GEOMETRY OF FEASIBLE SPACES OF TENSORS

A Dissertation

by

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ABSTRACT

Due to the exponential growth of the dimension of the space of tensors $V_1 \otimes \cdots \otimes V_n$, any naive method of representing these tensors is intractable on a computer. In practice, we consider feasible subspaces (subvarieties) which are defined to reduce the storage cost and the computational complexity. In this thesis, we study two such types of subvarieties: the third secant variety of the product of n projective spaces, and tensor network states.

For the third secant variety of the product of n projective spaces, we determine set-theoretic defining equations, and give an upper bound of the degrees of these equations.

For tensor network states, we answer a question of L. Grasedyck that arose in quantum information theory, showing that the limit of tensors in a space of tensor network states need not be a tensor network state. We also give geometric descriptions of spaces of tensor networks states corresponding to trees and loops.

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1. INTRODUCTION AND BACKGROUND

1.1 Motivation

Tensors are ubiquitous in mathematics and the sciences, and are especially important in algebraic statistics, biology, signal processing, and complexity theory [15, 16, 24, 29, 36, 38]. For example, in scientific computation the problem of determining the complexity of matrix multiplication can be viewed as decomposing a particular tensor (the matrix multiplication operator) according to its rank [28, 29]; in statistics, the problem of recovering the mixing matrix and source vector from the observation vector can be viewed as the symmetric tensor decomposition of the associated cumulants [25, 29, 37, 43]; in signal processing, CP decomposition, block term decomposition and other tensor decompositions are important [23, 27, 29]. In the study of tensors, the *rank* and *border rank* of a tensor are the standard measures of its complexity. Due to the geometric interpretations of rank and border rank, it is natural to study the secant varieties of Segre varieties since equations for these varieties produce tests for the border rank of a tensor. In practice, small secant varieties of Segre varieties play an important role as they correspond to tensors of low complexity. Another model defined to reduce the complexity of the spaces involved is tensor network states in quantum information theory. In this thesis we study both these models.

1.2 Equations for the secant varieties of Segre varieties

The study of equations for secant varieties of Segre varieties is a classical problem in algebraic geometry, but these equations are still far from being understood [29]. Before exploring the known results of these equations, let us review the basic definitions of rank, border rank and secant varieties of Segre varieties.

Definition 1. *A function* $f : A_1 \times \cdots \times A_n \to \mathbb{C}$ *is* multilinear *if it is linear in each factor* A_l*. The space of such multilinear functions is denoted by* $A_1^* \otimes \cdots \otimes A_n^*$ *and called the* tensor product *of the vector spaces* $A_1^*, \ldots, A_n^*$*. Elements* $T \in A_1^* \otimes \cdots \otimes A_n^*$ *are called tensors.*

Definition 2. *Given* $\beta \in A_1^*$*, ...,* $\beta_n \in A_n^*$*, define an element* $\beta_1 \otimes \cdots \otimes \beta_n \in A_1^* \otimes \cdots \otimes A_n^*$ *by* $\beta_1 \otimes \cdots \otimes \beta_n(u_1, \ldots, u_n) = \beta_1(u_1) \cdots \beta_n(u_n)$ *for any* $u_i \in A_i$*. An element of* $A_1^* \otimes \cdots \otimes A_n^*$ *is said to have* rank one *if it is of the form* $\beta_1 \otimes \cdots \otimes \beta_n$ *for some* $\beta_i \in A_i^*$*. The* rank *of a tensor* $T \in A_1 \otimes \cdots \otimes A_n$*, denoted by* $R(T)$*, is the minimum number* r *such that* $T = \sum_{u=1}^{r} Z_u$ *with each* Z_u *of rank one.*

Definition 3. *A tensor* T *has* border rank r*, denoted by* $\underline{R}(T)$*, if it is a limit of tensors of rank* r *but is not a limit of tensors of rank* s *for any* $s < r$*.*

Remark 1. *Note that* $R(T) \geq \underline{R}(T)$*. If* $T \in A_1 \otimes A_2$ *is a matrix, then* $R(T) = \underline{R}(T)$*. But this is not always true for* $T \in A_1 \otimes \cdots \otimes A_n$ *when* $n \geq 3$*. For example, let* $T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 \in A \otimes B \otimes C$*. One can check* T *has rank* 3*, but* $T = \lim_{t \to 0} \frac{1}{t}[(t-1)a_1 \otimes b_1 \otimes c_1 + (a_1 + ta_2) \otimes (b_1 + tb_2) \otimes (c_1 + tc_2)]$*, hence* $\underline{R}(T) = 2$*.*

Definition 4. *Define the* n*-factor Segre variety to be the image of the map*

$$Seg : \mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n \to \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$$

$$([v_1] \ldots, [v_n]) \mapsto [v_1 \otimes \cdots \otimes v_n]$$

Remark 2. $Seg(\mathbb{P}A_1 \times \mathbb{P}A_2)$ *is the set of rank one matrices, and* $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *is the set of rank one tensors.*

Definition 5. *The join of two varieties $Y, Z \in \mathbb{P}V$ is*

$$J(Y,Z) = \overline{\bigcup_{x\in Y, y\in Z, x\neq y} \mathbb{P}^1_{xy}},$$

where $\mathbb{P}^1_{xy}$ is the projective line through x and y.

Definition 6. *The join of k varieties $X_1, \ldots, X_k \subset \mathbb{P}V$ is defined by induction to be $J(X_1, \ldots, X_k) = J(X_1, J(X_2, \ldots, X_k))$, and the k-th secant variety of Y is defined to be the join of k copies of Y, $\sigma_k(Y) = J(Y, \ldots, Y)$.*

Remark 3. *$\sigma_k(Seg(\mathbb{P}A_1 \times \mathbb{P}A_2))$ is the set of matrices with rank at most k, and $\sigma_k(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is the set of tensors with border rank at most k.*

It is clear that the ideal of $Seg(\mathbb{P}A_1 \times \mathbb{P}A_2)$ is generated by all the 2×2 minors, denoted by $\wedge^2 A_1^* \otimes \wedge^2 A_2^*$, and the ideal of $\sigma_r(Seg(\mathbb{P}A_1 \times \mathbb{P}A_2))$ is generated by all the $(r+1) \times (r+1)$ minors, denoted by $\wedge^{r+1} A_1^* \otimes \wedge^{r+1} A_2^*$.

Given $W = A_1 \otimes \cdots \otimes A_n$, define a flattening $A_I \otimes A_J$ of W to be a decomposition $(A_{i_1} \otimes \cdots \otimes A_{i_p}) \otimes (A_{i_{p+1}} \otimes \cdots \otimes A_{i_n})$, where $I = \{i_1, \ldots, i_p\}$ and $J = \{i_{p+1}, \ldots, i_n\}$, $I \cup J = \{1, \ldots, n\}$, and $I \cap J = \varnothing$. Since $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ can be embedded in $Seg(\mathbb{P}A_I \times \mathbb{P}A_J)$, then $\wedge^2 A_I^* \otimes \wedge^2 A_J^*$ give equations for $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$. It turns out that $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ is ideal theoretically defined by all the 2×2 minors of flattenings, i.e. all $\wedge^2 A_I^* \otimes \wedge^2 A_J^*$ generate the ideal for $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$.

Since $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ can be embedded in $\sigma_r(Seg(\mathbb{P}A_I \times \mathbb{P}A_J))$, $\wedge^{r+1} A_I^* \otimes \wedge^{r+1} A_J^*$ give equations for $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$. When studying Bayesian networks, Garcia, Stillman and Sturmfels conjectured that all the 3×3 minors of flattenings give all the equations for $\sigma_2(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ [22]. Landsberg and Manivel showed the set theoretic version of this conjecture is true [30], and Raicu proved the ideal theoretic version is true [45]. For more history, see [3, 30, 35, 45].

It turns out that minors of flattenings are not enough to define higher secant varieties of Segre varieties. In 1983 Strassen discovered equations for $\sigma_3(Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \mathbb{P}A_3))$ beyond 4×4 minors of flattenings [48]. Landsberg and Manivel proved $\sigma_3(Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \mathbb{P}A_3))$ is set theoretically defined by Strassen's equations and 4×4 minors of flattenings [20, 31]. Landsberg and Weyman proved the ideal of $\sigma_3(Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \mathbb{P}A_3))$ is generated in degree 4 by the module which arises from Strassen's commutation condition [35].

For the fourth secant varieties of Segre varieties, Friedland showed $\sigma_4(Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \mathbb{P}A_3))$ is the zero set of certain equations of degree 5, 9 and 16 [20]. Bates and Oeding showed $\sigma_4(Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \mathbb{P}A_3))$ is the zero set of certain equations of degree 5, 6 and 9 by numerical methods [4]. Friedland and Gross gave this result a computer-free proof [21].

For higher secant varieties of Segre varieties, for example $\sigma_6(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, there are no equations known. On the other hand, there are some qualitative descriptions of equations of secant varieties of Segre varieties. Draisma and Kuttler proved that for arbitrary fixed r, there is an uniform bound $d(r)$ such that $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is set theoretically defined by equations of degree at most $d(r)$ for any n [17].

In this thesis, we determine set theoretic equations for the third secant variety of the Segre product of n projective spaces, and from the proof of this statement we derive an upper bound for the degrees of these equations. Given any partition $I \cup J \cup K = \{1, \ldots, n\}$, $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ can be embedded in $\sigma_3(Seg(\mathbb{P}A_I \times \mathbb{P}A_J \times \mathbb{P}A_K))$, thus Strassen's equations for all the partitions $I \cup J \cup K = \{1, \ldots, n\}$ and 4×4 minors for all the flattenings give us equations for $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$. Our main result is [44]:

Theorem 1. $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ *is set theoretically defined by Strassen's* *equations of all partitions* $I \cup J \cup K = \{1, \dots, n\}$ *and all* 4×4 *minors of flattenings.*

Corollary 1. $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ *is set theoretically defined by Strassen's equations of degree* 4 *for the partitions* $\{i\} \cup \{j\} \cup \{1, \dots, \widehat{i}, \cdots, \widehat{j}, \cdots, n\}$ *and all* 4×4 *minors of flattenings.*

1.3 Tensor network states

Tensor network states are interesting models in physics defined to reduce the complexity of the spaces involved. In physics, tensors describe states of quantum mechanical systems. If a system has n particles, its state is an element of $H_1 \otimes \cdots \otimes H_n$ with H_j Hilbert spaces. In numerical many-body physics, in particular solid state physics, one wants to simulate quantum states of thousands of particles, often arranged on a regular lattice (e.g., atoms in a crystal). Due to the exponential growth of the dimension of $H_1 \otimes \cdots \otimes H_n$ with n, any naive method of representing these tensors is intractable on a computer. Tensor network states were defined by restricting to a subset of tensors that is physically reasonable, in the sense that the corresponding spaces of tensors are only locally entangled because interactions (entanglement) in the physical world appear to just happen locally. These spaces are associated to graphs, i.e. for a fixed graph, we can associate complex vector spaces to each vertex and edge, and define a corresponding tensor network state. More precisely:

Let $V_1, \dots, V_n$ be complex vector spaces, let $\mathbf{v}_i = \dim V_i$. Let Γ be a graph with n vertices v_j, $1 \leq j \leq n$, and m edges e_s, $1 \leq s \leq m$, and let $\vec{e} = (e_1, \dots, e_m) \in \mathbb{N}^m$. Associate V_j to the vertex v_j and an auxiliary vector space E_s of dimension e_s to the edge e_s. Make Γ into a directed graph. (The choice of directions will not effect the end result.) Let $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$. For Γ, $s \in e(j)$ means e_s is incident to v_j, $s \in in(j)$ are the incoming edges and $s \in out(j)$ the outgoing edges.

Define a tensor network state $TNS(\Gamma, \vec{e}, \mathbf{V})$ to be:

$$TNS(\Gamma, \vec{e}, \mathbf{V}) := \{T \in \mathbf{V} \mid \exists T_j \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*), T = Con(T_1 \otimes \cdots \otimes T_n)\}, \tag{1.1}$$

where Con is the contraction of all the E_s's with all the E_s^*'s.

Such spaces have been studied since the 1980's, and go under different names: tensor network states, finitely correlated states (FCS), valencebond solids (VBS), matrix product states (MPS), projected entangled pairs states (PEPS), and multi-scale entanglement renormalization ansatz states (MERA), see, e.g., [14, 18, 19, 26, 46, 49] and the references therein. We will use the term tensor network states.

If Γ is a tree, then $TNS(\Gamma, \vec{e}, \mathbf{V})$ is closed [24]. Lars Grasedyck asked if every tensor network state is Zariski closed. In this thesis, we give a counterexample and show a tensor network state is not closed if the corresponding graph contains a cycle whose vertices have non-subcritical dimensions. We also give geometric descriptions of spaces of tensor networks states corresponding to trees and loops.

Grasedyck's question has a surprising connection to the area of Geometric Complexity Theory, in that the result is equivalent to the statement that the boundary of the Mulmuley-Sohoni type variety associated to matrix multiplication is strictly larger than the projections of matrix multiplication (and re-expressions of matrix multiplication and its projections after changes of bases). Tensor Network States are also related to graphical models in algebraic statistics [29].

2. PRELIMINARIES

2.1 Dimensions of secant varieties of Segre varieties

Terracini's lemma is a fundamental tool to compute the dimension of a join variety. Let Y, Z be projective varieties, and $\widehat{Y}, \widehat{Z}$ be the cones over Y, Z.

Lemma 1 (Terracini's lemma)**.** *Let $(v, w) \in \widehat{Y} \times \widehat{Z}$ be a general point, and $[u] = [v + w] \in J(Y, Z)$, then*

$$\widehat{T}_{[u]} J(Y, Z) = \widehat{T}_{[v]} Y + \widehat{T}_{[w]} Z,$$

where $\widehat{T}_{[v]} Y$ denotes the affine tangent space of Y at $[v]$.

Definition 7. *We call a variety $X \subset \mathbb{P}^n$* nondegenerate *if it spans $\mathbb{P}^n$, i.e. is not contained in any hyperplane. If $X \subset \mathbb{P}^n$ is an irreducible nondegenerate variety whose r-th secant variety $\sigma_r(X)$ has dimension strictly less than* $\min\{r \dim X + r - 1, n\}$, *we say that X is* defective, *and define the* defect $\delta_r(X) = r \dim X + r - 1 - \dim \sigma_r(X)$.

Here we list some known results on the dimensions of secant varieties of Segre varieties, for more results see [1, 8–11, 13].

Theorem 2 ([12])**.** *Consider $\sigma_r(Seg(\mathbb{P}^{a_1 - 1} \times \cdots \times \mathbb{P}^{a_n - 1}))$, and assume $a_n \geq \prod_{i=1}^{n-1} a_i - \sum_{i=1}^{n-1} a_i - n + 1$.*

1. *If $r \leq \prod_{i=1}^{n-1} a_i - \sum_{i=1}^{n-1} a_i - n + 1$, then $\sigma_r(Seg(\mathbb{P}^{a_1 - 1} \times \cdots \times \mathbb{P}^{a_n - 1}))$ has the expected dimension $r(a_1 + \cdots + a_n - n + 1) - 1$;*

2. *If $a_n > r \geq \prod_{i=1}^{n-1} a_i - \sum_{i=1}^{n-1} a_i - n + 1$, then $\sigma_r(Seg(\mathbb{P}^{a_1 - 1} \times \cdots \times \mathbb{P}^{a_n - 1}))$ has defect $\delta_r = r^2 - r(\prod_{i=1}^{n-1} a_i - \sum_{i=1}^{n-1} a_i - n + 1)$;*

3. *If $r \geq \min\{a_1, \ldots, a_n\}$, then $\sigma_r(Seg(\mathbb{P}^{a_1 - 1} \times \cdots \times \mathbb{P}^{a_n - 1})) = \mathbb{P}^{\prod_{i=1}^{n} a_i - 1}$.*

Theorem 3 ([13]). *The secant varieties of the Segre product of k copies of $\mathbb{P}^1$, $\sigma_r(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1))$, have the expected dimension except when $k = 2, 4$.*

2.2 Subspace varieties

Subspace varieties are important auxiliary varieties in the study of equations for secant varieties.

Definition 8. *The subspace variety $Sub_{b_1,\ldots,b_n}(A_1 \otimes \cdots \otimes A_n)$ is defined to be*

$$Sub_{b_1,\ldots,b_n}(A_1 \otimes \cdots \otimes A_n) := \mathbb{P}\{T \in A_1 \otimes \cdots \otimes A_n | \dim(T(A_j^*)) \leq b_j\}.$$

Proposition 1 ([29]). *The ideal of the subspace variety $Sub_{b_1,\ldots,b_n}(A_1 \otimes \cdots \otimes A_n)$ is generated in degrees $b_j + 1$ for $1 \leq j \leq n$ by the irreducible modules in $\wedge^{b_j+1} A_j^* \otimes \wedge^{b_j+1}(A_1^* \otimes \cdots \otimes A_{j-1}^* \otimes A_{j+1}^* \otimes \cdots \otimes A_n^*)$.*

The following *Kempf-Weyman desingularization* of $Sub_{b_1,\ldots,b_n}(A_1 \otimes \cdots \otimes A_n)$ is useful for finding equations, minimal free resolutions, and establishing properties of singularities [29, 50].

Proposition 2 ([50]). *Consider the product of Grassmannians*

$$B = G(b_1, A_1) \times \cdots \times G(b_n, A_n)$$

and the bundle

$$p : \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n \to B,$$

where $\mathcal{S}_j$ is the tautological rank b_j subspace bundle over $G(b_j, A_j)$. Assume that $b_1 \leq \cdots \leq b_n$. Then the total space $\tilde{Z}$ of $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ maps to $A_1 \otimes \cdots \otimes A_n$. The map $\tilde{Z} \to A_1 \otimes \cdots \otimes A_n$ gives a desingularization of $Sub_{b_1,\ldots,b_n}(A_1 \otimes \cdots \otimes A_n)$.

2.3 Strassen's equations

In 1983 V. Strassen [48] discovered equations for tensors of bounded border rank beyond minors of flattenings. We present a version of Strassen's equations due to G. Ottaviani, which is easy to generalize to higher cases.

Given $T \in A \otimes B \otimes C$, i.e. $T : B^* \to A \otimes C$, $Id_A \otimes T$ gives a linear map $A \otimes B^* \to A \otimes A \otimes C$, compose $Id_A \otimes T$ with the projection $A \otimes A \to \wedge^2 A$ to define $T^{\wedge}_{BA} : A \otimes B^* \to \wedge^2 A \otimes C$.

Theorem 4 ([42]). *Let $T \in A \otimes B \otimes C$, and assume $3 \le \dim A \le \dim B \le \dim C$. If $[T] \in \sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$, then $\operatorname{rank}(T^{\wedge}_{BA}) \le r(\dim A - 1)$. Thus the size $r(\dim A - 1) + 1$ minors of $T^{\wedge}_{BA}$ furnish equations for $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$, which are called Strassen's equations.*

Proof. If $T = a \otimes b \otimes c$, then the image of $T^{\wedge}_{BA}$ is $a \wedge A \otimes c$ and thus $\operatorname{rank}(T^{\wedge}_{BA}) = \dim A - 1$ and the theorem follows because $\operatorname{rank}((T_1 + T_2)^{\wedge}_{BA}) \le \operatorname{rank}(T_1{}^{\wedge}_{BA}) + \operatorname{rank}(T_2{}^{\wedge}_{BA})$ $\square$

Theorem 5 ([20,31]). *$\sigma_3(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ is the zero set of the size 4 minors of flattenings and Strassen's equations.*

2.4 Inheritance and prolongation

Inheritance is a general technique for studying equations of G-varieties.

Proposition 3 ([30]). *For all vector spaces B_j with $\dim B_j = b_j \ge \dim A_j = a_j \ge r$, a module $S_{\mu_1} B_1^* \otimes \cdots \otimes S_{\mu_n} B_n^*$ such that $l(\mu_j) \le a_j$ for all j, is in the ideal $I_d\left(\sigma_r(Seg(\mathbb{P}B_1 \times \cdots \times \mathbb{P}B_n))\right)$ if and only if $S_{\mu_1} A_1^* \otimes \cdots \otimes S_{\mu_n} A_n^*$ is in the ideal $I_d\left(\sigma_r(Seg(\mathbb{P}B_1 \times \cdots \times \mathbb{P}B_n))\right)$.*

Corollary 2 ([30]). *Let $\dim A_j \ge r$, $1 \le j \le n$. The ideal of $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is generated by the modules inherited from the ideal of $\sigma_r(Seg(\mathbb{P}^{r-1} \times \cdots \times$*

$\mathbb{P}^{r-1}$)) *and the modules generating the ideal of* $Sub_{r,\ldots,r}(A_1\otimes\cdots\otimes A_n)$. *The analogous scheme and set theoretic results hold as well.*

According to this corollary, when studying these equations we only need consider the small dimensional cases.

Prolongation is a general technique for finding equations of secant varieties. We list some basic facts about equations for secant varieties obtained by prolongation.

Proposition 4 ([29,47]). *Let* $X, Y \subset \mathbb{P}V$ *be subvarieties and assume that* $I_\delta(X) = 0$ *for* $\delta < d_1$ *and* $I_\delta(Y) = 0$ *for* $\delta < d_2$. *Then* $I_\delta(J(X, Y)) = 0$ *for* $\delta \le d_1 + d_2 - 2$.

Corollary 3 ([29, 47]). *Let* $X_1, \ldots, X_r \subset \mathbb{P}V$ *be varieties such that* $I_\delta(X_j) = 0$ *for* $\delta < d_j$. *Then* $I_\delta(J(X_1, \ldots, X_r)) = 0$ *for* $\delta \le d_1 + \cdots + d_r - r$.

As a special case we have:

Proposition 5 ([29]). *There are no nonzero degree* $d \le r$ *homogeneous polynomials vanishing on* $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$.

2.5 Normal forms of points in $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$

In this section we present how points of $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ are explicitly parametrized.

Proposition 6 ([5]). *Let* X *denote* $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *and* $p = [v] \in \sigma_2(X)$, *then* v *has one of the following normal forms:*

1, $p \in X$;

2, $v = x + y$ *with* $[x], [y] \in X$;

3, $v = x'$ *with* $x' \in \widehat{T}_{[x]}X$.

Theorem 6 ([5]). *Let* X *denote* $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *and* $p = [v] \in \sigma_3(X) \backslash \sigma_2(X)$, *then* v *has one of the following normal forms:*

1. $v = x + y + z$ *with* $[x], [y], [z] \in X$*;*

2. $v = x + x' + y$ *with* $[x], [y] \in X$ *and* $x' \in \widehat{T}_{[x]}X$*;*

3. $v = x + x' + x''$*, where* $[x(t)] \subset X$ *is a curve and* $x' = x'(0)$*,* $x'' = x''(0)$*;*

4. $v = x' + y'$*, where* $[x], [y] \in X$ *are distinct points that lie on a line contained in* X*,* $x' \in \widehat{T}_{[x]}X$*, and* $y' \in \widehat{T}_{[y]}X$*.*

Normal forms for Theorem 6 are as follows:

Theorem 7 ([5])**.** *Let* X *denote* $Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ *and* $p = [v] \in \sigma_3(X) \setminus \sigma_2(X)$*, then* v *has one of the following normal forms:*

1. $v = a_1^1 \otimes \cdots \otimes a_1^n + a_2^1 \otimes \cdots \otimes a_2^n + a_3^1 \otimes \cdots \otimes a_3^n$*;*

2. $v = \sum_{i=1}^{n} a_1^1 \otimes \cdots \otimes a_1^{i-1} \otimes a_2^i \otimes a_1^{i+1} \otimes \cdots \otimes a_1^n + a_3^1 \otimes \cdots \otimes a_3^n$*;*

3. $v = \sum_{i<j} a_1^1 \otimes \cdots \otimes a_1^{i-1} \otimes a_2^i \otimes a_1^{i+1} \otimes \cdots \otimes a_1^{j-1} \otimes a_2^j \otimes a_1^{j+1} \otimes \cdots \otimes a_1^n + \sum_{i=1}^{n} a_1^1 \otimes \cdots \otimes a_1^{i-1} \otimes a_3^i \otimes a_1^{i+1} \otimes \cdots \otimes a_1^n$*;*

4. $v = \sum_{s=2}^{n} a_2^1 \otimes a_1^2 \otimes \cdots \otimes a_1^{s-1} \otimes a_2^s \otimes a_1^{s+1} \otimes \cdots \otimes a_1^n + \sum_{i=1}^{n} a_1^1 \otimes \cdots \otimes a_1^{i-1} \otimes a_3^i \otimes a_1^{i+1} \otimes \cdots \otimes a_1^n$*,*

where $a_j^i \in A_i$*, and the vectors need not all be linearly independent.*

3. EQUATIONS FOR THE THIRD SECANT VARIETY OF THE SEGRE PRODUCT OF N PROJECTIVE SPACES

3.1 Outline of the proof of the main result

Our main result on equations of the third secant varieties of Segre varieties is:

Theorem 8. *$\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is set theoretically defined by Strassen's equations of all partitions $I \cup J \cup K = \{1, \ldots, n\}$ and all 4×4 minors of flattenings.*

Given $T \in A_1 \otimes \cdots \otimes A_n$, for each A_i we fix a basis $\{a^i_j\}$ and its dual basis $\{\alpha^i_j\}$. Let $X_k := Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k \times \mathbb{P}(A_{k+1} \otimes \cdots \otimes A_n))$, and $X := Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$.

Outline of the proof of the main result. If $T \in A_1 \otimes \cdots \otimes A_n$ satisfies all the equations given by 4×4 minors of flattenings, we may assume that $3 \geq \dim A_1 \geq \cdots \geq \dim A_n \geq 2$ [30]. If T satisfies Strassen's equations of the partition $\{1\} \cup \{2\} \cup \{3, \ldots, n\}$, then $T \in \sigma_3(X_2)$. We split our discussion into 4 cases to show $T \in \sigma_3(X)$.

Case 1: $T \in \sigma_3(X_2) \setminus \sigma_2(X_2)$ and $T \notin Sub_{3,2,\ldots,2}(A_1 \otimes \cdots \otimes A_n)$, then T has one of the four types of the normal forms in Theorem 7 for $\sigma_3(X_2)$. Because 4×4 minors of $T : A_1^* \otimes A_3^* \to A_2 \otimes A_4 \otimes \cdots \otimes A_n$ vanish, T has to have the same type of normal form for $\sigma_3(X_3)$. Similarly, by considering 4×4 minors of $T : A_1^* \otimes A_k^* \to A_2 \otimes \cdots \otimes \widehat{A_k} \otimes \cdots \otimes A_n$ we use induction to show that T has to maintain the same type of normal form for $\sigma_3(X)$.

Case 2: $T \in \sigma_3(X_2) \setminus \sigma_2(X_2)$ and $T \in Sub_{3,2,\ldots,2}(A_1 \otimes \cdots \otimes A_n) \setminus Sub_{2,2,\ldots,2}(A_1 \otimes \cdots \otimes A_n)$, then T has one of the normal forms in Theorem 7 for $\sigma_3(X_2)$. Because $\dim A_2 = \cdots = \dim A_n = 2$, the discussion of this case is more complicated than **Case 1**, and we split the argument into several subcases for each type of normal form. For each subcase, by considering 4×4 minors of $T : A_1^* \otimes A_3^* \to A_2 \otimes A_4 \otimes \cdots \otimes A_n$

and $T : A_2^* \otimes A_3^* \to A_1 \otimes A_4 \otimes \cdots \otimes A_n$, we show T has one of the normal forms for $\sigma_3(X_3)$. Note that the type of the normal form of T for $\sigma_3(X_2)$ could be different from the type of the normal form of T for $\sigma_3(X_3)$. By induction, we show that T has one of the normal forms of points in $\sigma_3(X)$.

Case 3: $T \in \sigma_3(X_2) \setminus \sigma_2(X_2)$ and $T \in Sub_{2,2,\ldots,2}(A_1 \otimes \cdots \otimes A_n)$. In this case, T has two types of normal forms, $T = (a_1^1 \otimes a_1^2 + a_2^1 \otimes a_2^2) \otimes b_1^3 + a_1^1 \otimes a_2^2 \otimes b_2^3 + a_2^1 \otimes a_1^2 \otimes b_3^3$ or $T = a_1^1 \otimes a_1^2 \otimes b_1^3 + a_1^1 \otimes a_2^2 \otimes b_2^3 + a_2^1 \otimes a_1^2 \otimes b_3^3$ for some $b_j^3 \in A_3 \otimes \cdots \otimes A_n$. For the generic normal form $T = (a_1^1 \otimes a_1^2 + a_2^1 \otimes a_2^2) \otimes b_1^3 + a_1^1 \otimes a_2^2 \otimes b_2^3 + a_2^1 \otimes a_1^2 \otimes b_3^3$, we show that there is a rank 2 matrix ϕ_{21} in the kernel of $T^{\wedge}_{A_2A_1} : A_1 \otimes A_2^* \to A_3 \otimes \cdots \otimes A_n$, and $\phi_{21}(T) \in S^2A_1 \otimes (A_3 \otimes \cdots \otimes A_n)$. So if for each $2 \leq i \leq n$, T has the generic type of normal form for $\sigma_3(Seg(\mathbb{P}A_1 \times \mathbb{P}A_i \times \mathbb{P}(A_2 \otimes \cdots \otimes \widehat{A_i} \otimes \cdots \otimes A_n)))$, then similarly we have a 2×2 matrix $\phi_{i1} \in Ker(T^{\wedge}_{A_iA_1})$ with full rank, and $\phi_{n1} \circ \cdots \circ \phi_{21}(T) \in S^nA_1$. Since each ϕ_{i1} is nonsingular, $T \in \sigma_3(X)$ if and only if $\phi_{n1} \circ \cdots \circ \phi_{21}(T) \in \sigma_3(\nu_n(\mathbb{P}A_1))$, where ν_n is the n-th Veronese embedding. Since the equations for $\sigma_3(\nu_n(\mathbb{P}^1))$ are known [33], we can check Strassen's equations and 4×4 minors of flattenings give equations for $\sigma_3(X)$ in this situation. If for some $2 \leq i \leq n$, say $i = 2$, T does not have the generic normal form for $\sigma_3(X_2)$, T must have the other type of normal form $T = a_1^1 \otimes a_1^2 \otimes b_1^3 + a_1^1 \otimes a_2^2 \otimes b_2^3 + a_2^1 \otimes a_1^2 \otimes b_3^3$. By considering 4×4 minors of $T : A_1^* \otimes A_3^* \to A_2 \otimes A_4 \otimes \cdots \otimes A_n$, $T : A_2^* \otimes A_3^* \to A_1 \otimes A_4 \otimes \cdots \otimes A_n$, and $T : A_1^* \otimes A_2^* \otimes A_3^* \to A_4 \otimes \cdots \otimes A_n$, we deduce $T \in \sigma_3(X_3)$. Then we use induction to show $T \in \sigma_3(X)$ by checking each type of the normal forms in Theorem 7, under the assumption that T is not of the generic normal form for $\sigma_3(X_2)$. When proceeding by induction, because $\dim T(A_3^* \otimes \cdots \otimes A_n^*) \leq 3$ we can view T as a tensor in $T(A_3^* \otimes \cdots \otimes A_n^*) \otimes A_3 \otimes \cdots \otimes A_n$ and reduce most cases to **Case 2**. For the remaining cases, we show directly $T \in \sigma_3(X)$.

Case 4: $T \in \sigma_2(X_2)$, then T has one of the three types of the normal forms in Proposition 6 for $\sigma_3(X_2)$. We verify by induction that for each normal form $T \in \sigma_3(X)$.

$\square$

3.2 Proof of the main theorem

We only need to show that if T satisfies Strassen's equations of all partitions $I \cup J \cup K = \{1, \ldots, n\}$ and 4×4 minors of all flattenings $I \cup J = \{1, \ldots, n\}$, then $T \in \sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$. For each A_i we fix a basis $\{a^i_j\}$ and its dual basis $\{\alpha^i_j\}$. Let $X_k := Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k \times \mathbb{P}(A_{k+1} \otimes \cdots \otimes A_n))$, and $X := Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$. For any flattening $I \cup J = \{1, \ldots, n\}$, 4×4 minors of $T : A_I^* \to A_J$ vanish if and only if $\dim T(A_I^*) \leq 3$. By Corollary 2, we can assume $3 \geq \dim A_1 \geq \cdots \geq \dim A_n \geq 2$. Since T satisfies Strassen's equations of the partition $\{1\} \cup \{2\} \cup \{3, \ldots, n\}$ and 4×4 minors of all flattenings, by Theorem 5 we have $T \in \sigma_3(X_2)$. We split our discussion into 4 cases to show $T \in \sigma_3(X)$.

3.2.1 *Case 1: $T \in \sigma_3(X_2) \setminus \sigma_2(X_2)$, $T \notin Sub_{3,2,\ldots,2}(A_1 \otimes \cdots \otimes A_n)$*

Since T has one of the normal forms in Theorem 7, we use induction to show $T \in \sigma_3(X)$ by verifying each normal form.

Type 1: Without loss of generality, let $T = a^1_1 \otimes a^2_1 \otimes u_1 + a^1_2 \otimes a^2_2 \otimes u_2 + a^1_3 \otimes a^2_3 \otimes u_3$, where $u_i \in A_3 \otimes \cdots \otimes A_n$. $\dim T(A_1^* \otimes A_3^*) \leq 3$ implies that $u_i : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank ≤ 1 for all i, say $u_i = b^3_i \otimes v_i$ for some $b^3_i \in A_3$ and $v_i \in A_4 \otimes \cdots \otimes A_n$. Therefore $T = a^1_1 \otimes a^2_1 \otimes b^3_1 \otimes v_1 + a^1_2 \otimes a^2_2 \otimes b^3_2 \otimes v_2 + a^1_3 \otimes a^2_3 \otimes b^3_3 \otimes v_3$, i.e. $T \in \sigma_3(X_3)$.

Now we use induction, assume $T = a^1_1 \otimes a^2_1 \otimes b^3_1 \otimes \cdots \otimes b^k_1 + a^1_2 \otimes a^2_2 \otimes b^3_2 \otimes \cdots \otimes b^k_2 + a^1_3 \otimes a^2_3 \otimes b^3_3 \otimes \cdots \otimes b^k_3$, then $\dim T(A_1^* \otimes A_k^*) \leq 3$ implies that $b^k_i : A_k^* \to A_{k+1} \otimes \cdots \otimes A_n$ has rank ≤ 1 for all $1 \leq i \leq 3$.

Type 2: $T = a^1_1 \otimes a^2_1 \otimes v^3_2 + a^1_1 \otimes a^2_2 \otimes v^3_1 + a^1_2 \otimes a^2_1 \otimes v^3_1 + a^1_3 \otimes a^2_3 \otimes v^3_3$, where $v^3_i \in A_3 \otimes \cdots \otimes A_n$. Since $T \notin \sigma_2(X_2)$, v^3_1 and v^3_3 are non-zero. $\dim T(A_1^* \otimes A_3^*) \leq 3$

implies v_1^3 and $v_3^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ have rank 1, say $v_i^3 = b_i^3 \otimes v_i^4$ for $i = 1, 3$ and some $b_i^3 \in A_3$, $v_i^4 \in A_4 \otimes \cdots \otimes A_n$, and for each $j = 2, 3$, $a_1^2 \otimes v_2^3(\alpha_j^3) + a_2^2 \otimes v_1^3(\alpha_j^3)$ is a linear combination of $a_1^2 \otimes v_2^3(\alpha_1^3) + a_2^2 \otimes v_1^3(\alpha_1^3)$ and $a_1^2 \otimes v_1^3(\alpha_1^3)$, then $v_2^3 = b_1^3 \otimes v_2^4 + b_2^3 \otimes v_1^4$ for some $b_2^3 \in A_3$ and $v_2^4 \in A_4 \otimes \cdots \otimes A_n$. Thus $T = a_2^1 \otimes a_1^2 \otimes b_1^3 \otimes v_1^4 + a_1^1 \otimes a_1^2 \otimes b_1^3 \otimes v_2^4 + a_1^1 \otimes a_1^2 \otimes b_2^3 \otimes v_1^4 + a_1^1 \otimes a_2^2 \otimes b_1^3 \otimes v_1^4 + a_3^1 \otimes a_3^2 \otimes b_3^3 \otimes v_3^4$.

Now we use induction, and assume that $T = \sum_{i=1}^{k} b_1^1 \otimes \cdots \otimes b_1^{i-1} \otimes b_2^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^k + b_3^1 \otimes \cdots \otimes b_3^k$, where $b_j^i = a_j^i$ for $i = 1, 2$ and $1 \le j \le 3$. The induction argument is similar to the case $k = 3$ above.

Type 3: $T = a_1^1 \otimes a_2^2 \otimes v_2^3 + a_2^1 \otimes a_1^2 \otimes v_2^3 + a_2^1 \otimes a_2^2 \otimes v_1^3 + a_1^1 \otimes a_1^2 \otimes v_3^3 + a_1^1 \otimes a_3^2 \otimes v_1^3 + a_3^1 \otimes a_1^2 \otimes v_1^3$, where $v_i^3 \in A_3 \otimes \cdots \otimes A_n$. If $v_1^3 = 0$, T has been discussed in **Case 1 Type 1**. If $v_2^3 = 0$, T has been discussed in **Case 1 Type 2**. So we assume v_1^3 and v_2^3 are non-zero. $\dim T(A_1^* \otimes A_3^*) \le 3$ implies $v_1^3 = u_1^3 \otimes u_1^4$, $v_2^3 = u_1^3 \otimes u_2^4 + u_2^3 \otimes u_1^4$ and $v_3^3 = u_1^3 \otimes u_3^4 + u_2^3 \otimes u_2^4 + u_3^3 \otimes u_1^4$ for some $u_1^3, u_2^3, u_3^3 \in A_3$, and $u_1^4, u_2^4, u_3^4 \in A_4 \otimes \cdots \otimes A_n$. Denote a_j^i by u_j^i when $i = 1, 2$, then $T = \sum_{1 \le i < j \le 4} u_1^1 \otimes \cdots \otimes u_2^i \otimes \cdots \otimes u_2^j \otimes \cdots \otimes u_1^4 + \sum_{i=1}^{4} u_1^1 \otimes \cdots \otimes u_3^i \otimes \cdots \otimes u_1^4$.

The induction argument is similar the above argument.

Type 4: $T = a_2^1 \otimes a_1^2 \otimes v_2^3 + a_2^1 \otimes a_2^2 \otimes v_1^3 + a_1^1 \otimes a_1^2 \otimes v_3^3 + a_1^1 \otimes a_3^2 \otimes v_1^3 + a_3^1 \otimes a_1^2 \otimes v_1^3$ for some $v_j^3 \in A_3 \otimes \cdots \otimes A_n$. Since $T \notin \sigma_2(X_2)$, $v_1^3 \neq 0$, then $\dim T(A_1^* \otimes A_3^*) \le 3$ implies $v_1^3 = u_1^3 \otimes u_1^4$, $v_2^3 = u_1^3 \otimes u_2^4 + u_2^3 \otimes u_1^4$, $v_3^3 = u_1^3 \otimes u_3^4 + u_3^3 \otimes u_1^4$ for some $u_j^3 \in A_3$, $u_j^4 \in A_4 \otimes \cdots \otimes A_n$. Denote a_j^i by u_j^i for $i = 1, 2$, then $T = \sum_{i=2}^{4} u_2^1 \otimes \cdots \otimes u_2^i \otimes \cdots \otimes u_1^4 + \sum_{i=1}^{4} u_1^1 \otimes \cdots \otimes u_3^i \otimes \cdots \otimes u_1^4$.

The induction argument is similar.

3.2.2 *Case 2: $T \in \sigma_3(X_2) \setminus \sigma_2(X_2)$,*

$T \in Sub_{3,2,\ldots,2}(A_1 \otimes \cdots \otimes A_n) \setminus Sub_{2,2,\ldots,2}(A_1 \otimes \cdots \otimes A_n)$

We show $T \in \sigma_3(X)$ by induction on each type of the normal forms.

Type 1: $T = a_1^1 \otimes b_1^2 \otimes b_1^3 + a_2^1 \otimes b_2^2 \otimes b_2^3 + a_3^1 \otimes b_3^2 \otimes b_3^3$, where $b_j^2 \in A_2$ and $b_j^3 \in A_3 \otimes \cdots \otimes A_n$. Without loss of generality, we can assume b_1^2 and b_2^2 are linearly independent, then $b_3^2 = b_1^2$ or $b_3^2 = b_1^2 + b_2^2$.

If $b_3^2 = a_1^2$, since $\dim T(A_2^* \otimes A_3^*) \leq 3$, then either $b_2^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank 1, or both b_1^3 and b_3^3 have rank 1 as maps $A_3^* \to A_4 \otimes \cdots \otimes A_n$.

When $b_2^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank 1, let $b_2^3 = a_2^3 \otimes b_2^4$ for some $b_2^4 \in A_4 \otimes \cdots \otimes A_n$. We only need to consider the case that at least one of b_1^3 and $b_3^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank 2. Without loss of generality we can assume $b_1^3 = u_1^3 \otimes b_1^4 + u_3^3 \otimes b_3^4$ for some $u_i^3 \in A_3$ and $b_i^4 \in A_4 \otimes \cdots \otimes A_n$ where $i = 1, 3$, then $\dim T(A_1^* \otimes A_3^*) \leq 3$ requires $b_3^3(\alpha_j^3) = x_j b_1^4 + y_j b_3^4$ for some x_j, y_j, where $j = 1, 2$. Consider $A_3 \otimes V_4$, where V_4 is spanned by b_1^4 and b_3^4, after a change of basis, we can assume $b_1^3 = u_1^3 \otimes b_1^4 + u_3^3 \otimes b_3^4$ and $b_3^3 = \lambda u_1^3 \otimes b_1^4 + u_1^3 \otimes b_3^4 + \lambda u_3^3 \otimes b_3^4$, or $b_3^3 = \mu u_1^3 \otimes b_1^4 + \nu u_3^3 \otimes b_3^4$. Then $T = T' + a_2^1 \otimes b_2^2 \otimes a_2^3 \otimes b_2^4$, where $T' = (a_1^1 + \lambda a_3^1) \otimes b_1^2 \otimes u_1^3 \otimes b_1^4 + (a_1^1 + \lambda a_3^1) \otimes b_1^2 \otimes u_3^3 \otimes b_3^4 + a_3^1 \otimes b_1^2 \otimes u_1^3 \otimes b_3^4 \in \widehat{T}_{(a_1^1+\lambda a_3^1)\otimes b_1^2 \otimes u_1^3 \otimes b_3^4} X_3$, or $T = (a_1^1 + \mu a_3^1) \otimes b_1^2 \otimes u_1^3 \otimes b_1^4 + (a_1^1 + \nu a_3^1) \otimes b_1^2 \otimes u_3^3 \otimes b_3^4 + a_2^1 \otimes b_2^2 \otimes a_2^3 \otimes b_2^4$.

When b_1^3 and $b_3^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ have rank 1, say $b_1^3 = a_1^3 \otimes b_1^4$ and $b_3^3 = u_3^3 \otimes b_3^4$ for some $u_3^3 \in A_3$ and $b_i^4 \in A_4 \otimes \cdots \otimes A_n$ where $i = 1, 3$, and assume $b_2^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank 2, $\dim T(A_2^* \otimes A_3^*) \leq 3$ requires $u_3^3 = a_1^3$ up to a scalar, and $\dim T(A_1^* \otimes A_3^*) \leq 3$ requires $b_1^4 = b_3^4$ up to a scalar, then $T = (a_1^1 + a_3^1) \otimes b_1^2 \otimes a_1^3 \otimes b_1^4 + a_2^1 \otimes b_2^2 \otimes a_1^3 \otimes b_2^3(\alpha_1^3) + a_2^1 \otimes b_2^2 \otimes a_2^3 \otimes b_2^3(\alpha_2^3)$.

If $b_3^2 = b_1^2 + b_2^2$, $\dim T(A_2^* \otimes A_3^*) \leq 3$ implies b_1^3 or $b_2^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank 1. If only one of them has rank 1, without loss of generality we assume that $b_2^3 = a_1^3 \otimes$

$u_1^4+a_2^3\otimes u_2^4$, and $b_1^3=u_1^3\otimes u_3^4$. $\dim T(A_2^*\otimes A_3^*)\leq 3$ implies $b_3^3=u_1^3\otimes u_4^4$ for some $u_4^4\in A_4\otimes\cdots\otimes A_n$. $\dim T(A_1^*\otimes A_3^*)\leq 3$ requires that u_3^4 and u_4^4 are linearly dependent, then we can assume $u_4^4=u_3^4$. $\dim T(A_1^*\otimes A_3^*)\leq 3$ also requires u_4^4 is a linear combination of u_1^4 and u_2^4. Consider $A_3\otimes V_4$, where V_4 is the subspace of $A_4\otimes\cdots\otimes A_n$ spanned by u_1^4 and u_2^4, after a change of basis, we can assume $b_2^3=a_1^3\otimes u_1^4+a_2^3\otimes u_2^4$ is still the identity matrix, and $b_1^3=b_3^3=a_1^3\otimes u_2^4$ or $a_1^3\otimes u_1^4$. Then $T=(a_1^1+a_3^1)\otimes b_1^2\otimes a_1^3\otimes u_2^4+T'$, where $T'=a_2^1\otimes b_2^2\otimes a_1^3\otimes u_1^4+a_2^1\otimes b_2^2\otimes a_2^3\otimes u_2^4+a_3^1\otimes b_2^2\otimes a_1^3\otimes u_2^4\in \widehat{T}_{a_2^1\otimes b_2^2\otimes a_1^3\otimes u_2^4}X_3$, or $T=(a_1^1+a_3^1)\otimes b_1^2\otimes a_1^3\otimes u_1^4+(a_2^1+a_3^1)\otimes b_2^2\otimes a_1^3\otimes u_1^4+a_2^1\otimes b_2^2\otimes a_2^3\otimes u_2^4$.

If both b_1^3 and b_2^3 have rank 1, let $b_1^3=a_1^3\otimes u_1^4$ and $b_2^3=u_2^3\otimes u_2^4$. If u_1^4 and u_2^4 are linearly independent, $\dim T(A_1^*\otimes A_3^*)\leq 3$ implies $b_3^3:A_3^*\to A_4\otimes\cdots\otimes A_n$ has rank 1. If u_1^4 and u_2^4 are dependent, say $u_1^4=u_2^4$, and if $u_2^3=a_1^3$ up to a scalar, since $\dim T(A_1^*\otimes A_3^*)\leq 3$, then $b_3^3(\alpha_1^3)=xb_3^3(\alpha_2^3)+yu_1^4$ for some x, y. So $T=(a_1^1+ya_3^1)\otimes b_1^2\otimes a_1^3\otimes u_1^4+(a_2^1+ya_3^1)\otimes b_2^2\otimes a_1^3\otimes u_1^4+a_3^1\otimes(b_1^2+b_2^2)\otimes(xa_1^3+a_2^3)\otimes b_3^3(\alpha_2^3)$. If u_2^3 and a_1^3 are independent, we can assume $u_2^3=a_2^3$, since $\dim T(A_2^*\otimes A_3^*)\leq 3$, then $b_3^3:A_3^*\to A_4\otimes\cdots\otimes A_n$ has rank 1.

Now we use induction. Assume $T=a_1^1\otimes b_1^2\otimes\cdots\otimes b_1^k+a_2^1\otimes b_2^2\otimes\cdots\otimes b_2^k+a_3^1\otimes b_3^2\otimes\cdots\otimes b_3^k$, without loss of generality we can assume $b_1^2=a_1^2$, $b_2^2=a_2^2$, then $b_3^2=a_1^2$ or $b_3^2=a_1^2+a_2^2$. The induction argument is similar to the case $k=3$.

Type 2: $T=a_1^1\otimes b_1^2\otimes b_2^3+a_1^1\otimes b_2^2\otimes b_1^3+a_2^1\otimes b_1^2\otimes b_1^3+a_3^1\otimes b_3^2\otimes b_3^3$, without loss of generality we can assume $b_1^2=a_1^2$ and $b_2^2=a_2^2$, then $b_3^2=a_1^2$, or $b_3^2=a_2^2+\lambda a_1^2$ for some $\lambda\in\mathbb{C}$.

When $b_3^2=a_1^2$, $\dim T(A_2^*\otimes A_3^*)\leq 3$ forces $b_1^3:A_3^*\to A_4\otimes\cdots\otimes A_n$ has rank 1, say $b_1^3=a_1^3\otimes b_1^4$. If $b_3^3:A_3^*\to A_4\otimes\cdots\otimes A_n$ has rank 2, say $b_3^3=a_1^3\otimes b_2^4+a_2^3\otimes b_3^4$, then $\dim T(A_1^*\otimes A_3^*)\leq 3$ requires that b_1^4 and $b_2^3(\alpha_2^3)$ are both in the subspace spanned by b_2^4 and b_3^4. After a change of basis, we can assume that $b_3^3=a_1^3\otimes b_2^4+a_2^3\otimes b_3^4$, and $b_1^3=a_1^3\otimes b_2^4$ or $b_1^3=a_1^3\otimes b_3^4$. We can assume $b_2^3(\alpha_2^3)=b_2^4+\lambda b_3^4$ or $b_2^3(\alpha_2^3)=b_3^4$. So we have four cases:

Case 1: If $b^3_1 = a^3_1 \otimes b^4_3$ and $b^3_2(\alpha^3_2) = b^4_2 + \lambda b^4_3$, $T = a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^3_2(\alpha^3_1) + a^1_1 \otimes a^2_1 \otimes (\lambda a^3_2) \otimes b^4_3 + a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes b^4_3 + a^1_2 \otimes a^2_1 \otimes a^3_1 \otimes b^4_3 + a^1_1 \otimes a^2_1 \otimes a^3_2 \otimes b^4_2 + a^1_3 \otimes a^2_1 \otimes a^3_1 \otimes b^4_2 + a^1_3 \otimes a^2_1 \otimes a^3_2 \otimes b^4_3$. Let $S(t) = (a^1_1 + ta^1_3 + t^2 a^1_2) \otimes (a^2_1 + t^2 a^2_2) \otimes (a^3_1 + ta^3_2 + t^2 \lambda a^3_2) \otimes (b^4_3 + tb^4_2 + t^2 b^3_2(\alpha^3_1))$, then $T = S''(0)$.

Case 2: If $b^3_1 = a^3_1 \otimes b^4_3$ and $b^3_2(\alpha^3_2) = b^4_3$, then $T = T' + T''$, where $T' = a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^3_2(\alpha^3_1) + a^1_1 \otimes a^2_1 \otimes a^3_2 \otimes b^4_3 + a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes b^4_3 + a^1_2 \otimes a^2_1 \otimes a^3_1 \otimes b^4_3 \in \widehat{T}_{a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^4_3} X_3$, and $T'' = a^1_3 \otimes a^2_1 \otimes a^3_1 \otimes b^4_2 + a^1_3 \otimes a^2_1 \otimes a^3_2 \otimes b^4_3 \in \widehat{T}_{a^1_3 \otimes a^2_1 \otimes a^3_1 \otimes b^4_3} X_3$.

Case 3: If $b^3_1 = a^3_1 \otimes b^4_2$ and $b^3_2(\alpha^3_2) = b^4_2 + \lambda b^4_3$, $T = T' + (\lambda a^1_1 + a^1_3) \otimes a^2_1 \otimes a^3_2 \otimes b^4_3$, where $T' = a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^3_2(\alpha^3_1) + a^1_1 \otimes a^2_1 \otimes a^3_2 \otimes b^4_2 + a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes b^4_2 + (a^1_2 + a^1_3) \otimes a^2_1 \otimes a^3_1 \otimes b^4_2 \in \widehat{T}_{a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^4_2} X_3$.

Case 4: If $b^3_1 = a^3_1 \otimes b^4_2$ and $b^3_2(\alpha^3_2) = b^4_3$, then $T = T' + (a^1_1 + a^1_3) \otimes a^2_1 \otimes a^3_2 \otimes b^4_3$, where $T' = a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^3_2(\alpha^3_1) + a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes b^4_2 + (a^1_2 + a^1_3) \otimes a^2_1 \otimes a^3_1 \otimes b^4_2 \in \widehat{T}_{a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^4_2} X_3$.

If $b^3_3 : A^*_3 \to A_4 \otimes \cdots \otimes A_n$ has rank 1, say $b^3_3 = (xa^3_1 + ya^3_2) \otimes b^4_3$, and b^4_1 and b^4_3 are linearly independent, $\dim T(A^*_1 \otimes A^*_3) \leq 3$ forces $b^3_2(\alpha^3_2)$ is a linear combination of b^4_1 and b^4_3. We can assume $b^3_2(\alpha^3_2) = b^4_1$ or $b^3_2(\alpha^3_2) = b^4_3 + \lambda b^4_1$. If $b^3_2(\alpha^3_2) = b^4_1$, $T = T' + a^1_3 \otimes a^2_1 \otimes (xa^3_1 + ya^3_2) \otimes b^4_3$, where $T' = a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^3_2(\alpha^3_1) + a^1_1 \otimes a^2_1 \otimes a^3_2 \otimes b^4_1 + a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes b^4_1 + a^1_2 \otimes a^2_1 \otimes a^3_1 \otimes b^4_1 \in \widehat{T}_{a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^4_1} X_3$. If $b^3_2(\alpha^3_2) = b^4_3 + \lambda b^4_1$, we can assume $b^3_3 = a^3_2 \otimes b^4_3$ or $b^3_3 = (a^3_1 + \mu a^3_2) \otimes b^4_3$. If $b^3_3 = a^3_2 \otimes b^4_3$, then $T = T' + (a^1_1 + a^1_3) \otimes a^2_1 \otimes a^3_2 \otimes b^4_3$, where $T' = a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^3_2(\alpha^3_1) + a^1_1 \otimes a^2_1 \otimes (\lambda a^3_2) \otimes b^4_1 + a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes b^4_1 + a^1_2 \otimes a^2_1 \otimes a^3_1 \otimes b^4_1 \in \widehat{T}_{a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^4_1} X_3$. If $b^3_3 = (a^3_1 + \mu a^3_2) \otimes b^4_3$, and if $\mu \neq 0$, let $\widetilde{a^3_2} = a^3_1 + \mu a^3_2$, then $T = T' + (1/\mu a^1_1 + a^1_3) \otimes a^2_1 \otimes \widetilde{a^3_2} \otimes b^4_3$, where $T' = a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes [b^3_2(\alpha^3_1) - 1/\mu(b^4_3 + \lambda b^4_1)] + a^1_1 \otimes a^2_1 \otimes (\lambda/\mu \widetilde{a^3_2}) \otimes b^4_1 + a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes b^4_1 + a^1_2 \otimes a^2_1 \otimes a^3_1 \otimes b^4_1 \in \widehat{T}_{a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^4_1} X_3$. If $\mu = 0$, $T = T' + T''$, where $T' = a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^3_2(\alpha^3_1) + a^1_1 \otimes a^2_1 \otimes (\lambda a^3_2) \otimes b^4_1 + a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes b^4_1 + a^1_2 \otimes a^2_1 \otimes a^3_1 \otimes b^4_1 \in \widehat{T}_{a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^4_1} X_3$, and $T'' = a^1_1 \otimes a^2_1 \otimes a^3_2 \otimes b^4_3 + a^1_3 \otimes a^2_1 \otimes a^3_1 \otimes b^4_3 \in \widehat{T}_{a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes b^4_3} X_3$. If b^4_1 and b^4_3 are linearly dependent, say $b^4_1 = b^4_3$, then $T = T' + T''$, where $T' =$

$a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^3(\alpha_1^3) + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + (a_2^1 + xa_3^1) \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$, and $T'' = a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes b_2^3(\alpha_2^3) + (ya_3^1) \otimes a_1^2 \otimes a_2^3 \otimes b_1^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes b_1^4} X_3$.

When $b_3^2 = a_2^2 + \lambda a_1^2$, $\dim T(A_2^* \otimes A_3^*) \leq 3$ implies three cases. Case 1: $b_1^3 = a_1^3 \otimes b_1^4$ and $b_2^3 = a_1^3 \otimes b_2^4$ for some $b_1^4, b_2^4 \in A_4 \otimes \cdots \otimes A_n$; Case 2: $b_1^3 = a_1^3 \otimes b_1^4$ and $b_3^3 = a_1^3 \otimes b_3^4$ for some $b_1^4, b_3^4 \in A_4 \otimes \cdots \otimes A_n$; Case 3: $b_1^3 = a_1^3 \otimes b_1^4$ and $b_3^3 = a_2^3 \otimes b_3^4$ for some $b_1^4, b_3^4 \in A_4 \otimes \cdots \otimes A_n$.

For case 1, if $b_3^3 = u_3^3 \otimes u_3^4$ for some $u_3^3 \in A_3$ and $u_3^4 \in A_4 \otimes \cdots \otimes A_n$, then $T = T' + a_3^1 \otimes (a_2^2 + \lambda a_1^2) \otimes u_3^3 \otimes u_3^4$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$. If $b_3^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank 2, $\dim T(A_1^* \otimes A_3^*) \leq 3$ requires $b_1^4 = b_2^4$ up to a scalar, and b_1^4 is a linear combination of $b_3^3(\alpha_1^3)$ and $b_3^3(\alpha_2^3)$, say $b_3^3(\alpha_1^3) = xb_3^3(\alpha_2^3) + yb_1^4$ or $b_1^4 = b_3^3(\alpha_2^3)$ up to a scalar, then $T = (a_1^1 + a_2^1 + y\lambda a_3^1) \otimes a_1^2 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 + (a_1^1 + ya_3^1) \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_3^1 \otimes (a_2^2 + \lambda a_1^2) \otimes (xa_1^3 + a_2^3) \otimes b_3^3(\alpha_2^3)$, or $T = T' + T''$, where $T' = a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_3^1 \otimes a_2^2 \otimes a_1^3 \otimes b_3^3(\alpha_1^3) + a_3^1 \otimes a_2^2 \otimes a_2^3 \otimes b_1^4 \in \widehat{T}_{a_3^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4} X_3$, and $T'' = (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 + a_3^1 \otimes a_1^2 \otimes a_1^3 \otimes \lambda b_3^3(\alpha_1^3) + a_3^1 \otimes a_1^2 \otimes (\lambda a_2^3) \otimes b_1^4 \in \widehat{T}_{a_3^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$.

For case 2, if $b_3^4 = b_1^4$ up to a scalar, then $b_1^3 = b_3^3$ up to a scalar, and $T = a_1^1 \otimes a_1^2 \otimes b_2^3 + (a_1^1 + a_3^1) \otimes a_2^2 \otimes b_1^3 + (a_2^1 + \lambda a_3^1) \otimes a_1^2 \otimes b_1^3$, which is discussed in **Case 2 Type 1**. Hence we assume b_1^4 and b_3^4 are linearly independent. $\dim T(A_1^* \otimes A_3^*) \leq 3$ implies $b_2^3(\alpha_2^3) = b_1^4$ up to a scalar, then $T = T' + a_3^1 \otimes (a_2^2 + \lambda a_1^2) \otimes a_1^3 \otimes b_3^4$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^3(\alpha_1^3) + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes b_1^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$.

For case 3, $\dim T(A_2^* \otimes A_3^*) \leq 3$ requires $b_2^3(\alpha_2^3) = b_1^4$ up to a scalar. Then $T = T' + a_3^1 \otimes (a_2^2 + \lambda a_1^2) \otimes a_2^3 \otimes b_3^4$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^3(\alpha_1^3) + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes b_1^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$.

Now we assume $T = \sum_{i=1}^{k} b_1^1 \otimes \cdots \otimes b_1^{i-1} \otimes b_2^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^k + b_3^1 \otimes \cdots \otimes b_3^k$. The induction argument is similar to the case $k = 3$.

Type 3: $T = a_1^1 \otimes b_2^2 \otimes b_2^3 + a_2^1 \otimes b_1^2 \otimes b_2^3 + a_2^1 \otimes b_2^2 \otimes b_1^3 + a_1^1 \otimes b_1^2 \otimes b_3^3 + a_1^1 \otimes b_3^2 \otimes b_1^3 + a_3^1 \otimes b_1^2 \otimes b_1^3$. Without loss of generality, we can assume $b_1^2 = a_1^2$, $b_2^2 = a_2^2$, and $b_3^2 = xa_1^2 + ya_2^2$. $\dim T(A_2^* \otimes A_3^*) \leq 3$ implies two cases. Case 1: $b_1^3 = a_1^3 \otimes b_1^4$ for some $b_1^4 \in A_4 \otimes \cdots \otimes A_n$, $b_2^3(\alpha_2^3) = b_1^4$ up to a scalar, and $b_2^3(\alpha_1^3) = b_3^3(\alpha_2^3) + \lambda b_1^4$ for some $\lambda \in \mathbb{C}$; Case 2: $b_1^3 = a_1^3 \otimes b_1^4$, and $b_2^3 = a_1^3 \otimes b_2^4$ for some $b_1^4, b_2^4 \in A_4 \otimes \cdots \otimes A_n$.

For case 1, $T = a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_3^3(\alpha_2^3) + a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_3^3(\alpha_2^3) + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes b_1^4 + a_2^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes b_3^3(\alpha_2^3) + a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_3^3(\alpha_1^3) + a_1^1 \otimes (y+\lambda)a_2^2 \otimes a_1^3 \otimes b_1^4 + (xa_1^1 + \lambda a_2^1 + a_3^1) \otimes a_1^2 \otimes a_1^3 \otimes b_1^4$. Let $S(t) = [a_1^1 + ta_2^1 + t^2(xa_1^1 + \lambda a_2^1 + a_3^1)] \otimes [a_1^2 + ta_2^2 + t^2(y+\lambda)a_2^2] \otimes (a_1^3 + ta_2^3) \otimes [b_1^4 + tb_3^3(\alpha_2^3) + t^2 b_3^3(\alpha_1^3)]$, then $T = S''(0)$.

For case 2, if $b_2^4 = \lambda b_1^4$ for some $\lambda \in \mathbb{C}$, then $b_2^3 = \lambda b_1^3$, $T = [(y+\lambda)a_1^1 + a_2^1] \otimes a_2^2 \otimes b_1^3 + (xa_1^1 + \lambda a_2^1 + a_3^1) \otimes a_1^2 \otimes b_1^3 + a_1^1 \otimes a_1^2 \otimes b_3^3$, which is discussed in **Case 2 Type 1**. Thus we assume b_1^4 and b_2^4 are independent. $\dim T(A_1^* \otimes A_3^*) \leq 3$ implies $b_3^3(\alpha_2^3) = b_1^4$ up to a scalar, so $T = a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_2^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^4 + a_2^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_3^3(\alpha_1^3) + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes b_1^4 + a_1^1 \otimes (xa_1^2 + ya_2^2) \otimes a_1^3 \otimes b_1^4 + a_3^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4$. Let $S(t) = [a_1^1 + ta_2^1 + t^2 a_3^1] \otimes [a_1^2 + ta_2^2 + t^2(xa_1^2 + ya_2^2)] \otimes (a_1^3 + t^2 a_2^3) \otimes [b_1^4 + tb_2^4 + t^2 b_3^3(\alpha_1^3)]$, then $T = S''(0)$.

Now we assume $T = \sum_{i<j} b_1^1 \otimes \cdots \otimes b_1^{i-1} \otimes b_2^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^{j-1} \otimes b_2^j \otimes b_1^{j+1} \otimes \cdots \otimes b_1^k + \sum_{i=1}^{k} b_1^1 \otimes \cdots \otimes b_1^{i-1} \otimes b_3^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^k$, and use induction to show $T \in \sigma_3(X)$. The induction argument is similar to the case $k = 3$.

Type 4: $T = a_2^1 \otimes b_1^2 \otimes b_2^3 + a_2^1 \otimes b_2^2 \otimes b_1^3 + a_1^1 \otimes b_1^2 \otimes b_3^3 + a_1^1 \otimes b_3^2 \otimes b_1^3 + a_3^1 \otimes b_1^2 \otimes b_1^3$. If $b_2^2 = b_1^2$, $T = a_2^1 \otimes b_1^2 \otimes b_2^3 + a_1^1 \otimes b_1^2 \otimes b_3^3 + a_1^1 \otimes b_3^2 \otimes b_1^3 + (a_2^1 + a_3^1) \otimes b_1^2 \otimes b_1^3$, which is discussed in **Case 2 Type 2**. Hence we can assume $b_i^2 = a_i^2$ for $1 \leq i \leq 2$. Assume $b_3^2 = xa_1^2 + ya_2^2$, then $T = (ya_1^1 + a_2^1) \otimes a_1^2 \otimes b_2^3 + (ya_1^1 + a_2^1) \otimes a_2^2 \otimes b_1^3 + a_1^1 \otimes a_1^2 \otimes (b_3^3 - yb_2^3) + (xa_1^1 + a_3^1) \otimes a_1^2 \otimes b_1^3$. Therefore after a change of basis, we only need to consider the case $T = a_2^1\otimes a_1^2\otimes b_2^3 + a_2^1\otimes a_2^2\otimes b_1^3 + a_1^1\otimes a_1^2\otimes b_3^3 + a_3^1\otimes a_1^2\otimes b_1^3$. $\dim T(A_2^*\otimes A_3^*)\leq 3$ implies $b_1^3: A_3^*\to A_4\otimes\cdots\otimes A_n$ has rank 1, say $b_1^3 = a_1^3\otimes b_1^4$ for some $b_1^4\in A_4\otimes\cdots\otimes A_n$.

If $b_3^3(\alpha_1^3)$ and $b_3^3(\alpha_2^3)$ are linearly independent, $\dim T(A_1^*\otimes A_3^*)\leq 3$ implies $b_1^4, b_2^3(\alpha_2^3)$ are in V_4, where V_4 is spanned by $b_3^3(\alpha_1^3)$ and $b_3^3(\alpha_2^3)$. For the subspace $A_3\otimes V_4$, after a change of basis, we can assume a_1^3 and $a_1^3\otimes b_3^3(\alpha_1^3)+a_2^3\otimes b_3^3(\alpha_2^3)$ are preserved, and $b_1^4 = b_3^3(\alpha_1^3)$, or $b_1^4 = b_3^3(\alpha_2^3)$. So we have two cases:

Case 1: If $b_1^4 = b_3^3(\alpha_1^3)$, assume $b_2^3(\alpha_2^3) = xb_3^3(\alpha_1^3)+yb_3^3(\alpha_2^3)$, then $T = T' + (ya_2^1 + a_1^1)\otimes a_1^2\otimes a_2^3\otimes b_3^3(\alpha_2^3)$, where $T' = a_2^1\otimes a_1^2\otimes a_1^3\otimes b_2^3(\alpha_1^3) + a_2^1\otimes a_1^2\otimes (xa_2^3)\otimes b_3^3(\alpha_1^3) + a_2^1\otimes a_2^2\otimes a_1^3\otimes b_3^3(\alpha_1^3) + (a_1^1+a_3^1)\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_1^3)\in \widehat{T}_{a_2^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_1^3)}X_3$.

Case 2: If $b_1^4 = b_3^3(\alpha_2^3)$, we can assume $b_2^3(\alpha_2^3) = b_3^3(\alpha_1^3)+\lambda b_3^3(\alpha_2^3)$ for some $\lambda\in\mathbb{C}$, or $b_2^3(\alpha_2^3) = \lambda b_3^3(\alpha_2^3)$. If $b_2^3(\alpha_2^3) = b_3^3(\alpha_1^3)+\lambda b_3^3(\alpha_2^3)$, $T = a_2^1\otimes a_1^2\otimes a_1^3\otimes b_2^3(\alpha_1^3) + a_2^1\otimes a_1^2\otimes(\lambda a_2^3)\otimes b_3^3(\alpha_2^3) + a_2^1\otimes a_2^2\otimes a_1^3\otimes b_3^3(\alpha_2^3) + a_3^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_2^3) + a_2^1\otimes a_1^2\otimes a_2^3\otimes b_3^3(\alpha_1^3) + a_1^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_1^3) + a_1^1\otimes a_1^2\otimes a_2^3\otimes b_3^3(\alpha_2^3)$. Let $S(t) = (a_2^1+ta_1^1+t^2a_3^1)\otimes(a_1^2+t^2a_2^2)\otimes(a_1^3+ta_2^3+t^2\lambda a_2^3)\otimes(b_3^3(\alpha_2^3)+tb_3^3(\alpha_1^3)+t^2b_2^3(\alpha_1^3))$, then $T = S''(0)$. If $b_2^3(\alpha_2^3) = \lambda b_3^3(\alpha_2^3)$, $T = T'+T''$, where $T' = a_2^1\otimes a_1^2\otimes a_1^3\otimes b_2^3(\alpha_1^3) + a_2^1\otimes a_1^2\otimes\lambda a_2^3\otimes b_3^3(\alpha_2^3) + a_2^1\otimes a_2^2\otimes a_1^3\otimes b_3^3(\alpha_2^3) + a_3^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_2^3)\in\widehat{T}_{a_2^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_2^3)}X_3$, and $T'' = a_1^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_1^3) + a_1^1\otimes a_1^2\otimes a_2^3\otimes b_3^3(\alpha_2^3)\in\widehat{T}_{a_1^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_2^3)}X_3$.

If $b_3^3(\alpha_2^3) = \lambda b_3^3(\alpha_1^3)$ for some $\lambda\in\mathbb{C}$, then we can assume $b_3^3 = a_1^3\otimes b_3^3(\alpha_1^3)$ or $b_3^3 = a_2^3\otimes b_3^3(\alpha_1^3)$. Thus we have four cases:

Case 1: If $b_3^3 = a_1^3\otimes b_3^3(\alpha_1^3)$, $b_3^3(\alpha_1^3)$ and b_1^4 are linearly independent, we can assume $b_2^3(\alpha_2^3) = xb_1^4 + yb_3^3(\alpha_1^3)$ for some $x,y\in\mathbb{C}$ due to $\dim T(A_1^*\otimes A_3^*)$, then $T = T'+T''$, where $T' = a_2^1\otimes a_1^2\otimes a_1^3\otimes b_2^3(\alpha_1^3) + a_2^1\otimes a_1^2\otimes xa_2^3\otimes b_1^4 + a_2^1\otimes a_2^2\otimes a_1^3\otimes b_1^4 + a_3^1\otimes a_1^2\otimes a_1^3\otimes b_1^4\in\widehat{T}_{a_2^1\otimes a_1^2\otimes a_1^3\otimes b_1^4}X_3$, and $T'' = a_2^1\otimes a_1^2\otimes ya_2^3\otimes b_3^3(\alpha_1^3) + a_1^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_1^3)\in\widehat{T}_{a_2^1\otimes a_1^2\otimes a_1^3\otimes b_3^3(\alpha_1^3)}X_3$.

Case 2: If $b_3^3 = a_1^3\otimes b_3^3(\alpha_1^3)$ and $b_3^3(\alpha_1^3) = \mu b_1^4$ for some $\mu\in\mathbb{C}$, $T = T' + a_2^1\otimes a_1^2\otimes$

$a_2^3 \otimes b_2^3(\alpha_2^3)$, where $T' = a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^3(\alpha_1^3) + a_2^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + (\mu a_1^1 + a_3^1) \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 \in \widehat{T}_{a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$.

Case 3: If $b_3^3 = a_2^3 \otimes b_3^3(\alpha_1^3)$, $b_3^3(\alpha_1^3)$ and b_1^4 are linearly independent, we can assume $b_2^3(\alpha_2^3) = x b_1^4 + y b_3^3(\alpha_1^3)$ due to $\dim T(A_1^* \otimes A_3^*)$, then $T = T' + (y a_2^1 + a_1^1) \otimes a_1^2 \otimes a_2^3 \otimes b_3^3(\alpha_1^3)$, where $T' = a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^3(\alpha_1^3) + a_2^1 \otimes a_1^2 \otimes x a_2^3 \otimes b_1^4 + a_2^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_3^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 \in \widehat{T}_{a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$.

Case 4: If $b_3^3 = a_2^3 \otimes b_3^3(\alpha_1^3)$ and $b_3^3(\alpha_1^3) = \mu b_1^4$ for some $\mu \in \mathbb{C}$, $T = T' + T''$, where $T' = a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^3(\alpha_1^3) + a_2^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_3^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 \in \widehat{T}_{a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$, and $T'' = a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes \mu b_1^4 + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes b_2^3(\alpha_2^3) \in \widehat{T}_{a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes b_1^4} X_3$.

Now assume $T = \sum_{i=2}^{k} b_2^1 \otimes b_1^2 \otimes \cdots \otimes b_1^{i-1} \otimes b_2^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^k + \sum_{i=1}^{k} b_1^1 \otimes \cdots \otimes b_1^{i-1} \otimes b_3^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^k$, and use induction to show $T \in \sigma_3(X)$. The induction argument is similar to the case $k = 3$.

3.2.3 Case 3: $T \in \sigma_3(X_2) \setminus \sigma_2(X_2)$, $T \in Sub_{2,2,\ldots,2}(A_1 \otimes \cdots \otimes A_n)$

Since $\dim T(A_3^* \otimes \cdots \otimes A_n^*) \leq 3$, then after a change of basis we can assume $T(A_3^* \otimes \cdots \otimes A_n^*) \subset V$, where V is spanned by $\{a_1^1 \otimes a_1^2 + a_2^1 \otimes a_2^2, a_1^1 \otimes a_2^2, a_2^1 \otimes a_1^2\}$ or $\{a_1^1 \otimes a_1^2, a_1^1 \otimes a_2^2, a_2^1 \otimes a_1^2\}$. So T has 2 types of normal forms.

Type 1: $T = (a_1^1 \otimes a_1^2 + a_2^1 \otimes a_2^2) \otimes b_1^3 + a_1^1 \otimes a_2^2 \otimes b_2^3 + a_2^1 \otimes a_1^2 \otimes b_3^3$, we reduce the problem to finding equations for $\sigma_3(\nu_n(\mathbb{P}^1))$, which has been settled.

Lemma 2. *Let $T \in A \otimes B \otimes C$, where $\dim A = \dim B$. If there is an element $\phi \in Ker(T_{BA}^{\wedge})$ with full rank, then $\phi(T) \in S^2 A \otimes C$.*

Proof of the lemma. Let $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ be bases for A, B, C respectively, and $\{a^i\}$, $\{b^j\}$, $\{c^k\}$ their dual bases. Let $T = \sum \alpha^{ijk} a_i \otimes b_j \otimes c_k$, then $T_{BA}^{\wedge} : a_l \otimes b^j \mapsto \sum_{i,k} \alpha^{ijk} (a_l \wedge a_i) \otimes c_k$. Let $\phi = \sum \beta_j^l a_l \otimes b^j \in Ker(T_{BA}^{\wedge})$, then $\sum \beta_j^l \alpha^{ijk} (a_l \wedge a_i) \otimes c_k = 0$, which means $\sum_j \beta_j^l \alpha^{ijk} = \sum_j \beta_j^i \alpha^{ljk}$. Since $\phi(T) = \sum \beta_j^l \alpha^{ijk} a_i \otimes a_l \otimes c_k$, then $\phi(T) \in S^2 A \otimes C$.

$\square$

Let V be a complex vector space. Given $\phi \in S^dV$, let $\phi_{a,d-a} \in S^aV \otimes S^{d-a}V$ denote the $(a, d-a)$-polarization of ϕ. As a linear map $S^aV^* \to S^{d-a}V$, $\operatorname{rank}(\phi_{a,d-a}) \leq r$ if $[\phi] \in \sigma_r(\nu_d(\mathbb{P}V))$ [33].

Theorem 9 ([33]). *$\sigma_3(\nu_3(\mathbb{P}^n))$ is ideal theoretically defined by Aronhold invariant and size 4 minors of $\phi_{1,2}$. $\sigma_3(\nu_d(\mathbb{P}^n))$ is scheme theoretically defined by size 4 minors of $\phi_{2,2}$ and $\phi_{1,3}$ when $d \geq 4$.*

Now given any $T \in A_1 \otimes \cdots \otimes A_n$, if there is some $1 \leq i \leq n$, and for any $j \neq i$, there is a $\phi_{ji} \in Ker(T^{\wedge}_{A_jA_i})$ with full rank, then $\widetilde{T} = \phi_{ni} \circ \cdots \circ \phi_{1i}(T) \in S^nA_i$ has the same rank with T. If T satisfies 4×4 minors of flattenings, $\widetilde{T}$ satisfies size 4 minors of symmetric flattenings, by Theorem 9 $\widetilde{T} \in \sigma_3(\nu_n(\mathbb{P}^1))$, then $T \in \sigma_3(X)$. If T is of Type 1, we always have $a^1_1 \otimes a^2_2 + a^1_2 \otimes a^2_1 \in Ker(T^{\wedge}_{A_2A_1})$ with full rank, hence if for any $2 \leq i \leq n$, T is of Type 1 when viewed as a tensor in $A_1 \otimes A_i \otimes (A_2 \otimes \cdots \otimes A_{i-1} \otimes \widehat{A_i} \otimes A_{i+1} \otimes \cdots \otimes A_n)$, then $T \in \sigma_3(X)$. If $T \in A_1 \otimes A_2 \otimes (A_3 \otimes \cdots \otimes A_n)$ is not of Type 1, then it must be of Type 2, and we will use induction to show that $T \in \sigma_3(X)$ in this situation.

Type 2: $T = a^1_1 \otimes a^2_1 \otimes b^3_1 + a^1_1 \otimes a^2_2 \otimes b^3_2 + a^1_2 \otimes a^2_1 \otimes b^3_3$, the dimension of $T(A^*_2 \otimes A^*_3)$ implies $b^3_3 : A^*_3 \to A_4 \otimes \cdots \otimes A_n$ has rank 1, or $b^3_2 : A^*_3 \to A_4 \otimes \cdots \otimes A_n$ has rank 1.

If $b^3_3 : A^*_3 \to A_4 \otimes \cdots \otimes A_n$ has rank 1, say $b^3_3 = a^3_1 \otimes b^4_3$, and $b^3_2 : A^*_3 \to A_4 \otimes \cdots \otimes A_n$ has rank 2, say $b^3_2 = a^3_1 \otimes b^4_1 + a^3_2 \otimes b^4_2$, then $\dim T(A^*_2 \otimes A^*_3) \leq 3$ implies $b^3_1(\alpha^3_2) = \lambda b^4_1 + \mu b^4_2$ for some $\lambda, \mu \in \mathbb{C}$. If b^4_3, b^4_1 and b^4_2 are linearly independent, then $\dim T(A^*_1 \otimes A^*_2 \otimes A^*_3) \leq 3$ forces $b^3_1(\alpha^3_1) = xb^4_3 + yb^4_1 + zb^4_2$ for some $x, y, z \in \mathbb{C}$, thus $T = a^1_1 \otimes a^2_1 \otimes (ya^3_1 \otimes b^4_1 + za^3_1 \otimes b^4_2 + \lambda a^3_2 \otimes b^4_1 + \mu a^3_2 \otimes b^4_2) + a^1_1 \otimes a^2_2 \otimes (a^3_1 \otimes b^4_1 + a^3_2 \otimes b^4_2) + (xa^1_1 + a^1_2) \otimes a^2_1 \otimes a^3_1 \otimes b^4_3$. For the subspace $A_3 \otimes V_4$, where $V_4 \subset A_4 \otimes \cdots \otimes A_n$ is spanned by b^4_1 and b^4_2, after a change of basis we can assume $a^3_1 \otimes b^4_1 + a^3_2 \otimes b^4_2$ is preserved, a^3_1

is mapped to $ua_1^3+va_2^3$ for some $u,v\in\mathbb{C}$, and $ya_1^3\otimes b_1^4+za_1^3\otimes b_2^4+\lambda a_2^3\otimes b_1^4+\mu a_2^3\otimes b_2^4$ is of the Jordan canonical form, i.e. $a_1^3\otimes b_1^4+a_2^3\otimes b_2^4$, or $a_1^3\otimes b_1^4$, or $a_1^3\otimes b_2^4$, or $\beta a_1^3\otimes b_1^4+a_1^3\otimes b_2^4+\beta a_2^3\otimes b_2^4$ for some $0\neq\beta\in\mathbb{C}$. Hence we have:

Subcase 1: $T=a_1^1\otimes(a_1^2+a_2^2)\otimes a_1^3\otimes b_1^4+a_1^1\otimes(a_1^2+a_2^2)\otimes a_2^3\otimes b_2^4+(xa_1^1+a_2^1)\otimes a_1^2\otimes(ua_1^3+va_2^3)\otimes b_3^4$.

Subcase 2: $T=a_1^1\otimes(a_1^2+a_2^2)\otimes a_1^3\otimes b_1^4+a_1^1\otimes a_2^2\otimes a_2^3\otimes b_2^4+(xa_1^1+a_2^1)\otimes a_1^2\otimes(ua_1^3+va_2^3)\otimes b_3^4$.

Subcase 3: $T=T'+(xa_1^1+a_2^1)\otimes a_1^2\otimes(ua_1^3+va_2^3)\otimes b_3^4$, where $T'=a_1^1\otimes a_1^2\otimes a_1^3\otimes b_2^4+a_1^1\otimes a_2^2\otimes a_1^3\otimes b_1^4+a_1^1\otimes a_2^2\otimes a_2^3\otimes b_2^4\in\widehat{T}_{a_1^1\otimes a_2^2\otimes a_1^3\otimes b_2^4}X_3$.

Subcase 4: $T=T'+(xa_1^1+a_2^1)\otimes a_1^2\otimes(ua_1^3+va_2^3)\otimes b_3^4$, where $T'=a_1^1\otimes(\beta a_1^2+a_2^2)\otimes a_1^3\otimes b_1^4+a_1^1\otimes(\beta a_1^2+a_2^2)\otimes a_2^3\otimes b_2^4+a_1^1\otimes a_1^2\otimes a_1^3\otimes b_2^4\in\widehat{T}_{a_1^1\otimes(\beta a_1^2+a_2^2)\otimes a_1^3\otimes b_2^4}X_3$.

If $b_3^4=pb_1^4+qb_2^4$ for some $p,q\in\mathbb{C}$, for $A_3\otimes V_4$, after a change of basis we can assume a_1^3 and $a_1^3\otimes b_1^4+a_2^3\otimes b_2^4$ are preserved, $b_3^4=b_1^4$ or b_2^4, and $a_2^3\otimes b_1^3(\alpha_2^3)$ is of the form $x_1^1a_1^3\otimes b_1^4+x_2^1a_1^3\otimes b_2^4+x_1^2a_2^3\otimes b_1^4+x_2^2a_2^3\otimes b_2^4$. If $b_3^4=b_1^4$ we have:

Subcase 5: $T=T'+a_1^1\otimes(x_2^2a_1^2+a_2^2)\otimes a_2^3\otimes b_2^4$, where $T'=a_1^1\otimes a_1^2\otimes a_1^3\otimes[b_1^3(\alpha_1^3)+x_1^1b_1^4+x_2^1b_2^4]+a_1^1\otimes a_1^2\otimes(x_1^2a_2^3)\otimes b_1^4+a_1^1\otimes a_2^2\otimes a_1^3\otimes b_1^4+a_2^1\otimes a_1^2\otimes a_1^3\otimes b_1^4\in\widehat{T}_{a_1^1\otimes a_1^2\otimes a_1^3\otimes b_1^4}X_3$.

If $b_3^4=b_2^4$, by changing a_2^3, b_2^4 and a_1^2, we can assume $x_1^2=1$ or 0. So we have:

Subcase 6: $T=a_1^1\otimes a_1^2\otimes a_1^3\otimes[b_1^3(\alpha_1^3)+x_1^1b_1^4+x_2^1b_2^4]+a_1^1\otimes a_1^2\otimes(x_2^2a_2^3)\otimes b_2^4+a_2^1\otimes a_1^2\otimes a_1^3\otimes b_2^4+a_1^1\otimes a_1^2\otimes a_2^3\otimes b_1^4+a_1^1\otimes a_2^2\otimes a_1^3\otimes b_1^4+a_1^1\otimes a_2^2\otimes a_2^3\otimes b_2^4$. Let $S(t)=(a_1^1+t^2a_2^1)\otimes(a_1^2+ta_2^2)\otimes(a_1^3+ta_2^3+t^2x_2^2a_2^3)\otimes[b_2^4+tb_1^4+t^2(b_1^3(\alpha_1^3)+x_1^1b_1^4+x_2^1b_2^4)]$, so $T=S''(0)$.

Subcase 7: $T=T'+T''$, where $T'=a_1^1\otimes a_1^2\otimes a_1^3\otimes[b_1^3(\alpha_1^3)+x_1^1b_1^4+x_2^1b_2^4]+a_1^1\otimes a_1^2\otimes(x_2^2a_2^3)\otimes b_2^4+a_2^1\otimes a_1^2\otimes a_1^3\otimes b_2^4\in\widehat{T}_{a_1^1\otimes a_1^2\otimes a_1^3\otimes b_2^4}X_3$, and $T''=a_1^1\otimes a_2^2\otimes a_1^3\otimes b_1^4+a_1^1\otimes a_2^2\otimes a_2^3\otimes b_2^4\in\widehat{T}_{a_1^1\otimes a_2^2\otimes a_1^3\otimes b_2^4}X_3$.

If $b_2^3:A_3^*\to A_4\otimes\cdots\otimes A_n$ has rank 1, say $b_2^3=a_1^3\otimes b_2^4$ for some $b_2^4\in A_4\otimes\cdots\otimes A_n$, and $b_3^3:A_3^*\to A_4\otimes\cdots\otimes A_n$ has rank 2, say $b_3^3=a_1^3\otimes b_1^4+a_2^3\otimes b_3^4$ for some

$b_1^4, b_3^4 \in A_4 \otimes \cdots \otimes A_n$, then $\dim T(A_1^* \otimes A_3^*) \leq 3$ implies $b_1^3(\alpha_2^3) = \lambda b_1^4 + \mu b_3^4$ for some $\lambda, \mu \in \mathbb{C}$. If b_3^4, b_1^4 and b_2^4 are linearly independent, then $\dim T(A_1^* \otimes A_2^* \otimes A_3^*) \leq 3$ forces $b_1^3(\alpha_1^3) = xb_1^4 + yb_2^4 + zb_3^4$ for some $x, y, z \in \mathbb{C}$. For the subspace $A_3 \otimes V_4$, where $V_4 \subset A_4 \otimes \cdots \otimes A_n$ is spanned by b_1^4 and b_3^4, after a change of basis we can assume $a_1^3 \otimes b_1^4 + a_2^3 \otimes b_3^4$ is preserved, a_1^3 is mapped to $ua_1^3 + va_2^3$ for some $u, v \in \mathbb{C}$ under the new basis, and $xa_1^3 \otimes b_1^4 + za_1^3 \otimes b_3^4 + \lambda a_2^3 \otimes b_1^4 + \mu a_2^3 \otimes b_3^4$ is of the Jordan canonical form, i.e. $a_1^3 \otimes b_1^4 + a_2^3 \otimes b_3^4$, or $a_1^3 \otimes b_1^4$, or $a_1^3 \otimes b_3^4$, or $\beta a_1^3 \otimes b_1^4 + a_1^3 \otimes b_3^4 + \beta a_2^3 \otimes b_3^4$ for some $0 \neq \beta \in \mathbb{C}$. Hence we have:

Subcase 8: $T = (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 + (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_2^3 \otimes b_3^4 + a_1^1 \otimes (ya_1^2 + a_2^2) \otimes (ua_1^3 + va_2^3) \otimes b_2^4$.

Subcase 9: $T = (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes b_3^4 + a_1^1 \otimes (ya_1^2 + a_2^2) \otimes (ua_1^3 + va_2^3) \otimes b_2^4$.

Subcase 10: $T = T' + a_1^1 \otimes (ya_1^2 + a_2^2) \otimes (ua_1^3 + va_2^3) \otimes b_2^4$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_3^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes b_3^4 \in \widehat{T}_{a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_3^4} X_3$.

Subcase 11: $T = T' + a_1^1 \otimes (ya_1^2 + a_2^2) \otimes (ua_1^3 + va_2^3) \otimes b_2^4$, where $T' = (\beta a_1^1 + a_2^1) \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 + (\beta a_1^1 + a_2^1) \otimes a_1^2 \otimes a_2^3 \otimes b_3^4 + a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_3^4 \in \widehat{T}_{(\beta a_1^1 + a_2^1) \otimes a_1^2 \otimes a_1^3 \otimes b_3^4} X_3$.

If $b_2^4 = pb_1^4 + qb_3^4$ for some $p, q \in \mathbb{C}$, for $A_3 \otimes V_4$, after a change of basis we can assume a_1^3 and $a_1^3 \otimes b_1^4 + a_2^3 \otimes b_3^4$ are preserved, $b_2^4 = b_1^4$ or b_3^4, and $a_2^3 \otimes b_1^3(\alpha_2^3)$ is of the form $x_1^1 a_1^3 \otimes b_1^4 + x_2^1 a_1^3 \otimes b_3^4 + x_1^2 a_2^3 \otimes b_1^4 + x_2^2 a_2^3 \otimes b_3^4$. If $b_2^4 = b_1^4$ we have:

Subcase 12: $T = T' + (x_2^2 a_1^1 + a_2^1) \otimes a_1^2 \otimes a_2^3 \otimes b_3^4$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes [b_1^3(\alpha_1^3) + x_1^1 b_1^4 + x_2^1 b_3^4] + a_1^1 \otimes a_1^2 \otimes x_1^2 a_2^3 \otimes b_1^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4} X_3$.

If $b_2^4 = b_3^4$, by changing a_2^3, b_3^4 and a_2^2, we can assume $x_1^2 = 1$ or 0. So we have:

Subcase 13: $T = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes [b_1^3(\alpha_1^3) + x_1^1 b_1^4 + x_2^1 b_3^4] + a_1^1 \otimes a_1^2 \otimes (x_2^2 a_2^3) \otimes b_3^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_3^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes b_3^4$. Let $S(t) = (a_1^1 + ta_2^1) \otimes (a_1^2 + t^2 a_2^2) \otimes (a_1^3 + ta_2^3 + t^2 x_2^2 a_2^3) \otimes [b_3^4 + tb_1^4 + t^2(b_1^3(\alpha_1^3) + x_1^1 b_1^4 + x_2^1 b_3^4)]$, so $T = S''(0)$.

Subcase 14: $T = T' + T''$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes [b_1^3(\alpha_1^3) + x_1^1 b_1^4 + x_2^1 b_3^4] + a_1^1 \otimes a_1^2 \otimes (x_2^2 a_2^3) \otimes b_3^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_3^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_3^4} X_3$, and $T'' = a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_1^4 + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes b_3^4 \in \widehat{T}_{a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_3^4} X_3$.

If both b_2^3 and $b_3^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ have rank 1, say $b_2^3 = a_1^3 \otimes b_2^4$ and $b_3^3 = u_3^3 \otimes b_3^4$ for some $u_3^3 \in A_3$ and $b_2^4, b_3^4 \in A_4 \otimes \cdots \otimes A_n$, and $b_1^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank 2, say $b_1^3 = a_1^3 \otimes u_1^4 + a_2^3 \otimes u_2^4$ for some $u_1^4, u_2^4 \in A_4 \otimes \cdots \otimes A_n$, b_2^4, u_1^4 and u_2^4 are linearly independent, then $b_3^4 = x b_2^4 + y u_1^4 + z u_2^4$ for some $x, y, z \in \mathbb{C}$. After a change of basis, we can assume $x = 0$ or 1, $u_3^3 = a_1^3$ or a_2^3. For the subspace $A_3 \otimes V_4$, where V_4 is spanned by u_1^4 and u_2^4, after a change of basis we can assume $a_1^3 \otimes u_1^4 + a_2^3 \otimes u_2^4$ and a_1^3 are preserved, and $y u_1^4 + z u_2^4 = u_1^4$ or u_2^4. Then we have:

Subcase 15: If $u_3^3 = a_1^3$, $x = 0$, $y u_1^4 + z u_2^4 = u_1^4$, then $T = (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_1^3 \otimes u_1^4 + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_2^4$.

Subcase 16: If $u_3^3 = a_1^3$, $x = 0$, $y u_1^4 + z u_2^4 = u_2^4$, then $T = T' + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_2^4$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_1^4 + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes u_2^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_2^4} X_3$.

Subcase 17: If $u_3^3 = a_1^3$, $x = 1$, $y u_1^4 + z u_2^4 = u_1^4$, then $T = (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_1^3 \otimes (u_1^4 + b_2^4) + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_1^1 \otimes (a_2^2 - a_1^2) \otimes a_1^3 \otimes b_2^4$.

Subcase 18: If $u_3^3 = a_1^3$, $x = 1$, $y u_1^4 + z u_2^4 = u_2^4$, then $T = T' + T''$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_1^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_2^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^4} X_3$, and $T'' = a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_2^1 \otimes a_1^2 \otimes a_1^3 \otimes u_2^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_2^4} X_3$.

If $u_3^3 = a_2^3$, for the subspace $A_3 \otimes V_4$, after a change of basis we can assume $a_1^3 \otimes u_1^4 + a_2^3 \otimes u_2^4$ and a_2^3 are preserved, $y u_1^4 + z u_2^4 = u_1^4$ or u_2^4, and a_1^3 is mapped to $\lambda a_1^3 + \mu a_2^3$ for some $\lambda, \mu \in \mathbb{C}$ under the new basis. Then we have:

Subcase 19: If $x = 0$, $y u_1^4 + z u_2^4 = u_1^4$, then $T = T' + a_1^1 \otimes a_2^2 \otimes (\lambda a_1^3 + \mu a_2^3) \otimes b_2^4$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_1^4 + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes u_1^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_1^4} X_3$.

Subcase 20: If $x = 0$, $y u_1^4 + z u_2^4 = u_2^4$, then $T = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_1^4 + (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_1^1 \otimes a_2^2 \otimes (\lambda a_1^3 + \mu a_2^3) \otimes b_2^4$.

By adjusting a_2^2, we can assume $\lambda a_1^3 + \mu a_2^3 = a_2^3$ or $a_1^3 + \mu a_2^3$. So we have:

Subcase 21: If $\lambda a_1^3 + \mu a_2^3 = a_2^3$, $x = 1$, $y u_1^4 + z u_2^4 = u_1^4$, $T = T' + T''$, where $T' = a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes b_2^4 + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes b_2^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes b_2^4} X_3$, and $T'' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_1^4 + a_2^1 \otimes a_1^2 \otimes a_2^3 \otimes u_1^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_1^4} X_3$.

Subcase 22: If $\lambda a_1^3 + \mu a_2^3 = a_2^3$, $x = 1$, $y u_1^4 + z u_2^4 = u_2^4$, $T = (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_2^3 \otimes (u_2^4 + b_2^4) + a_1^1 \otimes (a_2^2 - a_1^2) \otimes a_2^3 \otimes b_2^4 + a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_1^4$.

Subcase 23: If $\lambda a_1^3 + \mu a_2^3 = a_1^3 + \mu a_2^3$, $x = 1$, $y u_1^4 + z u_2^4 = u_1^4$, let $c_1^2 = a_1^2$, $c_2^2 = a_2^2 - a_1^2$, $v_1^4 = u_1^4 + b_2^4$ and $v_2^4 = b_2^4$, then $T = a_1^1 \otimes a_1^3 \otimes (c_1^2 \otimes v_1^4 + c_2^2 \otimes v_2^4) + a_1^1 \otimes a_2^3 \otimes (c_1^2 \otimes u_2^4 + \mu c_1^2 \otimes v_2^4 + \mu c_2^2 \otimes v_2^4) + a_2^1 \otimes a_2^3 \otimes c_1^2 \otimes v_1^4 = T' + a_1^1 \otimes c_2^2 \otimes (a_1^3 + \mu a_2^3) \otimes v_2^4$, where $T' = a_1^1 \otimes c_1^2 \otimes a_1^3 \otimes v_1^4 + a_1^1 \otimes c_1^2 \otimes a_2^3 \otimes (\mu v_2^4 + u_2^4) + a_2^1 \otimes c_1^2 \otimes a_2^3 \otimes v_1^4 \in \widehat{T}_{a_1^1 \otimes c_1^2 \otimes a_2^3 \otimes v_1^4} X_3$.

Subcase 24: If $\lambda a_1^3 + \mu a_2^3 = a_1^3 + \mu a_2^3$, $\mu \neq 0$, $x = 1$, $y u_1^4 + z u_2^4 = u_2^4$, let $c_1^2 = a_1^2$, $c_2^2 = \mu a_2^2 - a_1^2$, $v_1^4 = u_2^4 + b_2^4$ and $v_2^4 = b_2^4$, then $T = (a_1^1 + a_2^1) \otimes c_1^2 \otimes a_2^3 \otimes v_1^4 + a_1^1 \otimes c_2^2 \otimes (\frac{1}{\mu} a_1^3 + a_2^3) \otimes v_2^4 + a_1^1 \otimes c_1^2 \otimes a_1^3 \otimes (u_1^4 + \frac{1-\mu}{\mu} v_2^4)$.

Subcase 25: If $\lambda a_1^3 + \mu a_2^3 = a_1^3$, $x = 1$, $y u_1^4 + z u_2^4 = u_2^4$, then $T = T' + (a_1^1 + a_2^1) \otimes a_1^2 \otimes a_2^3 \otimes (u_2^4 + b_2^4)$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_1^4 + a_1^1 \otimes (a_2^2 - a_1^2) \otimes a_1^3 \otimes b_2^4 + a_1^1 \otimes a_1^2 \otimes (a_1^3 - a_2^3) \otimes b_2^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes b_2^4} X_3$.

If b_2^4 is in the subspace V_4 spanned by u_1^4 and u_2^4, after a change of basis of $A_3 \otimes V_4$ we can assume b_1^3 is preserved, and $b_2^4 = u_1^4$ or u_2^4. So we have:

Subcase 26: If $b_2^4 = u_1^4$, $T = a_1^1 \otimes (a_1^2 + a_2^2) \otimes a_1^3 \otimes u_1^4 + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_2^1 \otimes a_1^2 \otimes u_3^3 \otimes b_3^4$.

Subcase 27: If $b_2^4 = u_2^4$, $T = T' + a_2^1 \otimes a_1^2 \otimes u_3^3 \otimes b_3^4$, where $T' = a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_1^4 + a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes u_2^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes u_2^4 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_1^3 \otimes u_2^4} X_3$.

Subcase 28: If $b_1^3 : A_3^* \to A_4 \otimes \cdots \otimes A_n$ has rank 1, say $b_1^3 = u_1^3 \otimes b_1^4$ for some $u_1^3 \in A_3$ and $b_1^4 \in A_4 \otimes \cdots \otimes A_n$, then $T = a_1^1 \otimes a_1^2 \otimes u_1^3 \otimes b_1^4 + a_1^1 \otimes a_2^2 \otimes a_1^3 \otimes b_2^4 + a_2^1 \otimes a_1^2 \otimes u_3^3 \otimes b_3^4$.

Now we assume $T \in \sigma_3(X_{k-1})$, and T is of Type 2, but is not of Type 1 when viewed as a tensor in $A_1 \otimes A_2 \otimes (A_3 \otimes \cdots \otimes A_n)$. For each normal form, we show by induction that $T \in \sigma_3(X)$.

Subtype 1: $T = b_1^1 \otimes \cdots \otimes b_1^k + b_2^1 \otimes \cdots \otimes b_2^k + b_3^1 \otimes \cdots \otimes b_3^k$. Since we assume $T(A_3^* \otimes \cdots \otimes A_n^*) \subset V$, where V is spanned by $a_1^1 \otimes a_1^2$, $a_1^1 \otimes a_2^2$, and $a_2^1 \otimes a_1^2$, then $b_j^1 \otimes b_j^2 \in A_1 \otimes A_2$ has rank 1 for any $1 \leq j \leq 3$ implies $b_j^1 = a_1^1$ or $b_j^2 = a_1^2$. Hence we have two subcase: $T = a_2^1 \otimes a_1^2 \otimes b_1^3 \otimes \cdots \otimes b_1^k + a_1^1 \otimes a_2^2 \otimes b_2^3 \otimes \cdots \otimes b_2^k + (\lambda a_1^1 + \mu a_2^1) \otimes a_1^2 \otimes b_3^3 \otimes \cdots \otimes b_3^k$ or $T = a_1^1 \otimes a_2^2 \otimes b_1^3 \otimes \cdots \otimes b_1^k + a_2^1 \otimes a_1^2 \otimes b_2^3 \otimes \cdots \otimes b_2^k + a_1^1 \otimes (\lambda a_1^2 + \mu a_2^2) \otimes b_3^3 \otimes \cdots \otimes b_3^k$. Here we only show the first case since the argument for the second case is similar. For the first subcase, if $\lambda = 0$, T has been discussed in **Case 3 Type 1**, so we assume $\lambda \neq 0$. Now let $c_1^1 = a_2^1 \otimes a_1^2$, $c_2^1 = a_1^1 \otimes a_2^2$, and $c_3^1 = (\lambda a_1^1 + \mu a_2^1) \otimes a_1^2$. From the argument of **Case 2 Type 1**, we can deduce directly that $T \in \sigma_3(X_k)$ except for the following several subcases.

Exceptional Subcase 1: $b_j^3 = a_j^3$ for $1 \leq j \leq 2$, $b_3^3 = a_1^3 + a_2^3$, $b_2^k = a_1^k \otimes u_1^{k+1} + a_2^k \otimes u_2^{k+1}$ for some u_1^{k+1}, $u_2^{k+1} \in A_{k+1} \otimes \cdots \otimes A_n$, $b_1^k = b_3^k = a_1^k \otimes u_1^{k+1}$, $b_1^i = b_2^i = b_3^i$ for all $4 \leq i \leq k-1$, then there is no harm to assume $k = 4$. So $T = (c_1^1 + c_3^1) \otimes a_1^3 \otimes a_1^4 \otimes u_2^5 + c_2^1 \otimes a_2^3 \otimes a_1^4 \otimes u_1^5 + c_2^1 \otimes a_2^3 \otimes a_2^4 \otimes u_2^5 + c_3^1 \otimes a_2^3 \otimes a_1^4 \otimes u_2^5$. When $\mu \neq -1$, $T = [\lambda a_1^1 + (\mu + 1)a_2^1] \otimes a_1^2 \otimes (a_1^3 + \dfrac{\mu}{\mu+1} a_2^3) \otimes a_1^4 \otimes u_2^5 + T'$, where $T' = a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_1^4 \otimes u_1^5 + a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_2^4 \otimes u_2^5 + a_1^1 \otimes \dfrac{\lambda}{\mu+1} a_1^2 \otimes a_2^3 \otimes a_1^4 \otimes u_2^5 \in \widehat{T}_{a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_1^4 \otimes u_2^5} X_4$. When $\mu = -1$, $T = T' + T''$, where $T' = a_1^1 \otimes a_1^2 \otimes \lambda a_1^3 \otimes a_1^4 \otimes u_2^5 + (\lambda a_1^1 - a_2^1) \otimes a_1^2 \otimes a_2^3 \otimes a_1^4 \otimes u_2^5 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes a_1^4 \otimes u_2^5} X_4$, and $T'' = a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_1^4 \otimes u_1^5 + a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_2^4 \otimes u_2^5 \in \widehat{T}_{a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_1^4 \otimes u_2^5} X_4$.

Exceptional Subcase 2: $T = (c_1^1 + c_3^1) \otimes a_1^3 \otimes b_1^4 \otimes \cdots \otimes b_1^{k-1} \otimes a_1^k \otimes u_1^{k+1} + (c_2^1 + c_3^1) \otimes a_2^3 \otimes b_1^4 \otimes \cdots \otimes b_1^{k-1} \otimes a_1^k \otimes u_1^{k+1} + c_2^1 \otimes a_2^3 \otimes b_1^4 \otimes \cdots \otimes b_1^{k-1} \otimes a_2^k \otimes u_2^{k+1}$. It is harmless to assume $k = 4$. When $\mu \neq -1$, $T = [\lambda a_1^1 + (\mu+1)a_2^1] \otimes a_1^2 \otimes (a_1^3 + \dfrac{\mu}{\mu+1} a_2^3) \otimes a_1^4 \otimes u_1^5 + \dfrac{1}{\mu+1} a_1^1 \otimes [(\mu+1)a_2^2 + \lambda a_1^2] \otimes a_2^3 \otimes a_1^4 \otimes u_1^5 + a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_2^4 \otimes u_2^5$. When $\mu = -1$, $T = T' + a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_2^4 \otimes u_2^5$, where $T' = a_1^1 \otimes a_1^2 \otimes \lambda a_1^3 \otimes a_1^4 \otimes u_1^5 + a_1^1 \otimes a_2^2 \otimes a_2^3 \otimes a_1^4 \otimes u_1^5 + (\lambda a_1^1 - a_2^1) \otimes a_1^2 \otimes a_2^3 \otimes a_1^4 \otimes u_1^5 \in \widehat{T}_{a_1^1 \otimes a_1^2 \otimes a_2^3 \otimes a_1^4 \otimes u_1^5} X_4$.

Exceptional Subcase 3: $T = (c_1^1 + y c_3^1) \otimes a_1^3 \otimes b_1^4 \otimes \cdots \otimes b_1^{k-1} \otimes a_1^k \otimes u_1^{k+1} + (c_2^1 + y c_3^1) \otimes$

$a_2^3\otimes b_1^4\otimes\cdots\otimes b_1^{k-1}\otimes a_1^k\otimes u_1^{k+1}+c_3^1\otimes(a_1^3+a_2^3)\otimes b_1^4\otimes\cdots\otimes b_1^{k-1}\otimes(xa_1^k+a_2^k)\otimes b_3^k(\alpha_2^k)$ for some $x,\ y\in\mathbb{C}$. It is harmless to assume $k=4$. When $y\mu+1\neq 0$, $T=[y\lambda a_1^1+(y\mu+1)a_2^1]\otimes a_1^2\otimes(a_1^3+\dfrac{y\mu}{y\mu+1}a_2^3)\otimes a_1^4\otimes u_1^5+a_1^1\otimes(\dfrac{y\lambda}{y\mu+1}a_1^2+a_2^2)\otimes a_2^3\otimes a_1^4\otimes u_1^5+(\lambda a_1^1+\mu a_2^1)\otimes a_1^2\otimes(a_1^3+a_2^3)\otimes(xa_1^4+a_2^4)\otimes b_3^4(\alpha_2^4)$. When $y\mu+1=0$, $T=T'+(\lambda a_1^1+\mu a_2^1)\otimes a_1^2\otimes(a_1^3+a_2^3)\otimes(xa_1^4+a_2^4)\otimes b_3^4(\alpha_2^4)$, where $T'=a_1^1\otimes a_1^2\otimes y\lambda a_1^3\otimes a_1^4\otimes u_1^5+y(\lambda a_1^1+\mu a_2^1)\otimes a_1^2\otimes a_2^3\otimes a_1^4\otimes u_1^5+a_1^1\otimes a_2^2\otimes a_2^3\otimes a_1^4\otimes u_1^5\in\widehat{T}_{a_1^1\otimes a_1^2\otimes a_2^3\otimes a_1^4\otimes u_1^5}X_4$.

Subtype 2: $T=\sum_{i=1}^{k}b_1^1\otimes\cdots\otimes b_1^{i-1}\otimes b_2^i\otimes b_1^{i+1}\otimes\cdots\otimes b_1^k+b_3^1\otimes\cdots\otimes b_3^k$. Since $T(A_3^*\otimes\cdots\otimes A_n^*)\subset V$, where V is spanned by $a_1^1\otimes a_1^2$, $a_1^1\otimes a_2^2$ and $a_2^1\otimes a_1^2$, and $b_1^1\otimes b_1^2\in V$ has rank 1, we can assume $b_1^1=a_1^1$, $b_1^2=a_1^2$. If b_2^1 and b_1^1 are linearly independent, then assume $b_2^1=a_2^1$, otherwise assume $b_3^1=a_2^1$. If b_2^2 and b_1^2 are linearly independent, then assume $b_2^2=a_2^2$, otherwise assume $b_3^2=a_2^2$. Since $b_3^1\otimes b_3^2$ is a rank 1 matrix in V, then $b_3^1\otimes b_3^2=(xa_1^1+ya_2^1)\otimes a_1^2$ or $b_3^1\otimes b_3^2=a_1^1\otimes(xa_1^2+ya_2^2)$. Hence we have three subcases:

Subcase 1: $T=\sum_{i=3}^{k}a_1^1\otimes a_1^2\otimes b_1^3\otimes\cdots\otimes b_1^{i-1}\otimes(b_2^i+\dfrac{2}{k-2}b_1^i)\otimes b_1^{i+1}\otimes\cdots\otimes b_1^k+a_2^1\otimes a_2^2\otimes b_3^3\otimes\cdots\otimes b_3^k$, which is discussed in **Case 3 Type 1**.

Subcase 2: $T=(a_2^1\otimes a_1^2+a_1^1\otimes a_2^2)\otimes b_1^3\otimes\cdots\otimes b_1^k+\sum_{i=3}^{k}a_1^1\otimes a_1^2\otimes b_1^3\otimes\cdots\otimes b_1^{i-1}\otimes b_2^i\otimes b_1^{i+1}\otimes\cdots\otimes b_1^k+a_1^1\otimes a_1^2\otimes b_3^3\otimes\cdots\otimes b_3^k$, which has been discussed in **Case 3 Type 1** after a change of basis.

Subcase 3: $T=(a_2^1\otimes a_1^2+a_1^1\otimes a_2^2)\otimes b_1^3\otimes\cdots\otimes b_3^k+\sum_{i=3}^{k}a_1^1\otimes a_1^2\otimes b_1^3\otimes\cdots\otimes b_1^{i-1}\otimes b_2^i\otimes b_1^{i+1}\otimes\cdots\otimes b_1^k+b_3^1\otimes b_3^2\otimes b_3^3\otimes\cdots\otimes b_3^k$, where b_3^1 and a_1^1 are independent, or b_3^2 and a_1^2 are independent. Let $c_1^1=a_1^1\otimes a_1^2$, $c_2^1=a_2^1\otimes a_1^2+a_1^1\otimes a_2^2$, $c_3^1=b_3^1\otimes b_3^2$, and V_1 denote the subspace of $A_1\otimes A_2$ spanned by c_1^1, c_2^1 and c_3^1, since $b_3^1\otimes b_3^2=(xa_1^1+ya_2^1)\otimes a_1^2$ or $b_3^1\otimes b_3^2=a_1^1\otimes(xa_1^2+ya_2^2)$, by the argument of **Case 2 Type 2**, we have $T\in\sigma_3(X_k)$ directly except for a few subcases. From the argument of **Case 2 Type 2**, we can see it is harmless to assume $k=4$ when considering these exceptional subcases.

Exceptional Case 1: $b^3_j = a^3_j$ for $j = 1, 2$, $b^4_j = a^4_1 \otimes b^5_1$ for some $b^5_1 \in A_5 \otimes \cdots \otimes A_n$, $b^3_3 = a^3_2 + \lambda a^3_1$ for some $\lambda \in \mathbb{C}$, $b^4_3 : A^*_4 \to A_5 \otimes \cdots \otimes A_n$ has rank 2, say $b^4_3 = a^4_1 \otimes u^5_1 + a^4_2 \otimes u^5_2$, and b^5_1 is a linear combination of u^5_1 and u^5_2, then by redefining a^1_2 and a^2_2, we can assume $c^1_3 = a^1_2 \otimes a^2_1$ or $a^1_1 \otimes a^2_2$, $u^5_1 = b^5_1 - \mu u^5_2$ for some $\mu \in \mathbb{C}$ or $u^5_2 = b^5_1$. If $c^1_3 = a^1_2 \otimes a^2_1$, $u^5_1 = b^5_1 - \mu u^5_2$, then $T = a^1_1 \otimes [a^2_2 - (1+\lambda)a^2_1] \otimes a^3_1 \otimes a^4_1 \otimes b^5_1 + (a^1_1 + a^1_2) \otimes a^2_1 \otimes [(1+\lambda)a^3_1 + a^3_2] \otimes a^4_1 \otimes b^5_1 + a^1_2 \otimes a^2_1 \otimes (a^3_2 + \lambda a^3_1) \otimes (a^4_2 - \mu a^4_1) \otimes u^5_2$. If $c^1_3 = a^1_2 \otimes a^2_1$, $u^5_2 = b^5_1$, $T = T' + a^1_1 \otimes (a^2_2 - \lambda a^2_1) \otimes a^3_1 \otimes a^4_1 \otimes b^5_1$, where $T' = a^1_2 \otimes a^2_1 \otimes a^3_1 \otimes a^4_1 \otimes b^5_1 + a^1_1 \otimes a^2_1 \otimes (a^3_2 + \lambda a^3_1) \otimes a^4_1 \otimes b^5_1 + a^1_2 \otimes a^2_1 \otimes (a^3_2 + \lambda a^3_1) \otimes a^4_1 \otimes u^5_1 + a^1_2 \otimes a^2_1 \otimes (a^3_2 + \lambda a^3_1) \otimes a^4_2 \otimes b^5_1 \in \widehat{T}_{a^1_2 \otimes a^2_1 \otimes (a^3_2 + \lambda a^3_1) \otimes a^4_1 \otimes b^5_1} X_4$. If $c^1_3 = a^1_1 \otimes a^2_2$, $u^5_1 = x b^5_1 + y u^5_2$ for some $0 \neq x, y \in \mathbb{C}$, $T = a^1_1 \otimes (a^2_1 + a^2_2) \otimes [x a^3_2 + (x\lambda + 1)a^3_1] \otimes a^4_1 \otimes b^5_1 + (a^1_2 - \frac{x\lambda + 1}{x} a^1_1) \otimes a^2_1 \otimes a^3_1 \otimes a^4_1 \otimes b^5_1 + a^1_1 \otimes a^2_2 \otimes (a^3_2 + \lambda a^3_1) \otimes (y a^4_1 + a^4_2) \otimes u^5_2$. If $c^1_3 = a^1_1 \otimes a^2_2$, $u^5_2 = b^5_1$, $T = T' + (a^1_2 - \lambda a^1_1) \otimes a^2_1 \otimes a^3_1 \otimes a^4_1 \otimes b^5_1$, where $T' = a^1_1 \otimes a^2_2 \otimes a^3_1 \otimes a^4_1 \otimes b^5_1 + a^1_1 \otimes a^2_1 \otimes (a^3_2 + \lambda a^3_1) \otimes a^4_1 \otimes b^5_1 + a^1_1 \otimes a^2_2 \otimes (a^3_2 + \lambda a^3_1) \otimes a^4_1 \otimes u^5_1 + a^1_1 \otimes a^2_2 \otimes (a^3_2 + \lambda a^3_1) \otimes a^4_2 \otimes b^5_1 \in \widehat{T}_{a^1_1 \otimes a^2_2 \otimes (a^3_2 + \lambda a^3_1) \otimes a^4_1 \otimes b^5_1} X_4$. If $x = 0$, $b^4_1 = b^4_2$ and $b^4_3 : A^*_4 \to A_5 \otimes \cdots \otimes A_n$ all have rank 1.

Exceptional Case 2: If $c^1_3 = a^1_2 \otimes a^2_1$, $b^4_1 = b^4_3 = a^4_1 \otimes b^5_1$ for some $b^5_1 \in A_5 \otimes \cdots \otimes A_n$, $b^4_2 = a^4_1 \otimes u^5_1 + a^4_2 \otimes u^5_2$ for some $u^5_1, u^5_2 \in A_5 \otimes \cdots \otimes A_n$, and $u^5_1 = x u^5_2 + y b^5_1$ for some $x, y \in \mathbb{C}$, then $T = a^1_1 \otimes [(y - \lambda - 1)a^2_1 + a^2_2] \otimes a^3_1 \otimes a^4_1 \otimes b^5_1 + (a^1_1 + a^1_2) \otimes a^2_1 \otimes [a^3_2 + (\lambda + 1)a^3_1] \otimes a^4_1 \otimes b^5_1 + a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes [x a^4_1 + a^4_2] \otimes u^5_2$. If $c^1_3 = a^1_1 \otimes a^2_2$, then $T = a^1_1 \otimes (a^2_1 + a^2_2) \otimes [a^3_2 + (\lambda + 1)a^3_1] \otimes a^4_1 \otimes b^5_1 + [a^1_2 + (y - \lambda - 1)a^1_1] \otimes a^2_1 \otimes a^3_1 \otimes a^4_1 \otimes b^5_1 + a^1_1 \otimes a^2_1 \otimes a^3_1 \otimes (x a^4_1 + a^4_2) \otimes u^5_2$.

Subtype 3: $T = \sum_{i<j} b^1_1 \otimes \cdots \otimes b^{i-1}_1 \otimes b^i_2 \otimes b^{i+1}_1 \otimes \cdots \otimes b^{j-1}_1 \otimes b^j_2 \otimes b^{j+1}_1 \otimes \cdots \otimes b^k_1 + \sum_{i=1}^{k} b^1_1 \otimes \cdots \otimes b^{i-1}_1 \otimes b^i_3 \otimes b^{i+1}_1 \otimes \cdots \otimes b^k_1$. If $b^1_2 = b^1_1$, $b^2_2 = b^2_1$ up to a scalar, then we can assume $b^1_2 = b^1_1 = a^1_1$, $b^2_2 = b^2_1 = a^2_2$, $b^1_3 = a^1_2$, and $b^2_3 = a^2_1$. This has been discussed in **Case 3 Type 1**. Otherwise, Let $c^1_1 = b^1_1 \otimes b^2_1$, $c^1_2 = b^1_2 \otimes b^2_1 + b^1_1 \otimes b^2_2$,

and $c_3^1 = b_2^1 \otimes b_2^2 + b_3^1 \otimes b_1^2 + b_1^1 \otimes b_3^2$. By the argument of **Case 2 Type 3**, we can see $T \in \sigma_3(X_k)$ except only one subcase, and by the argument of **Case 2 Type 3** it is harmless to assume $k = 4$ for the exceptional subcase, $b_j^3 = a_j^3$ for $j = 1, 2$, $b_3^3 = xa_1^3 + ya_2^3$ for some $x, y \in \mathbb{C}$, $b_1^4 = b_2^4 = a_1^4 \otimes b_1^5$ for some $b_1^5 \in A_5 \otimes \cdots \otimes A_n$, $b_3^4 : A_4^* \to A_5 \otimes \cdots \otimes A_n$ has rank 2, say $b_3^4 = a_1^4 \otimes u_1^5 + a_2^4 \otimes u_2^5$ for some $u_1^5, u_2^5 \in A_5 \otimes \cdots \otimes A_n$, u_2^5 and b_1^5 are linearly independent, and $u_1^5 = \lambda u_2^5 + \mu b_1^5$ for some $\lambda, \mu \in \mathbb{C}$. So $T = [(x+\mu-5)c_1^1 + 2c_2^1 + c_3^1] \otimes a_1^3 \otimes a_1^4 \otimes b_1^5 + [(y+2)c_1^1 + c_2^1] \otimes a_2^3 \otimes a_1^4 \otimes b_1^5 + c_1^1 \otimes a_1^3 \otimes (\lambda a_1^4 + a_2^4) \otimes u_2^5$, which has been discussed in **Case 3 Type 1**.

Subtype 4: $T = \sum_{i=2}^{k} b_2^1 \otimes b_1^2 \otimes \cdots \otimes b_1^{i-1} \otimes b_2^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^k + \sum_{i=1}^{k} b_1^1 \otimes \cdots \otimes b_1^{i-1} \otimes b_3^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^k$. If $b_2^1 = b_1^1$ up to a scalar, T has been discussed in **Case 3 Type 1**. Otherwise, let $c_1^1 = b_1^1 \otimes b_1^2$, $c_2^1 = b_2^1 \otimes b_1^2$, $c_3^1 = b_2^1 \otimes b_2^2 + b_3^1 \otimes b_1^2$. From the argument of **Case 2 Type 4**, we can see $T \in \sigma_3(X_k)$.

3.2.4 Case 4: $T \in \sigma_2(X_2)$

We assume $T \in \sigma_2(X_{k-1})$, and show $T \in \sigma_3(X_k)$ by checking each type of the normal forms in Proposition 6.

Type 1: $T = b_1^1 \otimes \cdots \otimes b_1^k$. Then $T = b_1^1 \otimes \cdots \otimes b_1^{k-1} \otimes a_1^k \otimes b_1^k(\alpha_1^k) + b_1^1 \otimes \cdots \otimes b_1^{k-1} \otimes a_2^k \otimes b_1^k(\alpha_2^k) + b_1^1 \otimes \cdots \otimes b_1^{k-1} \otimes a_3^k \otimes b_1^k(\alpha_3^k)$.

Type 2: $T = b_1^1 \otimes \cdots \otimes b_1^k + b_2^1 \otimes \cdots \otimes b_2^k$. Since there is some $1 \le i \le k-1$ such that b_1^i and b_2^i are linearly independent, then $\dim T(A_i^* \otimes A_k^*) \le 3$ implies at least one of b_1^k and $b_2^k : A_k^* \to A_{k+1} \otimes \cdots \otimes A_n$ has rank 1, and the other one has rank at most 2, say $b_1^k = a_1^k \otimes b_1^{k+1}$ and $b_2^k = a_1^k \otimes b_2^{k+1} + a_2^k \otimes b_3^{k+1}$ for some $b_1^{k+1}, b_2^{k+1}, b_3^{k+1} \in A_{k+1} \otimes \cdots \otimes A_n$. Hence, $T = b_1^1 \otimes \cdots \otimes b_1^{k-1} \otimes a_1^k \otimes b_1^{k+1} + b_2^1 \otimes \cdots \otimes b_2^{k-1} \otimes a_1^k \otimes b_2^{k+1} + b_2^1 \otimes \cdots \otimes b_2^{k-1} \otimes a_2^k \otimes b_3^{k+1}$.

Type 3: $T = \sum_{i=1}^{k} b_1^1 \otimes \cdots \otimes b_1^{i-1} \otimes b_2^i \otimes b_1^{k+1} \otimes \cdots \otimes b_1^k$. Without loss of generality, we can assume b_1^1 and b_2^1 are linearly independent, and b_1^2 and b_2^2 are linearly independent, then $\dim T(A_1^* \otimes A_k^*) \le 3$ implies $b_1^k : A_k^* \to A_{k+1} \otimes \cdots \otimes A_n$ has rank 1, say

$b_1^k = a_1^k \otimes b_1^{k+1}$ for some $b_1^{k+1} \in A_{k+1} \otimes \cdots \otimes A_n$, and $\{b_1^{k+1}, b_2^k(\alpha_2^k), b_2^k(\alpha_3^k)\}$ spans an at most 2 dimensional subspace. Thus we can assume $b_2^k(\alpha_3^k) = xb_1^{k+1} + yb_2^k(\alpha_2^k)$ for some $x, y \in \mathbb{C}$, then $T = T' + a_1^1 \otimes \cdots \otimes b_1^{k-1} \otimes (a_2^k + ya_3^k) \otimes b_2^k(\alpha_2^k)$, where $T' = \sum_{i=1}^{k-2} b_1^1 \otimes \cdots \otimes b_1^{i-1} \otimes b_2^i \otimes b_1^{i+1} \otimes \cdots \otimes b_1^{k-1} \otimes a_1^k \otimes b_1^{k+1} + b_1^1 \otimes \cdots \otimes b_1^{k-1} \otimes a_1^k \otimes b_2^k(\alpha_1^k) + a_1^1 \otimes \cdots \otimes b_1^{k-1} \otimes xa_3^k \otimes b_1^{k+1} \in \widehat{T}_{b_1^1 \otimes \cdots \otimes b_1^{k-1} \otimes a_1^k \otimes b_1^{k+1}} X_k$.

4. ON THE GEOMETRY OF TENSOR NETWORK STATES *

In this chapter we study tensor network states, and answer a question of L. Grasedyck that arose in quantum information theory, showing that the limit of tensors in a space of tensor network states need not be a tensor network state. [34]

4.1 Definitions

Let $V_1, \ldots, V_n$ be complex vector spaces, let $\mathbf{v}_i = \dim V_i$. Let Γ be a graph with n vertices v_j, $1 \le j \le n$, and m edges e_s, $1 \le s \le m$, and let $\vec{e} = (e_1, \ldots, e_m) \in \mathbb{N}^m$. Associate V_j to the vertex v_j and an auxiliary vector space E_s of dimension e_s to the edge e_s. Make Γ into a directed graph. (The choice of directions will not effect the end result.) Let $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$. For Γ, $s \in e(j)$ means e_s is incident to v_j, $s \in in(j)$ are the incoming edges and $s \in out(j)$ the outgoing edges.

Define a tensor network state $TNS(\Gamma, \vec{e}, \mathbf{V})$ to be:

$$TNS(\Gamma, \vec{e}, \mathbf{V}) := \tag{4.1}$$
$$\{T \in \mathbf{V} \mid \exists T_j \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*), T = Con(T_1 \otimes \cdots \otimes T_n)\},$$

where Con is the contraction of all the E_s's with all the E_s^*'s.

Example 1. *Let Γ be a graph with two vertices and one edge connecting them, then $TNS(\Gamma, e_1, V_1 \otimes V_2)$ is just $\hat{\sigma}_{e_1}(Seg(\mathbb{P}V_1 \times \mathbb{P}V_2))$, the cone over the e_1-st secant variety of the Segre variety. To see this, let $\epsilon_1, \ldots, \epsilon_{e_1}$ be a basis of E_1 and $\epsilon^1, \ldots, \epsilon^{e_1}$ the dual basis of E^*. Assume, to avoid trivialities, that $\boldsymbol{v}_1, \boldsymbol{v}_2 \ge e_1$. Given $T_1 \in V_1 \otimes E_1$ we may write $T_1 = u_1 \otimes \epsilon_1 + \cdots + u_{e_1} \otimes \epsilon_{e_1}$ for some $u_\alpha \in V_1$. Similarly, given*

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$T_2 \in V_2 \otimes E_1^*$ *we may write* $T_1 = w_1 \otimes \epsilon^1 + \cdots + w_{e_1} \otimes \epsilon^{e_1}$ *for some* $w_\alpha \in V_2$. *Then* $Con(T_1 \otimes T_2) = u_1 \otimes w_1 + \cdots + u_{e_1} \otimes w_{e_1}$.

The graph used to define a set of tensor network states is often modeled to mimic the physical arrangement of the particles, with edges connecting nearby particles, as nearby particles are the ones likely to be entangled.

Remark 4. *The construction of tensor network states in the physics literature does not use a directed graph, because all vector spaces are Hilbert spaces, and thus self-dual. However the sets of tensors themselves do not depend on the Hilbert space structure of the vector space, which is why we omit this structure. The small price to pay is the edges of the graph must be oriented, but all orientations lead to the same set of tensor network states.*

4.2 Grasedyck's question

Lars Grasedyck asked:

*I*s $TNS(\Gamma, \vec{e}, \mathbf{V})$ Zariski closed? That is, given a sequence of tensors $T_\epsilon \in \mathbf{V}$ that converges to a tensor T_0, if $T_\epsilon \in TNS(\Gamma, \vec{e}, \mathbf{V})$ for all $\epsilon \neq 0$, can we conclude $T_0 \in TNS(\Gamma, \vec{e}, \mathbf{V})$?

He mentioned that he could show this to be true when Γ was a tree, but did not know the answer when Γ is a triangle. In the physics literature they were implicitly assuming tensor network states were closed, so he asked this question.

Definition 9. *A dimension* $\boldsymbol{v}_j$ *is critical, resp. subcritical, resp. supercritical, if* $\boldsymbol{v}_j = \Pi_{s \in e(j)} e_s$, *resp.* $\boldsymbol{v}_j \leq \Pi_{s \in e(j)} e_s$, *resp.* $\boldsymbol{v}_j \geq \Pi_{s \in e(j)} e_s$. *If* $TNS(\Gamma, \vec{e}, \boldsymbol{V})$ *is critical for all* j, *we say* $TNS(\Gamma, \vec{e}, \boldsymbol{V})$ *is critical, and similarly for sub- and super-critical.*

Theorem 10. $TNS(\Gamma, \vec{e}, \boldsymbol{V})$ *is not Zariski closed for any* Γ *containing a cycle whose vertices have non-subcritical dimensions.*

Notation 1. *$GL(V)$ denotes the group of invertible linear maps $V \to V$. $GL(V_1) \times \cdots \times GL(V_n)$ acts on $V_1 \otimes \cdots \otimes V_n$ by $(g_1, \ldots, g_n) \cdot v_1 \otimes \cdots \otimes v_n = (g_1 v_1) \otimes \cdots \otimes (g_n v_n)$. (Here $v_j \in V_j$ and the action on a tensor that is a sum of rank one tensors is the sum of the actions on the rank one tensors.) Let $\mathrm{End}(V)$ denote the set of all linear maps $V \to V$. We adopt the convention that $\mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_n)$ acts on $V_1 \otimes \cdots \otimes V_n$ by $(Z_1, \ldots Z_n) \cdot v_1 \otimes \cdots \otimes v_n = (Z_1 v_1) \otimes \cdots \otimes (Z_n v_n)$. Let $\mathfrak{gl}(V)$ denote the Lie algebra of $GL(V)$. It is naturally isomorphic to $\mathrm{End}(V)$ but it acts on $V_1 \otimes \cdots \otimes V_n$ via the Leibnitz rule: $(X_1, \ldots, X_n) \cdot v_1 \otimes \cdots \otimes v_n = (X_1 v_1) \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes (X_2 v_2) \otimes v_3 \otimes \cdots \otimes v_n + \cdots + v_1 \otimes \cdots \otimes v_{n-1} \otimes (X_n v_n)$. (This is because elements of the Lie algebra should be thought of as derivatives of curves in the Lie group at the identity.) If $X \subset V$ is a subset, $\overline{X} \subset V$ denotes its closure. This closure is the same whether one uses the Zariski closure, which is the common zero set of all polynomials vanishing on X, or the Euclidean closure, where one fixes a metric compatible with the linear structure on V and takes the closure with respect to limits.*

4.3 Connections to the GCT program

The triangle case is especially interesting because in the critical dimension case it corresponds to

$$End(V_1) \times End(V_2) \times End(V_3) \cdot Mmult_{e_3,e_2,e_1},$$

where $Mmult_{e_3,e_2,e_1} \in V_1 \otimes V_2 \otimes V_3$ is the matrix multiplication operator. In Geometric Complexity Theory (GCT) people study $Mmult$ and its $GL(V_1) \times GL(V_2) \times GL(V_3)$ orbit closure ([6]) which is a toy case of the varieties introduced by Mulmuley and Sohoni [7,39,40]. The varieties are $\overline{GL_{n^2} \cdot \det_n}$ and $\overline{GL_{n^2} \cdot l^{n-m}\,\mathrm{perm}_n}$, where $\det_n \in S^n\mathbb{C}^{n^2}$ is the determinant, $n > m$, $l \in S^1\mathbb{C}^1$, $\mathrm{perm}_m \in S^m\mathbb{C}^{m^2}$ is the permanent, and an inclusion $\mathbb{C}^{m^2+1} \subset \mathbb{C}^{n^2}$ has been chosen. It was shown that $End_{\mathbb{C}^{n^2}} \cdot \det_n \neq$

$\overline{GL_{n^2} \cdot \det_n}$ [32], and determining the difference between these sets is a subject of current research.

The critical loop case with $e_s = 3$ for all s is also related to the GCT program, as it corresponds to the multiplication of n matrices of size three. As a tensor, it may be thought of as a map $(X_1, \dots, X_n) \mapsto tr(X_1 \cdots X_n)$. This sequence of functions indexed by n, considered as a sequence of homogeneous polynomials of degree n on $V_1 \oplus \cdots \oplus V_n$, is complete for the class $\mathbf{VP}_e$ of sequences of polynomials of small formula size, see [41].

4.4 Critical loops

Proposition 7. *Let $\boldsymbol{v}_1 = e_2e_3, \boldsymbol{v}_2 = e_3e_1, \boldsymbol{v}_3 = e_2e_1$. Then $TNS(\triangle, (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3), V_1 \otimes V_2 \otimes V_3)$ consists of matrix multiplication and its degenerations (and their different expressions after changes of bases), i.e. $TNS(\triangle, (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3), V_1 \otimes V_2 \otimes V_3) = \mathrm{End}(V_1) \times \mathrm{End}(V_2) \times \mathrm{End}(V_3) \cdot M_{e_2,e_3,e_1}$. It has dimension $e_2^2e_3^2 + e_2^2e_1^2 + e_3^2e_1^2 - (e_2^2 + e_3^2 + e_1^2 - 1)$. More generally, if Γ is a critical loop, $TNS(\Gamma, (e_ne_1, e_1e_2, \dots, e_{n-1}e_n), V_1 \otimes \cdots \otimes V_n)$ is $\mathrm{End}(V_1) \times \cdots \times \mathrm{End}(V_n) \cdot M_{\vec{e}}$, where $M_{\vec{e}} : V_1 \otimes \cdots \otimes V_n \to \mathbb{C}$ is the matrix multiplication operator $(X_1, \dots, X_n) \mapsto \mathrm{trace}(X_1 \cdots X_n)$.*

Proof. For the triangle case, a generic element $T_1 \in E_2 \otimes E_3^* \otimes V_1$ may be thought of as a linear isomorphism $E_2^* \otimes E_3 \to V_1$, identifying V_1 as a space of $e_2 \times e_3$-matrices, and similarly for V_2, V_3. Choosing bases $e_s^{u_s}$ for E_s^*, with dual basis $e_{u_s,s}$ for E_s, induces bases $x_{u_3}^{u_2}$ for V_1 etc.. Let $1 \leq i \leq e_2$, $1 \leq \alpha \leq e_3$, $1 \leq u \leq e_1$. Then $con(T_1 \otimes T_2 \otimes T_3) = \sum x_\alpha^i \otimes y_u^\alpha \otimes z_i^u$ which is the matrix multiplication operator. The general case is similar. $\square$

Proposition 8. *The Lie algebra of the stabilizer of $M_{e_ne_1,e_1e_2,\dots,e_{n-1}e_n}$ in $GL(V_1) \times$*

$\cdots \times GL(V_n)$ is the image of $\mathfrak{sl}(E_1) \oplus \cdots \oplus \mathfrak{sl}(E_n)$ under the map

$$\alpha_1 \oplus \cdots \oplus \alpha_n \mapsto (Id_{E_n} \otimes \alpha_1, -\alpha_1^T \otimes Id_{E_2}, 0, \dots, 0) + (0, Id_{E_1} \otimes \alpha_2, -\alpha_2^T \otimes Id_{E_3}, 0, \dots, 0) + \cdots + (-\alpha_n^T \otimes Id_{E_1}, 0, \dots, 0, Id_{E_{n-1}} \otimes \alpha_n).$$

Here $\mathfrak{sl}(E_j) \subset \mathfrak{gl}(E_j)$ denotes the traceless endomorphisms and T as a superscript denotes transpose (which is really just cosmetic).

The proof is safely left to the reader.

Large loops are referred to as "1-D systems with periodic boundary conditions" in the physics literature and are often used in simulations. By Proposition 8, for a critical loop, $\dim(TNS(\Gamma, \vec{e}, \mathbf{V}) = e_1^2 e_2^2 + \cdots + e_{n-1}^2 e_n^2 + e_n^2 e_1^2 - (e_1^2 + \cdots + e_n^2 - 1)$, compared with the ambient space which has dimension $e_1^2 \cdots e_n^2$. For example, when $e_j = 2$ for all j, $\dim(TNS(\Gamma, \vec{e}, \mathbf{V})) = 12n + 1$, compared with $\dim \mathbf{V} = 4^n$.

4.5 Zariski closure

Theorem 11. *Let $\boldsymbol{v}_1 = e_2 e_3, \boldsymbol{v}_2 = e_3 e_1, \boldsymbol{v}_3 = e_2 e_1$. Then $TNS(\triangle, (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3), V_1 \otimes V_2 \otimes V_3)$ is not Zariski closed. More generally any $TNS(\Gamma, \vec{e}, \boldsymbol{V})$ where Γ contains a cycle with no subcritical vertex is not Zariski closed.*

Proof. Were $T(\triangle) := TNS(\triangle, (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), V_1 \otimes V_2 \otimes V_3)$ Zariski closed, it would be $\overline{GL(V_1) \times GL(V_2) \times GL(V_3) \cdot M_{e_2,e_3,e_1}}$. To see this, note that the $G = GL(V_1) \times GL(V_2) \times GL(V_3)$ orbit of matrix multiplication is a Zariski open subset of $T(\triangle)$ of the same dimension as $T(\triangle)$.

We need to find a curve $g(t) = (g_1(t), g_2(t), g_3(t))$ such that $g_j(t) \in GL(V_j)$ for all $t \neq 0$ and $\lim_{t \to 0} g(t) \cdot M_{e_2,e_3,e_1}$ is both defined and not in $\mathrm{End}(V_1) \times \mathrm{End}(V_2) \times \mathrm{End}(V_3) \cdot M_{e_2,e_3,e_1}$.

Note that for $(X,Y,Z) \in GL(V_1) \times GL(V_2) \times GL(V_3)$, we have

$$(X,Y,Z) \cdot M_{e_2,e_3,e_1}(P,Q,R) = \mathrm{trace}(X(P)Y(Q)Z(R)).$$

Here $X : E_2^* \otimes E_3 \to E_2^* \otimes E_3$, $Y : E_3^* \otimes E_1 \to E_3^* \otimes E_1$, $Z : E_1^* \otimes E_2 \to E_1^* \otimes E_2$.

Take subspaces $U_{E_2E_3} \subset E_2^* \otimes E_3$, $U_{E_3E_1} \subset E_3^* \otimes E_1$. Let $U_{E_1E_2} := Con(U_{E_2E_3}, U_{E_3E_1}) \subset E_2^* \otimes E_1$ be the images of all the $pq \in E_2^* \otimes E_1$ where $p \in U_{E_2E_3}$ and $q \in U_{E_3E_1}$ (i.e., the matrix multiplication of all pairs of elements). Take X_0, Y_0, Z_0 respectively to be the projections to $U_{E_2E_3}$, $U_{E_3E_1}$ and $U_{E_1E_2}^{\perp}$. Let X_1, Y_1, Z_1 be the projections to complementary spaces (so, e.g., $X_0 + X_1 = Id_{V_1^*}$). For $P \in V_1^*$, write $P_0 = X_0(P)$ and $P_1 = X_1(P)$, and similarly for Q, R.

Take the curve (X_t, Y_t, Z_t) with $X_t = \frac{1}{\sqrt{t}}(X_0 + tX_1)$, $Y_t = \frac{1}{\sqrt{t}}(Y_0 + tY_1)$, $Z_t = \frac{1}{\sqrt{t}}(Z_0 + tZ_1)$. Then the limiting tensor, as a map $V_1^* \times V_2^* \times V_3^* \to \mathbb{C}$, is

$$(P,Q,R) \mapsto \mathrm{trace}(P_0Q_0R_1) + \mathrm{trace}(P_0Q_1R_0) + \mathrm{trace}(P_1Q_0R_0).$$

Call this tensor $\tilde{M}$. First observe that $\tilde{M}$ uses all the variables (i.e., considered as a linear map $\tilde{M} : V_1^* \to V_2 \otimes V_3$, it is injective, and similarly for its cyclic permutations). Thus it is either in the orbit of matrix multiplication or a point in the boundary that is not in $\mathrm{End}(V_1) \times \mathrm{End}(V_2) \times \mathrm{End}(V_3) \cdot M_{e_2,e_3,e_1}$, because all such boundary points have at least one such linear map non-injective.

It remains to show that there exist $\tilde{M}$ such that $\tilde{M} \notin G \cdot M_{e_2,e_3,e_1}$. To prove some $\tilde{M}$ is a point in the boundary, we compute the Lie algebra of its stabilizer and show it has dimension greater than the the dimension of the stabilizer of matrix multiplication. One may take block matrices, e.g.,

$$X_0 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},\ X_1 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

and Y_0, Y_1 have similar shape, but Z_0, Z_1 have the shapes reversed. Here one takes any splitting $e_j = e_j' + e_j''$ to obtain the blocks.

For another example, if one takes $e_j = e$ for all j, X_0, Y_0, Z_1 to be the diagonal matrices and X_1, Y_1, Z_0 to be the matrices with zero on the diagonal, then one obtains a stabilizer of dimension $4e^2 - 2e > 3e^2 - 1$. (This example coincides with the previous one when all $e_j = 2$.)

To calculate the stabilizer of $\tilde{M}$, first write down the tensor expression of $\tilde{M} \in V_1 \otimes V_2 \otimes V_3$ with respect to fixed bases of V_1, V_2, V_3. Then set an equation $(X, Y, Z).\tilde{M} = 0$ where $X \in \mathfrak{gl}(V_1)$, $Y \in \mathfrak{gl}(V_2)$ and $Z \in \mathfrak{gl}(V_3)$ are unknowns. Recall that here the action of (X, Y, Z) on $\tilde{M}$ is the Lie algebra action, so we obtain a system of linear equations. Finally we solve this system of linear equations and count the dimension of the solution space. This dimension is the dimension of the stabilizer of $\tilde{M}$ in $GL(V_1) \times GL(V_2) \times GL(V_3)$.

To give an explicit example, let $e_1 = e_2 = e_3 = e$ and let $X_0 = \mathrm{diag}(x_1^1, \ldots, x_e^e)$, $Y_0 = \mathrm{diag}(y_1^1, ..., y_e^e)$, $Z_0 = \mathrm{diag}(z_1^1, ..., z_e^e)$, $X_1 = (x_j^i) - X_0$, $Y_1 = (y_j^i) - Y_0$, $Z_1 = (z_j^i) - Z_0$. Then

$$\tilde{M} = \sum_{i,j=1}^{e} (x_j^i y_j^j + x_i^i y_j^i) z_i^j.$$

Let $X = \sum a_{\binom{k}{l}}^{\binom{i}{j}} X_{\binom{i}{j}}^{\binom{k}{l}}$ be an element of $\mathfrak{gl}(V_1)$, where $\{X_{\binom{i}{j}}^{\binom{k}{l}}\}$ is a basis of $\mathfrak{gl}(V_1)$, and define Y and Z in the same pattern with coefficients $b_{\binom{k}{l}}^{\binom{i}{j}}$'s and $c_{\binom{k}{l}}^{\binom{i}{j}}$'s, respectively. Consider the equation $(X, Y, Z) \cdot T = 0$ and we want to solve this equation for $a_{\binom{k}{l}}^{\binom{i}{j}}$'s,

$b_{\binom{k}{l}}^{\binom{i}{j}}$'s and $c_{\binom{k}{l}}^{\binom{i}{j}}$'s. For these equations to hold, the coefficients of z_i^j's must be zero. That is, for each pair (j,i) of indices we have:

$$\sum_{k,l=1}^{e} a_{\binom{k}{l}}^{\binom{i}{j}} x_l^k y_j^j + b_{\binom{k}{l}}^{\binom{j}{j}} x_j^i y_k^l + a_{\binom{k}{l}}^{\binom{i}{i}} x_l^k y_j^i + b_{\binom{k}{l}}^{\binom{i}{j}} x_i^i y_l^k + c_{\binom{j}{i}}^{\binom{l}{k}} (x_l^k y_l^l + x_k^k y_l^k) = 0.$$

For these equations to hold, the coefficients of y_s^r's must be zero. For example, if $s \neq j$, $r \neq s$ then we have:

$$b_{\binom{r}{s}}^{\binom{j}{j}} x_j^i + b_{\binom{r}{s}}^{\binom{i}{j}} x_i^i + c_{\binom{j}{i}}^{\binom{s}{r}} x_r^r = 0$$

Now coefficients of x terms must be zero, for instance, if $i \neq j$ and $i \neq r$, then we have:

$$b_{\binom{r}{s}}^{\binom{j}{j}} = 0, \quad b_{\binom{r}{s}}^{\binom{i}{j}} = 0, \quad c_{\binom{j}{i}}^{\binom{s}{r}} = 0.$$

If one writes down and solves all such linear equations, the dimension of the solution is $4e^2 - 2e$.

The same construction works for larger loops and cycles in larger graphs as it is essentially local - one just takes all other curves the constant curve equal to the identity. $\square$

Remark 5. *When $e_1 = e_2 = e_3 = 2$ we obtain a codimension one component of the boundary. In general, the dimension of the stabilizer is much larger than the dimension of G, so the orbit closures of these points do not give rise to codimension one components of the boundary. It remains an interesting problem to find the codimension one components of the boundary.*

4.6 Algebraic geometry perspective

We recast the previous section in the language of algebraic geometry and put it in a larger context. This section also serves to motivate the proof of the previous section.

To make the parallel with the GCT program clearer, we describe the Zariski closure as the cone over the "closure" of the image of the rational map (i.e., the closure of the map defined on a Zariski open subset)

$$\begin{aligned} \mathbb{P}End(V_1) \times \mathbb{P}End(V_2) \times \mathbb{P}End(V_3) &\dashrightarrow \mathbb{P}(V_1 \otimes V_2 \otimes V_3) \\ ([X],[Y],[Z]) &\mapsto (X,Y,Z) \cdot [M_{e_2,e_3,e_1}]. \end{aligned} \tag{4.2}$$

(Compare with the map ψ in [7, §7.2].) A dashed arrow is used to indicate the map is not everywhere defined.

The indeterminacy locus (that is, points $([X],[Y],[Z])$ where the map is not defined), consists of $([X],[Y],[Z])$ such that for all triples of matrices P, Q, R,

$$\text{trace}(X(P)Y(Q)Z(R)) = 0.$$

In principle one can obtain (4.5) as the image of a map from a succession of blow-ups of $\mathbb{P}End(V_1) \times \mathbb{P}End(V_2) \times \mathbb{P}End(V_3)$.

One way to attain a point in the indeterminacy locus is to take $([X_0],[Y_0],[Z_0])$ as described in the proof. Taking a curve in G that limits to this point may or may not give something new. In the proof we gave two explicit choices that do give something new.

A more invariant way to discuss that $\tilde{M} \notin End(V_1) \times End(V_2) \times End(V_3) \cdot M_{e_2,e_3,e_1}$

is to consider an auxiliary variety, called a *subspace variety*,

$$Sub_{f_1,\dots,f_n}(\mathbf{V}) := \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists V_j' \subset V_j, \dim V_j' = f_j,\ and\ T \in V_1' \otimes \cdots \otimes V_n'\},$$

and observe that if $T \in \times_j \operatorname{End}(V_j) \cdot M_{\vec{\mathbf{e}}}$ and $T \notin \times_j GL(V_j) \cdot M_{\vec{\mathbf{e}}}$, then $T \in Sub_{f_1,\dots,f_n}(\mathbf{V})$ where $f_j < e_j$ for at least one j.

The statement that "$\tilde{M}$ uses all the variables" may be rephrased as saying that $\tilde{M} \notin Sub_{e_2e_3-1,e_2e_1-1,e_3e_1-1}(V_1 \otimes V_2 \otimes V_3)$.

4.7 Reduction from the supercritical case to the critical case with the same graph

For a vector space W, let $G(k, W)$ denote the Grassmannian of k-planes through the origin in W. Let $\mathcal{S} \to G(k, W)$ denote the tautological rank k vector bundle whose fiber over $E \in G(k, W)$ is the k-plane E. Assume $f_j \leq \mathbf{v}_j$ for all j with at least one inequality strict. Form the vector bundle $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ over $G(f_1, V_1) \times \cdots \times G(f_n, V_n)$, where $\mathcal{S}_j \to G(f_j, V_j)$ are the tautological subspace bundles. Note that the total space of $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$ maps to $\mathbf{V}$ with image $Sub_{\vec{f}}(\mathbf{V})$. Define a fiber sub-bundle, whose fiber over $(U_1 \times \cdots \times U_n) \in G(f_1, V_1) \times \cdots \times G(f_n, V_n)$ is $TNS(\Gamma, \vec{e}, U_1 \otimes \cdots \otimes U_n)$. Denote this bundle by $TNS(\Gamma, \vec{e}, \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n)$.

The supercritical cases may be realized, in the language of Kempf, as a "collapsing of a bundle" over the critical cases as follows:

Proposition 9. *Assume $f_j := \Pi_{s \in e(j)} e_s \leq \mathbf{v}_j$. Then $TNS(\Gamma, \vec{e}, \boldsymbol{V})$ is the image of the bundle $TNS(\Gamma, \vec{e}, \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n)$ under the map to $\boldsymbol{V}$. In particular*

$$\dim(TNS(\Gamma, \vec{e}, \boldsymbol{V})) = \dim(TNS(\Gamma, \vec{e}, \mathbb{C}^{f_1} \otimes \cdots \otimes \mathbb{C}^{f_n})) + \sum_{j=1}^{n} f_j(\boldsymbol{v}_j - f_j).$$

Proof. If $\Pi_{s \in e(j)} e_s \leq \mathbf{v}_j$, then any tensor $T \in V_j \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*)$, must lie in some $V_j' \otimes (\otimes_{s \in in(j)} E_s) \otimes (\otimes_{t \in out(j)} E_t^*)$ with $\dim V_j' = f_j$. The space $TNS(\Gamma, \vec{e}, \mathbf{V})$

is the image of this subbundle under the map to $\mathbf{V}$. $\square$

This type of bundle construction is standard, see [29, 50]. Using the techniques in [50], one may reduce questions about a supercritical case to the corresponding critical case.

4.8 Reduction of cases with subcritical vertices of valence one

The subcritical case in general can be understood in terms of projections of critical cases, but this is not useful for extracting information. However, if a subcritical vertex has valence one, one may simply reduce to a smaller graph as we now describe.

Proposition 10. *Let $TNS(\Gamma, \vec{e}, \boldsymbol{V})$ be a tensor network state, let v be a vertex of Γ with valence one. Relabel the vertices such that $v = v_1$ and so that v_1 is attached by e_1 to v_2. If $\boldsymbol{v}_1 \leq e_1$, then $TNS(\Gamma, \vec{e}, V_1 \otimes \cdots \otimes V_n) = TNS(\tilde{\Gamma}, \tilde{\vec{e}}, \tilde{V}_1 \otimes V_3 \otimes \cdots \otimes V_n)$, where $\tilde{\Gamma}$ is Γ with v_1 and e_1 removed, $\tilde{\vec{e}}$ is the vector $(e_2, \ldots, e_n)$ and $\tilde{V}_1 = V_1 \otimes V_2$.*

Proof. A general element in $TNS(\Gamma, \vec{e}, V_1 \otimes \cdots \otimes V_n)$ is of the form $\sum_{i,j=1}^{e_1,e_2} u_i \otimes v_{iz} \otimes w_z$, where $w_z \in V_3 \otimes \cdots \otimes V_n$. Obviously, $TNS(\Gamma, \vec{e}, V_1 \otimes \cdots \otimes V_n) \subseteq TNS(\tilde{\Gamma}, \tilde{\vec{e}}, \tilde{V}_1 \otimes V_3 \otimes \cdots \otimes V_n) =: TNS(\tilde{\Gamma}, \tilde{\vec{e}}, \tilde{\mathbf{V}})$. Conversely, a general element in $TNS(\tilde{\Gamma}, \tilde{\vec{e}}, \tilde{\mathbf{V}}))$ is of the form $\sum_z X_z \otimes w_z$, $X_z \in V_1 \otimes V_2$. Since $\mathbf{v}_1 \leq e_1$, we may express X_z in the form $\sum_{i=1}^{e_1} u_i \otimes v_{iz}$, where $u_1, \ldots, u_{v_1}$ is a basis of V_1. Therefore, $TNS(\Gamma, \vec{e}, \mathbf{V}) \supseteq TNS(\tilde{\Gamma}, \tilde{\vec{e}}, \tilde{\mathbf{V}})$. $\square$

4.9 Trees

With trees one can apply the two reductions successively to reduce to a tower of bundles where the fiber in the last bundle is a linear space. The point is that a critical vertex is both sub- and supercritical, so one can reduce at valence one vertices iteratively. Here are a few examples in the special case of chains. The result is similar to the Allman-Rhodes reduction theorem for phylogenetic trees [2].

Example 2. *Let* Γ *be a chain with* 3 *vertices. If it is supercritical,* $TNS(\Gamma, \vec{e}, \boldsymbol{V}) = V_1 \otimes V_2 \otimes V_3$. *Otherwise* $TNS(\Gamma, \vec{e}, \boldsymbol{V}) = Sub_{e_1, e_1 e_2, e_2}(V_1 \otimes V_2 \otimes V_3)$.

Example 3. *Let* Γ *be a chain with* 4 *vertices. If* $\boldsymbol{v}_1 \leq e_1$ *and* $\boldsymbol{v}_4 \leq e_3$*, then, writing* $W = V_1 \otimes V_2$ *and* $U = V_3 \otimes V_4$*, by Proposition 10,* $TNS(\Gamma, \vec{e}, \boldsymbol{V})$ *is the set of rank at most* e_2 *elements in* $W \otimes U$ *(the secant variety of the two-factor Segre). Other chains of length four have similar complete descriptions.*

Example 4. *Let* Γ *be a chain with* 5 *vertices. Assume that* $\boldsymbol{v}_1 \leq e_1$*,* $\boldsymbol{v}_5 \leq e_4$ *and* $\boldsymbol{v}_1 \boldsymbol{v}_2 \geq e_2$ *and* $\boldsymbol{v}_4 \boldsymbol{v}_5 \geq e_3$*. Then* $TNS(\Gamma, \vec{e}, \boldsymbol{V})$ *is the image of a bundle over* $G(e_2, V_1 \otimes V_2) \times G(e_3, V_4 \otimes V_5)$ *whose fiber is the set of tensor network states associated to a chain of length three.*

5. SUMMARY

In this thesis we study two feasible spaces of tensors, the third secant variety of the product of n projective spaces $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$, and tensor network states. These spaces arise in numerous applications such as signal processing and quantum information theory.

We determine the set theoretic defining equations of $\sigma_3(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$. For higher secant varieties of Segre varieties, it is known [17] that there is a uniform bound $d(r)$ such that $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ is defined by equations of degrees at most $d(r)$ for any n, and [30] when $\dim A_i \geq r$ for all $1 \leq i \leq n$, the equations of $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ can be obtained from the equations of the r-th secant variety of the Segre product of n copies of $\mathbb{P}^{r-1}$'s, i.e. $\sigma_r\left(Seg(\underbrace{\mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1}}_{n-\text{copies}})\right)$. We conjecture that when $\dim A_i \geq r$ for all i, the equations for $\sigma_r(Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n))$ can be obtained from the equations for the r-th secant variety of the Segre product of only r copies of $\mathbb{P}^{r-1}$'s, i.e. $\sigma_r\left(Seg(\underbrace{\mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1}}_{r-\text{copies}})\right)$.

We discuss under what conditions tensor network states are closed under the Zariski topology, equivalently (in our situation) the Euclidean topology. The research of the GL_{n^2} orbit closure of the determinant $\det_n$, $\overline{GL_{n^2} \cdot \det_n}$, in the GCT program provides additional motivation to study the geometry of tensor network states. In particular, when Γ is a triangle, the corresponding tensor network state is $\overline{GL(V_1) \times GL(V_2) \times GL(V_3) \cdot Mult}$, where $Mult$ is the matrix multiplication operator. Very little of the geometric properties even the triangle tensor network state are known, for example, the number of irreducible components of it is still unknown.

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