# APPLICATIONS OF GAME THEORY TO MULTI-AGENT COORDINATION PROBLEMS IN COMMUNICATION NETWORKS

A Dissertation

by

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# ABSTRACT

Recent years there has been a growing interest in the study of distributed control mechanisms for use in communication networks. A fundamental assumption in these models is that the participants in the network are willing to cooperate with the system. However, there are many instances where the incentives to cooperate is missing. Then, the agents may seek to achieve their own private interests by behaving strategically. Often, such selfish choices lead to inefficient equilibrium state of the system, commonly known as *the tragedy of commons* in Economics terminology. Now, one may ask the following question: how can the system be led to the socially optimal state in spite of selfish behaviors of its participants? The traditional control design framework fails to provide an answer as it does not take into account of selfish and strategic behavior of the agents. The use of game theoretical methods to achieve coordination in such network systems is appealing, as it naturally captures the idea of rational agents taking locally optimal decisions.

In this thesis, we explore several instances of coordination problems in communication networks that can be analyzed using game theoretical methods. We study one coordination problem each, from each layer of TCP/IP reference model - the network model used in the current Internet architecture. First, we consider societal agents taking decisions on whether to obtain content legally or illegally, and tie their behavior to questions of performance of content distribution networks. We show that revenue sharing with peers promote performance and revenue extraction from content distribution networks. Next, we consider a transport layer problem where applications compete against each other to meet their performance objectives by selfishly picking congestion controllers. We establish that tolling schemes that incentivize applications to choose one of several different virtual networks catering to particular needs yields higher system value. Hence, we propose the adoption of such virtual networks. We address a network layer question in third problem. How do the sources in a wireless network split their traffic over the available set of paths to

attain the lowest possible number of transmissions per unit time? We develop a two level distributed controller that attains the optimal traffic split. Finally, we study mobile applications competing for channel access in a cellular network. We show that the mechanism where base station conducting sequence of second price auctions and providing channel access to the winner achieves the benefits of the state of art solution, Largest Queue First policy.

DEDICATION

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# 1. INTRODUCTION

In recent years there has been a growing interest in the study of distributed control mechanisms for use in communication networks. A fundamental assumption in these models is that the participants in the network are willing to cooperate with the system in that their actions conform to the protocols stipulated by the system designer. However, there are many instances where the incentive to cooperate is missing. Consider, for example, routing between autonomous systems in the Internet. Ideally, the routing tables must be configured with shortest paths. However, ISPs who own these autonomous systems are profit driven and they prefer cheaper (profitable) routes to shorter ones (e.g *Hot Potato* routing). Such selfish behaviors of ISPs result in inefficient operation of the system. Often, as in the above example, it is true that selfish choices of the agents lead to *bad* equilibrium states of the system [23, 60, 61], which is known as *the tragedy of commons* in Economics. Now, one may ask the following question: how can the system be led to the socially optimal state in spite of selfish behaviors of its participants? The traditional control design framework fails to provide an answer as it does not take into account of selfish and strategic behavior of the agents. The use of game theoretical methods to achieve coordination in such network systems is appealing, as it naturally captures the idea of rational agents taking locally optimal decisions. In this thesis, we explore four instances of coordination problems in communication networks, choosing one problem from each layer of the Open Systems Interconnection (OSI) model. Below, we provide a summary of the work thus far, and present details in the sections following.

In Section 2, we consider a societal problem of ownership of content. We analyze the revenue loss incurring to a legitimate content distribution network that employs a centralized client-server model to sell content, while duplicate copies of the same content are freely available in the system. We ask the question: Can the content provider recover lost revenue through a more innovative approach to distribution? We evaluate the benefits of a hybrid revenue-sharing system that combines a legitimate Peer-to-Peer (P2P) swarm and a centralized client-server approach. In the hybrid revenue-sharing scheme, we develop reward schemes that incentivize legals, those clients who legally obtained the content, to act as agents of legal P2P swarm.

In Section 3, we study a resource allocation game in the Internet. A large number of congestion control protocols have been proposed in the last few years with all having the same purpose to divide available bandwidth among different flows in a fair manner. We study the interaction among numerous congestion control protocols in the Internet. We ask the question: Suppose that each flow has a number of congestion control protocols to choose from, which one (or combination) should it choose? We study both the socially optimal, as well as the selfish cases to determine the loss of system-wide value incurred through selfish decision making, so characterizing the price of heterogeneity. We also propose tolling schemes that incentivize flows to choose one of several different virtual networks catering to particular needs, and show that the total system value is greater, hence making a case for the adoption of such virtual networks.

In Section 4, we consider a problem of multipath routing in a wireless network. Here, each source makes a choice of traffic split among all of its available paths, to attain the lowest possible number of transmissions per unit time to support a given traffic matrix. Traffic bound in opposite directions over two wireless hops can utilize the "reverse carpooling" advantage of network coding in order to decrease the number of transmissions used. We call such coded hops as hyper-links. However, there is a dilemma among sources—the network coding advantage is realized only if there is traffic in both directions of a shared path. We develop a two level distributed control scheme that decouples user choices from each other by declaring a hyper-link capacity, allowing sources to split their traffic selfishly in a distributed fashion, and then changing the hyper-link capacity based on user actions.

Finally, in Section 5, we study an auction-theoretic mechanism for scheduling channel resources in cellular networks. In our setting, the players are smart phone apps that generate service requests, have costs associated with waiting, and bid against each other for service from base stations. We show that in a system in which we conduct a second-price auction at each base station and schedule the winner at each time, there exists a mean field equilibrium (MFE) that will schedule the user with highest value at each time. We further show that the scheme can be interpreted as a weighted longest queue first type policy. The result suggests that auctions can implicitly attain the same quality of service as queue-length based scheduling. In Section 6, we conclude the thesis and discuss future work.

# 2. APPLICATION LAYER : INCENTIVES FOR P2P ASSISTED CONTENT DISTRIBUTION*

The past decade has seen the rapid increase of content distribution using the Internet as the medium of delivery [31]. Users and applications expect a low cost for content, but at the same time require high levels of quality of service. However, providing content distribution at a low cost is challenging. The major costs associated with meeting demand at a good quality of service are (i) the high cost of hosting services on the managed infrastructure of CDNs such as Akamai [50, 76], and (ii) the lost revenue associated with the fact that digital content is easily duplicable, and hence can be shared in an illicit peer-to-peer (P2P) manner that generates no revenue for the content provider. Together, these factors have led content distributors to search for methods of defraying costs.

One technique that is often suggested for defraying distribution costs is to use legal peer-to-peer (P2P) networks to supplement provider distribution [52, 59]. It is well documented that the efficient use of P2P methods can result in significant cost reductions from the perspective of ISPs [24, 50]; however there are substantial drawbacks as well. Probably the most troublesome is that providers fear losing control of content ownership, in the sense that they are no longer in control of the distribution of the content and worry about feeding illegal P2P activity.

Thus, a key question that must be answered before we can expect mainstream utilization of P2P approaches is: *How can users that have obtained content legally be encouraged to reshare it legally?* Said in a different way, can mechanisms be designed that ensure legitimate P2P swarms will dominate the illicit P2P swarms?

In this paper, we investigate a "revenue sharing" approach to this issue. We suggest that users can be motivated to reshare the content legally by allowing them to share the

*Part of the data reported in this chapter is reprinted with permission from "Incentives for P2P-assisted content distribution: If you can't beat 'em, join 'em" by V. Ramaswamy, S. Adlakha, S. Shakkottai and A. Wierman. *50th Annual Allerton Conference on Communication, Control and Computing*, 2012, Copyright@2012 IEEE.

revenue associated with future sales. This can be accomplished through either a lottery scheme or by simply sharing a fraction of the sale price. Recent work on using lotteries to promote societally beneficial conduct [42] suggests that such schemes could potentially see wide spread adoption.

Such an approach has two key benefits: First, obviously, this mechanism ensures that users are incentivized to join the legitimate P2P network since they can profit from joining. Second, less obviously, this approach actually damages the illicit P2P network. Specifically, despite the fact that content is free in the illicit P2P network, since most users expect a reasonable quality of service, if the delay in the illegitimate swarm is large they may be willing to use the legitimate P2P network instead. Thus, by encouraging users to reshare legitimately, we are averting them from joining the illicit P2P network, reducing its capacity and performance; thus making it less likely for others to use it.

The natural concern about a revenue sharing approach is that by sharing profits with users, the provider is losing revenue. However, the key insight provided by the results in this paper is that by discouraging users from joining illicit P2P network, the increased share (possibly exponentially more) of legitimate copies makes up for the cost of sharing revenue with end-users.

More specifically, the contribution of this paper is to develop and analyze a model to explore the revenue sharing approach described above. Our model (see Section 2.1) is a fluid model that builds on work studying the capacity of P2P content distribution systems. The key novel component of the model is the competition for users among an illicit P2P system and a legal content distribution network (CDN), which may make use of a supplementary P2P network with revenue sharing. The main results of the paper (see Section 2.2) are Theorems 1-4, which highlight the order-of-magnitude gains in revenue extracted by the provider as a result of participating in revenue sharing. Further, In addition to the analytic results, to validate the insights provided by our asymptotic analysis of the fluid model we also perform numerical experiments of the underlying finite stochastic model. Tables 2.1 and 2.2 summarize these experiments, which highlight both that the results obtained in the fluid model are quite predictive for the finite setting and that there are significant beneficial effects of revenue sharing.

There is a significant body of prior work modeling and analyzing P2P systems. Perhaps the most related work from this literature is the work that focuses on server-assisted P2P content distribution networks [12, 36, 53, 65, 66, 77] in which a central server is used to "boost" P2P systems. This boost is important since pure P2P systems suffer poor performance during initial stages of content distribution. In fact, it is this initially poor performance that our revenue sharing mechanism exploits to ensure that the legitimate P2P network dominates.

Two key differentiating factors of the current work compared to this work are: (i) We model the impact of competition between legal and illegal swarms on the revenue extraction of a content provider. (ii) Unlike most previous works on P2P systems, we consider a time varying viral demand model for the evolution of demand in a piece of content based on the Bass diffusion model (see Section 2.1). Thus, we model the fact that interest in content grows as interested users contact others and make them interested.

With respect to (i), there has been prior work that focuses on identifying the relative value of content and resources for different users [5,44]. For instance, [5] deals with creating a content exchange that goes beyond traditional P2P barter schemes, while [44] attempts to characterize the relative value of peers in terms of their impact on system performance as a function of time. However, to the best of our knowledge, ours is the first work that considers the question of economics and incentives in hybrid P2P content distribution networks.

With respect to (ii), there has been prior work that considers fluid models of P2P systems such as [41,57,80]. However, these all focus on the performance evaluation of a P2P system with constant demand rate. As mentioned above, a unique facet of our approach is that we explicitly make use the transient nature of demand in our modeling. In the sense of explicitly accounting for transient demand, the closest work to ours is [66]. However, [66] focuses only on jointly optimizing server and P2P usage in the case of transient demand in order to obtain a target delay guarantee at the lowest possible server cost.

The remainder of the paper is organized as follows. We first introduce the details of our model in Section 2.1. Then, Section 2.2 summarizes analytic and numeric results. Finally, Section 2.4 provides concluding remarks.

## 2.1 Model overview

Our goal is to model the competition between illicit peer-to-peer (P2P) distribution and a legitimate content distribution network (CDN), which may make use of its own P2P network. Our model is a fluid model, and there are four main components:

1. The evolution of the demand for content. A key feature of this paper is that we consider a realistic model for the evolution of demand, specifically, the Bass diffusion model.

2. The model of user behavior, which allows the user to strategically choose between attaining content legally or illegally based on the price and performance of the two options.

3. The model of the illicit P2P system.

4. The model of the legal CDN and its possibility to use “revenue sharing”.

We discuss these each in turn in the following.

### *2.1.1 The evolution of demand*

The simplest possible model of demand is that the entire population gets interested in the content simultaneously at time $t = 0$. We call this the “Flash crowd model” due to the instantaneous appearance of all the demand. While the model is simplistic, it can serve as a foundation for developing performance results, and we will utilize it as our base case. More complex models of demand can be considered as well. Indeed, models of the dynamics of demand growth for innovations dates to the work of Griliches [19] and Bass [6]. The most widely used model for dynamics of demand growth is the Bass diffusion model which describes how new products get adopted as potential users interact with users that have already adopted the product. Such word of mouth interaction between users and potential users is very common in the Internet and we use a version of Bass diffusion model that only has word of mouth spreading. We describe both models formally below.

We define $N$ to be the total size of the population and $I(t)$ to be the number of users that are interested in the content at time $t$. In the Flash Crowd Model,

$$I(t) = N, \tag{2.1}$$

since all users are interested from the very beginning. In the Bass diffusion model, each interested user "attempts" to cause a randomly selected user to become interested in the content.[1] At any time $t$, there are $N - I(t)$ users that could potentially be interested in the content. Thus, the probability of finding such a users is $(N - I(t))/N$. Assuming that an interested user can interact with other users at rate 1 per unit time, we get that the rate at which interested users increase is given by the following differential equation:

$$\frac{dI(t)}{dt} = \left(\frac{N - I(t)}{N}\right) I(t). \tag{2.2}$$

The above differential equation can be easily solved and yields the so-called *logistic function* as its solution.

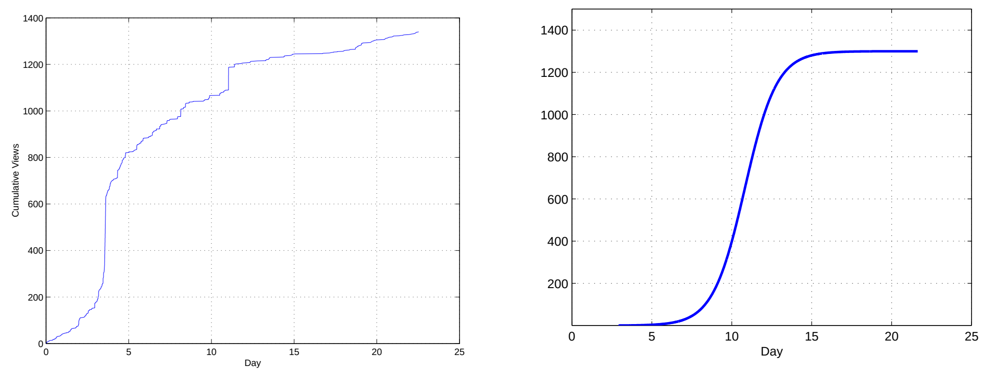
$$I(t) = \frac{I(0)e^t}{1 - (1 - e^t)\frac{I(0)}{N}}, \tag{2.3}$$

where $I(0)$ is the number of user that are interested in the content at time $t = 0$.

Though the Bass model is quite simple, it is a useful qualitative summary of the spread of content. To highlight this, Figure 2.1 (taken from [66]) highlights a similar behavior in a data trace from CoralCDN [17], a CDN hosted at different university sites. The figure shows the cumulative demand for a home video of the Asian Tsunami seen over a month in December 2005. For comparision, the figure on the right shows the model in equation

[1] Note that these "attempts" should not be interpreted literally, but rather as the natural diffusion of interest in the new content through the population.

(2.3). The qualitative usefulness of the Bass model has been verified empirically in many settings, and hence the Bass model is often considered as canonical [47].




(a) Single-file cumulative demand (b) Cumulative demand in Bass model

Figure 2.1: (a) shows the cumulative demand for a file over one month on Coral CDN (Dec 2005–Jan 2006). (b) shows the cumulative demand seen in a Bass diffusion.

### *2.1.2 The progression of a user*

In order to capture the strategic behavior of users in the face of competition between a legitimate CDN using P2P and an illicit P2P network our model is necessarily complex. Figure 2.2 provides a broad overview of the user behavior in the system, which we explain in detail in the following.

Let us explain the model through tracking the progression of a user. We term an initial user that wants, but has not yet attained, the content a *Wanter (W)*. When a Wanter arrives to the system, it has two options: get content from the illicit P2P system for free or get content from the legitimate system for a price $p$. We assume that the Wanter wishes to obtain content as quickly and cheaply as possible, and so she first approaches the illicit P2P swarm and then only attains the content from the legitimate system if the content is not attained a reasonable time interval (one infinitesimal clock tick in our model) from the illicit P2P. This cycle repeats, if necessary, until the content is attained. In some sense, this is the worst-case for the legitimate provider since the illicit source is tried first.

Once the Wanter has attained the content (legally or illegally), it could stay in the system and assist in content dissemination. We denote the probability of this event by $\kappa < 1$. Otherwise, it could simply *Quit (Q)* and leave the system with probability $1 - \kappa$. Now, if a Wanter obtains the content *legally* and decides to assist in dissemination, it has two options: (i) It might decide to use the content to assist the illicit P2P swarm, i.e., go *Rogue (R)*. We denote the probability this happens by $\rho < 1$. (ii) It might decide to assist the legitimate P2P swarm (if one exists) as a *Booster (B)*. We denote the probability of this event by $\beta < 1$. Note that $\beta = 0$ if no legal P2P is used. Clearly $\rho + \beta = \kappa$. However, if a Wanter obtains content *illegally* and chooses to stay in the system, it can only aid the illicit swarm as a *Fraudster (F)*. The probability of this event is simply $\kappa$.

Note that the goal of revenue sharing is to incentivize Wanters to become Boosters after attaining content legally, rather than going Rogue. The hope is that the revenue invested toward reducing the number of "early adopters" that go Rogue keeps the illicit P2P swarm from growing enough to provide good enough quality of service to dominate the legitimate swarm.

To model this system more formally, we introduce the following notation. Let $N_w(t)$ be the number of Wanters at time $t$, i.e., the number of users who have not yet attained the content, and assume $N_w(0) = 0$. Further, let $N_l(t)$ and $N_i(t)$ be the number of users with legal and illegal copies of the content at time $t$. Note that the total number of interested users at any time $t$ satisfies the following equation

$$I(t) = N_w(t) + N_l(t) + N_i(t) \tag{2.4}$$

We can break this down further by noting that the number of Rogues, Fraudsters, and

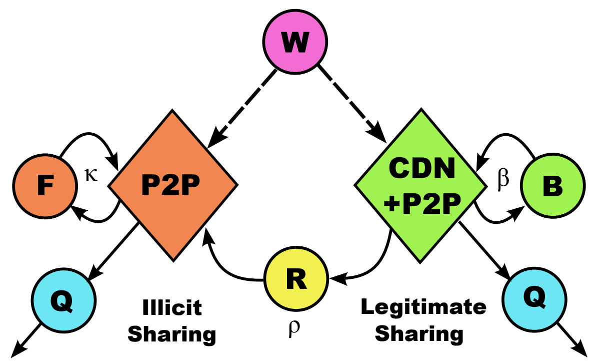


Figure 2.2: An overview of the progression of a user through the systems. The labels are defined as follows: W - Wanter, F - Fraudster, R - Rogue, B - Booster, and Q - Quit.

Boosters in the system at time $t$ (denoted by $N_r(t)$, $N_f(t)$, and $N_b(t)$ respectively) is:

$$N_r(t) = \rho N_l(t) \tag{2.5}$$

$$N_f(t) = \kappa N_i(t) \tag{2.6}$$

$$N_b(t) = \beta N_l(t), \tag{2.7}$$

with $\rho + \beta < 1$. The rest of legal and illegal users leave the system.

The key remaining piece of the model is to formally define the transition of Wanters to holders of illegal/legal content, i.e., the evolution of $N_i(t)$ and $N_l(t)$. However, this evolution depends critically on the model of the two systems, and so we describe it in the next section.

### *2.1.3 System models*

We discuss in detail the illicit and legitimate system models below. The factors in these models are key determinants of the choice of a Wanter to get the content legally or illegally. When modeling the two systems, we consider a fluid model, and so the performance is determined primarily by the capacity of each system, i.e., the combination of the initial seeds and the Fraudsters/Boosters that choose to join (and add capacity). However, other factors also play a role, as we describe below. Throughout, we model the upload capacity of a user as being one.

#### 2.1.3.1 The illicit P2P system

There are two components to the model of the illicit P2P network: (i) the efficiency of the network in terms of finding content, and (ii) the initial size of the network and its growth.

Let us start with (i). To capture the efficiency of the P2P system, we take a simple qualitative model. When attaining the content illegally, a Wanter must contact either a Rogue or a Fraudster. We let $\eta(t)$ capture the probability of a Wanter finding a Rogue or a Fraudster when looking for one instantaneous time slot. We consider two cases: an efficient P2P and an inefficient P2P. In an ***efficient P2P***, we model

$$\eta(t) = 1,$$

with the understanding the the P2P allows easy lookup of content and all content is truthfully represented. In contrast, for an ***inefficient P2P***, we model

$$\eta(t) = (N_r(t) + N_f(t))/N,$$

where recall that $N$ is the total population size. This corresponds to looking randomly within the user population for a Rogue or Fraudster. Neither of these models is completely realistic, but they provide lower and upper bounds to the true efficiency of an illicit P2P system.

Next, with respect to (ii), we model the initial condition for the illicit network with $N_i(0) = 0$, since the assumption is that the content has not yet been released, and therefore is not yet available in the illicit P2P swarm. From this initial condition, $N_i(0)$ evolves as follows:

$$\frac{dN_i(t)}{dt} = \min\left\{\eta(t)\left(N_w(t) + \frac{dI(t)}{dt}\right), N_r(t) + N_f(t)\right\}, \tag{2.8}$$

The interpretation of the above is that $N_r(t) + N_f(t)$ is the current capacity of the illicit

P2P and $\eta(t)\left(N_w(t)+\frac{dI(t)}{dt}\right)$ is the fraction of the Wanters (newly arriving and remaining in the system) that find the content in the illicit P2P network. The min operator then ensures that no more than the capacity is used.

#### *2.1.3.2 The legitimate CDN*

As discussed in the introduction, our goal in this work is to contrast the revenue attained by a CDN that uses P2P and revenue sharing with one that does not use P2P. Thus, there are two key factors in modeling the legitimate CDN: (i) the rate at which users that possess content copies become fraudsters or boosters, and (ii) the initial size of the CDN and its growth, which depends on the presence/absence of the legal P2P.

Let us start with (i). From a performance standpoint, the most important parameter is $\kappa$, since it determines what fraction of users stay in the system and act as servers. These users could either support the legal system as boosters, or the illegal one as fraudsters. The question that we wish to answer is that of how much of an impact the division of those who stay into fraudsters and boosters would have on revenue obtained. As we saw earlier,

$$\rho+\beta=\kappa,$$

and our key result will be on their relative impact on obtainable revenue. How we might attempt to control the booster factor $\beta$ through different amounts of revenue sharing requires further modeling of user motivation, which we will consider in greater detail in Section 2.3. But initially we are more concerned with the impact of $\rho$ and $\beta$, rather than how to socially engineer their values.

Next, with respect to (ii), unlike for the illicit P2P swarm, the legitimate network does not start empty. This is because it has a set of dedicated servers at the beginning which are then (possibly) supplemented using a P2P network. We denote by $C_N$ be the capacity of the dedicated CDN servers when the total population size is $N$. Note that this capacity must scale with the total population size to ensure that the average wait time for the users is small. As shown in [66], a natural scaling that ensures no more that $O(\ln\ln N)$ delay is to have the capacity $C_N = \Theta\left(N/\ln N\right)$. Based on this, we adopt

$$C_N = \frac{N}{\ln N}$$

in this work. Additionally, we assume $N_l(0) = 0$ in the case of Flash Crowd model and $N_l(0) = I(0)$ in the case of Bass model.

Given these initial conditions, $N_l(t)$ evolves as follows:

$$\frac{dN_l(t)}{dt} = \begin{cases} C_N + \beta N_l(t), & N_w(t) > 0, \\ \min\left\{C_N + \beta N_l(t), \frac{dI(t)}{dt} - \frac{dN_i(t)}{dt}\right\} & N_w(t) = 0. \end{cases} \tag{2.9}$$

The interpretation for the above is that if there are a positive number of Wanters remaining in the system, then the full current capacity of the CDN can be used to serve them, i.e., $C_N + \beta N_l(t)$. However, if there are no "leftover" Wanters, arriving Wanters that are not served by the illicit P2P ($\frac{dI(t)}{dt} - \frac{dN_i(t)}{dt}$) are served up to the capacity of the CDN.

## 2.2 Results

To characterize the performance of the CDN against the illicit P2P distribution, we use *fractional legitimate copies*, which is defined as follows:

**Definition 1.** *The* ***fractional legitimate copies****, $L$, is defined as*

$$L = \frac{N_l(T_\infty)}{N}, \tag{2.10}$$

*where $T_\infty$ is defined as the time after which only $\Omega(\ln N)$ users are left in the system without a copy of the content*

Using this metric, we look at the performance of the CDN in two settings: when the CDN competes against inefficient illicit P2P sharing and when it competes against efficient illicit P2P sharing. Recall, that our models for these two cases are meant to serve as upper and lower bounds on the true efficiency of an illicit P2P system. We start by considering the case of an inefficient, illicit P2P. Note that the theorems stated below characterize only the asymptotic growth of the fractional legitimate copies.

### *2.2.1 Inefficient illicit P2P*

As discussed before, we look at the performance of CDN, under two simple models of demand evolutions, namely Flash Crowd Model (2.1) and Bass model (2.3).

First, we state the result for Flash Crowd model.

**Theorem 1.** *Suppose $I(t)$ satisfies* (2.1)*. The fractional legitimate copies attained by the content provider in the presence an inefficient, illicit P2P is*

$$L \in \Omega\left(\frac{\ln\ln N + (\ln N)^{\frac{\beta}{\kappa}}}{\ln N}\right). \tag{2.11}$$

*Further, when $\beta = 0$,*

$$L \in \Theta\left(\frac{\ln\ln N}{\ln N}\right). \tag{2.12}$$

*Proof.* To prove theorem we analyze two processes $\bar{N}_l(t)$ and $\bar{N}_i(t)$ which bounds the actual evolutions $N_l(t)$ and $N_i(t)$. Importantly, the bounding processes are equivalent to the original processes when $\beta = 0$.

Before stating the results, we introduce a few notation. Let

$$\begin{aligned}
\theta_1 &= \frac{\kappa}{2} + \frac{\kappa}{2}\sqrt{1 + \frac{4}{\kappa \ln N}}, \quad \theta_2 = \frac{\kappa}{2} - \frac{\kappa}{2}\sqrt{1 + \frac{4}{\kappa \ln N}}, \\
b &= -\frac{\theta_1}{\theta_2}, \quad \Delta\theta = \theta_1 - \theta_2, \qquad (2.13) \\
\bar{\tau} &= \frac{2}{\Delta\theta} \ln\left(\frac{\sqrt{1 + \frac{4}{\kappa \ln N}} + 1}{\sqrt{1 + \frac{4}{\kappa \ln N}} - 1}\right), \qquad (2.14) \\
\bar{N}_l &= \frac{\kappa C_N}{\beta\theta_1}\left(\frac{1}{1+b}\right)^{\frac{\beta}{\kappa}} \left(1 - e^{\left(-\frac{\beta\theta_1\bar{\tau}}{2\kappa}\right)}\right) e^{\left(\frac{\beta\theta_1}{\kappa}\bar{\tau}\right)} \\
&- \frac{\kappa C_N}{\beta\theta_2}\left(\frac{1}{1+b}\right)^{\frac{\beta}{\kappa}} e^{\left(\frac{\bar{\tau}\beta}{2}\right)} \left(1 - e^{\frac{\beta\theta_2\bar{\tau}}{2\kappa}}\right). \qquad (2.15)
\end{aligned}$$

Finally, we are ready to define the bounding processes used in the proof, $\bar{N}_l(t)$ and $\bar{N}_i(t)$. Let $\bar{N}_i(0) = N_i(0)$. Furthermore, let

$$\frac{d\bar{N}_i(t)}{dt} = \frac{\rho\bar{N}_l(t) + \kappa\bar{N}_i(t)}{N}(N - (\bar{N}_l(t) + \bar{N}_i(t))). \tag{2.16}$$

Similarly, let $\bar{N}_l(0) = N_l(0)$ and

$$\frac{d\bar{N}_l(t)}{dt} = \begin{cases} C_N + \beta\bar{N}_l(t)\frac{N-(\bar{N}_l(t)+\bar{N}_i(t))}{N}, & \bar{N}_w(t) > 0, \\ 0, & \bar{N}_w(t) = 0. \end{cases} \tag{2.17}$$

where $\bar{N}_w(t) = N - (\bar{N}_i(t) + \bar{N}_l(t))$.

We can now state our result characterizing the number of legal and illegal copies.

**Lemma 1.** *In the presence of an inefficient, illicit P2P, the number of illegal and legal copies at the end of evolution is*

$$N_l(T_\infty) \geq \bar{N}_l,$$

*where equality holds when* $\beta = 0$.

*Proof.* Recall that the efficiency factor of an inefficient illicit P2P, $\eta(t)$, is given by

$$\eta(t) = \frac{N_r(t) + N_f(t)}{N} = \frac{\rho N_l(t) + \kappa N_i(t)}{N}. \tag{2.18}$$

The second equality follows from (2.5) and (2.6). From (2.8), the illegal growth rate is

$$\frac{dN_i(t)}{dt} \stackrel{(a)}{=} \eta(t)N_w(t) \tag{2.19}$$

$$\stackrel{(b)}{=} \frac{(\rho N_l(t)+\kappa N_i(t))(N-(N_l(t)+N_i(t)))}{N}. \tag{2.20}$$

(a) follows from the definition of $\eta(t)$ and the fact that $N_w(t) \leq N$. (b) follows from (2.18) and (2.4). From equation (2.9), the growth rate of legal copies is given by

$$\frac{dN_l(t)}{dt} = \begin{cases} C_N + \beta N_l(t), & N_w(t) > 0, \\ 0, & N_w(t) = 0. \end{cases} \tag{2.21}$$

Let $U(t)$ be the total copies of the content in the system. Then, $U(t) = N_l(t) + N_i(t)$.

Now, we claim that,

$$N_l(T_\infty) \geq \bar{N}_l(T_\infty), \tag{2.22}$$

and the equality holds when $\beta = 0$.

The proof is as follows: First, we define, $\bar{U}(t) = \bar{N}_l(t) + \bar{N}_i(t)$. We can obtain $\frac{dN_i}{dU}$ and $\frac{d\bar{N}_i}{d\bar{U}}$ from the pair of equations (2.19), (2.21) and (2.16), (2.17) respectively. Then, it can be shown that

$$\frac{dN_i}{dU}|_{N_i=x,U=y} \leq \frac{d\bar{N}_i}{d\bar{U}}|_{\bar{N}_i=x,\bar{U}=y}, \tag{2.23}$$

and the equality holds when $\beta = 0$. Note that the range space of functions $U(t)$ and $\bar{U}(t)$ are identical. Since, the initial values $N_i(0)$ and $\bar{N}_i(0)$ are equal by definition, we get the result in (2.22).

Now, we derive $\bar{N}_l(t)$. Let $\bar{\tau}$ be the time at which the number of wanters in the system vanishes to zero. Then, $\bar{N}_w(t) = 0$ and $\bar{U}(t) = N$ for $t \in [\bar{\tau}, T_\infty]$. Adding (2.17) and (2.16), for $t \in (0, \bar{\tau}]$, we get,

$$\begin{aligned} \frac{d\bar{U}}{dt} &= \left((\beta + \rho)\bar{N}_l(t) + \kappa \bar{N}_i(t)\right) \frac{(N - (\bar{N}_l(t) + \bar{N}_i(t)))}{N} \\ &\overset{(f)}{=} \kappa \bar{U}(t) \frac{N - \bar{U}(t)}{N}. \end{aligned}$$

(f) follows from the fact that $\rho + \beta = \kappa$ and the definition of $\bar{U}(t)$.

The above differential equation is in the form of a standard Riccatti equation, and it's solution can be written as

$$\bar{U}(t) = \frac{N\theta_2}{\kappa} + \frac{N\Delta\theta/\kappa}{1 + be^{-\Delta\theta t}}, \tag{2.24}$$

where $\Delta\theta = \theta_1 - \theta_2$. $\theta_1, \theta_2$ and $b$ are given by equation (2.13). From the relation, $\bar{U}(\bar{\tau}) = N$, we get (2.14).

Now, from (2.17), for $t \in (0, \bar{\tau}]$, we get

$$\frac{d\bar{N}_l(t)}{dt} = C_N + \beta \bar{N}_l(t) \frac{N - (\bar{N}_l(t) + \bar{N}_i(t))}{N}.$$

A lower bound on the solution of the above differential equation is provided by Lemma 8 in Section 2.5. From the defintions of $b$ and $\bar{\tau}$, given by (2.13) and (2.14), it is clear that $b > 1$ and $\bar{\tau} > \ln b/\Delta\theta$. Then, by evaluating (2.147) at $t = \bar{\tau}$ with $\bar{N}_l(0) = I(0)$, we get $\bar{N}_l$ in (2.15). Also, when $\beta = 0$, the lemma yields an exact solution of the above differential equation. Hence proved. □

As mentioned in the statement of Lemma 1, the inequality is exact in the case of $\beta = 0$. Additionally, in this case, the form of $N_l(T_\infty)$ simplifies.

**Corollary 1.** *Let $\beta = 0$. In the presence of an inefficient, illicit P2P, the number of illegal and legal copies is given by*

$$N_l(T_\infty) = \frac{2C_N}{\Delta\theta} \ln \left( \frac{\sqrt{1 + \frac{4}{\kappa \ln N}} + 1}{\sqrt{1 + \frac{4}{\kappa \ln N}} - 1} \right). \tag{2.25}$$

Now that we have characterized the number of legal and illegal copies precisely, attaining the statement in the theorem is accomplished by studying the asymptotics of the results in Lemma 1 and Corollary 1.

To begin, recall from (2.10) that,

$$L = \frac{N_l(T_\infty)}{N} \geq \frac{\bar{N}_l}{N}, \tag{2.26}$$

where $\bar{N}_l$ is defined by (2.15). Following a few algebraic steps, from the above equation, we get that

$$L \in \Omega\left(\frac{\ln\ln N + (\ln N)^{\frac{\beta}{\kappa}}}{\ln N}\right) \tag{2.27}$$

and $L \in \Theta\left(\frac{\ln\ln N}{\ln N}\right)$ if $\beta = 0$, which completes the proof. $\square$

The interpretation of this theorem is striking. When booster factor, $\beta$, is zero, the fractional legitimate copies is exponentially small, $\Theta\left(\frac{\ln\ln N}{\ln N}\right)$. However, as $\beta$ increases, the fractional legitimate copies grows by orders of magnitude.

Now, we consider the second model for demand evolution, Bass model. For analytic reasons, we are not able to work with the exact Bass model. Thus, we approximate the logistic curve, (2.3), as follows:

$$I(t) = \begin{cases} \frac{NI(0)e^t}{N-I(0)+I(0)e^t} & 0 \le t \le T_1 \quad : \text{Phase 1} \\ I_2 = N/\ln N & T_1 < t \le T_2 \quad : \text{Phase 2} \\ I_3 = \frac{N}{2} & T_2 < t \le T_3 \quad : \text{Phase 3} \\ I_4 = N & T_3 < t < T_4 \quad : \text{Phase 4,} \end{cases} \tag{2.28}$$

where we have $T_1 = \ln(N/(I(0)\ln N))$, $T_2 = \ln(N/I(0))$, $T_3 = 2\ln(N/I(0))$ and $T_4 = 3\ln(N/I(0))$.[2] Notice that the first stage is the exact Bass diffusion, while the other stages are order sense approximations of the actual expression. Though this model is approximate, it yields the same qualitative insight as the original model. Now, we are ready to state the result.

**Theorem 2.** *Suppose $I(t)$ satisfies* (2.28)*. The fractional legitimate copies attained by the content provider in the presence an inefficient, illicit P2P is*

$$L \in \Omega\left(\frac{\ln\ln N + (\ln N)^{\frac{\beta}{\kappa}}}{\ln N}\right) \tag{2.29}$$

[2]Note that the value of $T_1$ has been chosen such that $\lim_{N\to\infty} I(T_1) = N/\ln N$.

*Further, when* $\beta = 0$,

$$L \in \Theta\left(\frac{\ln\ln N}{\ln N}\right). \tag{2.30}$$

*Proof.* To prove the theorem, we will go through a sequence of intermediate results characterizing the number of legal/illegal copies at the transition points of the approximate Bass model.

We start by characterizing the number of legal and illegal copies at the end of Phase 1.

**Lemma 2.** *In the presence of an inefficient, illicit P2P, the number of illegal and legal copies at the end of Phase* 1 *of the approximate Bass model are given by*

$$\begin{aligned} N_i(T_1) = &\left(\frac{\rho I(0)}{\kappa-\rho} + \frac{N\rho}{(\kappa-\rho)^2}\right)\exp\left(B_N\right) \\ &- \frac{I(T_1)\rho}{\kappa-\rho} - \frac{N\rho}{(\kappa-\rho)^2} \end{aligned} \tag{2.31}$$

$$N_l(T_1) = I(T_1) - N_i(T_1), \tag{2.32}$$

*where*

$$\begin{aligned} I(T_1) &= \frac{N}{\ln N}\frac{N}{N - I(0) + (N/\ln N)} \\ B_N &= \left(\frac{(\kappa-\rho)}{N}(I(T_1) - I(0))\right). \end{aligned}$$

Note that in the above, we have allowed $\kappa$, $\rho$, and $\beta$ to be arbitrary. In fact, in this case, $\beta$ is inconsequential since the full amount of interested copies can be served by the dedicated capacity of the CDN. Note that in the case when $\rho = \kappa$, things simplify considerably.

**Corollary 2.** *Let* $\rho = \kappa$*. In the presence of an inefficient, illicit P2P, the number of illegal and legal copies at the end of Phase* 1 *of the approximate Bass model are given by*

$$N_i(T_1) = \frac{\kappa(I^2(T_1) - I^2(0))}{2N}$$

$$N_l(T_1) = I(T_1) - N_i(T_1),$$

*where* $I(T_1) = \frac{N}{\ln N} \frac{N}{N - I(0) + (N/\ln N)}$.

We now prove the lemma.

*Proof of Lemma 2.* From equation (2.28), the population of interested copies in phase $I$ is given by

$$I(t) = \frac{NI(0)e^t}{N - I(0) + I(0)e^t}. \tag{2.33}$$

From the above equation, it is easy to verify that the rate of growth of interested copies is less than the server capacity $C_N$, i.e., $dI(t)/dt \leq C_N$. Thus, any interested user is served instantaneously either by a legal or illegal mechanism. Hence, the number of Wanters in the system is zero, i.e, $N_w(t) = 0$. Therefore, it follows from equation (2.4) that $N_l(t) + N_i(t) = I(t)$.

Next, from equation (2.8), we get that

$$\begin{aligned} \frac{dN_i(t)}{dt} &= \min\left\{\eta(t)\frac{dI(t)}{dt}, N_r(t) + N_f(t)\right\} \\ &\overset{(a)}{=} \eta(t)\frac{dI(t)}{dt}, \end{aligned} \tag{2.34}$$

where the equality (a) follows from the definition of $\eta(t)$ and the fact that $dI(t)/dt \leq C_N < N$.

Because we are considering an inefficient P2P, we have

$$\begin{aligned} \eta(t) &= \frac{N_r(t) + N_f(t)}{N}, \\ &\overset{(b)}{=} \frac{\rho N_l(t) + \kappa N_i(t)}{N}, \\ &\overset{(c)}{=} \frac{\rho(I(t) - N_i(t))}{N} + \frac{\kappa N_i(t)}{N}, \\ &= \frac{\rho I(t)}{N} + \frac{(\kappa - \rho)N_i(t)}{N}. \end{aligned}$$

where equality (b) follows from (2.5), (2.6) and the equality (c) follows from the fact that $N_l(t) = I(t) - N_i(t)$. Substituting the above result in equation (2.34), we get

$$\frac{dN_i(t)}{dt} = \frac{dI(t)}{dt}\frac{\rho I(t)}{N} + \frac{dI(t)}{dt}\frac{(\kappa-\rho)N_i(t)}{N}.$$

The solution of the above differential equation is given by

$$N_i(t) = K\exp\left(\frac{I(t)(\kappa-\rho)}{N}\right) - \frac{\rho I(t)}{\kappa-\rho} - \frac{N\rho}{(\kappa-\rho)^2},$$

where the constant $K$ can be obtained from the fact that $N_i(0) = 0$. Thus, the evolution of illegal copies is given by

$$\begin{aligned} N_i(t) = &\left(\frac{\rho I(0)}{\kappa-\rho} + \frac{N\rho}{(\kappa-\rho)^2}\right)\exp\left(\frac{(\kappa-\rho)}{N}(I(t)-I(0))\right) \\ &- \frac{\rho I(t)}{\kappa-\rho} - \frac{N\rho}{(\kappa-\rho)^2}. \end{aligned}$$

The number of illegal copies at the end of Phase 1 can be obtained by evaluating the above expression at $t = T_1$. The remaining population get the content legally, i.e, $N_l(T_1) = I(T_1) - N_i(T_1)$. □

Now that we have characterized the number of legal and illegal copies at the end of Phase 1, we can move to Phases 2-4. Unfortunately, the resulting number of legal and illegal copies at the end of these phases is much more complicated. However, much of this complicated form is only necessary to specify the exact analytic values. Once we focus on the asymptotic form (as in Theorem 1), it simplifies considerably.

Before stating the result, we need to introduce a considerable amount of notation. This notation stems from the fact that we do not analyze the exact process of $N_l(t)$ and $N_i(t)$. Instead, we define a processes $\bar{N}_l(t)$ and $\bar{N}_i(t)$ which bounds $N_l(t)$ and $N_i(t)$ and analyze these processes. Importantly, the bounding processes are equivalent to the original processes when $\beta = 0$, i.e., the case of no revenue sharing. Before defining $\bar{N}_l$ and $\bar{N}_i$, Let

$$\Delta\bar{\tau}_2 = \frac{1}{\kappa \ln N Z_1} \ln\left(\frac{Z_1 + 1 - \frac{2I(T_1)}{(N/\ln N)}}{Z_1 - 1 + \frac{2I(T_1)}{(N/\ln N)}}\right) + \frac{1}{\kappa \ln N Z_1} \ln\left(\frac{Z_1+1}{Z_1-1}\right), \tag{2.35}$$

$$\Delta\bar{\tau}_3 = \frac{2}{\kappa Z_2} \ln\left(\frac{Z_2 + 1 - \frac{4}{\ln N}}{Z_2 - 1 + \frac{4}{\ln N}}\right) + \frac{2}{\kappa Z_2} \ln\left(\frac{Z_2+1}{Z_2-1}\right), \tag{2.36}$$

$$\Delta\bar{\tau}_4 = \frac{1}{\kappa Z_3} \ln\left(\frac{Z_3+1}{Z_3-1}\right), \tag{2.37}$$

where $Z_1 = \sqrt{1 + \frac{4\ln N}{\kappa}}$, $Z_2 = \sqrt{1 + \frac{16}{\kappa \ln N}}$, $Z_3 = \sqrt{1 + \frac{4}{\kappa \ln N}}$ and $I(T_1) = \frac{N}{\ln N}\frac{N}{N - I(0) + (N/\ln N)}$. In addition, let

$$\theta_1^j = \kappa\frac{I_j}{2N} + \frac{1}{2}\sqrt{\left(\frac{\kappa I_j}{N}\right)^2 + \frac{4\kappa}{\ln N}}, \tag{2.38}$$

$$\theta_2^j = \kappa\frac{I_j}{2N} - \frac{1}{2}\sqrt{\left(\frac{\kappa I_j}{N}\right)^2 + \frac{4\kappa}{\ln N}}, \tag{2.39}$$

$\Delta\theta_j = \theta_1^j - \theta_2^j$ and

$$b_j = \frac{N\theta_{1,j} - \kappa I(T_{j-1})}{\kappa I(T_{j-1}) - N\theta_{2,j}}. \tag{2.40}$$

Note that, in the above definition, in fact $I(T_{j-1}) = I_{j-1}$ for $j = 3$ and 4.

Furthermore, for $j = 2, 3$ and 4, let

$$d_j = (b_j + \exp(\Delta\theta_j \Delta\bar{\tau}_j)) \tag{2.41}$$

$$q_1^j = \left(\frac{\beta\theta_2^j}{\kappa} - \frac{\beta I_j}{N}\right) \tag{2.42}$$

$$q_2^j = \frac{\beta\theta_1^j}{\kappa} - \frac{\beta I_j}{N} \tag{2.43}$$

Finally, we are ready to define the bounding processes used in the proof, $\bar{N}_l(t)$ and $\bar{N}_i(t)$. Let $\bar{N}_i(T_1) = N_i(T_1)$. Furthermore, during Phase $j$, let

$$\frac{d\bar{N}_i(t)}{dt} = \frac{\rho\bar{N}_l(t) + \kappa\bar{N}_i(t)}{N}(I_j - (\bar{N}_l(t) + \bar{N}_i(t))). \tag{2.44}$$

Similarly, let $\bar{N}_l(T_1) = N_l(T_1)$ and, during Phase $j$,

$$\frac{d\bar{N}_l(t)}{dt} = \begin{cases} C_N + \beta\bar{N}_l(t)\frac{I_j-(\bar{N}_l(t)+\bar{N}_i(t))}{N}, & \bar{N}_w(t) > 0, \\ 0, & \bar{N}_w(t) = 0. \end{cases} \tag{2.45}$$

where $\bar{N}_w(t) = I_j - (\bar{N}_i(t) + \bar{N}_l(t))$. Finally, let

$$\bar{U}(t) = \bar{N}_l(t) + \bar{N}_i(t).$$

To state the result, we use a bit more notation about these processes. Let $\bar{N}_l^1 = N_l(T_1)$ and for $j = 2, 3$, and 4 define $\bar{N}_l(T_j)$ recursively as follows:

$$\begin{aligned} \bar{N}_l^j = {} & \bar{N}_l^{j-1}\left(\frac{1+b_j}{d_j}\right)^{\frac{\beta}{\kappa}} e^{\left(-q_1^j\Delta\bar{\tau}_j\right)} + \\ & + C_N\left(\frac{b_j}{d_j}\right)^{\frac{\beta}{\kappa}} e^{\left(-q_1^j\Delta\bar{\tau}_j\right)}\left(\frac{e^{\left(q_1^j\frac{\ln b_j}{\Delta\theta_j}\right)}}{q_1^j} - \frac{1}{q_1^j}\right)\mathbf{1}_{b\geq 1} \\ & + C_N\left(\frac{1}{d_j}\right)^{\frac{\beta}{\kappa}} e^{\left(-q_1^j\Delta\bar{\tau}_j\right)}\left(\frac{e^{\left(q_2^j\Delta\bar{\tau}_j\right)}}{q_2^j} - \frac{e^{\left(\frac{q_2^j\ln b_j}{\Delta\theta_j}\right)}\mathbf{1}_{b\geq 1}}{q_2^j}\right) \\ & - C_N\left(\frac{1}{d_j}\right)^{\frac{\beta}{\kappa}} e^{\left(-q_1^j\Delta\bar{\tau}_j\right)}\frac{1}{q_2^j}(1-\mathbf{1}_{b\geq 1}), \end{aligned} \tag{2.46}$$

where $\mathbf{1}_{b\geq 1}$ is given by

$$\mathbf{1}_{b\geq 1} = \begin{cases} 1 & b \geq 1, \\ 0 & b < 1. \end{cases} \tag{2.47}$$

We can now state our result characterizing the number of legal and illegal copies at the end of Phases 2-4.

**Lemma 3.** *In the presence of an inefficient, illicit P2P, the number of illegal and legal copies at the end of Phase $j$, $j \in \{2, 3, 4\}$ of the approximate Bass model are given by*

$$N_l(T_j) \geq \bar{N}_l^j,$$

*where equality holds when $\beta = 0$.*

From the approximate Bass model (2.28), the evolution of demand in Phase $j$, for $j = 2, 3$ and 4, is given by,

$$I(t) = I_j, \quad \text{where} \quad t \in [T_{j-1}, T_j).$$

Note that in these three phases, a change in the number of interested copies occurs only at the beginning of the phase and then, it remains constant throughout the phase. That means, the dynamics of evolutions of $N_l(t)$ and $N_i(t)$ in these phases are similar to that of Flash Crowd model discussed in Lemma 1. Also, it can be shown that each of these phases is long enough so that every interested user appearing at the beginning of a phase is being served by the end of that phase. Therefore, we can analyaze each of these phases independently. Now, by recursively applying the analysis of Lemma 1 for each of the three phases, we get Lemma 3. A detailed proof of the above lemma is given below.

*Proof.* From the approximate Bass model (2.28), the evolution of demand in Phase $j$ is,

$$I(t) = I_j, \quad \text{where} \quad t \in (T_{j-1}, T_j],$$

and the number of Wanters in Phase $j$ is $N_w(t) = I_j - (N_l(t) + N_i(t))$.

Recall that the efficiency factor of an inefficient illicit P2P, $\eta(t)$, is given by

$$\eta(t) = \frac{N_r(t) + N_f(t)}{N} \qquad = \frac{\rho N_l(t) + \kappa N_i(t)}{N}. \tag{2.48}$$

The second equality follows from (2.5) and (2.6).

From equation (2.8), the illegal growth rate in Phase $j$ is

$$\begin{aligned} \frac{dN_i(t)}{dt} &\stackrel{(a)}{=} \min\left\{\eta(t)N_w(t), N_r(t) + N_f(t)\right\}, \\ &\stackrel{(b)}{=} \eta(t)N_w(t) &(2.49) \\ &\stackrel{(c)}{=} \frac{\rho N_l(t) + \kappa N_i(t)}{N}(I_j - (N_l(t) + N_i(t))). &(2.50) \end{aligned}$$

Here (a) follows from the fact that $I(t)$ is constant in the last three phases. (b) follows from the definition of $\eta(t)$ and the fact that $N_w(t) \leq N$. (c) follows from (2.48).

From equation (2.9), the growth rate of legal copies in Phase $j$ is given by

$$\frac{dN_l(t)}{dt} = \begin{cases} C_N + \beta N_l(t), & N_w(t) > 0, \\ 0, & N_w(t) = 0. \end{cases} \tag{2.51}$$

The second equality follows from the fact that $\frac{dN_i}{dt} = 0$ when there are no Wanters in the system (from (2.49)) and $I(t)$ is constant.

Let $U(t)$ be the total copies of the content in the system. Then,

$$U(t) = N_l(t) + N_i(t).$$

Note that the growth rate $N_l(t)$ is at least equal to $C_N$ when $N_w(t) > 0$. In that case, it can be shown that

$$C_N \times (T_j - T_{j-1}) > (I(T_j) - I(T_{j-1})).$$

since $I(0) << C_N$, by assumption. This means that every interested user generated in any one of the last three phases can be served within that phase itself. Furthermore, Lemma 2 shows that no Wanters are left unserved after Phase 1. Therefore, we can conclude that

$$N_l(T_j) + N_i(T_j) = U(T_j) = I(T_j) = I_j. \tag{2.52}$$

The same arguments hold true in the case of $\bar{N}_l(t)$, i.e,

$$\bar{N}_l(T_j) + \bar{N}_i(T_j) = \bar{U}(T_j) = I(T_j) = I_j. \tag{2.53}$$

Now, we claim that,

$$N_l(T_j) \geq \bar{N}_l(T_j), \tag{2.54}$$

and the equality holds when $\beta = 0$.

We can derive $\frac{dN_i}{dU}$ and $\frac{d\bar{N}_i}{d\bar{U}}$ from the pair of equations (2.49), (2.51) and (2.44), (2.45) respectively. Then, it can be shown that

$$\frac{dN_i}{dU}|_{N_i=x,U=y} \leq \frac{d\bar{N}_i}{d\bar{U}}|_{\bar{N}_i=x,\bar{U}=y}, \tag{2.55}$$

and the equality holds when $\beta = 0$. Note that the range space of functions $U(t)$ and $\bar{U}(t)$ are identical; in fact they are equal to $[I(T_{j-1}), I(T_j)]$ in Phase $j$ which follows from (2.52) and (2.53). Furthermore, recall that the initial values of $N_i(T_1)$ and $\bar{N}_i(T_1)$ are equal by definition. Hence, the conclusion is

$$N_i(T_j) \leq \bar{N}_i(T_j).$$

Then, the claim in (2.54) is true from the facts that $N_l(T_j) = I(T_j) - N_i(T_j)$ and $\bar{N}_l(T_j) = I(T_j) - \bar{N}_i(T_j)$.

Our objective is to derive an expression of $\bar{N}_l(t)$. Then, evaluate the expression at $t = T_j$ in order to obtain a lower bound on the number of legal copies at the end of each

Phase $j$.

Let $\bar{\tau}_j$ be the time such that $\bar{U}(\bar{\tau}_j) = I_j$. This event happens within Phase $j$ itself (from (2.53)). i.e, $\bar{\tau}_j \in (T_{j-1}, T_j]$. In addition,

$$\bar{N}_w(t) = 0 \text{ when } t \in (\bar{\tau}_j, T_j].$$

Adding (2.45) and (2.44), for $t \in (T_{j-1}, \bar{\tau}_j]$, we get,

$$\begin{aligned}\frac{d\bar{U}}{dt} &= \big((\beta+\rho)\bar{N}_l(t) + \kappa\bar{N}_i(t)\big)\,\frac{(I_j - (\bar{N}_l(t)+\bar{N}_i(t)))}{N}\\ &\overset{(e)}{=} \big(\kappa\bar{N}_l(t) + \kappa\bar{N}_i(t)\big)\,\frac{(I_j - (\bar{N}_l(t)+\bar{N}_i(t)))}{N}\\ &\overset{(f)}{=} \kappa\bar{U}(t)\frac{I_j - \bar{U}(t)}{N}.\end{aligned}$$

(e) follows from the fact that $\rho + \beta = \kappa$. (f) follows from the definition of $\bar{U}(t)$ in Phase $j$.

The differential equation given above is a standard Riccatti equation. Its solution is given by

$$\bar{U}(t) = \frac{N\theta_{2,j}}{\kappa} + \frac{N\Delta\theta_j/\kappa}{1 + b_j e^{-\Delta\theta_j(t-T_{j-1})}}, \tag{2.56}$$

where $\Delta\theta_j = \theta_{1,j} - \theta_{2,j}$. $\theta_{1,j}, \theta_{2,j}$ and $b_j$ are given by equations (2.38), (2.39) and (2.40) respectively.

Let $\Delta\bar{\tau}_j = \bar{\tau}_j - T_{j-1}$. Recall that $\bar{\tau}_j$ is the solution of the equation $\bar{U}(\bar{\tau}_j) = I_j$. Hence, from the above result, we get,

$$\begin{aligned}\bar{\tau}_j - T_{j-1} = &\frac{1}{\Delta\theta_j}\ln\left(\frac{\sqrt{1+\frac{4}{\kappa\ln N}}j + 1 - \frac{2I(T_{j-1})}{I(T_j)}}{\sqrt{1+\frac{4}{\kappa\ln N}}j - 1 + \frac{2I(T_{j-1})}{I(T_j)}}\right)\\ &+ \frac{1}{\Delta\theta_j}\ln\left(\frac{\sqrt{1+\frac{4}{\kappa\ln N}}j + 1}{\sqrt{1+\frac{4}{\kappa\ln N}}j - 1}\right).\end{aligned} \tag{2.57}$$

The above expression yields (2.35), (2.36) and (2.37) respectively, when $I(T_j)$ is substituted by actual values from the bass model.

Now, applying the above expression in (2.45), for $t \in (T_{j-1}, \bar{\tau}_j]$, we get

$$\frac{d\bar{N}_l(t)}{dt} = C_N + \beta \bar{N}_l(t) \frac{I_j - (\bar{N}_l(t) + \bar{N}_i(t))}{N}.$$

A lower bound on the solution of the above differential equation is provided by Lemma 8 in Section 2.5. It can be shown that $b \exp(-\Delta\theta_j \Delta\bar{\tau}_j) << 1$. Then $\bar{\tau}_j$ satisfies the condition stipulated by that lemma and a lower bound on the number of legal at the end of Phase $j$ can be obtained by evaluating (2.147) at $t = \bar{\tau}_j$, which yields $\bar{N}_l^j$ in (2.46). In case $\beta = 0$, (2.147) is an exact solution of the above differential equation. □

As mentioned in the statement of Lemma 3, the inequality is exact in the case of $\beta = 0$. Additionally, in this case, the form of $N_l(T_4)$ simplifies.

**Corollary 3.** *Let $\beta = 0$. In the presence of an inefficient, illicit P2P, the number of illegal and legal copies at the end of Phase 4 of the approximate Bass model is given by*

$$N_l(T_4) = N_l(T_1) + C_N \sum_{j=2}^{4} \Delta\bar{\tau}_j \tag{2.58}$$

*where $N_l(T_1)$ is given by Corollary 2.*

Now that we have characterized the number of legal and illegal copies at the end of Phase 4 precisely, attaining the statement in theorem is accomplished by taking studying the asymptotics of the results in Lemma 3 and Corollary 3. Throughout, we use $A_N \sim B_N$ to denote $\lim_{N\to\infty} \frac{A_N}{B_N} = 1$.

To begin, recall from (2.10) that,

$$L = \frac{N_l(T_\infty)}{N} = \frac{N_l(T_\infty)}{N} \tag{2.59}$$

$$\geq \frac{\bar{N}_l^4}{N}, \tag{2.60}$$

where $\bar{N}_l^4$ is recursively defined by (2.46) in terms of $\bar{N}_l^1, \bar{N}_l^2$ and $\bar{N}_l^3$. As $N$ goes larger, from the above equation, we get that

$$L \in \Omega\left(\frac{\ln\ln N + (\ln N)^{\frac{\beta}{\kappa}}}{\ln N}\right) \tag{2.61}$$

and $L \in \Theta\left(\frac{\ln\ln N}{\ln N}\right)$ if $\beta = 0$, which completes the proof.

□

Note that the results of the above theorem match with that of Theorem 1. That means, the fractional legitimate copies attained by the CDN under Bass model of evolution is no different from that of Flash Crowd model in asymptotic sense.

Next, let us consider the case of an efficient, illicit P2P system.

### *2.2.2 Efficient illicit P2P*

As before, we first consider the case of Flash Crowd model.

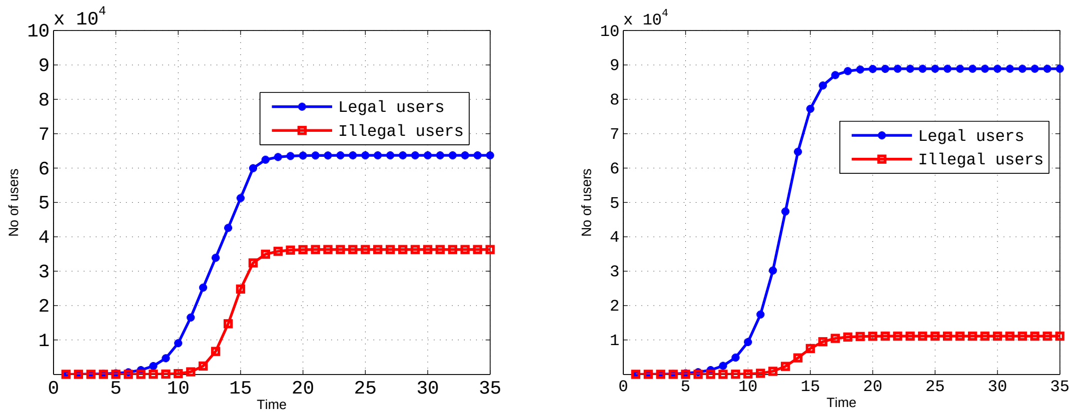
**Theorem 3.** *Suppose $I(t)$ satisfies* (2.1)*. Let $\kappa \in (0, 1 - I(0)/N)$. The fractional legitimate copies attained by the content provider in the presence an efficient, illicit P2P is*

$$L \in \Omega\left(\frac{1}{\ln N}\frac{(\ln N)^{\frac{\beta}{\kappa}} - 1}{\left(\frac{\beta}{\kappa}\right)}\right). \tag{2.62}$$

*Further, when $\beta = 0$,*

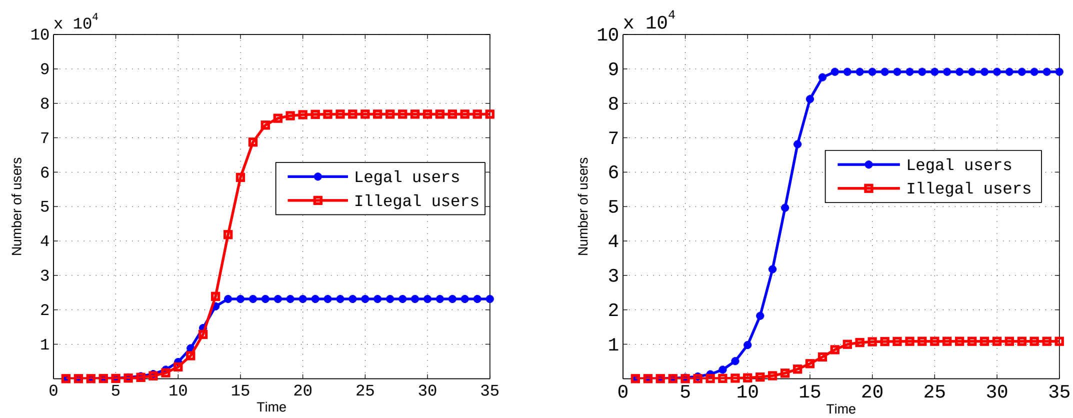
$$L \in \Theta\left(\frac{\ln\ln N}{\ln N}\right). \tag{2.63}$$

*Proof.* The proof parallels to that of Theorem 1.We mimick the approach of the proof of Theorem 3 and define two processes $\bar{N}_l(t)$ and $\bar{N}_i(t)$ that bound $N_l(t)$ and $N_i(t)$ and analyze these processes. Importantly, the bounding processes are equivalent to the original processes when $\beta = 0$.




(a) $\kappa = 0.75, \beta = 0$ (b) $\kappa = 0.75, \beta = 0.52$

Figure 2.3: Evolution of usage in the presence of inefficient illicit P2P sharing.




(a) $\kappa = 0.4, \beta = 0$ (b) $\kappa = 0.4, \beta = 0.38$

Figure 2.4: Evolution of usage in the presence of efficient illicit P2P sharing.

Let $\bar{U}(t) = \bar{N}_l(t) + \bar{N}_i(t)$. Further, let $\bar{N}_l(0) = N_l(0) = 0$ and

$$\frac{d\bar{N}_l(t)}{dt} = = \begin{cases} C_N + \beta \bar{N}_l(t) & \bar{N}_w(t) > 0, \\ 0 & \bar{N}_w(t) = 0. \end{cases} \tag{2.64}$$

where $\bar{N}_w(t) = N - \bar{U}(t)$. Furthermore, we define $\bar{N}_i(0) = N_i(0) = 0$ and

$$\frac{d\bar{N}_i(t)}{dt} = \begin{cases} \rho\bar{N}_l(t) + \kappa\bar{N}_i(t) & 0 \leq \bar{U}(t) \leq \frac{N}{1+\rho}, \\ N - \bar{N}_l(t) - \bar{N}_i(t) & \frac{N}{1+\rho} \leq \bar{U}(t) \leq N. \end{cases} \tag{2.65}$$

Finally, let $\bar{N}_i(0) = N_i(0) = 0$. To state the results, we may need a bit more notation. Let

$$\bar{N}_l = \frac{N}{\ln N\beta}\left(e^{\beta\bar{\tau}} - 1\right). \tag{2.66}$$

Furthermore, $\bar{\tau} = \frac{1}{1+\beta}\ln\left(1 + \frac{\ln N(1+\beta)H^{\frac{-\beta}{\kappa}}}{1+\rho}\right) + \frac{1}{\kappa}\ln(H)$, where $H = 1 + \frac{\kappa \ln N}{(1+\rho)}$. Now, we characterize the number of legal copies and illegal copies in the following lemma.

**Lemma 4.** *In the presence of an efficient, illicit P2P, the number of illegal copies is given by*

$$N_l(T_\infty) \geq \bar{N}_l, \tag{2.67}$$

*and the equality holds when* $\beta = 0$.

*Proof.* From equation (2.8), the growth rate of illegal copies is given by

$$\frac{dN_i}{dt} \stackrel{a}{=} \min\{N_w(t), \rho N_l(t) + \kappa N_i(t)\} \tag{2.68}$$

$$\stackrel{b}{=} \min\{I(t) - U(t), \rho N_l(t) + \kappa N_i(t)\} \tag{2.69}$$

where (a) follows from equations (2.5), (2.6) along with the facts that $\eta = 1$ and $I(t)$ is constant. (b) follows from the definition of the number of wanters in the system.

From equation (2.9), the growth rate of legal copies in Phase $j$ is given by

$$\frac{dN_l(t)}{dt} \stackrel{c}{=} C_N + \beta N_l(t) \quad \text{if} \quad N_w(t) > 0,$$

$$\stackrel{d}{=} 0 \quad \text{if} \quad N_w(t) = 0. \tag{2.70}$$

(d) follows from the facts that $\frac{dN_i}{dt} = 0$ when there are no wanters in the system (from (2.68)) and $I(t)$ is constant.

As defined before, let $U(t)$ be the total copies of the content in the system. Then, $U(t) = N_l(t) + N_i(t)$.

Now, we claim that,

$$N_l(T_j) \geq \bar{N}_l(T_j). \tag{2.71}$$

and the equality holds when $\beta = 0$.

Note that

$$\frac{d\bar{N}_l(t)}{dt}|_{\bar{U}=x,\bar{N}_i=y} \stackrel{e}{=} \frac{dN_l(t)}{dt}|_{U=x,N_i=y}, \tag{2.72}$$
$$\frac{d\bar{N}_i(t)}{dt}|_{\bar{U}=x,\bar{N}_i=y} \stackrel{f}{\geq} \frac{dN_i(t)}{dt}|_{U=x,N_i=y}. \tag{2.73}$$

and (f) is an equality when $\beta = 0$. (e) follows from (2.64) and (2.70). And (f) is due to (2.68) and (2.65). From the above equations, we can deduce that

$$\frac{d\bar{N}_l}{d\bar{U}}|_{\bar{U}=x,\bar{N}_i=y} \leq \frac{dN_l}{dU}|_{U=x,N_i=y}. \tag{2.74}$$

Note that the range of functions $U(t)$ and $\bar{U}(t)$ are identical, $[I(0), N]$. Since $N_l(0) = \bar{N}_l(0)$, from the above equation, we get that $N_l(T_j) \geq \bar{N}_i(T_j)$, Also, equality holds when $\beta = 0$.

Let $\bar{\tau}$ be the instant at which $\bar{N}_w(\bar{\tau}) = 0$. Then, the number of legal copies, $N_l(t)$, is given by

$$\bar{N}_l(t) = \begin{cases} \left(\frac{C_N}{\beta}\right) e^{\beta t} - \frac{C_N}{\beta} & t \in (0, \bar{\tau}], \\ \bar{N}_l(\bar{\tau}) & t > \bar{\tau}. \end{cases} \tag{2.75}$$

The above result follows from (2.64) and the initial condition $N_l(0) = 0$. Now, we resort to find $\bar{\tau}$. Note that, $\bar{N}_w(\bar{\tau}) = 0$ implies $\bar{U}(\bar{\tau}) = N$. Therefore, first we derive $\bar{U}(t)$ and then, finds the time at which $\bar{U}(t)$ reaches $N$.

Note that $\bar{U}(0) < \frac{N}{1+\rho}$, by assumption. Then, from (2.64) and (2.65), we get that

$$\frac{d\bar{U}(t)}{dt} = \rho\bar{U}(t) + C_N, \quad \text{if} \quad t \in [0, \nu],$$

where $\nu$ is defined as $\bar{U}(\nu) = \frac{N}{1+\rho}$. Solving the above equation with the initial condition $\bar{U}(0) = 0$ yields

$$\bar{U}(t) = \frac{C_N}{\kappa} e^{\kappa t} - \frac{C_N}{\kappa}, \quad \text{if} \quad t \in [0, \nu]. \tag{2.76}$$

Then, from the above result $\nu$ can shown to be $\nu = \frac{1}{\kappa}\ln(H)$, where $H = 1 + \frac{\kappa \ln N}{1+\rho}$.

Now, consider the case $t \in [\nu, \bar{\tau}]$. Then, $\frac{N}{1+\rho} \leq \bar{U}(t) \leq N$ and hence, from (2.65),

$$\frac{dN_i}{dt} = N - \bar{N}_l(t) - \bar{N}_i(t), \quad \text{if} \quad t \in [\nu, \bar{\tau}].$$

Solving the above equation, we get

$$\begin{aligned}
\bar{N}_i(t) &= N - \left(\bar{N}_l(\nu) + \frac{C_N}{\beta}\right)\frac{e^{\beta(t-\nu)}}{1+\beta} + \frac{C_N}{\beta} \\
&\quad + \left(\bar{N}_i(\nu) + \frac{\bar{N}_l(\nu)}{1+\beta} - \frac{C_N}{1+\beta} - N\right) e^{-(t-\nu)}, \\
&= N - \frac{C_N}{\beta}\frac{e^{\beta(t)}}{1+\beta} + \frac{C_N}{\beta} \\
&\quad - \left(\frac{N\rho}{1+\rho} + \frac{C_N e^{\beta\nu}}{1+\beta}\right) e^{-(t-\nu)},
\end{aligned}$$

for $t \in [\nu, \bar{\tau}]$. Here, the second equality is obtained by replacing $\bar{N}_i(\nu)$ with $\bar{U}(\nu) - \bar{N}_l(\nu)$ and by substituting $\bar{N}_l(\nu)$ from (2.75). Then, $\bar{U}(t)$, which is eqaul to $\bar{N}_l(t) + \bar{N}_i(t)$, is given by

$$\bar{U}(t) = N + \frac{C_N e^{\beta t}}{1+\beta} - \left(\frac{N\rho}{1+\rho} + \frac{C_N e^{\beta\nu}}{1+\beta}\right) e^{-(t-\nu)}.$$

Now, solving for $t$, from $\bar{U}(t) = N$, we get that

$$\bar{\tau} = \nu + \frac{1}{1+\beta} \ln \left(1 + \frac{\ln N(1+\beta)e^{-\beta\nu}}{1+\rho}\right) \tag{2.77}$$

$$= \frac{1}{\kappa} \ln H + \frac{1}{1+\beta} \ln \left(1 + \frac{\ln N(1+\beta)H^{\frac{-\beta}{\kappa}}}{1+\rho}\right). \tag{2.78}$$

The second result follows by susbtituting $\nu = \frac{1}{\kappa} \ln H$, where $H = 1 + \frac{\kappa \ln N}{1+\rho}$.

Finally, substituting $\bar{\tau}$ in (2.75) yields $\bar{N}_l$, which completes the proof. □

As mentioned in the statement of Lemma 4, the inequality is exact in the case of $\beta = 0$. Additionally, in this case, the form of $N_l(T_\infty)$ simplifies.

**Corollary 4.** *Let $\beta = 0$. Then, the number of legal copies at the end of Phase 4 is given by $N_l(T_\infty) = C_N \bar{\tau}$,*

Now that we have characterized the number of legal and illegal copies precisely, attaining the statement in theorem is accomplished by studying the asymptotics of the results in Lemma 4 and Corollary 4. From (2.10), Lemma 4, Corollary 4 and equation (2.66), we can show that

$$L \in \Omega\left(\frac{1}{\ln N} \frac{(\ln N)^{\frac{\beta}{\kappa}} - 1}{\left(\frac{\beta}{\kappa}\right)}\right), \tag{2.79}$$

and $L \in \Theta\left(\frac{\ln \ln N}{\ln N}\right)$ if $\beta = 0$, which completes the proof.

□

Again, the fractional legitimate copies rises by an order of magnitude as the booster factor, $\beta$, increases. Interestingly, the efficiency of the illicit P2P does not impact the asymptotic order of the fractional revenue when $\beta = 0$, since in both the efficient and inefficient case it is $\Theta\left(\frac{\ln \ln N}{\ln N}\right)$. However, the efficiency of the illicit P2P does affect the fractional legitimate copies attained for positive values of booster factor. In particular, it causes a $(1 - \frac{\beta}{\kappa})$ factor change in the fractional legitimate copies attained; however this has almost no effect on the asymptotic growth.

Now, we consider the second case, Bass model of evolution.

**Theorem 4.** *Suppose $I(t)$ satisfies (2.3). Let $\kappa \in (0, 1 - I(0)/N)$. The fractional legitimate copies attained by the content provider in the presence an efficient, illicit P2P is*

$$L \in \Omega\left(\frac{1}{\ln N}\frac{(\ln N)^{\frac{\beta}{\kappa}} - 1}{\left(\frac{\beta}{\kappa}\right)}\right). \tag{2.80}$$

*Further, when $\beta = 0$,*

$$L \in \Theta\left(\frac{\ln \ln N}{\ln N}\right). \tag{2.81}$$

*Proof.* In our model, an efficient illicit P2P is characterized by efficiency parameter, $\eta(t)$, equal to one. Then, from (2.8), the evolution of illegal copies of content in the system, $N_i(t)$, is given by

$$\frac{dN_i(t)}{dt} = \min\left\{N_w(t) + \frac{dI(t)}{dt}, \rho N_l(t) + \kappa N_i(t)\right\}. \tag{2.82}$$

And, the evolution of legal copies of the content in the system, $N_i(t)$, is given by,

$$\frac{dN_l(t)}{dt} = \begin{cases} C_N + \beta N_l(t) & N_w(t) > 0, \\ \min\{C_N + \beta N_l(t), \frac{dI}{dt} - \frac{dN_i}{dt}\} & N_w(t) = 0. \end{cases} \tag{2.83}$$

As the interest for the content evolves according to the Bass demand model, the evolution of $N_l(t)$ and $N_i(t)$ traverses along multiple stages of dynamics as shown in Figure 2.5. Below, we discuss these stages of evolution in detail.

**Stage 1**: By assumption, $N_l(0) = I(0)$, $N_i(0) = 0$ and $N_w(0) = 0$ where $I(0)$ is the initial demand in the system. Then,

$$N_w(0) + \frac{dI(t)}{dt}|_{t=0} > \rho N_l(0) + \kappa N_i(0).$$

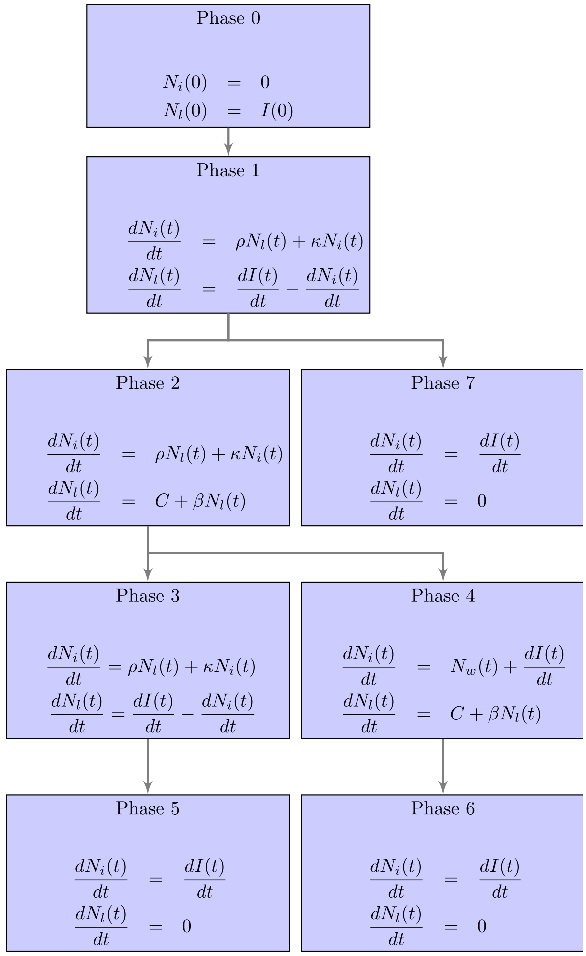



Figure 2.5: Evolutionary phases of the growth of legal and illegal copies of content in the presence of an efficient illicit P2P

The above result follows from our assumption that $\kappa < 1 - \frac{I(0)}{N}$. Therefore, at $t = 0$, from (2.82),

$$\frac{dN_i(t)}{dt} = \rho N_l(t) + \kappa N_i(t). \tag{2.84}$$

From (2.83), the evolution of $N_l(t)$ at time $t = 0$ is,

$$\begin{aligned}
\frac{dN_l(t)}{dt} &= \frac{dI(t)}{dt} - \frac{dN_i(t)}{dt}, && (2.85)\\
&= \frac{dI(t)}{dt} - (\rho N_l(t) + \kappa N_i(t)). && (2.86)
\end{aligned}$$

The first equality follows from the facts that $N_w(0) = 0$ and $\frac{dI(t)}{dt}|_{t=0} < C_N$. Also, from the above equations, we get that $N_l(t) + N_i(t) = I(t)$.

The evolution exits Stage 1 when any one of the following conditions is attained,

$$\text{C1}: \qquad \frac{dI}{dt}(t) - \frac{dN_i}{dt} \geq C_N + \beta N_l(t), \tag{2.87}$$

$$\text{C2}: \qquad \frac{dI}{dt}(t) \leq \rho N_l(t) + \kappa N_i(t). \tag{2.88}$$

Here, C1 occurs when the number of wanters approaching the legitimate CDN exceeds its current capacity, Then, from (2.83), the dynamics of evolution of $N_l(t)$ changes. C2 happens when the number of users attempting to download from the illicit P2P reduces below the current capacity of the illicit P2P. Then, from (2.82), the dynamics of evolution of $N_i(t)$ changes. Next, we show if $\kappa < 1 - \frac{2}{\sqrt{\ln N}}$, C1 occurs before C2 and the evolution proceeds to Stage 2. Otherwise, Stage 1 is followed by Stage 7.

Now, let $T_2$, be the time at which C1 is attained, i.e,

$$\frac{dI(t)}{dt}|_{t=T_2} - \frac{dN_i(t)}{dt}|_{t=T_2} = C_N + \beta N_l(T_2), \tag{2.89}$$

$$\Rightarrow \quad \frac{dI(t)}{dt}|_{t=T_2} - \kappa I(T_2) = C_N \tag{2.90}$$

$$\Rightarrow \quad I(T_2) = \frac{N(1-\kappa)}{2}\left[1 - \sqrt{1 - \frac{4}{\ln N(1-\kappa)^2}}\right] \tag{2.91}$$

The second equality follows from (2.84) along with the facts that $\kappa = \rho + \beta$ and $N_l(t) + N_i(t) = I(t)$. Equation (2.91) follows from the definition of $I(t)$. In the above equation, $T_2$ has a real positive solution iff $\kappa < 1 - \frac{2}{\sqrt{\ln N}}$. Also, let $T_7$ be the time at which C2 is attained, i.e,

$$\frac{dI(t)}{dt}|_{t=T_7} = \rho N_l(T_7) + \kappa N_i(T_7)$$

$$\Rightarrow \frac{dI(t)}{dt}|_{t=T_7} - \kappa I(T_7) = -\beta N_l(T_7). \tag{2.92}$$

The second equality follows from the facts that $\kappa = \rho + \beta$ and $N_l(t) + N_i(t) = I(t)$. From

(2.90), (2.92) and the definition of $I(t)$, it can be shown that, if $T_2$ has a real valued solution, then $T_2 < T_7$. Therefore, Stage 1 is followed by Stage 2 if $\kappa < 1 - \frac{2}{\sqrt{\ln N}}$ and, Stage 7 otherwise.

**Stage 2** : The evolution enters Stage 2 from Stage 1 due to the condition C1 given by (2.87). Then, the dynamics of $N_i(t)$ does not change from that of Stage 1,

$$\frac{dN_i}{dt} = \rho N_l(t) + \kappa N_i(t), \tag{2.93}$$

but the dynamics of $N_l(t)$ changes to,

$$\frac{dN_l}{dt} = C_N + \beta N_l(t). \tag{2.94}$$

Also, from the above equations and (2.87), $N_l(t) + N_i(t) \leq I(t)$.

A transition from this stage occurs when any one of the following conditions is satisfied,

$$\begin{aligned} \text{C3 :} \qquad & C_N + \beta N_l(t) \geq \frac{dI(t)}{dt} - \frac{dN_i(t)}{dt}, \\ & N_w(t) = 0, && (2.95) \\ \text{C4 :} \qquad & \frac{dI(t)}{dt} + N_w(t) \leq \rho N_l(t) + \kappa N_i(t). && (2.96) \end{aligned}$$

Here, C3 occurs when the number of wanters in the system goes to zero and the rate at which newly generated population approaching the legitimate CDN falls below its current capacity. Then, from (2.83), the dynamics of evolution of $N_l(t)$ changes. C2 happens when the number of users attempting to download from the illicit P2P reduces below the current capacity of the illicit P2P. Then, from (2.82), the dynamics of evolution of $N_i(t)$ changes. The evolution enters Stage 3, if C3 is attained before C4. Otherwise, it proceeds to Stage 4.

Let $T_3$ mark the time at which the evolution enters Stage 3. Then, from C3 and (2.93),

$$C_N + \beta N_l(T_3) \geq \frac{dI(t)}{dt} |_{t=T_3} - (\rho N_i(T_3) + \kappa N_l(T_3)), \tag{2.97}$$

$$\text{and} \quad N_w(T_3) = 0. \tag{2.98}$$

Also, let Stage 4 start at time $t = T_4$. Then, from C4,

$$\frac{dI(t)}{dt}|_{t=T_4} + N_w(T_4) = \rho N_l(T_4) + \kappa N_i(T_4). \tag{2.99}$$

**Stage 3**: The evolution enters Stage 3 from Stage 2 due to the condition C3 given by (2.95). Then, the dynamics $N_i(t)$ does not change from that of Stage 2,

$$\frac{dN_i(t)}{dt} = \rho N_l(t) + \kappa N_i(t), \tag{2.100}$$

but, the evolution of $N_l(t)$ changes to,

$$\begin{aligned} \frac{dN_l(t)}{dt} &= \frac{dI(t)}{dt} - \frac{dN_i(t)}{dt}, & (2.101) \\ &= \frac{dI(t)}{dt} - (\rho N_l(t) + \kappa N_i(t)). & (2.102) \end{aligned}$$

This stage starts at $t = T_3$, which is defined by (2.97) and (2.98). From the above dynamics equations and (2.98), we get $N_l(t) + N_i(t) = I(t)$.

We show that the evolution of $N_l(t)$, given by (2.101), does not change as long as the evolution of $N_i(t)$ does not deviate from (2.100). This claim holds true if

$$\begin{aligned} C_N + \beta N_l(t) &\geq \frac{dI(t)}{dt} - (\rho N_l(t) + \kappa N_i(t)), \\ \Rightarrow \quad & \frac{dI(t)}{dt} - \kappa I(t) \leq C_N, & (2.103) \end{aligned}$$

for all $t \geq T_3$. The second inequality follows from the facts $\kappa = \rho + \beta$ and $N_l(t) + N_i(t) = I(t)$. At $t = T_3$ the above requirement is met, which follows from (2.97). Then, we get

$$I(T_3) \geq \frac{N(1-\kappa)}{2}, \tag{2.104}$$

from the definition of $I(t)$ and (2.103). The function $\frac{dI(t)}{dt} - \kappa I(t)$ is monotonically decreasing if $I(t) > \frac{N(1-\kappa)}{2}$. Then, (2.103) holds for all $t > T_3$ and that proves our claim.

The above discussion implies that a transition from this stage happens only when the dynamics of evolution of $N_i(t)$ changes. From (2.82) and (2.100), the dynamics of $N_i(t)$ changes, when the number of users downloading from the illicit P2P reduces below the current capacity of illicit P2P,

$$\text{C5}: \quad \frac{dI(t)}{dt} \leq \rho N_l(t) + \kappa N_i(t). \tag{2.105}$$

When C5 occurs, evolution enters Stage 5. Let this occurs at $t = T_5$. Then,

$$\frac{dI(t)}{dt}\,|_{t=T_5} = \rho N_l(T_5) + \kappa N_i(T_5). \tag{2.106}$$

**Stage 4**: The evolution enters Stage 3 from Stage 2 due to the condition C4 given by (2.96). Then, the dynamics of $N_l(t)$ does not change from that of Stage 2,

$$\frac{dN_l(t)}{dt} = C_N + \beta N_l(t), \tag{2.107}$$

but the evolution of $N_i(t)$ changes to,

$$\frac{dN_i(t)}{dt} = N_w(t) + \frac{dI(t)}{dt}, \tag{2.108}$$

This stage starts at time $t = T_4$ defined by (2.99).

We claim that the evolution of $N_i(t)$ follows (2.108) for all $t \geq T_4$. This claim holds true if

$$\left(N_w(t) + \frac{dI(t)}{dt}\right) \leq \rho N_l(t) + \kappa N_i(t), \tag{2.109}$$

for all $t \geq T_4$. Note that Equation (2.109) holds true at $t = T_4$. Since, $N_w(t) = I(t) - (N_l(t) + N_i(t))$ by definition, from Equation (2.108), we get that $\frac{dN_w(t)}{dt} < 0$. Also, using the definition of $N_w(t)$ in (2.99), we can show that

$$\frac{dI(t)}{dt}\,|_{t=T_4} - \kappa I(T_4) = -(1+\kappa)N_w(T_4) - \beta N_l(T_4) < 0.$$

Then, from the definition of $I(t)$, the above result holds for all $t \geq T_4$. Then, we get

$$\frac{d}{dt}\left(N_w(t) + \frac{dI}{dt}\right) < \frac{d}{dt}(\rho N_l(t) + \kappa N_i(t)),$$

which along with (2.99) proves (2.109).

The above discussion implies that a transition from this stage occurs when the evolution of $N_l(t)$ changes. From (2.107) and (2.83), the evolution of $N_l(t)$ changes when the number of wanters goes to zero. Then,

$$N_w(T_6) = 0. \tag{2.110}$$

where $T_6$ marks the beginning of Stage 6.

**Stage $\mathbf{5}, \mathbf{6}, \mathbf{7}$:**

These are the final stages of evolution. Stage 5 is preceded by Stage 3, Stage 6 is preceded by Stage 4, and Stage 7 is preceded by Stage 1. The dynamics of all these stages are identical,

$$\frac{dN_l(t)}{dt} = 0, \tag{2.111}$$

$$\frac{dN_i(t)}{dt} = \frac{dI(t)}{dt}. \tag{2.112}$$

It is easy to see that the evolutions of $N_l(t)$ and $N_l(t)$ stay in these stages forever once they reach here.

In summary, if $\kappa \geq 1 - \frac{2}{\sqrt{lnN}}$, the evolution of $N_i(t)$ and $N_l(t)$ traverse along the sequence of phases, *Stage* 1 $\rightarrow$*Stage* 7. Otherwise, they proceed along the sequence of phases, *Stage* 1 $\rightarrow$ *Stage* 2 $\rightarrow$*Stage* 3(*Stage* 4) $\rightarrow$*Stage* 5(*Stage* 6). In the next section, we analyze these two cases separately and obtain a lower bound on number of legal copies of the content in the system at the end of evolution.

### *2.2.3 Analysis*

We first consider the case, $\kappa \geq 1 - \frac{2}{\sqrt{\ln N}}$. Let us introduce a few notation before stating the result. We define

$$\Phi(x) = \left(\frac{I(0)}{N}\right)^{\beta} N \left[(1-\kappa)\psi\left(\beta, \frac{x}{N}\right) - \kappa\psi\left(\beta - 1, \frac{x}{N}\right)\right], \tag{2.113}$$

and $\psi(\beta, x) = \int_{I(0)/N}^{x} \left(\frac{1-u}{u}\right)^{\beta} du$. Also, let

$$\bar{T} = \ln\left[\frac{N(1-\kappa)G}{I(0)\left(2-(1-\kappa)G\right)}\right], \tag{2.114}$$

where $G = 1 + \sqrt{1 + \frac{4\beta D}{N(1-\kappa)^2}}$ and $D = \Phi(N(1-\kappa))\left(\frac{N(1-\kappa)}{I(0)\kappa}\right)^{\beta}$. Now, we are ready to provide the result.

**Lemma 5.** *Assume $\kappa \geq 1 - \frac{2}{\sqrt{\ln N}}$. Then, a lower bound on the number of legal copies of the content in the system at $t = T_\infty$ is given by,*

$$N_l(T_\infty) \geq (\Phi(I(\bar{T})) + I(0))e^{\beta\bar{T}}. \tag{2.115}$$

*where $I(t)$ is given by* (2.3)*.*

*Proof.* Recall that, when $\kappa \geq 1 - \frac{2}{\sqrt{\ln N}}$, the evolution of $N_l(t)$ and $N_i(t)$ takes place in two stages, namely Stage 1 and Stage 7. Solving the dynamics of evolution in Stage 1, given by (2.85) and (2.84), we get

$$\begin{aligned} N_l(t) &= (\Phi(I(t)) - \Phi(I(0))e^{\beta t} + I(0)e^{\beta t}, \\ &= (\Phi(I(t)) + I(0))e^{\beta t}, \end{aligned} \tag{2.116}$$

where $\Phi(x)$ is defined by (2.113). The second equality follows since $\Phi(I(0)) = 0$.

Stage 7 starts at $t = T_7$. Recall from (2.92) that $T_7$ is a solution to the equation,

$$\frac{dI(t)}{dt} - \kappa I(t) = -\beta N_l(t)$$

. It is not easy to solve the above equation exactly . Hence, here, we obtain a lower bound on $T_7$. Let $r = \ln(\frac{N(1-\kappa)}{I(0)\kappa})$. Note that, at $t = r$,

$$\frac{dI}{dt}(t) - \kappa I(t) = 0.$$

Also, the function $\frac{dI}{dt}(t) - \kappa I(t)$ is positive for $t < r$ and, it is monotonically decreasing for $t \geq r$. Then, $r \leq T_7$. Then, $N_l(r) \leq N_l(T_7)$. That implies the solution of the equation,

$$\frac{dI}{dt} - \kappa I(t) = -\beta N_l(r),$$

must be less than or equal to $T_7$. Now, substituting $N_l(r)$ from Equation (2.116) in the above equation, and then, solving for $t$ yields $\bar{T}$, which is defined by (2.114), as the unique solution. Since no legals are generated in Stage 7 according to (2.111), and $T_7 \geq \bar{T}$, we have

$$N_l(T_\infty) = N_l(T_7) \geq N_l(\bar{T}).$$

Now, obtain $N_l(\bar{T})$ from (2.116) and substitute in the above inequality to prove the lemma.

$\square$

Now, we consider the second case where $\kappa < 1 - \frac{2}{\sqrt{\ln N}}$. We introduce a few notation before stating the result. Let

$$I_2 = \tfrac{N(1-\kappa)}{2}\left[1 - \sqrt{1 - \tfrac{4}{\ln N(1-\kappa)^2}}\right], \tag{2.117}$$

$$T_2 = \ln\left[\tfrac{NI_2}{I(0)(N-I_2)}\right], \tag{2.118}$$

$$I_3 = \tfrac{I_2 e^{\Delta T_1}}{1-\frac{I_2}{N}+\frac{I_2}{N}e^{\Delta T_1}}, \tag{2.119}$$

$$\Delta T_1 = \tfrac{1}{\kappa}\ln\left[\tfrac{\frac{c}{\kappa}+\frac{N(1-\kappa)}{2}[1+H]}{\frac{c}{\kappa}+\frac{N(1-\kappa)}{2}[1-H]}\right], \tag{2.120}$$

$$\Delta T_2 = \frac{1}{\kappa}\ln\left[\frac{\frac{c}{\kappa}+I_3}{\frac{c}{\kappa}+I_2}\right], \tag{2.121}$$

$$\bar{T}_3 = T_2 + \Delta T_2 \tag{2.122}$$

$$L_3 = \frac{C}{\beta}(e^{\beta \Delta T_2} - 1) + (\Phi(I_2) + I(0))e^{\beta \bar{T}_3},$$

where $H = \sqrt{1 - \frac{4}{\ln N(1-\kappa)^2}}$.

Also, let

$$I_4 = I(\bar{T}_3) = \frac{I(0)e^{\bar{T}_3}}{1-\frac{I(0)}{N}+\frac{I(0)}{N}e^{\bar{T}_3}}, \tag{2.123}$$

$$I_5 = \frac{N(1-\kappa)}{2}\left[1+\sqrt{1+\frac{4\beta L_3}{N(1-\kappa)^2}}\right], \tag{2.124}$$

$$\bar{T}_5 = \ln\left[\frac{NI_5}{I(0)(N-I_5)}\right], \tag{2.125}$$

$$L_4 = (\Phi(I_5) - \Phi(I_4))e^{\beta \bar{T}_5} + L_3 e^{\beta(\bar{T}_5 - \bar{T}_3)},$$

where $I(t)$ is the Bass demand function.

**Lemma 6.** *Assume $\kappa < 1 - \frac{2}{\sqrt{\ln N}}$. Then, a lower bound on the number of legals at $t = T_\infty$ is given by,*

$$N_l(T_\infty) \geq \begin{cases} L_3 & \text{if} \quad \bar{T}_5 \leq \bar{T}_3 \\ L_4, & \text{else.} \end{cases} \tag{2.126}$$

*Proof.* When $\kappa < 1 - \frac{2}{\sqrt{\ln N}}$, the evolution of of $N_l(t)$ and $N_i(t)$ takes place along a sequence of stages, which is given by, '*Stage* 1 $\rightarrow$ *Stage* 2 $\rightarrow$*Stage* 3(*or Stage* 4)$\rightarrow$*Stage* 5(or Stage 6)'. An exact characterization of $N_l(t)$ and $N_i(t)$ might be quite difficult as the analysis involves solving many complex differential equations. Therefore, we define two processes $\bar{N}_l(t)$ and $\bar{N}_i(t)$; $\bar{N}_l(t)$ bounds $N_l(t)$ from below and $\bar{N}_i(t)$ bounds $N_i(t)$ from above. We analyze these bounding processes instead of the actual processes.

We go through a sequence of intermediate steps to prove this lemma.

**Step 1**: *Define $\bar{N}_l(t)$ and $\bar{N}_i(t)$*

First of all, let $\bar{N}_l(0) = N_l(0)$ and $\bar{N}_i(0) = N_i(0)$. Let $\bar{N}_l(t)$ evolves as follows,

$$\frac{d\bar{N}_l(t)}{dt} = \begin{cases} \frac{dI}{dt} - (\rho\bar{N}_l(t) + \kappa\bar{N}_i(t)), & [0, T_2], \\ C_N + \beta\bar{N}_l(t), & [T_2, \bar{T}_3], \\ \frac{dI}{dt} - (\rho\bar{N}_l(t) + \kappa\bar{N}_i(t)), & [\bar{T}_3, \max\{\bar{T}_3, \bar{T}_5\}], \\ 0, & [\max\{\bar{T}_3, \bar{T}_5\}, T_\infty]. \end{cases} \tag{2.127}$$

Also, let

$$\frac{d\bar{N}_i(t)}{dt} = \begin{cases} (\rho\bar{N}_l(t) + \kappa\bar{N}_i(t)), & [0, T_2], \\ (\rho\bar{N}_l(t) + \kappa\bar{N}_i(t)) & \\ \quad +R\delta(t - \bar{T}_3), & [T_2, \bar{T}_3], \\ (\rho\bar{N}_l(t) + \kappa\bar{N}_i(t)), & (\bar{T}_3, \max\{\bar{T}_3, \bar{T}_5\}], \\ \frac{dI}{dt} & [\max\{\bar{T}_3, \bar{T}_5\}, T_\infty]. \end{cases} \tag{2.128}$$

where $T_2$ is given by (2.118), $\bar{T}_3$ is defined by (2.122), $\bar{T}_5$ is defined by (2.125), $R = I(\bar{T}_3) - (N_l(\bar{T}_3) + N_i(\bar{T}_3))$ and $\delta(t)$ is Kronecker delta function. It can be verified that $\bar{T}_3 > T_2$. Also, the following equations can be verified:

$$\frac{dI(t)}{dt}\Big|_{t=\bar{T}_3} - \kappa I(\bar{T}_3) \leq C_N, \tag{2.129}$$

$$\bar{N}_l(t) + \bar{N}_i(t) < I(t) \text{for} \quad T_2 < t < \bar{T}_3, \tag{2.130}$$

$$\frac{dI(t)}{dt}\Big|_{t=\bar{T}_5} - \kappa I(\bar{T}_5) = \beta\bar{N}_l(\bar{T}_3). \tag{2.131}$$

Also, we define $\bar{N}_w(t) = I(t) - (\bar{N}_l(t) + \bar{N}_i(t))$. In the next step, we show that $\bar{N}_l(t) \leq N_l(t)$ for all $t$.

**Step 2**: *We claim that $\bar{N}_l(t) \leq N_l(t)$:*

Recall that, the actual processes may pass through either Stages 3 and 5 or Stages 4 and 6. We analyze these two cases separately.

*Case 1: The evolution of $N_l(t)$ and $N_i(t)$ takes place along Stages 3 and 5*

First of all, we have $N_l(0) = \bar{N}_l(0)$ and $N_i(0) = \bar{N}_i(0)$ from the definition of the bounding processes. Now, suppose $\bar{T}_3 \leq T_3$. Then, comparing Stage 1 dynamics, (2.85, 2.84), and Stage 2 dynamics (2.94, 2.93) with the bounding process dynamics (2.127, 2.128), we get that, for $t \in [0, \bar{T}_3]$,

$$\frac{d\bar{N}_l(t)}{dt} = \frac{dN_l(t)}{dt} \quad \text{and} \quad \frac{d\bar{N}_i(t)}{dt} \geq \frac{dN_i(t)}{dt}.$$

Then,

$$\bar{N}_l(t) = N_l(t) \quad \text{if} \quad t \in [0, \bar{T}_3]. \tag{2.132}$$

Also, suppose $\bar{T}_5 \leq T_5$. Then, comparing Stage 2 dynamics, (2.94, 2.93), Stage 3 dynamics (2.101, 2.100) and Stage 5 dynamics (2.111, 2.112) with the bounding process dynamics (2.127, 2.128), we get that, for $t \in [\bar{T}_3, T_\infty]$,

$$\frac{d\bar{N}_l(t)}{dt} \leq \frac{dN_l(t)}{dt} \quad \text{and} \quad \frac{d\bar{N}_i(t)}{dt} \geq \frac{dN_i(t)}{dt}.$$

Then, $\bar{N}_l(t) \leq N_l(t)$ for $t > \bar{T}_3$. To complete the proof, we must show that $\bar{T}_3 \leq T_3$ and $\bar{T}_5 \leq T_5$.

*Show that* $\bar{T}_3 \leq T_3$: Recall that Stage 3 begins at $T_3$ in the evolution of the original processes. From the definition of $T_3$, given by (2.97),

$$\begin{aligned} &\tfrac{dI(t)}{dt}\,|_{t=T_3} - (\rho N_l(T_3) + \kappa N_i(T_3)) \leq C_N + \beta N_l(T_3), \\ &\quad \Rightarrow \tfrac{dI(t)}{dt}\,|_{t=T_3} - \kappa I(T_3) \leq C_N. \end{aligned} \tag{2.133}$$

The second inequality follows from the facts that $\kappa = \rho + \beta$ and $N_i(T_3) + N_l(T_3) = I(T_3)$ (since $N_w(T_3) = 0$ from (2.98)).

First, we guess a lower bound for $T_3$. Suppose, at time $t = r$,

$$I(r) = \frac{N(1-\kappa)}{2}\left[1 + \sqrt{1 + \frac{4}{\ln N(1-\kappa)^2}}\right],$$

is satisfied. Note that $I(r) > I(T_2)$ and hence, $r > T_2$. It can be shown that if $t \in [T_2, r]$,

$$\frac{dI}{dt}(t) - \kappa I(t) \geq C_N,$$

with equality at $t = T_2$ and $t = r$. Also, the function, $\frac{dI}{dt}(t) - \kappa I(t)$ strictly decreasing if $t \geq r$. Then, from (2.133) and the fact that $T_3 > T_2$, we conclude that $r \leq T_3$.

Now, obtain a better lower bound for $T_3$. Let us define $U(t) = N_l(t) + N_i(t)$. From (2.98), we have $N_w(T_3) = 0$, which implies that $U(T_3) = I(T_3)$. We know that $U(r) \leq I(r)$. Find $t'$ such that $U(t') = I(r)$. Then, $U(t') \leq I(t')$. Then, get $s$ such that $U(s) = I(t')$. Since $U(t)$ and $I(t)$ are monotonically increasing, we have $r \leq t' \leq s \leq T_3$.

From the dynamics of evolution of Stage 2, given by (2.93) and (2.94), we can show that during the interval $[T_2, T_3]$,

$$U(t) \;\; = \;\; \left(\frac{C}{\kappa} + I_2\right) e^{\kappa(t-T_2)} - \frac{C}{\kappa}.$$

Then, it can be shown that $t' = T_2 + \Delta T_1, I_3 = I(t')$ and $s = \bar{T}_3$. Hence, $\bar{T}_3 \leq T_3$.

*Show that* $\bar{T}_5 \leq T_5$: Recall that Stage 5 begins at $T_5$. From (2.106),

$$\frac{dI(t)}{dt}\,|_{t=T_5} - \kappa I(T_5) = -\beta N_l(T_5).$$

The above result is due to the facts that $\kappa = \rho + \beta$ and $N_i(t) + N_l(t) = I(t)$ in Stage 3 and 5.

We guess a lower bound for $T_5$. From, (2.131),

$$\frac{dI(t)}{dt}\Big|_{t=\bar{T}_5} - \kappa I(\bar{T}_5) = -\beta \bar{N}_l(\bar{T}_3).$$

is satisfied. If $\bar{T}_5 \leq \bar{T}_3$, then $\bar{T}_5 \leq T_3 \leq T_5$. Suppose $\bar{T}_5 > \bar{T}_3$. Recall that $\bar{T}_3 \leq T_3 \leq T_5$ and $\bar{N}_l(\bar{T}_3) = N_l(\bar{T}_3)$ (from (2.132)). Then, $\bar{N}_l(\bar{T}_3) \leq N_l(T_5)$. Also, $\frac{dI(t)}{dt} - \kappa I(t)$ is a decreasing function of $t$ when its value is negative. Combining these facts with the definitions of $T_5$

and $\bar{T}_5$, we can assert that $\bar{T}_5 \leq T_5$.

*Case 2: The evolution of $N_l(t)$ and $N_i(t)$ takes place along Stage 4 and Stage 6.*

We have to consider two cases, $T_4 < \bar{T}_3$ and $T_4 \geq \bar{T}_3$ respectively.

Suppose $T_4 < \bar{T}_3$: First, we show that,

$$\bar{N}_l(\bar{T}_3) = N_l(\bar{T}_3). \tag{2.134}$$

Note that the dynamics of actual and the bounding processes are identical untill $t = T_4$. Then, $N_w(T_4) = \bar{N}_w(T_4)$. Also, during $T_4 < t \leq \min\{T_6, \bar{T}_3\}$, $\bar{N}_i(t)$ grows faster than $N_i(t)$, while $\bar{N}_l(t)$ grows at the same rate as that of $N_l(t)$. Therefore, to prove (2.134) holds true, we just need to show that $T_6 \geq \bar{T}_3$, which is done as follows: Note that, when $t \in [T_4, \min\{T_6, \bar{T}_3\}]$, the growth rate of $N_l(t) + N_i(t)$ is less than that of $\bar{N}_l(t) + \bar{N}_i(t)$, and hence $\bar{N}_w(t) \leq N_w(t)$. Then, from (2.130) and the definition of $\bar{N}_w(t)$, we get $N_w(t) > 0$ when $T_4 < t < \bar{T}_3$ (since $T_4 > T_2$ by definition). Then, from (2.110), we get that $T_6$ cannot be less than $\bar{T}_3$.

Now, suppose $\bar{T}_5 \leq \bar{T}_3$. Then, from (2.134) and (2.127),

$$\bar{N}_l(T_\infty) = \bar{N}_l(\bar{T}_3) = N_l(\bar{T}_3) \leq N(T_\infty),$$

which proves our claim. Now, we show that $\bar{T}_5 \leq \bar{T}_3$ as follows: For all $t > T_4$, (2.109) is satisfied. Then, we get

$$\frac{dI(t)}{dt}\Big|_{t=\bar{T}_3} - \kappa I(\bar{T}_3) \leq -\beta N_l(\bar{T}_3).$$

due to the assumption, $T_4 < \bar{T}_3$ and the definition of $N_w(t)$. But, from (2.131) and (2.134),

$$\frac{dI(t)}{dt}\Big|_{t=\bar{T}_5} - \kappa I(\bar{T}_5) = -\beta N_l(\bar{T}_3).$$

Therefore, $\bar{T}_5 \leq \bar{T}_3$ since $\frac{dI}{dt} - \kappa I(t)$ is decreasing in $t$ once it goes negative.

Suppose $T_4 \geq \bar{T}_3$: Note that the dynamics of actual and the bounding processes are identical untill $t = \bar{T}_3$. To prove the claim, we show that

$$\frac{dN_l(t)}{dt} \geq \frac{d\bar{N}_l(t)}{dt} \quad \text{when} \quad t \geq \bar{T}_3. \tag{2.135}$$

At $t = \bar{T}_3$, from (2.129), the dynamics of actual and the bounding processes, the above expression holds true. Also, during $t \in [\bar{T}_3, T_6]$, $\frac{dN_l(t)}{dt}$ and $\frac{d\bar{N}_l(t)}{dt}$ are increasing and decreasing functions respectively. Hence, (2.135) holds true until $t \leq T_6$. Now, we show that $\bar{T}_5 < T_6$, and hence the growth rate of $\bar{N}_l(t)$ is zero for $t \geq T_6$. This asserts that (2.135) holds for $t \geq T_6$. The proof is as follows: From (2.99) and the definition of $N_w(t)$, we get

$$\frac{dI(t)}{dt}\,|_{t=T_4} - \kappa I(T_4) = -\beta N_l(T_4) - (1+\kappa)N_w(T_4). \tag{2.136}$$

Then, $\bar{T}_5 \leq T_4$ due to these reasons: 1) $\bar{T}_5$ satisfies (2.131), 2) $\beta\bar{N}_l(\bar{T}_3) = \beta N_l(\bar{T}_3 < \beta N_l(T_4) + (1+\kappa)N_w(T_4)$ since $\bar{T}_3 < T_4$ by assumption, 3) $\frac{dI(t)}{dt} - \kappa I(t)$ is decreasing once its value goes negative. Now, since $T_4 < T_6$, we have $\bar{T}_5 < T_6$, and hence (2.135) is attained.

Having shown that $\bar{N}_l(t)$ bounds $N_l(t)$ from below, we evaluate $\bar{N}_l(T_\infty)$ in the next step.

**Step 5**: *Evaluate the bounding process, $\bar{N}_l(T_\infty)$:*

*Find $\bar{N}_l(T_2)$*: The evolution of the bounding processes during $[0, T_2]$ are given by (2.127) and (2.128). Solving them, we get

$$\begin{aligned} \bar{N}_l(t) &= (\Phi(I(t)) - \Phi(I(0))e^{\beta t} + I(0)e^{\beta t}, \\ &= (\Phi(I(t)) + I(0))e^{\beta t}, \end{aligned}$$

where $\Phi(x)$ is defined by (2.113). The second equality holds true since $\Phi(I(0)) = 0$.

Substituting $T_2$ from (2.118) in the above result,

$$\bar{N}_l(T_2) = (\Phi(I_2) + I(0))e^{\beta T_2},$$

where $I_2 = I(T_2)$.

*Find* $\bar{N}_l(\bar{T}_3)$: Solving the growth equations given by (2.127) and (2.128), for the interval $[T_2, \bar{T}_3]$, we get

$$\bar{N}_l(t) \quad = \quad \left(\frac{C}{\beta} + \bar{N}_l(T_2)\right) e^{\beta(t-T_2)} - \frac{C}{\beta}.$$

Substituting, $\bar{T}_3$ from (2.122), and $\bar{N}_l(T_2)$ in the above expression, we get

$$N_l(\bar{T}_3) = \frac{C}{\beta}(e^{\beta \Delta T_2} - 1) + (\Phi(I_2) + I(0))e^{\beta \bar{T}_3} = L_3.$$

where $L_3$ is given by (2.123).

*Let* $\bar{T}_3 < \bar{T}_5$. *Find* $\bar{N}_l(\bar{T}_5)$: Solving the growth equations given by (2.127) and (2.128), for the interval $[\bar{T}_3, \bar{T}_5]$, we get

$$\bar{N}_l(t) = (\Phi(I(t)) - \Phi(I(\bar{T}_3))e^{\beta t} + \bar{N}_l(\bar{T}_3)e^{\beta(t-\bar{T}_3)},$$

Substituting $\bar{T}_3, \bar{T}_5$ and $\bar{N}_l(\bar{T}_3)$ in the above equation, we get

$$\bar{N}_l(t) = (\Phi(I_5) - \Phi(I_4))e^{\beta t} + L_3 e^{\beta(\bar{T}_5 - \bar{T}_3)} = L_4,$$

where $I_5, I_4, L_3$ and $L_4$ are given by (2.124), (2.123), (2.123) and (2.126) respectively.

*Find* $\bar{N}_l(T_\infty)$: From (2.127), we have $\frac{d\bar{N}_l(t)}{dt} = 0$, for $t \geq \max\{\bar{T}_3, \bar{T}_5\}$. Therefore, we have $\bar{N}_l(T_\infty) = \bar{N}_l(\max\{\bar{T}_3, \bar{T}_5\})$. Then,

$$N_l(T_\infty) \geq \bar{N}_l(T_\infty) = \begin{cases} \bar{N}_l(\bar{T}_3) = L_3 & \text{if} \quad \bar{T}_5 \leq \bar{T}_3 \\ \bar{N}_l(\bar{T}_5) = L_4, & \quad \text{else.} \end{cases}$$

□

We have characterized the number of legal copies generated in the system in the presence of an efficient illicit P2P in the previous two lemmas. Attaining the statement in the theorem is accomplished by studying the asymptotics of the results in Lemma 5 and 6. We start by introducing a few notation.

$$\begin{aligned}
\Delta T_3 &= \frac{1}{\kappa}\ln\left[\kappa(1-\kappa)\ln N + (1-\kappa)\right], \\
\tilde{T}_3 &= T_2 + \Delta T_3, \qquad (2.137) \\
\Delta T_4 &= \frac{1}{\kappa}\ln\left[\frac{\kappa(1-\kappa)}{1+\kappa}\ln N + (1-\kappa)\right], \qquad (2.138) \\
\tilde{T}_4 &= T_2 + \Delta T_4. \qquad (2.139)
\end{aligned}$$

Also, we say, $A_N \sim B_N$, if $\lim_{N\to\infty}\frac{A_N}{B_N} = 1$, $A_N \preceq B_N$, if $\lim_{N\to\infty}\frac{A_N}{B_N} \leq 1$. and, $A_N \succeq B_N$, if $\lim_{N\to\infty}\frac{A_N}{B_N} \geq 1$. Now, we are ready to prove the theorem.

As $N$ goes large, for any given $\kappa$, the assumption of Lemma 6 that $\kappa < 1 - \frac{2}{\sqrt{\ln N}}$ is attained. Therefore, in the asymptotic case, we use the result of Lemma 6. That lemma says,

$$N_l(T_\infty) \geq \begin{cases} L_3, & \text{if} \quad \bar{T}_5 \leq \bar{T}_3 \\ L_4, & \quad \text{else.} \end{cases} \qquad (2.140)$$

where $\bar{T}_3, L_3, \bar{T}_5$ and $L_4$ are given by (2.122), (2.123), (2.125) and (2.126) respectively. The proof is done in two steps. First, we evaluate $L_3$. Next, we show that $\bar{T}_3 \succeq \bar{T}_5$. Then, from the above equation, we get that $N_l(T_\infty) \succeq L_3$.

*Evaluate $L_3$:* As $N$ goes larger, it can be shown that,

$$\begin{aligned}
I_2 \sim \frac{N}{\ln N(1-\kappa)}, \quad \Delta T_2 &\sim \frac{1}{\kappa}\ln\left(\kappa(1-\kappa)\log N\right), \\
T_2 &\sim \ln\left(\frac{N}{I(0)(1-\kappa)\ln N}\right), \\
\bar{T}_3 &\sim \ln\left[\frac{N(\kappa(1-\kappa)\ln N)^{\frac{1}{\kappa}}}{I(0)(1-\kappa)\ln N}\right]. \\
\Phi(I_2) \sim \left(\frac{I(0)}{N}\right)^{\beta} N\frac{(1-\kappa)}{(1-\beta)}&\left(\frac{1}{(1-\kappa)\ln N}\right)^{1-\beta}.
\end{aligned}$$

The above results follows from (2.117), (2.121), (2.118), (2.122) and (2.113) respectively.

Substituting the above results in (2.123), we get that

$$L_3 \sim \frac{N}{\ln N\beta}\left(\frac{(\ln N\kappa(1-\kappa))^{\frac{\beta}{\kappa}}}{(1-\beta)} - 1\right). \tag{2.141}$$

*Show that* $\bar{T}_3 \succeq \bar{T}_5$*:* First of all, from (2.125) and (2.124), note that, $I(\bar{T}_5) = I_5$ and $I_5 \leq N$. Also, for large values of $N$, from (2.122) and the definition of $I(t)$, we can show that, $I(\bar{T}_3) \sim N$. Combining these two results, we get $I(\bar{T}_5) \preceq I(\bar{T}_3)$ This result in turn implies that $\bar{T}_5 \preceq \bar{T}_3$, since $I(t)$ is monotonically increasing.

Hence, from (2.140),

$$N_l(T_\infty) \succeq L_3.$$

From (2.141), the above equation, and (2.10), we get (2.62), which completes the first part of theorem.

The second part of the theorem deals with the case $\beta = 0$. From, (2.62), we have,

$$L \in \Omega\left(\frac{\ln\ln N}{\ln N}\right). \tag{2.142}$$

Now, to complete the proof, it suffices to prove the following lemma.

**Lemma 7.** *When* $\beta = 0$*,*

$$L \in o\left(\frac{\ln\ln N}{\ln N}\right)$$

.

*Proof.* Recall that when $\kappa < 1 - \frac{2}{\sqrt{\ln N}}$, which holds for any $\kappa$ when $N$ is large, the evolution of $N_l(t)$ and $N_i(t)$ takes place along the sequence of phases,'*Stage* 1 $\rightarrow$ *Stage* 2 $\rightarrow$*Stage* 3 ( or *Stage* 4)$\rightarrow$*Stage* 5 ( or Stage 6)'. We analyze each of these phases and obtain an upper bound on $N_l(T_\infty)$ as follows.

*Stage* 1*:* An upper bound on the number of legal copies at the end of this stage is given by,

$$N_l(T_2) \preceq \frac{N}{\ln N(1-\kappa)}. \tag{2.143}$$

which follows from the facts that $N_l(t) \leq I(t)$ for all $t$ and $I(T_2) \sim \frac{N}{\ln N(1-\kappa)}$. *Stage 2:* First we show that as $N$ goes large, $T_4 \preceq T_3$ and hence, in the asymptotic case Stage 2 is followed by Stage 4. The proof of this claim proceeds as follows. Let, $U(t) = N_l(t) + N_i(t)$. From the dynamics of evolution of Stage 2, given by (2.93) and (2.94),

$$U(t) \quad = \quad \left(\frac{C}{\kappa} + I_2\right) e^{\kappa(t-T_2)} - \frac{C}{\kappa}, \tag{2.144}$$

where $I_2$ is given (2.117) and $T_2$ is given by (2.118). Now, substituting $\tilde{T}_3$ from (2.137) in the above equation, we get

$$U(\tilde{T}_3) \sim I(\tilde{T}_3).$$

Also, it is easy to verify that $\bar{T}_3$ satisfies (2.97). These results along with the definition of $T_3$, given by (2.97-2.98), implies that $\tilde{T}_3 \sim T_3$. Similarly, substituting $\tilde{T}_4$ in (2.144), we can show that

$$U(\tilde{T}_4) \sim \frac{1}{1+\kappa}\left(I(\tilde{T}_4) + \frac{dI}{dt}(\tilde{T}_4)\right).$$

This result along with the definition of $T_4$, given by (2.99), implies that $\tilde{T}_4 \sim T_4$.

We have, $\tilde{T}_4 \preceq \tilde{T}_3$, since

$$U(\tilde{T}_4) = \frac{N}{1+\kappa} < N = U(\tilde{T}_3),$$

and $U(t)$ is monotonically increasing. Therefore, we conclude that $T_4 \preceq T_3$. And hence, this stage is always followed by Stage 4.

Then, from the dynamics of $N_l(t)$, given by (2.94),

$$N_l(T_4) = N_l(T_2) + C_N(T_4 - T_2).$$

Now, from (2.143) and the definitions of $\tilde{T}_4$ and $T_2$, we get

$$N_l(T_4) \preceq \frac{N}{\ln N(1-\kappa)} + \frac{N}{\kappa \ln N} \ln\left(\ln N \frac{\kappa(1-\kappa)}{1+\kappa} + 1 - \kappa\right). \tag{2.145}$$

*Stage* 4*:* This stage starts at time $t = T_4$. From the discussion given above (in Stage 3 analysis), $T_4 \sim \tilde{T}_4$. Then, from (2.139), $I(T_4) \sim I(\tilde{T}_4) \sim N$ and $\frac{dI}{dt}(T_4) \sim \frac{dI}{dt}(\tilde{T}_4) \sim 0$. Also, $N_w(T_4) = I(\tilde{T}_4) - U(\tilde{T}_4) \sim \frac{N\kappa}{1+\kappa}$. Recall that $U(t) = N_l(t) + N_i(t)$. And $U(\tilde{T}_4)$ is obtained from (2.144) and (2.139).

Using these facts and the dynamics of $N_i(t)$ and $N_l(t)$ given by (2.108) and (2.107) respectively, we show that,

$$U(t) = (C_N + N)(1 - e^{-t}) + U(\tilde{T}_4)e^{-(t-\tilde{T}_4)}.$$

This stage terminates, when no Wanters are left to be served, i.e $U(t) \sim N$. Let $\tilde{T}_6$ marks this event. Then,

$$\tilde{T}_6 \sim \ln\left(\frac{\ln N}{1+\kappa}\right).$$

The legal copies of content generated in this phase is $C_N \times (\tilde{T}_6 - \tilde{T}_4)$ from the dynamics of $N_l(t)$ given by (2.107). Then, from the above result and (2.145), we get

$$N_l(T_\infty) \preceq \frac{N}{\ln N} \ln\left[\frac{(\ln N)^{(\frac{1}{\kappa}+1)}}{1+\kappa}\left(\frac{(1-\kappa)\kappa}{(1+\kappa)}\right)^{\frac{1}{\kappa}}\right],$$

which completes the proof. □

□

The above theorem along with Theorem 3 asserts that the fractional legitimate copies attained by the CDN under Bass model of evolution is no different from that of Flash Crowd model in asymptotic order.

Since Theorems 1 and 3 rely on a fluid model, and characterize only the asymptotic growth rate of the fractional legitimate copies produced in the system, we present numerical simulations to verify the qualitative insights in discrete systems with finite $N$.

To simulate the underlying discrete stochastic system, we assume time is discrete and that there are $N = 100,000$ users in the system. A Bass model based interest evolution is assumed. That means, at each time slot, each user picks a Poisson distributed number (with mean 1) of other users to spread interest to. The server has a FIFO policy with service rate $C = 8000 \approx N/\ln N$.

Figure 2.3 illustrates the evolution of legal and illegal copies of the content in the case of an inefficient illicit P2P system with $\kappa = 0.75$. In Figure 2.3(a), where $\beta = 0$, the final number of legal copies produced in the system is $63,000$. When the booster factor increases, as shown in Figure 2.3(b) where $\beta = 0.52$, the number of legal copies increases to $88,888$; In fact, the fractional legitimate copies increases by more than 25%.

Table 2.1: Fractional revenue ratio of inefficient illicit P2P

| $\frac{\beta}{\kappa}$ | $\kappa = 0.75$ | | $\kappa = 0.5$ | |
|---|---|---|---|---|
| | Simulation | Analytical | Simulation | Analytical |
| 0 | 0.64 | 0.60 | 0.69 | 0.67 |
| 0.10 | 0.71 | 0.71 | 0.77 | 0.75 |
| 0.24 | 0.77 | 0.72 | 0.82 | 0.77 |
| 0.41 | 0.81 | 0.75 | 0.86 | 0.79 |
| 0.63 | 0.87 | 0.79 | 0.92 | 0.80 |
| 0.92 | 0.97 | 0.85 | 0.98 | 0.82 |

In Table 2.1, we compare the simulation results against our analytical results from Lemma 3 and Corollary 3, for various combinations of $\kappa$ and $\beta$. As expected from Corollary 3, our analytical predictions closely match with the simulation results in the case, $\beta = 0$. In the case, $\beta > 0$, the predicted values are less than those obtained using simulation, which agrees with Lemma 3; nevertheless, the differences are quite small. Also observe that, as $\beta$ increases, the fractional legitimate copies improves significantly. Especially, in the case, $\kappa = 0.75$, as booster factor increases from $\beta = 0$ to $\beta = 0.92\kappa$, the fractional legitimate copies increases by 150%.

Next, we move to the case of an efficient illicit P2P. Figure 2.4 illustrates the case of an efficient illicit P2P system. In Figure 2.4(a), where $\beta = 0$, the final number of legal copies produced in the system is $45,920$. When the booster factor increases, as shown in Figure 2.4(b) where $\beta = 0.38$, the number of legal copies increases to $96,380$; In fact, the fractional legitimate copies increases by more than 100%.

Table 2.2: Fractional revenue ratio of efficient illicit P2P

| $\frac{\beta}{\kappa}$ | $\kappa = 0.75$ | | $\kappa = 0.5$ | | $\kappa = 0.25$ | |
|---|---|---|---|---|---|---|
| | Simulation | Analytical | Simulation | Analytical | Simulation | Analytical |
| 0 | 0.03 | 0.03 | 0.15 | 0.15 | 0.42 | 0.37 |
| 0.48 | 0.07 | 0.07 | 0.28 | 0.26 | 0.56 | 0.50 |
| 0.69 | 0.18 | 0.14 | 0.40 | 0.38 | 0.67 | 0.59 |
| 0.84 | 0.30 | 0.24 | 0.54 | 0.52 | 0.77 | 0.68 |
| 0.95 | 0.55 | 0.41 | 0.78 | 0.69 | 0.9 | 0.78 |

In Table 2.2, we tabulate the simulation results and the analytical results. The analytical results are obtained from Lemma 5 and Lemma 6. The simulation results are in agreement with our analytical predictions. Also note that, the improvement attained in the fractional legitimate copies, as $\beta$ increase, is phenomenal. For example, in the case, $\kappa = 0.75$, as booster factor increases from $\beta = 0$ to $\beta = 0.95\kappa$, the fractional legitimate copies increases by 1833%.

## 2.3 Revenue sharing model

In the previous sections, we studied the impact of the three parameters $\rho$, $\beta$ and $\kappa$ on the eventual number of legal content copies in the system. We made the assumption that $\rho + \beta = \kappa$, following the intuition that $\kappa$ is the fixed probability of a user who has the content being willing to redistribute it, and *which* P2P swarm is joined affects the number of legal copies. We now consider the motivation behind the users' decisions on which swarm to join.

Suppose that the purchase price of a copy of the content is $p$. Hence, a user that wishes to obtain a legal copy of the content must pay the content generator the sum $p$ through some kind of online banking system. Suppose that the content owner utilizes a simple model for revenue sharing, where a user receives $\epsilon p$ for each piece of content it distributes when taking part in the legitimate network as a Booster. Thus, $\epsilon = 0$ corresponds to no revenue sharing. Note that this could potentially be implemented on a system such as BitTorrent by simply keeping track of amount uploaded by each peer[3]. The value $\epsilon$ can be viewed either as a share of the revenue from each download or as the expected payoff of a lottery scheme operated by the CDN.

While it is difficult to exactly predict the effect of revenue sharing, it seems reasonable that increased revenue sharing should limit the likelihood of a Wanter going rogue after attaining the content legally. To qualitatively capture this effect, we model $\rho$ as a decreasing function of $\epsilon$. A specific form could be

$$\rho = \kappa\phi(\epsilon),$$

where $\phi(.)$ is a decreasing function with $\phi(0) = 1$ and $\phi(1) = 0$.

Recall that we defined the parameter $R$ as the fractional revenue, also the fraction of legitimate copies in the system at $T_\infty$. It is clear that the profit obtained by the content owner also depends on the amount of revenue shared with the boosters, which in turn depends on the exact form of $\phi(\epsilon)$. Hence, the content owner would have to determine the optimal amount of revenue sharing in order to maximize profit. For illustration, let us choose

$$\phi(\epsilon) = N^{-\epsilon},$$

in our simulations. The results are shown in Figure 2.6, which illustrates the impact of the amount of revenue sharing on the fractional revenue ratio of the CDN in the cases of inefficient and efficient illicit P2Ps. We use $\kappa = 0.75$ in the simulation. The key point to observe in the figure is that there is a clear optimal amount of revenue sharing for the provider. In both cases, this amount is fairly small, however, it is clearly desirable to share more revenue in the presence of an efficient illicit P2P than in the presence of an inefficient

[3]BitTorrent Trackers already collect such information in order to gather performance statistics.

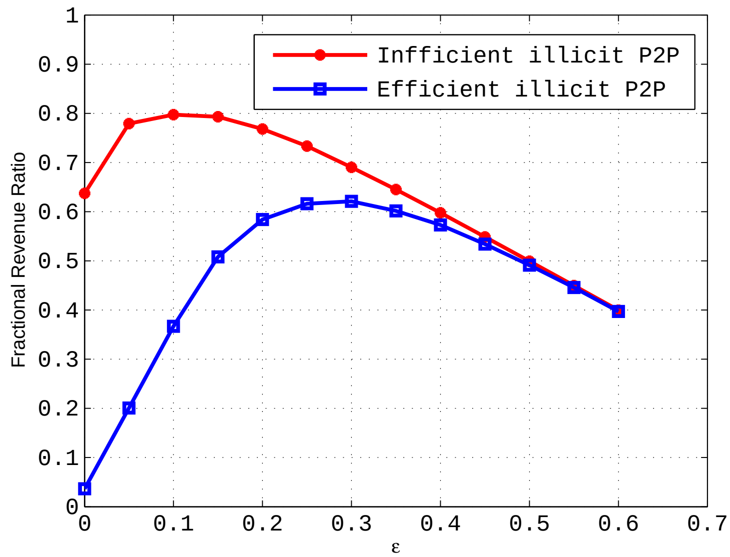



Figure 2.6: Impact of the amount of revenue sharing on the fractional revenue attained by the CDN.

illicit P2P. In fact, sharing nearly zero percent of the revenue still provides fairly close to the optimal fractional revenue in the inefficient case, while one must share more than 10% of the revenue to be near-optimal in the case of an efficient, illicit P2P.

## 2.4 Conclusion

Our goal in this work is to quantify the ramifications of coopting legal P2P content sharing, not only as a means of reducing costs of content distribution, but, more importantly, as a way of hurting the performance of illegal P2P file sharing. The model that we propose internalizes the idea that demand for any content is transient, and that all content will eventually be available for free through illegal file sharing. The objective then is not to cling to ownership rights, but to extract as much revenue from legal copies as possible within the available time. We develop a revenue sharing scheme that recognizes the importance of early adopters in extending the duration of time that revenue may be extracted. In particular, keeping users from "going rogue" (becoming seeds in illegal networks) by allowing them to extract some revenue for themselves (and so defray part of their expense in purchasing the content in the first place), provides *order sense improvements* in the extractable revenue. We realize that our paradigm is contrary to the "conventional wisdom" of charging *more* rather than *less* to early adopters, and also to discourage file sharing using legal threats. However, as many recent studies have demonstrated, incentives work better than threats in human society, and adoption of our revenue sharing approach might result in a cooperative equilibrium between content owners, distributors and end-users. Future work includes a characterization of the exact value of users based on their times of joining the system, as well as considering content streaming, which requires strict quality of service guarantees.

In the next chapter, we study a transport layer control problem. Recently a number of congestion control protocols has been proposed for use in the Internet. These protocols differ in the way they indicate congestion to the sources. For example, TCP Reno uses packet loss as the congestion indicator, while TCP Vegas uses end to end delay to mark congestion. However, the relative value of one protocol against another is not well understood. For instance, when flows choose distinct protocols, they may not receive the same throughput. We study a scenario where a group of applications compete for network resources to achieve their service requirement ( may be a function of delay, throughput or both) by strategically choosing protocols. Then, we ask the following questions: How should applications choose protocols? Should a delay sensitive application pick a delay based congestion controller? Does the selfish interaction among these applications lead to an equilibrium ? If so, what is the efficiency of the equilibrium relative to the socially optimal case? We try to answer these questions in the following chapter.

## 2.5 Supplemental

**Lemma 8.** *Consider a differential equation given*

$$\frac{dy}{dt} = C_N + \frac{\beta y}{N}(I - U(t)) \tag{2.146}$$

*where*

$$U(t) = \frac{N\theta_2}{\kappa} + \frac{N\Delta\theta/\kappa}{1 + be^{-\Delta\theta(t-T)}}.$$

*Then for all* $t - T > \frac{\ln b}{\Delta\theta}$, *the solution to the above differential equation satisfies the inequality*

$$\begin{aligned}
y(t) \quad & \geq y(T)\left(\tfrac{1+b}{d}\right)^{\frac{\beta}{\kappa}} e^{(-q_1(t-T))} \\
& +C_N\left(\tfrac{b}{d}\right)^{\frac{\beta}{\kappa}} e^{(-q_1(t-T))}\left(\frac{e^{\left(q_1\frac{\ln b}{\Delta\theta}\right)}}{q_1} - \frac{1}{q_1}\right)\mathbf{1}_{b\geq 1} \\
& +C_N\left(\tfrac{1}{d}\right)^{\frac{\beta}{\kappa}} e^{(-q_1(t-T))}\left(\frac{e^{\left(q_2\Delta\tau_j\right)}}{q_2} - \frac{e^{\left(q_2\frac{\ln b}{\Delta\theta}\right)}}{q_2}\mathbf{1}_{b\geq 1}\right) \\
& -C_N\left(\tfrac{1}{d}\right)^{\frac{\beta}{\kappa}} e^{(-q_1(t-T))}\tfrac{1}{q_2}(1-\mathbf{1}_{b\geq 1}),
\end{aligned} \tag{2.147}$$

*where* $d = (b + \exp(\Delta\theta(t-T)))$, $q_1 = \left(\frac{\beta\theta_2}{\kappa} - \frac{\beta I}{N}\right)$ *and* $q_2 = \frac{\beta\theta_1}{\kappa} - \frac{\beta I}{N}$. *Furthermore, for* $\beta = 0$, *equality holds.*

*Proof.* A general solution to the above differential equation is

$$y(t) = \frac{\int C_N \exp(\int P dt) + M}{\int P dt} \tag{2.148}$$

where $P(t) = -\frac{\beta}{N}(I - U(t))$. We have

$$\int P dt = -\frac{\beta I t}{N} + \frac{\beta\theta_2 t}{\kappa} \qquad\qquad + \frac{\beta}{\kappa}\ln\left(1 + (1/b)\exp(\Delta\theta(t-T))\right).$$

Then,

$$C_N e^{\int P dt} = C_N B(t)\exp\left(\frac{\beta\theta_2}{\kappa} - \frac{\beta I t}{N}\right)t,$$

where

$$B(t) = \left(1 + (1/b)\exp(\Delta\theta(t-T))\right)^{\frac{\beta}{\kappa}}.$$

For $b \geq 1$, we can lower bound $B(t)$ as

$$B(t) \geq \begin{cases} 1 & t \leq \frac{\ln b}{\Delta\theta} + T \\ \left(\frac{1}{b}\right)^{\frac{\beta}{\kappa}} \exp\left(\frac{\beta}{\kappa}\Delta\theta(t-T)\right) & t > \frac{\ln b}{\Delta\theta} + T. \end{cases} \tag{2.149}$$

On the other hand, if $b < 1$,

$$B(t) \geq \left(\frac{1}{b}\right)^{\frac{\beta}{\kappa}} \exp\left(\frac{\beta}{\kappa}\Delta\theta(t-T)\right), \quad \forall t. \tag{2.150}$$

Let us now evaluate $A(t)$. We have

$$A(t) = \int C_N e^{\int P dt} dt.$$

Initially consider the case $b \geq 1$. For $t < \frac{\ln b}{\Delta\theta} + T$, it is easy to verify that

$$A(t) \geq C_N \frac{\exp\left(\left(\frac{\beta\theta_2}{\kappa} - \frac{\beta I}{N}\right)t\right)}{\frac{\beta\theta_2}{\kappa} - \frac{\beta I}{N}} \tag{2.151}$$

where the inequality follows from (2.149). For $t > \frac{\ln b}{\Delta\theta} + T$, we have

$$\begin{aligned} A(t) &\geq A(\frac{\ln b}{\Delta\theta} + T) + \int_{\frac{\ln b}{\Delta\theta}+T}^{t} C_N e^{\int P dt} \\ &\geq C_N \exp(q_1 T) \exp\left(q_1 \frac{\ln b}{\Delta\theta}\right) \frac{1}{q_1} \\ &+ C_N \exp(q_1 t) \left(\frac{1}{b}\right)^{\frac{\beta}{\kappa}} \frac{\exp\left(\frac{\beta\Delta\theta}{\kappa}(t-T)\right)}{q_2} \\ &- C_N \exp(q_1 T) \left(\frac{1}{b}\right)^{\frac{\beta}{\kappa}} \frac{\exp\left(q_2 \frac{\ln b}{\Delta\theta}\right)}{q_2}. \end{aligned} \tag{2.152}$$

where $q_1 = \left(\frac{\beta\theta_2}{\kappa} - \frac{\beta I}{N}\right)$ and $q_2 = \frac{\beta\theta_1}{\kappa} - \frac{\beta I}{N}$.

In the second case, in which $b < 1$, for all values of $t$, we have,

$$A(t) \geq C_N \exp(q_1 t) \left(\frac{1}{b}\right)^{\frac{\beta}{\kappa}} \frac{\exp\left(\frac{\beta\Delta\theta}{\kappa}(t-T)\right)}{q_2}.$$

where the inequality follows from (2.150).

Then, combining the expressions of $A(t)$ in both cases, for $t > \frac{\ln b}{\Delta\theta} + T$, we have,

$$\begin{aligned} A(t) \geq\ & C_N \exp\left(q_1 T\right) \exp\left(q_1 \frac{\ln b}{\Delta\theta}\right) \frac{1}{q_1} \mathbf{1}_{b\geq 1} \\ &+ C_N \exp\left(q_1 t\right) \left(\frac{1}{b}\right)^{\frac{\beta}{\kappa}} \frac{\exp\left(\frac{\beta\Delta\theta}{\kappa}(t-T)\right)}{q_2} \\ &- C_N \exp\left(q_1 T\right) \left(\frac{1}{b}\right)^{\frac{\beta}{\kappa}} \frac{\exp\left(q_2 \frac{\ln b}{\Delta\theta}\right)}{q_2} \mathbf{1}_{b\geq 1}. \end{aligned} \tag{2.153}$$

where $\mathbf{1}_{b\geq 1}$ is the indicator function defined by (2.47).

Using the above result in equation (2.148), we get that for $t > \frac{\ln b}{\Delta\theta} + T$,

$$y(t) = \frac{M}{\exp(\int P dt)} + \frac{A(t)}{\exp(\int P dt)} \tag{2.154}$$

$$\begin{aligned} &\geq M \left(\frac{b}{d}\right)^{\frac{\beta}{\kappa}} \exp\left(-q_1 t\right) \\ &+ C_N \left(\frac{b}{d}\right)^{\frac{\beta}{\kappa}} \exp\left(-q_1(t-T)\right) \exp\left(q_1 \frac{\ln b}{\Delta\theta}\right) \frac{1}{q_1} \mathbf{1}_{b\geq 1} \\ &+ C_N \left(\frac{1}{d}\right)^{\frac{\beta}{\kappa}} \frac{\exp\left(\frac{\beta\Delta\theta}{\kappa}(t-T)\right)}{q_2} \\ &- C_N \left(\frac{1}{d}\right)^{\frac{\beta}{\kappa}} \exp\left(-q_1(t-T)\right) \frac{\exp\left(q_2 \frac{\ln b}{\Delta\theta}\right)}{q_2} \mathbf{1}_{b\geq 1}. \end{aligned} \tag{2.155}$$

where $d = (b + \exp(\Delta\theta(t-T)))$. Using boundary conditions, we can show that

$$\begin{aligned} M =\ & \left(\frac{1+b}{b}\right)^{\frac{\beta}{\kappa}} \exp\left(q_1 T\right) \left(y(T) - C_N \left(\frac{b}{1+b}\right)^{\frac{\beta}{\kappa}} \frac{1}{q_1} \mathbf{1}_{b\geq 1}\right) \\ &- \left(\frac{1+b}{b}\right)^{\frac{\beta}{\kappa}} \left(C_N \left(\frac{1}{1+b}\right)^{\frac{\beta}{\kappa}} \frac{1}{q_2} (1 - \mathbf{1}_{b\geq 1})\right). \end{aligned}$$

Substituting the above equation in equation (2.155) and rearranging yields (2.147). For $\beta = 0$, the inequalities in equations (2.149) and (2.150) become equalities and we get the lemma. $\square$

# 3. TRANSPORT LAYER: MUTUAL INTERACTION OF HETEROGENEOUS CONGESTION CONTROLLERS*

Recent years have seen the design of a large number of congestion control protocols for use on the Internet. Their designs all revolve around the idea that link congestion is indicated by some notion of "price", which the source can respond to. Different congestion price metrics include packet loss, packet marks, packet delays or some combination thereof. However, the relative value of one protocol versus another is not well understood. For example, it might be conjectured that a delay sensitive application would consider using a protocol that has a delay-based congestion metric, and a throughput maximizing application might favor a loss-based metric. How should applications choose the protocol to use?

An analytical framework for network resource allocation was developed in seminal work by Kelly *et al.* [26]. If the flow $i$ has a rate $x_i \geq 0$ and the utility associated with such a flow is represented by a concave, increasing function $U_i(x_i)$, the objective is

$$\max \quad \sum_{i \in \mathcal{N}} U_i(x_i) \tag{3.1}$$

$$\text{s.t.} \quad y_l \leq c_l, \ \forall\, l \in \mathcal{L} \tag{3.2}$$

where $\mathcal{N}$ is the set of sources, $\mathcal{L}$ the set of links, $c_l$ the capacity of link $l \in \mathcal{L}$. Also let $R$ be the routing matrix with $R_{li} = 1$ if the route associated with source $i$ uses link $l$. The load on link $l$ is $y_l = \sum_{r \in \mathcal{N}} R_{lr} x_r$. The problem can be solved using ideas based on *Primal-Dual* system dynamics [26, 30, 37, 67, 69] to yield a set of controllers. At the source we have

$$\textbf{Source:} \quad \dot{x}_i(t) = \kappa_i \left( U_i'(x_i(t)) - \sum_{l: l \in \mathcal{L}} R_{li} p_l(t) \right)_{x_i}^{+}, \tag{3.3}$$

*Part of the data reported in this chapter is reprinted with permission from "Which protocol? Mutual interaction of heterogeneous congestion controllers" by V. Ramaswamy, D. Choudhury and S. Shakkottai. *Proc. of IEEE INFOCOM*, 2011, Copyright@2011 IEEE.

where $k_i > 0$,, and the notation $(\phi)^+_\xi$ is used to denote the function

$$(\phi)^+_\xi = \begin{cases} \phi & \xi > 0 \\ \max\{\phi, 0\} & \xi = 0. \end{cases} \tag{3.4}$$

(3.4) ensures that $x$ is non-negative. The controller in (3.3) has an attractive interpretation that the source rate of flow $i$ responds to feedback in the form of link prices $p_l(t)$, with the end-to-end price being calculated as the sum of prices on all links that the flow traverses—something that is common to all congestion control protocols. Source rate is always non-negative, which is enforced by the definition of the function in (3.4). The price $p_l(t)$ at link $l$ is calculated using

$$\textbf{Link:} \quad \dot{p}_l(t) = \rho(p_l(t)) \left( \sum_{j \in \mathcal{N}} R_{lj} x_j(t) - c_l \right)^+_{p_l(t)}. \tag{3.5}$$

(3.5) ensures that the price is non-negative. Each link has a buffer in which packets are queued. If the total load at a link $l$ given by $\sum_{j \in \mathcal{N}} R_{lj} x_j(t)$ is greater than the capacity $c_l$, the queue length increases, while if it is less than $c_l$, the queue length decreases as seen in (3.5). The queue length is always non-negative, as enforced by the definition in (3.4). The gain parameter $\rho(p_l)$ is any positive function. Thus, the *link-price* $p_l(t)$ can be identified with the *queue length* at link $l$. It has been shown [26,30,37,67,69] that the above control scheme converges to the optimal solution to the problem in (3.1).

While this framework indicates that the fundamental price of a link is proportional to queue length, congestion control protocols use several different congestion metrics. For example, TCP Reno [70] uses packet drops (or marks) as its price metric, while TCP Vegas uses end-to-end delay [37]. Other protocols include Scalable TCP [27] (that uses loss-feedback, and allows scaling of rate increases/decreases based on network characteristics), FAST-TCP [78] (that uses delay-feedback, and is meant for high bandwidth environments), and TCP-Illinois [35] (that uses loss and delay signals to attain high throughput). However, drops, marks, and delays are all functions of the queue length. Thus, *a key difference between protocols is their way of interpreting queue length information.*

A fall out of different price-interpretations is that when flows choose distinct congestion control protocols, they do not obtain the same throughput on shared links. For example, studies such as [45, 71–73] study inter-protocol as well as intra-protocol fairness, while [4] considers a game of choosing between protocols, assuming that a certain throughput would be guaranteed per combination.

Throughput alone does not fully capture the performance of an application, since it might also be impacted by queueing effects such as delay and packet loss. We consider applications that might have different sensitivities to queueing. Indeed, a large fraction of Internet traffic consists of file transfers (less delay sensitive) and buffered video streams (more delay sensitive) from data centers or content distribution networks. We model these flows as having (possibly different) utilities for throughput, and disutilities for the queueing encountered on their respective paths.

We anticipate for a future Internet architecture where multiple congestion controlling schemes are available to cater the needs of different service classes and the flows are allowed choose the ones according to their service preferences. Hence, we assume that flows play "fair" in that they choose to follow the constraints imposed by employing *some* form of congestion control. Thus, the flows choose from a set of "reasonable" congestion control mechanisms, for example variants of TCP, so as to maximize their payoff that is utility minus disutility.

Our objective is similar to the proposal in [55], where a system design for virtual links tailored for flows that are rate sensitive (R) and delay sensitive (D) is presented. The idea is that an R-flow would pick the virtual link where it is guaranteed higher rate, whereas a D-flow would pick one where it is guaranteed a lower delay. However, unlike that work, we have two basic differences. First, we explicitly model utility (for throughput) and disutility (for queuing) for all kinds of flows, rather than assume that D-type flows would be willing to live with smaller rate. This enables us to explore the space of multiple classes of service with tolling, since it gives an objective measure on the choice made by the flow. Second, we allow a choice between TCP flavors (i.e., interpretation of queue length by congestion controllers) according to the application in question. However, in [55] the only way to reduce delay is to have short buffers for the D service class, which might also result in more losses.

Our finding is that if the number of flows in the system is large, the optimal strategy of a flow is to choose a price interpretation from among the space of available ones that is most similar to its disutility function. Using this finding, we can characterize the total system value to all flows, and we show that the ratio of this value to the optimum value can be arbitrarily small. Finally, we consider the situation in which we create multiple virtual networks with tolling, with each flow having a choice between networks and between protocols. We show that we can fix the tolls such that the overall system value can be increased significantly, in-spite of the toll. We next present our model and summarize our main results.

## 3.1 Model and main results

We consider a system in which each flow $i \in \mathcal{N}$ has a so-called $\alpha-$*fair* utility function [46],

$$U_i(x_i) \triangleq w_i x_i^{1-\alpha_i}/(1-\alpha_i), \tag{3.6}$$

with $\alpha_i \geq 1$, and a disutility that depends on the vector of link prices $p$ as

$$\tilde{U}_i(x_i, p) \triangleq \sum_{l \in \mathcal{L}} R_{li} (p_l/\tau_i)^{\beta} x_i, \tag{3.7}$$

where $\beta > 1$ is a constant. The overall payoff is the difference of the two, given by

$$F_i(x_i, p) \triangleq U_i(x_i) - \tilde{U}_i(p). \tag{3.8}$$

The $\alpha-$fair utility function was proposed by Mo et. al [46] as a method of capturing a large class of fairness measures based on the value of $\alpha$ used. For instance, they showed that $\alpha \to 1$ results in *proportional* fairness, while $\alpha \to \infty$ results in *max-min* fairness. The form of the disutility function is such that based on $\beta$, the disutility can be (almost) linear in queue length (which in turn is proportional to delay, weighted by the parameter $T$), to gradually increasing convexity as $\beta$ rises, to a sharp cutoff for large $\beta$. The threshold parameter $\tau_i$ in (3.7) models the flow's sensitivity to queue length, with a small value of $\tau_i$ indicating high sensitivity (*e.g.,* delay sensitive applications need short queue lengths) and a large value indicating low sensitivity (*e.g.,* loss sensitive applications are affected only by buffer overflow).

We define a set of protocols $\mathcal{T}$, with cardinality $T = |\mathcal{T}|$. Each protocol $z \in \mathcal{T}$ is associated with a price-interpretation function $m^z(p_l) \triangleq (p_l/T_z)^{\beta}$. Note that these price-interpretation functions take the same form as disutilities, and model the way in which a particular protocol $z \in \mathcal{T}$ interprets link prices[1]. Again, a loss-based protocol would have a high value of $T_z$, while a dealy-based protocol would have a low value. This corresponds to the fact that in a protocol that is modulated by buffer over flows such as TCP Reno, the queue length has no impact until a maximum threshold (buffer size) is reached, after which the price is very high ($T_z =$ buffer size here). Similarly, TCP Vegas (approximately) decides on whether the achieved throughput is too high or too low as compared to a threshold, which in turn can be related to a threshold on the per-packet delay seen by the flow ($T_z$ is less than the buffer size here). Now, while a flow $i$ cannot change its disutility function parameterized by $\tau_i$ it can choose to use a combination of protocols as it finds appropriate. A particular flow $i$'s choice could take the form

$$q_i(p) \triangleq \sum_{z \in \mathcal{T}} \epsilon_i^z \sum_{l=1}^{L} R_{li} m^z(p_l) \tag{3.9}$$

where $\sum_{z \in \mathcal{T}} \epsilon_i^z = 1$, and $\epsilon_i^z \geq 0$. The convex combination models the idea that a flow

[1] We will refer to "price-interpretation functions" and "protocols" interchangeably.

sometimes measures price in one way (*e.g.,* delay-based) and sometimes in another way (*e.g.,* loss-based). $\epsilon_i^z$ can be thought of as the probability with which flow $i$ uses protocol $z$. For example, this situation might correspond to a flow using delay and loss measurements simultaneously, and responding to congestion signals (loss or delay) probabilistically. We refer to the choice $[\epsilon_i^1, \epsilon_i^2, \cdots \epsilon_i^T]$, made by flow $i$ as $\epsilon_i \in \mathbf{E}_i \triangleq \{\epsilon_i : \sum_{z \in \mathcal{T}} \epsilon_i^z = 1,\ \epsilon_i^z \geq 0\}$. Further, we denote aggregate choices of all flows by $\epsilon \in \mathcal{E} \triangleq \Pi_{i \in \mathcal{N}} \mathbf{E}_i$, and will refer to $\epsilon \in \mathcal{E}$ as a *protocol-profile.*

We first show in Section 3.2 that for a given protocol-profile, the bandwidth allocations (and hence the payoffs) are unique. Further, a primal-dual type control will converge to this unique bandwidth allocation. The result is essentially a consistency check that allows us to analytically determine the payoffs as a function of the protocol-profile chosen.

We show in Section 3.3 that all bandwidth allocations that are attainable by a protocol-profile over $T$ protocols with $m^1(p) \geq m^2(p) \geq \cdots \geq m^T(p)$ are attainable by a protocol-profile over just the two protocols $m^1(p)$ and $m^T(p)$. The result has the appealing interpretation that when $m^z(p) = (p/T_z)^\beta$, it is sufficient to only consider the "strictest" interpretation (smallest $T_z$, which can be thought of as delay-based feedback) and the most "lenient" (largest $T_z$, associated with loss-based feedback). We next show that with two protocols with $T_s < T_l$, the bandwidth allocation received by a flow $i$ is decreasing in the weight it places on the strict protocol. Although the proof is involved, the result is intuitive since a strict protocol would always interpret $p$ as a larger congestion than the lenient protocol. However, since payoffs are the sum of utility and disutility, it does not follow that all flows would choose the protocol with the higher threshold.

We show in Sections 3.4 and 3.5 that in many cases, the total system value is maximized when all flows choose to use only $m^1(p) = (p/T_s)^\beta$. On the one hand if flows have price-insensitive payoffs, the protocol-profile used does not matter as long as all of them use the same profile. On the other hand, if there is a mix of flows, some of which have a large disutility function (price-sensitive) and others which do not (price-insensitive), using the strict price-interpretation $m^1(p) = (p/T_s)^\beta$, ensures that the price does not become too large for all flows, which maximizes system value.

In Sections 3.4 and 3.5, we also consider the case flows use selfish optimizations to choose their protocol-profiles and study the Nash equilibrium. If all flows have price-insensitive payoffs, then they all choose the lenient price-interpretation $m^2(p) = (p/T_l)^\beta$. This case can be mapped to throughput maximizing flows all choosing TCP Reno. If we have a mix of flow types sharing a link, it turns out that the price-sensitive flows with disutility function parametrized by $\tau \leq T_s$, choose the strict price-interpretation $m^1(p) = (p/T_s)^\beta$, regardless of the choice of others. Similarly, the price-sensitive flows with disutility threshold $\tau \geq T_l$, choose the lenient price-interpretation $m^2(p) = (p/T_l)^\beta$. While the other flows may employ mixed strategies. When the number of flows in the system is large, a flow with disutility threshold $\tau$ picks a mixed strategy that yields an effective price interpretation $(p/\tau)^\beta$. The result is interesting since it suggests that a delay sensitive application cannot do any better in terms of overall payoff even if it chooses a more lenient protocol. We also characterize the ratio of system value in the game versus the social optimum for the single-link case to determine an efficiency ratio, which can be quite high.

Finally, in Section 3.7 we introduce virtual networks, each of which is assigned a certain fraction of the capacity, and chooses a toll. Flows can choose a network and protocols. The idea is similar to Paris Metro Pricing (PMP) [11, 51, 68], and we show that the system value at Nash equilibrium can be higher overall in spite of tolling. The result suggests that the Internet might benefit by having separate tiers of service for delay-sensitive and loss-sensitive flows.

## 3.2 Problem formulation

We assume that for each link, there exists at least one flow that uses only that link. The assumption implies that all links have a non-zero price. We hypothesize from (3.3) and (3.5) that the payoffs should be determined by the protocol-profile $\epsilon$ as

$$x_i^*(p^*, \epsilon_i) = (U_i')^{-1}\left(\sum_{z=1}^{T} \epsilon_i^z \sum_{l=1}^{L} R_{li} m^z(p_l^*)\right), \tag{3.10}$$

with $\epsilon_i \in \mathbf{E}_i$ and for all $l \in \mathcal{L}$.

$$\sum_{i=1}^{N} R_{li} x_i^*(p^*, \epsilon_i) = c_l \quad p_l^* > 0, \tag{3.11}$$

Note that although we have denoted $x^*$ as depending on both $\epsilon$ and $p^*$, the prices themselves depend on $\epsilon$ through $x^*$, and the solution $(x^*(\epsilon), p^*(\epsilon))$ (if it exists) is solely a function of $\epsilon$. We show that the equilibrium exists, and can be reached using Primal-Dual dynamics. We have the following proposition.

**Proposition 1.** *Given any protocol-profile $\epsilon$, Primal-Dual dynamics converge to the unique solution $(x^*, p^*)$ of the conditions (3.10) and (3.11).*

*Proof.* For price-interpretation functions of the form $(p/T_z)^\beta$, the source dynamics in (3.3) can be re-written as

$$\dot{x}_i(t) = \kappa_i \left( U_i'(x_i) - \left( \sum_{z=1}^{T} \epsilon_i^z \left( \frac{T_1}{T_z} \right)^\beta \right) \sum_{l=1}^{L} R_{li} m^1(p_l) \right)_{x_i}^{+}$$

where $m^1(p_l) = (\frac{p_l}{T_1})^\beta$. Let $\mathcal{U}_i(x_i) = \frac{1}{\zeta_i} U_i(x_i)$ where $\zeta_i = \sum_{z=1}^{T} \epsilon_i^z (\frac{T_1}{T_z})^\beta$, and let $\kappa_i = \zeta_i$. Then the above equation can be modified as

$$\dot{x}_i(t) = \zeta_i \left( \mathcal{U}_i'(x_i(t)) - \sum_{l=1}^{L} R_{li} m^1(p_l(t)) \right)_{x_i}^{+}. \tag{3.12}$$

Now, in (3.5) choose $\rho(p_l) = \frac{1}{m'^1(p_l)}$, where $m'^1$ is derivative of $m^1$. Then the price-update equation can be re-written as,

$$\dot{m}^1(p_l(t)) = \left( \sum_{i=1}^{N} R_{li} x_i(t) - c_l \right)_{p_l}^{+}. \tag{3.13}$$

Equations (3.12) and (3.13) correspond to the primal-dual dynamics of the following convex maximization problem

$$\begin{aligned} \max_{x>0} \quad & \sum_{i=1}^{N} \mathcal{U}_i(x_i) \\ \text{subject to} \quad & \sum_{i=1}^{N} R_{li} x_i \le c_l, \ \forall l \in \mathcal{L}. \end{aligned}$$

The above is a convex optimization problem with a unique solution satisfying (3.10) and (3.11). Thus, by the usual Lyapunov argument [30, 37, 67, 69] Primal-Dual dynamics converge to this solution. Note that our choice of price interpretation makes it a special case of the result in Appendix A Case-1 of [72]. □

We are now in a position to ask questions about what the flows' payoffs would look like at such an equilibrium, and how this would impact the choice of the protocol-profile. Recall that the payoff obtained by a flow when the system state is at $x^*(\epsilon), p^*(\epsilon)$ is given by

$$F_i(\epsilon) = U_i(x_i^*(\epsilon)) - \tilde{U}_i(p^*(\epsilon)). \tag{3.14}$$

We define a system-value function $V$, which is equal to the sum of payoff functions of all flows in the network,

$$V(\epsilon) = \sum_{i=1}^{N} F_i(\epsilon). \tag{3.15}$$

Our first objective is to find an optimal protocol-profile that maximizes the system-value function.

$$\textbf{Opt}: \qquad \max_{\epsilon \in \mathcal{E}} V(\epsilon). \tag{3.16}$$

Let $\epsilon_S^*$ be an optimal profile vector for the above problem. Then we refer to $V_S = V(\epsilon_S^*)$ as the **value of the social optimum**.

An alternative would be for flows to individually maximize their own payoffs. However, such a proceeding might not not lead to an optimal system state that maximizes the value function (3.15). We characterize the equilibrium state of such a selfish behavior by modeling it as a strategic game.

Let $\mathcal{G} =< \mathcal{N}, \mathcal{E}, \mathcal{F} >$ be a strategic game, where $\mathcal{N}$ is the set of flows (players), $\mathcal{E}$ is the set of all protocol profiles (action sets) and $\mathcal{F} = \{F_1, F_2, \cdots, F_N\}$, where $F_i : \mathcal{E} \to \mathbb{R}$ is the payoff function of user $i$ defined in (3.14). Define $\epsilon_{-i} = [\epsilon_1, \epsilon_2, \cdots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_N]$, *i.e.*, this represents the choices of all flows except $i$. Then $\epsilon = [\epsilon_i, \epsilon_{-i}]$. For any fixed $\epsilon_{-i}$, flow $i$ maximizes its payoff as shown below.

$$\textbf{Game}: \qquad \max_{\epsilon_i \in \mathbf{E}_i} F_i(\epsilon_i, \epsilon_{-i}) \quad \forall i \in \mathcal{N}. \tag{3.17}$$

The game is said to be at a Nash equilibrium when flows do not have any incentive to unilaterally deviate from their current state. We define $\epsilon_G^*$ as a Nash equilibrium of the game $\mathcal{G}$ if

$$(\epsilon_G)_i^* = \arg\max_{\epsilon_i \in \mathbf{E}_i} F_i(\epsilon_i, (\epsilon_G)_{-i}^*), \quad \forall i \in N$$

We refer to $V_G = V(\epsilon_G^*)$ as the **value of the game**. Finally, we define the "Efficiency Ratio ($\eta$)" as

$$\eta = \frac{V_G}{V_S}. \tag{3.18}$$

## 3.3 Basic results

We first show that a $T$-protocol network can be replaced with an equivalent 2-protocol network. Consider a $T$-protocol network with price interpretation functions $[m^1, m^2, \cdots, m^T]$. Let $\epsilon \in \mathcal{E}_T$ be a profile state in the $T$-network. Then the equilibrium rate vector $x^*(\epsilon)$ and price vector $p^*(\epsilon)$ satisfy the equilibrium conditions (3.10) and (3.11). Now, consider a 2-protocol network with price interpretation functions $m^1$ and $m^T$. Note that $m^1 \geq m^z \geq m^T, z = 2, \cdots, T-1$. Let $\mu \in \mathcal{E}_2$ be a profile state in the 2-protocol network.

**Proposition 2.** *For any equilibrium $(x^*(\epsilon), p^*(\epsilon))$ in a $T$-protocol network, $\exists$ a protocol-profile $\mu$ s.t. $(x^*(\epsilon), p^*(\epsilon))$ is also an equilibrium for the 2-protocol network.*

*Proof.* For any given $\epsilon \in \mathcal{E}_T$, let $(x^*(\epsilon), p^*(\epsilon))$ be an equilibrium pair that satisfies the equilibrium conditions (3.10) and (3.11), which are reproduced below for clarity.

$$x_i^*(\epsilon) = (U_i')^{-1}\left(\textstyle\sum_{z=1}^{T} \epsilon_i q_i^{z*}\right), \forall i \in \mathcal{N},$$
$$Rx^*(\epsilon) = c, \quad p_l^* > 0, \forall l \in \mathcal{L}.$$

where $q_i^{z*} = \sum_{l=1}^{L} R_{li} m^z(p_l^*(\epsilon))$. The fact that $m^T \leq m^z \leq m^1$, implies, $q_i^{T*} \leq q_i^{z*} \leq q_i^{1*}, \quad \forall i \in \mathcal{N}, Z \in \mathcal{T}$. Since both $m^1$ and $m^T$ are strictly increasing functions, there exists a unique $\mu_i \in [0,1]$, such that,

$$\sum_{z=1}^{T} \epsilon_i^z q_i^{z*} = \mu_i q_i^{1*} + (1-\mu_i) q_i^{T*}.$$

Now, we have

$$\begin{aligned} x_i^*(\epsilon) &= (U_i')^{-1}\left(\sum_{z=1}^{T} \epsilon_i^z q_i^{z*}\right) \\ &= (U_i')^{-1}\left(\mu_i q_i^{1*} + (1-\mu_i) q_i^{T*}\right), \forall i \in \mathcal{N}, \\ & \quad Rx^*(\epsilon) = c, \quad p_l^* > 0, \forall l \in \mathcal{L}. \end{aligned}$$

The above equations correspond to the equilibrium conditions of a 2-protocol network with price interpretation functions $m^1$ and $m^T$. Therefore, there exists a protocol-profile $\mu = [\mu_1, \cdots, \mu_N]$ such that $(x^*(\epsilon), p^*(\epsilon))$ is an equilibrium pair of 2-protocol network. □

The above proposition shows that any equilibrium state of a $T$-protocol network can be obtained with an equivalent 2-protocol network. Therefore we restrict our study to 2-protocol networks with a "strict" price interpretation $m^s = (\frac{p}{T_s})^\beta$ and a "lenient" price interpretation $m^l = (\frac{p}{T_l})^\beta$, *i.e.*, $T_s < T_l$. Also, we redefine the protocol profile of flow $i$, $\epsilon_i$, as is $\epsilon_i = \epsilon_i^1$, where $\epsilon_i^1$ is the weight applied on the strict price interpretation. Finally, the equilibrium rate of flow $i$ can be written in terms of $m^s$ and $m^l$ as follows:

$$\begin{aligned} x_i^*(\epsilon) &= (U_i')^{-1}\left(\sum_{l=1}^{L} R_{li}\left(\epsilon_i m^s(p_l^*) + (1-\epsilon_i)m^l(p_l^*)\right)\right) \\ &= (U_i')^{-1}\left((\epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^{\beta})\sum_{l=1}^{L} R_{li} m^s(p_l^*)\right). \end{aligned} \tag{3.19}$$

where $\epsilon = [\epsilon_1, \epsilon_2, \cdots, \epsilon_N]$ is the system protocol-profile. The above result follows from (3.10).

We next show that the bandwidth allocation received by a flow $i$ is decreasing in the weight it places on the strict protocol $m^s(p) = (p/T_s)^{\beta}$.

**Proposition 3.** *Let $x_i^*(\epsilon)$ be the equilibrium rate of flow $i$ for any $\epsilon \in \mathcal{E}_2$. Then,*

$$\frac{\partial x_i^*}{\partial \epsilon_i} \leq 0, \forall i \in \mathcal{N},$$

*Proof.* From (3.19), we have

$$U_i'(x_i^*) = \sum_{l=1}^{L} R_{li} m^s(p_l^*)\left(\epsilon_i + (1-\epsilon_i)\left(\frac{T_s}{T_l}\right)^{\beta}\right).$$

Then, differentiating above equation with respect to $\epsilon_j$, we get,

$$\frac{\partial x_i^*}{\partial \epsilon_j} = A_{ij} + \sum_{l=1}^{L} \frac{\partial p_l^*}{\partial \epsilon_j} B_{il}, \tag{3.20}$$

where

$$\begin{aligned} A_{ij} &= \frac{(1-(\frac{T_s}{T_l})^{\beta})\left(\sum_{l=1}^{L} R_{li} m^s(p_l^*)\right)}{U_i''(x_i^*)}\delta_{ij}, \quad \text{and} \\ B_{il} &= \frac{R_{li}(m^s)'(p_l^*)(\epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^{\beta})}{U_i''(x_i^*)}. \end{aligned}$$

Also, $\delta_{ij} = 1$ if $i = j$, and zero otherwise. At equilibrium, $\sum_{i=1}^{N} R_{li} x_i^*(\epsilon) = c_l, \forall l \in \mathcal{L}$. Now, differentiating this equation with respect to $\epsilon_j$, we get

$$\sum_{i=1}^{N} R_{li} \frac{\partial x_i^*}{\partial \epsilon_j} = 0, \quad \forall l \in \mathcal{L}. \tag{3.21}$$

Replacing $\frac{\partial x_i^*}{\partial \epsilon_j}$ with (3.20), we obtain

$$\sum_{i=1}^{N} R_{li} \frac{(\epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^\beta)}{U_i''(x_i^*)} \sum_{k=1}^{L} R_{ki} (m^s)'(p_k^*) \frac{\partial p_k^*}{\partial \epsilon_j}$$
$$+ R_{lj} \frac{(1-(\frac{T_s}{T_l})^\beta)\left(\sum_{k=1}^{L} R_{kj} m^s(p_k^*)\right)}{U_j''(x_j^*)} = 0.$$

Now, rearranging terms in the above expression, we get,

$$\sum_{k=1}^{L} (m^s)'(p_k^*) \frac{\partial p_k^*}{\partial \epsilon_j} \sum_{i=1}^{N} R_{li} R_{ki} \frac{(\epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^\beta)}{-U_i''(x_i^*)}$$
$$= R_{lj} \frac{(1-(\frac{T_s}{T_l})^\beta)\left(\sum_{k=1}^{L} R_{kj} m^s(p_k^*)\right)}{U_j''(x_j^*)}.$$

We can represent the above in a matrix form as

$$RWR^T \zeta = r,$$

where

$$\begin{aligned} W &= \operatorname{diag} \frac{(\epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^\beta)}{-U_i''(x_i^*)} \\ \zeta &= \left[ (m^s)'(p_1^*) \frac{\partial p_1^*}{\partial \epsilon_j} \quad (m^s)'(p_2^*) \frac{\partial p_2^*}{\partial \epsilon_j} \cdots (m^s)'(p_L) \frac{\partial p_L^*}{\partial \epsilon_j} \right]^T \\ r &= \frac{(1-(\frac{T_s}{T_l})^\beta)\left(\sum_{k=1}^{L} R_{kj} m^s(p_k^*)\right)}{U_j''(x_j^*)} [R_{1j} \quad \cdots \quad R_{Lj}]^T. \end{aligned}$$

Note that $U_i$ is a strictly concave function and hence $U_i''(x_i^*) < 0$. Therefore, $RWR^T$ is a positive definite matrix. Now, we have

$$\zeta = (RWR^T)^{-1} r. \tag{3.22}$$

Let $H = (RWR^T)^{-1}$, where $H$ is an $L \times L$ matrix. Let us represent its elements using $h_{lm}$. Thus, from (3.22), we have

$$\frac{\partial p_l^*}{\partial \epsilon_j} = \frac{\sum_{k=1}^{L} R_{kj} h_{lk}}{(m^s)'(p_l^*)} \frac{(1 - (\frac{T_s}{T_l})^\beta) \left( \sum_{k=1}^{L} R_{kj} m^s(p_k^*) \right)}{U_j''(x_j^*)}. \tag{3.23}$$

Let $V = WR^T(RWR^T)^{-1}R$. Then, from (3.20) and (3.23), we get

$$\frac{\partial x_j^*}{\partial \epsilon_j} = \frac{(1-(\frac{T_s}{T_l})^\beta)\left(\sum_{l=1}^{L} R_{kj} m^s(p_k^*)\right)}{U_j''(x_j^*)} (1 - v_{jj}), \tag{3.24}$$

$$\frac{\partial x_i^*}{\partial \epsilon_j} = -\frac{(1-(\frac{T_s}{T_l})^\beta)\left(\sum_{k=1}^{L} R_{kj} m^s(p_k^*)\right)}{U_j''(x_j^*)} v_{ij}, \tag{3.25}$$

where $v_{ij}$ represent elements of $V$.

Now, we show that $\frac{\partial x_j^*}{\partial \epsilon_j}$ is negative given the assumption in the lemma. Note that $V$ is a projection matrix. The diagonal elements of a projection matrix are positive and less than or equal to unity. i.e, $v_{jj} \leq 1$. Then, from (3.24), we conclude that $\frac{\partial x_j^*}{\partial \epsilon_j} \leq 0$ and hence have proved the proposition.

□

The above proposition is intuitive in that a strict protocol would force the flow to cut down its rate for the same price as a lenient protocol.

**Corollary 5.** *In the single link case, the link-price $p^*$ and the rate vector $x^*$ satisfies,* $\frac{\partial p^*}{\partial \epsilon_j} < 0$ *and* $\frac{\partial x_i^*}{\partial \epsilon_j} > 0$ *if* $i \neq j, \forall i, j \in \mathcal{N}$.

*Proof.* From (3.23), (3.24) and (3.25), we have

$$\frac{\partial p^*}{\partial \epsilon_j} = \frac{(1 - (\frac{T_s}{T_l})^\beta) m^s(p^*)}{(m^s)'(p^*) U_j''(x_j^*)} \frac{1}{\sum_{r=1}^{N} \nu_r}, \tag{3.26}$$

$$\frac{\partial x_i^*}{\partial \epsilon_j} = \frac{(1-(\frac{T_s}{T_l})^\beta) m^s(p^*)}{U_j''(x_j^*)} \left( \delta_{ij} - \frac{\nu_j}{\sum_{r=1}^N \nu_r} \right), \tag{3.27}$$

where

$$\nu_i = -\frac{\epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^\beta}{U_i''(x_i^*)} = \frac{x_i^*}{\alpha_i m^s(p^*)}.$$

The above result follows from (3.19) and the fact that $U_i''(x_i^*) = \frac{-\alpha_i}{x_i^*} U_i'(x_i^*)$. Note that $U_i''(x) < 0$ since $U_i$ is strictly concave. Now, the corollary is straightforward from the above results. □

Now, we now study different mixes of flow types in order to understand the system value in each case.

## 3.4 Flows with price-insensitive payoff

We associate each flow $i \in \mathcal{N}$ to a class, based on its disutility function of the form $\sum_{l\in\mathcal{L}} R_{li}(p_l/\tau_i)^\beta x_i$. We begin by considering a system of flows that have a price-insensitive payoff, *i.e.,* $\tau_i = \infty\ \forall i \in \mathcal{N}$. This means that payoff is solely a function of bandwidth, and we have $F_i(\epsilon) = U_i(x^*(\epsilon))$. However, even in this situation, flows must employ congestion control, *i.e.,* they must choose a protocol-profile. From Section (3.3), recall that since we only have two protocols, the flow $i$'s choice of protocol profile is defined by a scalar value $\epsilon_i$. Also note that $T_z \neq \infty$ for each protocol $z = 1, 2$. The system-value is equal to the sum of user payoffs, $V(\epsilon) = \sum_{i=1}^N U_i(x^*(\epsilon))$. We then have the following result.

**Proposition 4.** *The system-value is maximized when the protocol choices made by all users are the same. Thus, if $\epsilon_S^* = arg\max_{\epsilon\in\mathcal{E}} V(\epsilon)$, and $(\epsilon_S^*)_i$ is used to denote the protocol choice made by-profile of user $i$, then $(\epsilon_S^*)_i = (\epsilon_S^*)_j, \forall i, j \in \mathcal{N}$.*

*Proof.* We first derive an upper bound for system-value $V(\epsilon)$ and then show that the upper bound is achieved when all sources choose the same protocol. Suppose that $\mathcal{X} = \{x|Rx = c\}$. Let $\hat{x} = \arg\max_{Rx=c} \sum_{i=1}^N U_i(x_i)$. Note that equilibrium rate $x^*(\epsilon) \in \mathcal{X}$, since $Rx^* = c$.

Then the value of $\sum_{i=1}^{N} U_i(x)$ evaluated at $x^*(\epsilon)$ satisfies

$$V(\epsilon) = \sum_{i=1}^{N} U_i(x_i^*(\epsilon)) \leq \sum_{i=1}^{N} U_i(\hat{x}_i).$$

We showed in Proposition 2 that the equilibrium rate $x^*(\epsilon)$, is the unique maximizer of the convex problem $\max_{x>0, Rx=c} \sum_{i=1}^{N} \frac{1}{\zeta_i} U_i(x_i)$, where $\zeta_i = \epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^\beta$. Then, $x^*(\epsilon)$ can be made equal to $\hat{x}$, the optimal point in set $\mathcal{X}$, by choosing $\zeta_i = \zeta_j \; \forall i, j \in \mathcal{N}$. Such a choice means that

$$\begin{aligned} \zeta_i = \zeta_j \quad &\Rightarrow \quad \epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^\beta = \epsilon_j + (1-\epsilon_j)(\frac{T_s}{T_l})^\beta, \\ &\Rightarrow \quad \epsilon_i = \epsilon_j. \end{aligned}$$

Thus, if $\epsilon_S^* = \arg\max_{\epsilon \in \mathcal{E}} V(\epsilon) \Rightarrow (\epsilon_S^*)_i = (\epsilon_S^*)_j, \forall i, j \in \mathcal{N}$. Therefore, the system value is maximized when the protocol choices made by all the users are identical. Also, the maximum value does not depend on the parameters of the selected protocol. $\square$

We next consider the game in which flows are allowed to choose their protocols selfishly.

**Proposition 5.** *Let $G =< \mathcal{N}, \mathcal{E}, \mathcal{F} >$ be a strategic game with payoff function of user $i$ is given as $F_i(\epsilon) = U_i(x_i^*(\epsilon))$. Then there exists a Nash equilibrium for game $G$, and the equilibrium profile for any user $i \in \mathcal{N}$ is $(\epsilon_G^*)_i = 0$.*

*Proof.* Differentiating $F_i$ w.r.t $\epsilon_i$, and using Proposition 3

$$\frac{\partial F_i}{\partial \epsilon_i} = U'(x_i^*(\epsilon)) \frac{\partial x_i^*(\epsilon)}{\partial \epsilon_i} \leq 0$$

Hence, $F_i(\epsilon)$ is maximized when $\epsilon_i = 0$. Therefore, $(\epsilon_G^*)_i = 0, \forall i \in \mathcal{N}$. $\square$

**Efficiency Ratio:** We showed in Proposition 4 that the value function is maximized when all flows pick the same protocol-profile. In Proposition 5 we saw that when each flow selfishly maximizes its own payoff, there exists a Nash equilibrium under which every source chooses the lowest priced protocol, *i.e.,* the protocol with the higher value of $T$. Such a profile is a special case of all flows choosing the same protocol-profile. Thus, value of the social optimum and the value of the game are identical and Efficiency Ratio ($\eta$) is unity.

**Example-1**: Consider the case in which a single link with capacity $c = 10$ is shared by 2 price-insensitive flows. Users have $\alpha$-fair utility functions with $\alpha = 2$, $w_1 = 100$ and $w_2 = 100$. We use price-interpretation functions $(\frac{p}{2})^2$ and $(\frac{p}{5})^2$. Note that the simulation parameters $\alpha, \beta$ and threshold values are chosen arbitrarily. These parameters may not correspond to any particular protocol used in practice. Nevertheless, the observations made here hold true for any values of $\alpha \geq 1, \beta > 1$ and $T_s, T_l, \tau_i > 0$.

In Figure (3.1) we show the system value for different choices of protocol profiles. The plot illustrates that system value is maximized when both flows choose the same profile. Figure (3.2) shows how the payoff function of a flow varies with its protocol profile. We find that regardless of the value of the protocol profile chosen by the other flow, the payoff function is maximized when it picks the lower price protocol.

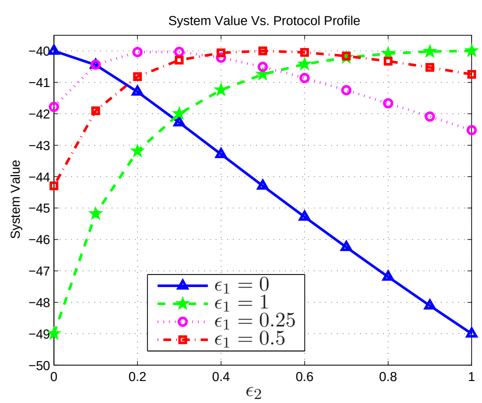



Figure 3.1: System Value with price-insensitive flows as a function of the protocol-profile. We observe that the system value is maximized when both flows choose the same protocol-profile.

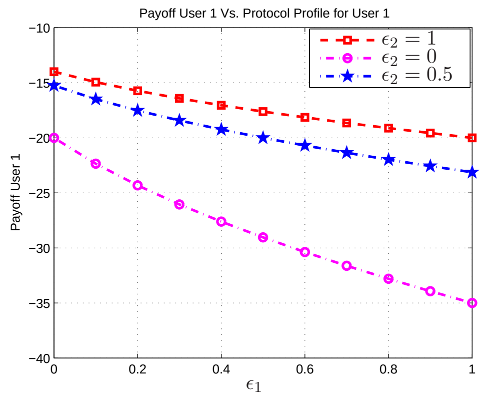



Figure 3.2: Payoff of a price-insensitive flow as a function of its protocol-profile. We observe that payoff is maximized when the flow chooses the more lenient price interpretation, regardless of the other flow.

## 3.5 Mixed environment

We now consider the case where a network is shared by flows with different disutilities. We identify the optimal protocol profile that maximizes the system value, and compare it with and the Nash equilibrium. We first study the case of a network consisting of a single link.

### *3.5.1 Single link case*

Consider a single link system with capacity $c$ shared by $N$ flows. The payoff of user $i \in \mathcal{N}$ is $F_i(\epsilon) = U_i(x_i^*(\epsilon)) - \left(\frac{p^*(\epsilon)}{\tau_i}\right)^\beta x_i^*(\epsilon)$. Then, the system value is $V(\epsilon) = \sum_{i=1}^N F_i(\epsilon)$.

**Proposition 6.** *The system- value is maximized when all users pick the protocol with lowest threshold, i.e., if $\epsilon_S^* = \arg\max_{\epsilon \in \mathcal{E}} V(\epsilon)$, then $(\epsilon_S^*)_i = 1, \forall i \in \mathcal{N}$.*

*Proof.* (Sketch) Recall that $\alpha_i \geq 1$ by our assumption. Given this assumption, it can be shown through straightforward differentiation that $\tilde{U}_i(\epsilon_i)$ is a monotonically decreasing function of $\epsilon_i$. Now, the value function $V$ is maximum when $U(\epsilon)$ is maximized and $\tilde{U}(\epsilon)$ is minimized. We already know from Proposition 4 that $U(\epsilon)$ is maximized when all flows choose the same protocol-profile. Coupling this result with the fact that $\tilde{U}_i(\epsilon_i)$ is decreasing in $\epsilon_i$, we see that system value is maximized when $\epsilon_i = 1, \forall i \in \mathcal{N}$. $\square$

We now study the strategic game in which users individually maximize their payoff as in (3.17). We show that there exists a Nash equilibrium and characterize the protocol-profile.

**Proposition 7.** *Let $G =< \mathcal{N}, \mathcal{E}, \mathcal{F} >$ be a strategic game with payoff of user $i$ is $F_i(\epsilon) = U_i(x_i^*(\epsilon)) - (\frac{p^*(\epsilon)}{\tau_i})^\beta x_i^*(\epsilon)$. Then there exists a Nash equilibrium (NE) for Game G. At NE, flows with greatest sensitivity to price choose the strict protocol, i.e., if $\tau_i = T_s$, then $\epsilon_i = 1$.*

*Proof.* We will show that $F_i(\epsilon)$ is quasi-concave, and use the Theorem of Nash to show existence of a NE. Differentiating $F_i$ w.r.t $\epsilon_i$,

$$\frac{\partial F_i}{\partial \epsilon_i} = (U_i'(x_i^*) - d_i(p^*))\frac{\partial x_i^*}{\partial \epsilon_i} - d_i'(p^*)x_i^*\frac{\partial p^*}{\partial \epsilon_i}, \tag{3.28}$$

where $d_i(p^*) = (\frac{p^*}{\tau_i})^\beta$ and $d_i'(p^*)$ is its derivative. Now, substituting the results from (3.26) and (3.27), in the above equation, we get

$$\frac{\partial F_i}{\partial \epsilon_i} = B(U_i'(x_i^*) - d_i(p^*))\left(1 - \frac{\nu_j}{\sum_{r=1}^N \nu_r}\right) \tag{3.29}$$

$$-B\frac{d_i'(p^*)x_i^*}{(m^s)'(p^*)\sum_{r=1}^N \nu_r}, \tag{3.30}$$

where $B = \frac{(1-(\frac{T_s}{T_l})^\beta m^s(p^*)}{U_i''(x_i^*)}$ and $\nu_i = \frac{x_i^*}{\alpha_i m^s(p^*)}$. Note that $B < 0$ since $U_i''$ is a negative function.

From (3.19) along with the definitions of $\nu_i$ and $d_i(p^*)$, the above expression can be simplified as follows:

$$\frac{\partial F_i}{\partial \epsilon_i} =$$

$$\frac{Bm^s(p^*)\sum_{r=1,r\neq i}^N \frac{x_r^*}{\alpha_r}}{\sum_{r=1}^N \frac{x_r^*}{\alpha_r}}\left(\epsilon_i + (1-\epsilon_i)\left(\frac{T_s}{T_l}\right)^\beta - \left(\frac{T_s}{\tau_i}\right)^\beta\right)$$

$$-\frac{Bm^s(p^*)}{\sum_{r=1}^N \frac{x_r^*}{\alpha_r}}\left(\frac{T_s}{\tau_i}\right)^\beta x_i^*. \tag{3.31}$$

We show that if the above expression has a root, then it is unique. The roots are characterized by

$$\epsilon_i + (1-\epsilon_i)(\frac{T_s}{T_l})^\beta = (\frac{T_s}{\tau_i})^\beta \left(1 + \frac{x_i^*}{\sum_{r=1,r\neq i}^{N} \frac{x_r^*}{\alpha_r}}\right). \tag{3.32}$$

First observe that the left side of the above expression is strictly increasing in $\epsilon_i$ (since $T_s < T_l$). Since $\frac{\partial x_i^*}{\partial \epsilon_i} < 0$ and $\frac{\partial x_r^*}{\partial \epsilon_i} > 0$ if $r \neq i$ (from Proposition 3 and Corollary 5), the right side of the above expression is strictly decreasing. Therefore, the set of roots of the equation, $\frac{\partial F_i}{\partial \epsilon_i}(x) = 0$ is a singleton or null set. Thus, $F_i$ is unimodal or monotonic in $\epsilon_i$ for any fixed $\epsilon_{-i}$ and hence quasi concave.

Since $\epsilon_i \in [0,1]$ is a non-empty compact convex set, by the theorem of Nash, the quasi concavity of $F_i(\epsilon_i, \epsilon_{-i})$ guarantees that there exists a $\epsilon_G^*$, such that for all $i = 1, \cdots, N$,

$$(\epsilon_G^*)_i = \arg \max_{\epsilon_i \in [0,1]} F_i(\epsilon_i, (\epsilon_G^*)_{-i}).$$

Hence, the first part of the proof is complete.

Now, consider a flow with disutility (per unit rate) $(\frac{p}{\tau_i})^\beta$, where $\tau = T_s$. Replacing $\tau_i$ with $T_s$ in (3.31), we observe that $\frac{\partial F_i}{\partial \epsilon_i} > 0$ (Note that $B < 0$). Therefore, payoff is maximized when $\epsilon_i = 1$. □

In the next section, we study the characteristics of the NE and show that it is unique.

### *3.5.2 Nash equilibrium characteristics*

We have established the existence of NE of the strategic game (3.17) in the previous section. We conduct further studies on the properties of NE in this section. First, we derive conditions for the NE system protocol profile. Then, in Proposition 8, we show that the game has a unique NE. Finally, in Proposition 9, we derive the NE strategies of flows when there are large number of flows in the system.

Let $\hat{\epsilon}$ be a Nash equilibrium system protocol profile (action profile). Then, by definition, it must satisfy the condition that

$$\hat{\epsilon}_i = \arg\max_{\epsilon_i \in [0,1]} F_i(\epsilon_i, (\hat{\epsilon}_i)_{-i}), \forall i \in \mathcal{N}.$$

Then, from the first order optimality condition, we have

$$\frac{\partial F_i(\hat{\epsilon})}{\partial \epsilon_i}(\epsilon_i - \hat{\epsilon}_i) \leq 0.$$

Consequently, from (3.31), we get that, $\forall i \in \mathcal{N}$,

$$\gamma(\hat{\epsilon}_i) = \left( \frac{1}{T_s^\beta} \wedge \frac{1}{T_i^\beta} \left( 1 + \frac{x_i^*(\hat{\epsilon})}{\sum_{r=1, r\neq i}^N \frac{x_r^*(\hat{\epsilon})}{\alpha_r}} \right) \right) \vee \frac{1}{T_l^\beta}. \tag{3.33}$$

where $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ and $\gamma(\epsilon_i) = \epsilon_i(\frac{1}{T_s})^\beta + (1 - \epsilon_i)(\frac{1}{T_l})^\beta$. In addition, the Nash equilibrium profile must also satisfy,

$$x_i^*(\hat{\epsilon}) = \left( \frac{w_i}{\gamma(\hat{\epsilon}_i)(p^*)^\beta} \right)^{\frac{1}{\alpha_i}}, \tag{3.34}$$

$$\sum_{i=1}^N x_i^*(\hat{\epsilon}) = c \quad . \tag{3.35}$$

Here, (3.34) follows from (3.19) and the definition of $U_i(x)$. Also, (3.35) follows from the assumption that every link has one flow using that link alone. Now, we show that the set of Nash equilibria, characterized by (3.33)-(3.35), is singleton.

**Proposition 8.** *The strategic game, $G =< \mathcal{N}, \mathcal{E}, \mathcal{F} >$, has a unique Nash equilibrium.*

*Proof.* To prove by contradiction, assume multiple Nash equilibria exist. Let two distinct NE system protocol profiles be $\hat{\epsilon}^1$ and $\hat{\epsilon}^2$. Also, let $x_i^1 = x_i^*(\hat{\epsilon}^1), x_i^2 = x_i^*(\hat{\epsilon}^2), p^1 = p^*(\hat{\epsilon}^1)$, $p^2 = p^*(\hat{\epsilon}^2)$, $\gamma_i^1 = \gamma(\hat{\epsilon}^1)$ and $\gamma_i^2 = \gamma(\hat{\epsilon}^2)$. Then, by reordering the flow indices, we get that, for some $k \in \{0, 1, \cdots, N\}$,

$$\gamma_i^1 > \gamma_i^2 \quad \text{for} \quad i = 1, 2, \cdots, k, \tag{3.36}$$

$$\gamma_i^1 \leq \gamma_i^2 \quad \text{for} \quad i = k+1, \cdots, N. \tag{3.37}$$

Also, if $k = 0$, there exist a flow $i \in \mathcal{N}$ such that $\gamma_i^1 < \gamma_i^2$. We show that the above condition are infeasible for all values of $k$, under the NE conditions given by (3.33-3.34).

Initially, consider the case when $k = N$. Then, from (3.33), for $i = 1, 2, \cdots, N$, we have

$$\frac{x_i^1}{\sum_{r=1, r\neq i}^{N} \frac{x_r^1}{\alpha_r}} > \frac{x_i^2}{\sum_{r=1, r\neq i}^{N} \frac{x_r^2}{\alpha_r}} \Rightarrow \frac{x_i^1}{\sum_{r=1}^{N} \frac{x_r^1}{\alpha_r}} > \frac{x_i^2}{\sum_{r=1}^{N} \frac{x_r^2}{\alpha_r}} \tag{3.38}$$

$$\Rightarrow \frac{\sum_{r=1}^{N} \frac{x_r^1}{\alpha_r}}{\sum_{r=1}^{N} \frac{x_r^1}{\alpha_r}} > \frac{\sum_{r=1}^{N} \frac{x_r^2}{\alpha_r}}{\sum_{r=1}^{N} \frac{x_r^2}{\alpha_r}} \tag{3.39}$$

which is a contradiction. Hence, this case is not feasible. Similarly, we can show that the case when $k = 0$ is also not feasible.

Now, consider the case when $1 \leq k < N$. Also, suppose that $p^1 \geq p^2$. Then, from (3.34), we have

$$x_i^1 < x_i^2, \quad \text{for} \quad i = 1, 2, \cdots, k.$$

Let

$$i^* = \arg\max_i \frac{x_i^1}{x_i^2}.$$

Note that $i^* > k$ and hence, $\gamma_{i^*}^1 \leq \gamma_{i^*}^2$. Also, from (3.35), note that $x_{i^*}^1 > x_{i^*}^2$.

Observe that,

$$\frac{x_i^1}{x_{i^*}^1} = \frac{x_i^1}{x_i^2}\frac{x_{i^*}^2}{x_{i^*}^1}\frac{x_i^2}{x_{i^*}^2} \leq \frac{x_i^2}{x_{i^*}^2},$$

and strict inequality holds if $i \leq k$. It follows from the above result that,

$$\frac{x_{i^*}^1}{\sum_{r=1}^{N} \frac{x_r^1}{\alpha_r}} > \frac{x_{i^*}^2}{\sum_{r=1}^{N} \frac{x_r^2}{\alpha_r}} \Rightarrow \frac{x_{i^*}^1}{\sum_{r=1, r\neq i^*}^{N} \frac{x_r^1}{\alpha_r}} > \frac{x_{i^*}^2}{\sum_{r=1, r\neq i^*}^{N} \frac{x_r^2}{\alpha_r}}.$$

Finally, from (3.33) and the above result, we get $\gamma_{i^*}^1 \geq \gamma_{i^*}^2$. But, from the definition of $i^*$, we know that $\gamma_{i^*}^1 \ngtr \gamma_{i^*}^2$. In case $\gamma_{i^*}^1 = \gamma_{i^*}^2$, then, from (3.34) and the assumption that $p^1 \geq p^2$, we get $x_{i^*}^1 \leq x_{i^*}^2$, which also raises a contradiction. Hence, this case is also not feasible. In similar fashion, we can show that the case in which $p^1 < p^2$ is also not feasible.

Hence, our assumption that multiple NE exist is not true. Therefore, NE is unique. $\square$

Next, we characterize the NE in the asymptotic regime.

**Proposition 9.** *When the number of flows in the system, $N$, is large, the protocol profile of flow $i$ at NE, $\hat{\epsilon}_i$, satisfies*

$$\hat{\epsilon}_i \left(\frac{1}{T_s}\right)^\beta + (1-\hat{\epsilon}_i)\left(\frac{1}{T_l}\right)^\beta = \left(\left(\frac{1}{T_s}\right)^\beta \wedge \left(\frac{1}{\tau_i}\right)^\beta\right) \vee \left(\frac{1}{T_l}\right)^\beta.$$

*Proof.* Recall from (3.33) that, the NE protocol profile of flow $i$, satisfies,

$$\gamma(\hat{\epsilon}_i) = \left(\frac{1}{T_s^\beta} \wedge \left(\frac{1}{\tau_i}\right)^\beta \left(1 + \frac{x_i^*(\hat{\epsilon})}{\sum_{r=1,r\neq i}^N \frac{x_r^*(\hat{\epsilon})}{\alpha_r}}\right)\right) \vee \frac{1}{T_l}^\beta.$$

In order to prove the proposition, we claim that,

$$\lim_{N\to\infty} \frac{x_i^*(\hat{\epsilon}_i)}{\sum_{r=1,r\neq i}^N \frac{x_r^*(\hat{\epsilon}_i)}{\alpha_r}} = 0, \tag{3.40}$$

holds true. Before proving the above result, we introduce a few notations: Let $\alpha_{max} = \max_i \alpha_i, \alpha_{min} = \min_i \alpha_i, w_{max} = \max_i w_i$ and $w_{min} = \min_i w_i$.

Now, the proof of the claim (3.40) is as follows: From (3.35), we can show that,

$$\frac{x_i^*(\hat{\epsilon})}{\sum_{r=1,r\neq i}^N \frac{x_r^*(\hat{\epsilon})}{\alpha_r}} \leq \frac{\alpha_{max}}{\frac{c}{x_i^*(\hat{\epsilon})} - 1}.$$

Also, from (3.34), we have,

$$x_i^*(\hat{\epsilon}) = \left(\frac{w_i}{\gamma(\hat{\epsilon}_i)(p^*(\hat{\epsilon}))^\beta}\right)^{\frac{1}{\alpha_i}} \leq \left(\frac{w_{\max} T_l^\beta}{(p^*(\hat{\epsilon}))^\beta}\right)^{\frac{1}{\alpha_{min}}}. \tag{3.41}$$

The above result follows from the fact that $\gamma(\hat{\epsilon}_i) \geq (\frac{1}{T_l})^\beta$.

From Corollary 5, we observe that the link-price is a decreasing function of protocol profile of each flow and hence, the system protocol profile $\epsilon$. Therefore, the link price achieves the lowest value, when every flow adopts the strict protocol. Then, from (3.34) and (3.35), it is easy to show that

$$(p^*(\hat{\epsilon}))^\beta \geq w_{min} \left( \frac{N^{\alpha_{min}}}{c^{\alpha_{max}}} \right) T_s^\beta. \tag{3.42}$$

Finally, from (3.41) and (3.42), we have

$$\frac{x_i^*}{\sum_{r=1, r\neq i}^{N} \frac{x_r^*}{\alpha_r}} \leq \frac{\alpha_{max}}{\frac{c}{x_i^*} - 1} \leq \frac{\alpha_{max}}{NK - 1}$$

where $K$ is a constant. The upper bound in the above expression goes to zero for large values of $N$. Therefore, the claim in (3.40) holds true and hence, the proof is completed. $\square$

**Example-2**: We consider a link with capacity $c = 10$ shared by two flows with disutilities $(\frac{p}{2})^2$ and $(\frac{p}{5})^2$, respectively, and $w_1 = w_2 = 1$. The other parameters are unchanged from Example-1. We show the system value for different choices of protocol-profiles in Figure 3.3. The value is maximized when both flows choose the strict protocol. Figure (3.4) shows how the payoff of each flow varies with its choice of protocol profile, given other's is fixed. We find that for the first (sensitive) flow, the payoff function is maximized when it chooses the strict protocol, regardless of the other flow. But the payoff of the second (less-sensitive) flow is maximized for some combination of protocols. The results validate our findings.

**Example-3**: We consider a link with capacity $c = 1000$. There are 40 flows sharing the link. The strict and lenient thresholds are $T_s = 2$ and $T_l = 7$ respectively. In our simulations, we have set $\beta = 2$, $\alpha = 2$ for half of users and $\alpha = 3$ for the other half. There are 10 classes of flows, with each class containing 4 flows. The disutility threshold of a Class $i$ flow, given by $\tau_i$, is chosen according to the following relation: $(\frac{1}{\tau_i})^\beta = (\frac{1}{T_l})^\beta + ((\frac{1}{T_s})^\beta - (\frac{1}{T_l})^\beta)(i/10)$.

We choose a candidate flow that belongs to Class 4. We assume that every other flow has chosen their NE protocol profile. That means, the effective price interpretation of a flow

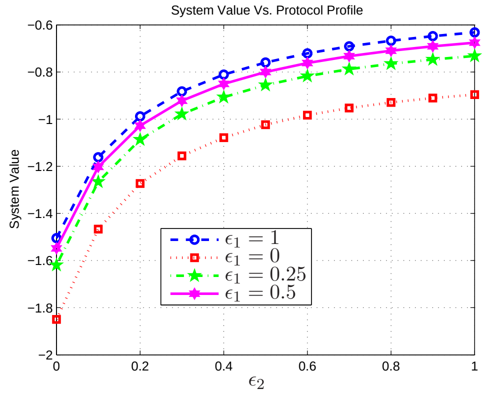



Figure 3.3: System value against protocol choices ($\epsilon_i$): Two flows sharing a link.

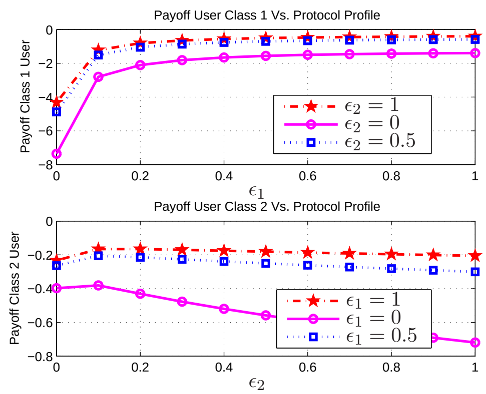



Figure 3.4: Payoff against protocol choice ($\epsilon_i$): Two flows sharing a link.

belonging to Class $i$ is $(\frac{p}{\tau_i})^\beta$. Figure (3.5) plots the payoff of the candidate flow as a function of its effective protocol choice $\gamma(\epsilon_4) = \epsilon_4(1/T_s)^\beta + (1-\epsilon_4)(1/T_l)^\beta$, where $\epsilon_4$ is its protocol profile. As claimed by Proposition 9, the payoff is maximized when $\gamma(\epsilon_4) = (\frac{1}{\tau_4})^\beta = 0.17$.

### *3.5.3 Network case*

We consider a system of flows with log utility functions, which is a special class of an $\alpha$-fair utility function with $\alpha \to 1$. The payoff of flow $i \in \mathcal{N}$ is $F_i(\epsilon) = w_i \log(x_i^*(\epsilon)) - \sum_{l=1}^{L} R_{li}(\frac{p_l^*(\epsilon)}{\tau_i})^\beta x_i^*(\epsilon)$. Then the system-value is $V(\epsilon) = \sum_{i=1}^{N} F_i(\epsilon)$.

**Proposition 10.** *The System-Value function is maximized when all flows pick the higher*

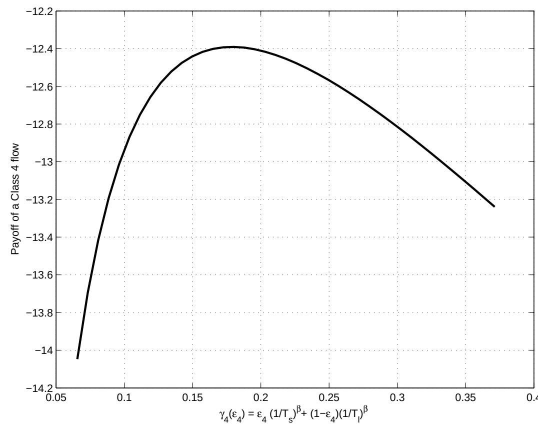



Figure 3.5: Payoff of a Class 4 flow is maximized when $\gamma(\epsilon_4) = (\frac{1}{T_4})^\beta = 0.17$.

*priced protocol, namely* $m^1 = \left(\frac{p}{T_s}\right)^\beta$. *Let* $\epsilon_S^* = arg\max_\epsilon V(\epsilon)$, *then* $(\epsilon_S^*)_i = 1, \forall i = 1, \cdots, N$,

*Proof.* We can show through straightforward differentiation that, the disutility function, $\tilde{U}_i(\epsilon_i)$, is a monotonically decreasing function of $\epsilon_i$. The rest of the proof is similar to that of Proposition 6. □

We now consider a game with two types of flows: price-insensitive flows with zero disutilities, and price-sensitive flows with disutility (per unit rate) $(\frac{p_l^*}{T_s})^\beta$. In this special case, there exists a unique Nash equilibrium. In Proposition 5 we saw that price-insensitive flows pick the lenient protocol at Nash equilibrium irrespective of the choices of the other players. We will now show that price-sensitive flows pick the strict protocol at Nash equilibrium.

**Proposition 11.** *Any flow $i$ with disutility (per rate) $(p/T_s)^\beta$ (i.e. $\tau_i = T_s$) picks $\epsilon_i = 1$ is the Nash equilibrium.*

*Proof.* It can be shown through straightforward differentiation that $\frac{\partial F_i}{\partial \epsilon_i} > 0$ for any flow $i \in \mathcal{N}$ with disutility (per rate) $(p/T_s)^\beta$, which completes the proof. □

## 3.6 Efficiency ratio

We now characterize the loss of system value at Nash equilibrium, as compared to the value of the social optimum. We focus on the case of a single link with capacity $c$.

**Proposition 12.** *Assume $\alpha_i > 1, \forall i \in \mathcal{N}$. When the number of flows in the system is large,*

$$\eta = \frac{V_G}{V_S} < \hat{\alpha}(\frac{T_l}{T_s})^{\beta}.$$

*where $\hat{\alpha} = \max_i \alpha_i$.*

*Proof.* Let $\epsilon^* = [\epsilon_1^*, \epsilon_2^*, \cdots, \epsilon_N^*]$ be the system protocol profile at social optimum. From Proposition 6, every user chooses the strict protocol at social optimum, i.e $\epsilon_i^* = 1, \forall i$. Hence, from (3.19), and the definition of $U_i$, we have

$$x_i^*(\epsilon^*) = \left( w_i \left( \frac{T_s}{p^*(\epsilon^*)} \right)^{\beta} \right)^{\frac{1}{\alpha_i}}, \quad \sum_i x_i^*(\epsilon^*) = c. \tag{3.43}$$

Interpreting $\left(\frac{p^*(\epsilon^*)}{T_s}\right)^{\beta}$ as the dual variable, the above equations can be identified as the KKT conditions of the optimization problem given below:

$$\max_x \quad \sum_i \frac{w_i x_i^{1-\alpha_i}}{1-\alpha_i}, \quad \text{subject to} \quad \sum_i x_i = c.$$

And, $x^*(\epsilon^*)$ is the unique maximizer of the above problem. The payoff of a flow at social optimum, from (3.8) and the above results, is given by

$$F_i(\epsilon^*) \quad = U_i(x_i^*(\hat{\epsilon^*})) \left(1 + \mathbf{1}_i(\alpha_i - 1)(\tfrac{T_s}{T_i})^{\beta}\right). \tag{3.44}$$

where $\mathbf{1}_i = 1$ if flow $i$ is a price sensitive flow and zero otherwise. The system value at social optimum is $V_S = \sum_i F_i(\epsilon^*)$.

Now, let $\hat{\epsilon} = [\hat{\epsilon}_1, \hat{\epsilon}_2, \cdots, \hat{\epsilon}_N]$ be the system protocol profile at Nash equilibrium. From

Proposition 9, equation (3.19) and the definition of $U_i$, we have

$$x_i^*(\hat{\epsilon}) = \left( w_i \left( \frac{T_l \wedge (\tau_i \vee T_s)}{p^*(\hat{\epsilon})} \right)^{\beta} \right)^{\frac{1}{\alpha_i}}, \quad \sum_i x_i^*(\hat{\epsilon}) = c. \tag{3.45}$$

Recall that $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. Interpreting $\left(\frac{p^*(\hat{\epsilon})}{T_s}\right)^{\beta}$ as the dual variable, the above equations can be identified as the KKT conditions of the optimization problem given below:

$$\max_x \quad \sum_i \frac{w_i(\frac{T_l \wedge (\tau_i \vee T_s)}{T_s})^{\beta} x_i^{1-\alpha_i}}{1-\alpha_i}, \quad \text{subject to} \quad \sum_i x_i = c.$$

Also, $x^*(\hat{\epsilon})$ is the unique maximizer of the above problem. Finally, the payoff of a flow is

$$F_i(\hat{\epsilon}) \quad = U_i(x_i^*(\hat{\epsilon})) \left( 1 + \mathbf{1}_i(\alpha_i - 1) \left( \frac{T_l \wedge (\tau_i \vee T_s)}{\tau_i} \right)^{\beta} \right). \tag{3.46}$$

The system value at NE is $V_G = \sum_i F_i(\hat{\epsilon})$.

Now, from the above results and the fact that $U_i$'s are negative, since $\alpha_i > 1$ by the assumption of this proposition, we can show that

$$\begin{aligned} & V_G \geq \hat{\alpha} \sum_i \left( \frac{\tilde{T}}{T_s} \right)^{\beta} U_i(x_i^*(\hat{\epsilon})) \geq \hat{\alpha} \sum_i \left( \frac{\tilde{T}}{T_s} \right)^{\beta} U_i(x_i^*(\epsilon_i^*)) \\ & > \hat{\alpha} \left( \frac{T_l}{T_s} \right)^{\beta} \sum_i U_i(x_i^*(\epsilon_i^*)) \left( 1 + \mathbf{1}_i(\alpha_i - 1)(\frac{T_s}{T_i})^{\beta} \right) \\ & = \hat{\alpha}(\frac{T_l}{T_s})^{\beta} V_S, \end{aligned} \tag{3.47}$$

where $\hat{\alpha} = \max_i \alpha_i$ and $\tilde{T} = T_l \wedge (\tau_i \vee T_s)$. Since $V_G$ and $V_S$ are negative, the efficiency ratio $\eta$, can be bounded as

$$\eta = \frac{V_G}{V_S} < \hat{\alpha} \left( \frac{T_l}{T_s} \right)^{\beta},$$

which completes the proof. □

**Example-4:** The exact expression for efficiency ratio is derived for the following special case: We assume that every flow has the same utility function, i.e, in (3.6), $w_i = w$ and $\alpha_i = \alpha, \forall i \in \mathcal{N}$. We associate the flows, having disutility functions of the form $(\frac{p}{\tau_j})^{\beta} x$ with Class-$j$. Assume that there are $J-1$ such classes with $\tau_1 < \tau_2 < .. < \tau_{J-1}$ and $\tau_j \in [T_l, T_s], \forall j$. The flows having zero disutility function is classified as Class $J$. For algebraic convenience, we define $\tau_j = \infty$. Let $N_i$ be the number of flows belonging to Class $i$ and $n_i = N_i/N$. Then, the Value of social optimum ($V_S$) and value of game equilibrium ($V_G$) are given by

$$V_S = \frac{N}{1-\alpha}\left(\frac{c}{N}\right)^{1-\alpha} \sum_{j=1}^{J} n_j(1 + \mathbf{1}_j(\alpha-1)\left(\frac{T_s}{\tau_j}\right)^{\beta}), \tag{3.48}$$

and

$$V_G = \frac{N(\frac{c}{N})^{1-\alpha} S_1}{(1-\alpha)S_2}, \tag{3.49}$$

respectively, where

$$S_1 = \left(\alpha \sum_{j=1}^{J-1} n_i \left(\frac{\tau_j}{T_s}\right)^{(\frac{\beta}{\alpha})(1-\alpha)} + n_J \left(\frac{T_l}{T_s}\right)^{(\frac{\beta}{\alpha})(1-\alpha)}\right)$$

and

$$S_2 = \left(\sum_{j=1}^{J-1} n_j \left(\frac{\tau_j}{T_s}\right)^{\frac{\beta}{\alpha}} + n_J \left(\frac{T_l}{T_s}\right)^{\frac{\beta}{\alpha}}\right)^{1-\alpha}.$$

Also, $\mathbf{1}_j = 0$ when $j = J$ and one otherwise. The efficiency ratio, $\eta$, is given by

$$\eta = \frac{S_1}{S_2 \sum_{j=1}^{J} n_j(1 + \mathbf{1}_j(\alpha-1)(\frac{T_s}{\tau_j})^{\beta})}. \tag{3.50}$$

Now, we plot the efficiency ratio for the following case. Let two classes of flows, namely Class 1 and Class 2, are sharing a link. Also, let their disutility thresholds be $\tau_1 = T_s$ and $\tau_2 = T_l$ respectively. Letting $\alpha = 2$ and $\beta = 3$, we plot the efficiency ratio ($\eta$), given by (3.50), in Figure 3.6. The Figure 3.6 shows that $\eta$ increases with $(\frac{T_l}{T_s})$. Note that a higher

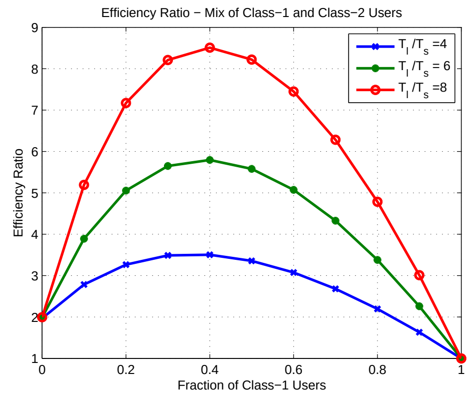



Figure 3.6: Efficiency Ratio ($\eta$) in the single link case, plotted against the fraction of Class-1 flows for different ratios of $T_l/T_s$. Since $V_S$ and $V_G$ were negative in this example, a higher ratio is worse.

ratio is worse. Hence, the performance deteriorates with $(\frac{T_l}{T_s})$.

## 3.7 Paris metro pricing

We have shown in the previous section that when the flows selfishly choose protocols to maximize their own payoff, the system performance at the resulting equilibrium, compared to the socially optimal case, can be much worse. This is due to the fact that, as shown by Proposition 9, the flows with relatively lower disutility functions choose relatively lenient protocols, and hence capture a larger fraction of channel bandwidth leaving not enough for the ones with larger disutility functions who choose stricter protocols. As a solution to the aforementioned problem, we propose a scheme in which the network is partitioned into virtual subnetworks each having its own queuing buffer, independent price (queue-length) dynamics and fixed entrance toll. A flow is free to choose a protocol along with a subnetwork so as to maximize his own payoff. This scheme is similar to *Paris Metro Pricing* (PMP) [51]. We show that the efficiency of this scheme is superior to the conventional, untolled, single network scheme.

We characterize the performance of the proposed scheme in a single link case. The single link, with capacity $c$ (bits/sec), is partitioned into $J$ virtual subnetworks. Let $S_j$ represent the $j^{th}$ sub-network. The bandwidth and toll associated with $S_j$ are denoted by

$c_j$ and $\lambda_j$ respectively. Also, let $\mathbf{c} = [c_1, \cdots, c_J]$ and $\lambda = [\lambda_1, \cdots, \lambda_J]$. We refer to $\mathbf{c}$ and $\lambda$ as bandwidth vector and toll vector respectively.

We assume that every flow has the same utility function, i.e, in (3.6), $w_i = w$ and $\alpha_i = \alpha, \forall i \in \mathcal{N}$. We associate the flows having disutility functions of the form $(\frac{p}{\tau_j})^\beta x$ to Class-$j$. We assume that there are $J-1$ such classes and $\tau_1 < \tau_2 < .. < \tau_{J-1}$ with $\tau_j \in [T_s, T_l]$. The price insensitive flows are classified as Class-$J$. For algebraic convenience, we define $\tau_J = \infty$. We also assume that there are a large number of flows in each class. Let $N_j$ represent the number of flows in Class-$j$.

A flow that seeks to maximize its payoff picks a subnetwork that yields the maximum payoff. Thus, if $\hat{k}$ is the subnetwork chosen by flow $i$,

$$\hat{k} = \arg \max_{k \in \{1, \cdots, J\}} F_{jk} \quad j = 1 \cdots, J$$

where $F_{jk}$ is the payoff of a Class-$j$ flow in $S_k$. A Nash equilibrium (NE) here is a state from which none of the flows has an incentive to deviate from its current choice of subnetwork. Note that we already know the flow's choices of protocols in each network so no deviations in protocol are possible. The desired NE is one in which all Class-$j$ flows select $S_j$, i.e

$$F_{jj} \geq F_{jk}, \quad \forall j, \forall k. \tag{3.51}$$

Note that the payoffs received are uniquely determined by the PMP system parameters $\mathbf{c}$ and $\lambda$. Now, we derive sufficient conditions on the pair, $\mathbf{c}$ and $\tau$, so that (3.51) holds true.

Assume that the system is at the desired equilibrium, i.e, every Class-$j$ flow is sending its traffic over $S_j$. Let $p_k^*$ be the equilibrium price (per unit rate) in $S_k$. The throughput received by a Class $j$ flow (or anticipated by a Class $j$ flow if it shifted to $S_k$) is given by,

$$x_{jk}^* = \left(\frac{\tau_j}{p_k^*}\right)^{\frac{\beta}{\alpha}} \quad \text{and} \quad x_{Jk}^* = \left(\frac{T_l}{p_k^*}\right)^{\frac{\beta}{\alpha}}, \forall k. \tag{3.52}$$

The above results are due to the fact that the entry of a Class-$i$ flow into $S_k$ may not significantly change its price, $p_k^*$, since there are large number of flows in $S_k$. In (3.52), the first result follows from Proposition 9, (3.34) and the assumption that $T_j \in [T_s, T_l]$ when $j < J$, while the second one follows from Proposition 5. The link price $p_k^*$ in $S_k$, follows from the above results and the fact that rates of flows sharing a sub-network add up to its bandwidth allocation, is given by

$$p_k^* = \left(\frac{N_k}{c_k}\right)^{\frac{\alpha}{\beta}} \tau_k, \text{if} \quad k < J, \quad \text{and} \quad p_J^* = \left(\frac{N_J}{c_J}\right)^{\frac{\alpha}{\beta}} T_l \tag{3.53}$$

The payoff of Class $j$ flow in $S_k$, from (3.8), is given by

$$\begin{aligned} F_{jk}(\mathbf{c}, \lambda) &= \tfrac{(x_{jk}^*)^{1-\alpha}}{1-\alpha} - \left(\tfrac{p_k^*}{\tau_j}\right)^{\beta} x_{jk}^* - \lambda_k, \\ &= A_{ik}\left(\tfrac{c_i}{N_i}\right)^{1-\alpha} - \lambda_i, \quad \forall k, \end{aligned} \tag{3.54}$$

where $A_{ik} = \frac{\alpha}{1-\alpha}(\frac{\tau_i}{\tau_k})^{(\frac{\beta}{\alpha})(1-\alpha)}$ for $i, k < J$, $A_{iJ} = \frac{\alpha}{1-\alpha}(\frac{\tau_i}{T_l})^{(\frac{\beta}{\alpha})(1-\alpha)}$, $A_{Jk} = \frac{1}{1-\alpha}(\frac{T_l}{\tau_k})^{(\frac{\beta}{\alpha})(1-\alpha)}$, $k < J$ and $A_{JJ} = \frac{1}{1-\alpha}$. Also, (3.54) follows from (3.52) and (3.53).

The following lemma derives conditions on the pair $(\mathbf{c}, \lambda)$ for (3.51) to hold true. Before stating the lemma, we introduce some notation. Let

$$l_{ik}(\mathbf{c}) = A_{ki}(\tfrac{c_i}{N_i})^{1-\alpha} - A_{kk}(\tfrac{c_k}{N_k})^{1-\alpha}. \tag{3.55}$$

$$u_{ik}(\mathbf{c}) = A_{ii}(\tfrac{c_i}{N_i})^{1-\alpha} - A_{ik}(\tfrac{c_k}{N_k})^{1-\alpha}, \tag{3.56}$$

**Lemma 9.** *Suppose the pair* $(\mathbf{c}, \lambda)$ *satisfy the following conditions: if* $1 \leq k < J$*,*

$$\frac{c_{k+1}}{c_k} \leq \tfrac{N_{k+1}}{N_k}\left(\tfrac{\tau_{k+1}}{\tau_k}\right)^{\frac{\beta}{\alpha}}, \tag{3.57}$$

$$\frac{c_J}{c_{J-1}} \leq \tfrac{N_J}{N_{J-1}}\left(\tfrac{\tau_l}{\tau_{J-1}}\right)^{\frac{\beta}{\alpha}}, \textstyle\sum_{j=1}^{J} c_j = c, \tag{3.58}$$

$$l_{k(k+1)}(\mathbf{c}) \leq \lambda_k - \lambda_{(k+1)} \leq u_{k(k+1)}(\mathbf{c}), \tag{3.59}$$

*Then,* (3.51) *hold true and the state where all the Class-*$j$ *flows choosing* $S_j$, $\forall j$, *is a Nash equilibrium.*

*Proof.* The Nash equilibrium conditions, (3.51), are equivalent to

$$l_{ik}(\mathbf{c}) \leq \lambda_i - \lambda_k \leq u_{ik}(\mathbf{c}), \quad k > i, \forall i, \tag{3.60}$$

which follows from the definition of $F_{ik}$ given by (3.54). Recall the definitions of, $l_{ik}$ and $u_{ik}$ from (3.55) and (3.56) respectively. Therefore, we prove the lemma by showing that (3.60) hold true when (3.57)-(3.59) are satisfied.

Suppose (3.57)-(3.59) are true. Then, it is easy to observe that $l_{ik} \leq u_{ik}, \forall k > i$. Also, we have

$$\sum_{t=k}^{m-1} l_{t(t+1)} \leq \lambda_k - \lambda_m, \forall m > k, \forall k. \tag{3.61}$$

From the definitions of $l_{ik}$'s and the fact that $\tau_i < \tau_k$ if $i < k$, it is easy to show that

$$l_{k(k+j)} - l_{k(k+j-1)} \leq l_{(k+j-1)(k+j)}, \tag{3.62}$$

for $k < J$ and $1 < j \leq J - k$. Then, we have,

$$\begin{aligned} l_{km} &= l_{k(k+1)} + (l_{k(k+2)} - l_{k(k+1)}) + \cdots + (l_{km} - l_{k(m-1)}) \\ &\leq \sum_{t=k}^{m-1} l_{t(t+1)} \leq \lambda_k - \lambda_m. \end{aligned} \tag{3.63}$$

In similar fashion, we can show that $u_{km} \geq \lambda_k - \lambda_m$. Then, (3.60) is proved and hence the lemma. □

The system-value is sum of payoffs of all the flows, which is given by,

$$V_T(\mathbf{c}, \lambda) = \sum_{i}^{J} N_i F_{ii} = \sum_{i=0}^{J} N_i \left( A_{ii} \left( \frac{c_i}{N_i} \right)^{1-\alpha} - \lambda_i \right). \tag{3.64}$$

We must choose $\mathbf{c}$ and $\lambda$ that maximize (3.64) satisfying the NE conditions, (3.57) -(3.59). Let $(\hat{\mathbf{c}}, \hat{\lambda})$ be one such optimal pair. Note that (3.64) is a decreasing function of toll vector,

$\lambda$. Hence, from (3.58) and (3.68), we get

$$\hat{\lambda}_J = 0, \quad \text{and} \quad \hat{\lambda}_k = \sum_{i=k}^{J} l_{i(i+1)}. \tag{3.65}$$

Substituting the optimal toll values in (3.64), we get

$$\begin{aligned} V_T(\mathbf{c}) \;=\; & \frac{\bar{N}_J}{1-\alpha}\left(\frac{c_J}{N_J}\right)^{1-\alpha} + \\ & \frac{\bar{N}_{J-1}\alpha}{1-\alpha}\left(\frac{c_{J-1}}{N_{J-1}}\right)^{1-\alpha}\left(1-\frac{1}{\alpha}\left(\frac{T_l}{\tau_{J-1}}\right)^{\frac{\beta}{\alpha}(1-\alpha)}\right) + \\ & \sum_{k=1}^{J-2}\frac{\alpha\bar{N}_k}{1-\alpha}\left(\frac{c_k}{N_k}\right)^{1-\alpha}\left(1-\left(\frac{\tau_{k+1}}{\tau_k}\right)^{\frac{\beta}{\alpha}(1-\alpha)}\right), \end{aligned} \tag{3.66}$$

where $\bar{N}_k = \sum_{i=1}^{k} N_i$. Then, define,

$$V_T = \max_{\mathbf{c}} V_T(\mathbf{c}) \quad \text{subject to} \quad (3.57) - (3.58). \tag{3.67}$$

We refer to $V_T$ as *System value with tolling.* Now, we have the following proposition, which asserts that the system value achieved by the tolled multi-tier regime is superior to that of the untolled single tier regime.

**Proposition 13.** *The system value with tolling is no less than the value of single tier network game. i.e, $V_T \geq V_G$. Also, the strict inequality holds if there exists a $k < J$ such that*

$$\left(\frac{\bar{N}_J N_k}{N_J \bar{N}_k}\right)^{\frac{1}{\alpha}} \leq \left(\frac{T_l}{\tau_k}\right)^{\frac{\beta}{\alpha}}\left(1-\left(\frac{\tau_{k+1}}{\tau_k}\right)^{\frac{\beta}{\alpha}(1-\alpha)}\right), \tag{3.68}$$

*Proof.* Suppose $\mathbf{c}$ attains equality in (3.57)-(3.58), i.e a corner point of the constraint set. Note that the elements of $\mathbf{c}$, the bandwidths allocated to each subnetwork, that means to each flow class, is equal to the total bandwidth received by the corresponding flow class at the NE of the un-tolled single network game. Also, from (3.65) and (3.55), the optimal entrance toll in each subnetwork drops to zero. Then, $V_T(\mathbf{c}) = V_G$. Hence, we conclude that $V_T \geq V_G$.

Note that $V_T(\mathbf{c})$ is strictly concave and hence, (3.67) has a unique maximizer. When (3.68) holds true, the unique maximizer lies in the interior of the constraint set of (3.67). Then, $V_T > V_G$ which completes the proof. $\square$

Next, we derive a bound on the efficiency of the multi-tier tolling scheme. Let

$$\bar{\eta} = 1 + \alpha \sum_{k=1}^{J-1} \sum_{i=1}^{k} n_i. \tag{3.69}$$

where $n_i = \frac{N_i}{N}$. Then, we claim that

$$\eta_T = \frac{V_T}{V_S} \leq \min\{\eta_G, \bar{\eta}\}. \tag{3.70}$$

where $\eta_G$ is the efficiency of single tier scheme without tolling. The claim can be proved as follows: Let $\bar{c}_j = N_j \frac{c}{N}$ for all $1 \leq j \leq J$. Then, $\bar{c} = [\bar{c}_1, \cdots, \bar{c}_J]$ lies in the feasible set of the optimization problem, (3.67). Then, $V_T(\bar{c}) \leq V_T$. It can be shown that $\frac{V_T(\bar{c})}{V_S} < \bar{\eta}$ where $V_S$ is given by (3.48). Therefore, $\eta_T < \bar{\eta}$. Also, from Proposition 13, we get that $\eta_T \leq \eta_G$. Together, we get the claim.

Note that, $\bar{\eta}$, does not depend on the ratio, $\frac{T_s}{T_l}$; but it scales up with the number of classes in the system. Nevertheless, $\eta_T$ is no more than the efficiency of the single tier networks without tolling. Therefore, we conclude that when the number of classes in the system is not arbitrarily large, the efficiency of multi-tier tolling schemes are superior to the single tier networks and, it does not scale up with the ratio, $\frac{T_s}{T_l}$. Note that there might be Nash equilibria other than the one stated by Lemma 9. Therefore, (3.70) may be better than the efficiency of the worst Nash equilibrium. Now, we present a numerical example to validate our analytical observations.

**Example-6**: Let two flow classes, namely Class 1 and Class 2, with disutility thresholds $\tau_1 = T_s$ and $\tau_2 = T_l$ are sharing a link with capacity $c$ units. The link is partitioned into

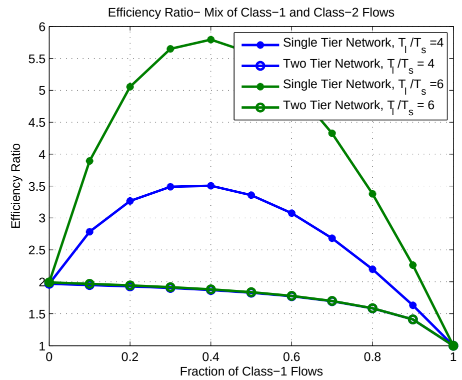



Figure 3.7: Comparison of Efficiency Ratio ($\eta$) between PMP scheme and Game in a network with price-insensitive flows and delay sensitive flows. Since $V_S$ and $V_G$ were negative in this example, a higher ratio is worse.

two subnetworks, namely $S_1$ and $S_2$. Let $N_i$ be the number of flows in Class $i$ and define $n_i = N_i/(N_1 + N_2)$, for $i = 1, 2$. The optimal bandwidth allocation to subnetwork $S_1$, that maximizes the system value with tolling, is given by

$$\hat{c}_1 = \frac{c}{1 + \frac{n_2}{n_1 n_2^{\frac{1}{\alpha}}}\left(1 - \left(\frac{T_l}{T_s}\right)^{(\frac{\beta}{\alpha})(1-\alpha)}\right)^{-\frac{1}{\alpha}}} \vee \frac{c}{1 + \frac{n_2}{n_1}(\frac{T_l}{T_s})^{\frac{\beta}{\alpha}}}.$$

Also, the optimal toll in $S_1$ is given by $\hat{\lambda}_1 = \left[\left(\frac{N_2}{c-\hat{c}_1}\right)^{\alpha-1} - \left(\frac{N_1}{\hat{c}_1}(\frac{T_s}{T_l})^{\frac{\beta}{\alpha}}\right)^{\alpha-1}\right]\frac{\alpha}{\alpha-1}$. Note that $S_2$ has no entrance toll and the optimal allocation to $S_2$ is $\hat{c}_2 = c - \hat{c}_1$. We define Efficiency Ratio ($\eta_T$) here as the ratio of System-Value with tolling ($V_T$) to Social optimum ($V_S$). From (3.64) and $V_S$,(from (3.48)), we can show that

$$\eta_T = \frac{V_T}{V_S} = \frac{\alpha\left((n_1+n_2)\left(\frac{\hat{c}_1}{cn_1}\right)^{1-\alpha} + n_1\left(\frac{\hat{c}_1}{cn_1}\right)^{1-\alpha}K\right)}{\left(1 + (\alpha-1)(n_1 + n_2(\frac{T_s}{T_l})^{\beta})\right)},$$

where $K = \left(1 - \left(\frac{T_l}{T_s}\right)^{(\frac{\beta}{\alpha})(1-\alpha)}\right)$.

In Figure (3.7), we have compared $\eta$ attained using the PMP scheme versus that of a single-tier. We have used $\alpha = 2$, $\beta = 3$ and $(\frac{T_l}{T_s}) = 4$ in our simulation. We observe that in-spite of tolling, the PMP scheme always performs better than the single-tier scheme.

Also, note that, unlike the single tier scheme, the efficiency of the PMP scheme does not scale with $\frac{T_l}{T_s}$.

## 3.8 Conclusion

In this work we examined the consequences of the idea that a protocol is simply a way of interpreting Lagrange multipliers. We showed that flows could choose the interpretations, based on criteria such as delay or loss sensitivity. We determined the socially optimal protocol, as well as the choice that would result by flows taking their own selfish decisions. We showed that the social good is maximized by using the strictest possible price interpretation. However, based on different mixes of flow types a mix of interpretations could be the Nash equilibrium state. We characterized the loss of efficiency for some specific cases, and showed that a multi-tier network with tolling is capable of achieving superior system value. The result suggests the consideration of multiple tolled virtual networks, each geared towards a particular kind of flow. In the future we propose to explore the idea of virtual, tolled subnetworks further.

Having studied a transport layer control problem, we move to a routing problem that arises in wireless networks. We consider a scenario in which multiple paths are available between each source and destination. How do the sources split their traffic over the available set of paths so as to attain the lowest possible number of transmissions per unit time ? The question becomes more difficult when certain routes can utilize the "reverse carpooling" advantage of network coding to decrease the number of transmissions used. We call the coded links as "Hyper-links". Due to network coding longer paths may become cheaper. However, the network coding advantage is realized only if there is traffic in both directions of such routes. When the sources are allowed to choose their paths selfishly, they may not prefer these paths as the first mover may see a disadvantage. Then, how do we incentivize sources to use the routes with hyper-links ? Can we develop a distributed controller that attains the lowest system cost in spite of the incentives provided to the sources ? We answer these questions in the next chapter.

# 4. NETWORK LAYER: A POTENTIAL GAME APPROACH TO MULTI-PATH WIRELESS NETWORK CODING*

There has recently been significant interest in multihop wireless networks, both as a means for basic Internet access, as well as for building specialized sensor networks. However, limited wireless spectrum together with interference and fading pose significant challenges for network designers. The technique of network coding has the potential to improve the throughput and reliability of multihop wireless networks by taking advantage of the broadcast nature of wireless medium.

For example, consider a wireless network coding scheme depicted in Figure 4.1(a). In this example, two wireless nodes need to exchange packets $x_1$ and $x_2$ through a relay node. A simple *store-and-forward* approach needs four transmissions. However, the network coding approach uses a *store-code-and-forward* technique in which the two packets from the clients are combined by means of an XOR operation at the relay and broadcast to both clients simultaneously. The clients can then decode this coded packet (using information stored at clients) to obtain the packets they need.

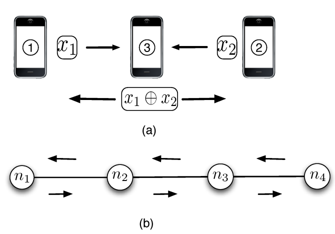



Figure 4.1: (a) Wireless Network Coding (b) Reverse carpooling.

Katti *et al.* [25] presented a practical network coding architecture, referred to as *COPE*,

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that implements the above idea while also making use of overheard packets to aid in decoding. Experimental results shown in [25] indicate that the network coding technique may result in a significant improvement in the network throughput.

Effros *et al.* [14] introduced the strategy of *reverse carpooling* that allows two information flows traveling in opposite directions to share a path. Figure 4.1(b) shows an example of two connections, from $n_1$ to $n_4$ and from $n_4$ to $n_1$ that share a common path $(n_1, n_2, n_3, n_4)$. The wireless network coding approach results in a significant (up to 50%) reduction in the number of transmissions for two connections that use reverse carpooling. In particular, once the first connection is established, the second connection (of the same rate) can be established in the opposite direction with little additional cost.

The key challenge in the design of network coding schemes is to maximize the number of *coding opportunities*, where a coding opportunity refers to an event in which at least one transmission can be saved by transmitting a combination of the packets. Insufficient number of coding opportunities may affect the performance of a network coding scheme and is one of the major barriers in realizing the coding advantage. Accordingly, the goal of this work is to design, analyze, and validate network mechanisms and protocols that improve the performance of the network coding schemes through increasing the number of coding opportunities.

Consider the scenario depicted in Figure 4.2. We have two sources with equal traffic, each of which is aware of two paths leading to its destination. Each has one path that costs 6 units, while the other path costs 7 units. If both flows use their individually cheaper paths, the total cost is 12 units. However, if both use the more expensive path, since network coding is possible at the node $n_2$, the total cost is reduced to 11 units. Thus, we see that there is a dilemma here—savings can only be obtained if there is sufficient bi-directional traffic on $(n_1, n_2, n_3)$.

A commonly used framework in the study of routing problems is that of *potential games.* Here, there exits a so-called *potential function*—a scalar value that can be thought of as representing the global utility or cost of the system. The potential function is such that the

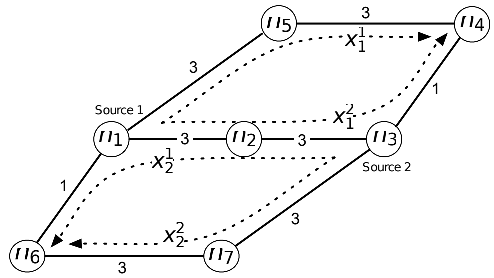


Figure 4.2: Each flow has two routes available, one of which permits network coding. The challenge is to ensure that both sources are able to discover the low cost solution.

marginal difference in the payoff received by an agent following from a unilateral change in action is equal to the marginal change in the potential function. Intuitively, it seems that the coupling between an individual agent's payoff and that of the whole system ought to ensure that the system state should converge under myopic learning dynamics. Indeed Sandholm et al. present results under which potential games converge to the optimal solution when it is unique [63], or when the number of players is sufficiently large and a probabilistic approach can be taken [7]. Extensions in the context of systems with inertia [38], as well as finding near-potential games with boundable error [10] have been studied more recently.

However, the problem that we consider presents the issue of a game with a finite number of players that has multiple equilibria, some which have lower cost than others. We can think of the system in Figure 4.2 as a potential game, with the potential function being the total cost given the traffic splits. However, if each source attempts to learn its optimal traffic split based on the marginal cost that it observes, it could easily choose the inefficient solution. The first mover here is clearly at a disadvantage as it essentially creates the route that the other can piggyback upon (in a reverse direction). Our challenge in this work is to extend the potential game framework to eliminate the first-mover disadvantage. A main contribution of this work is the development of the idea of *state space augmentation* in potential games as a way of promoting optimal coordination in such situations.

Network coding was initiated by a seminal work by Ahlswede *et al.* [3] and since then has attracted a significant interest from the research community. The network coding technique was utilized in a wireless system developed by Katabi *et al.* [25]. The proposed architecture, referred to as COPE, contains a special network coding layer between the IP and MAC layers. Sagduyu and Ephremides [62] focused on the applications of network coding in simple path topologies (referred to in [62] as *tandem* networks) and formulated several related cross-layer optimization problems. Similarly, [21] considered the problem of utility maximization when network coding is possible. However, their focus is on opportunistic coding as opposed to *creating* coding opportunities that we focus on. The practicality of utilizing network coding over multiple paths for low latency applications was demonstrated by Feng *et al.* [16].

Sengupta *et al.* [64] consider a very similar problem to ours, and present a general linear programming formulation to solve it. However, their objective was to find a centralized solution, as opposed to the distributed learning dynamics that we seek. Das *et al.* [13] proposed a new framework called "context based routing" in multihop wireless networks that enables sources to choose routes that increase coding opportunities. They proposed a heuristic algorithm that measures the imbalance between flows in opposite directions, and if this imbalance is greater than 25%, provides a discount of 25% to the smaller flow. This has the effect of incentivizing equal bidirectional flows, resulting in multiple coding opportunities. Our objective is similar, but we develop iterated distributed decision making methods that trade off a potential increase in cost of longer paths, with the potential cost reduction due to enhanced coding opportunities.

Marden *et al.* [39] considered a similar problem to ours, but unlike our focus on how to align user incentives, their attention was largely on the efficiency loss of the Nash equilibrium attained. Thus, they considered the system as a potential game, and considered the worst case and best case equilibria that the system might converge to. They showed that under the potential game framework, the best case Nash equilibrium can be optimal, while the cost of the worst case Nash equilibrium can be unboundedly large. To the best of our knowledge, the initial version of our work that was presented at a conference [58] was the first to propose a distributed algorithm that attains the optimal solution. The underlying idea of state-space augmentation was presented in that work. In parallel with our work, Marden *et al.* [40] described a "state-based game," which also augments the potential game framework with additional state, and later used the framework in the context of consensus formation in networks [33]. Also in parallel work, ParandehGheibi et al. [54] presented an optimal solution specific to the network coding problem using classical Lagrange multiplier ideas. In contrast to their work, we present a new technique whereby we modify the potential function seen by players in order to ensure that they take system-wide optimal decisions. From a methodological standpoint, we believe that our approach can find application in equilibrium selection in a wide range of coordination problems *(eg.* in understanding how altruistic behavior can alter the set of achievable equilibria).

The key contribution of this research is a distributed two-level control scheme that would iteratively lead the sources to discover the appropriate splits for their traffic among multiple paths. In a traditional potential game approach, the matrix of traffic splits of the different flows would be the state of the system. In our work, we introduce the idea of *augmenting* the state space with additional variables that are controlled separately by *augmented agents*. Unlike Lagrange multipliers, the additional state variables need not correspond to a constraint set. Instead, these augmented variables are used to modify the potential function seen by the original agents in such a way that they are directed towards the optimal equilibrium. In this sense, the idea can be thought of as a generalized Lagrange multiplier. We also illustrate that our approach can coexist with the usual Lagrange multiplier approach to handle constraints.

We explore the idea of state space augmentation using the network coding problem. Here, at one timescale we have sources that selfishly choose to split their traffic across available multiple paths using marginal costs on each path to direct their actions. The learning dynamics that they use are consistent with a potential game approach. However, the costs that they see are set by augmented agents as well as Lagrange multipliers, both of which operate at a different timescale from the source dynamics. The augmented agents in our problem are so-called *hyper-links* that consist of a node and two links over which the node can broadcast using network coding, as exemplified by the node $n_2$ in Figure 4.2. These hyperlinks provide a *rebate* for usage of the coded path in order to incentivize flows to explore their usage. The rebate takes the form of a *hyper-link capacity,* which simply means that the the hyper-link does not charge the flows for usage up to its chosen capacity. Besides the need to encourage flows to explore codable paths, we also impose a constraint that each link has a maximum rate that it can support due to scheduling or spectrum limitations. This constraint is realized via a Lagrange multiplier approach.

Hence, our approach consists of two control loops, with the inner employing well-studied learning algorithms such as BNN dynamics [9] assuming a fixed rebate by hyperlinks, as well as a price that corresponds to the Lagrange multiplier. The outer loop consists of gradient-type controllers that modify the rebate and price, respectively. All controllers only use local information for their decisions. The process of iteration continues until the entire network has reached local minimum which, since our formulation is convex, is also the socially optimal solution. We prove that this process is globally asymptotically stable. Note, however, that our optimality result involves two nested asymptotic results, so we cannot implement the idea directly. In practice, we have can only run each loop for a finite number of steps before switching to the other.

We illustrate this approach using numerical experiments. For comparison, we numerically solve the problem as a linear program to find the optimal solution. The experiments indicate that: the convergence of the augmented potential game is fast; the costs are reduced significantly upon using network coding; more expensive paths before network coding became cheaper and shortest paths were not necessarily optimal. Thus, the iterative algorithm that we develop performs well in practice.

This work is organized as follows: Section-4.1 develops a system model and problem formulation assuming no scheduling constraints on the maximum number of transmissions at each node. In Section-4.2, we introduce the concept of *hyper-links.* In Section-4.3 we reformulate the problem with constraints on peak transmissions from each node and present a bi-level distributed controller - a combination of rate controller and hyper-link controller- to solve the problem. The rate controller is presented Section-4.4 and the hyper-link controller is presented in Section-4.5. Section-4.6 contains simulation results and Section-4.7 concludes the work.

## 4.1 System overview

Our objective is to design a distributed multi-path network coding system for multiple unicast flows traversing a shared wireless network. We model the communication network as a graph $G(\mathcal{N}, E)$, where $\mathcal{N}$ is a set of network nodes and $E$ is a set of wireless links. For each link $(n_i, n_j) \in E$, where $n_i$ and $n_j$ are any two nodes, there exists a wireless channel that allows the node $n_i$ to transmit information to the node $n_j$. Each link $(n_i, n_j)$ is associated with a cost $\alpha_{ij}$. The value of $\alpha_{ij}$ captures the cost (in expected number of required transmissions) of sending a packet successfully from $n_i$ to $n_j$. Due to the broadcast nature of the wireless channels, the node $n_i$ can transmit to two neighbors $n_j$ and $n_k$ simultaneously at a cost $\max\{\alpha_{ij}, \alpha_{ik}\}$.

In wireless networks, even though broadcasting enables simultaneous transmission to neighboring nodes, it also acts as interference at those nodes which are listening to some node other than the broadcasting node. This type of interference in wireless networks, called Co-Channel Interference, is handled by upper MAC protocols (for example CSMA) which schedules transmission periods of links in the network such that interference is minimized. We assume that a perfect schedule of wireless links is given to us and, therefore, there is no interference at the receivers. However, this imposes a constraint on the maximum number of transmissions per unit time on the nodes. In this section, we develop a basic framework, while ignoring these scheduling constraints. We will include these constraints in Section 4.3.

We assume that the network supports flows $\{1, 2, \ldots, \}$, where each flow is associated with a source and destination node. Each flow $i$ is also associated with several paths $\{P_i^1, P_i^2, \ldots\}$ that connect its source and destination nodes. Our goal is to build a distributed traffic management scheme in which the source node of each flow $i$ can split its traffic, $x_i$ (packets per unit time), among multiple different paths, so as to reduce the *total number of transmissions per unit time* required to support given traffic demands. Note that on some of these paths there might be a possibility of network coding.

We will first examine a simple network with coding opportunities and derive system cost associated with the network, in terms of the total number of transmissions required. Then we will study how the coding helps in reducing the system cost.

**Example** Consider the network depicted on Figure 4.2. The network supports three flows: (i) flow 1 from $n_1$ to $n_4$, (ii) flow 2 from $n_4$ to $n_6$, and (iii) flow 3 from $n_5$ to $n_1$. We denote by $x_i$ the traffic associated with flow $i$, $1 \leq i \leq 3$. Suppose that the packets that belong to flow 1 can be sent over two paths $(n_1, n_2, n_3, n_4)$ and $(n_1, n_2, n_5, n_4)$. We denote these paths by $P_1^1$ and $P_1^2$. The traffic split on paths $P_1^1$ and $P_1^2$ is given by $x_1^1$ and $x_1^2$, respectively, such that $x_1^1 + x_1^2 = x_1$. Similarly, flow 2 can be sent over two paths $P_2^1 = (n_4, n_3, n_2, n_6)$ and $P_2^2 = (n_4, n_8, n_6)$ at rates $x_2^1$ and $x_2^2$, such that $x_2^1 + x_2^2 = x_2$. Finally, flow 3 can be sent over two paths $P_3^1 = (n_5, n_7, n_1)$ and $P_3^2 = (n_5, n_2, n_1)$, at rates $x_3^1$ and $x_3^2$, with sum $x_3$.

Note that path $P_1^2 = (n_1, n_2, n_5, n_4)$ of flow 1 and path $P_3^2 = (n_5, n_2, n_1)$ of flow 3 share two links $(n_1, n_2)$ and $(n_2, n_5)$ in the opposite directions. Thus, the packets sent along these two paths can benefit from reverse carpooling. Specifically, the node $n_2$ can combine packets of flow 1 received from the node $n_1$ and packets of flow 3 received from the node $n_5$. Similarly, the node $n_3$ can combine packets of flow 1 received from the node $n_2$ and packets of flow 2 received from the node $n_4$. Note that the cost saving at the node $n_2$ is proportional to $\min\{x_1^2, x_3^2\}$, while the saving at the node $n_3$ is proportional to $\min\{x_1^1, x_2^1\}$. Recall that we are ignoring scheduling constraints in this section.

The cost (transmissions per unit time) at the node $n_2$ when coding is enabled is

$$\begin{aligned} C_{n_2}(x_1^2, x_3^2) = \quad & \max\{\alpha_{21}, \alpha_{25}\} \min\{x_1^2, x_3^2\} \\ & + \alpha_{25}(x_1^2 - \min\{x_1^2, x_3^2\}) \\ & + \alpha_{21}(x_3^2 - \min\{x_1^2, x_3^2\}). \end{aligned} \tag{4.1}$$

Here, the first term on the right is the cost incurred due to coding at the node $n_2$. This is because a coded packet from $n_2$ is broadcast to both destination nodes, $n_1$ and $n_5$, and so the cost per packet is $\max\{\alpha_{21}, \alpha_{25}\}$. The second and third term are "overflow" terms. Since it is possible that $x_1^2 \neq x_3^2$, the remaining flow of the larger (that cannot be encoded because of the lack of flow in the opposite direction) is sent without coding at the regular link cost.

The cost at the node $n_2$, given by (4.1), can be re-written as shown below:

$$\begin{aligned} C_{n_2}(x_1^2, x_3^2) &= \alpha_{25}x_1^2 \quad +\alpha_{21}x_3^2 + \Big\{ \max\{\alpha_{21}, \alpha_{25}\} \\ &\qquad\qquad -(\alpha_{21} + \alpha_{25})\Big\} \min\{x_1^2, x_3^2\}. \end{aligned}$$

Using the fact that $\max\{x_1, x_2\} + \min\{x_1, x_2\} = x_1 + x_2$, we obtain

$$\begin{aligned} C_{n_2}(x_1^2, x_3^2) &= \alpha_{25}x_1^2 + \alpha_{21}x_3^2 \\ &- \min\{\alpha_{21}, \alpha_{25}\} \min\{x_1^2, x_3^2\}. \end{aligned} \tag{4.2}$$

The above equation can be interpreted as the cost at the node $n_2$ without coding minus the savings obtained when coding is used. Thus, the cost saved at the node $n_2$ due to network coding is $\min\{\alpha_{21}, \alpha_{25}\} \min\{x_1^2, x_3^2\}$ . Similarly, for the node $n_3$ the cost saved is $\min\{\alpha_{32}, \alpha_{34}\} \min\{x_1^1, x_2^1\}$.

The total system cost can be expressed as:

$$\begin{aligned} C(X) = \sum_{i=1}^{3}\sum_{j=1}^{2} \beta_i^j x_i^j \; - &\quad \min\{\alpha_{21}, \alpha_{25}\} \min\{x_1^2, x_3^2\} \\ - &\quad \min\{\alpha_{32}, \alpha_{34}\} \min\{x_1^1, x_2^1\}, \end{aligned} \tag{4.3}$$

where $X = \{x_1^1, x_1^2, x_2^1, x_2^2, x_3^1, x_3^2\}$ is the state of the system and $\beta_i^j$ is the uncoded path cost (equal to the sum of the link costs on the path) $j$ used by flow $i$. For example, $\beta_1^1 = \alpha_{12} + \alpha_{23} + \alpha_{34}$, for path $P_1^1 = (n_1, n_2, n_3, n_4)$. Thus, the first term on the right in

(4.3) is the total cost of the system without any coding, while the second and third terms are the savings obtained by coding at nodes $n_2$ and $n_3$.

In the next subsection, we present a system model and derive a general expression for system cost. Then we formulate an optimization problem which minimizes system cost by finding an optimal traffic split of each flow, over the multiple paths available to them.

### *4.1.1 System model*

Our system model consists of a set of nodes $\mathcal{N} = \{n_1, \ldots, n_N\}$ and a set of flows $\mathcal{F} = \{1, \ldots, F\}$. Each flow, $f \in \mathcal{F}$ is defined as a tuple $(n_f^s, n_f^d, x_f)$, where $n_f^s \in \mathcal{N}$ is the source node, $n_f^d \in \mathcal{N}$ is the destination node, and $x_f$ *packets/sec* is its traffic demand. A flow may be associated with multiple paths connecting its source and destination nodes. Let $P_f$ be the number of such paths available to flow $f$ and $x_f^s$ be the traffic sent by the flow over path $s$ associated with it. Then, $\sum_{s=1}^{P_f} x_f^s = x_f$. Let $\mathbf{x}_f = \{x_f^1, \cdots, x_f^{T_f}\}$ represent a traffic split of flow $f$. Then, the state of the system $X$ is defined as a set of traffic splits of all flows in the system. i.e $X = \{\mathbf{x}_1, \cdots, \mathbf{x}_F\}$.

A node participating in more than one path may have the opportunity to combine traffic and save on transmission if the paths traverse the node in reverse directions. Suppose paths $q$ and $r$, associated with flow $i$ and $j$ respectively, traverse the node $n_k$ in reverse directions. Assume the node $n_k$ receives packets belonging to flow $i$ which are sent over path $q$ and transmits those packets to the node $n_i$. Similarly, it collects packets belonging to flow $j$ traversing over path $r$ and forwards them to the node $n_j$. Thus, the packets sent along these paths can benefit from reverse carpooling and there exists a coding opportunity for flows $i$ and $j$ at the node $n_k$. We represent this coding opportunity at the node $n_k$, which is associated with two neighboring nodes and two flows, as $h = n_k[(i, q, n_i), (j, r, n_j)]$[1]. For example, consider the network shown on Figure 4.2. In this network, the coding opportunity available at the node $n_2$ can be represented as $n_2[(1, P_1^2, n_3), (2, P_2^1, n_1)]$. Finally, we

[1]In all the futute references of $h$, we may assume that it is associated with $n_k(h)[(i(h), q(h), n_i(h)), (j(h), r(h), n_j(h))]$. For notational convenience, we may drop the reference to $h$ in the previous representation and simply use $n_k[(i, q, n_i), (j, r, n_j)]$

assume that $H$ such coding opportunities are present in the system.

From (4.2), the cost (transmissions per unit time) at the node $n_k$ after coding enabled is given by

$$\begin{aligned} C_{n_k}(x_i^q, x_j^r) &= \alpha_{ki} x_i^q + \alpha_{kj} x_j^r \\ &- \min\{\alpha_{ki}, \alpha_{kj}\} \min\{x_i^q, x_j^r\}. \end{aligned} \tag{4.4}$$

The total system cost can be expressed as:

$$\begin{aligned} \mathbf{C}(X) = & \sum_{f=1}^{F} \sum_{p=1}^{P_f} \beta_f^p x_f^p \; - \\ & \sum_{h=1}^{H} \min\{\alpha_{ki}, \alpha_{kj}\} \min\{x_i^q, x_j^r\} \end{aligned} \tag{4.5}$$

where $X$ is the state of the system and $\beta_f^p$ is the uncoded path cost (equal to the sum of the link costs on the path) $p$ used by flow $f$.

Our goal is to build a distributed traffic management scheme in which the source node of each flow $f$ can split its traffic, $x_i$ (packets per unit time), among multiple different paths, so as to reduce the system cost (4.5), *total number of transmissions per unit time* required to support a given traffic demands. We formulate the objective of minimizing cost, subject to the traffic requirements of each flow, as an optimization problem given below:

$$\begin{aligned} \min_{X \geq 0} \quad & \mathbf{C}(X), \\ \text{subject to} \quad & \sum_{p=1}^{P_f} x_f^p = x_f \qquad f = 1, \ldots, F. \end{aligned} \tag{4.6}$$

The problem poses major challenges due to the need to achieve a certain degree of coordination among the flows. For example, for the network depicted in Figure 4.2, increasing of the value of $x_3^2$ (the decision made by the node $n_5$) will result in a system-wide cost reduction only if it is accompanied by the increase in the value of $x_1^2$. In the next section, we develop a distributed traffic management scheme, that does not require any coordination among flows on deciding their traffic splits.

## 4.2 Augmented state space and hyper-links

The optimization problem in (4.6) can be solved efficiently in a centralized manner. But centralized implementations are not practical in large and complex systems. In this section, we propose a simple way of decomposing it into subproblems that can be solved in a decentralized fashion. We do this by means of adding extra state variables to the system, which we refer to as *state-space augmentation.*

It can be observed from (4.5) that decisions of flows $i$ and $j$ are coupled through the term $\min(x_i^q, x_j^r)$. In general, for any given $x_i^q$ and $x_j^r$, this term can be expressed as an optimal value of the following optimization problem,

$$\begin{aligned}\min\{x_i^q, x_j^r\} \quad &= \max_{y>0} \Big(y - \lambda_1(y - \min\{y, x_i^q\}) \\ &\qquad -\lambda_2(y - \min\{y, x_j^r\})\Big),\end{aligned} \tag{4.7}$$

where $\lambda_1, \lambda_2 \geq 1$ are any arbitrary constants. Note that the right hand side of the above equality does not have any coupling term, due to the presence of the augmented variable $y$. Therefore, we can convert the coupled problem (4.6) into a decoupled one by replacing each 'coupled' term ($\min\{x_i^q, x_j^r\}$) with an equivalent 'de-coupled' expression from (4.7). Since each coupling term is associated with a coding opportunity $h$, the augmented variable $y_h$ is introduced in association with each coding opportunity. Let $Y = \{y_1, y_2, \cdots, y_H\}$. Now, define $\mathbf{C}(X, Y)$ as

$$\begin{aligned}\mathbf{C}(X,Y) &= \textstyle\sum_{f=1}^{F} \sum_{p=1}^{P_f} \beta_f^s x_f^s - \sum_{h=1}^{H} (\min\{\alpha_{ki}, \alpha_{kj}\}) y_h \\ &+ \textstyle\sum_{h=1}^{H} \Big(\omega_{1h}(y_h - \min\{y_h, x_i^q\}) + \omega_{2h}(y_h - \min\{y_h, x_j^r\})\Big),\end{aligned}$$

where $\omega_{1h}, \omega_{2h} \geq \min\{\alpha_{ki}, \alpha_{kj}\}$ are any arbitrary constants. It can be seen that the cost function (4.5) can be re-written as

$$\mathbf{C}(X) = \min_{Y \geq 0} \mathbf{C}(X, Y). \tag{4.8}$$

Choosing $\omega_{1h} = \alpha_{ki}$ and $\omega_{2h} = \alpha_{kj}$, we get

$$\begin{aligned}\mathbf{C}(X, Y) = \sum_{f=1}^{F} \sum_{p=1}^{P_f} \beta_f^s x_f^s - \sum_{h=1}^{H} (\min\{\alpha_{ki}, \alpha_{kj}\}) y_h \\ + \sum_{h=1}^{H} \big(\alpha_{ki}(y_h - \min\{y_h, x_i^q\}) \\ + \alpha_{kj}(y_h - \min\{y_h, x_j^r\})\big) .\end{aligned} \tag{4.9}$$

The cost function has thus been augmented using the variables $y_h$. For any fixed value of $Y$, the cost function only depends on $X$, and the sources can attempt to modify $X$ find their individually lowest cost solution. The augmented variables $Y$ can then be modified to change the cost function. In Sections 4.4–4.5 we will formally show how this is accomplished. We now show that our choices for $\omega$'s lead to an appealing interpretation for the function $\mathbf{C}(X, Y)$.

Consider coding opportunity $h = n_k[(i, q, n_i), (j, r, n_j)]$, where the node $n_k$ encodes packets coming from $i^{th}$ and $j^{th}$ flows, and then broadcast them to nodes $n_i$ and $n_j$ respectively. Grouping the terms associated with coding opportunity $h$ in (4.9), we get

$$\begin{aligned} C(h) &= \alpha_{ki} x_i^q + \alpha_{kj} x_j^r - \min\{\alpha_{ki}, \alpha_{kj}\} y_h + \\ &\quad \alpha_{ki}(y_h - \min\{y_h, x_i^q\}) + \alpha_{kj}(y_h - \min\{y_h, x_j^r\}), \\ &= \max\{\alpha_{ki}, \alpha_{kj}\} y_h + \alpha_{ki}(x_i^q - \min\{x_i^q, y_h\}) \\ &\quad + \alpha_{kj}(x_j^r - \min\{x_j^r, y_h\}). \end{aligned} \tag{4.10}$$

In the above expression, $C(h)$, the first term corresponds to the cost of broadcasting coded traffic, if we restrict the total coded (broadcast) traffic between the two flows at the node $n_k$

to be less or equal to $y_h$, and the last two terms are the transmission costs associated with the remaining uncoded traffic. This leads to the concept of *hyper-link*, which can be thought of as a broadcast link with capacity $y_h$. It is composed of physical links $(n_k, n_i)$ and $(n_k, n_j)$ and carries only encoded traffic from flows $i$ and $j$. And the remaining uncoded traffic is sent through uni-cast links $(n_k, n_i)$ and $(n_k, n_j)$ respectively. Formally, a hyper-link and a hyper-path are defined as follows:

**Definition 2.** *A* hyper-link *is a broadcast-link composed of three nodes and two flows. A hyper-link $h = n_k[(i, q, n_i), (j, r, n_j)]$ at the node $n_k$ can encode packets belonging to flow $i$ (sending packets on path $q$) with flow $j$ (sending packets on path $r$). Here, the nodes $n_i$ and $n_j$ are the next-hop neighbors of $n_k$; for flow $i$ along path $q$ and for flow $j$ along path $r$, respectively. Also, $y_h$ denotes capacity of the hyper-link (in packets per unit time).*

*A* hyper-path *$p \in \mathcal{S}_i$ between source $n_i^s$ and destination $n_i^d$ is a virtual path over a physical path between $n_i^s$ and $n_i^d$. A hyper-path contains zero or more hyper-links on it and at* each node *on the underlying physical path there can be atmost one hyper-link. It follows that the set of all paths are a subset of the hyper-paths.*

The cost at hyper-link $h$, given by (4.10), can be re-written as:

$$\begin{aligned} C(h) &= \alpha_{ki} x_i^q + \alpha_{kj} x_j^r - T(h), \text{ where} && (4.11) \\ T(h) &= \alpha_{ki} \min\{x_i^q, y_h\} + \alpha_{kj} \min\{x_j^r, y_h\} \\ &\quad - \max\{\alpha_{ki}, \alpha_{kj}\} y_h. && (4.12) \end{aligned}$$

Recall that the first two cost terms are the total cost at the node $n_k$ when coding is disabled. The remaining cost, $T(h)$ can be thought of as the *rebate* obtained by using hyper-link $h = n_k[(i, q, n_i), (j, r, n_j)]$. Note that the rebate could be *negative* (hence adding to the total cost), which might happen when one of the flow rate is 0 and the other flow rate is less than the hyper-link capacity.

Now the function $\mathbf{C}(X,Y)$ in (4.9) can be written as follows:

$$C(X,Y) = \sum_{f=1}^{F}\sum_{p=1}^{P_f}\beta_f^s x_f^s - \sum_{h=1}^{H} T(h), \tag{4.13}$$

which represents the total system cost without coding minus the total rebate of all the hyper-links. Here, $C(X,Y)$ - total number of transmissions per unit time required to support a given traffic load- is the system cost given the system state $(X,Y)$, where $X$ is the set of traffic vectors of all flows in the system and $Y$ is set of hyper-link capacities. Our objective is to minimize the cost function which can be formally stated as

$$\begin{aligned} \min_{X,Y\geq 0} \quad & \mathbf{C}(X,Y) \\ \text{subject to} \quad & \sum_{p=1}^{P_f} x_f^p = x_f \quad \forall f = 1,\cdots,F. \end{aligned} \tag{4.14}$$

In the next section, we will also account for the fact that the transmission rate of each node is limited due to scheduling constraints.

## 4.3 Peak transmission constraints

In a practical scenario, the maximum number of transmissions per unit time from a wireless node is limited by scheduling. In this section, we assume that the schedule has been predetermined, and imposes a constraint on the maximum amount of traffic that can be accommodated on any particular link. In doing so, we will illustrate the fact that the state space augmentation can be used in conjunction with Lagrange multiplier that enforces a constraint. reformulate problem (4.14) taking into account the transmission constraints at each node.

Let $R_{ki}^{fp}$ be a routing variable. It takes a value equal to 1 if any path $p$ associated with flow $f$ passes through link $(n_k, n_i)$ and otherwise 0. Similarly, define $Z_k^h$ which takes 1 if hyper-link $h$ is associated with the node $n_k$ and otherwise 0. Let $T_k$ be the maximum number of allowable transmissions per unit time at the node $n_k$. Then, at each node $n_k$, the total number of uncoded transmissions minus the saved number of transmissions (using hyper-links) should be less than or equal to $T_k$. Therefore,

$$\sum_{i=1}^{N}\sum_{f=1}^{F}\sum_{p=1}^{P_f} R_{ki}^{fp}\alpha_{ki}x_f^p - \sum_{h=1}^{H} Z_k^h T(h) \leq T_k. \quad \forall n_k \in \mathcal{N}.$$

Now, incorporating these constraints on transmission rate, the problem (4.14) can be rewritten as

$$\begin{aligned}
\min_{X\geq 0, Y\geq 0} \quad & \mathbf{C}(X,Y) = \sum_{f=1}^{F}\sum_{p=1}^{P_f} \beta_f^p x_f^p - \sum_{h=1}^{H} T(h), \\
\text{subject to} \quad & \sum_{p=1}^{P_f} x_f^p = x_f, \qquad \forall f = 1, \cdots, F, \qquad (4.15) \\
& \sum_{i=1}^{N}\sum_{f=1}^{F}\sum_{p=1}^{P_f} R_{ki}^{fp}\alpha_{ki}x_f^p - \sum_{h=1}^{H} Z_k^h T(h) \leq T_k, \\
& \forall k = 1, \cdots, N, \qquad (4.16)
\end{aligned}$$

where $X$ is the set of traffic vectors of all flows in the system and $Y$ is set of hyper-link capacities. Note that the augmented cost $C(X,Y)$ is jointly convex in $X$ and $Y$. The constraint sets are also convex. Therefore, the above problem is convex. We assume that the feasible sets of the above problem -set of traffic vectors $X$ and set of hyper-link capacities $Y$ which satisfy both traffic demands (4.15) and peak transmission constraints (4.16)- is nonempty. We can use dual decomposition techniques to construct a distributed algorithm to solve this problem. The Lagrangian function is

$$\begin{aligned}
\mathbb{C}(X,Y,\Sigma) &= \sum_{f=1}^{F}\sum_{p=1}^{P_f} \beta_f^p x_f^p - \sum_{h=1}^{H} T(h) + \sum_{k=1}^{N} \sigma_k V_k \\
\text{where} \quad V_k &= \left( \sum_{f=1}^{F}\sum_{p=1}^{P_f} R_{ki}^{fp}\alpha_{ki}x_f^s - \sum_{h=1}^{H} Z_k^h T(h) - T_k \right). \qquad (4.17)
\end{aligned}$$

Note that $\sigma_k$ is a non-negative Lagrange multiplier associated with the transmission constraint of the node $n_k$. We can interpret $\sigma_k$ as the 'price' charged by the node $n_k$ for each transmission. Let $\Sigma = [\sigma_1, \cdots, \sigma_N]$ be a set of node-prices.

We define $\mathbb{C}(X, Y, \Sigma)$ as our new system function given the system state $(X, Y, \Sigma)$, where $X$ is the set of traffic vectors of all flows in the system, $Y$ is the set of hyper-link capacities and $\Sigma$ is the set of node-prices. Our objective is find an optimal state of the problem given below.

$$\begin{aligned} \max_{\Sigma \geq 0} \quad & \min_{X,Y \geq 0} \mathbb{C}(X, Y, \Sigma), \\ & \sum_{f=1}^{F} x_f^p = x_f, \quad \forall f = 1, \cdots, F. \end{aligned}$$

We propose a bi-level distributed iterative algorithm to find an optimal state for the above problem.

1. **Traffic Splitting:** In this phase, each source node finds the optimum traffic assignment given the hyper-link capacities and node-prices. For any given $(Y, \Sigma)$,

   $$\textbf{TS}: \quad \min_{X \geq 0} \mathbb{C}(X, Y, \Sigma), \qquad \sum_{f=1}^{F} x_f^p = x_f \quad f = 1, \cdots, F.$$

   We model this part as a traditional potential game. The reason for our choice is that there exist several simple, well-studied controllers for routing in potential games. Thus, for any fixed value of the augmented variables and Lagrange multipliers, we can use any of these controllers to obtain convergence. Details of our game model and the payoffs used are discussed in Section 4.4. Note that signalling is required to ensure feedback of node-prices and hyper-link rebates to the source nodes, but this overhead is small.

2. **Node Control:** In this phase, we adjust the augmented variables (hyper-link capacities) and Lagrange multipliers (node-prices) assuming that potential game of the

sources has attained equilibrium.

$$\textbf{NC}: \quad \max_{\Sigma \geq 0} \min_{Y \geq 0} \mathbb{C}(X^*, Y, \Sigma),$$

where $X^*$ is the assignment matrix at equilibrium. We use gradient decent controllers to modify the optimal hyper-link state and node-price. Details are discussed in Section 4.5.

We call our controller as *Decoupled Dynamics*. The two phases operate at different time scales. Traffic splitting is done at every *small* time scale and the node-control is done at every *large* time scale. Thus, sources attain equilibrium for given hyper-link capacities and prices, then the hyper-link capacities and prices are adjusted, and this in turn forces the sources to change their splits. This process continues until the source splits, hyper-link capacities and prices converge.

## 4.4 Traffic splitting: multi-path network coding game

We model the traffic-splitting process of decoupled dynamics as a potential game with continuous action space, which we refer to as the *Multi-Path Network Coding Game* (MPNC Game). A potential game with continuous action space is defined by,

1. a set of players, $\mathcal{F}$,
2. an action space, $\mathbb{X} = \{X_i, \forall i \in \mathcal{F} | X_i \subset R^M, M \in \mathbb{N}\}$, where $X_i$ is an action set of player $i$,
3. a set of continuously differentiable payoff functions of players, $\mathcal{C} = \{C_i : \mathbb{X} \to R, \forall i \in \mathcal{F}\}$,
4. a continuously differentiable potential function, $\Phi : \mathbb{X} \to R$, such that

$$\nabla_{a_i} \Phi(a_i, a_{-i}) = \nabla_{a_i} C_i(a_i, a_{-i}), \tag{4.18}$$

where $a_i \in X_i, a_{-i} \in \mathbb{X} \backslash X_i$.

Now, having defined the components of a potential game, we identify the corresponding entities in the case of MPNC game.

First of all, the flows are the players in the MPNC game. Then, the set of players is given by $\mathcal{F} = \{1, 2, \cdots, F\}$. The action set of player $i$ (flow $i$) is defined as

$$X_i = \{\vec{x}_i = (x_i^1, x_i^2, \cdots, x_i^{P_i}) | \sum_j x_i^j = x_i\},$$

where $x_i$ is the traffic demand of flow $i$ and $P_i$ is the number of hyper paths available to it. Note that each action $\vec{x}_i$ corresponds to, an instance of distribution of traffic demand seen by flow $i$, over the set of available hyperpaths. Then, the action space, $\mathbb{X}$, is given by $\mathbb{X} = \{X_1, \cdots, X_F\}$.

Finally, the payoff function of a player $i$ is defined as

$$C_i(\vec{x}_i, \vec{x}_{-i}) = \mathbb{C}((\vec{x}_i, \vec{x}_{-i}), Y, \Sigma) - \mathbb{C}((\vec{0}, \vec{x}_{-i}), Y, \Sigma) \tag{4.19}$$

where $\mathbb{C}$ is the system cost function given by (4.17). In the above definition, $\vec{x}_i$ is the action of player $i$, $\vec{x}_{-i}$ is a set of actions of other players and $\vec{0}$ is a null vector. Also $Y$ is the set of hyper link capacities and $\Sigma$ is the set of node prices which remain invariant during each realization of MPNC game. The utility defined above is sometimes referred to as the *Wonderful life utility (WLU)* [18]. It is well known that payoff as in (4.19) results in a potential game with potential function $\Phi = \mathbb{C}$ [18].

In the context of MPNC game, it is clear that the payoff function, given by (4.19), is equal to the total transmission cost incurred by player $i$, while sending its own traffic over the set of available hyperpaths. Hence, in this game, the objective of each player is to *minimize* its own payoff.

But there is a caveat in using the system cost function $\mathbb{C}$ as the potential function and $C_i$'s as the payoff functions. Recall from the conditions (3) and (4) of the definition of potential game that, the potential function and the utility functions must be differentiable. But, from (4.17) and (4.12) note that, the system cost function contains "min" terms over the hyper-link capacity and the flow rates, which makes the function non-differentiable. In order to have a continuously differentiable cost function we approximate these "min" terms using a generalized mean-valued function.

Let $a = \{a_1, \ldots, a_n\}$ be the set of positive real numbers and let $t$ be some non-zero real number. Then the generalized *t-mean* of $a$ is given by:

$$M_t(a) = \left( \frac{1}{n} \sum_{i=1}^{n} a_i^t \right)^{\frac{1}{t}} \tag{4.20}$$

The "min" function over the set $a$ is approximated using $M_t(a)$ as:

$$\min\{a_1, \ldots, a_n\} = \lim_{t \to -\infty} M_t(a) \tag{4.21}$$

Substituting for $M_t$ (4.20), instead of the "min" function in (4.17) we get the approximated total system function as:

$$\tilde{C}(X, Y, \Sigma) = \sum_{f=1}^{F} \sum_{s=1}^{S_f} \beta_f^s x_f^s - \sum_{h=1}^{H} \tilde{T}(h) + \sum_{k=1}^{N} \sigma_k \tilde{V}_k, \tag{4.22}$$

where for a hyper-link $h = n_k[(i, q, n_i), (j, r, n_j)] \in \mathcal{H}$:

$$\begin{aligned} \tilde{T}(h) \quad &= \alpha_{ki} \left( \tfrac{(x_i^q)^t + (y_h)^t}{2} \right)^{\frac{1}{t}} + \alpha_{kj} \left( \tfrac{(x_i^r)^t + (y_h)^t}{2} \right)^{\frac{1}{t}} \\ &\quad - \max\{\alpha_{ki}, \alpha_{kj}\} y_h \end{aligned} \tag{4.23}$$

and

$$\tilde{V}_k = \sum_{f=1}^{F} \sum_{s=1}^{S_f} \sum_{m=1}^{N} R_{km}^{fp} \alpha_{ki} x_f^s - \sum_{h=1}^{H} Z_k^h \tilde{T}(h) - T_k. \tag{4.24}$$

The system function $\tilde{C}(X, Y, \Sigma)$ is continuous and differentiable. So, we use the approximated function as our potential function. Similarly, the payoff of player $i$, given by

(4.19), is approximated as follows:

$$\tilde{C}_i(\vec{x}_i, \vec{x}_{-i}) = \tilde{C}((\vec{x}_i, \vec{x}_{-i}), Y, \Sigma) - \tilde{C}((\vec{0}, \vec{x}_{-i}), Y, \Sigma). \tag{4.25}$$

The marginal payoff obtained by flow $i \in \mathcal{F}$, given his action, $\vec{x}_i$, and the set of actions of other players, $\vec{x}_{-i}$, is

$$F_i(X, Y, \Sigma) = \nabla_{\vec{x}_i} \tilde{C}_i(X, Y, \Sigma) = \nabla_{\vec{x}_i} \tilde{C}(X, Y, \Sigma), \tag{4.26}$$

where $X = (\vec{x}_i, \vec{x}_{-i})$. The above result follows from definition of potential function and (4.18). Note that $F_i$ is a vector and let its $p^{th}$ component be $F_i^p$. Then,

$$\begin{aligned} F_i^p(X, Y, \Sigma) &= \frac{\partial \tilde{C}(X, Y, \Sigma)}{\partial x_i^p}, \ \forall i \in \mathcal{F}, \ p \in \mathcal{P}_i & (4.27) \\ &= \beta_i^p - \sum_{h \in \mathcal{H}_i^p} \frac{\partial \tilde{T}(h)}{\partial x_i^p} + \sum_{k=1}^{N} \sum_{m=1}^{N} R_{km}^{ip} \sigma_k \alpha_{km} \\ &- \sum_{h \in \mathcal{H}_i^p} \sum_{k=1}^{N} Z_k^h \sigma_k \frac{\partial \tilde{T}(h)}{\partial x_i^p}. & (4.28) \end{aligned}$$

where, $\mathcal{H}_i^p$ the set of all hyper-links associated with flow $f_i^p$. From (4.23)

$$\frac{\partial \tilde{T}(h)}{\partial x_i^p} = \frac{1}{2} \alpha_{ki} \left( \frac{x_i^p}{M_t(x_i^p, y_h)} \right)^{t-1}, \tag{4.29}$$

and we have the min-approximation

$$M_t(x_i^p, y_h) = \left( \frac{(x_i^p)^t + (y_h)^t}{2} \right). \tag{4.30}$$

As we will show below, our algorithm will converge to the optimal state for any given value of $t < 0$. Thus, we can attain a solution that is arbitrarily close to the original problem by choosing $|t|$ as large as desired. Also note that the payoff is the marginal cost incurred in using an option, so the players try to minimize their cost. The source node of each flow,

$i \in \mathcal{F}$ observes the marginal cost, $F_i^p$, obtained in using a particular option (particular hyperpath), $p \in \mathcal{P}_i$, and changes the mass on that particular option, $x_i^p$, so as to attain equilibrium.

Next, we define the concept of equilibrium in potential games. A commonly used concept in non-cooperative games, is the Nash equilibrium. The game is said to be at Nash equilibium, if flows do not have any incentive to unilaterally deviate from their current action states. An action profile, $\hat{X} = (\hat{\vec{x}}_i, \hat{\vec{x}}_{-i}) \in \mathbb{X}$, results in a Nash equilibrium of MNPC game if

$$C_i(\hat{\vec{x}}_i, \hat{\vec{x}}_{-i}) \leq C_i(\vec{x}_i, \hat{\vec{x}}_{-i}), \forall \vec{x}_i \in X_i, \forall i \in \mathcal{F}.$$

The above NE condition also implies that

$$F_i^p(\hat{X}) \leq F_i^{p'}(\hat{X}) \quad \forall p, p' \in \mathcal{P}_i, \forall i \in \mathcal{F},$$

where $F_i^p$ is the marginal payoff given by (4.27). The above result can be interpreted as follows: At NE, for any player $i \in \mathcal{F}$, all the options (hyper paths) being used by that player, yield the same marginal payoff. Also, the marginal payoff that would have been obtained is higher for all those unused options.

The above concept refers to an *equilibrium condition*; the question arises as to how the system actually arrives at such a state. A commonly used kind of population dynamics is *Brown-von Neumann-Nash (BNN) Dynamics* [9]. The source nodes use BNN dynamics to control the mass on each option. But since each source tries to *minimize* its payoff, we use a modified version of BNN dynamics:

$$\dot{x}_i^p = \left( x_i \gamma_i^p - x_f^i \sum_{j=1}^{P_i} \gamma_i^j \right), \tag{4.31}$$

$$\text{where, } \gamma_i^p = \max \left\{ \frac{1}{x_i} \sum_{j=1}^{P_i} F_i^j x_f^j - F_i^p, 0 \right\}$$

where $F_i^p$ is the marginal payoff of player $i$ given by (4.27). In the next subsection, we prove the stability of our inner loop contorol.

### *4.4.1 Convergence of MPNC game*

We show in this susection that the multi-path network coding game converges to a stationary point when each source uses BNN dynamics. We will use the theory of Lyapunov functions [28] to show that our population game $\mathcal{G}$, is stable for a given hyper-link state $\breve{Y}$ and node-price state $\breve{\Sigma}$. We use the approximated system function (4.22) as our candidate Lyapunov function.

**Theorem 5.** *The system of flows $\mathcal{F}$ that use* BNN dynamics *with payoffs given by (4.27) is globally asymptotically stable for a given hyper-link state $\breve{Y}$ and node-price state $\breve{\Sigma}$.*

*Proof.* We use the approximated system function $\tilde{C}(X, Y, \Sigma)$ (4.22) as our Lyapunov function. It is simple to verify that the cost function $\tilde{C}(X, \breve{Y}, \breve{\Sigma})$, is non-negative and convex, and hence is a valid candidate. For a given hyper-link state, $\breve{Y}$, and node-price state, $\breve{\Sigma}$, we define our Lyapunov function as:

$$\mathcal{L}_{\breve{Y}\breve{\Sigma}}(X) = \tilde{C}(X, \breve{Y}, \breve{\Sigma}).$$

From (4.27)

$$\frac{\partial \mathcal{L}_{\breve{Y}\breve{\Sigma}}(X)}{\partial x_f^p} = \frac{\partial \tilde{C}(X, \breve{Y}, \breve{\Sigma})}{\partial x_f^p} = F_f^p(X, \breve{Y}, \breve{\Sigma}).$$

Hence,

$$\begin{aligned} \dot{\mathcal{L}}_{\breve{Y}\breve{\Sigma}}(X) \quad &= \textstyle\sum_{f=1}^{F} \sum_{p=1}^{S_f} \frac{\partial \mathcal{L}_{\breve{Y}\breve{\Sigma}}(X)}{\partial x_f^p} \dot{x}_f^p, \\ &= \textstyle\sum_{f=1}^{F} \sum_{p=1}^{S_f} F_f^p(X, \breve{Y}, \breve{\Sigma}) \dot{x}_f^p. \end{aligned}$$

From (4.31) we can substitute the value for $\dot{x}_f^p$ and we have

$$\dot{\mathcal{L}}_{\breve{Y}\breve{\Sigma}}(X) = \sum_{f=1}^{F}\sum_{p=1}^{S_f} F_f^p(x_f\gamma_f^p - x_f^p\sum_{j=1}^{S_f}\gamma_f^j),$$

$$= \sum_{f=1}^{F} x_f \left(\sum_{p=1}^{S_f} F_f^P\gamma_f^p - \left(\frac{1}{x_f}\sum_{p=1}^{S_f}F_f^p x_f^p\right)\sum_{j=1}^{S_f}\gamma_f^j\right).$$

We define

$$\bar{F}_f \triangleq \frac{1}{x_f}\sum_{p=1}^{S_f}F_f^p x_f^p,$$

$$\Longrightarrow \sum_{f=1}^{F} x_f\left(\sum_{p=1}^{S_f}F_f^P\gamma_f^p - \sum_{j=1}^{S_f}\bar{F}_f\gamma_f^j\right),$$

$$= \sum_{f=1}^{F} x_f\left(\sum_{p=1}^{S_f}\gamma_f^p(F_f^P - \bar{F}_f)\right),$$

$$\leq -\sum_{f=1}^{F} x_f\left(\sum_{p=1}^{S_f}(\gamma_f^p)^2\right) \leq 0.$$

Thus,

$$\dot{\mathcal{L}}_{\breve{Y}\breve{\Sigma}}(X) \quad \leq \quad 0, \quad \forall\, X \in \mathcal{X}.$$

where equality exists when the state $X$ corresponds to the stationary point of BNN dynamics. Hence, the system is globally asymptotically stable. □

### *4.4.2 Efficiency*

The objective of our system is to minimize the system function for a given load vector $\vec{x} = [x_1, \ldots, x_Q]$ and given hyper-link state $\breve{Y}$ and node-price state $\breve{\Sigma}$. Here the system function $\tilde{C}(X, \breve{Y}, \breve{\Sigma})$ and is defined in (4.22). This can be represented as the following constrained minimization problem:

$$\min_{X} \tilde{C}(X, \breve{Y}, \breve{\Sigma}) \tag{4.32}$$

subject to:

$$\begin{aligned} \sum_{p=1}^{S_i} x_i^p &= x_i \quad \forall\, i \in \mathcal{F} \\ x_i^p &\geq 0. \end{aligned} \tag{4.33}$$

The Lagrange dual associated with the above minimization problem, for a given $\breve{Y}$ and $\breve{\Sigma}$ is

$$\begin{aligned} \mathcal{L}_{\breve{Y}\breve{\Sigma}}(\lambda, h, X) = \max_{\lambda,h} \min_{X} \Bigg( \tilde{C}(X, \breve{Y}, \breve{\Sigma}) \quad - \\ \sum_{i=1}^{F} \lambda_i \Big( \sum_{p=1}^{S_i} x_i^p - x_i \Big) \quad - \quad \sum_{i=1}^{F} \sum_{p=1}^{S_i} h_i^p x_i^p \Bigg) \end{aligned} \tag{4.34}$$

where $\lambda_i$ and $h_p^i \geq 0$ , $\forall\, i \in \mathcal{F}$ and $p \in \mathcal{S}_i$, are the dual variables. Now the above dual problem gives the following Karush-Kuhn-Tucker first order conditions:

$$\frac{\partial \mathcal{L}_{\breve{Y}\breve{\Sigma}}}{\partial x_i^p}(\lambda, h, X^{\star}) = 0 \quad \forall\, i \in \mathcal{F} \text{ and } p \in \mathcal{S}_i \tag{4.35}$$

and

$$h_i^p x_i^{\star p} = 0 \quad \forall\, i \in \mathcal{F} \text{ and } p \in \mathcal{S}_i \tag{4.36}$$

where $X^{\star}$ is the global minimum for the primal problem (4.32). Hence from (4.35) we have, $\forall\, i \in \mathcal{F}$ and $\forall\, p \in \mathcal{S}_i$,

$$\begin{aligned} \frac{\partial \tilde{C}}{\partial x_i^p}(X^{\star}, \breve{Y}, \breve{\Sigma}) - \lambda_i \frac{\partial (\sum_{p=1}^{S_i} x_i^{\star p} - x^{\star} i)}{\partial x_i^p} + h_i^p &= 0 \\ \Rightarrow \frac{\partial \tilde{C}}{\partial x_i^p}(X^{\star}, \breve{Y}, \breve{\Sigma}) &= \lambda_i + h_i^p \qquad (4.37) \\ \Rightarrow F_i^p(X^{\star}, \breve{Y}, \breve{\Sigma}) &= \lambda_i + h_i^p \qquad (4.38) \end{aligned}$$

where the last equation follows from (4.26).

From (4.36), it follows that

$$F_i^p(X^\star, \breve{Y}, \breve{\Sigma}) \;=\; \lambda_i \quad \text{when } x_i^{\star p} > 0 \tag{4.39}$$

and

$$F_i^p(X^\star, \breve{Y}, \breve{\Sigma}) \;=\; \lambda_i + h_i^p \quad \text{when } x_i^{\star p} = 0 \tag{4.40}$$

$\forall\ i \in \mathcal{F}$ and $\forall\ p \in \mathcal{S}_i$. The above condition (4.39, 4.40), implies that the payoff on all the options used is identical and for options not in use the payoff is more, which is equivalent to the NE condition given by (4.31). Notice that we use a modified definition of Nash equilibrium, since each source tries to minimize it's cost (or payoff). The following theorem proves the efficiency of our system.

**Theorem 6.** *The solution of the minimization problem in (4.32) is identical to the Nash equilibrium of MPNC game.*

*Proof.* Consider the BNN dynamics (4.31), at stationary point, $\tilde{X}$, we have $\dot{x}_i^p = 0$, which implies that either,

$$\hat{F}_i = \; F_i^p(\tilde{X}, \breve{Y}, \breve{\Sigma}) \tag{4.41}$$

$$\text{or} \qquad \hat{x}_i^p = 0,$$

$$\text{where,} \quad \hat{F}_i \triangleq \tfrac{1}{\hat{x}_i} \textstyle\sum_{r=1}^{Q} \hat{x}_i^r F_i^r(\tilde{X}, \breve{Y}, \breve{\Sigma}) \quad \forall\ i \in \mathcal{F}, \tag{4.42}$$

The above expressions imply that all hyper-paths used by a particular flow $i \in \mathcal{F}$ yield same payoff, $\hat{F}_i$, while hyper-paths not used ($x_i^p = 0$) yield a payoff higher than $\hat{F}_i$.

We observe that the conditions required for Nash equilibrium are identical to the KKT first order conditions (4.39)-(4.40) of the minimization problem (4.32) when

$$\hat{F}_i = \lambda_i \quad \forall\ i \in \mathcal{F}$$

It follows from the convexity of the total system cost that, there is no duality-gap between the primal (4.32) and the dual (4.34) problems. Thus, the optimal primal solution is equal to optimal dual solutions, which is identical to the Nash equilibrium. □

## 4.5 Node control

Thus far we have designed a distributed scheme that would result in minimum cost for a given hyper-link state or capacities $Y$, node-price state $\Sigma$ and for a given load vector $\vec{x} = \{x_1, \ldots, x_f\}$. In this phase of Decoupled Dynamics, the hyper-link capacities and node-prices are adjusted based on the current value of system function. This phase runs at a larger time-scale as compared to the traffic splitting phase described in Section 4.4. It is assumed that during this phase all the flows instantly reach equilibrium, i.e., changing the hyper-link capacities and node-prices would force all the source nodes to attain Wardrop equilibrium instantaneously.

The node control can be formulated as a convex optimization problem as follows:

$$\begin{aligned} \max_{\Sigma} \min_{Y} \quad & Q(Y, \Sigma), \\ \text{subject to,} \quad & y_h, \sigma_k \geq 0, \ \forall y_h \in Y \text{ and } \forall \sigma_k \in \Sigma. \end{aligned} \tag{4.43}$$

where, $Q(Y, \Sigma)$ is the minimum value of the system function for a given hyper-link state $Y$ and node-price state $\Sigma$, i.e., $Q(Y, \Sigma) = \tilde{C}(X^\star, Y, \Sigma)$, where, for a given $Y$ and $\Sigma$, $X^\star$ is an optimal state of the flows that results in minimum cost.[2] We use simple gradient descent:

$$\dot{y}_h = -\kappa \frac{\partial Q(Y, \Sigma)}{\partial y_h} \ \forall y_h \in Y, \tag{4.44}$$

$$\dot{\sigma}_k = \rho \frac{\partial Q(Y, \Sigma)}{\partial \sigma_k} \ \forall \sigma_k \in \Sigma. \tag{4.45}$$

The partial derivative, $\frac{\partial Q}{\partial y_h}$, is over the variables $y_h \in Y$. Keeping $\Sigma$ fixed and changing the hyper-link capacity $y_h$, of some hyper-link $h \in \mathcal{H}$, would result in a different state of

[2] Notice, there could be many different states, $X^\star$, which result in a minimum cost but the minimum value, $\tilde{C}(X^\star, Y, \Sigma)$, is unique.

the flows, $X_h^\star$ and hence a different minimum cost, $\tilde{C}(X_h^\star, Y_h, \Sigma)$, where $Y_h$ corresponds to the changed hyper-link capacity of $y_h$ while other capacities are fixed, as compared to $Y$. Thus for a hyper-link, $h = n_k[(i,q,n_i),(j,t,n_j)]$ with capacity $y_h$,

$$\begin{aligned}\frac{\partial Q(Y,\Sigma)}{\partial y_h} &= \tfrac{\partial \tilde{C}}{\partial y_h}(X^\star, Y, \Sigma) \\ &\quad + \textstyle\sum_{i=1}^{F}\sum_{p=1}^{P_i} \frac{\partial \tilde{C}}{\partial x_i^p}(X^\star, Y, \Sigma)\frac{\partial x_i^{*p}}{\partial y_h} \\ &= \tfrac{\partial \tilde{C}}{\partial y_h}(X^\star, Y, \Sigma) + \textstyle\sum_{i=1}^{F} F_i \sum_{p=1}^{S_i} \frac{\partial x_i^{*p}}{\partial y_h},\end{aligned} \tag{4.46}$$

where the last expression follows from the definition of $F_i^p$ (Definition 4.27) and the fact that for changes in the hyper-link state, the sources attain Wardrop equilibrium instantaneously. In other words, before and after a small change in $y_h$ the system is in Wardrop equilibrium. Hence, $F_i^p = F_i \ \forall i \in \mathcal{F}$ and $\forall p \in \mathcal{S}_i$. Finally, $\sum_{p=1}^{S_i} \frac{\partial x_i^{*p}}{\partial y_h} = 0$, since the total load $x_i^* = \sum_{p=1}^{S_i} x_i^{*p}$ is fixed. For hyper-link $h = n_k[(i,q,n_i),(j,t,n_j)]$,

$$\frac{\partial Q(Y,\Sigma)}{\partial y_h} = \tfrac{\partial \tilde{C}}{\partial y_h}(X^\star, Y, \Sigma) = -(1+\sigma_k)\tfrac{\partial \tilde{T}}{\partial y_h}(h), \tag{4.47}$$

where from (4.23),

$$\begin{aligned}\frac{\partial \tilde{T}}{\partial y_h}(h) &= \tfrac{\alpha_{ki}}{4}\left(\tfrac{y_h}{M_t(x_i^q, y_h)}\right)^{t-1} + \tfrac{\alpha_{kj}}{4}\left(\tfrac{y_h}{M_t(x_j^r, y_h)}\right)^{t-1} \\ &\quad - \max\{\alpha_{ki}, \alpha_{kj}\}, \\ \text{and} \qquad M_t(x_i^q, y_h) &= \left(\tfrac{(x_i^q)^t + (y_h)^t}{2}\right)^{\frac{1}{t}}.\end{aligned}$$

Similarly, we can show that

$$\frac{\partial Q(Y,\Sigma)}{\partial \sigma_k} = \tfrac{\partial \tilde{C}}{\partial \sigma_k}(X^\star, Y, \Sigma) = \tfrac{\partial \tilde{V}_k}{\partial \sigma_k}, \tag{4.48}$$

where, from (4.24)

$$\frac{\partial \tilde{V}_k}{\partial \sigma_k} = \left( \sum_{m=1}^{N} \sum_{f=1}^{F} \sum_{p=1}^{S_f} R_{km}^{fp} \alpha_{ki} \hat{x}_f^s - \sum_{h=1}^{H} Z_k^h \tilde{T}(h) - T_k \right).$$

**Theorem 7.** *At the large time-scale, the hyper-link capacity control with dynamics (4.44) and node price control with dynamics (4.45) is globally asymptotically stable.*

*Proof.* We use the following Lyapunov function

$$G(Y, \Sigma) \quad = \quad \frac{1}{2\kappa} \sum_{h=1}^{H} (y_h - \hat{y}_h)^2 + \frac{1}{2\rho} \sum_{k=1}^{N} (\sigma_k - \hat{\sigma}_k)^2 \tag{4.49}$$

where $\hat{y}_h \in \hat{Y}$ and $\hat{\sigma}_k \in \hat{\Sigma}$ are optimizers of (4.43). We will use LaSalle's invariance principle [28] to show stability.

Differentiating $G$ we obtain

$$\dot{G} = \frac{1}{\kappa} \sum_{h=1}^{H} (y_h - \hat{y}_h) \dot{y}_h + \frac{1}{\rho} \sum_{k=1}^{N} (\sigma_k - \hat{\sigma}_k) \dot{\sigma}_k.$$

Now from (4.44) and (4.45),

$$\dot{G} = -\sum_{h=1}^{H} (y_h - \hat{y}_h) \frac{\partial Q}{\partial y_h} + \sum_{k=1}^{N} (\sigma_k - \hat{\sigma}_k) \frac{\partial Q}{\partial \sigma_k}. \tag{4.50}$$

We will show that $\dot{G} \leq 0, \forall Y, \forall \Sigma$.

Note that $Q(Y, \Sigma) = \tilde{C}(X^*, Y, \Sigma)$, where $X^*$ is a minimizer of approximated cost function defined in (4.22) for fixed $Y$ and $\Sigma$. Also, for any fixed node-price state $\Sigma$, the approximated cost function is jointly convex in $X$ and $Y$. Therefore, minimizing it over a convex set of $X$ yields a convex function. In essence, $Q(Y, \Sigma)$ is convex in $Y$ for any fixed $\Sigma$. It can be observed that for any fixed hyper-link state $Y$ and rate vector $X$, the approximated cost function defined in (4.22) is a linear function of $\Sigma$. Then the minimization of $\tilde{C}(X, Y, \Sigma)$ over $X$ can be thought of as a point-wise minimization of infinite number of linear functions of $\Sigma$ which results in a concave function of $\Sigma$. Therefore, $Q(Y, \Sigma)$ is concave in $\Sigma$ for any fixed $Y$. Therefore, from the convex-concave nature of $Q(Y,\Sigma)$ we can show that

$$Q(\hat{Y},\Sigma) \leq Q(\hat{Y},\hat{\Sigma}) \leq Q(Y,\hat{\Sigma}), \forall Y, \forall \Sigma. \tag{4.51}$$

where $\hat{Y}$ and $\hat{\Sigma}$ are optimizers of the problem (4.43). Now, using the first order properties of convex and concave functions,

$$Q(\hat{Y},\Sigma) \geq Q(Y,\Sigma) + \sum_{h=1}^{H}(\hat{y}_h - y_h)\frac{\partial Q}{\partial y_h}, \tag{4.52}$$

$$Q(Y,\hat{\Sigma}) \leq Q(Y,\Sigma) + \sum_{k=1}^{N}(\hat{\sigma}_k - \sigma_k)\frac{\partial Q}{\partial \sigma_k}. \tag{4.53}$$

From equations (4.50-4.53), we can write

$$\dot{G} = -\sum_{h=1}^{H}(y_h - \hat{y}_h)\frac{\partial Q}{\partial y_h} + \sum_{k=1}^{N}(\sigma_k - \hat{\sigma}_k)\frac{\partial Q}{\partial \sigma_k} \leq 0$$

In order to apply La Salle's invariance priniciple, let us consider a set of points $\mathcal{E}$ for which the condition $\dot{G} = 0$ is satisfied. The largest invariant set $\mathcal{M}$ is a subset of points such that $\frac{\partial Q}{\partial y_h} = 0, \forall y_h \in Y$ and $\frac{\partial Q}{\partial \sigma_k} = 0, \forall \sigma_k \in \Sigma$. Pick any point $(\tilde{Y},\tilde{\Sigma}) \in \mathcal{M}$. We can show from the properties convex-concave nature of function $Q(Y,\Sigma)$ that $Q(\tilde{Y},\tilde{\Sigma}) \leq Q(Y,\tilde{\Sigma}), \forall Y$ and $Q(\tilde{Y},\tilde{\Sigma}) \geq Q(\tilde{Y},\Sigma), \forall \Sigma$. Therefore, the pair $(\tilde{Y},\tilde{\Sigma})$ satisfies the condition (4.51) and it is an optimizer of (4.43). From La Salle's principle, the dynamics converge to the largest invariant set $\mathcal{M}$ and therefore the convergent point is an optimal state of (4.43). Hence the system is globally asymptotically stable [28].

$\square$

## 4.6 Simulations

We simulated our system in Matlab to show system convergence. We first performed our simulations for our simple network shown in Figure 4.3(a). The load at the source nodes 1, 2 and 3 is given as 4.73, 2.69 and 3.56 respectively, which are randomly generated values. We use the following costs on the individual links ($\alpha_{ij}$): $\alpha_{12} = 2.8$, $\alpha_{23} = 1.6$, $\alpha_{34} = 1.8$, $\alpha_{25} = 1.3$, $\alpha_{54} = 2.1$, $\alpha_{26} = 1.7$, $\alpha_{48} = 2.9$, $\alpha_{86} = 2.2$, $\alpha_{57} = 1.9$, $\alpha_{71} = 2.6$; we assume the costs on the links are symmetric. We use the approximated cost function (4.22), with a value of $t = -30$ for the approximation parameter (4.21) for our simulations. We have assumed that the maximum number of transmissions (per unit time) from each node is limited to 15. The simulation is run for 50 large time units, and in each large time scale we have 20 small time units.

We compare the total cost of the system for the following:

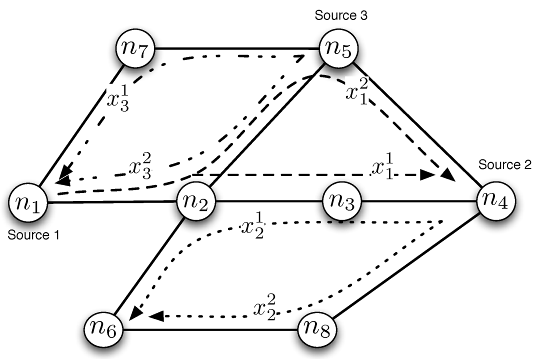
1. Decoupled Dynamics (DD): This is the algorithm that we developed under the augmented potential game framework; we use our hyper-links to decouple the flows that participate in coding.

2. Coupled Dynamics (no hyper-link) (CD): Here, there is coupling between individual flows and coding happens at the minimum rate of the constituent flows. In other words, this is the original potential game without augmentation. We use similar game dynamics as that was used in DD. The total cost is specified in Equation (4.5).

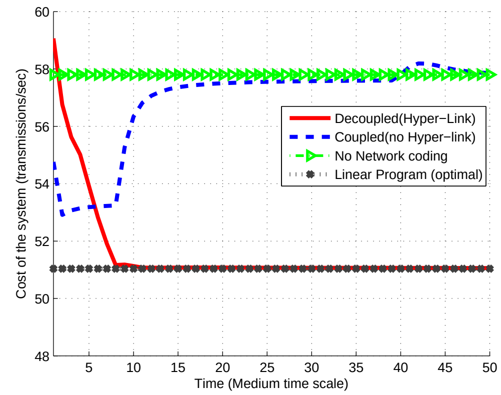
3. No Coding: In this system no network coding is used. This gives an baseline with respect to which the gains attained by coding can be quantified.

4. LP Optimal (LP): This is a centralized solution. We formulated our system as a Linear Program (LP) of minimizing cost (4.17) over $X$ and $Y$ for a given load vector that we obtain using an LP-solver.

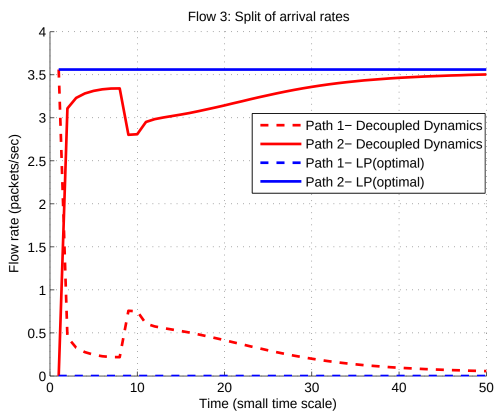
As seen in the Figure 4.3(b), the total cost of the system (number of transmissions per unit time) for our model (decoupled using hyper-link) is close to the optimal solution obtained by solving it in a centralized fashion. We compared the final system state of DD and CD with that of the solution obtained using LP. We observe from Table 4.1 that the values for the split ($X$) and the hyper-link capacities ($Y$) generated by DD are near-optimal



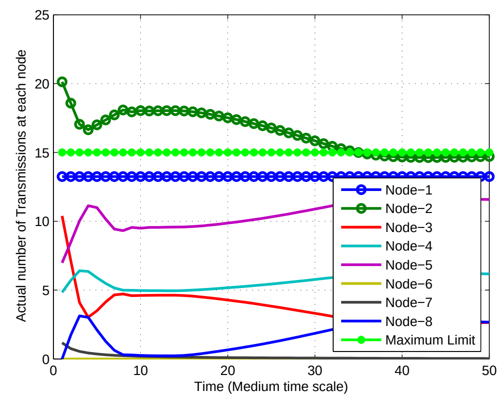

(a) Network topology




(b) Total system cost (per unit rate)




(c) Traffic (packets/sec) from flow-3 using DD converges to optimum




(d) Transmissions (per unit time) of each node converges to below maximum

Figure 4.3: Performance evaluation of simple network topology

(LP results), but CD is very different. We have plotted time evolution of traffic splits of flow 3, over options 1 and 2, in the Figure 4.3(c), which shows that they converge to the optimal values obtained by LP solver. In Figure 4.3(d), we have shown that the number of transmissions from all the nodes is less than or equal to the maximum threshold.

Next, we perform our simulations on a bigger topology shown in Figure 4.4. This network consists of 30 nodes shared by 6 flows. Flows 1, 2, 3 and 6 have two hyper-paths each and flows 4 and 5 have three hyper-paths each. There are 6 hyper-links in the system. Table 4.2 describes the source, destination nodes and the hyper-paths for each

Table 4.1: Comparison of state variables for LP, DD and CD

| Variable | $x_1^1$ | $x_1^2$ | $x_2^1$ | $x_2^2$ | $x_3^1$ | $x_3^2$ | $y_2$ | $y_3$ |
|---|---|---|---|---|---|---|---|---|
| LP | 1.52 | 3.2 | 1.52 | 1.16 | 0.00 | 3.56 | 3.20 | 1.52 |
| DD | 1.6 | 3.12 | 1.71 | 0.97 | 0.09 | 3.46 | 3.29 | 1.58 |
| CD | 4.70 | 0.00 | 0.01 | 2.68 | 0.62 | 2.93 | N/A | N/A |

flows. Notice, options 2 and 3 of flow 4 have the same physical path but different hyper-links, $y_1$ and $y_2$ at node $n_7$. This is because the sub-flow of $x_4$ traversing the physical path $(16, 15, 11, 6, 7, 8)$ can be encoded with two different flows, $x_1^2$ and $x_2^2$ traversing in the reverse direction at node 7.

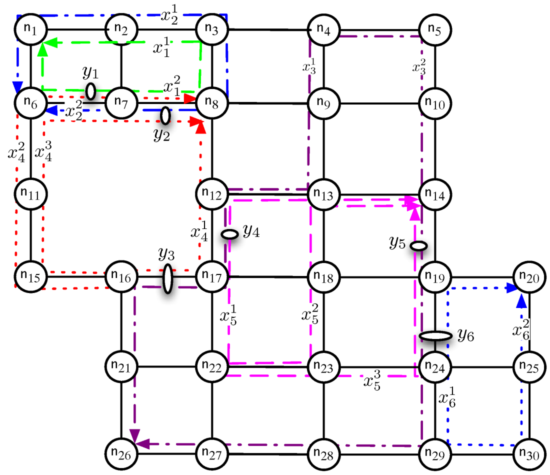



Figure 4.4: Complex network

We ran our algorithms on this network with random link costs. The simulation is run for 150 large time units, and in each large time scale we have 50 small time units. As seen in Figure 4.5, the total system cost for decoupled dynamics converges to the optimal solution which is obtained by solving the problem in a centralized fashion. We observe from Table 4.3 that the values for the split $(X)$ and the hyper-link capacities $(Y)$ generated by DD are near-optimal (LP results), but CD is very different.

Table 4.2: Source, destination nodes and hyper-paths corresponding to each flow.

| Id | Src Node | Dest. Node | Hyper-Paths |
|---|---|---|---|
| 1 | 8 | 1 | (8,3,2,1) & (8,7,6,1) |
| 2 | 8 | 6 | (8,3,2,1,6) & (8,7,6) |
| 3 | 5 | 26 | (5,4,9,13,12,17,16,21,26) & (5,10,14,19,24,29,28,27,26)) |
| 4 | 16 | 8 | (16,17,12,8), (16,15,11,6,7,8) & (16,15,11,6,7,8) |
| 5 | 23 | 14 | (23,22,17,12,13,14), (23,18,13,14) & (23,24,19,14) |
| 6 | 29 | 20 | (29,24,19,20) & (29,30,25,20) |

Table 4.3: Comparison of state variables for no coding, LP, DD and CD.

| | No Coding | LP | DD | CD |
|---|---|---|---|---|
| $x_1^1$ | 19.10 | 19.10 | 19.09 | 19.09 |
| $x_1^2$ | 0 | 0 | 0.01 | 0.01 |
| $x_2^1$ | 0 | 0 | 0.01 | 0.04 |
| $x_2^2$ | 21.08 | 21.07 | 21.07 | 21.07 |
| $x_3^1$ | 15.32 | 12.42 | 12.99 | 15.32 |
| $x_3^2$ | 0 | 2.90 | 2.33 | 0 |
| $x_4^1$ | 14.97 | 15.10 | 15.02 | 15.08 |
| $x_4^2$ | 0.06 | 0 | 0.0087 | 0.0087 |
| $x_4^3$ | 0 | 0 | 0 | 0 |
| $x_5^1$ | 0 | 8.69 | 8.8 | 0.05 |
| $x_5^2$ | 0 | 0 | 0.05 | 9.19 |
| $x_5^3$ | 11.6 | 2.90 | 2.79 | 11.54 |
| $x_6^1$ | 18.43 | 18.43 | 18.43 | 18.43 |
| $x_6^2$ | 0 | 0 | 0 | 0 |
| $y_1$ | N/A | 0 | 0 | N/A |
| $y_2$ | N/A | 0.17 | 0.63 | N/A |
| $y_3$ | N/A | 12.47 | 13.87 | N/A |
| $y_4$ | N/A | 8.69 | 9.15 | N/A |
| $y_5$ | N/A | 2.9 | 2.68 | N/A |
| $y_6$ | N/A | 2.9 | 3.98 | N/A |

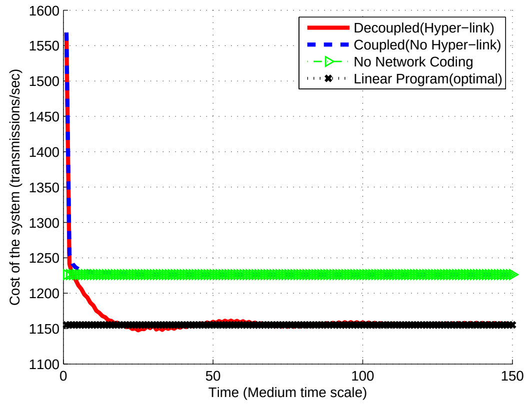



Figure 4.5: Comparison of total system cost (per unit rate), for different systems: DD and non-coded against LP.

## 4.7 Conclusion

We considered a wireless network with given costs on arcs, traffic matrix and multiple paths. The objective was to find the splits of traffic for each source across its multiple paths in a distributed manner leveraging the reverse carpooling technique where the peak transmissions (per unit time) at each node is limited. For this we split the problem into two sub-problems, and propose a two-level distributed control scheme set up as a game between the sources and the hyperlink nodes. On one level, given a set of hyperlink capacities and node-prices, the sources selfishly choose their splits and attain a Nash equilibrium. On the other level, given the traffic splits, the hyperlinks and nodes may slightly increase or decrease their capacities and prices using a steepest descent algorithm. We constructed a Lyapunov function argument to show that this process asymptotically converges to the minimum cost solution, although performed in a distributed fashion.

In designing the two level controller, we came up with an interesting formulation that we believe might be useful in other coordination games. The idea is to augment the state space of the system using additional variables that are controlled by unselfish agents. Although these agents only have local information at their disposal, they are able to modify the potential function of the system as a whole, and hence change the actions taken by the selfish routing agents. Essentially, these agents take on some of the system cost on themselves in order to redistribute the overall costs. The system wide cost is minimized as a result. We also showed that the idea can be coupled with a Lagrange multiplier approach to enforce constraints as well.

We performed several numerical studies and found that our two-level controller converges fast to the optimal solutions. Some of the bi-products of our experiments were that: more expensive paths before network coding became cheaper and shortest paths were not necessarily optimal. In conclusion, from a methodological standpoint we have a distributed controller that achieves a near-optimal solution when the individuals are self-interested.

In the next chapter, we explore the benefits of an auction based scheduling mechanism that allocates channel resources to a large number of competing mobile applications in a cellular network. We model the apps as queues that arrive and depart as they are turned on and off. Conventional wisdom suggest to use Longest Queue First (LQF) policy in which the server awards its service at each instant to the longest of queues at that instant. LQF has many nice properties like achieving throughput optimality, fairness etc. However, this policy requires knowledge of queues at the scheduler (base station), which may not be possible in the case of cellular networks. The applications may be asked to provide queue length information. In that case, the applications, being selfish, may attempt to obtain unfair amount of resources by providing false informations to mislead the scheduler. Then, how can we enforce the apps to report their queue lengths truthfully? One solution is second price auction based scheduling. When the resource becomes available, the base station conducts a second-price auction in which one unit of service is awarded to the highest bidder at the payment of second highest bid. Now, the question we are interested in answering is whether conducting such an auction repeatedly over time with queues arriving and departing would result in some form of equilibrium? Would the scheduling decisions resulting from such auctions resemble that of LQF? We attempt to answer these questions in the next chapter.

# 5. MAC LAYER: A MEAN FIELD GAMES APPROACH TO SCHEDULING IN CELLULAR SYSTEMS

There has been a rapid increase in the usage of smart hand held devices for Internet access. These devices are supported by cellular data networks, with the usage of these data networks taking the form of packets generated by apps running on the smart devices. The users of the apps terminate and start new ones every so often, and move around to different cells as they do so. Scheduling uplink and downlink packets in a "fair" manner under these circumstances is a topic of much recent research.

In this work, we consider a system consisting of smart phone users whose apps are modeled as queues that arrive when the user starts the app, and depart when the user terminates that app and starts a new one. Apps may generate packets (uplink) or might request packets from else where (downlink), and these processes are captured by considering jobs of different sizes that arrive to these queues. Users move around in an area that is divided up into cells that each has a cellular base station, and scheduling a particular user in a cell implies providing a unit of service to the queue that represents his/her currently running app. At any time, the user might terminate the app with a fixed probability, giving rise to a geometric lifetime for each app. Note that the app may be terminated even if it has packets queued up, *i.e.,* the lifetime of a queue is unrelated to the amount of service performed on it or the jobs waiting for service.

The problem of scheduling in wired and wireless systems has been a topic of much recent research. Most have focused on the case where a finite number of infinitely log lived flows exist in the system, and the objective is the maximize the total throughput of the system as a whole. A seminal piece of work in this regime is [74], in which the so called *max-weight algorithm* was introduced. Essentially, the argument consisted of minimizing the drift of quadratic Lyapunov function by maximizing the queue-length weighted sum of acceptable schedules. Follow on works [15, 32, 34, 48, 49, 75] have illustrated its validity in a variety of network scenarios.

If queues arrive and depart in the system, then a natural scheduling policy in the single server case is a *Longest Queue First (LQF)* scheme, in which each server picks the longest of the queues requesting service from it, and awards it one unit of service. LQF has many attractive properties, such a minimizing the expected value of the longest queue in the system. It has also been shown [1] that with Bernoulli arrivals it minimizes the probability of the shortest queue being shorter than a target value. In other words, it minimizes the longest queue, and maximizes the shortest queue, effectively giving rise to queue that at similar in length.

Critical to all the above work is the assumption that the queue length values are available to the scheduler. While the downlink queues would naturally be available at a cellular base station, the only way to get the uplink queue information is to ask the users themselves. However, reporting a larger value of queue length implies a higher probability of being scheduled under all the above policies, implying a strong incentive to lie about one's queue length. How are we to design a scheduling scheme that possesses the good qualities of LQF, while relying on self-reported values from the users?

An appealing idea is to use some kind of pricing or auction scheme to take scheduling decisions for cellular data access. For instance, [20] describes an experimental trial of a system in which day-ahead prices are announced to users, who then decide on whether or not to use their 3G service based on the price at that time. However, these prices should be have to be determined empirically.

The key objective of this work is to design an incentive compatible scheduling scheme that behaves in an LQF-like fashion. Thus, we aim to systematically analyze an auction theoretic framework in which each app bids for service from the cellular base station that the device is currently located in. The auction is conducted in a second-price fashion, with the winner being the one that bids highest, and the charge being the second highest bid. It is well known that such an auction promotes truth-telling [29]. The question we are interested in answering is whether conducting such an auction repeatedly over time with queues arriving and departing would result in some form of equilibrium? Would the scheduling decisions resulting from such auctions resemble that of LQF?

In this work, we investigate the existence of such an equilibrium using the theory of Mean Field Games (MFG). MFG has received a lot of attention in the recent years [2,22,79]. MFG offers a mathematical framework to approximate Perfect Bayesian Equilibrium (PBE) in large player dynamic games which is otherwise intractable. PBE requires each player to keep track of their beliefs on the future plays of every other opponent in the system. This makes the computation of PBE computationally intractable when the number of players is large. Henceforth, Mean Field Equilibrium (MFE), an equilibrium concept in MFG is used to approximate PBE. In MFG, the players model their opponents through an assumed distribution over their action spaces, and play the best response action against this distribution. We say that the system is at MFE if this best response action turns out to be a sample drawn from the assumed distribution.

Our main result is that the dynamic auction based scheduling mechanism has a MFE. Also, we show that the equilibrium bidding strategy of each player is montone in their queue length. That means, at each time the service is awarded to the longest of queues, a policy that resembles LQF. Hence, we believe that auction theoretic scheduling mechanism may attain the same benefits as that of LQF policy.

## 5.1 Model

Consider a large geographical area that is uniformly partitioned into $N$ cells each having one base station. We assume that there are a large number of mobile users and assume that they are randomly moving around the region passing from one cell to another cell. At every unit interval, the mobile users are uniformly and randomly distributed across $N$ cells such that each cell contains exactly $M$ users. Each base station conducts second price auction among the users within its cell territory at unit intervals. And the winner receives one unit worth of service. Let $Q_{i,k}$ represents the residual workload of agent (mobile user) $i$, just before $k^{th}$ auction. We assume that $Q_{i,k} \in [0, \infty)$ and note that it completely represents the state of the queue at time $k$. Agent $i$'s workload is influenced by the following three processes.

1. Arrivals: After every auction, an arrival $A_{i,k}$ occurs at agent $i$, where $A_{i,k}$ is a random variable independent of every other parameter and distributed according to $\Phi_A$.

2. Service: $D_{i,k}$ is the random variable representing the amount of service delivered at the $k$-th time instant. We assume that the server serves at-most a unit amount of workload of the winner in any auction. $D_{i,k} = \min\{1, Q_{i,k}\} \times W_{i,k}$, where $W_{i,k} = \mathbf{1}(i$ wins at time $k)$.

3. Regeneration: We assume that after participating in an auction agent $i$ may regenerate its workload with probability $1-\beta$ where $0 < \beta < 1$. We assume that the new workload is a random variable distributed according to $\Psi_R$.

Hence, the state of agent $i$ at time $k+1$ is,

$$Q_{i,k+1} = \begin{cases} Q_{i,k} - D_{i,k} + A_{i,k} & \text{agent } i \text{ does not regenerate at } k \\ R_{i,k} & \text{otherwise,} \end{cases} \tag{5.1}$$

where $A_{i,k} \sim \Phi_A$ and $R_{i,k} \sim \Psi_R$. Below we state the assumptions on the arrival and regeneration processes.

**Assumption 1.** *The arrivals $\{A_{i,k}\}$ are i.i.d random variables distributed according to $\Phi_A$. We assume that $A_{i,k} \in [0, \bar{A}]$. Also, these random variables have a bounded density function, $\phi_A$. ($\|\phi_A\| < c_\phi$.)*

**Assumption 2.** *The regeneration values $\{R_{i,k}\}$ are i.i.d are random variables distributed according to $\Psi_R$ and they have a bounded density $\psi_R$. ($\|\psi_R\| < c_\psi$.)*

Each agent bears a *holding* cost at every instant, that corresponds to the dis-utility due to unserved workload. The holding cost of agent $i$ at time $k$ is $C(Q_{i,k})$, where $C : R^+ \to R^+$. The agent also pays for for service if it wins the auction. This is called *bidding* cost. Let

$X_{i,k}$ is the bid submitted by agent $i$ in the $k$-th auction and

$$\bar{X}_{-i,k} = \max_{j \in M_{i,k}} X_{j,k},$$

where $M_{i,k}$ is the set consisting of all other agents participating in at time $k$ with agent $i$. Then, the bidding cost of agent $i$ is $\bar{X}_{-i,k} \times W_{i,k}$. We make some assumptions on the holding cost function as stated below.

**Assumption 3.** *The holding cost function $C : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is continuous, increasing and strictly convex. We also assume that $C$ is $O(q^m)$ for some integer $m$.*

### *5.1.1 Optimal bidding strategy*

In this section we begin to understand the strategy space available to an agent. We note that the information available with any agent, about the market at any time prior to the auction, only includes the following:

1. The bids it made in each of the previous auction from point of last regeneration.

2. The auctions it won.

3. The payments made for the auctions won.

Let, $H_{i,k}$ be the vector containing the above information available to agent $i$ at time $k$. An agent is unaware of any information concerning other agents. Each agent holds a belief that is a distribution over future trajectories which gets updated via Baye's rule as new information arrives at the occurrence of each auction event. Let $\mu_{i,k}$ be the belief of agent $i$ at time $k$.

Let *pure strategy* $\theta_i$ be the history dependent strategy of agent $i$, i.e $\theta_i(H_{i,k}) = X_{i,k}$. We define $\boldsymbol{\theta}_{-i}$ to be the vector of strategies of all agents except agent $i$ and $\boldsymbol{\theta} = [\theta_i, \boldsymbol{\theta}_{-i}]$. We refer to $\boldsymbol{\theta}$ as *strategy profile.*

Given a strategy profile $\boldsymbol{\theta}$, a history vector $H_{i,k}$ and a belief vector $\mu_{i,k}$, expected cost is,

$$V_{i,\mu_{i,k}}(H_{i,k};\boldsymbol{\theta}) = \mathbb{E}_{\theta,\mu_{i,k}}\left[\sum_{t=k}^{T_i^{(k)}}\left[C(Q_{i,t}) + \bar{X}_{-i,t}\mathbf{1}(W_{i,t}=1)\right]\right], \tag{5.2}$$

where $T_i^{(k)}$ is the time at which player $i$ regenerates after time $k$.

We are now ready to introduce the notion of Nash equilibrium in dynamic games, called *Perfect Bayesian Equilibrium* (PBE).

**Definition 3** (Perfect Bayesian equilibrium)**.** *A strategy profile $\boldsymbol{\theta}$ is said to be a Perfect Bayesian Equilibrium if*

1. *For each agent $i$, after any history $H_{i,k}$, $\theta_i(H_{i,k}) \in \arg\max_{\theta_i'} V_{i,\mu_{i,k}}(H_{i,k}, \theta_i', \boldsymbol{\theta}_{-i})$*
2. *The belief vectors $\mu_{i,k}$ are updated via Bayes' rule for all agents.*

The above equilibrium requires each agent to keep track of complex beliefs over other agents and update them using Bayes' rule at each time. As the number of agents grow large, this imposes large computational constraints on the agents. Also, the equilibrium bid calculation of an agent depends on the entire histories and the strategies of all the agents. So this equilibrium characterization is intractable.

## 5.2 Mean field model

In the mean field model we approximate the model parameters of the above stochastic game as the number of agents in the game approaches infinity. According to the belief of a single agent, as the number of the other agents increases, we conjecture that the distribution of a random agent's state does not change under Bayesian updates. Further, we can also conjecture that the bid distributions of the $m-1$ agents in an auction are independent, as it is unlikely that they would have interacted from the point of earliest regeneration of the all the agents in the auction. Since, in any auction the identity of other agents is unimportant, agent $i$ needs to only maintain belief over the bid of a random agent.

In the following sections, we formalize these ideas and define the concept of mean field equilibrium (MFE).

### *5.2.1 Agent's decision problem*

In this section, we address a single agent's decision problem. Let the candidate be agent $i$. As described above, the agent needs to maintain a belief over the bids of a random agent. Suppose this cumulative distribution is $\rho$. We assume that $\rho \in \mathcal{P}$ where,

$$\mathcal{P} = \{\rho | \rho \text{ is a continuous c.d.f,} \int (1 - \rho(x))dx < E\},$$

where $E < \infty$ and independent of $\rho$. Under this belief model, the expected cost of the agent (5.2), can be re-written as,

$$V_{i,\rho}(H_{i,k}; \boldsymbol{\theta}) = \mathbf{E}\left[\sum_{t=k}^{T_i^k} [C(Q_{i,t}) + r_\rho(X_{i,k}))]\right] \tag{5.3}$$

where the expectation is over $T_i^k$ and future state evolutions. Note that $X_{i,k} = \theta_i(H_{i,k})$. Also, $r_\rho(x) = E[\bar{X}_{-i,k}\mathbf{I}\{\bar{X}_{-i,k} \leq x\}]$ is the *expected bidding cost* when the agents bids $x$ under the assumption that the bids of other agents are distributed according to $\rho$. We see that in replacing the belief with $\rho$, we have made an agent's decision problem independent of other agents' strategies, hence we represent the cost by $V_{i,\rho}(H_{i,k}; \theta_i)$.

We now give the expression for $r_\rho$ in terms of $\rho$. Given $\rho$, the winning probability in a second price auction is

$$p_\rho(x) = \mathbf{Pr}(\bar{X}_{-i,k} \leq x) = \rho(x)^{M-1}. \tag{5.4}$$

where $M$ is the number of agents selected for participating in an auction. The expected payment when bidding $x$ is

$$r_\rho(x) = \mathbf{E}[\bar{X}_{-i,k}\mathbf{I}\{\bar{X}_{-i,k} \leq x\}] = xp_\rho(x) - \int_0^x p_\rho(u)du. \tag{5.5}$$

Since, $T_i^k$ is a geometric random variable, the above expression reduces to

$$V_{i,\rho}(H_{i,k};\theta_i) = \mathbb{E}\left[\sum_{t=k}^{\infty} \beta^t [C(Q_{i,t}) + r_\rho(X_{i,t})]\right]. \tag{5.6}$$

Here, the state process $Q_{i,k}$ is Markov; the future state is independent of past states and past actions given the current state and current action. The transition kernel of the process is

$$\begin{aligned} \mathbf{Pr}(Q_{i,k+1} \in B | Q_{i,k} = q, X_{i,k} = x) =& \beta p_\rho(x) \mathbf{Pr}((q-1)^+ + A_k \in B) \qquad (5.7) \\ &+ \beta(1 - p_\rho(x)) \mathbf{Pr}(q + A_k \in B) + (1-\beta)\Psi_R(B). \\ & \qquad (5.8) \end{aligned}$$

where $B \subseteq R^+$ is a Borel set and $x^+ \triangleq \max(x, 0)$. Recall that $A_k \sim \Phi_A$ is the arrival between $(k)^{th}$ and $(k+1)^{th}$ auction and $\Psi_R$ is density function of the regeneration process. In the above expression, the first two terms correspond to the event that the agent does not regenerate. In particular the first corresponds to the event that agent wins the auction at time $k$. The last term captures the event that the agent regenerates after auction $k$. Also, note that the transition kernel is time invariant. Therefore, the agent's decision problem, which is to find a policy that minimizes the cost given above, can be modeled as an infinite horizon discounted cost MDP. From Theorem 5.5.3 in [56], there exists an optimal Markov deterministic policy to a discounted cost MDP. Then, from (5.6), the optimal value function of the agent can be written as

$$\hat{V}_{i,\rho}(q) = \inf_{\theta_i \in \Theta} \mathbb{E}\left[\sum_{t=1}^{\infty} \beta^t [C(Q_{i,t}) + r_\rho(X_{i,t})] \,|Q_{i,0} = q\right]. \tag{5.9}$$

where $\Theta$ is the space of Markov deterministic policies.

Note that user index is redundant in the above expression as we are concerned with a single agent's decision problem, In future notations, we will omit the user subscript $i$.

### *5.2.2 Stationary distribution*

Given cumulative bid distribution $\rho$ and a Markov policy $\theta \in \Theta$, the transition kernel given by (5.7) can be re-written as,

$$\begin{aligned}\mathbf{Pr}(Q_{k+1} \in B | Q_k = q) =& \beta p_\rho(\theta(q)) \mathbf{Pr}((q-1)^+ + A_k \in B) + \\ & \beta(1 - p_\rho(\theta(q))) \mathbf{Pr}(q + A_k \in B) + (1-\beta)\Psi_R(B). \quad (5.10)\end{aligned}$$

Then, we have an important result in the following lemma:

**Lemma 10.** *The Markov chain described by the transition probabilities in* (5.10) *is positive Harris recurrent and has a unique stationary distribution.*

*Proof.* From eq. (5.10) we note that,

$$\mathbf{Pr}(Q_{k+1} \in B | Q_k = q) \geq (1-\beta)\Psi_R(B)$$

where $0 < \beta < 1$ and $\Psi_R$ is a probability measure. The result then follows from results in Chapter 12, Meyn and Tweedie [43]. □

We denote the unique stationary distribution by $\Pi_{\rho,\theta}$.

### *5.2.3 Mean field equilibrium*

In this section, we define the mean field equilibrium for our stochastic game. Assume that all agents conjecture the same bid distribution $\rho$ and the decision problem in eq. (5.9) has an optimal policy $\hat{\theta}_\rho$. This induces a dynamics with transition probabilities as in eq. (5.10). We have shown in the previous section that the dynamics induced by the transition kernel eq. (5.10) has a stationary distribution which we denote by $\Pi_\rho = \Pi_{\rho,\hat{\theta}_\rho}$.

The mean field equilibrium requires the consistency check, that the bid distribution induced by the stationary distribution $\Pi_\rho$ be equal to the bid distribution conjectured by the agent, i.e., $\rho$. In other words we require,

$$\rho(x) = \Pi_\rho(\theta_\rho^{-1}([0,x])). \tag{5.11}$$

Thus, we have the following definition of MFE:

**Definition 4** (Mean field equilibrium)**.** *Let $\rho$ be a bid distribution and $\theta_\rho$ be a stationary policy for an agent. Then, we say that $(\rho, \theta_\rho)$ constitutes a mean field equilibrium if*

1. *$\theta_\rho$ is an optimal policy of the decision problem in eq. (5.9), given bid distribution $\rho$; and*

2. *$\rho(x) = \Pi_\rho(\theta_\rho^{-1}([0,x])), \forall x \in \mathbb{R}^+$.*

We prove the existence of an MFE in Section 5.4. Before that, in the following section, we establish monotonicity and continuity the optimal bid function. These properties are essential in showing the existence of an MFE.

## 5.3 Properties of optimal bid function

In this section, we state the optimality equation for the single agent's decision problem given in eq. (5.9) and describe an optimal strategy. We subsequently list some useful properties of this optimal strategy. In this section we have a fixed bid distribution $\rho$, and hence, omit $\rho$ from the subscripts.

Note that the decision problem given by eq. (5.9) is an infinite horizon, discounted Markov decision problem. The optimality equation or Bellman equation corresponding to the decision problem is

$$\begin{aligned}\hat{V}_\rho(q) &= C(q) + \beta \mathbf{E}_A(\hat{V}_\rho(q+A)) \\ &\qquad + \inf_{x \in R^+} [r_\rho(x) - p_\rho(x)\beta \mathbf{E}_A \left( \hat{V}_\rho(q+A) - \hat{V}\rho((q-1)^+ + A) \right)],\end{aligned} \tag{5.12}$$

where $A$ is the arrival process. In the following lemma we show that there exists a unique solution to the above optimality equation and derive an optimal Markov stationary strategy to the decision problem.

We first introduce some necessary notation.Let,

$$\mathcal{V} = \left\{ f : \mathbb{R}^+ \mapsto \mathbb{R}^+ : \sup_{q \in \mathbb{R}^+} \left| \frac{f(q)}{w(q)} \right| < \infty \right\}, \tag{5.13}$$

where $w(q) = \max C(q), 1$. Note that $\mathcal{V}$ is a Banach space with *$w$-norm*,

$$\|f\|_w = \sup_{q \in \mathbb{R}^+} \left| \frac{f(q)}{w(q)} \right| < \infty.$$

Also, define the operator $T_\rho$ as

$$\begin{aligned}(T_\rho f)(q) &= C(q) + \beta \mathbf{E}_A f(q + A) \\ &\quad + \inf_{x \in \mathbb{R}^+} \left[ r_\rho(x) - p_\rho(x) \beta (\mathbf{E}_A (f(q + A) - f((q - 1)^+ + A))) \right], \end{aligned} \tag{5.14}$$

where $f \in \mathcal{V}$. Lemma 17 shows that infimum in the above operator occurs at $\max\{0, \beta \Delta f(q)\}$, where $\Delta f(q) = \mathbf{E}_A (f(q + A) - f((q - 1)^+ + A))$. Then, substituting $r_\rho$ and $p_\rho$ from (5.4) and (5.5), the above expression can be rewritten as,

$$(T_\rho f)(q) = C(q) + \beta \mathbf{E}_A f(q + A) - \int_0^{\max\{0, \beta \Delta f(q)\}} p_\rho(u) du. \tag{5.15}$$

Now, we are ready to state the lemma.

**Lemma 11.** *Given a cumulative bid distribution $\rho$,*

1. *There exists a unique $\hat{f}_\rho \in \mathcal{V}$ such that $T_\rho \hat{f}_\rho = \hat{f}_\rho$. Also, for any $f \in \mathcal{V}$, $T_\rho^n f \to \hat{f}_\rho$ (as $n \to \infty$).*

2. *The unique fixed point $\hat{f}_\rho$ of operator $T_\rho$ is a unique solution to the optimality equation (5.12), i.e., $\hat{f}_\rho = \hat{V}_\rho$.*

3. *Let $\hat{\theta}_\rho(q) = \max \left\{ 0, \mathbf{E}_A \left[ \hat{V}_\rho(q + A) - \hat{V}_\rho((q - 1)^+ + A) \right] \right\}$. Then, $\hat{\theta}_\rho$ is an optimal policy.*

*Proof.* First and second statement in the lemma follows from Theorem 6.10.4 in [56] if the following conditions are satisfied. Let $Q_k$ be the random variable denoting queue length at time $k$. Then, the conditions to be satisfied are,

$$T_\rho f \in \mathcal{V}, \forall f \in \mathcal{V}, \tag{5.16}$$

$$\sup_{x \in \mathbb{R}^+} |C(q) + r(x)| \leq K_1 w(q), \text{ for some } K_1 > 0, \forall q \in \mathbb{R}^+, \tag{5.17}$$

$$\mathbf{E}_{Q_1}[f(Q_1)|Q_0 = q] \leq K_2 w(q), \text{for some } K_2 > 0, \forall q \in \mathbb{R}^+, \forall f \in \mathcal{V}, \tag{5.18}$$

and

$$\beta^j \mathbf{E}_{Q_j}(w(Q_j)|Q_0 = q) \leq K_3 w(q), \text{ for some } 0 < K_3 < 1, \text{ for some } j, \ \forall q \in \mathbb{R}^+. \tag{5.19}$$

To prove (5.16), one may observe from (5.15) that

$$C(q) \leq (T_\rho f)(q) \leq C(q) + \beta \mathbf{E}_A f(q + A). \tag{5.20}$$

Here, the left most expression is positive. And, the rightmost expression is bounded by some multiple of $w(q)$ since $A$ is a bounded random variable by Assumption 1. Together, we get (5.16). Further, (5.17) holds true from the definition of $w(q)$ and from the fact that

$$r(x) \leq \lim_{y \to \infty} r(y) < (M - 1) \int (1 - \rho(x)) dx < (m - 1)E.$$

Here, the last inequality is due to $\rho \in \mathcal{P}$. Equation (5.18) holds true since

$$\begin{aligned} \mathbf{E}_{Q_1}[f(q_1)|Q_0 = q] &\leq \|f\|_w \mathbf{E}_{Q_1}[w(Q_1)|Q_0 = q] \\ &= \|f\|_w \left[ p(b) \mathbf{E}_A w((q-1)^+ + A) + (1 - p(b)) \mathbf{E}_A w(q + A) \right] \end{aligned}$$

$$\le \|f\|_w \left[\mathbf{E}_A w(q+A)\right]$$
$$\le \|f\|_w K_2 w(q).$$

for some large enough $K_2$ due to Assumption 3. Finally, we have eq. (5.19) since,

$$\begin{aligned}\beta^j \mathbf{E}_{Q_j}[w(Q_j)|Q_0 = q] &= \beta^j \mathbf{E}_{Q_j}[C(Q_j)|Q_0 = q] \\ &\le \beta^j C(q + j\bar{A}) \\ &\le \beta C(q),\end{aligned}$$

for large enough $j$. Here $\bar{A}$. as defined in Assumption 1, is the maximum arrival possible between any two adjacent auctions.

Since all the conditions of Theorem 6.10.4 are met, the first result in the lemma holds true. The second result can be obtained by comparing (5.14) and (5.12). The last part of the lemma follows from Lemma 17. □

Now, we establish that $\hat{V}_\rho$ and $\hat{\theta}_\rho$ are continuous and increasing functions.

**Lemma 12.** *Given a cumulative bid distribution function $\rho$, we have*

1. *$\hat{V}_\rho$ is a continuous monotone increasing function.*
2. *$\hat{\theta}_\rho$ is a continuous strictly monotone increasing function.*

*Proof.* Let $f \in \mathcal{V}$. Suppose $f$ is a continuous monotone increasing function. Now, we prove that $T_\rho f$ is also continuous monotone increasing function. Since, $T_\rho^n f \to \hat{V}_\rho$ according to statement 2 of the previous lemma, we can conclude that $\hat{V}_\rho$ also holds the same property.

First we prove that $T_\rho f$ is a monotone increasing function. Let $q > q'$. Then,

$$\begin{aligned}T_\rho f(q) - T_\rho f(q') =& C(q) - C(q') + \beta \mathbf{E}_A(f(q+A) - f(q'+A)) \\ &+ \beta \inf_x [r_\rho(x) - p_\rho(x)\mathbf{E}_A(f(q+A) - f((q-1)^+ + A))] \\ &- \beta \inf_b [r_\rho(x) - p_\rho(x)\mathbf{E}_A(f(q'+A) - f((q'-1)^+ + A))]\end{aligned}$$

$$
\begin{aligned}
&\geq \beta \mathbf{E}_A(f(q+A) - f(q'+A)) \\
&\quad + \beta \inf_b [p_\rho(x)\mathbf{E}_A(f(q'+A) - f((q'-1)^+ + A)) \\
&\qquad - \mathbf{E}_A(f(q+A) + f((q-1)^+ + A))] \\
&\geq \beta \min \left\{ \mathbf{E}_A(f(q+A) - f(q'+A)), \right. \\
&\quad \left. \mathbf{E}_A(f((q-1)^+ + A) - f((q'-1)^+ + A)) \right\} \geq 0.
\end{aligned}
$$

The second inequality follows from the assumption that $C(.)$ is an increasing function. And the last inequality follows from the assumption that $f(.)$ is an increasing function.

To prove that $T_\rho f$ is continuous consider a sequence $\{q_n\}$ such that $q_n \to q$. Since $f$ is a continuous function, $f(q_n+a) \to f(q+a)$. Then, by using dominated convergence theorem, we have $\mathbf{E}_A f(q_n+A) \to \mathbf{E}_A f(q+A)$ and $\mathbf{E}_A f((q_n-1)^+ + A) \to \mathbf{E}_A f((q-1)^+ + A)$. Also, $\Delta f(q_n) \geq 0$ as $f$ is an increasing function. Then, from (5.15), we get that

$$
\begin{aligned}
T_\rho f(q_n) &= C(q_n) + \beta \mathbf{E}_A f(q_n + A) - \int_0^{\beta \Delta f(q_n)} p_\rho(u) du && (5.22) \\
&\to C(q) + \beta \mathbf{E}_A f(q+A) - \int_0^{\beta \Delta f(q)\}} p_\rho(u) du = T_\rho f(q). && (5.23)
\end{aligned}
$$

Hence, $Tf$ is a continuous function. This yields statement 1 in the lemma.

Now, to prove second part of the lemma, assume that $\Delta f$ is an increasing function. First, we show that $\Delta T_\rho f$ is an increasing function. Let $q > q'$. From (5.15), for any $a < \bar{A}$ we can write

$$
\begin{aligned}
& (T_\rho f)(q+a) - (T_\rho f)((q-1)^+ + a) - (T_\rho f)(q'+a) + (T_\rho f)((q'-1)^+ + a) \\
= \; & C(q+a) - C((q-1)^+ + a) - C(q'+a) + C((q'-1)^+ + a) \\
& \quad + \beta \mathbf{E}_A f(q+a+A) - \beta \mathbf{E}_A f((q-1)^+ + a + A) \\
& \quad - \beta \mathbf{E}_A f(q'+a+A) + \beta \mathbf{E}_A f((q'-1)^+ + a + A) \\
& \quad - \int_{\beta \Delta f(q'+a)}^{\beta \Delta f(q+a)} p_\rho(u)\ du + \int_{\beta \Delta f((q'-1)^+ + a)}^{\beta \Delta f((q-1)^+ + a)} p_\rho(u)\ du
\end{aligned}
$$

$$
\begin{aligned}
= \quad & C(q+a) - C((q-1)^+ + a) - C(q'+a) + C((q'-1)^+ + a) \\
& + \mathbf{E}_A f((q+a-1)^+ + A) - \mathbf{E}_A f((q-1)^+ + a + A) \\
& - \mathbf{E}_A f((q'+a-1)^+ + A) + \mathbf{E}_A f((q'-1)^+ + a + A) \\
& + \int_{\beta\Delta f(q'+a)}^{\beta\Delta f(q+a)} 1 - p_\rho(u)\ du + \int_{\beta\Delta f((q'-1)^+ + a)}^{\beta\Delta f((q-1)^+ + a)} p_\rho(u)\ du
\end{aligned}
$$

It can be easily verified that $\mathbf{E}_A(f(q+a-1)^+ + A) - \mathbf{E}_A(f(q-1)^+ + a + A) - \mathbf{E}_A(f(q'+a-1)^+ + A) + \mathbf{E}_A(f(q'-1)^+ + a + A) \geq 0$ as $f$ is increasing (due to statement 1 of this lemma). From the assumption that $\Delta f$ is increasing, the last two terms in the above expression are also non-negative. Now, taking expectation on both sides, we obtain $\Delta T_\rho f(q) - \Delta T_\rho f(q') \geq \Delta C(q) - \Delta C(q') > 0$. Therefore, from Statement 2 and 3 of the previous lemma, we have

$$
\theta_\rho(q) - \theta_\rho(q') = \Delta \hat{V}_\rho(q) - \Delta \hat{V}_\rho(q') \geq \Delta C(q) - \Delta C(q') > 0.
$$

Here, the last inequality holds since $C$ is a strictly convex increasing function. □

We state a useful Corollary that defines the optimal policy of the agent.

**Corollary 6.** *An optimal policy of the agent's decision problem* (5.9) *is given by*

$$
\hat{\theta}_\rho(q) = \beta \mathbf{E}_A \left[ \hat{V}_\rho(q+A) - \hat{V}_\rho((q-1)^+ + A) \right]
$$

The proof follows from Statement 3 of Lemma 11 and Statement 1 of Lemma 12

## 5.4 Existence of MFE

Now, we have the main result showing the existence of MFE.

**Theorem 8.** *There exists an MFE* $(\rho, \hat{\theta}_\rho)$ *such that*

$$
\rho(x) = \Pi_\rho \left( \hat{\theta}_\rho^{-1}[0, x] \right), \forall x \in R^+.
$$

We prove theorem in the next section. Before moving to the proof, let us introduce some useful notation. Let $\Theta = \{\theta : \mathbb{R} \mapsto \mathbb{R}, \|\theta\|_w < \infty\}$. Note that $\Theta$ is a normed space with $w$-norm. Also, let $\Omega$ be the space of absolutely continuous probability measures on $\mathbb{R}^+$. We endow this probability space with the topology of weak convergence. Note that this is same as the topology of point-wise convergence of continuous cumulative distribution functions.

We define $\theta^* : \mathcal{P} \mapsto \Theta$ as $(\theta^*(\rho))(q) = \hat{\theta}_\rho(q)$, where $\hat{\theta}_\rho(q)$ is the optimal bid given by Corollary 6. It can easily verified that $\hat{\theta}_\rho \in \Theta$. Also, define the mapping $\hat{\Pi}$ that takes a bid distribution $\rho$ to the invariant workload distribution $\Pi_\rho(\cdot) = \Pi_{\rho,\hat{\theta}_\rho}(\cdot)$. Later, using Lemma 13 we will show that $\Pi_\rho(\cdot) \in \Omega$. Therefore, $\hat{\Pi} : \mathcal{P} \to \Omega$. Finally, let $\mathcal{F}$ be a mapping from $\mathcal{P}$. We define $\mathcal{F}$ as $(\mathcal{F}(\rho))(x) = \Pi_\rho(\hat{\theta}_\rho^{-1}([0, x]))$.

Now to prove the above theorem we show that $\mathcal{F}$ has a fixed point, i.e $\mathcal{F}(\rho) = \rho$. Schauder's fixed point theorem, stated below, yields the sufficient conditions for the existence of a fixed point to the mapping $\mathcal{F}$.

**Theorem 9** (Schauder's fixed point theorem)**.** *Suppose $\mathcal{F}(\mathcal{P}) \subset \mathcal{P}$. Then, $\mathcal{F}(.)$ has a fixed point, if $\mathcal{F}$ is continuous, $\mathcal{F}(\mathcal{P})$ is contained in a convex and compact subset of $\mathcal{P}$.*

In subsequent sections, we show that the mapping $\mathcal{F}$ satisfies the conditions of the above theorem, and hence it has a fixed point. Note that $\mathcal{P}$ is a convex set. Therefore, we just need to show that the other two conditions are satisfied.

## 5.5 MFE existence: proof

### *5.5.1 Continuity of the map $\mathcal{F}$*

To prove the continuity of mapping $\mathcal{F}$, we first show that $\theta^*$ and $\hat{\Pi}$ are continuous mappings. To that end, we will show that for any sequence $\rho_n \to \rho$ in uniform norm, we have $\theta^*(\rho_n) \to \theta^*(\rho)$ in $w$-norm and $\hat{\Pi}(\rho_n) \Rightarrow \hat{\Pi}(\rho)$ ( $\Rightarrow$ implies weak convergence). Then, we show that $\mathcal{F}(\mathcal{P}) \in \mathcal{P}$. Finally, we use the continuity of $\theta^*$ and $\hat{\Pi}$ to prove that $\mathcal{F}(\rho_n) \to \mathcal{F}(\rho)$ which completes the proof.

#### *5.5.1.1 Step 1: continuity of $\theta^*$*

**Theorem 10.** *The map $\theta^*$ is continuous.*

*Proof.* Define the map $V^* : \mathcal{P} \mapsto \mathcal{V}$ that takes $\rho$ to $\hat{V}_\rho(\cdot)$. From Corollary 6,

$$\begin{aligned}
0 <& |\hat{\theta}_{\rho_1}(q) - \hat{\theta}_{\rho_2}(q)| \\
=& |\beta[\mathbf{E}_A(\hat{V}_{\rho_1}(q+A) - \hat{V}_{\rho_1}((q-1)^+ + A) - \hat{V}_{\rho_2}(q+A) + \hat{V}_{\rho_2}((q-1)^+ + A))]| && (5.24)\\
\leq& \beta\mathbf{E}_A|\hat{V}_{\rho_1}(q+A) - \hat{V}_{\rho_2}(q+A)| + \beta\mathbf{E}_A|\hat{V}_{\rho_1}((q-1)^+ + A) - \hat{V}_{\rho_2}((q-1)^+ + A)| && (5.25)\\
\leq& \beta\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\|_w \mathbf{E}_A(w(q+A) + w((q-1)^+ + A)) && (5.26)\\
\leq& K\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\|_w w(q) && (5.27)
\end{aligned}$$

for some large $K$ independent of $q$. The last inequality follows from the fact that the random variable $A$ has bounded support. Hence, $\|\theta^*_{\rho_1} - \theta^*_{\rho_2}\|_w \leq K\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\|_w$ and continuity of the map $V^*$ implies the continuity of the map $\theta^*$.

For any $\rho \in \mathcal{P}$ and $f_1, f_2 \in \mathcal{V}$, from (5.15), we have

$$\begin{aligned}
|T_\rho f_1(q) - T_\rho f_2(q)| \leq& \beta|\mathbf{E}_A(f_1(q+A) - f_2(q+A))| \\
&+ \left| \int_0^{\beta\Delta f_1(q)} \rho^{M-1}(u)\, du - \int_0^{\Delta f_2(q)} \rho^{M-1}(u)\, du \right| \\
\leq& \beta\|f_1 - f_2\|K_1 w(q) + \left| \int_{\beta\Delta f_2(q)}^{\beta\Delta f_1(q)} |\rho^{M-1}(u)|\, du \right| \\
\leq& \beta\|f_1 - f_2\|K_1 w(q) + \beta|\Delta f_1(q) - \Delta f_2(q)| \\
\leq& \beta(K_1 + K_2)\|f_1 - f_2\|w(q)
\end{aligned}$$

Therefore,

$$\|T_\rho f_1 - T_\rho f_2\|_w \leq \hat{K}\|f_1 - f_2\|_w \Rightarrow \quad \text{(A)} \tag{5.28}$$

for some large $\hat{K}$, independent of $\rho$.

Now, let $T_{\rho_1}$ and $T_{\rho_2}$ be the Bellman operators corresponding to $\rho_1$ and $\rho_2$. We will bound $|T_{\rho_1} f - T_{\rho_2} f|$. From (5.4), we have

$$
\begin{aligned}
|p_{\rho_1}(x) - p_{\rho_2}(x)| =& |\rho_1^{M-1}(x) - \rho_2^{M-1}(x)| \\
=& |\rho_1^{M-1}(x) - \rho_2(x)\rho_1^{M-2}(x) + \rho_2(x)\rho_1^{M-2}(x) - \rho_2^{M-1}(x)| \\
\leq& |\rho_1(x) - \rho_2(x)| + |\rho_1^{M-2}(x) - \rho_2^{M-2}(x)| \quad \text{(since } \rho_1(x) \leq 1\text{)}
\end{aligned}
$$

Hence by induction, $|p_{\rho_1}(x) - p_{\rho_2}(x)| \leq (M-1)|\rho_1(x) - \rho_2(x)| \leq (M-1)\|\rho_1 - \rho_2\|$. Also, from (5.5)

$$
|r_{\rho_1}(x) - r_{\rho_2}(x)| \leq x|p_{\rho_1}(x) - p_{\rho_2}(x)| + \int_0^x |p_{\rho_1}(u) - p_{\rho_2}(u)| du \leq 2x(M-1)\|\rho_1 - \rho_2\|
$$

Now, using the definition of $T_\rho$ from 5.15,

$$
\begin{aligned}
|T_{\rho_1} f(q) - T_{\rho_2} f(q)| &= |\int^{\beta \Delta f(q)} p_{\rho_1}(u) du - \int^{\beta \Delta f(q)} p_{\rho_2}(u) du| && (5.29) \\
&\leq 2(M-1)\Delta f(q) ||\rho_1 - \rho_2|| \\
&\leq 2(M-1) K_1 \|f\|_w w(q) \|\rho_1 - \rho_2\|, \Rightarrow \text{ (B)} && (5.30)
\end{aligned}
$$

where the last statement is due to the fact that $f \in \mathcal{V}$.

Now, let $j$ be such that $T_{\rho_1}^j$ is a $\alpha$-contraction.

$$
\begin{aligned}
\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\|_w =& \|T_{\rho_1}^j \hat{V}_{\rho_1} - T_{\rho_2}^j \hat{V}_{\rho_2}\|_w \\
\leq& \|T_{\rho_1}^j \hat{V}_{\rho_1} - T_{\rho_1}^j \hat{V}_{\rho_2}\|_w + \|T_{\rho_1}^j \hat{V}_{\rho_2} - T_{\rho_2}^j \hat{V}_{\rho_2}\|_w \\
\implies (1-\alpha)\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\|_w \leq& \|T_{\rho_1}^j \hat{V}_{\rho_2} - T_{\rho_2}^j \hat{V}_{\rho_2}\|_w \qquad (5.31)
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
\|T_{\rho_1}^j \hat{V}_{\rho_2} - T_{\rho_2}^j \hat{V}_{\rho_2}\|_w \leq& \|T_{\rho_1}^j \hat{V}_{\rho_2} - T_{\rho_1}^{j-1} T_{\rho_2} \hat{V}_{\rho_2}\|_w + \|T_{\rho_1}^{j-1} T_{\rho_2} \hat{V}_{\rho_2} - T_{\rho_1}^{j-2} T_{\rho_2}^2 \hat{V}_{\rho_2}\|_w \\
&+ \cdots + \|T_{\rho_1} T_{\rho_2}^{j-1} \hat{V}_{\rho_2} - T_{\rho_2}^j \hat{V}_{\rho_2}\|_w
\end{aligned}
$$

$$\leq \hat{K}^{j-1}\|T_{\rho_1}\hat{V}_{\rho_2} - T_{\rho_2}\hat{V}_{\rho_2}\|_w + \cdots + \|T_{\rho_1}T_{\rho_2}^{j-1}\hat{V}_{\rho_2} - T_{\rho_2}^{j}\hat{V}_{\rho_2}\|_w \quad (5.32)$$

$$\leq (\hat{K}^{j-1} + \cdots + 1)\|T_{\rho_1}\hat{V}_{\rho_2} - T_{\rho_2}\hat{V}_{\rho_2}\|_w \quad (5.33)$$

$$\leq 2(m-1)K\|\rho_1 - \rho_2\|(\hat{K}^{j-1} + \cdots + 1)\|\hat{V}_{\rho_2}\|_w \quad (5.34)$$

Here (5.32) and (5.34) are due to (B) and (A) respectively. Now, from (5.31) and (5.34), we get

$$\begin{aligned}\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\|_w \leq & \frac{2(m-1)K(\hat{K}^{j-1} + \cdots + 1)}{1-\alpha}\|\rho_1 - \rho_2\|\|\hat{V}_{\rho_2}\|_w & (5.35)\\ \leq & \frac{2(m-1)K(\hat{K}^{j-1} + \cdots + 1)}{1-\alpha}\|\rho_1 - \rho_2\|(\|\hat{V}_{\rho_1}\|_w + \|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\|_w) & (5.36)\\ & & (5.37)\end{aligned}$$

Therefore, if $\frac{2(m-1)K(\hat{K}^{j-1}+\cdots+1)}{1-\alpha}\|\rho_1 - \rho_2\| < \frac{1}{2}$, then

$$\|\hat{V}_{\rho_1} - \hat{V}_{\rho_2}\|_w \leq \frac{4(m-1)K(\hat{K}^{j-1} + \cdots + 1)}{1-\alpha}\|\hat{V}_{\rho_1}\|_w\|\rho_1 - \rho_2\| \quad (5.38)$$

Hence, the map $\hat{V}$ and $\hat{\theta}$ are continuous. □

#### *5.5.1.2 Step 2: continuity of the map* $\hat{\Pi}$

Recall that $\hat{\Pi}$ takes $\rho \in \mathcal{P}$ to probability measure $\Pi_\rho(.) = \Pi_{\rho,\hat{\theta}_\rho}(.)$. First we show that $\Pi_\rho(.) \in \Omega$, where $\Omega$, as defined before, is the space of absolutely continuous (with respect to Lebesgue measure) measures on $\mathbb{R}^+$.

**Lemma 13.** *For any $\rho \in \mathcal{P}$ and any $\theta \in \Theta$, $\Pi_{\rho,\theta}(\cdot)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+$.*

*Proof.* $\Pi_{\rho,\theta}(\cdot)$ is the invariant queue-length distribution of the dynamics

$$q \to \begin{cases} q + A & \text{with probability } \beta p_\rho(\theta(q)) \\ (q-1)^+ + A & \text{with probability } \beta(1 - p_\rho(\theta(q)) \\ R & \text{with probability } (1-\beta), \end{cases} \tag{5.39}$$

where, $A \sim \Phi_A$ and $R \sim \Psi_R$. This is the same as the dynamics

$$q \to \begin{cases} q' + A & \text{with probability } \beta \\ R & \text{with probability } (1-\beta), \end{cases}$$

where $q'$ is a random variable with distribution generated by the conditional probabilities

$$\begin{aligned} p(q' = q|q) =& p_\rho(\theta(q)) \\ p(q' = (q-1)^+|q) =& 1 - p_\rho(\theta(q)) \end{aligned}$$

Let $\Pi'$ be the distribution of $q'$. Then for any Borel set $B$,

$$\begin{aligned} \Pi_{\rho,\theta}(B) =& \beta(\Phi_A * \Pi')(B) + (1-\beta)\Psi_R(B) \\ =& \beta \int_{-\infty}^{\infty} \Phi_A(B-y) d\Pi'(y) + (1-\beta)\Psi_R(B) \end{aligned} \tag{5.40}$$

If $B$ is a Lebesgue null-set, then so is $B - y$ $\forall y$. So, $\Phi_A(B-y) = 0$ and $\Psi_R(B) = 0$ and therefore $\pi(B) = 0$. $\square$

We now develop a useful characterization of $\Pi_{\rho,\theta}$. Let

$$\Upsilon^{(k)}_{\rho,\theta}(B|q) = \mathbf{Pr}(Q_k \in B|\text{no regeneration }, Q_0 = q)$$

be the distribution of queue length $Q_k$ at time $k$ induced by the transition probabilities (5.10) conditioned on the event that $Q_0 = q$ and that there are no regenerations until time

$k$. We can now express the invariant distribution $\Pi_{\rho,\theta}(\cdot)$ in terms of $\Upsilon^{(k)}_{\rho,\theta}(\cdot|q)$ as in the following lemma.

**Lemma 14.** *For any bid distribution $\rho \in \mathcal{P}$ and for any stationary policy $\theta \in \Theta$, the Markov chain described by the transition probabilities in eq.* (5.10) *has a unique invariant distribution $\Pi_{\rho,\theta}(\cdot)$ given by,*

$$\Pi_{\rho,\theta}(B) = \sum_{k\geq 0}(1-\beta)\beta^k \mathbf{E}_{\Psi_R}(\Upsilon^{(k)}_{\rho,\theta}(B|Q)), \tag{5.41}$$

*where $\mathbf{E}_{\Psi_R}(\Upsilon^{(k)}_{\rho,\theta}(B|Q)) = \int \Upsilon^{(k)}_{\rho,\theta}(B|q)d\Psi(q)$.*

*Proof.* For brevity, denote $\Pi_{\rho,\theta}(\cdot)$ be $\Pi(\cdot)$ and $\Upsilon^{(k)}_{\rho,\theta} = \Upsilon^{(k)}$ . Let $-\tau$ be the last time before 0 the chain regenerated. We have

$$\Pi(B) = \sum_{k=0}^{\infty} \mathbf{Pr}(B, \tau = k) \tag{5.42}$$

$$= \sum_{k=0}^{\infty} \mathbf{Pr}(\tau = k)\mathbf{Pr}(B|\tau = k) \tag{5.43}$$

Since the regeneration events are independent of the queue-length and occur geometrically with probability $(1-\beta)$, $\mathbf{Pr}(\tau = k) = (1-\beta)\beta^k$. Hence,

$$\Pi(B) = \sum_{k=0}^{\infty}(1-\beta)\beta^k \mathbf{Pr}(Q_0 \in B|\tau = k) \tag{5.44}$$

$$= \sum_{k=0}^{\infty}(1-\beta)\beta^k \mathbf{E}(\mathbf{E}(\mathbf{1}_{Q_0\in B}|\tau = k, Q_{-k} = Q)|\tau = k) \tag{5.45}$$

$$= \sum_{k=0}^{\infty}(1-\beta)\beta^k \mathbf{E}(\Upsilon^{(k)}(B|Q)|\tau = k) \tag{5.46}$$

$$= \sum_{k=0}^{\infty}(1-\beta)\beta^k \mathbf{E}_{\Psi_R}(\Upsilon^{(k)}(B|Q)). \tag{5.47}$$

since $Q_{-k} \sim \Psi_R$ given $\tau = k$. □

We shall now prove the continuity of $\hat{\Pi}$ in $\rho$. Let $\Upsilon^{(k)}_{\rho} = \Upsilon^{(k)}_{\rho,\hat{\theta}_\rho}$.

**Theorem 11.** *The mapping $\hat{\Pi} : \mathcal{P} \mapsto \Omega$ is continuous.*

*Proof.* To prove continuity of the mapping $\hat{\Pi}$, we just need to show that for any sequence $\rho_n \to \rho$ in $w$-norm and for any open set $B$, $\liminf_{n\to\infty} \Pi_{\rho_n}(B) \geq \Pi_\rho(B)$. By Fatou's lemma,

$$\begin{aligned}\liminf_{n\to\infty} \Pi_{\rho_n}(B) &= \liminf_{n\to\infty} \sum_{k=0}^{\infty}(1-\beta)\beta^k \mathbf{E}_{\Psi_R}[\Upsilon^{(k)}_{\rho_n}(B|Q)) \\ &\geq \sum_{k=0}^{\infty}(1-\beta)\beta^k \mathbf{E}_{\Psi_R}[\liminf_{n\to\infty} \Upsilon^{(k)}_{\rho_n}(B|Q)]\end{aligned} \tag{5.48}$$

where $Q \sim \Psi_R$.

Recursively, define functions $\Upsilon^{(0)}_{B,\rho}(q) = \mathbf{1}_{(q\in B)}$ and $\Upsilon^{(k)}_{B,\rho}(q) = \mathbf{E}[\Upsilon^{(k-1)}_{B,\rho}(Q')|q]$, where

$$\mathbf{Pr}_\rho(Q' \in C|q) = p_\rho(\hat{\theta}_\rho(q))\Phi_A(C-(q-1)^+) + (1-p_\rho(\hat{\theta}_\rho(q)))\Phi_A(C-q). \tag{5.49}$$

Using backward equations, it is easy to see that $\mathbf{E}_{\Psi_R}[\Upsilon^{(k)}_{\rho}(B|Q)] = \mathbf{E}_{\Psi_R}[\Upsilon^{(k)}_{B,\rho}(Q)]$, where $Q \sim \Psi_R$.

We now prove that $\liminf_{n\to\infty} \Upsilon^{(k)}_{B,\rho_n}(q) \geq \Upsilon^{(k)}_{B,\rho}(q)$ for every $q \in \mathbb{R}^+$. In fact we prove a stronger result: if $q_n \to q$ is any converging sequence, then $\liminf_{n\to\infty} \Upsilon^{(k)}_{B,\rho_n}(q_n) \geq \Upsilon^{(k)}_{B,\rho}(q)$ for every $k$.

We show the above result by mathematical induction on $k$. For $k = 0$, we have $\Upsilon^{(0)}_{B,\rho_n}(q_n) = \mathbf{1}_{(q_n\in B)}$ and, one can easily check that for any open set $B$, $\liminf_{n\to\infty} \mathbf{1}_{(q_n\in B)} \geq \mathbf{1}_{(q\in B)}$. Hence, our hypothesis holds true for $k = 0$. Suppose that the hypothesis is true till $k = m-1$. To prove the lemma, we just need to verify that the hypothesis holds for $k = m$. Verify that $\mathbf{Pr}_{q_n,\rho_n}(\cdot) \implies \mathbf{Pr}_{q,\rho}(\cdot)$ by considering the integrals of a bounded continuous function. Then, by Skorokhod representation theorem, there exists $X_n$ and $X$ on common probability space such that $X_n \sim \mathbf{Pr}_{q_n,\rho_n}$, $X \sim \mathbf{Pr}_{q,\rho}$ and $X_n \to X$ a.s. We have,

$$\liminf \Upsilon^{(m)}_{B,\rho_n}(q_n) = \liminf \mathbf{E}(\Upsilon^{(m-1)}_{B,\rho_n}(X_n)) \tag{5.50}$$

$$\geq \mathbf{E}(\liminf \Upsilon^{(m-1)}_{B,\rho_n}(X_n)) \qquad \text{(by Fatou's lemma)} \tag{5.51}$$

$$\geq \mathbf{E}(\Upsilon^{(m-1)}_{B,\rho}(X)) \qquad \text{(by induction hypothesis)} \tag{5.52}$$

$$= \Upsilon^{(m)}_{B,\rho}(q) \tag{5.53}$$

which completes the proof. □

*5.5.1.3 Step 3: continuity of the mapping $\mathcal{F}$*

Now, using the results from Step 1 and Step 2, we establish continuity of the mapping $\mathcal{F}$. First we show that $\mathcal{F}(\rho) \in \mathcal{P}$.

**Lemma 15.** *For any $\rho \in \mathcal{P}$, let $\hat{\rho}(x) = (\mathcal{F}(\rho))(x) = \Pi_\rho(\hat{\theta}_\rho^{-1}([0,x])), x \in \mathbb{R}^+$. Then, $\hat{\rho} \in \mathcal{P}$.*

*Proof.* From the definition of $\Pi_\rho$, it is easy to note that $\hat{\rho}$ is a distribution function. Since $\hat{\theta}_\rho$ is continuous and strictly increasing function as shown in Lemma 12, $\hat{\theta}_\rho^{-1}(\{x\})$ is either empty or a singleton. Then, from Lemma 13, we get that $\Pi_\rho(\hat{\theta}_\rho^{-1}(\{x\})) = 0$. Together, we get that $\hat{\rho}(x)$ has no jumps at any $x$ and hence it is continuous.

To complete the proof, we need to show that the expected bid under the cumulative distribution function $\hat{\rho}$ is bounded from above by a constant that is independent of $\hat{\rho}$. To that end, define a new Markov random process $\tilde{Q}_k$ with the probability transition matrix

$$\mathbf{Pr}(\tilde{Q}_{k+1} \in B | \tilde{Q}_k = q) = \beta \mathbf{1}_{(q+\bar{A} \in B)} + (1-\beta)\Psi_R(B) \tag{5.54}$$

where $\bar{A}$ is the maximum possible arrival between any two consecutive auction instants. The process $\tilde{Q}_k$ has an invariant distribution which is given by,

$$\tilde{\Pi}(B) = \sum_{k=0}^{\infty} (1-\beta)\beta^k \mathbf{E}_{\Psi_R}(\mathbf{1}_{(q+k\hat{A}) \in B}). \tag{5.55}$$

The proof of the above result is identical to that of Lemma 14. For any $q$ given, the above probability measure (5.54) stochastically bounds the probability measure in eq. (5.10), Therefore, it can be shown that $\tilde{\Pi}$ stochastically dominates $\Pi_\rho$ for all $\rho \in \mathcal{P}$, i.e, $\Pi_\rho \preccurlyeq \tilde{\Pi}$.

Now, the expected value of the optimal bid function $\hat{\theta}_\rho(q)$ under $\Pi_\rho$ satisfies,

$$\mathbf{E}_{\Pi_\rho}[\hat{\theta}_\rho(q)] \leq \mathbf{E}_{\tilde{\Pi}}[\hat{\theta}_\rho(q)] \tag{5.56}$$

$$\leq \mathbf{E}_{\tilde{\Pi}}[\hat{V}_\rho(q+\bar{A})] \tag{5.57}$$

$$\leq \sum_{k=0}^{\infty}(1-\beta)\beta^k \mathbf{E}_{\Psi_R}(\hat{V}_\rho(q+(k+1)\bar{A})) \tag{5.58}$$

Above, the first inequality follows from stochastic dominance of $\tilde{\Pi}$ and the second inequality is due to the definition of optimal bid function.

From (5.12), we can observe that for any $\rho$, $\hat{V}_\rho(q) \leq \sum_{k=0}^{\infty} \beta^k C(q+k\bar{A})$ independent of $\rho$. Since $C(q) \in O(q^m)$ for some $m$, we have $\hat{V}_\rho(q) \in O(q^m)$. Then, $\mathbf{E}_{\Psi_R}(\hat{V}_\rho(q+(k+1)\hat{A})) \in O(k^m)$ as the moments of $\Psi_R$ are bounded. This directly gives that $\mathbf{E}_{\Pi_\rho}[\hat{\theta}_\rho(q)]$ is bounded by the some constant that is independent of $\rho$ and, hence independent of $\hat{\rho}$. This completes the proof. □

Now, we have the main theorem showing continuity of the map $\mathcal{F}$.

**Theorem 12.** *The mapping $\mathcal{F} : \mathcal{P} \mapsto \mathcal{P}$ given by $(\mathcal{F}(\rho))(x) = \Pi_\rho(\hat{\theta}_\rho^{-1}([0,x]))$ is continuous.*

*Proof.* Let $\rho_n \to \rho$ in uniform norm. From previous steps, we have $\hat{\theta}_{\rho_n} \to \hat{\theta}_\rho$ in $w$-norm and $\Pi_{\rho_n} \Rightarrow \Pi_\rho$. Then, using Theorem 5.5 of Billingsley [8], one can show that

$$\Pi_{\rho_n}(\hat{\theta}_{\rho_n}^{-1}(B)) \Rightarrow \Pi_\rho(\hat{\theta}_\rho^{-1}(B)),$$

for any Borel set $B$. Then, $\mathcal{F}(\rho_n)$ converges point-wise to $\mathcal{F}(\rho)$ as it is continuous at every $x$, i.e., $(\mathcal{F}(\rho_n))(x) \to (\mathcal{F}(\rho_n))(x)$ for all $x \in \mathbb{R}^+$.

Now, we complete the proof by showing that in the norm space $\mathcal{P}$, point wise convergence implies convergence in uniform norm. Let $\rho_n, \rho \in \mathcal{P}$ and $F_n \to F$ point-wise. Given $\epsilon > 0$, choose $L$ large enough so that $\rho(L) > 1-\epsilon$. Since $\rho$ is continuous function by definition, it is uniformly continuous on the compact set $[0, L]$. Therefore, we can construct a sequence $0 = x_1 < x_2 < \cdots < x_k = L$ such that and $\rho(x_{i+1}) - \rho(x_i) < \epsilon$. Let $J$ be large enough so that for all $n > J$, $|\rho(x_i) - \rho_n(x_i)| < \epsilon$ for all $i$. For any $y$ such that $x_i < y < x_{i+1}$,

$$|\rho(y) - \rho_n(y)| < \rho(y) - \rho(x_i) + |\rho(x_i) - \rho_n(x_i)| + |\rho_n(y) - \rho_n(x_i)| \tag{5.59}$$

$$< |\rho(x_{i+1}) - \rho(x_i)| + |\theta(x_i) - \rho_n(x_i)| + |\rho_n(x_{i+1}) - \rho_n(x_i)| \tag{5.60}$$

$$< 2|(\rho(x_{i+1}) - \rho(x_i))| + |\rho(x_i) - \rho_n(x_i)| + 2\epsilon \tag{5.61}$$

$$< 5\epsilon \tag{5.62}$$

While if $L < y$, then

$$|\rho(y) - \rho_n(y)| < |\rho_n(y) - \rho_n(L) + |\rho_n(L) - \rho(L)| + |\rho(y) - \rho(L)| \tag{5.63}$$

$$< 1 - \rho(L) + \epsilon + \epsilon + 1 - \rho(L) \tag{5.64}$$

$$< 4\epsilon. \tag{5.65}$$

Therefore, $|\rho(y) - \rho_n(y)| < 5\epsilon$ for all $n > J$ and hence $\rho_n$ converges to $\rho$ uniformly. This completes the proof. □

### *5.5.2 $\mathcal{F}(\mathcal{P})$ contained in a compact subset of $\mathcal{P}$*

We show that the closure of the image of the mapping $\mathcal{F}$, denoted by $\overline{\mathcal{F}(\mathcal{P})}$, is compact and is contained in $\mathcal{P}$. As $\mathcal{P}$ is a normed space, sequential compactness of any subset of $\mathcal{P}$ implies that the subset is compact. Henceforth, we just need to show that $\overline{\mathcal{F}(\mathcal{P})}$ is sequentially compact. Sequential compactness of a set $\overline{\mathcal{F}(\mathcal{P})}$ means the following: if $\{\rho_n\} \in \overline{\mathcal{F}(\mathcal{P})}$ is a sequence, then there exists a subsequence $\{\rho_{n_j}\}$ and $\rho \in \overline{\mathcal{F}(\mathcal{P})}$ such that $\rho_{n_j} \to f$. We use Arzela-Ascoli theorem and uniform tightness of the measures in $\mathcal{F}(\mathcal{P})$ to show the sequential compactness. The version of Arzela-Ascoli theorem that we will use is stated below:

**Theorem 13** (Arzela-Ascoli Theorem)**.** *Let $X$ be a $\sigma$-compact metric space. Let $\mathcal{G}$ be a* *family of continuous real valued functions on $X$. Then the following two statements are equivalent:*

1. *Every sequence $\{g_n\} \subset \mathcal{G}$ there exists a subsequence $g_{n_j}$ which converges uniformly on every compact subset of $X$.*

2. *The family $\mathcal{G}$ is equicontinuous on every compact subset of $X$ and for any $x \in X$, there is a constant $C_x$ such that $|g(x)| < C_x$ for all $g \in \mathcal{G}$.*

Say the family of functions $\mathcal{F}(\mathcal{P})$ satisfies the conditions of Arzela-Ascoli theorem. Also, let they satisfy the uniform tightness property, i.e, $\forall f \in \mathcal{F}(\mathcal{P})$, there exists an $x_\epsilon$ such that $1 \geq f(x_\epsilon) > 1 - \epsilon$. Then, for any sequence $\{\rho_n\} \subset \mathcal{F}(\mathcal{P})$, there exists a subsequence $\{\rho_{n_j}\}$ that converges uniformly on every compact sets to a continuous increasing function $\rho$. As these functions are uniformly tight, uniform convergence on compact sets imply uniform convergence. i.e. $\rho_{n_j} \to \rho$. Therefore, $\mathcal{F}(\mathcal{P})$ is totally bounded and hence so is its closure.

Finally, we have to show that $\overline{\mathcal{F}(\mathcal{P})} \subset \mathcal{P}$. From the tightness property, the limit function $\rho$ satisfies that $1 \geq \rho(x_\epsilon) \geq (1 - \epsilon)$ and therefore $\rho(\infty) = 1$. Also, we have

$$\int (1 - \rho(x))dx \leq \liminf_{n_j \to \infty} \int (1 - \rho_{n_j}(x))dx < \infty. \tag{5.66}$$

The first inequality is due to Fatou's lemma. And, the second inequality holds since $\{\rho_{n_j}\} \in \mathcal{P}$. Therefore $\rho \in \mathcal{P}$ and hence $\overline{\mathcal{F}(\mathcal{P})} \subset \mathcal{P}$.

Now we just need to verify $\mathcal{F}(\mathcal{P})$ satisfies the conditions of Arzela-Ascoli theorem and tightness property. First we verify the conditions of the Arzela-Ascoli theorem. Note that the functions in consideration are uniformly bounded by 1. To prove equicontinuity, consider an $\hat{\rho} = \mathcal{F}(\rho)$ and let $x > y$.

$$\hat{\rho}(x) - \hat{\rho}(y) = \Pi_\rho(\theta_\rho(q) \leq x) - \Pi_\rho(\theta_\rho(q) \leq y) = \Pi_\rho(y < \theta_\rho(q) \leq x) \tag{5.67}$$

**Lemma 16.** *For any interval $[a, b]$, $\Pi_\rho([a, b]) < c \cdot (b - a)$, for some large enough $c$.*

*Proof.* We know that $\Pi([a,b]|\rho,\theta) = \sum_{k\geq 0}(1-\beta)\beta^k \mathbf{E}_{\Psi_R}(\Upsilon_\rho^{(k)}([a.b]|Q_0))$. Let $A_k$ be the net arrivals and $D_k$ be the net departures till time $k$. Then,

$$\Upsilon_\rho^{(k)}([a,b]|Q_0) = \mathbf{E}(\mathbf{1}_{(Q_0+A_k-D_k\in[a,b])}|Q_0) \tag{5.68}$$

$$= \mathbf{E}(\mathbf{E}(\mathbf{1}_{(Q_0+A_k-D_k\in[a,b])}|D_k,Q_0)|Q_0) \tag{5.69}$$

$$= \mathbf{E}(\mathbf{E}(\mathbf{1}_{(A_k\in[a-Q_0+D_k,b-Q_0+D_k])}|Q_0,D_k)|Q_0) \tag{5.70}$$

$$\leq c_1\cdot(b-a). \tag{5.71}$$

Above results holds since the random variable $A_k$ is independent of $Q_0$ and $D_k$ for any $k$ and it has a bounded density function. Therefore, $\mathbf{E}_{\Psi_R}(\Upsilon_\rho^{(k)}([a.b]|Q_0)) \leq c.(b-a)$ for all $k>0$. For $k=0$, we know that $\Psi_R$ has a bounded density which implies $\Psi_R([a,b]) \leq c_1\psi\cdot(b-a)$. These two results prove that there is a large enough $c$ such that $\Pi_\rho([a,b]) < c\cdot(b-a)$. $\square$

The above lemma and equation eq. (5.67) imply that $\hat\rho(x)-\hat\rho(y) \leq c(\theta_\rho^{-1}(x)-\theta_\rho^{-1}(y))$. To show equicontinuity, it is enough to show that $\limsup_{y\uparrow x}\frac{\hat\rho(x)-\hat\rho(y)}{x-y} \leq K(x)$ for some $K$ independent of $\hat\rho$. We have

$$\limsup_{y\uparrow x}\frac{\hat\rho(x)-\hat\rho(y)}{x-y} \leq c\limsup_{y\uparrow x}\frac{\theta_\rho^{-1}(x)-\theta_\rho^{-1}(y)}{x-y} \tag{5.72}$$

$$= c\limsup_{y\uparrow x}\frac{\theta_\rho^{-1}(x)-\theta_\rho^{-1}(y)}{\theta_\rho\theta_\rho^{-1}(x)-\theta_\rho\theta_\rho^{-1}(y)} \tag{5.73}$$

$$\leq c\limsup_{y'\to x'}\frac{x'-y'}{\theta_\rho(x')-\theta_\rho(y')} \qquad (x'=\theta_\rho^{-1}(x)) \tag{5.74}$$

$$\leq c\limsup_{y'\to x'}\frac{x'-y'}{\beta(\Delta C(x')-\Delta C(y'))} \tag{5.75}$$

$$\leq c\frac{1}{H(x')} \tag{5.76}$$

where eq. (5.74) due to strict monotonicity of $\theta_\rho$ and where

$$0 < H(x') = \begin{cases} \mathbf{E}_A[C'(x'+A) - C'(\overline{x-1}+A)] & x'>1 \\ \mathbf{E}_A[C'(x'+A)] & x'\leq 1 \end{cases}$$

and $C'(x) = \frac{dC(x)}{dx}$.

Now, we show uniform tightness property of $\mathcal{F}(\mathcal{P})$. We have already shown that $F(\mathcal{P}) \subset \mathcal{P}$. Hence, the expected value of the bids distributed according to the functions in $F(\mathcal{P})$ are uniformly bounded. Now, using Markov inequality, it can be shown that functions in consideration are uniformly tight.

## 5.6 Conclusion

We studied an auction theoretic scheduling mechanism for use in cellular networks where the base station allocates resources to mobile applications via repeatedly conducting second price auctions and serving the winner at each time with one unit of service. Here, we have a dynamic game in which each app play against his opponents by choosing a bidding strategy so as to minimize his expected cost. We established that the game has a MFE (of strategies) that closely approximates its Bayesian equilibrium. Also, we have shown that the equilibrium bidding strategy of each player is montone in their queue length. It implies that at each time the service is awarded to the longest of all queues - a policy that resembles LQF. Hence, we propose that auction theoretic scheduling mechanism can be used as an alternative to LQF policy when the queue lengths not known at the scheduler.

## 5.7 Supplemental: Technical lemma

**Lemma 17.** *Define* $g(b, v) = r(b) - p(b)v$. *Then* $v \in \arg\min_{b \in \mathbb{R}^+} g(b, v)$.

*Proof.* If $v \leq 0$, then $-p(b|\rho)v \geq 0$ and $g(b, v)$ is increasing in $b$. Hence the minimum occurs at $b = 0$. If $v > 0$, then consider

$$\begin{aligned} g(b, v) - g(v, v) =& bp(b) - \int_0^b p(u)du - vp(b) + \int_0^v p(u)du \\ =& (b - v)p(b) + \int_b^v p(u)du \\ =& \int_b^v p(u) - p(b)du \\ \geq& 0 \end{aligned}$$

with equality at $b = v$. Hence we have the desired result $\square$

# 6. CONCLUSION

In this thesis, we studied several instances of coordination problems in communication networks using game theoretical tools. A summary can be found in Table 6.1. The scenarios we considered involve interaction among a group of agents who compete for network resources to attain their own private interests. For example, the agents may selfishly choose congestion controllers to maximize their payoff that may be a function of the throughput and the delay incurred in the network, or strategically choose routes that to yield the minimum transmission cost. In most of such scenarios, selfish interactions among the agents lead to chaos or to bad equilibrium states. We characterized the *price of selfishness*; and devised mechanisms or rules that encourage cooperative behavior among these agents.

There are several extensions possible to the work presented here. The incentive design problem for P2P systems can be further explored to consider heterogeneous classes of users with varying degree of sensitivity to delay and price. Also, the current work can be extended to study incentive schemes for streaming contents. In future, we may study the protocol selection problem considering a larger set of protocol choices that contains UDP, RTCP etc. along with TCP and its variations. Also, we propose to further explore the idea of tolled, virtual networks. The benefits of auction based scheduling can be further investigated in the case of heterogeneous classes of applications.

One of the goals of the thesis is to create tractable analytical models of complex network systems. As far as possible, I have validated the accuracy of these models with real-time measured data. The final objective, both in my thesis and my future research, is to work towards an analytical approach to network design.

Table 6.1: A summary of coordination problems studied

| Coordination problems | OSI layer | Game structure | Incentive schemes |
|---|---|---|---|
| P2P incentive problem | Application layer | static game, action space, payoff structure are known | Booster incentives |
| Protocol game | Transport layer | static game, action space payoff structure are known | link tolls (delay) |
| Multipath routing game | Routing layer | repeated game, action space is known, payoffs are learned | rebate (link capacity) |
| Scheduling game | MAC layer | dynamic game, action space is known, payoffs are learned | second price auction |

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