$

Thus, we have the equation $-\int_{P_t} \mathbf{w} dv = \mathbf{0}$. On taking the cross product with $\mathbf{p}$ and utilizing the anti-communtative property of the product we have:

$$\int_{P_t} \mathbf{w} \times \mathbf{p} dv = \int_{P_t} (\mathbf{T} - \mathbf{T}^T)\mathbf{p} dv = \mathbf{0}. \tag{3.3f}$$

Thus, we have symmetry in the Cauchy stress:

$$\mathbf{T} = \mathbf{T}^T \tag{3.3g}$$

since (3.3f) must hold for all $\mathbf{p} \in \mathcal{V}$. Note that we have assumed that the body is free of body couples.

## 3.3 Balance of Energy and The Second Law of Thermodynamics

Finally, we derive the balance of energy. The rate of change of the total energy of a subpart of the body is equal to the sum of the following quantities: the rate at which work is done by tractions on the surface, the total heat flux at the surface, energy in thermal form (heat) supplied to the body, and the rate at which the body force does work. We shall assume the total energy of the body is composed of two parts: the potential energy associated with that stored in atomic bonds and kinetic energy of the motion. Thus we have:

$$\frac{d}{dt} \int_{P_t} \left( \varepsilon + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) \varrho dv = \int_{\partial P_t} \mathbf{t_n} \cdot \mathbf{v} da - \int_{\partial P_t} \mathbf{q} \cdot \mathbf{n} da + \int_{P_t} \varrho r dv + \int_{P_t} \varrho \mathbf{b} \cdot \mathbf{v} dv, \tag{3.4a}$$

where $\varepsilon$ is the internal energy, $r$ is the external heating (both per unit mass), and $\mathbf{q}$ is the heat flux vector. We once again transform the integral so that the time derivative can be calculated for the integrand:

$$\begin{aligned}
&\frac{d}{dt}\int_{P_t}\left(\varepsilon+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\varrho dv=\frac{d}{dt}\int_{P_R}\left(\varepsilon+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\varrho\det(\mathbf{F})dV\\
&=\int_{P_R}\left\{\left(\frac{d\varepsilon}{dt}+\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\right)\varrho\det(\mathbf{F})+\left(\varepsilon+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\frac{d}{dt}\Big[\varrho\det(\mathbf{F})\Big]\right\}dV\\
&=\int_{P_t}\varrho\frac{d\varepsilon}{dt}dv+\int_{P_t}\varrho\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dv.
\end{aligned}\tag{3.4b}$$

Also, we use the divergence theorem and the associative property of the scalar product to find:

$$\int_{\partial P_t}\mathbf{q}\cdot\mathbf{n}da=\int_{P_t}\mathrm{div}(\mathbf{q})dv\tag{3.4c}$$

and

$$\begin{aligned}
&\int_{\partial P_t}\mathbf{t}_{\mathbf{n}}\cdot\mathbf{v}da=\int_{\partial P_t}\mathbf{T}^T\mathbf{n}\cdot\mathbf{v}da=\int_{\partial P_t}\mathbf{T}\mathbf{v}\cdot\mathbf{n}da=\int_{P_t}\mathrm{div}(\mathbf{T}\mathbf{v})dv\\
&=\int_{P_t}\mathbf{T}\cdot\frac{\partial\mathbf{v}}{\partial\mathbf{x}}dv+\int_{P_t}\mathrm{div}(\mathbf{T})\cdot\mathbf{v}dv=\int_{P_t}\mathbf{T}\cdot\mathbf{D}dv+\int_{P_t}\mathrm{div}(\mathbf{T})\cdot\mathbf{v}dv.
\end{aligned}\tag{3.4d}$$

Note that the last term of (3.4a), (3.4b), and (3.4d) can be collected and on using the balance of linear momentum (3.2d) we find:

$$\int_{P_t}\mathbf{v}\cdot\left[\varrho\frac{d\mathbf{v}}{dt}-\mathrm{div}(\mathbf{T})-\varrho\mathbf{b}\right]dv=0.\tag{3.4e}$$

Thus, the balance of energy is given by:

$$\varrho\frac{d\varepsilon}{dt}=\mathbf{T}\cdot\mathbf{D}-\mathrm{div}(\mathbf{q})+\varrho r.\tag{3.4f}$$

The second law of thermodynamics is also utilized herein. We shall require the strong assumption that it is locally valid and utilize the $2^{nd}$ law of thermodynamics in the Clausius-Duhem local form [16]:

$$\mathbf{T}\cdot\mathbf{D}-\varrho\frac{d\varepsilon}{dt}+\varrho\theta\frac{d\eta}{dt}-\frac{\mathbf{q}}{\theta}\cdot\frac{\partial\theta}{\partial\mathbf{x}}=\varrho\theta\zeta:=\xi\geq 0,\tag{3.4g}$$

where $\theta$ represents the absolute temperature, $\eta$ the entropy, $\zeta$ the rate of entropy production, and $\xi$ the rate of dissipation. In the next section, we shall reintroduce various energy storage mechanisms, and this will lead to a restatement of this law in a slightly different form.

# 4. IMPLICIT CONSTITUTIVE EQUATIONS

## 4.1 The Gibbs Potential

As stated in the introduction, much of the constitutive development herein follows that of Rajagopal et al. [8]. However, we make a slight generalization in that the temperature is injected into the formulation, and the work is then extended in the following section through a linearization process. Let us first begin this generalization by introducing the Gibbs potential $\Phi$. Within the context of classical thermodynamics, the relevant intensive "properties" of the system are the pressure and the density when one chooses to use the Gibbs potential. A natural generalization of this idea is to allow the Gibbs potential to depend upon the Cauchy stress $\mathbf{T}$ and the density $\varrho$ along with any internal variables appropriate for the investigation under consideration; we need not incorporate any internal variables in the current study. The absolute temperature $\theta$ is also incorporated into the formulation as it is well accepted that the moduli of many materials depend upon the temperature. Thus, we shall the assume the Gibbs potential has the following form:

$$\Phi = \bar{\Phi}(\mathbf{T}, \varrho, \theta). \tag{4.1}$$

We shall assume that $\bar{\Phi}$ must be invariant under Galilean transformations which implies:

$$\bar{\Phi}(\mathbf{T}, \varrho, \theta) = \bar{\Phi}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T, \varrho, \theta), \tag{4.2}$$

where the Galilean transformation $\mathbf{Q}$ belongs to the full orthogonal group. Equation (4.2) reveals that, at most, $\bar{\Phi}$ has the form:

$$\Phi = \bar{\Phi}(\mathrm{I}_{\mathbf{T}}, \mathrm{II}_{\mathbf{T}}, \mathrm{III}_{\mathbf{T}}, \varrho, \theta). \tag{4.3}$$

Obviously the relationship (4.3) implies that the body is isotropic (with respect to the current configuration), and therefore we obtain the same result of Rajagopal et al.: A generalization of the starting point (4.1) is warranted in order to develop models which are capable of describing the response of anisotropic bodies.

In order to devise a richer class of models which have the ability to capture the response of anisotropic bodies, we shall allow (4.1) to also depend upon the deformation gradient $\mathbf{F}$. As before, we shall stipulate that the Gibbs potential must satisfy Galilean invariance. Thus, on noting that the deformation gradient transforms as $\mathbf{QF}$ under Galilean transformations, we find the Gibbs potential must satisfy the relation:

$$\Phi = \bar{\Phi}(\mathbf{T}, \mathbf{F}, \varrho, \theta) = \grave{\Phi}(\mathbf{QTQ}^T, \mathbf{QF}, \varrho, \theta). \tag{4.4}$$

On choosing the orthogonal transformation to be the transpose of the rotation tensor from the decomposition of the deformation gradient, i.e., $\mathbf{Q} = \mathbf{R}^T$, we can restate (4.4) as follows:

$$\Phi = \tilde{\Phi}(\mathbf{T}^*, \mathbf{V}^*, \varrho, \theta), \tag{4.5}$$

where $\mathbf{T}^* := \mathbf{R}^T\mathbf{TR}$ and $\mathbf{V}^*$ is defined in a similar fashion. Note that $\tilde{\Phi}$ will depend on the (separate and joint) invariants of $\mathbf{T}^*$ and $\mathbf{V}^*$, and the set of invariants which $\tilde{\Phi}$ shall depend upon will depend on the nature of the anisotropy. At this point, we shall make assumptions regarding the structure of the Gibbs potential $\Phi$:

- We assume that the Gibbs potential is identically zero when the stress is zero.
- We further assume that as the body tends to a stress-free state, the Gibbs potential is a smooth convex function of the stress. This implies that $\Phi$, as well as its first derivative with respect to $\mathbf{T}$, tend to zero as $\mathbf{T}$ tends to $\mathbf{O}$ while the second derivative is positive definite.

It is easy to verify that the constitutive equation (4.5) can now be stated in the form:

$$\Phi_{\mathrm{g}} = \|\mathbf{T}^*\|^2 \mathrm{g}(\mathbf{T}^*, \mathbf{V}^*, \varrho, \theta), \tag{4.6}$$

where $\|\cdot\|$ denotes the Frobenius norm. Note the similarity in structure to that used in [8]. Also, note that g must also be finite as $\|\mathbf{T}^*\|$ tends to zero. Thus, our starting point here is with the constitutive assumption (4.6) for the Gibbs potential instead of the Helmholtz potential $\Psi$ as in the case of Hyperelasticity (Green elasticity), and we shall illustrate the efficacy with such an approach with examples in the next section.

Now, within the realm of Hyperelasticity, one usually introduces the relationship between the internal energy, the Helmholtz potential, the temperature, and the entropy via the Legendre transformation:

$$\varepsilon = \Psi - \theta\eta, \tag{4.7}$$

and the $2^{nd}$ law of thermodynamics becomes:

$$\mathbf{T}\cdot\mathbf{D} - \varrho\frac{d\Psi}{dt} - \varrho\eta\frac{d\theta}{dt} - \frac{\mathbf{q}}{\theta}\cdot\frac{\partial\theta}{\partial\mathbf{x}} = \xi \geq 0. \tag{4.8}$$

We shall follow the same process, and note that the Helmholtz and Gibbs potentials are connected through the relationship:

$$\Psi := \frac{\partial\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*}\cdot\mathbf{T}^* - \Phi_{\mathrm{g}}. \tag{4.9}$$

Since equation (4.8) involves the total time derivative of the Helmholtz potential, ultimately our goal is to calculate this rate in terms of the Gibbs potential. Thus, on utilizing equations (4.6) and (4.9), we find the following relationships:

$$\frac{d\Psi}{dt} = \frac{d}{dt}\left(\frac{\partial\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*}\right)\cdot\mathbf{T}^* + \frac{\partial\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*}\cdot\frac{d\mathbf{T}^*}{dt} - \frac{d\Phi_{\mathrm{g}}}{dt}, \tag{4.10a}$$

where

$$\frac{d}{dt}\left(\frac{\partial\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*}\right) := \frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\mathbf{T}^*}\frac{d\mathbf{T}^*}{dt} + \frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\mathbf{V}^*}\frac{d\mathbf{V}^*}{dt} + \frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\varrho}\frac{d\varrho}{dt} + \frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\theta}\frac{d\theta}{dt}, \tag{4.10b}$$

and

$$\frac{d\Phi_{\mathrm{g}}}{dt} := \frac{\partial\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*}\cdot\frac{d\mathbf{T}^*}{dt} + \frac{\partial\Phi_{\mathrm{g}}}{\partial\mathbf{V}^*}\cdot\frac{d\mathbf{V}^*}{dt} + \frac{\partial\Phi_{\mathrm{g}}}{\partial\varrho}\frac{d\varrho}{dt} + \frac{\partial\Phi_{\mathrm{g}}}{\partial\theta}\frac{d\theta}{dt}. \tag{4.10c}$$

On combining (4.10a), (4.10b), and (4.10c) we find that the total time derivative of $\Psi$ is given by the equation:

$$\begin{aligned}\frac{d\Psi}{dt} &= \left(\frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\mathbf{T}^*}\frac{d\mathbf{T}^*}{dt} + \frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\mathbf{V}^*}\frac{d\mathbf{V}^*}{dt} + \frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\varrho}\frac{d\varrho}{dt} + \frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\theta}\frac{d\theta}{dt}\right)\cdot\mathbf{T}^* \\ &- \left(\frac{\partial \mathrm{g}}{\partial\mathbf{V}^*}\cdot\frac{d\mathbf{V}^*}{dt} + \frac{\partial \mathrm{g}}{\partial\varrho}\frac{d\varrho}{dt} + \frac{\partial \mathrm{g}}{\partial\theta}\frac{d\theta}{dt}\right)\|\mathbf{T}^*\|^2. \qquad (4.10\text{d})\end{aligned}$$

On substituting equation (4.10d) into the $2^{nd}$ law of thermodynamics (4.8), and utilizing the balance of mass in the Eulerian form (3.1c) we obtain[§]:

$$\begin{aligned}&\left[\mathbf{V}^{*-1}\frac{d\mathbf{V}^*}{dt} - \varrho\frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^{*2}}\frac{d\mathbf{T}^*}{dt} - \varrho\frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\mathbf{V}^*}\frac{d\mathbf{V}^*}{dt} + \varrho^2\frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\varrho}\mathrm{tr}\left(\mathbf{V}^{*-1}\frac{d\mathbf{V}^*}{dt}\right)\right. \\ &\left. + \varrho\left(\mathbf{T}^*\otimes\frac{\partial \mathrm{g}}{\partial\mathbf{V}^*}\right)\frac{d\mathbf{V}^*}{dt} - \varrho^2\frac{\partial \mathrm{g}}{\partial\varrho}\left(\mathbf{T}^*\otimes\mathbf{1}\right)\mathbf{V}^{*-1}\frac{d\mathbf{V}^*}{dt}\right]\cdot\mathbf{T}^* \\ &- \varrho\left(\eta + \mathbf{T}^*\cdot\frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\theta} - \|\mathbf{T}^*\|^2\frac{\partial \mathrm{g}}{\partial\theta}\right)\frac{d\theta}{dt} - \frac{\mathbf{q}}{\theta}\cdot\frac{\partial\theta}{\partial\mathbf{x}} = \xi. \qquad (4.11)\end{aligned}$$

As stated previously, we are interested in developing models for bodies which are elastic in the sense that the only dissipation mechanism present is that associated with the conduction in the material, i.e.:

$$\xi = \xi_c = -\frac{\mathbf{q}}{\theta}\cdot\frac{\partial\theta}{\partial\mathbf{x}}. \qquad (4.12\text{a})$$

Thus, *sufficient conditions* under which equation (4.11) is satisfied are i) the total time derivative of the left stretch tensor is given in terms of the Gibbs potential by the following relationship:

$$\begin{aligned}\frac{d\mathbf{V}^*}{dt} = \varrho\mathbf{V}^*&\left[\frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^{*2}}\frac{d\mathbf{T}^*}{dt} + \frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\mathbf{V}^*}\frac{d\mathbf{V}^*}{dt} - \varrho\frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\varrho}\mathrm{tr}\left(\mathbf{V}^{*-1}\frac{d\mathbf{V}^*}{dt}\right)\right. \\ &\left. - \left(\mathbf{T}^*\otimes\frac{\partial \mathrm{g}}{\partial\mathbf{V}^*}\right)\frac{d\mathbf{V}^*}{dt} + \varrho\frac{\partial \mathrm{g}}{\partial\varrho}\left(\mathbf{T}^*\otimes\mathbf{1}\right)\mathbf{V}^{*-1}\frac{d\mathbf{V}^*}{dt}\right], \qquad (4.12\text{b})\end{aligned}$$

and ii) the entropy is prescribed in the following manner:

$$\eta = -\mathbf{T}^*\cdot\frac{\partial^2\Phi_{\mathrm{g}}}{\partial\mathbf{T}^*\partial\theta} + \|\mathbf{T}^*\|^2\frac{\partial \mathrm{g}}{\partial\theta} = -\frac{\partial\Psi}{\partial\theta}. \qquad (4.12\text{c})$$

[§]Note that the transitive property of the scalar product along with the symmetry of the linear transformation **D** have been used repeatedly in this calculation.

Note that equation (4.12c) falls in line with the classical theory. Furthermore, the relationship (4.12b) can be written in the more compact form:

$$\mathcal{A}\frac{d\mathbf{T}^*}{dt} + \mathcal{B}\frac{d\mathbf{V}^*}{dt} = \mathbf{O}, \tag{4.13}$$

where $\mathcal{A}$ and $\mathcal{B}$ are fourth order tensor valued operators. In the following sections, we wish to explore special circumstances in which the abstract equation (4.13) is transformed to a more amenable form in order to understand the nature of the types of constitutive equations we are capable of developing.

### 4.2 Linearization of a Solution to the Rate Equation

Equation (4.13) is an implicit relationship between the stress and stretch, and on prescribing the constitutive equation $\Phi$, the integrability conditions can be analyzed given appropriate initial values. This task along with the resolution of an appropriate weak formulation of the equations (3.1b), (3.2d), and (3.4f) in conjunction with the constitutive equation (4.13) is an open area of research; however, such an issue is not the crux of this work. We wish to investigate the various classes of models that are possible within this framework, and more importantly, the phenomena that they are capable of describing. In particular, we shall recount the linearization process used to develop Hooke's law using Hyperelasticity as a starting point and compare the result to the same obtained from beginning with the Gibbs potential, wherein the stress takes center stage. So that these ideas can be made more clear, we shall assume that the body is isotropic, initially of uniform temperature, and undergoes an isotermal process. Now, if these assumptions are made for the Helmholtz potential, as is the case in Hyperelasticity, and the $2^{nd}$ law of thermodynamics is utilized, one finds that the Cauchy stress can be derived from the Helmholtz potential via the formula [13]:

$$\mathbf{T} = \phi_0\mathbf{1} + \phi_1\mathbf{B} + \phi_2\mathbf{B}^2, \tag{4.14}$$

where

$$\phi_0 = \mathrm{III}_{\mathbf{B}}\frac{\partial\Psi}{\partial\mathrm{III}_{\mathbf{B}}}, \quad \phi_1 = 2\left(\frac{\partial\Psi}{\partial\mathrm{I}_{\mathbf{B}}} + \mathrm{I}_{\mathbf{B}}\frac{\partial\Psi}{\partial\mathrm{II}_{\mathbf{B}}}\right), \quad \phi_2(\mathrm{I}_{\mathbf{B}}, \mathrm{II}_{\mathbf{B}}, \mathrm{III}_{\mathbf{B}}) = -2\frac{\partial\Psi}{\partial\mathrm{II}_{\mathbf{B}}}. \quad (4.15)$$

Note that equation (4.14) can be written in terms of $\mathbf{V}$ in view of the relationship (2.5a). Now, if we specify *any* form of $\Psi$ (i.e., any function of the principal invariants of $\mathbf{B}$), make the assumption that the trace-norm of the displacement gradient is small, i.e.,

$$\max_{\mathbf{X}\in\kappa_R(\mathbb{B}),t\in R}\left\|\frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right\| = \mathrm{O}(\delta), \delta \ll 1, \quad (4.16)$$

and finally neglect terms $\mathbf{O}(\delta^2)$, then we obtain the relationship for a linearly elastic solid:

$$\mathbf{T} = \lambda\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}, \quad (4.17a)$$

where the Lamé constants are given as follows:

$$\lambda := 2\left(\frac{\partial}{\partial\mathrm{I}_{\mathbf{B}}} + \frac{\partial}{\partial\mathrm{II}_{\mathbf{B}}} + \frac{\partial}{\partial\mathrm{III}_{\mathbf{B}}}\right)(\phi_0 + \phi_1 + \phi_2)\bigg|_{(3,3,1)} \quad (4.17b)$$

and

$$\mu := \phi_1(3,3,1) + 2\phi_2(3,3,1). \quad (4.17c)$$

The problem with such an approach is that is does not allow for the separation of non-linearities associated wtih the kinematics from those of the constitutive equation, i.e., if one assumes that the displacement gradient (strain) is small, then this automatically leads to a linear relationship between the stress and strain. It is well known that such a constitutive relation is incapable of describing many phenomena, even when the body remains in the elastic regime. For example, Inglis [12] was the first to study the boundary value problem of a plate with a centered crack, and he found that on applying even the smallest load that one could imagine, the stress blows up at the crack tip. Not only is an infinite stress at the crack tip physically untenable, but if the stress is infinite then so must be the strain, which contradicts our *a priori* assumption that the displacement gradient is small. Furthermore, Hooke's Law is incapable of describing the limiting or threshold type behavior illustrated in

Figures 4.1 and 4.2. Thus, let us use the Gibbs potential to illustrate the method for developing models which are capable of addressing such type of behavior. Before we go forward, let us make a note about figures 4.1 and 4.2. Of course, one familiar with the tensile testing of metals until failure would be quite confused by Figure 4.2. However, we must remember that we are considering bodies that remain in the elastic regime, and the linearity of the load-extension data would be characterized in the $\varepsilon \leq \varepsilon_{1d}$ portion of the plot. The region of the plot with $\varepsilon \geq \varepsilon_{1d}$ is the response one would expect in the area of stress concentration, e.g., a crack.

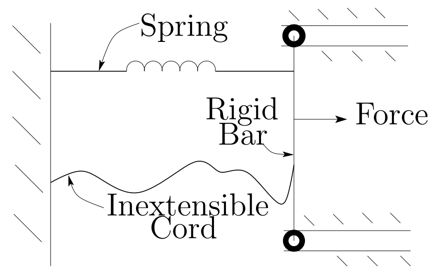


Figure 4.1. One-dimensional mechanical analog depicting the threshold type behavior which is capable within the proposed framework.

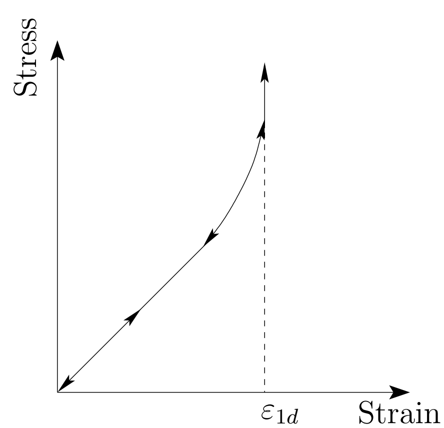


Figure 4.2. Cartoon illustrating the relationship between the one-dimensional stress and one-dimensional linearized strain.

We shall make the same assumptions as we did for the Helmholtz potential, but write the Gibbs potential in terms of the principal invariants of the stress:

$$\Phi = \Phi_i(\mathrm{I}_{\mathbf{T}}, \mathrm{II}_{\mathbf{T}}, \mathrm{III}_{\mathbf{T}}). \tag{4.18}$$

On using the relationship between the Helmholtz and Gibbs potential (4.9), calculating the material time derivative:

$$\frac{d\Psi}{dt} = \frac{d}{dt}\left(\frac{\partial \Phi_i}{\partial \mathbf{T}}\right) \cdot \mathbf{T}, \tag{4.19}$$

and substituting the result into the $2^{nd}$ law of thermodynamics we obtain:

$$\left[\mathbf{V}^{-1}\frac{d\mathbf{V}}{dt} - \frac{d}{dt}\left(\frac{\partial \Phi_i}{\partial \mathbf{T}}\right)\right] \cdot \mathbf{T} = \xi. \tag{4.20}$$

Thus, a *sufficient* condition that there be no dissipation during the process is:

$$\ln(\mathbf{V}) - \ln(\mathbf{V}_0)^{\P} = \frac{\partial \Phi_i}{\partial \mathbf{T}} - \left.\frac{\partial \Phi_i}{\partial \mathbf{T}}\right|_0, \tag{4.21}$$

where $\left.\frac{\partial \Phi_i}{\partial \mathbf{T}}\right|_0$ is the free energy associated with the strain in the elastic body in the reference configuration. If we assume that the body is free of strain in the reference configuration, then we have:

$$\mathbf{E}_H := \ln(\mathbf{V}) = \frac{\partial \Phi_i}{\partial \mathbf{T}}, \tag{4.22}$$

where $\mathbf{E}_H$ is the Hencky strain. Finally, if we assume that the displacement gradient is small as before:

$$\ln(\mathbf{V}) = \ln\left[\sqrt{\mathbf{1} + 2\boldsymbol{\varepsilon} + \mathbf{O}(\delta^2)}\right] = \ln\left[\mathbf{1} + \boldsymbol{\varepsilon} + \mathbf{O}(\delta^2)\right] = \boldsymbol{\varepsilon} + \mathbf{O}(\delta^2), \tag{4.23}$$

and neglect small terms, we obtain the following asymptotic relationship:

$$\ln(\mathbf{V}) \sim \boldsymbol{\varepsilon} = \frac{\partial \Phi_i}{\partial \mathbf{T}}. \tag{4.24}$$

¶The natural logarithm "ln" of the stretch tensor $\mathbf{V}$ exists since this linear transformation is symmetric and therefore diagonalizable in the spectral basis. Also note that the symmetry of this tensor has been exploited in this calculation.

Note that on specifying $\Phi_i$, we obtain a relationship wherein the linearized measure of strain is given by a non-linear function of stress, i.e., the linearization process does not cause the entire class of models to dissolve into a single model. One can clearly see that the models that can be obtained under equation (4.24) are capable of describing a wide range of responses. For example, one can develop models so that on solving a classical problem of a plate with a crack, the stress can be as large as need be at the crack tip, but the strain remains small which is completely consistent with the framework presented here. There have been models proposed in the past wherein the strain is written explicitly as a function of the stress, however, no general framework has been put in place to develop such models and they are typically a result of fitting stress-strain data from tensile test type experiments (initially by G. F. Bülfinger [4] and later by Ramberg and Osgood [17]). Perhaps even more important is that the author's purpose for the models is to describe the stress-strain curve for certain metals undergoing elastic-plastic deformations which is in vain since we have shown here that such models fall within the realm of elasticity and as such are incapable of describing dissipative phenomena such as plasticity.

# 5. A BOUNDARY VALUE PROBLEM

In order to make more concrete the abstract ideas of the preceding sections, we shall consider a number of different applications of the framework that has been put in place. First, let us consider a simple, yet insightful, boundary value problem with not only the purpose of illustrating the efficacy of the type of models which can be derived in the proposed framework, but also to show how one may go about solving classical plane stress type problems with these models wherein the (linearized) strain is obtained as a function of the stress.

## 5.1 Plane Stress Formulation of a Loaded Annulus

In this subsection, we shall investigate the response of a body which is characterized by the constitutive equation:

$$\Phi_i(\mathbf{T}) = \frac{\alpha\mu}{\gamma^2}\left[\sqrt{1+\gamma^2\mathrm{tr}(\mathbf{T}^2)} - 1\right] + \frac{\alpha}{\beta}\Big\{\ln\big[1+\beta\mathrm{tr}(\mathbf{T})\big] - \beta\mathrm{tr}(\mathbf{T})\Big\}, \tag{5.1}$$

where $\alpha$, $\beta$, and $\gamma$ are non-negative constants. Thus, on using the formula (4.24) we have:

$$\boldsymbol{\varepsilon} = \alpha\left[\frac{\mu\mathbf{T}}{\sqrt{1+\gamma^2\mathrm{tr}(\mathbf{T}^2)}} - \frac{\beta\mathrm{tr}(\mathbf{T})\mathbf{1}}{1+\beta\mathrm{tr}(\mathbf{T})}\right]. \tag{5.2}$$

We shall study the problem illustrated in Figure 5.1 and therefore assume a special structure for the Cauchy stress:

$$\mathbf{T} = T_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{r\theta}(r)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + T_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta. \tag{5.3}$$

Since assumptions have been made for the stress, and therefore the linearized strain through (5.2), we must satisfy the compatibility equation which for the special deformation under consideration reduces to:

$$\frac{d}{dr}\left(r\varepsilon_{\theta\theta}\right) = \varepsilon_{rr}. \tag{5.4a}$$

On utilizing the constitutive relation (5.2), we find the compatibility equation can

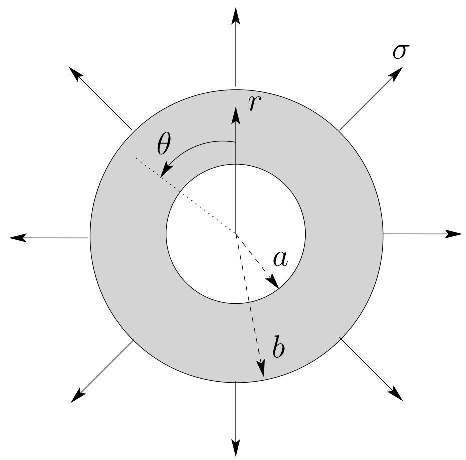


Figure 5.1. Schematic of the loaded cylindrical annulus along with a polar coordinate system and the dimension labels relevant to the boundary value problem under consideration.

be written in terms of the stresses in the following manner:

$$\frac{d}{dr}\left\{r\left[\frac{\mu T_{\theta\theta}}{\sqrt{d_2}} - \frac{\beta(T_{rr}+T_{\theta\theta})}{d_1}\right]\right\} = \frac{\mu T_{rr}}{\sqrt{d_2}} - \frac{\beta(T_{rr}+T_{\theta\theta})}{d_1}, \tag{5.4b}$$

where

$$d_1 := 1 + \beta(T_{rr}+T_{\theta\theta}), \tag{5.4c}$$

and

$$d_2 := 1 + \gamma^2(T_{rr}^2 + T_{\theta\theta}^2 + 2T_{r\theta}^2). \tag{5.4d}$$

We must also satisfy balance of linear momentum, which on application of equation (5.3) and neglect of the body force and acceleration terms, reduces to:

$$\frac{dT_{rr}}{dr} + \frac{T_{rr}-T_{\theta\theta}}{r} = 0, \tag{5.5a}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 T_{r\theta}\right) = 0. \tag{5.5b}$$

We briefly note that equation (5.5b) implies that if there is no shear applied at the boundary, then no shear can exist in the interior and therefore $T_{r\theta} \equiv 0$. Thus, equations (5.4b) and (5.5a) constitute a system of non-linear differential equations for the two normal stresses $T_{rr}$ and $T_{\theta\theta}$. Eliminating the hoop stress $T_{\theta\theta}$ from the equations, we find $T_{rr}$ is governed by the single non-linear differential equation:

$$\frac{d}{dr}\left[\frac{\mu}{\sqrt{d_2}}\left(r\frac{dT_{rr}}{dr}+T_{rr}\right)-\frac{\beta}{d_1}\left(r\frac{dT_{rr}}{dr}+2T_{rr}\right)\right]$$
$$=\frac{\mu T_{rr}}{\sqrt{d_2}}-\frac{\beta}{d_1}\left(r\frac{dT_{rr}}{dr}+2T_{rr}\right), \tag{5.6}$$

which through Cauchy's theorem is subject to the following two boundary conditions:

$$T_{rr}(a)=0 \quad \text{and} \quad T_{rr}(b)=\sigma. \tag{5.7}$$

If we introduce the following non-dimensional variables:

$$\hat{r}=\frac{r-a}{b-a} \quad \text{and} \quad \hat{T}_{rr}=\frac{T_{rr}}{\sigma}, \tag{5.8}$$

the governing equation changes to:

$$(\hat{r}+p_g)^2\left\{\frac{p_\mu}{\sqrt{\hat{d}_2}}-\frac{p_\beta}{\hat{d}_1}+\frac{p_\beta^2}{\hat{d}_1^2}\left[(\hat{r}+p_g)\frac{d\hat{T}_{rr}}{d\hat{r}}+2\hat{T}_{rr}\right]-\frac{p_\mu p_\gamma^2}{\hat{d}_2^{\frac{3}{2}}}\left[(\hat{r}+p_g)\frac{d\hat{T}_{rr}}{d\hat{r}}\right.\right.$$
$$\left.\left.+\hat{T}_{rr}\right]^2\right\}\frac{d^2\hat{T}_{rr}}{d\hat{r}^2}+(\hat{r}+p_g)\left\{\frac{3p_\mu}{\sqrt{\hat{d}_2}}-\frac{3p_\beta}{\hat{d}_1}+\frac{3p_\beta^2}{\hat{d}_1^2}\left[(\hat{r}+p_g)\frac{d\hat{T}_{rr}}{d\hat{r}}+2\hat{T}_{rr}\right]\right.$$
$$\left.-\frac{p_\mu p_\gamma^2}{\hat{d}_2^{\frac{3}{2}}}\left[(\hat{r}+p_g)\frac{d\hat{T}_{rr}}{d\hat{r}}+\hat{T}_{rr}\right]\left[2(\hat{r}+p_g)\frac{d\hat{T}_{rr}}{d\hat{r}}+3\hat{T}_{rr}\right]\right\}\frac{d\hat{T}_{rr}}{d\hat{r}}=0 \tag{5.9a}$$

where

$$p_g=\frac{a}{b-a}, \quad p_\gamma=\sigma\gamma, \quad p_\mu=\sigma\mu, \quad p_\beta=\sigma\beta, \tag{5.9b}$$

$$\hat{d}_1:=1+p_\beta\left[(\hat{r}+p_g)\frac{d\hat{T}_{rr}}{d\hat{r}}+2\hat{T}_{rr}\right], \tag{5.9c}$$

and

$$\hat{d}_2:=1+p_\gamma^2\left[2\hat{T}_{rr}^2+2(\hat{r}+p_g)\hat{T}_{rr}\frac{d\hat{T}_{rr}}{d\hat{r}}+(\hat{r}+p_g)^2\left(\frac{d\hat{T}_{rr}}{d\hat{r}}\right)^2\right]. \tag{5.9d}$$

The boundary conditions are transformed to require:

$$\hat{T}_{rr}(0)=0 \quad \text{and} \quad \hat{T}_{rr}(1)=1. \tag{5.10}$$

Once we have calculated the normal stress $\hat{T}_{rr}$, we can compute the normalized hoop stress using the formula:

$$\hat{T}_{\theta\theta} := \frac{T_{\theta\theta}}{\sigma} = (\hat{r} + p_g)\frac{d\hat{T}_{rr}}{d\hat{r}} + \hat{T}_{rr}. \tag{5.11}$$

Note that if $p_\beta$ and $p_\gamma$ are sufficiently small, we obtain the following Cauchy-Euler equation for the stresses under the assumptoin of Hooke's law (hence the superscript designation "$(H)$"):

$$(p_g + \hat{r})^2 \frac{d^2\hat{T}_{rr}^{(H)}}{d\hat{r}^2} + 3(p_g + \hat{r})\frac{d\hat{T}_{rr}^{(H)}}{d\hat{r}} = 0. \tag{5.12}$$

This equation can be solved exactly, subject to the boundary conditions (5.10), and the radial and hoop stress are given be the relationships:

$$\hat{T}_{rr}^{(H)} = \frac{(1+p_g)^2}{1+2p_g}\left[1 - \frac{p_g^2}{(p_g + \hat{r})^2}\right], \tag{5.13}$$

and

$$\hat{T}_{\theta\theta}^{(H)} = \frac{(1+p_g)^2}{1+2p_g}\left[1 + \frac{p_g^2}{(p_g + \hat{r})^2}\right], \tag{5.14}$$

respectively.

## 5.2 Results

Now, this problem is pregnant with possiblities with regard to illustrating the types of responses possible within the proposed framework, and therefore we could solve this equation for a variety of special cases. We shall choose two scenarios: equibiaxial compression of a large plate with a centered hole and the radial extension of a thin ring. The governing differential equation is non-linear, and therefore it must be solved numerically. We show that if the parameters take on certain values ($p_\beta = 0.1$, $p_\gamma = 0.0$, and $p_\mu = 1.0$) so that the constitutive equation is essentially that corresponding to Hooke's law, and therefore the governing equation is linear, then the solution obtained by MATLAB via the bvp4c subroutine is in very good agreement with the exact solution. However, in order to ensure that the approximate solution to the full non-linear equation is calculated correctly, a simple one-dimensional finite element routine, based on the weak formulation developed by Rajagopal and Bustamante [10], is developed in order to corroborate the solution obtained with the MATLAB subroutine. For both the plate and ring problems, quadratic interpolation functions are used for the radial stress; this provides a better estimate of the hoop stress through equation (5.11). We find that for the plate problem, increasing the number of elements to ten results in excellent agreement between the two methods, while the ring problem requires fifteen elements. For the ring, the variability in $T_{rr}$ for the case corresponding to $p_g = 0.01$ causes the number of elements required to achieve agreement to increase, even though $T_{\theta\theta}$ is essentially constant. Note that for all calculations we set $p_\alpha = 1.0$.

The first problem we study is the equibiaxial compression of an "infinite plate" with a centered hole. We know that for an infinite plate subject to equibiaxial compression, Hooke's law predicts the normalized stress $\hat{T}_{\theta\theta}$ (or "$K_t$") of 2 at the hole edge. We first determine the $p_g$ which predicts this (normalized) stress, and we use this value of $p_g = 0.01$ for the calculations to follow for this problem. Of course, since the problem is solved numerically, there can be no such thing as infinity. However, this value for $p_g$ implies that the radius to thickness ratio is sufficiently small, and the numerical solution to equation (5.9), with the aforementioned parameter values, and exact solution (5.13) are essentially identical under the assumption of Hooke's Law.

Figures 5.2-5.5 illustrate the approximate stresses and strains from solving this problem numerically. Here we have set the following parametes: $p_\gamma = 0.5$ and $p_\beta = 0.01$, and we vary $p_\mu$. Let us begin the study by observing the response in Figure 5.2. Note that both boundary conditions are satisfied, and the response is monotonically increasing away from the hole. Furthermore, as the parameter $p_\mu$ is varied ($= 0.01$-blue diamonds, $= 0.1$-red circles) the stress may be higher or lower than that associated with Hooke's law. Figure 5.3 shows that the strain associated with Hooke's law is significantly higher. The hoop stress, or $K_t$, is plotted in Figure 5.4. Note that away from the hole, the stress markers nearly lie on top of one another, but at the hole, the stress can be significantly more compressive or less compressive; this ultimately depends on the material. This is exactly what one should expect from equation (5.2): When the stresses are sufficiently small, the response will be similar to that predicted by Hooke's law, but near areas of stress concentration, the constitutive equations are capable of predicting more or less concentration, and ultimately this will depend upon the material. The hoop strains are plotted in Figure 5.5, and note that the response associated with Hooke's law is dramatically different than that shown from equation (5.2). In particular, the magnitude of the (non-dimensional) Hooke's law strain tends to increase dramatically; it is far greater than 1 near the hole, while the other strains all remain close to zero. This behavior is exacerbated in the next problem which involves solving for the stresses associated with a thick ring, then drastically shrinking the thickness of the ring and comparing the two solutions.

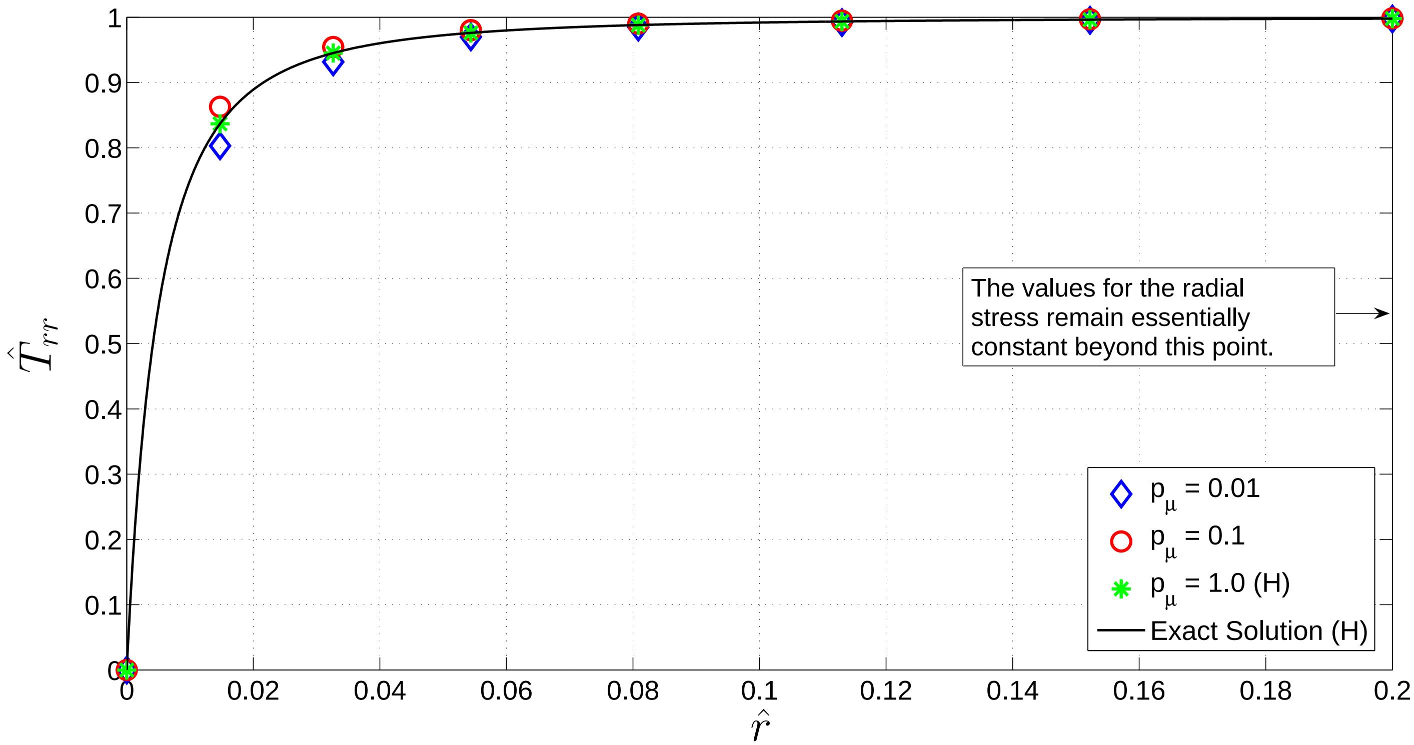


Figure 5.2. Plot of the radial stress in the plate as a function of the radial position.

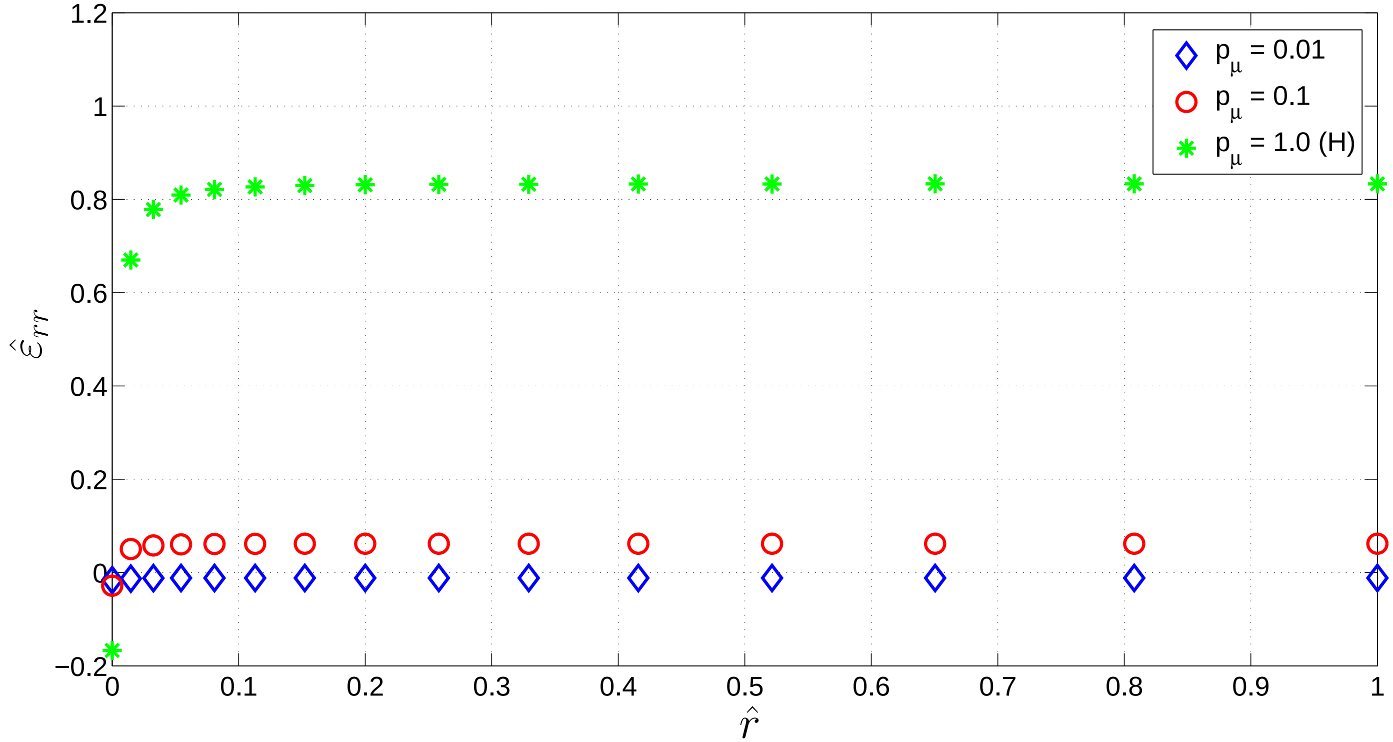


Figure 5.3. Plot of the radial strain in the plate as a function of the radial position.

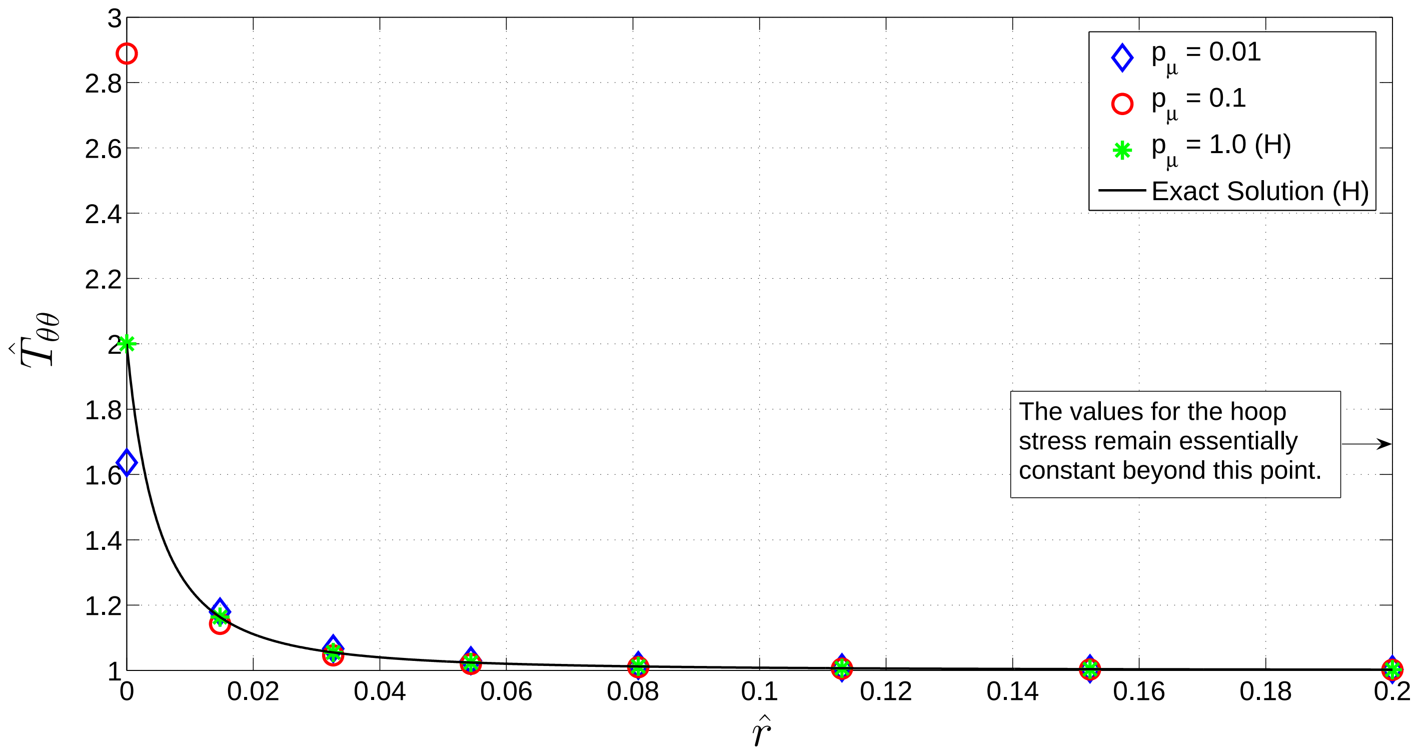


Figure 5.4. Plot of the hoop stress in the plate as a function of the radial position.

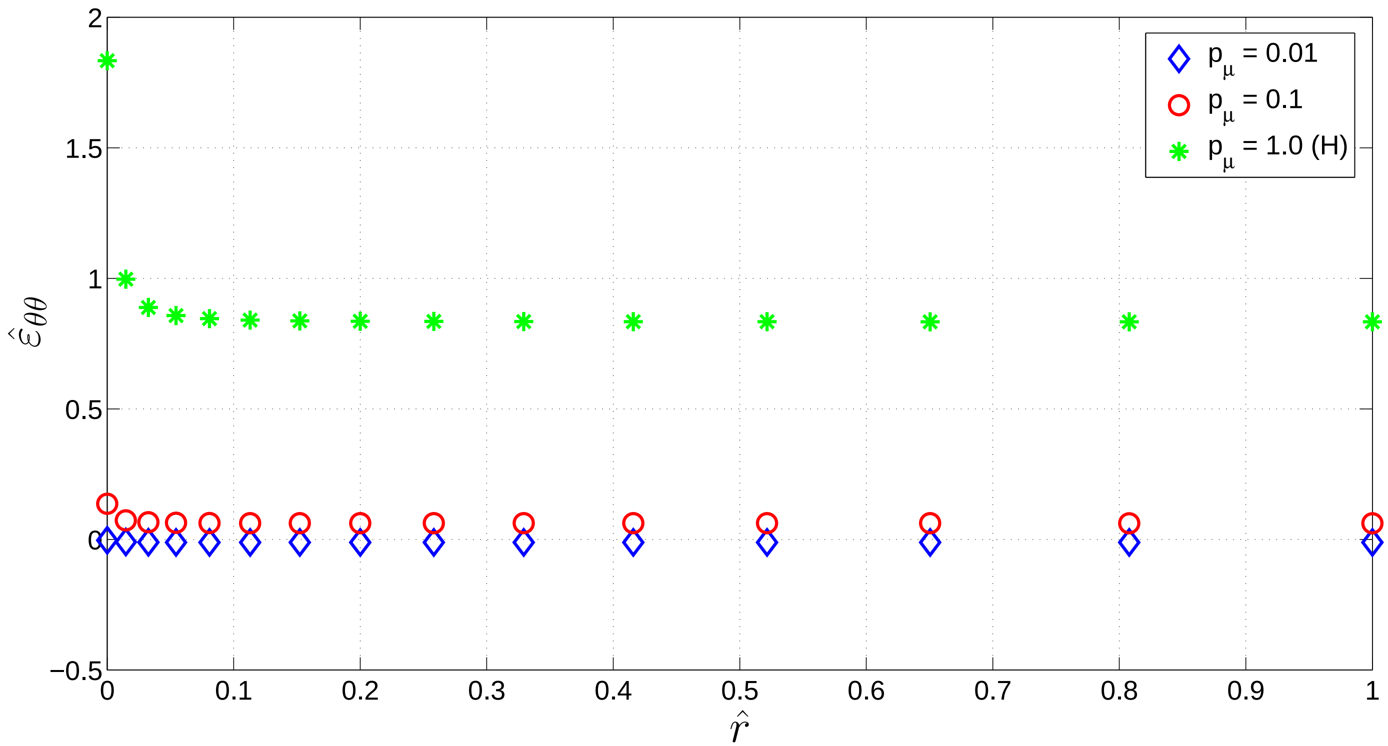


Figure 5.5. Plot of the hoop strain in the plate as a function of the radial position.

For the second problem, we study the radial tension of a ring, i.e., all other parameters being set ($p_\gamma = 0.5$, $p_\beta = 0.01$, and $p_\mu = 0.1$), we take the parameter $p_g$ to be small and then very large to simulate the collapse of the (area) ligament. This process can be viewed as an analogue to the process of how one may go about studying the response at a crack tip. For instance, if one assumes that a crack can be modeled as an ellipse, at least in a two-dimensional sense, then essentially what one has is an ellipse with a very small aspect ratio, and this leads to a dramatic increase of the stress at the crack tip. Of course, in order to study the response of an actual crack tip, one must, at the least, study a two-dimensional plane stress or plane stress problem, and we are not advocating that the numerical values for our problem are representative of those associated with the solution of a boundary value problem with a crack. We are simply stating that qualitatively the shrinking of the ligment area will cause the stresses to dramatically increase, and this is the qualitative response one would expect if the aspect ratio of an ellipse, which to some extent represents a crack, is taken to tend to zero.

The radial stresses and strains are plotted in Figures 5.6 and 5.7 for a material characterized by (5.2) and also for Hooke's Law "($H$)". For the radial stresses, we observe once again that there are very few differences in the solution for fixed $p_g$, and that the solution for the "infinite plate" ($p_g = 0.01$) varies drastically in a boundary layer region near the hole and remains nearly constant outide the layer while the radial stress associated with the thin ring ($p_g = 100$) varies linearly with the (non-dimensional) thickness. An interesting observation can be gathered from Figures 5.8 and 5.9. For the infinite plate, the hoop stresses are very small, regardless of the material, but when the ligament is thinned, both materials predict a dramatic rise in the stress level. However, a cursory look at Figure 5.9 shows that Hooke's Law predicts that the hoop strain will *also* drastically increase to a large value, while the hoop strain associated with the material given by (5.2) is very small, entirely consistent with the linearization process.

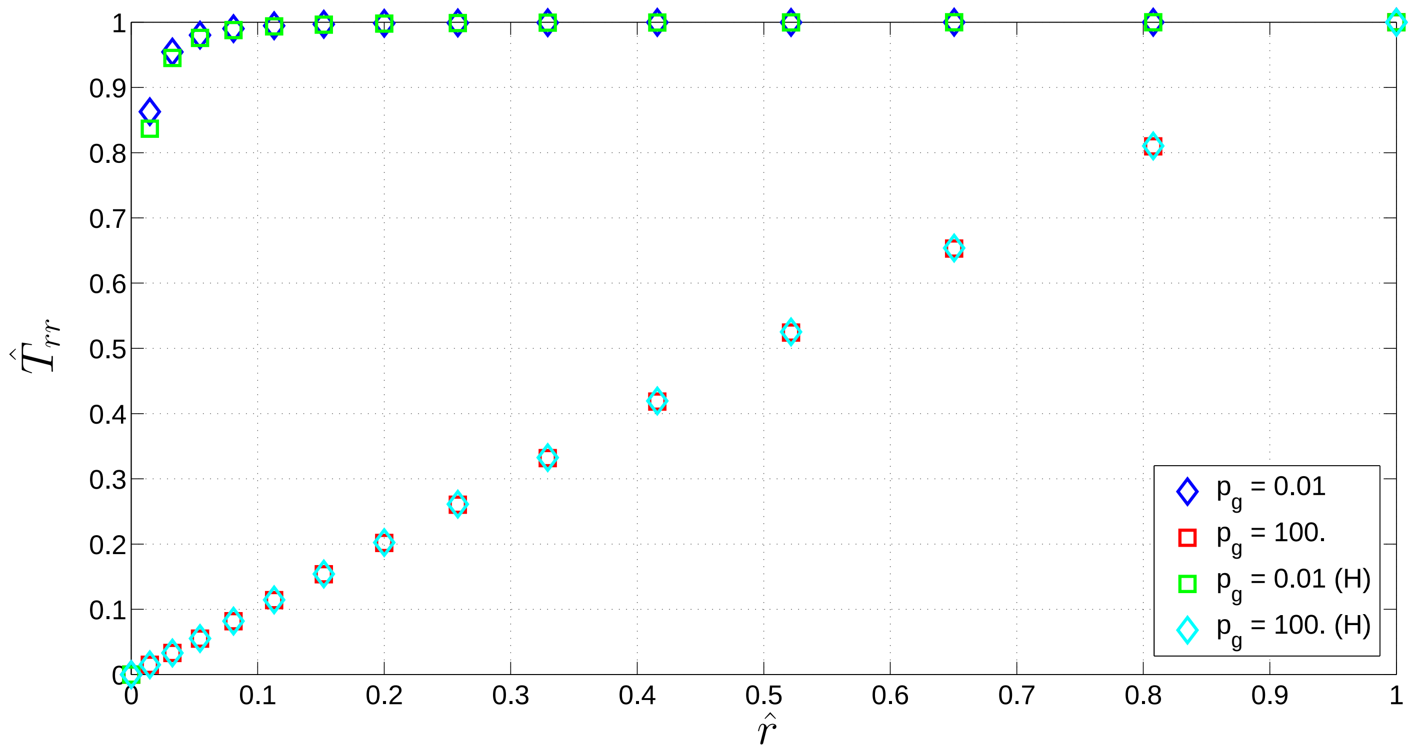


Figure 5.6. Plot of the radial stress in the ring as a function of the wall thickness.

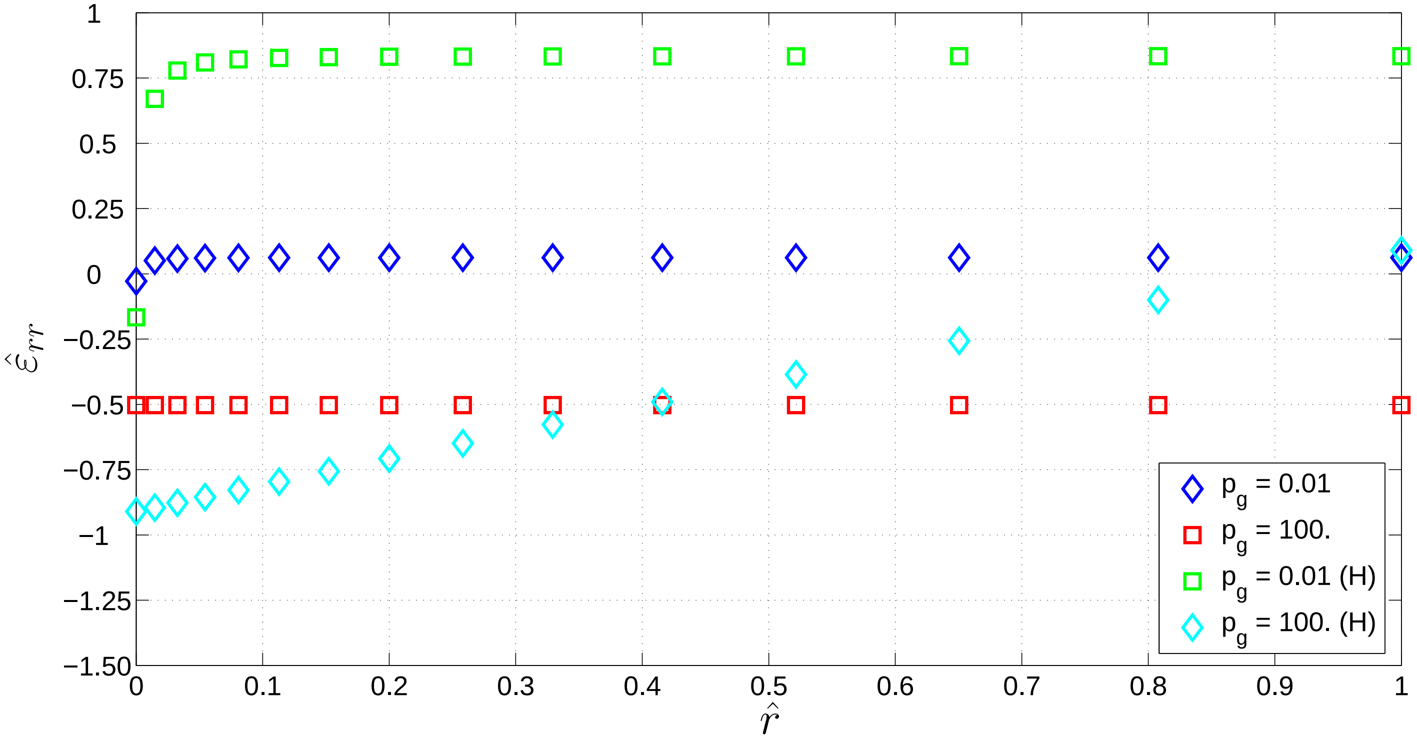


Figure 5.7. Plot of the radial strain in the ring as a function of the wall thickness.

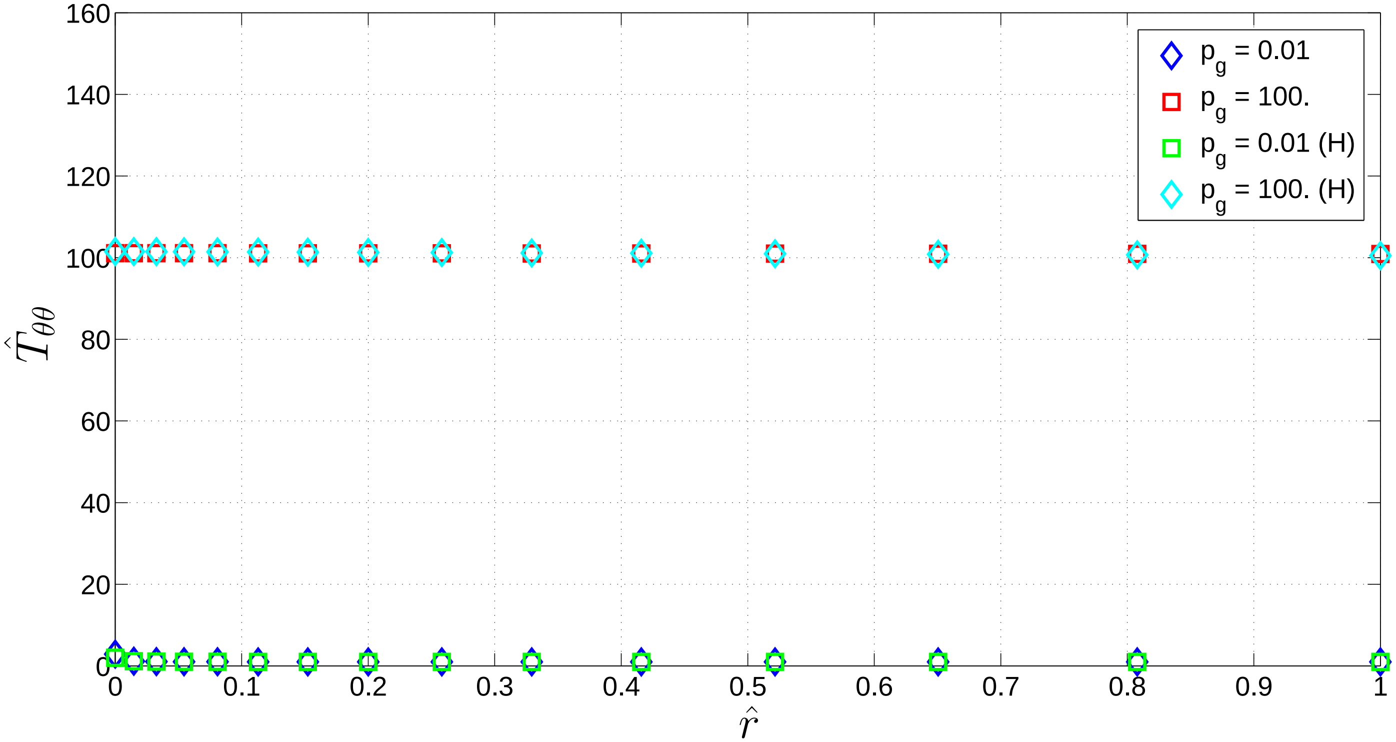


Figure 5.8. Plot of the hoop stress in the ring as a function of the wall thickness.

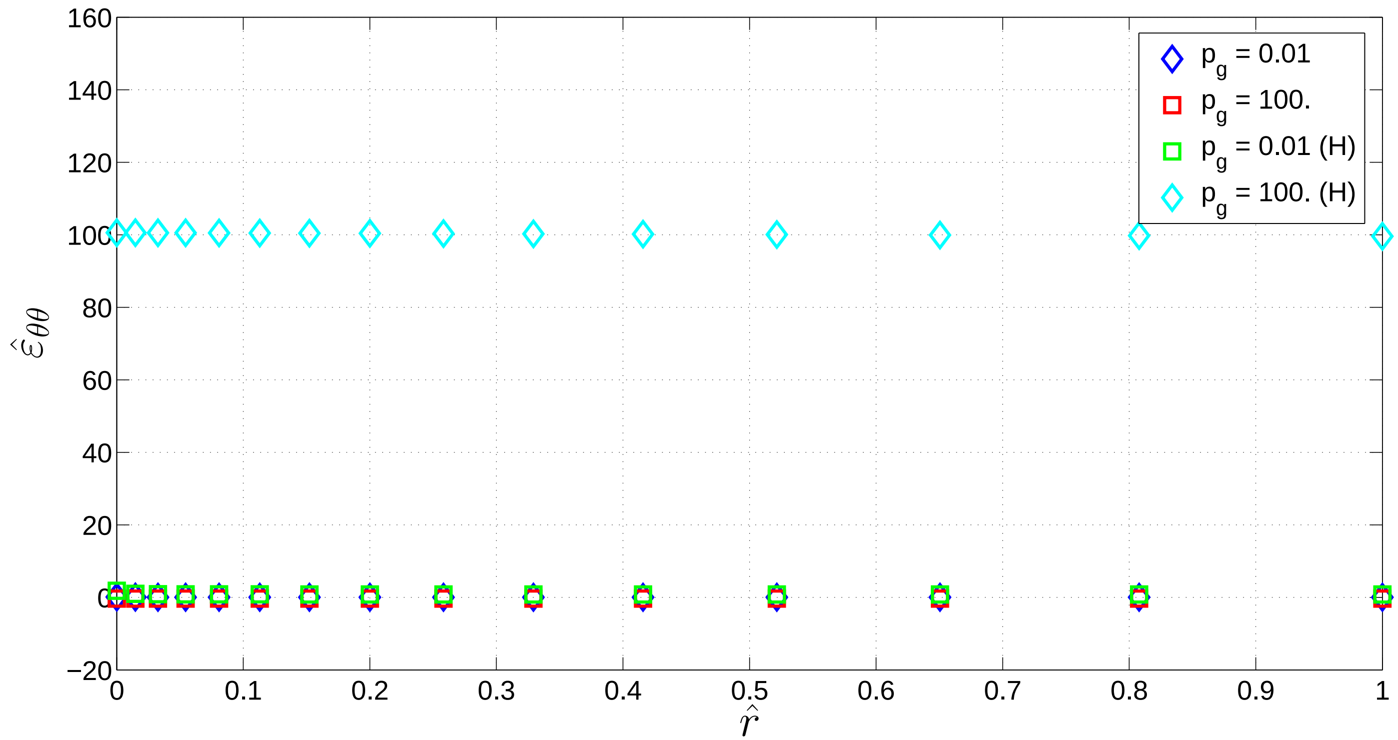


Figure 5.9. Plot of the hoop strain in the ring as a function of the wall thickness.

# 6. SUMMARY AND FUTURE WORK

## 6.1 A Problem of Finite Deformation

The majority of this paper has been concerned with the development of a thermodynamic framework wherein models can be developed to describe various types of behavior under the small displacement gradient assumption, and a boundary value problem has been solved to illustrate one of these applications. However, there is also a necessity to study these implicit models within the context of finite deformation since the constitutive equations provided by the Helmholtz potential lead to anamolous results in some cases. For example, the analysis of an elastic material with an elliptical rigid inclusion undergoing large deformations shows that the strain "blows up," which is still unfeasible, even in the context of finite elasticity [18]. A more simple problem to consider is the radial expansion of a spherical annulus with a rigid spherical inclusion centered at the core of the annulus (Figure 6.1). Now, one could choose to use either a plane stress or plane strain formulation to study this problem. However, the plane stress formulation will lead to a system of non-linear differential equations for the stresses, and one must satisfy the displacement-free condition $\mathbf{u} = \mathbf{0}$ at the interface $R = A$. Thus, one must contend with not only solving the aforementioned equations for the stresses, but then substituting these stresses into the constitutive relationship and integrating once again to find the displacement field $\mathbf{u}$. Instead, one could begin with assumptions for the motion, and this is the path we recommend. In view of the symmetry of the geometry and loading, it is reasonable to seek a mapping of the form:

$$r = f(R), \quad \theta = \Theta, \quad \varphi = \Phi. \tag{6.1}$$

Then the left stretch tensor takes the form:

$$(\mathbf{V})_{R\Theta\Phi} = \frac{df}{dR}\mathbf{e}_R \otimes \mathbf{e}_R + \frac{f}{R}\left(\mathbf{e}_\Theta \otimes \mathbf{e}_\Theta + \mathbf{e}_\Phi \otimes \mathbf{e}_\Phi\right). \tag{6.2}$$

Now, if one utilizes (4.22) and chooses a form for the Gibbs potential similar to (5.1), this implies that $T_{r\theta} = T_{r\varphi} = T_{\varphi\theta} = 0$, and $T_{\theta\theta} = T_{\varphi\varphi}$. We must also satisfy the balance of linear momentum which in a spherical coordinate system takes the form:

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial T_{r\varphi}}{\partial \varphi} + \frac{1}{r}\big[2T_{rr} - (T_{\theta\theta} + T_{\varphi\varphi}) + T_{\theta r}\cot(\theta)\big] + \varrho b_r = \varrho\frac{dv_r}{dt}, \tag{6.3a}$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial T_{\theta\varphi}}{\partial \varphi} + \frac{1}{r}\left[3T_{r\theta} + (T_{\theta\theta} - T_{\varphi\varphi})\cot(\theta)\right] + \varrho b_\theta = \varrho\frac{dv_\theta}{dt}, \tag{6.3b}$$

$$\frac{\partial T_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\varphi\theta}}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial T_{\varphi\varphi}}{\partial \varphi} + \frac{1}{r}\left[3T_{r\varphi} + 2T_{\theta\varphi}\cot(\theta)\right] + \varrho b_\varphi = \varrho\frac{dv_\varphi}{dt}. \tag{6.3c}$$

Note that we have assumed symmetry of the stress in recording (6.3). On utilizing the assumption for the motion, ignoring the body force, and making use of the knowledge of $\mathbf{T}$ we have the only non-trivial component of the balance of linear momentum:

$$\frac{dT_{rr}}{dR} + \frac{2}{f}\frac{df}{dR}\left(T_{rr} - T_{\theta\theta}\right) = 0. \tag{6.4}$$

The other equations relating the stress and stretch must also be satisfied:

$$\ln(V_{rr}) = \frac{\partial \Phi_i}{\partial T_{rr}} \quad \text{and} \quad \ln(V_{\theta\theta}) = \frac{\partial \Phi_i}{\partial T_{\theta\theta}}, \tag{6.5}$$

which altogether produces 3 equations for the three unknowns $T_{rr}$, $T_{\theta\theta}$, and $f$ (recall that $T_{\theta\theta} = T_{\varphi\varphi}$). As stated previously, we must satisfy $f(A) = A$ at the interface of the two materials $R = A$, and the traction condition $T_{rr} = \sigma$ at the boundary $R = B$. Let us make a few notes regarding our system of differential equations. Since the equations are non-linear, it is unlikely that nice closed form solutions exist. Thus, one should naturally develop an optimal numerical routine scheme for approximating the solutions. One could choose to use the finite element method, which seems to be the default numerical algorithm these days. However, Galerkin's method which is often employed to solve second order equations is not well suited for first order equations. Thus, one may have to resort to minimization methods or Galerkin-Least Squares (GaLS) type methods. Such a problem should be viewed as a starting point for studying stress concentration in implicit elastic bodies undergoing large deformations.

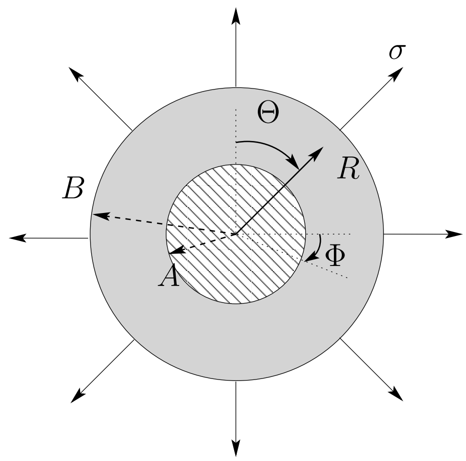


Figure 6.1. Depiction of the spherical annulus with the centered rigid inclusion and the spherical coordinate system $(r, \theta, \phi)$.

## 6.2 Consideration for Dissipative Mechanisms

Until this point, we have been concerned with modelling threshold type behavior or limiting strain response for elastic bodies. However, with some work, a plethora of dissipative phenomena can be incoporated into the theory. Herein, we shall utilize ideas developed in [19] to illustrate how one can model tearing. Such behavior has a multitude of applications, including delamination of composites and crack growth in metals. It is well known that a body can experience stress concentration in a small region, and after a certain number of cycles, experience failure. In this section, we show how to incoporate this type of effect into the current framework and also show how one can go about developing constitutive relations for the heat flux which is required for solving full thermo-mechanical problems. Finally, we note that ideas given in works by Rajagopal et al. ([20], [21]) lead the way to extending implicit theories for modelling plasticity, but we shall not go into detail here on this topic.

### 6.2.1 Tearing

Much of the work in this section follows that created in [19]. The idea is that the body is characterized by the Helmholtz potential *and* the rate of dissipation, and the main notion is that in a closed system, the rate of dissipation should be maximized. Furthermore, there exists an infinity of “natural configurations” that the body occupies on unloading, and these configurations evolve due to the dissipative process. For our special case of tearing, each natural configuration simply corresponds to the reference configuration with a different tear length. To this end we shall assume that the body is initially of uniform temperature and undergoes an isothermal process. Furthermore, let us define a quantity $\tau$ which represents the number of bonds broken per unit length. For reasons already discussed in the current work, we shall use the Gibbs potential (per unit volume) as opposed to the Helmholtz potential, and assume it may be decomposed as follows:

$$\Phi_{total} = \Phi_i(\mathbf{T}) + \Phi_\tau(\mathbf{T}_\tau, \tau). \tag{6.6}$$

Here $\Phi_i := \|\mathbf{T}\|^2 \mathrm{g}_i$ denotes the free energy associated with the bulk of the material and $\Phi_\tau := \|\mathbf{T}_\tau\|^2 \mathrm{g}_\tau$ is the energy which can be released once a bond has been broken. Furthermore, note that $\mathbf{T}_\tau$ denotes the Cauchy stress required to break $\tau$ bonds at time $t$. As before, we have the following relationship between the Helmholtz potential and the Gibbs potential:

$$\Psi_{total} = \Psi_i + \Psi_\tau, \tag{6.7}$$

where

$$\Psi_i := \frac{\partial \Phi_i}{\partial \mathbf{T}} \cdot \mathbf{T} - \Phi_i \quad \text{and} \quad \Psi_\tau := \frac{\partial \Phi_\tau}{\partial \mathbf{T}_\tau} \cdot \mathbf{T}_\tau - \Phi_\tau. \tag{6.8}$$

Now, one can go through the same process as before to show that the Hencky strain is derivable from the Gibbs potential via:

$$\mathbf{E}_H = \frac{\partial \Phi_i}{\partial \mathbf{T}}, \tag{6.9}$$

however, for this case, the reduced energy-dissipation inequality shows the rate of dissipation must satisfy:

$$-\frac{d}{dt}\left(\frac{\partial \Phi_\tau}{\partial \mathbf{T}_\tau}\right)\cdot \mathbf{T}_\tau + \frac{\partial \mathrm{g}_\tau}{\partial \tau}\frac{d\tau}{dt}\|\mathbf{T}_\tau\|^2 = \xi. \tag{6.10}$$

Now, the procedure outlined in [19] involves prescribing the rate of dissipation, i.e.:

$$\hat{\xi} = \hat{\xi}\left(\mathbf{T}_\tau, \tau, \frac{d\tau}{dt}\right), \tag{6.11}$$

and using the method of Lagrange multipliers to ensure that the rate of dissipation is maximized, i.e., we wish to maximize the rate of dissipation subject to the constraint that equation (6.10) and (6.11) are the same. Thus, let us introduce an auxilary function $H$:

$$H = \hat{\xi} - \hat{\lambda}(\xi - \hat{\xi}) = \hat{\xi} - \hat{\lambda}\left[-\frac{d}{dt}\left(\frac{\partial \Phi_\tau}{\partial \mathbf{T}_\tau}\right)\cdot \mathbf{T}_\tau + \frac{\partial \mathrm{g}_\tau}{\partial \tau}\frac{d\tau}{dt}\|\mathbf{T}_\tau\|^2 - \hat{\xi}\right], \tag{6.12}$$

where $\hat{\lambda}$ is the Lagrange multiplier. On maximizing this function, we find:

$$\frac{\partial H}{\partial \dot{\tau}} = 0 \quad \Rightarrow \quad \frac{\partial \hat{\xi}}{\partial \dot{\tau}} - \hat{\lambda}\left\{-\frac{\partial}{\partial \dot{\tau}}\left[\frac{d}{dt}\left(\frac{\partial \Phi_\tau}{\partial \mathbf{T}_\tau}\right)\right]\cdot \mathbf{T}_\tau + \frac{\partial \mathrm{g}_\tau}{\partial \tau}\|\mathbf{T}_\tau\|^2 - \frac{\partial \hat{\xi}}{\partial \dot{\tau}}\right\} = 0. \tag{6.13}$$

This equation essentially tells us what $\dot{\tau}$ will result in the rate of dissipation being maximized, i.e., we obtain an evolution equation for the tear. If enough microstructural data is known about a given material, one may associate with the number of torn bonds a particular length, at least in one dimension; this may be more difficult for a two dimensional tear.

In order to make these ideas more clear, we shall consider a simple yet very insightful problem which is the time-dependent peeling of a thin strip that is bonded to a rigid surface as shown in Figure 6.2. At some time $\hat{t}$, the stress exceeds a critical value, and a portion of the strip begins to separate from the rigid surface. We shall not concern ourselves with the deformation of the bonded portion.

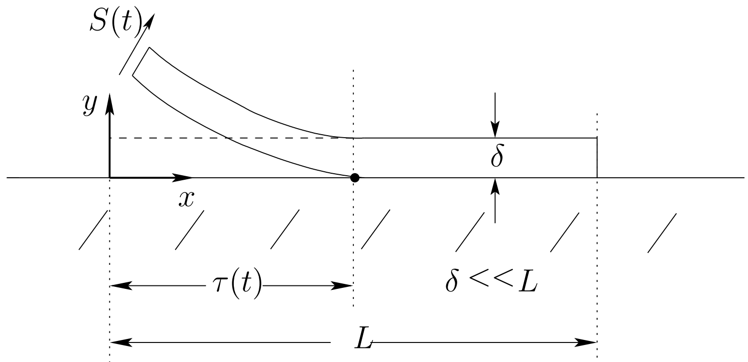


Figure 6.2. Illustration of the tearing of a thin membrane.

An interesting structure to study for $\Phi_i$ takes the following form:

$$\Phi(\mathrm{I}_{\mathbf{T}}, \mathrm{II}_{\mathbf{T}}) = \alpha\left[\frac{1}{\beta}\left(\mathrm{e}^{-\beta \mathrm{tr}(\mathbf{T})} - 1\right) + \frac{\mu}{\gamma^2}\sqrt{1+\gamma^2 \mathrm{tr}(\mathbf{T}^2)} + \mathrm{tr}(\mathbf{T}) - \frac{\mu}{\gamma^2}\right], \tag{6.14}$$

and if one assumes that the strains are small, we have [22]:

$$\boldsymbol{\varepsilon}(\mathrm{I}_{\mathbf{T}}, \mathrm{II}_{\mathbf{T}}) = \alpha\left[\left(1 - e^{-\beta \mathrm{tr}(\mathbf{T})}\right)\mathbf{1} + \frac{\mu \mathbf{T}}{\sqrt{1+\gamma^2 \mathrm{tr}(\mathbf{T}^2)}}\right]. \tag{6.15}$$

Furthermore, we assume that the displacement field takes the following form for this special motion:

$$\mathbf{u}(x, y, z, t) = \begin{cases} u(x,t)\mathbf{e}_y, & x < \tau(t) \\ \mathbf{0}, & x \geq \tau(t) \end{cases}, \tag{6.16}$$

and for this displacement field, the linearized strain reduces to:

$$\boldsymbol{\varepsilon}(x, t) = \frac{1}{2}\frac{\partial u}{\partial x}(\mathbf{e}_y \otimes \mathbf{e}_x + \mathbf{e}_x \otimes \mathbf{e}_y). \tag{6.17}$$

No information is known *a priori* for the stress, and the balance of linear momentum takes the following form in a Cartesian coordinate system:

$$\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} = 0, \tag{6.18a}$$

$$\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} = \varrho\frac{\partial^2 u}{\partial t^2}, \tag{6.18b}$$

$$\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} = 0. \tag{6.18c}$$

On writing out the constitutive equations, we find:

$$0 = 1 - e^{-\beta(T_{xx}+T_{yy}+T_{zz})} + \frac{\mu T_{xx}}{\sqrt{1+\gamma^2\left[T_{xx}^2+T_{yy}^2+T_{zz}^2+2\left(T_{xy}^2+T_{yz}^2+T_{zx}^2\right)\right]}}, \tag{6.19a}$$

$$0 = 1 - e^{-\beta(T_{xx}+T_{yy}+T_{zz})} + \frac{\mu T_{yy}}{\sqrt{1+\gamma^2\left[T_{xx}^2+T_{yy}^2+T_{zz}^2+2\left(T_{xy}^2+T_{yz}^2+T_{zx}^2\right)\right]}}, \tag{6.19b}$$

$$0 = 1 - e^{-\beta(T_{xx}+T_{yy}+T_{zz})} + \frac{\mu T_{zz}}{\sqrt{1+\gamma^2\left[T_{xx}^2+T_{yy}^2+T_{zz}^2+2\left(T_{xy}^2+T_{yz}^2+T_{zx}^2\right)\right]}}, \tag{6.19c}$$

$$\frac{1}{2}\frac{\partial u}{\partial x} = \frac{\alpha\mu T_{xy}}{\sqrt{1+\gamma^2\left[T_{xx}^2+T_{yy}^2+T_{zz}^2+2\left(T_{xy}^2+T_{yz}^2+T_{zx}^2\right)\right]}}, \tag{6.19d}$$

$$0 = \frac{\alpha\mu T_{yz}}{\sqrt{1+\gamma^2\left[T_{xx}^2+T_{yy}^2+T_{zz}^2+2\left(T_{xy}^2+T_{yz}^2+T_{zx}^2\right)\right]}}, \tag{6.19e}$$

$$0 = \frac{\alpha\mu T_{zx}}{\sqrt{1+\gamma^2\left[T_{xx}^2+T_{yy}^2+T_{zz}^2+2\left(T_{xy}^2+T_{yz}^2+T_{zx}^2\right)\right]}}. \tag{6.19f}$$

Equation (6.17) states that the strain is independent of $y$ and $z$, and therefore, through the relationship (6.15), so must be the stress. Thus, terms such as $\frac{\partial T_{xy}}{\partial y}$ and $\frac{\partial T_{yy}}{\partial y}$ must vanish. Then equation (6.18) implies that $T_{xx}$ does not change with $x$ which means that the only way in which we satisfy the traction condition $T_{xx} = 0$ at $x = 0$ is for $T_{xx} = 0 \ \forall \ x$. Also note that a sufficient condition to satisfy equations (6.19a)-(6.19c), (6.19e), (6.19f) is for $T_{xx} = T_{yy} = T_{zz} = T_{yz} = T_{xz} = 0$. Thus, the only non-zero component of stress is $T_{xy}$, and we can solve equation (6.19d) for this component:

$$T_{xy} = \pm\frac{\frac{\partial u}{\partial x}}{\sqrt{4(\mu\alpha)^2 - 2\gamma^2\left(\frac{\partial u}{\partial x}\right)^2}}. \tag{6.20}$$

We shall take the positive root. Note also that since the motion is isochoric, equation (3.1b) implies that $\varrho = \varrho_R$ and equation (6.18b) reduces to:

$$\frac{\partial}{\partial x}\left\{\left[4(\mu\alpha)^2 - 2\gamma^2\left(\frac{\partial u}{\partial x}\right)^2\right]^{-\frac{1}{2}}\frac{\partial u}{\partial x}\right\} = \varrho_R\frac{\partial^2 u}{\partial t^2} \tag{6.21}$$

This is a quasillinear PDE, and may be solved subject to the following initial and (moving) boundary conditions [23]:

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < \tau(t), \tag{6.22}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{2\alpha\mu S(t)}{1+2\gamma^2 S^2(t)}, \quad 0 < t \leq T, \tag{6.23}$$

$$u(\tau(t),t) = 0, \quad 0 < t \leq T, \tag{6.24}$$

If we prescribe $\Phi_\tau$ and $\xi$ as follows:

$$\Phi_\tau = \hat{\Phi}_\tau(\tau) \quad \text{and} \quad \xi = \hat{\xi}(\mathbf{T}_\tau, \tau, \dot{\tau}), \tag{6.25}$$

then we can obtain an evolution equation that is congruent with traditional fatigue laws:

$$\frac{d\tau}{dt} = \mathfrak{F}\left[\left.\frac{\partial u}{\partial x}\right|_{x=\tau}, \tau\right], \quad 0 < t \leq T, \quad \tau(0) = 0. \tag{6.26}$$

Of course, once the stress reaches some critical value $T_{xy}^{crit}$, the strip can be assumed to fail. With some effort, this type of formulation can be used to view crack growth from a new perspective.

### 6.2.2 Constitutive Equations for the Heat Flux

For the majority of this work, we have assumed that the body is initially of uniform temperature and undergoes an isothermal process. However, the resolution of a full thermomechanical problem requires constitutive specifications for the internal energy, the radiant heating, and the heat flux vector in addition to the Gibbs potential. On specifying the Gibbs potential, one can determine the internal energy and entropy using (4.7) and (4.12c), respectively, and for some practical problems, the radiant heating can be neglected. Thus, the remaining task is to determine the heat flux changes within the material. We shall use the procedure as in the last section which maximizes the rate of dissipation. Recall from Section 4, we have the following relationship for the rate of dissipation associated with the conduction within the material:

$$\xi_c = -\frac{\mathbf{q}}{\theta} \cdot \frac{\partial \theta}{\partial \mathbf{x}}. \tag{6.27}$$

Now, if we make the following constitutive specification for the rate of dissipation:

$$\hat{\xi}(\theta, \mathbf{q}) = k(\theta)\|\mathbf{q}\|^2, \quad k(\theta) > 0 \; \forall \; \theta \in [0, \theta_{max}], \tag{6.28}$$

and maximize the augmented function:

$$\Xi = \hat{\xi} - \hat{\beta}(\xi_c - \hat{\xi})] = k(\theta)\|\mathbf{q}\|^2 - \hat{\beta}\Big[ - \frac{\mathbf{q}}{\theta} \cdot \frac{\partial \theta}{\partial \mathbf{x}} - k(\theta)\|\mathbf{q}\|^2 \Big], \tag{6.29}$$

where $\hat{\beta}$ is a Lagrange mutliplier which enforces the contraint, we find a generalization of Fourier's law of heat conduction:

$$\frac{\partial \Xi}{\partial \mathbf{q}} = \mathbf{0} \quad \Rightarrow \quad \mathbf{q} = -\hat{k}(\theta)\frac{\partial \theta}{\partial \mathbf{x}}. \tag{6.30}$$

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