6D (2, 0) THEORY AND M5 BRANES: A KK MODE APPROACH

A Dissertation

by

SHAN HU

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Chair of Committee,	Dimitri V. Nanopoulos
Committee Members,	Bhaskar Dutta
	Christopher Pope
	Stephen Fulling
Head of Department,	George R. Welch

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ABSTRACT

6d (2, 0) theory on M5 branes is investigated by considering its KK modes on a 2d space. Selecting KK modes on different 2d spaces amounts to choosing different set of selfdual strings as the perturbative degrees of freedom thus will give the 6d theories related to each other by U-duality. The 4d effective theory for the KK modes is studied via the M5-D3 duality. Except for the (p, q) open strings, which is the KK mode arising from the selfdual strings, the 3-string junction should also be added since it is the bound state of the (p, q) open strings. The quantization of the 3-string junctions gives the fields, which, when lifted to 6d, may account for the conformal anomaly of the 6d (2, 0) theory. The interaction between the open strings and the 3-string junctions is also considered. The Lagrangian and the corresponding N=4 supersymmetry transformation is obtained up to some additional terms to be added. Although the original 6d (2, 0) theory is not constructed directly, the 4d effective theory for the KK modes gives an equivalent description, from which the 6d S-matrix can be calculated.

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURE Page

1. INTRODUCTION

String/M theory offers a natural framework to construct the various quantum field theories. For example, the U (N) Super Yang-Mills could be realized as the effective theory on N coincident D branes. The quantization of the open strings ending on D branes gives the super Yang-Mills field, while the Chan-Paton factors carried by open strings account for the U (N) gauge asymmetry. Aside from D branes in string theory, there are also M2 branes and M5 branes in M-theory, from which, the interesting quantum field theory may be obtained. The low energy effective theory on multiple M2 branes has been well understood in recent years. However, the low energy effective theory on multiple M5 branes is still an open problem. All we know is it is a 6d conformal field theory with the (2, 0) supersymmetry and the A_{N-1} gauge symmetry. When reduced on one dimension, the 6d (2, 0) theory becomes the ordinary 5d U (N) Super Yang-Mills. AdS/CFT also predicts that the theory should have the N^3 other than N^2 degrees of freedom. The twisted reduction of the 6d (2, 0) theory on the 2d surfaces gives the fruitful N=2 gauge theories related to each other by S-duality, which is the peculiar property of this theory. In chapter, we will introduce the string/M theory embedding of the 6d $(2, 0)$ theory. Section A and section B gives the type IIB and the M theory realization of the 6d $(2, 0)$ theory respectively. Section C is about the little string theory, a close relative of the 6d $(2, 0)$ theory.

1.1 The type IIB realization of the 6d $(2, 0)$ theory

In [1], Witten considered the type IIB string theory compactified on $R^6 \times K3$ and discovered a new superconformal six-dimensional string theory without the dynamical gravity, invariant under the $(2, 0)$ supersymmetry algebra, classified by the ADE group.

$K3$ space is a four-dimensional hyper-Kahler surface [2]. Let M represents the $K3$ manifold, then the relevant properties of the K3 geometry is the homology $H_2(M)$ of two-cycles, which is also dual to the cohomology of 2-forms $H_2(M)$. When the volume of the 2-cycles embedded in $K3$ shrink to zero, K3 reaches a singular point in moduli space, in which, the gauge symmetry develops [1]. The singularities obey an ADE classiffication, i.e. there are two infinite series, A_r, $r = 1, 2 \cdots$ and D_r, $r = 3, 4 \cdots$ and three exceptional cases $E6$, $E7$ and $E8$. Suppose $\{\omega^a | \omega^a \in H_2(M)\}$ is a basis of 2-forms, while $\{C_2^a\}$ is the dual 2-cycles, then there is a one-to-one correspondence between the node of each simply laced Dynkin diagram and C_2^a.

Type IIA theory on $R^6 \times K3$ gives another 6d theory, which is invariant under the $(1,1)$ supersymmetry, and is also classified by the ADE group [1]. The $(2,0)$ and $(1,1)$ supersymmetries comes from type IIA and type IIB string theories respectively. The particle states in the 6d $(1,1)$ theory arising from the $K3$ compactification of type IIA has two sources. Uncharged fields come from the dimensional reduction of the ten-dimensional type IIA supergravity. For example, the moduli parameterizing the volume of C_2^a becomes the scalars in 6d, while the RR 3-form A_3 becomes 1-forms via

$$A_3 \to A^a \wedge \omega^a. \tag{1.1}$$

The charged states arise from D2-branes wrapped on the two-cycles C_2^a. The D2-A_3 coupling then becomes

$$\int_{C_2^a} d^3\sigma A_3 \to \int d\tau A^a. \tag{1.2}$$

The mass of the charged states is proportional to the volume of C_2^a. When C_2^a collapses to zero size, charged states are massless, and then the nonabelian gauge symmetry is obtained. This is in agreement with the duality between the heterotic string on the space $R_6 \times T_4$ and the Type IIA string on $R^6 \times K3$.

For the 6d $(2,0)$ theory from the type IIB string theory compactified on $R^6 \times K3$, the particle states still has two sources. Chargeless fields come from the dimensional reduction of the ten-dimensional Type IIB supergravity. Now the the RR 4-form A_4 becomes 2-forms via

$$C_4 \to B_2^a \wedge \omega^a, \tag{1.3}$$

D3-branes wrapping on the two-cycle C_2^a give the 1d charged strings in 6d. The D3-C_4 coupling in 10d reduces to the string-B_2^a coupling, i.e.

$$\int_{C_2^a} d^4\sigma C_4 \to \int d^2\sigma B_2^a. \tag{1.4}$$

The tension of teh string is still proportional to the volume of C_2^a and will become tensionless when the volume of C_2^a approaches 0. Both C_4 and $D3$ are selfdual, so in 6d, the tensor B_2^a and the strings are also selfdual.

In summary, Type IIA supergravity compactified on K3 gives the 6d (1, 1) vector multiplet, while the type IIB supergravity on K3 gives the 6d (2, 0) tensor multiplet. For the charged states, D2 wrapping 2-cycles are 0d states in 6d, while D3 wrapping 2-cycles give the 1d states in 6d. The neutral 6d (1, 1) vector multiplets together with the charged point states compose the 6d (1, 1) vector multiplet in the representation of the ADE algebra. 6d (1, 1) theory is then a Super-Yang Mills theory. The same conclusion does not hold for the 6d (2, 0) theory because the charged states are 1d strings.

1.2 6*d* $(2,0)$ theory on M5 branes

The A-series of the 6d (2, 0) theory also has an interpretation in M-theory, which is supposed to be the unification of all five different string theories. The theory is still under the developing, but it is very clear that M theory should reduce to the eleven dimensional supergravity in low energy limit and contains the two dimensional branes and the five dimensional branes, both are 1/2 BPS states preserving 16 supersymmetries. With respect to the 3-form field in 11d supergravity, M2 brane and the M5 brane are electric and the magnetic states respectively. The A_{N-1} type 6d $(2,0)$ theory are low energy effective theory on N coincident M5 banes [3]. The type IIB picture and the M theory realization are equivalent, as is shown in [4, 5].

The five scalars are then the Goldstone bosons from the spontaneous symmetry breaking of translational invariance induced by the five brane and parameters as the relative transverse positions of the M5-branes in the eleven-dimensional space. Just as open strings may end on D-branes, open M2 branes may end on M5 branes. From the M5 branes' point of view, these open M2 branes will appear as the strings. At the origin of moduli space, the M5-branes coincide and the strings become tensionless, in parallel with the Type IIB picture. Aside from the A-series, the D-series 6d (2, 0) theory can also be described in M theory. The parallel orientifold plane should be added in this case. The exotic series E6, E7 and E8 have no known M-theory realization.

The existence of the 6d SCFT is also predicted by AdS/CFT [6]. The near horizon limit of the black M5 solutions is $AdS_7 \times S^4$. M theory on $AdS_7 \times S^4$ is suppose to be dual to a 6d superconformal field theory. AdS/CFT duality offers some clues about this unknown 6d SCFT. For example, the entropy of the 6d (2, 0) theory should have a N^3 scaling [7], in contrast to the N^2 scaling entropy of the 4d super-yang mills theory.

1.3 Little string theory

M theory compactified on S^1 gives the type IIA string theory. Correspondingly, M5 brane with a transverse dimension compactified becomes the NS5 brane. Type IIA little string theory [8, 9], which is the low energy theory on NS5, could be taken as the 6d (2, 0) theory with a transverse dimension compactified. The compactification of the transverse dimension will add more degrees of freedom, namely, the winding mode. Especially, closed membrane wrapping S gives the closed fundamental string on NS5 with the tension $T_{F1} = 2\pi R T_{M2}$. 6d (2, 0) tensor multiplet then appears as the lowest oscillation mode of these noncritical strings [10, 11, 12]. However, one thing to notice is that these strings are all chargeless and is essentially the fundamental string arising from the closed membrane other than the selfdual string from the open membrane.

The low energy effective theory on type IIB NS5 branes is the type IIB little string theory. Here, the string refers to the type IIB string. Under the S-duality transformation, NS5 brane becomes the D5 brane, while the fundamental strings on NS5 becomes the D string on D5. In type IIB the NS5-F1 and the D5-D1 bound state both exist. The effective theory on N coincident D5 branes is the $U(N)$ super-yang-mills theory. Correspondingly, the type IIB little theory is just a 6d (1, 1) $U(N)$ super yang-mills theory.

Type IIA NS5 branes compactified on a circle of radius R is T-dual to type IIB NS5 branes compactified on a circle of radius $1/M_s^2 R$. In fact, in the type IIB realization, type IIA string theory on $R^6 \times K3 \times S^1$ is also T-dual to the type IIB string on $R^6 \times K3 \times S^1$, since $K3$ is the self-mirror space.

2. HOPF-WESS-ZUMINO TERM IN THE EFFECTIVE ACTION OF THE 6D $(2,0)$ FIELD THEORY REVISITED*

We discuss the Hopf-Wess-Zumino term in the effective action of the 6d $(2,0)$ theory of the type A_{N-1} in a generic Coulomb branch. For such terms, the supergravity calculation could be trusted. We calculate the WZ term on supergravity side and show that it could compensate the anomaly deficit, as is required by the anomaly matching condition. In contrast with the SYM theory, in which each WZ term involves one root $e_i - e_j$, here, the typical WZ term involves two roots $e_i - e_j$ and $e_k - e_j$. Such kind of triple interaction may come from the integration out of the massive objects carrying three indices. A natural candidate is the recently proposed 1/4 BPS objects in the Coulomb phase of the 6d $(2,0)$ theories. The WZ term could be derived from the field theory by a 1-loop calculation. Without the 6d $(2,0)$ theory at hand, we take the supersymmetric equations for 3-Lie algebra-valued (2, 0) tensor multiplet as the starting point to see how far we can go. Although the $H_3 \wedge A_3$ part of the WZ term may be produced under the suitable assumptions, the $A_3 \wedge F_4$ part is far more difficult to obtain, indicating that some key ingredient is still missing.

Low energy effective action of the field theory in the Coulomb branch may contain the Wess-Zumino term arising from the integration out of massive fermions getting masses via the Yukawa coupling with the vacuum expectation value of the scalar fields [13, 14]. The existence of the Wess-Zumino term is also required by the anomaly matching condition [15]. At a generic point of the moduli space, the gauge symmetry is broken, and then, the 't Hooft anomaly produced by massless degrees of freedom is

*Part of this section is reprinted with permission from "Hopf-Wess-Zumino term in the effective action of the 6d, $(2,0)$ field theory revisited" by S. Hu and D. V. Nanopoulos, JHEP 1110, 054, (2011), http://link.springer.com/article/10.1007/JHEP10(2011)054.

different from the anomaly at the origin. On the other hand, the anomaly matching condition states that the 't Hooft anomaly should be the same everywhere on the moduli space of vacua. As a result, away from the origin, the integrating out of the massive degrees of freedom should generate the Wess-Zumino term in the low energy effective action compensating the deficit so that the total anomaly remains the same [15, 16].

The WZ term is a topological term that does not depend on the metric nor the coupling, so it is protected without the need of invoking any supersymmetric non-renormalization theorems. For such terms, we may expect that the 1-loop calculation in field theory and the supergravity calculation would match. On supergravity side, the Wess-Zumino term is associated with the magnetic-electric coupling. For Dp-branes with $p \geq 3$, it is given by $\int_{W_{p+2}} F_{8-p}(\wedge dA)^{p-3} = (-1)^p \int_{W_{p+1}} F_{8-p} \wedge A(\wedge dA)^{p-4}$ [17, 18], where F_{8-p} is the magnetic field strength, while $(\wedge dA)^{p-3}$ offers the electric charge. When $p = 3$, $\int_{W_{p+2}} F_{8-p}(\wedge dA)^{p-3} \to \int_{W_5} F_5$, because the D3-brane carries magnetic as well as the electric charge [18]. For M5 branes, the WZ term is composed by $\int_{W_6} db_2 \wedge A_3$ and $\int_{W_7} A_3 \wedge F_4$, which are discussed in [19] and [16] respectively. $\int_{W_6} db_2 \wedge A_3$ does not contribute to the anomaly. It is $\int_{W_7} A_3 \wedge F_4$ that accounts for the anomaly deficit. [16] considered the situation when the gauge symmetry is broken from $SU(N+1)$ to $SU(N) \times U(1)$ by the vacuum expectation value ϕ^a. The corresponding WZ term takes the form of $\int_{W_7} \sigma_3(\hat{\phi}) \wedge d\sigma_3(\hat{\phi})$, where $d\sigma_3(\hat{\phi})$ is the pullback of the 4-form field strength generated by a single M5 brane while $\sigma_3(\hat{\phi})$ is the corresponding 3-form potential. $\hat{\phi} = \phi/|\phi|$. It was shown that with the coefficient given by $N(N+1)/2$, the WZ term could indeed reproduce the anomaly deficit between $SU(N+1)$ and $SU(N) \times U(1)$. In this note, we will extend the discussion to the generic Coulomb branch $(\phi_1^a, \cdots, \phi_N^a)$. We will show that the supergravity calculation could give the right coefficient, while the WZ term, although takes the form of $\int_{W_7} \sigma_3(\hat{\phi}_{ij}) \wedge d\sigma_3(\hat{\phi}_{kj})$ with $\hat{\phi}_{ij} = (\phi_i - \phi_j)/|\phi_i - \phi_j|$, could produce the same amount of anomaly as that of $\int_{W_7} \sigma_3(\hat{\phi}) \wedge d\sigma_3(\hat{\phi})$. So the WZ term obtained from the supergravity calculation indeed compensates the anomaly deficit.

In the generic Coulomb branch, the WZ term in SYM theory is $(-1)^p \int_{W_{p+1}} F_{8-p}(\hat{\phi}_{ij}) \wedge A_{ij}[\wedge dA_{ij}]^{p-4}$,[1] which is the typical pair-wise interaction arising from the the integration out of massive fermions carrying index (i, j), or open strings connecting the i_{th} and the j_{th} D-brane [17, 18]. The term $\int_{W_7} \sigma_3(\hat{\phi}_{ij}) \wedge d\sigma_3(\hat{\phi}_{kj})$ for M5 branes seems indicate some kind of triple interaction: three M5 branes could interact simultaneously. One may naturally expect that such term comes from the integration out of massive fermions with (i, j, k) index, or open M2 branes connecting the i_{th}, the j_{th}, and the k_{th} M5 branes. In [20] and more recently, [21], the 1/4 BPS objects in the Coulomb phase of the ADE-type 6d $(2, 0)$ superconformal theories are considered. They are made of waves on selfdual strings and junctions of selfdual strings. In [21], it was shown that the number of 1/4 BPS objects matches exactly one third of the anomaly constant $c_G = d_G h_G$ for all ADE types, indicating that the anomaly may be produced by these 1/4 BPS objects. Moreover, the tension of the string junctions is characterized by $(|\phi_i - \phi_j|, |\phi_j - \phi_k|, |\phi_k - \phi_i|)$, which is just what is needed to produce the WZ term, since the selfdual string with tension $|\phi_i - \phi_j|$ is not enough to give the coupling like $\sigma_3(\hat{\phi}_{ij}) \wedge d\sigma_3(\hat{\phi}_{kj})$.

For SYM theories, the WZ term could be derived by a standard 1-loop calculation [17, 18, 22]. It is natural to expect that the WZ term for $6d$ $(2, 0)$ theories could also be derived in a similar way [16]. However, the $6d$ $(2, 0)$ theory is not constructed yet. Moreover, since the theory is strongly-coupled, it is possible that the Lagrangian formulation does not exist at all. Nevertheless, in [23], the supersymmetric equations of motion for the 3-algebra valued $(2, 0)$ tensor multiplet were found. The 3-algebra

[1] $A_{ij} = A_i - A_j$. It is the relative flux that makes sense.

may play an important role in M-theory, as is shown in the construction of the field theory for multiple M2 branes [24, 25, 26]. For M5 branes, progress has been made in the reconstruction of the single M5 brane from the BLG model and the ABJM theory via the Nambu-Poisson algebra [27, 28, 29, 30]. If the fundamental degrees of freedom for M5 branes carry three indices, the most natural algebra structure for them is the 3-algebra. A serious problem is that there is no finite dimensional Euclidean 3-algebra satisfying the fundamental identity except for A_4. In [23], the fundamental identity is imposed for the closure of the supersymmetry. We will show that a weaker version of the fundamental identity could also ensure the supersymmetry. The equations in [23] could be naturally interpreted as the supersymmetric gauge field equations on loop space, as is shown in [31]. We will give a slightly different set of gauge field equations, in which the loop parameter R does not appear, while the auxiliary field C^μ does not need to be factorized to the vector and the 3-algebra part. Even if the fundamental identity is relaxed to a weaker version, it is still difficult to find a definite representation of the 3-algrbra with N^3 scaling generators. We will neglect this problem and go ahead to discuss the WZ term based on equations in [23]. It may be possible to get the $H_3 \wedge A_3$ part of the WZ term, but it's more difficult to obtain the $A_3 \wedge F_4$ part. To relate the Hopf-Wess-Zumino term derived from the field theory and that obtained from supergravity, we made special assumptions on both sides. The final forms of the two could at most have the similar structure, but are still different, so there must be some key ingredient missing. Besides, to get $H_3 \wedge A_3$, we need $d = 5$, which is consistent with [23], in which, the constraint reduces the dynamics from $6d$ to $5d$. However, for $A_3 \wedge F_4$, we still need $d = 6$.

This chapter is organized as follows: In section 2, we get the WZ term for $6d$ A_{N-1} $(2, 0)$ theory in a generic Coulomb branch from the supergravity calculation. Part of the details is given in appendix A. In section 3, we will show that the WZ terms obtained from the supergravity calculation could indeed compensate the anomaly deficit thus guarantee the anomaly matching condition. In section 4, we will discuss the degrees of freedom in M5 branes producing the WZ term. In section 5, we will review the equations of motion for the 3-algebra valued $(2,0)$ tensor multiplet obtained in [23], and rewrite it as the supersymmetric gauge field equations. Based on the equations in [23], we will derive the WZ term on the field theory side to see how close it could approach the WZ term obtained from supergravity. In section 6, we will discuss the possible representation of the 3-algebra with the relaxed fundamental identity. The conclusion is in section 7.

2.1 The Hopf-Wess-Zumino term from the supergravity calculation

Consider the $6d$ $(2,0)$ field theory describing N M5 branes. On a generic Coulomb branch $(\phi_1^a, \cdots, \phi_N^a)$ with $a = 1 \cdots 5$ and $\phi_i \neq \phi_j$, for $i \neq j$, the gauge symmetry is broken to $U(1)^N$. On field theory side, N copies of $(2,0)$ tensor multiplets remain massless, while the rest fields get mass. Integrating out these massive degrees of freedom, one gets the effective action of the $6d$, $(2,0)$ field theory on Coulomb branch. On supergravity side, we have N M5 branes locating at $(\phi_1, \cdots, \phi_N)$. At least for WZ terms, the calculation on both sides should coincide. $6d$ $(2,0)$ field theory is still mysterious to us, while the multi-centered supergravity solution of M5 branes is more tractable, so we will try to get the WZ term in the effective action through the supergravity calculation.

The action for the coupling of M5 branes with the 11d supergravity could be written as[2] [32, 33, 34]

$$S = S_g + S_{M5}$$

[2]Just as the (6.15) in [34], $*F_4$ could be added into the worldvolume of the M5 brane, but then an equal term will appear in the bulk.

$$
\begin{aligned}
&= \frac{1}{2\kappa^2}\int_{M_{11}} *R - \frac{1}{2} * \hat{F}_4 \wedge \hat{F}_4 - \frac{1}{6} F_4 \wedge F_4 \wedge A_3 \\
&- T_5 \int_{W_6} d^6\xi \sqrt{-\det(g_{\mu\nu} + (i_{v_1} \tilde{*} h_3)_{\mu\nu})} + \frac{1}{2} v_1 \wedge h_3 \wedge \tilde{*}(v_1 \wedge \tilde{*} h_3) \\
&+ \frac{T_5}{2}\int_{W_6} db_2 \wedge A_3 + \frac{T_5}{2}\int_{W_7} A_3 \wedge F_4
\end{aligned}
\tag{2.1}
$$

where $F_4 = dA_3$, $W_6 = \partial W_7$,

$$
\hat{F}_4 = F_4 + 2\kappa^2 T_5 * G_7, \tag{2.2}
$$

$$
h_3 = db_2 - A_3. \tag{2.3}
$$

$d * G_7 = *J_6$. $*J_6$ is the the M5-brane current. The last term in (2.1) is just the Hopf-Wess-Zumino term proposed in [16]. The field equations for $\hat{F}_4$ are

$$
d\hat{F}_4 = 2\kappa^2 T_5 * J_6, \tag{2.4}
$$

$$
d * \hat{F}_4 + \frac{1}{2}\hat{F}_4 \wedge \hat{F}_4 = -2\kappa^2 T_5 \; h_3 \wedge *J_6. \tag{2.5}
$$

$*G_7$, A_3, and F_4 have the dependence on gauge, while $*J_6$, h_3, and $\hat{F}_4$ are gauge independent. (2.4) and (2.5) only contain gauge invariant quantities.

Suppose the vacuum expectation values of b_2 are equal to zero, consider N M5 branes locating at $(\phi_1, \cdots, \phi_N)$. One can choose the slowly varying configuration $(\phi_1, \cdots, \phi_N)$, for which, $\hat{F}_4 \wedge \hat{F}_4 = 0$. Since $db_2 = 0$, one can further assume $A_3 \wedge *J_6 = 0$, thus $d * \hat{F}_4 = 0$. $\hat{F}_4$ then becomes the magnetic field generated by N M5 branes in the configuration $(\phi_1, \cdots, \phi_N)$.

$$
\hat{F}_4 = \sum_{i=1}^{N} \hat{F}_{4i}, \qquad A_3 = \sum_{i=1}^{N} A_{3i}. \tag{2.6}
$$

Note that M5 brane with h_3 flux carries the magnetic as well as the electric charge, and so $h_3 \wedge F_4$ involves the electric-electric interaction as well as the electric-magnetic interaction. Under the present assumption, $h_3 \wedge F_4$ only contains the electric-magnetic coupling, which is our expectation for the WZ term. To calculate the WZ term on the j_{th} M5 brane, we need the pullback of A_3 and F_4 on the corresponding W_7. The pullback of $*G_7$ on W_7 vanishes, so we simply have

$$\int_{W_{7j}} A_3 \wedge F_4 = \int_{W_{7j}} A_3 \wedge \hat{F}_4 = Q_1^2 \int_{W_{7j}} \sum_{i=1}^{N} \sigma_{3ij} \wedge \sum_{k=1}^{N} \omega_{4kj}, \tag{2.7}$$

where $d\sigma_{3ij} = \omega_{4ij}$. ω_{4ij} is the pullback of ω_{4i} on W_{7j}. ω_{4i} is the unite volume form of S^4 surrounding the i_{th} brane. Altogether, we have

$$\frac{T_5}{2} \sum_{j=1}^{N} \int_{W_{7j}} A_3 \wedge F_4 = \frac{Q_1^2 T_5}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \int_{W_{7j}} \sigma_{3ij} \wedge \omega_{4kj} \tag{2.8}$$

Aside from the $F_4 \wedge F_4 \wedge A_3$ term in supergravity, $\int A_3 \wedge F_4$ is the other term which has the N^3 scaling.

However, (2.8) is still not exactly the WZ term in the effective action. First, when $i = j = k$, we get a self-interaction term. There are totally N such terms. These self-interaction terms will be produced only when the N massless tensor multiplets are also integrated out. Since we only integrate massive degrees of freedom, these terms will not appear in the effective action. Second, for the given brane configuration, or equivalently, the Coulomb branch, the more accurate expression for the effective action on supergravity side should be [35]

$$S_{eff} = S_g + S_{M5}, \tag{2.9}$$

where S_g is the action of the supergravity fields generated by M5 branes, while S_{M5} is the action of M5 branes on the background generated by themselves. S_{eff} is on-shell with respect to supergravity as it should be. (2.8) comes from S_{M5}. As is shown in the appendix A, S_g contains a term which is $-2/3$ of (2.8), so altogether, we conclude that

$$\Gamma_{WZ} = \frac{Q_1^2 T_5}{6} (\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \int_{W_{7j}} \sigma_{3ij} \wedge \omega_{4kj} - \sum_{i=1}^{N} \int_{W_{7i}} \sigma_{3ii} \wedge \omega_{4ii}) \tag{2.10}$$

or

$$2\kappa^2 \Gamma_{WZ} = \frac{Q_1^3}{6} (\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \int_{W_{7j}} \sigma_{3ij} \wedge \omega_{4kj} - \sum_{i=1}^{N} \int_{W_{7i}} \sigma_{3ii} \wedge \omega_{4ii}), \tag{2.11}$$

where $Q_1 = 2\kappa^2 T_5$.

2.2 The anomaly matching

At the origin of the moduli space, 6d, A_{N-1} (2, 0) field theory has the following form of anomaly when coupled to a background $SO(5)_R$ gauge field 1-form A, and in a general gravitational background [36, 37].

$$I_8(N) = (N-1) I_8(1) + \frac{1}{24}(N^3 - N) p_2(F). \tag{2.12}$$

$I_8(1)$ is the anomaly polynomial for a single, free, (2, 0) tensor multiplet [38, 39]:

$$I_8(1) = \frac{1}{48}\left[p_2(F) - p_2(R) + \frac{1}{4}(p_1(F) - p_1(R))^2 \right]. \tag{2.13}$$

$p_2(F)$ is the second Pontryagin class for the background $SO(5)_R$ field strength F:

$$p_2(F) = \frac{1}{8}(\frac{i}{2\pi})^4 \left[(trF^2) \wedge (trF^2) - 2trF^4 \right]. \tag{2.14}$$

At a generic point of the moduli space, the only massless degrees of freedom are $N-1$ copies of tensor multiplets giving rise to the anomaly of $(N-1)I_8(1)$. However, based on 't Hooft anomaly matching condition, the integration out of the massive degrees of freedom will produce the WZ term in the effective action, which will offer the missing $(N^3 - N)p_2(F)/24$ part so that the total anomaly is still the same as before [16]. In the following, we will show that the WZ term in (2.11) could indeed give the $(N^3 - N)p_2(F)/24$ part of the normal bundle anomaly.

Turn on the background $SO(5)_R$ gauge field A on W_6. $\partial W_7 = W_6$, so A could be smoothly extended to W_7. On W_7, we have gauge field $A_i^{ab} = -A_i^{ba}$, with $a, b = 1 \cdots 5$, $i = 1 \cdots 7$. In presence of the background field A, the pullback of the S^4 unite volume form on W_7 becomes

$$\begin{aligned}\omega_4(\hat{\phi}, A) &= \frac{1}{2} e_4(\hat{\phi}, A) = \frac{1}{64\pi^2} \epsilon_{a_1 \cdots a_5} [(D_{i_1}\hat{\phi})^{a_1} (D_{i_2}\hat{\phi})^{a_2} (D_{i_3}\hat{\phi})^{a_3} (D_{i_4}\hat{\phi})^{a_4} \\ &\quad -2F_{i_1 i_2}^{a_1 a_2} (D_{i_3}\hat{\phi})^{a_3} (D_{i_4}\hat{\phi})^{a_4} + F_{i_1 i_2}^{a_1 a_2} F_{i_3 i_4}^{a_3 a_4}] \hat{\phi}^{a_5} dx^{i_1} \wedge \cdots \wedge dx^{i_4}, \end{aligned} \tag{2.15}$$

$(D_i\hat{\phi})^a = \partial_i \hat{\phi}^a - A_i^{ab} \hat{\phi}^b$, F_{ij}^{ab} is the field strength. $\hat{\phi}$ is a unite vector in the transverse space R^5. If $\omega_4(\hat{\phi}, A)$ represents the pullback of the 4-form field strength generated by the i_{th} M5 brane on W_{7j}, $\hat{\phi}$ is determined by the relative position of W_{6i} and W_{7j}. Especially, at the boundary of W_{7j}, $\hat{\phi}$ is simply determined by the relative position of W_{6i} and W_{6j} in the transverse space, i.e.

$$\hat{\phi}^a = \frac{\phi_i^a - \phi_j^a}{|\phi_i - \phi_j|}, \tag{2.16}$$

where ϕ_i^a is the vacuum expectation value of scalar field for the i_{th} M5 brane.

$e_4(\hat{\phi}, A)$ is the global angular form defined over the sphere bundle with fiber S^4 and base space W_7.

$$de_4 = 0. \tag{2.17}$$

Under the $SO(5)$ transformation,

$$\begin{aligned}\hat{\phi}^a &\to \hat{\phi}^a + \Lambda^{ab}\hat{\phi}^b \\ A^{ab} &\to A^{ab} + d\Lambda^{ab} + [\Lambda, A]^{ab}.\end{aligned} \tag{2.18}$$

$D\hat{\phi}^a$ and F^{ab} transform covariantly under (2.18), while $e_4(\hat{\phi}, A)$ is $SO(5)$ invariant. For the present problem, we have $N^2 - N$ global angular forms $e_4(\hat{\phi}, A)$ with different $\hat{\phi}$ but the same A. Since e_4 is $SO(5)$ invariant, they can also be equivalently represented by $e_4(\hat{\phi}, A)$ with the same $\hat{\phi}$ but different A. $p_2(F)$ is the second Pontryagin class of a rank 5 real vector bundle, nevertheless, we still have

$$p_2(F) = \chi(F)^2, \tag{2.19}$$

where $\chi(F)$ is the Euler class of a rank 4 subbundle with the orthogonal line bundle trivial. One can always choose particular $\hat{\phi}_0$ so that

$$e_4(\hat{\phi}_0, A) = \chi(F) \tag{2.20}$$

Actually, for such $\hat{\phi}_0$, $D\hat{\phi}_0 = 0$, so $e_4(\hat{\phi}_0, A)$ reduces to the Euler class. We take this $\hat{\phi}_0$ as the standard and transform all of the angular forms into the form of $e_4(\hat{\phi}_0, \tilde{A})$, where $\tilde{A}$ are different connections defined on the same normal bundle. Just as the invariant polynomials, if $\tilde{A}$ and $\tilde{A}'$ are two different connections,

$$e_4(\hat{\phi}_0, \tilde{A}) - e_4(\hat{\phi}_0, \tilde{A}') = de_3(\hat{\phi}_0, \tilde{A}) - de_3(\hat{\phi}_0, \tilde{A}') = dR(\hat{\phi}_0, \tilde{A}, \tilde{A}'), \tag{2.21}$$

with e_3 the corresponding Chern-Simons forms.

$$R(\hat{\phi}_0, \tilde{A}, \tilde{A}') = -\frac{1}{32\pi^2}\int_0^1 dt \;\; \epsilon_{a_1\cdots a_5}[(D_t\hat{\phi}_0)^{a_1}(D_t\hat{\phi}_0)^{a_2} - F_t^{a_1 a_2}]\eta^{a_3 a_4}\hat{\phi}_0^{a_5}, \tag{2.22}$$

where

$$\eta = \tilde{A} - \tilde{A}', \quad A_t = \tilde{A}' + t\eta, \quad F_t = dA_t - A_t^2, \quad D_t = (d - A_t). \tag{2.23}$$

$R(\hat{\phi}_0, \tilde{A}, \tilde{A}')$ is $SO(5)$ invariant.

$$e_3(\hat{\phi}_0, \tilde{A}) - e_3(\hat{\phi}_0, \tilde{A}') = R(\hat{\phi}_0, \tilde{A}, \tilde{A}'). \tag{2.24}$$

For different connections, e_3 only differ by a $SO(5)$ invariant term.

Return to the original global angular form $e_4(\hat{\phi}, A)$, we will have

$$e_4(\hat{\phi}, A) = \chi(F) + d\alpha(\hat{\phi}, A), \tag{2.25}$$

$$e_4(\hat{\phi}, A) \wedge e_4(\hat{\phi}', A) = p_2(F) + d\beta(\hat{\phi}, \hat{\phi}', A), \tag{2.26}$$

where both α and β are $SO(5)$ invariant.

$$p_2(F) = d[e_3(\hat{\phi}, A) \wedge e_4(\hat{\phi}', A) - \beta(\hat{\phi}, \hat{\phi}', A)]. \tag{2.27}$$

By descent equations,

$$\delta[e_3(\hat{\phi}, A) \wedge e_4(\hat{\phi}', A)] = \delta p_2^0(A) = dp_2^1(A). \tag{2.28}$$

Now, consider the $SO(5)$ gauge transformation of (2.11). For each term,

$$\begin{aligned}\delta[\frac{1}{6}\int_{W_7}\sigma_3(\hat{\phi},A)\wedge\omega_4(\hat{\phi}',A)] &= \delta[\frac{1}{24}\int_{W_7}e_3(\hat{\phi},A)\wedge e_4(\hat{\phi}',A)] \\ &= \frac{1}{24}\int_{W_7}dp_2^1(A)=\frac{1}{24}\int_{W_6}p_2^1(A).\end{aligned} \qquad (2.29)$$

There are totally N^3-N such terms, so Γ_{WZ} could indeed reproduce the $(N^3-N)p_2(F)/24$ part of the anomaly.

If the $SU(N)$ group is broken to some subgroup like $U(N_1)\times U(N_2)\times SU(N_3)$ with $N_1+N_2+N_3=N$, the deficit of the anomaly produced by massless degrees of freedoms is $(N^3-N_1^3-N_2^3-N_3^3)p_2(F)/24$. Corresponding, Γ_{WZ} will contain $N^3-N_1^3-N_2^3-N_3^3$ terms exactly compensating the deficit.

2.3 The degrees of freedom in M5 branes producing the WZ term

The supergravity interaction between D-branes is pairwise. This is consistent with the fact that the WZ term for N D-branes in a generic Coulomb branch could be written as the sum of $N(N-1)/2$ terms labeled by (ij) index [17, 18]. On the other hand, the $\int A_3\wedge F_4$ term in the action of M5 branes gives a triple interaction. That is, three M5 branes could interact simultaneously. The supergravity interaction for D-branes is produced by open strings connecting two D-branes. Similarly, one may expect that the triple interaction $\int A_3\wedge F_4$ could be produced by open M2 branes connecting three M5 branes.

Another example of N^3 interaction is given by M theory compactified on a Calabi-Yau threefold with M5-branes wrapping 4-cycles, giving rise to $N=1$ 5d supergravity along with the chiral strings [40, 41]. In the bulk, we have Chern-Simons term $C_1\wedge dC_1\wedge dC_1$, while in the worldsheet of chiral strings, $\int C_1\wedge dC_1$ may exist [16]. These N^3 degrees of freedom in entropy are explained as states living at the triple-intersection of M5 branes [42, 43].

In [20] and more recently, [21], the 1/4 BPS objects in the Coulomb phase of the ADE-type 6d $(2,0)$ superconformal theories are explored. They are made of waves on selfdual strings and junctions of selfdual strings. Especially, in [21], it is shown that the number of 1/4 BPS objects matches exactly one third of the anomaly constant $c_G = d_G h_G$ for all ADE types, which strongly indicates that the anomaly may be produced by these 1/4 BPS objects. In A_{N-1} case, there are $N(N-1)/2$ 1/2 BPS selfdual strings with tension $T_{ij} \propto |\phi_i - \phi_j|$. On each selfdual string, there are left and right 1/4 BPS waves. Turning on these BPS waves, we get $N(N-1)$ 1/4 BPS objects. For every three M5 branes ijk, 1/4 BPS junction exists. The tension of the string junctions is characterized by $(|\phi_i - \phi_j|, |\phi_j - \phi_k|, |\phi_k - \phi_i|)$. The junction forms a dual lattice to the triangle Δ_{ijk}, if one indentify the $SO(5)$ in W_6 with the $SO(5)$ in the transverse space. For such configuration, the tension of selfdual strings is balanced and the junction is 1/4 BPS. There are totally $N(N-1)(N-2)/3$ such objects because of the junction and anti-junction. Altogether, the 1/4 BPS objects on N M5 branes in a generic Coulomb branch is $N(N^2-1)/3$.

Let us rewrite (2.11) in a more symmetric way.

$$2\kappa^2 \Gamma_{WZ} = \sum_{i\neq j, j\neq k, k\neq i} \Omega_{ijk} + \sum_{i\neq j} \Omega_{ij}, \tag{2.30}$$

where

$$\begin{aligned} \Omega_{ijk} &= \frac{Q_1^3}{6}\Big[\int_{W_{7i}} (\sigma_{3ji} \wedge \omega_{4ki} + \sigma_{3ki} \wedge \omega_{4ji}) + \int_{W_{7j}} (\sigma_{3ij} \wedge \omega_{4kj} + \sigma_{3kj} \wedge \omega_{4ij}) \\ &+ \int_{W_{7k}} (\sigma_{3ik} \wedge \omega_{4jk} + \sigma_{3jk} \wedge \omega_{4ik})\Big]. \end{aligned} \tag{2.31}$$

$$\Omega_{ij}=\frac{Q_1^3}{6}[\int_{W_{7i}}(\sigma_{3ji}\wedge\omega_{4ii}+\sigma_{3ii}\wedge\omega_{4ji}+\sigma_{3ji}\wedge\omega_{4ji})+\int_{W_{7j}}(\sigma_{3ij}\wedge\omega_{4jj}+\sigma_{3jj}\wedge\omega_{4ij}+\sigma_{3ij}\wedge\omega_{4ij})]. \tag{2.32}$$

It seems that junction and anti-junction may produce the term Ω_{ijk}, while left and right waves on selfdual strings could give Ω_{ij}. Recall that in D-brane case, the WZ term arising from the integration out of massive fermions ψ_{ij} is expressed in terms of the vector $\phi_i-\phi_j$ [17, 18]; here, the WZ term produced by string junctions (ijk) could be calculated from the vectors $(\phi_i-\phi_j,\phi_j-\phi_k,\phi_k-\phi_i)$. When $i=k$, the three string junctions degenerate to one selfdual string with tension $T_{ij}\propto|\phi_i-\phi_j|$ and the other tensionless selfdual string perpendicularly ending on it. So, in some sense, selfdual string with waves is a degeneration of the string junction. $\Omega_{ij}=\frac{1}{2}(\Omega_{iji}+\Omega_{jij})$.

Except for $\int_{W_7}A_3\wedge F_4$, M5 brane action contains another term $\int_{W_6}db_2\wedge A_3=-\int_{W_6}b_2\wedge F_4=\int_{W_6}H_3\wedge A_3$. Now suppose the vacuum expectation value of b_2 on the i_{th} M5 brane is b_{2i}, $\int_{W_6}H_3\wedge A_3$ part of the WZ term should also enter into the low energy effective action, although it does not contribute to the anomaly since $dH_3=0$. In [19], based on the supergravity calculation, it is shown that

$$\Gamma_H=\int_{W_6}H_3\wedge A_3\propto-\sum_{i\neq j}\int_{W_6}b_{2ij}\wedge\omega_{4ij}=\sum_{i\neq j}\int_{W_6}H_{3ij}\wedge\sigma_{3ij}, \tag{2.33}$$

where $b_{2ij}=b_{2i}-b_{2j}$, $H_{3ij}=db_{2ij}=H_{3i}-H_{3j}$. This is the typical pairwise interaction. The reduction of $\int_{W_6}H_3\wedge A_3$ on S^1 gives $\int_{W_5}F_2\wedge A_3$, the WZ term of the 5d SYM theory. In 5d SYM theory, $\int_{W_5}F_2\wedge A_3$ is generated by the integration out of massive fermions coming from the selfdual strings wrapping S^1, so it is quite possible that (2.33) is produced by 1/2 BPS selfdual strings.

The non-abelian part of the R-symmetry anomaly are all accounted for by 1/4 BPS objects. R-symmetry anomaly and Weyl anomaly are related by supersymmetry. In [44], the conformal anomaly of 6d (2, 0) SCFT of A_{N-1} type is calculated as

$$A_{2,0}(N) = (N-1)A_{tens} + (N^3 - N)A, \tag{2.34}$$

where A_{tens} is the conformal anomaly of the free $(2,0)$ tensor multiplet. It is expected that the 1/4 BPS objects could produce $(N^3 - N)A$, if they could give the corresponding part in R-symmetry anomaly. Note that $N-1$ 1/2 BPS massless particles and $N(N-1)(N-2)/3$ junctions of selfdual strings always contribute to the anomaly and entropy. However, the $N(N-1)/2$ selfdual strings have no contribution to the anomaly nor entropy unless the BPS waves are turned on thus the supersymmetry is reduced to 1/4. Once the selfdual strings become 1/4 BPS, the anomaly polynomial of them is the same as that of the string junctions, since the 1/4 BPS selfdual strings could be taken as the degeneration of the string junctions.

Then the question is why the 1/2 BPS selfdual strings have no contribution to the anomaly and the entropy. In $N = 4$ SYM theory, 1/4 BPS states arising from string junctions ending on three D3 branes also exist [45], however, the anomaly and entropy are both give by 1/2 BPS particles. The general form of the anomaly for a 6d $(2,0)$ SCFT of the ADE type G is

$$A_{2,0} = r_G A_{tens} + c_G A_X, \tag{2.35}$$

where $c_G = d_G h_G = r_G h_G(h_G + 1)$. r_G, d_G and h_G are the rank, the dimension, and the Coxeter number of the Lie algebra of type G. The theory contains r_G 1/2 BPS massless particles, $r_G h_G$ 1/2 BPS selfdual strings, and $c_G/3$ 1/4 BPS objects. The anomaly of the single M5 brane does not have the A_X part, so r_G 1/2 BPS massless particles only contribute to A_{tens}. Then A_X should be generated by 1/2 BPS selfdual strings or 1/4 BPS objects. If one wants to interpret it in terms of selfdual strings, each selfdual string should give the anomaly of $(h_G+1)A_X$, which in $SU(N)$ case, is $(N+1)A_X$. It is difficult to explain this h_G+1 factor. Otherwise, since the total number of the 1/2 BPS states is d_G, they can account for the A_X part if each one contributes $h_G A_X$. This looks more reasonable, but the problem is that the r_G 1/2 BPS massless particles will contribute to A_{tens} as well as A_X. The most natural possibility is that A_X is produced by 1/4 BPS objects, which are intrinsically three selfdual string junctions.

Finally, notice that for $N=4$ SYM theory, the anomaly takes the form $A_4 = (N^2-1)A_{vec}$, where A_{vec} is the anomaly of a free vector multiplet. The anomaly is not renormalized from weak to strong coupling, so we can calculate it from the free field value. Besides, N^2-1 elements in the Lie algebra give the same contribution to the anomaly, indicating that they are allowed to transform into each other. On the other hand, for $6d$ (2, 0) SCFT, the anomaly polynomial is of the form $A_{2,0} = (N-1)A_{tens} + (N^3-N)A_X$ other than $(N^3-1)A_{tens}$, which seems indicate that there are something special about the non-abelian part.

$6d$ $(2,0)$ SCFT compactified on S^1 gives $5d$ SYM theory. Selfdual strings wrapping on S^1 become 1/2 BPS particles. The unwrapped selfdual strings and 1/4 BPS string junctions in $6d$ descend to the corresponding string-like objects in $5d$ [20, 21]. String junctions could also appear as point-like particles in the compactified theory. Consider the $6d$ SCFT compactified on a Riemann surface Σ_g with $g>1$ [46, 47, 48], the T part of Σ_g is the natural place for string junctions to wrap. Σ_g is built from $2(g-1)$ T_N blocks and $3(g-1)$ I_N blocks. T_N and I_N are spheres with 3 and 2 full punctures respectively. The dimension of the Coulomb branch for T_N and I_N are

$$d_c T_N = \frac{(N-1)(N-2)}{2}, \qquad d_c I_N = N-1. \tag{2.36}$$

The effective number of vector multiplets for T_N and I_N are

$$n_v T_N = \frac{2N^3}{3} - \frac{3N^2}{2} - \frac{N}{6} + 1, \qquad n_v I_N = N^2 - 1. \tag{2.37}$$

Note that

$$2(g-1)d_c T_N + 3(g-1)d_c I_N = (g-1)(N^2-1) \tag{2.38}$$

and

$$2(g-1)n_v T_N + 3(g-1)n_v I_N = (g-1)(\frac{4N^3}{3} - \frac{N}{3} - 1) \tag{2.39}$$

are the dimension of the Coulomb branch and the effective number of vector multiplets for the Σ_g theory. Especially, when $g = 1$, Σ_1 is simply constructed from one I_N. The degrees of freedom arising from strings could be calculated as

$$n_{T_N} = n_v T_N - d_c T_N = 4C_N^3, \qquad n_{I_N} = n_v I_N - d_c I_N = 2C_N^2. \tag{2.40}$$

n_{T_N} could be naturally accounted for by string junctions[3], while n_{I_N} is associated with the selfdual strings. In the generic point of the Coulomb branch of T_N, the Seiberg-Witten curve is a Riemann surface with $(N-1)(N-2)/2$ genus and $3N$ simple punctures [47, 49]. In that case, n_{T_N} could also be explained as the number of M2 branes with two boundaries. However, at the origin of the moduli space, the only nontrivial configurations are M2 branes with three boundaries. The anomaly polynomial $I_6(N)$ in $4d$ is obtained from $I_8(N)$ in $6d$ by the integration over Σ_g [50, 51]. Both of them have a N^3 scaling part.

[3]String junctions could give $2C_N^3$, but the extra factor 2 is a little difficult to explain.

2.4 WZ term from the integration of massive fermions

The WZ term could be derived by a 1-loop calculation in field theory. For $SU(N)$ gauge theories, this has been done in [17, 18, 22]. In Coulomb branch, fermions get mass due to the Yukawa coupling. The integration of the fermion loop gives WZ terms in the low energy effective action. We don't know the structure of the $6d$ $(2,0)$ field theory. A recent calculation on scattering amplitudes [52] indicates that an interacting $6d$ Lagrangian with classical $OSp(8|4)$ symmetry cannot be constructed using only $(2,0)$ tensor multiplets, even if the Lagrangian is non-local. A Lagrangian description may exist, however, if one includes additional degrees of freedom, for example, the selfdual strings. If the Lagrangian could be constructed, then it may contain the fermionic degrees of freedom, and the corresponding Dirac operator and Yukawa couplings. We may expect that the WZ term could be obtained by the similar fermion-loop integration. Of course, if the Lagrangian does not exist at all no matter what kinds of degrees of freedom are added, such calculation is not valid. In the following, we will discuss the possible ingredients of this theory and the relevance with the WZ term. It is just a speculation and the final answer is still far.

First, let us consider the possible algebra structure. The $SU(N)$ gauge theories are constructed through the 2-algebra. The Yukawa coupling is $\Gamma_I[X^I,\psi]$. In Coulomb branch, it becomes $\Gamma_I(\phi_i^I-\phi_j^I)\psi_{ij}$, giving mass $|\phi_i-\phi_j|$ to ψ_{ij}. $|\phi_i-\phi_j|$ is the length of the string connecting the i_{th} and the j_{th} D-branes. On the other hand, the Lagrangian of the M2 branes has the 3-algebra structure [24, 25, 26]. The ABJM theory could also be written in the 3-algebra language and has a sextic potential [53]. The Yukawa coupling takes the form $\Gamma_I\Gamma_J[X^I,X^J,\psi]$. The fermion mass scales as the area other than the length, reflecting the fact that M2 branes are connected by M2 branes other than strings [54, 55]. M5 branes are also connected by M2 branes, so we may expect that the interaction could also be constructed through the 3-algebra. However, there is a difference. The M2 branes connecting parallel M2 branes are totally located in the transverse space. The endpoint is simply a point. As a result, we get $\Gamma_I\Gamma_J[X^I, X^J, \psi]$, where X^I and X^J carry the transverse index. Conversely, the M2 branes connecting parallel M5 branes have one dimension living in the worldvolume of M5. The endpoint is a string. Correspondingly, we may have $\Gamma_\mu\Gamma_I[C^\mu, X^I, \psi]$, where $\mu = 0\cdots 5$, $I = 6\cdots 10$.

Actually, in [23], an attempt to find the $6d$ $(2,0)$ theory with the 3-algebra structure has already been made. It was shown that for the closure of the supersymmetry, an additional vector C^μ must be introduced, while the 3-brackets appearing in the equations of motion always take the form $[C^\mu, A, B]$. Later, in [31, 56], the equations of motion found in [23] get a natural interpretation as the supersymmetric gauge field equations in loop space. Especially, in [31], it was shown that the BPS solutions to these equations yield the non-abelian extension of the selfdual string solitons proposed by [57], indicating that these equations may capture some aspects of the dynamics of M5 branes. C^μ is associated with the vector tangential to the loop in the worldvolume of M5 branes.

In the following, we will discuss the relevance of the equations in [23] with the WZ term. With the equations of motion at hand, we may try to derive the WZ term in a similar way as that in [17, 18].

The field content in [23] includes the tensor multiplet composed by X^I with $I = 6\cdots 10$, ψ, and $H^{\mu\nu\lambda}$ with $\mu,\nu,\lambda = 0\cdots 5$, an auxiliary gauge field A^μ, and a vector field C^μ. X^I, ψ, $H^{\mu\nu\lambda}$, and C^μ take values in a vector space Λ with the basis t^a, i.e. $X^I = X^I_a t^a$, etc. As a 3-algebra, Λ has an associated Lie algebra g_Λ spanned by the transformations $[t^a, t^b, *]$, where $*$ stands for an arbitrary element of Λ. A^μ takes values in g_Λ. $A^\mu X^I = A^\mu_{ab}[t^a, t^b, X^I]$, etc. A^μ and $H^{\mu\nu\lambda}$ are related by

$F_{ab}^{\mu\nu}[t^a, t^b, *] = [C_\lambda, H^{\mu\nu\lambda}, *]$, with $F_{ab}^{\mu\nu}$ the field strength of A_{ab}^{μ}. So, for the given C_λ, $F^{\mu\nu}$ is actually a transgression of $H^{\mu\nu\lambda}$ [31].

The dimension of Λ is denoted by $d(\Lambda)$, which is not specified at present. $\forall A, B \in \Lambda$, there is an inner product $\langle A|B\rangle = \langle B|A\rangle^*$. The fundamental identity is

$$[A, B, [X, Y, Z]] = [[A, B, X], Y, Z] + [X, [A, B, Y], Z] + [X, Y, [A, B, Z]]. \qquad (2.41)$$

As is well known, the only nontrivial finite-dimensional 3-Lie algebra with positive definite metric is A_4, so we have to either accept a Euclidean 3-algebra with (2.41) violated, or a Lorentzian 3-algebra satisfying (2.41). In [23], the supersymmetry transformation is discussed, and the fundamental identity is imposed for the closure of the supersymmetry. In appendix B, we will show that such restriction could be relaxed in some degree. Actually, since all 3-bracket takes the form of $[C^\mu, A, B]$, the fundamental identity only needs to hold for such kind of gauge transformations. More concretely, we need a subspace $\bar{\Lambda} \subset \Lambda$, $\forall\ C_1, C_2 \in \bar{\Lambda}$, $[C_1, C_2, *] = 0$.[4] $\forall\ X_1, X_2, Y \in \Lambda$,

$$\begin{aligned} & [C_2, X_2, [C_1, X_1, Y]] - [C_1, X_1, [C_2, X_2, Y]] \\ = \ & [[C_2, X_2, C_1], X_1, Y] + [C_1, [C_2, X_2, X_1], Y] \\ = \ & [C_1, [C_2, X_2, X_1], Y] \\ = \ & [C_2, [C_1, X_2, X_1], Y]. \end{aligned} \qquad (2.42)$$

The fundamental identity only needs to be satisfied for gauge transformations generated by $[C, X, *]$. $[\delta_1, \delta_2] = \delta_3$, where $\delta_1 * = [C_1, X_1, *]$, $\delta_2 * = [C_2, X_2, *]$, $\delta_3 * = [C_1, [C_2, X_2, X_1], *] = [C_2, [C_1, X_2, X_1], *]$. One can define a Lie-algebra g generated

[4] Obviously, $\forall\ A \in \Lambda$, $[A, A, *] = 0$. In [31], $\bar{\Lambda}$ is a one-dimensional space generated by C. Here, we want to keep more possibilities, so $\bar{\Lambda}$ may contain more than one generators.

by $[C, X, *]$, $\forall\ C \in \bar{\Lambda}$, $\forall\ X \in \Lambda$, for which, the Jacobi identity is satisfied. The theory only has the relevance with g. We even don't need to know the definition of $[X, Y, Z]$ if none of them belong to $\bar{\Lambda}$.

Similar with [31], we may try to describe selfdual strings by fields valued in g.

$$\tilde{\psi}_\mu = [C_\mu, \psi, *], \tag{2.43}$$

$$\tilde{X}^I_\mu = [C_\mu, X^I, *], \tag{2.44}$$

$$\tilde{F}^{\mu\nu} = [C_\lambda, H^{\mu\nu\lambda}, *]. \tag{2.45}$$

The original tensor multiplet, when combined with C^μ, becomes fields associated with selfdual strings. ψ, X and H are converted to the vector spinor, the 1-form and the 2-form respectively. From $\tilde{\psi}_\mu$, One can also construct a spinor $\tilde{\psi} = \Gamma_\mu \tilde{\psi}^\mu$.

The equations of motion for 3-algebra valued (2, 0) tensor multiplets found in [23] are

$$[C^\mu, C^\nu, *] = 0, \tag{2.46}$$

$$\nabla_\nu C^\mu = 0, \tag{2.47}$$

$$[C^\rho, \nabla_\rho X^I, *] = 0, \quad [C^\rho, \nabla_\rho \psi, *] = 0, \quad [C^\rho, \nabla_\rho H_{\mu\nu\lambda}, *] = 0, \tag{2.48}$$

$$\tilde{F}^{\mu\nu} - [C_\lambda, H^{\mu\nu\lambda}, *] = 0, \tag{2.49}$$

$$\Gamma_\mu \nabla^\mu \psi + [C^\mu, X^I, \Gamma_\mu \Gamma_I \psi] = 0, \tag{2.50}$$

$$\nabla^2 X^I - \frac{i}{2}[C^\mu, \bar{\psi}, \Gamma_\mu \Gamma^I \psi] + [C^\mu, X^J, [C_\mu, X_J, X^I]] = 0, \tag{2.51}$$

$$\nabla_{[\mu} H_{\nu\kappa\lambda]} + \frac{1}{4}\epsilon_{\mu\nu\kappa\lambda\sigma\tau}[C^\sigma, X^I, \nabla^\tau X_I] + \frac{i}{8}\epsilon_{\mu\nu\kappa\lambda\sigma\tau}[C^\sigma, \bar{\psi}, \Gamma^\tau \psi] = 0, \tag{2.52}$$

where $\nabla^\mu = \partial^\mu - iA^\mu$, $X^I, \psi, H_{\mu\nu\lambda} \in \Lambda$, $C^\mu \in \bar{\Lambda}$.

$$\tilde{F}^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu + [A^\nu, A^\mu]. \tag{2.53}$$

For generic $A^\mu \in g_\Lambda$, $\tilde{F}^{\mu\nu} \in g_\Lambda$. However, (2.49) shows that $\tilde{F}^{\mu\nu} \in g$, so we may also require $A^\mu \in g$, and then $\nabla_\nu C^\mu = \partial_\nu C^\mu = 0$. The supersymmetry transformations are [23]

$$\begin{aligned}
&\delta X^I = i\bar{\varepsilon}\Gamma^I\psi,\\
&\delta\psi = \Gamma_\mu\Gamma_I\nabla^\mu X^I\varepsilon + \frac{1}{12}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}\varepsilon - \frac{1}{2}\Gamma_{IJ}\Gamma_\lambda[X^I, X^J, C^\lambda]\varepsilon,\\
&\delta H_{\mu\nu\lambda} = 3i\bar{\varepsilon}\Gamma_{[\mu\nu}\nabla_{\lambda]}\psi + i\bar{\varepsilon}\Gamma_I\Gamma_{\mu\nu\lambda\kappa}[X^I, \psi, C^\kappa],\\
&\delta A_\mu = i\bar{\varepsilon}\Gamma_{\mu\lambda}[C^\lambda, \psi, *],\\
&\delta C_\mu = 0.
\end{aligned} \tag{2.54}$$

(2.46) and (2.49) are automatically satisfied by definition. From (2.46)-(2.52), we can derive the equations of motion for $\tilde{\psi}_\mu$, $\tilde{X}^I_\mu$, and $\tilde{F}^{\mu\nu}$. For $\tilde{Y}_\mu = [C_\mu, Y, *]$, the covariant derivative is defined as

$$\begin{aligned}
\nabla^\nu\tilde{Y}_\mu &= \partial^\nu\tilde{Y}_\mu - i[A^\nu, \tilde{Y}_\mu]\\
&= [\partial^\nu C_\mu, Y, *] + [C_\mu, \partial^\nu Y, *] - i[A^\nu_{ab}[t^a, t^b, C_\mu], Y, *] - i[C_\mu, A^\nu_{ab}[t^a, t^b, Y], *]\\
&= [C_\mu, \nabla^\nu Y, *].
\end{aligned} \tag{2.55}$$

The obtained equations are

$$\Gamma_\nu\nabla^\nu\tilde{\psi}_\mu - \Gamma_I\Gamma^\nu[\tilde{X}^I_\nu, \tilde{\psi}_\mu] = 0, \tag{2.56}$$

$$\nabla^2\tilde{X}^I_\mu + \frac{i}{2}[\bar{\tilde{\psi}}_\mu, \Gamma^I\tilde{\psi}] + [\tilde{X}^\nu_J, [\tilde{X}^J_\nu, \tilde{X}^I_\mu]] = 0, \tag{2.57}$$

$$\nabla_{[\mu}\tilde{F}_{\nu\kappa]} = 0, \tag{2.58}$$

$$\nabla^{\mu}\tilde{F}_{\mu\nu} - \frac{1}{24}[\tilde{X}^I_{\lambda}, \nabla_{\nu}\tilde{X}^{\lambda}_I] + \frac{i}{48}([\bar{\tilde{\psi}}_{\nu}, \tilde{\psi}] - [\bar{\tilde{\psi}}^{\lambda}, \Gamma_{\nu}\tilde{\psi}_{\lambda}]) = 0, \tag{2.59}$$

together with the constraints

$$\nabla^{\rho}\tilde{\psi}_{\rho} = \nabla^{\rho}\tilde{X}^I_{\rho} = \nabla^{\rho}G_{\mu\nu\lambda,\,\rho} = 0, \tag{2.60}$$

where $G_{\mu\nu\lambda,\,\rho} = [C_{\rho}, H_{\mu\nu\lambda}, *]$, $g^{\rho\lambda}G_{\mu\nu\lambda,\,\rho} = \tilde{F}_{\mu\nu}$. From (2.56), we also have

$$\Gamma_{\mu}\nabla^{\mu}\tilde{\psi} - \Gamma_I\Gamma^{\mu}[\tilde{X}^I_{\mu}, \tilde{\psi}] = 0. \tag{2.61}$$

The corresponding supersymmetry transformations read as

$$\begin{aligned}
&\delta\tilde{X}^I_{\mu} = i\bar{\varepsilon}\Gamma^I\tilde{\psi}_{\mu},\\
&\delta\tilde{\psi}_{\mu} = \Gamma_{\nu}\Gamma_I\varepsilon\nabla^{\nu}\tilde{X}^I_{\mu} + \frac{1}{12}\Gamma^{\rho\nu\lambda}\varepsilon G_{\rho\nu\lambda,\,\mu} - \frac{1}{2}\Gamma_{IJ}\Gamma^{\lambda}\varepsilon[\tilde{X}^I_{\mu}, \tilde{X}^J_{\lambda}],\\
&\delta\tilde{F}_{\mu\nu} = 3i\bar{\varepsilon}\Gamma_{[\mu\nu}\nabla_{\lambda]}\tilde{\psi}^{\lambda} + i\bar{\varepsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}[\tilde{X}^{\lambda}_I, \tilde{\psi}^{\kappa}],\\
&\delta A_{\mu} = i\bar{\varepsilon}\Gamma_{\mu\lambda}\tilde{\psi}^{\lambda}.
\end{aligned} \tag{2.62}$$

The constraints (2.48), or equivalently, (2.60), are quite crucial for equations (2.56)-(2.59) to take a neat form.

We will derive (2.58) and (2.59) as an example. Since $[C^{\lambda}, \nabla_{\lambda}H_{\mu\nu\kappa}, *] = 0$,

$$\begin{aligned}
&\nabla_{[\mu}\tilde{F}_{\nu\kappa]} = [C^{\lambda}, \nabla_{[\mu}H_{\nu\kappa\lambda]}, *]\\
= \quad &-\frac{1}{4}\epsilon_{\mu\nu\kappa\lambda\sigma\tau}[C^{\lambda}, [C^{\sigma}, X^I, \nabla^{\tau}X_I], *] - \frac{i}{8}\epsilon_{\mu\nu\kappa\lambda\sigma\tau}[C^{\lambda}, [C^{\sigma}, \bar{\psi}, \Gamma^{\tau}\psi], *].
\end{aligned} \tag{2.63}$$

From (2.42), we have

$$[C^\lambda, [C^\sigma, X^I, \nabla^\tau X_I], *] = [C^\sigma, [C^\lambda, X^I, \nabla^\tau X_I], *], \tag{2.64}$$

$$[C^\lambda, [C^\sigma, \bar{\psi}, \Gamma^\tau \psi], *] = [C^\sigma, [C^\lambda, \bar{\psi}, \Gamma^\tau \psi], *]. \tag{2.65}$$

As a result, $\nabla_{[\mu}\tilde{F}_{\nu\kappa]} = 0$. The Bianchi identity is satisfied, so $\tilde{F}_{\mu\nu}$ is indeed the field strength of A_μ. Since $H = *H$, from (2.52),

$$\nabla^\mu H_{\mu\nu\lambda} + \frac{1}{24}([C_\nu, X^I, \nabla_\lambda X_I] - [C_\lambda, X^I, \nabla_\nu X_I]) + \frac{i}{48}([C_\nu, \bar{\psi}, \Gamma_\lambda \psi] - [C_\lambda, \bar{\psi}, \Gamma_\nu \psi]) = 0. \tag{2.66}$$

So

$$\begin{aligned} & \nabla^\mu \tilde{F}_{\mu\nu} = [C^\lambda, \nabla^\mu H_{\mu\nu\lambda}, *] \\ = & -\frac{1}{24}[C^\lambda, [C_\nu, X^I, \nabla_\lambda X_I] - [C_\lambda, X^I, \nabla_\nu X_I], *] \\ - & \frac{i}{48}[C^\lambda, [C_\nu, \bar{\psi}, \Gamma_\lambda \psi] - [C_\lambda, \bar{\psi}, \Gamma_\nu \psi], *] \\ = & \frac{1}{24}[\tilde{X}^I_\lambda, \nabla_\nu \tilde{X}^\lambda_I] - \frac{i}{48}([\bar{\tilde{\psi}}_\nu, \tilde{\psi}] - [\bar{\tilde{\psi}}^\lambda, \Gamma_\nu \tilde{\psi}_\lambda]) = \tilde{J}_\nu, \end{aligned} \tag{2.67}$$

where we have used $\nabla_\lambda \tilde{X}^\lambda_I = 0$.

(2.56)-(2.59) looks quite like the equations of motion of the SYM theory. Fields $\tilde{\psi}_\mu$, $\tilde{X}^I_\mu$, and A_μ take values in a 2-Lie-algebra g, while the corresponding theory becomes a gauge theory. The Bianchi identity and the field equation for $\tilde{F}$ could be derived from the equation for H, if the selfdual condition is imposed. These equations are extension of the supersymmetric gauge field equations in loop space obtained in [31], for which, $C^\mu = C\dot{x}^\mu(\tau)$, $|\dot{x}(\tau)| = R$.

The auxiliary field C^μ is now absorbed in $\tilde{\psi}_\mu$, $\tilde{X}^I_\mu$, and $\tilde{F}_{\mu\nu}$. Note that when $C^\mu = 0$, the original (2.46)-(2.52) reduce to the equations of motion for free tensor multiplets, while the new-defined fields become zero. These new fields may represent selfdual strings. The length of C^μ characterizes the length of strings. Especially, when the strings shrink to points, which are described by tensor multiplets, the interaction disappears. A particular C^μ corresponds to a particular set of selfdual strings. We want to take C^μ as the new degrees of freedom added, so we will not specify it. The path integral may cover all possible configurations of C^μ.

Now, consider the Coulomb branch of the theory. The supersymmetry transformation (2.54) for the original fields suggests that the vacuum configuration is given by constant X^I satisfying $[C^\mu, X^I, X^J] = 0$, while the supersymmetry transformation (2.62) for the new fields shows that the vacuum is given by constant $\tilde{X}^I_\mu$ with $[\tilde{X}^I_\mu, \tilde{X}^J_\nu] = 0$. From (2.42) it is easy to see if $[C^\mu, X^I, X^J] = 0$, $[\tilde{X}^I_\mu, \tilde{X}^J_\nu] = 0$. Choose a maximal subspace Λ_0, $\Lambda_0 \subset \Lambda$, $\forall A, B \in \Lambda_0$, $\forall C \in \bar{\Lambda}$, $[C, A, B] = 0$. $\Lambda = \Lambda_0 \oplus \Lambda_1$, there is a special set of basis $\left\{t^1 \cdots t^M\right\}$ for Λ_1, $\forall C \in \bar{\Lambda}$, $\forall A \in \Lambda_0$, $[C, A, t^m] \propto t^m$. Λ_0 and $\left\{t^1 \cdots t^M\right\}$ could be taken as the Cartan subalgebra and the roots respectively. Suppose the vacuum expectation value of X^I is given by $\bar{X}^I$, $\bar{X}^I \in \Lambda_0$. $[C_\mu, \bar{X}^I, t^m] = \phi^I_{m\mu} t^m$. Let $[C_\mu, \bar{X}^I, *] = \tilde{\bar{X}}{}^I_\mu$, $[C_\nu, t^m, *] = \tilde{t}^m_\nu$, then $[\tilde{\bar{X}}{}^I_\mu, \tilde{t}^m_\nu] = \phi^I_{m\mu} \tilde{t}^m_\nu$. $\tilde{t}^m_\nu$ is the corresponding root in g. Similarly, suppose the vacuum expectation value of $H^{\mu\nu\lambda}$ is $\bar{H}^{\mu\nu\lambda}$, $\bar{H}^{\mu\nu\lambda} \in \Lambda_0$, then $[C_\lambda, \bar{H}^{\mu\nu\lambda}, t^m] = f^{\mu\nu}_m t^m$. For $[C_\lambda, \bar{H}^{\mu\nu\lambda}, *] = \tilde{\bar{F}}{}^{\mu\nu}$, $[\tilde{\bar{F}}{}^{\mu\nu}, \tilde{t}^m_\nu] = f^{\mu\nu}_m \tilde{t}^m_\nu$.

To calculate the WZ term, we need to plug the vacuum expectation values into the Dirac equation for fermions. Let ψ^m denote fermions taking values in the root t^m. From (2.50),

$$\Gamma_\mu \nabla^\mu \psi^m + \phi^I_{m\mu} \Gamma^\mu \Gamma_I \psi^m = 0. \tag{2.68}$$

From (2.61),

$$\Gamma_\mu \nabla^\mu \tilde{\psi}^m + \phi^I_{m\mu} \Gamma^\mu \Gamma_I \tilde{\psi}^m = 0. \tag{2.69}$$

Besides,

$$[C_\lambda, \bar{H}^{\mu\nu\lambda}, \Gamma_\mu\Gamma_\nu\psi^m] = f_m^{\mu\nu}\Gamma_\mu\Gamma_\nu\psi^m, \quad [\tilde{\bar{F}}^{\mu\nu}, \Gamma_\mu\Gamma_\nu\tilde{\psi}^m] = f_m^{\mu\nu}\Gamma_\mu\Gamma_\nu\tilde{\psi}^m. \tag{2.70}$$

ψ^m and $\tilde{\psi}^m$ are the $6d$ anti-chiral and chiral fermions respectively. Written as the $11d$ Majorana spinors, $\Gamma_7\psi^m = -\psi^m$, $\Gamma_7\tilde{\psi}^m = \tilde{\psi}^m$, where $\Gamma_7 = \Gamma_{012345}$. One can either choose ψ^m or $\tilde{\psi}^m$ to do the calculation. Here, we use ψ^m. Just as that in [17, 18], the WZ term could be written as

$$\Gamma_m = \mathtt{Tr}\left\{\mathtt{ln}\,[i\Gamma_0\Gamma_\mu\partial^\mu + \Gamma_0\Gamma_\mu A_m^\mu + i\Gamma_0\Gamma^\mu\Gamma_I\phi_{m\mu}^I]\frac{1-\Gamma_7}{2}\right\}, \tag{2.71}$$

$$\frac{\delta\Gamma_m}{\delta\phi_{m\mu}^I(x)} = Sp[\langle x|\,\frac{1}{i\Gamma_\nu\partial^\nu + \Gamma_\nu A_m^\nu + i\Gamma^\nu\Gamma_I\phi_{m\nu}^I}\,|x\rangle\, i\Gamma^\mu\Gamma_I(\frac{1-\Gamma_7}{2})], \tag{2.72}$$

where Sp is the trace in spinor indices. WZ term comes from the imaginary part of the effective action. Taking the difference of (2.72) with its complex conjugate,

$$\begin{aligned}\frac{\delta Im\Gamma_m}{\delta\phi_{m\mu}^I(x)} &= -\frac{1}{2}Sp[\langle x|\,\frac{1}{i\Gamma_\nu\partial^\nu + \Gamma_\nu A_m^\nu + i\Gamma^\nu\Gamma_I\phi_{m\nu}^I}\,|x\rangle\,\Gamma^\mu\Gamma_I\Gamma_7] \\ &= -\frac{1}{2}Sp[\langle x|\,\frac{1}{\not{D}}\,|x\rangle\,\Gamma^\mu\Gamma_I\Gamma_7] \\ &= -\frac{1}{2}Sp[\langle x|\,\frac{\not{D}}{\not{D}^2}\,|x\rangle\,\Gamma^\mu\Gamma_I\Gamma_7].\end{aligned} \tag{2.73}$$

(2.73) may be expanded as the sum of the terms proportional to 1 or $Tr(\Gamma_{M_1}\cdots\Gamma_{M_k})$, where $M_1,\ldots,M_k$ are distinct indices. Note that $Tr(\Gamma_{M_1}\cdots\Gamma_{M_k}) = 0$ unless $k = 11$, so we need to extract the term proportional to $Tr(\Gamma_0\cdots\Gamma_{10}) \propto \epsilon_{0\cdots10}$. In the numerator, $i\Gamma^\nu\Gamma_I\phi_{m\nu}^I$ in $\not{D}$ will be kept, while in the denominator,

$$\not{D}^2 = -\partial^2 + \phi_{m\mu}^I\phi_{mI}^\mu + \frac{i}{2}\Gamma_\mu\Gamma_\nu f_m^{\mu\nu} - \Gamma_\mu\Gamma^\nu\Gamma_I\partial^\mu\phi_{m\nu}^I + \Gamma^{\mu\nu}\Gamma_{IJ}\phi_{m\mu}^I\phi_{m\nu}^J + \cdots, \tag{2.74}$$

where $f_m^{\mu\nu} = \partial^\mu A_m^\nu - \partial^\nu A_m^\mu + [A_m^\mu, A_m^\nu]$,

$$\frac{1}{\not{D}^2} = -\sum_{n=0}^{\infty} \frac{[\frac{i}{2}\Gamma_\mu\Gamma_\nu f_m^{\mu\nu} - \Gamma_\mu\Gamma^\nu\Gamma_I\partial^\mu\phi_{m\nu}^I + \Gamma^{\mu\nu}\Gamma_{IJ}\phi_{m\mu}^I\phi_{m\nu}^J + \cdots]^n}{(\partial^2 - \phi_{m\mu}^I\phi_{mI}^\mu)^{n+1}}. \tag{2.75}$$

The integral that needs to be performed is

$$\begin{aligned} \langle x| \frac{1}{(\partial^2 - \phi_{m\mu}^I\phi_{mI}^\mu)^{n+1}} |x\rangle &= (-1)^{n+1}\int \frac{d^dp}{(2\pi)^d}\frac{1}{(p^2 + \phi_{m\mu}^I\phi_{mI}^\mu)^{n+1}} \\ &= \frac{i\Gamma(n+1-\frac{d}{2})}{(2\sqrt{\pi})^d\Gamma(n+1)}\frac{1}{(\sqrt{\phi_{m\mu}^I\phi_{mI}^\mu})^{2n+2-d}}. \end{aligned} \tag{2.76}$$

The constraint $[C^\mu, \nabla_\mu\psi^m, *] = 0$ reduces the dynamics from $6d$ to $5d$. If C^μ is taken as the vector tangential to the selfdual string, the constraint means that the physical momentum of the string is along the transverse direction. As a result, $d = 5$.

In analogy with the $5d$ SYM theory, we may let $n = 4$, and then $2n + 2 - d = 5$.

$$\frac{\delta Im\Gamma_m}{\delta\phi_{m\mu}^I(x)} \propto \frac{Sp\left\{[\frac{i}{2}\Gamma_\rho\Gamma_\nu f_m^{\rho\nu} - \Gamma_\rho\Gamma^\nu\Gamma_J\partial^\rho\phi_{m\nu}^J + \Gamma^{\rho\nu}\Gamma_{KJ}\phi_{m\rho}^K\phi_{m\nu}^J + \cdots]^4\phi_{m\lambda}^L\Gamma^\lambda\Gamma_L\Gamma^\mu\Gamma_I\Gamma_7\right\}}{(\sqrt{\phi_{m\mu}^I\phi_{mI}^\mu})^5} \tag{2.77}$$

The term containing one $f_m^{\rho\nu}$ and three $\partial^\rho\phi_{m\nu}^J$ is

$$\begin{aligned} \frac{\delta L_1}{\delta\phi_{m\mu}^I(x)} &\propto -\frac{f_{m\rho\nu}\partial_{\rho_1}\phi_{m\nu_1}^{J_1}\partial_{\rho_2}\phi_{m\nu_2}^{J_2}\partial_{\rho_3}\phi_{m\nu_3}^{J_3}\phi_{m\lambda}^L}{(\sqrt{\phi_{m\mu}^I\phi_{mI}^\mu})^5} \\ &\quad Sp[\Gamma^\rho\Gamma^\nu\Gamma^{\rho_1}\Gamma^{\nu_1}\Gamma^{\rho_2}\Gamma^{\nu_2}\Gamma^{\rho_3}\Gamma^{\nu_3}\Gamma^\lambda\Gamma^\mu\Gamma_7\Gamma_{J_1}\Gamma_{J_2}\Gamma_{J_3}\Gamma_L\Gamma_I] \end{aligned} \tag{2.78}$$

The counterpart of L_1 in the low energy effective action is $L_1' \propto \int_{W_6} H_3 \wedge A_3$,

$$\frac{\delta L_1'}{\delta\phi_m^I(x)} \propto \frac{\epsilon^{\rho\nu\mu\rho_1\rho_2\rho_3}\epsilon_{J_1J_2J_3LI}h_{m\rho\nu\mu}\partial_{\rho_1}\phi_m^{J_1}\partial_{\rho_2}\phi_m^{J_2}\partial_{\rho_3}\phi_m^{J_3}\phi_m^L}{(\sqrt{\phi_m^I\phi_{mI}})^5}, \tag{2.79}$$

Here m denotes a particular root. If $m \sim (i,j)$, $\phi_m^I = \phi_i^I - \phi_j^I = \phi_{ij}^I$, $h_{m\rho\nu\mu} =$

$h_{i\rho\nu\mu} - h_{j\rho\nu\mu} = h_{ij\rho\nu\mu}$. In general, one may expect that ϕ^I_μ could be expanded as $\phi^I_\mu = \sum_k c_{k\mu} a^I_k$, where $c_{k\mu}$ and a^I_k are vectors along the longitudinal and transverse directions respectively. The simplest situation is $\phi^J_{m\nu} = c_{m\nu}\phi^J_m$. $\partial_\rho \phi^J_{m\nu} = \partial_\rho c_{m\nu}\phi^J_m + c_{m\nu}\partial_\rho \phi^J_m = c_{m\nu}\partial_\rho\phi^J_m$. Therefore, $\partial_{\rho_3}\phi^{J_3}_{m\nu_3}\Gamma^{\rho_3}\Gamma^{\nu_3}\Gamma^\lambda \phi^L_{m\lambda} = |c_m|^2\Gamma^{\rho_3}\partial_{\rho_3}\phi^{J_3}_m\phi^L_m$. Similarly, $\partial_{\rho_1}\phi^{J_1}_{m\nu_1}\partial_{\rho_2}\phi^{J_2}_{m\nu_2}\Gamma^{\rho_1}\Gamma^{\nu_1}\Gamma^{\rho_2}\Gamma^{\nu_2} \sim -|c_m|^2\partial_{\rho_1}\phi^{J_1}_m\partial_{\rho_2}\phi^{J_2}_m\Gamma^{\rho_1}\Gamma^{\rho_2}$, where we have neglected the term involving $c^\nu_m\partial_\nu\phi^J_m$. (2.78) is simplified to

$$\frac{\delta L_1}{\delta \phi^I_m(x)} \propto \frac{f_{m\rho\nu}c_{m\mu}\partial_{\rho_1}\phi^{J_1}_m\partial_{\rho_2}\phi^{J_2}_m\partial_{\rho_3}\phi^{J_3}_m\phi^L_m}{|c_m|(\sqrt{\phi^I_m\phi_{mI}})^5}$$
$$Sp[\Gamma^\rho\Gamma^\nu\Gamma^{\rho_1}\Gamma^{\rho_2}\Gamma^{\rho_3}\Gamma^\mu\Gamma_7\Gamma_{J_1}\Gamma_{J_2}\Gamma_{J_3}\Gamma_L\Gamma_I] \tag{2.80}$$

To get the nonzero result, the first six Gamma matrices should multiply to 1, while the last five Gamma matrices should cover $\Gamma_6\cdots\Gamma_{10}$. The trace then becomes $Sp[\Gamma_0\cdots\Gamma_{10}] \sim \epsilon_{0\cdots 10}$. ρ_1, ρ_2 and ρ_3 must be different, so $\{\rho,\nu,\mu\} = \{\rho_1,\rho_2,\rho_3\}$. One can similarly expand $f_{m\rho\nu}$ as $f_{m\rho\nu} = h_{m\rho\nu\sigma}c^\sigma_m$. $f_{m\rho\nu}c_{m\mu} = h_{m\rho\nu\sigma}c^\sigma_m c_{m\mu}$. As is mentioned before, c_m is not fixed but should also be integrated in the path integral. The orientation of the selfdual string in $6d$ spacetime is arbitrary. Summing over all possible directions, $\sum c^\sigma_m c_{m\mu} = |c_m|^2 g^\sigma_\mu$, so $\sum f_{m\rho\nu}c_{m\mu} = h_{m\rho\nu\sigma}\sum c^\sigma_m c_{m\mu} = |c_m|^2 h_{m\rho\nu\mu}$. (2.80) then becomes

$$\frac{\delta L_1}{\delta \phi^I_m(x)} \propto \frac{|c_m|(*h_m)_{\rho\nu\mu}\partial_{\rho_1}\phi^{J_1}_m\partial_{\rho_2}\phi^{J_2}_m\partial_{\rho_3}\phi^{J_3}_m\phi^L_m}{(\sqrt{\phi^I_m\phi_{mI}})^5}\epsilon^{\rho\nu\mu\rho_1\rho_2\rho_3}\epsilon_{J_1J_2J_3LI}. \tag{2.81}$$

Compared with (2.79), (2.81) contains $*$ which is resulted from Γ_7 inside the trace. This is not quite satisfactory, but luckily, since $*h = h$, (2.79) and (2.81) still coincide up to a $|c_m|$ factor.

Notice that to get L_1 which is close to L'_1, $d = 5$ is quite crucial. For an ordinary $6d$ theory without the constraint, the denominator is $(\sqrt{\phi^I_{m\mu}\phi^\mu_{mI}})^{2n-4}$, so one cannot get L'_1 no matter which n is taken. In other words, to get the WZ term $H_3 \wedge A_3$, the basic degrees of freedom should be the $1d$ object with $5d$ momentum other than the $0d$ object with $6d$ momentum. For 1/2 BPS selfdual strings, the momentum is along the transverse direction, since the momentum along the longitudinal direction may reduce the selfdual string to a 1/4 BPS state [58]. So the constraint in [23] may indicate that the selfdual strings involved in equations are 1/2 BPS states, while the WZ term $H_3 \wedge A_3$ is generated by 1/2 BPS selfdual strings.

The next problem is to get the WZ term corresponding to $A_3 \wedge F_4 \sim \sigma_3 \wedge \omega_4$, which is much more difficult. First, we need to find an explicit expression for σ_3. Consider the $5d$ transverse space with coordinate ϕ^{a_i}, $a_i = 6 \cdots 10$,

$$\omega_4 = \epsilon_{a_1 a_2 a_3 a_4 a_5} \frac{1}{|\phi|^5} \phi^{a_1} d\phi^{a_2} \wedge d\phi^{a_3} \wedge d\phi^{a_4} \wedge d\phi^{a_5} = *\hat{\phi}. \tag{2.82}$$

σ_3 could be constructed in analogy with the gauge field describing the magnetic monopole in $3d$ space. Select an arbitrary vector v, $v \cdot \phi = |v||\phi| \cos\theta$,

$$\begin{aligned}
\sigma_3 &= \frac{(\cos^3\theta - 3\cos\theta + 2) * (v \wedge \phi)}{3\sin^4\theta |v||\phi|} \\
&= \frac{(v\cdot\phi)^3 - 3(v\cdot\phi)|v|^2|\phi|^2 + 2|v|^3|\phi|^3}{3\sin^4\theta|v|^4|\phi|^7} \\
&\quad \epsilon_{a_1 a_2 a_3 a_4 a_5} v^{a_1}\phi^{a_2} d\phi^{a_3} \wedge d\phi^{a_4} \wedge d\phi^{a_5}.
\end{aligned} \tag{2.83}$$

σ_3 is singular on a ray OV starting from the origin and extending in $-v$ direction. In $11d$ spacetime, $OV \times W_6 = W_7$. W_7 is the Dirac brane similar to the Dirac string [32, 34]. Replace v by $-v$, we get another σ_3. Take the average of these two σ_3's, the last term in (2.83) could be dropped. Of course, in this case, the singularity exists in a straight line. The typical WZ term is $\sigma_{3ji} \wedge \omega_{4ki}$. $d\sigma_{3ji} = \omega_{4ji}$.

$$\sigma_{3ji} = \frac{(v \cdot \phi_{ji})^3 - 3(v \cdot \phi_{ji})|v|^2|\phi_{ji}|^2}{3\sin^4\theta |v|^4 |\phi_{ji}|^7} \epsilon_{a_1a_2a_3a_4a_5} v^{a_1} \phi_{ji}^{a_2}\ d\phi_{ji}^{a_3} \wedge d\phi_{ji}^{a_4} \wedge d\phi_{ji}^{a_5}, \tag{2.84}$$

$$\begin{aligned} \sigma_{3ji} \wedge \omega_{4ki} \quad = \quad & \frac{(v \cdot \phi_{ji})^3 - 3(v \cdot \phi_{ji})|v|^2|\phi_{ji}|^2}{3\sin^4\theta |v|^4 |\phi_{ji}|^7 |\phi_{ki}|^5} \\ & \epsilon_{a_1a_2a_3a_4a_5} \epsilon_{b_1b_2b_3b_4b_5} \\ & v^{a_1} \phi_{ji}^{a_2} \phi_{ki}^{b_1}\ d\phi_{ji}^{a_3} \wedge d\phi_{ji}^{a_4} \wedge d\phi_{ji}^{a_5} \wedge d\phi_{ki}^{b_2} \wedge d\phi_{ki}^{b_3} \wedge d\phi_{ki}^{b_4} \wedge d\phi_{ki}^{b_5}. \end{aligned} \tag{2.85}$$

The field theory calculation gives $\delta\Gamma_{WZ}/\delta\phi^I$, which is the counterpart of the Lorentz force on supergravity side. Consider a M5 brane in a background field $\hat{F}_4$, the action contains the term $S = -\int_{W_6} A_6$, $dA_6 = *\hat{F}_4 + A_3 \wedge \hat{F}_4/2$.

$$\begin{aligned} & \frac{\delta S}{\delta \phi^I} \\ = \quad & -\frac{1}{6!} \epsilon^{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6} \partial_{\rho_1} Y^{n_1} \partial_{\rho_2} Y^{n_2} \partial_{\rho_3} Y^{n_3} \partial_{\rho_4} Y^{n_4} \partial_{\rho_5} Y^{n_5} \partial_{\rho_6} Y^{n_6} (dA_6)_{In_1n_2n_3n_4n_5n_6} \\ = \quad & -\frac{1}{6!} \epsilon^{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6} \partial_{\rho_1} Y^{n_1} \partial_{\rho_2} Y^{n_2} \partial_{\rho_3} Y^{n_3} \partial_{\rho_4} Y^{n_4} \partial_{\rho_5} Y^{n_5} \partial_{\rho_6} Y^{n_6} \\ & (*\hat{F}_4 + \frac{1}{2} A_3 \wedge \hat{F}_4)_{In_1n_2n_3n_4n_5n_6} \\ = \quad & f_{6I} + g_{6I}. \end{aligned} \tag{2.86}$$

Y^{n_i} are embedding coordinates. $n_i = 0 \cdots 10$, $Y^I = \phi^I$. f_{6I} and g_{6I} are forces related with the magnetic-magnetic interaction and the electric-magnetic interaction respectively. g_{6I} is the Lorentz force derived from the WZ term. If db_2 is also taken into account,

$$g_{6I} = \frac{1}{6!}\epsilon^{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6}\partial_{\rho_1}Y^{n_1}\partial_{\rho_2}Y^{n_2}\partial_{\rho_3}Y^{n_3}\partial_{\rho_4}Y^{n_4}\partial_{\rho_5}Y^{n_5}\partial_{\rho_6}Y^{n_6}(\frac{1}{2}h_3 \wedge \hat{F}_4)_{In_1n_2n_3n_4n_5n_6}. \tag{2.87}$$

For the WZ term in (2.31),[5]

$$\begin{aligned} \frac{\delta \int_{W_{7i}} \sigma_{3ji} \wedge \omega_{4ki}}{\delta\phi_i^I(x)} &= -\int_{W_{7i}} [\frac{\delta\sigma_{3ji}}{\delta\phi_{ji}^I(x)} \wedge \omega_{4ki} + \sigma_{3ji} \wedge \frac{\delta\omega_{4ki}}{\delta\phi_{ki}^I(x)}] \\ &= (\sigma_{3ji} \wedge F_{3kiI})(x) - (\omega_{4ji} \wedge F_{3kiI} + \omega_{4ki} \wedge F_{3jiI})(x). \end{aligned} \tag{2.88}$$

$\partial W_{7i} = W_{6i}$, $x \in W_{6i}$.

$$F_{3kiI} = \alpha \, \frac{\epsilon_{ILJ_1J_2J_3}\phi_{ki}^L d\phi_{ki}^{J_1} \wedge d\phi_{ki}^{J_2} \wedge d\phi_{ki}^{J_3}}{|\phi_{ki}|^5} \tag{2.89}$$

up to a total derivative. α is a constant. Except for the 6-form, (2.88) also contains the 7-form because $\sigma_3 \wedge \omega_4$ alone is not closed.

$$\begin{aligned} \frac{\delta \int_{W_{7i}}(\sigma_{3ji} \wedge \omega_{4ki} + \sigma_{3ki} \wedge \omega_{4ji})}{\delta\phi_i^I(x)} &= (\sigma_{3ji} \wedge F_{3kiI} + \sigma_{3ki} \wedge F_{3jiI})(x) \\ &\quad -2(\omega_{4ji} \wedge F_{3kiI} + \omega_{4ki} \wedge F_{3jiI})(x), \end{aligned} \tag{2.90}$$

$$\begin{aligned} &\frac{\delta \int_{W_7} \Omega_{ijk}}{\delta\phi_i^I(x)} \\ \propto\ &(\sigma_{3ji} \wedge F_{3kiI} + \sigma_{3ki} \wedge F_{3jiI} - \sigma_{3kj} \wedge F_{3ijI} - \sigma_{3jk} \wedge F_{3ikI})(x) \\ &-2(\omega_{4ji} \wedge F_{3kiI} + \omega_{4ki} \wedge F_{3jiI} - \omega_{4kj} \wedge F_{3ijI} - \omega_{4jk} \wedge F_{3ikI})(x). \end{aligned} \tag{2.91}$$

We need to get the 6-form of (2.91) from the field theory calculation. Unfortu-

[5] Here, for simplicity, the last line in (2.88) is denoted as the differential form, but it should be more accurately written in the form like that in (2.86) and (2.87).

nately, there are several problems. The only input on field theory side is the vacuum expectation value $(\phi_1, \cdots, \phi_N)$ on W_6. As a result, we have to construct the 6-form from $(\phi_{ij}, \phi_{jk}, \phi_{ki})$. σ_{3ji} may differ from (2.84) by an exact 3-form. However, (2.84) contains a constant vector v, which has no relevance with the vacuum expectation value. Nevertheless, to preserve the supersymmetry, the vacuum expectation value should be constant, so we may take $(\phi_1, \cdots, \phi_N)$ as constant and replace the v in (2.85) by $\bar{\phi}_{ki} = \phi_k - \phi_i$. $\bar{\phi}_{ki}$ should be distinguished from ϕ_{ki}. The former is only defined at the boundary W_6, while the latter is defined in the entire W_7. For forms living on W_6, $\bar{\phi}_{ji} = \phi_{ji}$, so one may write $\sigma_{3ji} \wedge F_{3kiI}$ as

$$\begin{aligned} \sigma_{3ji} \wedge F_{3kiI} &= \alpha \, \frac{(\phi_{ki} \cdot \phi_{ji})^3 - 3(\phi_{ki} \cdot \phi_{ji})|\phi_{ki}|^2|\phi_{ji}|^2}{3 \sin^4 \theta |\phi_{ki}|^9 |\phi_{ji}|^9} \epsilon_{Ib_1b_2b_3b_4} \epsilon_{La_1a_2a_3a_4} |\phi_{ji}|^2 \phi^L_{ki} \phi^{a_1}_{ji} \\ & \quad \phi^{b_1}_{ki} \, d\phi^{a_2}_{ji} \wedge d\phi^{a_3}_{ji} \wedge d\phi^{a_4}_{ji} \wedge d\phi^{b_2}_{ki} \wedge d\phi^{b_3}_{ki} \wedge d\phi^{b_4}_{ki}. \end{aligned} \tag{2.92}$$

$\phi_{ki} \cdot \phi_{ji} = |\phi_{ki}||\phi_{ji}| \cos \theta$. Of course, in this case, the pullback of $\sigma_{3ji} \wedge F_{3kiI}$ on W_6 actually vanishes.

Now, return to the field theory calculation. To get a 6-form like (2.92), we may let $n = 7$.

$$\begin{aligned} & \frac{\delta Im \Gamma_m}{\delta \phi^I_{m\mu}(x)} \\ \propto \; & \frac{Sp\left\{ [\frac{i}{2}\Gamma_\rho \Gamma_\nu f^{\rho\nu}_m - \Gamma_\rho \Gamma^\nu \Gamma_J \partial^\rho \phi^J_{m\nu} + \Gamma^{\rho\nu} \Gamma_{KJ} \phi^K_{m\rho} \phi^J_{m\nu} + \cdots]^7 \phi^L_{m\lambda} \Gamma^\lambda \Gamma_L \Gamma^\mu \Gamma_I \Gamma_7 \right\}}{(\sqrt{\phi^I_{m\mu} \phi^\mu_{mI}})^{16-d}} \end{aligned} \tag{2.93}$$

The term containing six $\partial^\rho \phi^J_{m\nu}$ is

$$\frac{\delta L_2}{\delta \phi^I_{m\mu}(x)} \propto - \frac{\partial_{\rho_1} \phi^{J_1}_{m\nu_1} \partial_{\rho_2} \phi^{J_2}_{m\nu_2} \partial_{\rho_3} \phi^{J_3}_{m\nu_3} \partial_{\rho_4} \phi^{J_4}_{m\nu_4} \partial_{\rho_5} \phi^{J_5}_{m\nu_5} \partial_{\rho_6} \phi^{J_6}_{m\nu_6} \phi^K_{m\rho} \phi^J_{m\nu} \phi^L_{m\lambda}}{(\sqrt{\phi^I_{m\mu} \phi^\mu_{mI}})^{16-d}}$$

$$Sp[\Gamma^{\rho_1}\Gamma^{\nu_1}\Gamma^{\rho_2}\Gamma^{\nu_2}\Gamma^{\rho_3}\Gamma^{\nu_3}\Gamma^{\rho_4}\Gamma^{\nu_4}\Gamma^{\rho_5}\Gamma^{\nu_5}\Gamma^{\rho_6}\Gamma^{\nu_6}\Gamma^{\rho\nu}\Gamma^{\lambda}\Gamma^{\mu}$$

$$\Gamma_7\Gamma_{J_1}\Gamma_{J_2}\Gamma_{J_3}\Gamma_{J_4}\Gamma_{J_5}\Gamma_{J_6}\Gamma_{KJ}\Gamma_L\Gamma_I] \tag{2.94}$$

Except for $\delta\phi$ on the left-hand side, there are nine ϕ in the numerator, so to get a scale-invariant result, we must have $d = 6$. The result is in contrast with the previous discussion on $H_3 \wedge A_3$, for which, the scale invariant result needs $d = 5$. So it seems that for N^3 degrees of freedom, the $6d$ dynamics is still necessary. Note that the near-extremal entropy of the M5 branes scales as the entropy of the $6d$ massless ideal gas [59].

To related (2.94) with (2.92), we need to make a further assumption on $\phi^I_{m\mu}$. Consider the $6d$ (2, 0) theory of the A_{N-1} type. $m \sim (i, j)$ is not enough to produce (2.92), so one may let $m \sim (i, j, k)$. It is natural to expand $\phi^I_{m\mu}$ as

$$\phi^I_{m\mu} = c_{i\mu}\phi^I_{jk} + c_{j\mu}\phi^I_{ki} + c_{k\mu}\phi^I_{ij} = c_{ij\mu}\phi^I_{jk} - c_{jk\mu}\phi^I_{ij} = c_{ki\mu}\phi^I_{ij} - c_{ij\mu}\phi^I_{ki} = c_{jk\mu}\phi^I_{ki} - c_{ki\mu}\phi^I_{jk}, \tag{2.95}$$

where $c_{ij\mu} = c_{i\mu} - c_{j\mu}$.

$$\delta\phi^I_{m\mu} = c_{kj\mu}\delta\phi^I_i + c_{ik\mu}\delta\phi^I_j + c_{ji\mu}\delta\phi^I_k. \tag{2.96}$$

In analogy with the previous calculation,

$$\begin{aligned}
& \Gamma^{\rho_1}\Gamma^{\nu_1}\Gamma^{\rho_2}\Gamma^{\nu_2}\partial_{\rho_1}\phi^{J_1}_{m\nu_1}\partial_{\rho_2}\phi^{J_2}_{m\nu_2} \\
\sim\ & \Gamma^{\rho_1}\Gamma^{\rho_2}[-|c_{ki}|^2\partial_{\rho_1}\phi^{J_1}_{ji}\partial_{\rho_2}\phi^{J_2}_{ji} + (c_{ki}\cdot c_{ji})\partial_{\rho_1}\phi^{J_1}_{ki}\partial_{\rho_2}\phi^{J_2}_{ji} \\
& +(c_{ji}\cdot c_{ki})\partial_{\rho_1}\phi^{J_1}_{ji}\partial_{\rho_2}\phi^{J_2}_{ki} - |c_{ji}|^2\partial_{\rho_1}\phi^{J_1}_{ki}\partial_{\rho_2}\phi^{J_2}_{ki} \\
& +\frac{1}{2}\Gamma^{\nu_1\nu_2}(c_{ji\nu_1}c_{ki\nu_2} - c_{ji\nu_2}c_{ki\nu_1})\partial_{\rho_1}\phi^{J_1}_{ki}\partial_{\rho_2}\phi^{J_2}_{ji}
\end{aligned}$$

$$-\frac{1}{2}\Gamma^{\nu_1\nu_2}(c_{ji\nu_1}c_{ki\nu_2}-c_{ji\nu_2}c_{ki\nu_1})\partial_{\rho_1}\phi^{J_1}_{ji}\partial_{\rho_2}\phi^{J_2}_{ki}]+\cdots \tag{2.97}$$

To get an expression which has the similar structure as (2.92), we have to assume $|c_{ji}|^2=|\phi_{ji}|^2$, $c_{ki}\cdot c_{ji}=\phi_{ki}\cdot\phi_{ji}$. This could be realized if one defines an inner product preserving linear mapping g from the $5d$ transverse space to the $6d$ longitudinal spacetime. For example, $g(x_n)=x_{n-5}$, $n=6\cdots 10$. $c_i=g(\phi_i)$. Each g gives an identification of the internal $SO(5)$ group and the spatial $SO(5)$ rotational group. Recall that in [20, 21], the locking of the internal $SO(5)$ R-symmetry and the spatial $SO(5)$ rotational symmetry is quite crucial for the construction of 1/4 BPS string junctions, while the junction composed by three half selfdual strings should form a lattice dual to the triangle $(\phi_{ij},\phi_{jk},\phi_{ki})$. Obviously, (c_{ij},c_{jk},c_{ki}) is just a lattice dual to $(\phi_{ij},\phi_{jk},\phi_{ki})$, so it seems that the orientation of the three half selfdual strings could be characterized by (c_{ij},c_{jk},c_{ki}).

$$\begin{aligned}\sqrt{\phi^I_{m\mu}\phi^{\mu}_{mI}} &= [|c_{ki}|^2|\phi_{ij}|^2+|c_{ij}|^2|\phi_{ki}|^2-2(c_{ki}\cdot c_{ij})(\phi_{ki}\cdot\phi_{ij})]^{\frac{1}{2}}\\ &= 2^{\frac{1}{2}}[|\phi_{ki}|^2|\phi_{ij}|^2-(\phi_{ki}\cdot\phi_{ij})^2]^{\frac{1}{2}}=2^{\frac{1}{2}}|\phi_{ki}||\phi_{ij}|\sin\theta=2^{\frac{3}{2}}s_{ijk},\end{aligned} \tag{2.98}$$

where s_{ijk} is the area of the triangle Δ_{ijk}. Replace $\delta L_2/\delta\phi^I_{m\mu}(x)$ by $\delta L_2/\delta\phi^I_i(x)$, due to (2.96), there will be an extra $c_{kj\mu}$ entering into the numerator of (2.94).

$$\Gamma^{\lambda}\Gamma^{\mu}c_{kj\mu}\phi^L_{m\lambda}=\frac{1}{2}\Gamma^{\lambda\mu}(c_{ji\lambda}c_{ki\mu}-c_{ji\mu}c_{ki\lambda})\phi^L_{kj}-|c_{ki}|^2\phi^L_{ji}+(c_{ji}\cdot c_{ki})\phi^L_{ki}+(c_{ki}\cdot c_{ji})\phi^L_{ji}-|c_{ji}|^2\phi^L_{ki} \tag{2.99}$$

$$\Gamma^{\rho\nu}\Gamma_{KJ}\phi^K_{m\rho}\phi^J_{m\nu}=\frac{1}{2}\Gamma^{\rho\nu}\Gamma_{KJ}(c_{ji\rho}c_{ki\nu}-c_{ji\nu}c_{ki\rho})(\phi^K_{ji}\phi^J_{ki}-\phi^J_{ji}\phi^K_{ki}) \tag{2.100}$$

so

$$\Gamma^{\rho\nu}\Gamma^{\lambda}\Gamma^{\mu}\Gamma_{KJ}c_{kj\mu}\phi^K_{m\rho}\phi^J_{m\nu}\phi^L_{m\lambda}=2\Gamma_{KJ}\sin^2\theta|c_{ji}|^2|c_{ki}|^2\phi^L_{kj}(\phi^J_{ji}\phi^K_{ki}-\phi^K_{ji}\phi^J_{ki})+\cdots$$

$$= \quad 2\Gamma_{KJ}\sin^2\theta|\phi_{ji}|^2|\phi_{ki}|^2\phi^L_{kj}(\phi^J_{ji}\phi^K_{ki} - \phi^K_{ji}\phi^J_{ki}) + \cdots \tag{2.101}$$

Plug (2.97), (2.99) and (2.100) into (2.94), the rest Gamma matrices carrying longitudinal indices are $\Gamma^{\rho_1}\Gamma^{\rho_2}\Gamma^{\rho_3}\Gamma^{\rho_4}\Gamma^{\rho_5}\Gamma^{\rho_6}$, which should multiply to 1, so there will be at most three matrices with different indices. Aside from the trace, each term also contains a factor $\partial_{\rho_1}\phi\partial_{\rho_2}\phi\partial_{\rho_3}\phi\partial_{\rho_4}\phi\partial_{\rho_5}\phi\,\partial_{\rho_6}\phi$, where ϕ can be ϕ_{ji} or ϕ_{ki}. Notice that ∂ acting on the same ϕ must carry different indices, otherwise, the anti-symmetrization of the transverse indices of ϕ may give zero. As a result, there must be three ϕ_{ji} and three ϕ_{ki}. Altogether, there are 16 terms with the coefficient $(\phi_{ji}\cdot\phi_{ki})^3$ and 28 terms with the coefficient $(\phi_{ji}\cdot\phi_{ki})|\phi_{ji}|^2|\phi_{ki}|^2$. The result is still far from (2.92). Just as the previous discussion for $H_3 \wedge A_3$, due to the Γ_7, an extra $*$ exists. Of course, the selfdual condition should also be imposed on $h_3 = db_2 - A_3$, which may solve the problem. However, at present stage, $\sigma_3 \neq *\sigma_3$. Moreover, there are ten Gamma matrices inside the trace carrying the transverse indices, so the final result is zero. This is not quite unacceptable. Under the present assumption, the Lorentz force will vanish. If we neglect this problem and simply assume the trace in (2.94) gives $\epsilon^{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6}(\epsilon_{JKJ_1J_2J_3}\epsilon_{ILJ_4J_5J_6} - \epsilon_{JKJ_4J_5J_6}\epsilon_{ILJ_1J_2J_3})$,

$$\begin{aligned}\frac{\delta L_2}{\delta\phi^I_i(x)} \quad\propto\quad & \frac{\beta(\phi_{ji}\cdot\phi_{ki})^3 + \gamma(\phi_{ji}\cdot\phi_{ki})|\phi_{ji}|^2|\phi_{ki}|^2}{\sin^8\theta|\phi_{ki}|^8|\phi_{ji}|^8}\phi^J_{ji}\phi^K_{ki}(\phi^L_{ki} - \phi^L_{ji}) \\ & \epsilon^{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6}(\epsilon_{JKJ_1J_2J_3}\epsilon_{ILJ_4J_5J_6} - \epsilon_{JKJ_4J_5J_6}\epsilon_{ILJ_1J_2J_3}) \\ & \partial_{\rho_1}\phi^{J_1}_{ji}\partial_{\rho_2}\phi^{J_2}_{ji}\partial_{\rho_3}\phi^{J_3}_{ji}\partial_{\rho_4}\phi^{J_4}_{ki}\partial_{\rho_5}\phi^{J_5}_{ki}\partial_{\rho_6}\phi^{J_6}_{ki}.\end{aligned} \tag{2.102}$$

β and γ are two integers that cannot be determined because of the ambiguities of the trace. In the above calculation, $\phi^I_{m\mu} = c_{ki\mu}\phi^I_{ij} - c_{ij\mu}\phi^I_{ki}$ is used. One can also plug $\phi^I_{m\mu} = c_{ij\mu}\phi^I_{jk} - c_{jk\mu}\phi^I_{ij}$ or $\phi^I_{m\mu} = c_{jk\mu}\phi^I_{ki} - c_{ki\mu}\phi^I_{jk}$ into (2.94) and the results should be equal. However, from (2.94) to (2.102), several terms are neglected, so we still need to add the terms from the other two expressions and then take the average. The final result is

$$\begin{aligned}
\frac{\delta L_2}{\delta\phi_i^I(x)} \propto\; & \epsilon^{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6}(\epsilon_{JKJ_1J_2J_3}\epsilon_{ILJ_4J_5J_6} - \epsilon_{JKJ_4J_5J_6}\epsilon_{ILJ_1J_2J_3}) \\
& [\frac{\beta(\phi_{ji}\cdot\phi_{ki})^3 + \gamma(\phi_{ji}\cdot\phi_{ki})|\phi_{ji}|^2|\phi_{ki}|^2}{3\sin^8\theta|\phi_{ki}|^8|\phi_{ji}|^8}\phi_{ji}^J\phi_{ki}^K(\phi_{ki}^L - \phi_{ji}^L) \\
& \partial_{\rho_1}\phi_{ji}^{J_1}\partial_{\rho_2}\phi_{ji}^{J_2}\partial_{\rho_3}\phi_{ji}^{J_3}\partial_{\rho_4}\phi_{ki}^{J_4}\partial_{\rho_5}\phi_{ki}^{J_5}\partial_{\rho_6}\phi_{ki}^{J_6} \\
& +\frac{\beta(\phi_{ij}\cdot\phi_{kj})^3 + \gamma(\phi_{ij}\cdot\phi_{kj})|\phi_{ij}|^2|\phi_{kj}|^2}{3\sin^8\rho|\phi_{kj}|^8|\phi_{ij}|^8}\phi_{kj}^J\phi_{ij}^K\phi_{kj}^L \\
& \partial_{\rho_1}\phi_{kj}^{J_1}\partial_{\rho_2}\phi_{kj}^{J_2}\partial_{\rho_3}\phi_{kj}^{J_3}\partial_{\rho_4}\phi_{ij}^{J_4}\partial_{\rho_5}\phi_{ij}^{J_5}\partial_{\rho_6}\phi_{ij}^{J_6} \\
& +\frac{\beta(\phi_{jk}\cdot\phi_{ik})^3 + \gamma(\phi_{jk}\cdot\phi_{ik})|\phi_{jk}|^2|\phi_{ik}|^2}{3\sin^8\sigma|\phi_{ik}|^8|\phi_{jk}|^8}\phi_{ik}^J\phi_{jk}^K\phi_{kj}^L \\
& \partial_{\rho_1}\phi_{ik}^{J_1}\partial_{\rho_2}\phi_{ik}^{J_2}\partial_{\rho_3}\phi_{ik}^{J_3}\partial_{\rho_4}\phi_{jk}^{J_4}\partial_{\rho_5}\phi_{jk}^{J_5}\partial_{\rho_6}\phi_{jk}^{J_6}],
\end{aligned} \tag{2.103}$$

where $(\phi_{ij}\cdot\phi_{kj}) = |\phi_{ij}||\phi_{kj}|\cos\rho$, $(\phi_{jk}\cdot\phi_{ik}) = |\phi_{jk}||\phi_{ik}|\cos\sigma$, $\sin\theta|\phi_{ji}||\phi_{ki}| = \sin\rho|\phi_{ij}|$ $|\phi_{kj}| = \sin\sigma|\phi_{ik}||\phi_{jk}| = 2s_{ijk}$. Obviously, (2.103) is not the expected 6-form in (2.91), the most serious problem is the appearance of $\sin^8$ other than $\sin^4$ in the denominator.

Recall that to derive $H_3 \wedge A_3$, $m \sim (i,j)$, while to get $A_3 \wedge F_4$, $m \sim (i,j,k)$. It is better if the two could be obtained in a unified way. If one replaces the σ_{3ji} in (2.31) and (2.32) by $h_{3ij} = h_{3i} - h_{3j}$,

$$\begin{aligned}
\Omega'_{ijk} &= -\frac{Q_1^3}{6}\int_{W_6} b_{2ji}\wedge\omega_{4ki} + b_{2ki}\wedge\omega_{4ji} + b_{2ij}\wedge\omega_{4kj} \\
& \quad +b_{2kj}\wedge\omega_{4ij} + b_{2ik}\wedge\omega_{4jk} + b_{2jk}\wedge\omega_{4ik} \\
&= -\frac{Q_1^3}{6}\int_{W_6} b_{2ki}\wedge\omega_{4ki} + b_{2ij}\wedge\omega_{4ij} + b_{2jk}\wedge\omega_{4jk} \\
&= -\frac{Q_1^3}{6}\int_{W_7} h_{3ki}\wedge\omega_{4ki} + h_{3ij}\wedge\omega_{4ij} + h_{3jk}\wedge\omega_{4jk}.
\end{aligned} \tag{2.104}$$

$$\Omega'_{ij} = -\frac{Q_1^3}{3}\int_{W_7} h_{3ij}\wedge\omega_{4ij}. \tag{2.105}$$

If $h_{3i} = H_{3i}/N$,

$$\sum_{i\neq j, j\neq k, k\neq i} \Omega'_{ijk} + \sum_{i\neq j} \Omega'_{ij} = -\frac{Q_1^3}{6} \sum_{i\neq j} \int_{W_7} H_{3ij} \wedge \omega_{4ij}, \tag{2.106}$$

where $H_{3ij} = H_{3i} - H_{3j}$ is the 3-form appearing in the low energy effective action. Correspondingly,

$$\frac{\delta \int_{W_7} \Omega'_{ijk}}{\delta \phi_i^I(x)} \propto (h_{3ik} \wedge F_{3ikI} + h_{3ij} \wedge F_{3ijI})(x) \tag{2.107}$$

If we expand $f_m^{\mu\nu}$ in a similar way as that in (2.95),

$$\begin{aligned} f_m^{\mu\nu} &= c_{i\lambda} h_{jk}^{\mu\nu\lambda} + c_{j\lambda} h_{ki}^{\mu\nu\lambda} + c_{k\lambda} h_{ij}^{\mu\nu\lambda} = c_{ij\lambda} h_{jk}^{\mu\nu\lambda} - c_{jk\lambda} h_{ij}^{\mu\nu\lambda} \\ &= c_{jk\lambda} h_{ki}^{\mu\nu\lambda} - c_{ki\lambda} h_{jk}^{\mu\nu\lambda} = c_{ki\lambda} h_{ij}^{\mu\nu\lambda} - c_{ij\lambda} h_{ki}^{\mu\nu\lambda}. \end{aligned} \tag{2.108}$$

Plug $f_m^{\mu\nu} = c_{ki\lambda} h_{ij}^{\mu\nu\lambda} - c_{ij\lambda} h_{ki}^{\mu\nu\lambda}$ and $\phi_{m\mu}^I = c_{ki\mu} \phi_{ij}^I - c_{ij\mu} \phi_{ki}^I$ into (2.78), one may get the undesired terms involving both ϕ_{ij}^I and ϕ_{ki}^I. Even if we neglect these terms, the extra $\sin^5\theta$ in the denominator is still a serious problem.

2.5 The representation of the 3-algebra

Until now, a definite representation of the 3-algebra Λ is still not given. Consider Λ associated with the $6d$ $(2,0)$ theory of the type A_{N-1}. The dimension of Λ is denoted by $d(N)$. Based on the counting of the degrees of freedom, one may expect $d(N) - N = (N^3 - N)/3$. A natural candidate is the cubic matrices introduced in [60], which is the extension of the Hermitian matrix. The elements are T_{ijk} satisfying

$$T_{ijk} = T_{jki} = T_{kij} = T_{jik}^* = T_{kji}^* = T_{ikj}^* \tag{2.109}$$

$i, j, k = 1 \cdots N$. The 3-product is defined as $(ABC)_{ijk} = A_{ijl}B_{ilk}C_{ljk}$. The 3-bracket is given by

$$[A, B, C] = ABC + BCA + CAB - BAC - CBA - ACB. \tag{2.110}$$

The gauge transformation is realized as $\delta X = i\alpha^{ab}[t_a, t_b, X]$. There is a natural metric in Λ:

$$\langle A|B\rangle = A_{ijk}B^*_{ijk} = A_{ijk}B_{jik}, \tag{2.111}$$

which is invariant under the gauge transformation, i.e. $\langle \delta A|B\rangle + \langle A|\delta B\rangle = 0$. The Cartan subalgebra is Λ_0,

$$\Lambda_0 = \{X|X_{ijk} = \delta_{ij}c_{jk} + \delta_{jk}c_{ki} + \delta_{ki}c_{ij}\}. \tag{2.112}$$

$\bar{\Lambda}$ is given by $\bar{\Lambda} = \{C|C \in \Lambda_0, C_{ijj} = c_j\}$. The fundamental identity does not hold in general. It is hoped that it will be satisfied for gauge transformations in the form of $[C, X, *]$, $\forall\, C \in \bar{\Lambda}$, $\forall\, X \in \Lambda$, but unfortunately, this is not the case.

Another natural 3-algebra is the Nambu-Poisson algebra [61], the elements of which generate the volume-preserving diffeomorphism transformations of the 3d manifold M. The Nambu bracket is defined as

$$[f, g, h] = \epsilon^{ijk}\partial_i f \partial_j g \partial_k h, \qquad i, j, k = 1, 2, 3. \tag{2.113}$$

For $M = T^3$, the basis of functions could be selected as $\left\{\chi^{\vec{n}} = e^{2\pi i n_a x^a}\right\}$, where x^a are the Cartesian coordinates on T^3 with the equivalence relations $x^a \sim x^a + k^a$, $k^a \in Z$.

$$[\chi^{\vec{l}}, \chi^{\vec{m}}, \chi^{\vec{n}}] = (2\pi i)^3 \epsilon^{abc} l_a m_b n_c \chi^{\vec{l}+\vec{m}+\vec{n}}. \tag{2.114}$$

Obviously, if $\vec{l} = \alpha\vec{m}$, $[\chi^{\vec{l}}, \chi^{\vec{m}}, *] = 0$. $\chi^{\alpha\vec{m}}$ generate a subspace $\bar{\Lambda}$, $\bar{\Lambda} = \{f(m_a x^a)\}$. f is only the function of $m_a x^a$.

$$[f(m_a x^a), g, h] = f'[m_1(\partial_2 g\partial_3 h - \partial_3 g\partial_2 h) + m_2(\partial_3 g\partial_1 h - \partial_1 g\partial_3 h) + m_3(\partial_1 g\partial_2 h - \partial_2 g\partial_1 h)]. \tag{2.115}$$

In special case like $m = (1, 0, 0)$, the Nambu bracket reduces to the Poisson bracket, which has the finite dimensional matrix realization with the gauge group effectively taken as $U(N) \times U(N) \times \cdots \times U(N)$, the direct product of N $U(N)$ groups. However, the problem is that the three indices (i, j, k) are not in equal footing, which is not reasonable. One can also let $m = (1, 1, 1)$, but in that case, it is difficult to get a finite realization.

It seems that even if the fundamental identity is relaxed to a weaker version, it is still hard to find a satisfactory finite dimensional Euclidian 3-algebra. Another problem deserving discussion is the Cartan subalgebra which is directly related with the Coulomb branch. For cubic matrices, $\forall\ A, B, C \in \Lambda_0$, $[A, B, C] = 0$. For Nambu algebra, $\forall\ \vec{l}$,

$$\Lambda_{\vec{l}} = \left\{\chi | \chi = \sum a_n \chi^{\vec{k}_n}, \vec{l} \cdot \vec{k}_n = 0\right\}, \tag{2.116}$$

$\forall\ f, g, h \in \Lambda_{\vec{l}}$, $[f, g, h] = 0$. Obviously, both $dim\Lambda_0$ and $dim\Lambda_{\vec{l}}$ have the N^2 scaling. It is likely that for 3-algebra Λ with $dim\Lambda \sim N^3$, the Cartan subalgebra will have the N^2 scaling dimension. On the other hand, the vacuum configurations of the N M5 branes are only characterized by N vectors in transverse space, so the dimension of the Coulomb branch can only be $N - 1$. Nevertheless, to get the Coulomb branch, we also need to mod out the gauge equivalent configurations. For Nambu algebra, $\forall\ \chi \in \Lambda_{\vec{l}}$, $\vec{l} \cdot \nabla\chi = 0$, so χ is the collection of the functions invariant under the translation along the $\vec{l}$ direction. Select an arbitrary vector $\vec{m}$, $\vec{l} \cdot \vec{m} = 0$. $\forall\ \chi \in \Lambda_{\vec{l}}$, one can always make a suitable volume-preserving diffeomorphism transformation, under which, $\chi \to \chi(m_a x^a)$. The gauge inequivalent configurations are then parameterized by 1d functions $\chi(m_a x^a)$ with the N scaling dimension. Actually, under the appropriate gauge transformation, any function $f(x_1, x_2, x_3)$ in Λ could be converted into the form of $f(x_1)$, because f is a scalar function. The gauge inequivalent class is characterized by $\{f(x_1)\}$. This is quite like the Hermitian matrices. One can always diagonalize a Hermitian matrix by a unitary transformation, so the gauge inequivalent class is parameterized by eigenvalues, which, in continuous limit, becomes the $1d$ function. We have seen that the dimension of the Coulomb branch has the N scaling, but there is still a problem. In Coulomb branch, fermions taking values in elements of the Cartan subalgebra will remain massless, so the massive fermions will have the $N^3 - N^2$ other than $N^3 - N$ scaling. In (2.95), if $i = j$, $\phi^I_{m\mu} = 0$, but $(\phi^I_{ij}, \phi^I_{jk}, \phi^I_{ki}) \longrightarrow (0, \phi^I_{ik}, \phi^I_{ki})$. $\phi^I_{m\mu}$ is something related with the area, while it is $(\phi^I_{ij}, \phi^I_{jk}, \phi^I_{ki})$ that is directly related with the string tension. For elements in Cartan algebra, the triangle is degenerate, but the distances do not vanish unless $i = j = k$. It is better if the fermion mass is directly related with $(\phi^I_{ij}, \phi^I_{jk}, \phi^I_{ki})$ other than $\phi^I_{m\mu}$.

Finally, notice that for cubic matrices, if $C^\mu \in \bar{\Lambda}$, $C^\mu_{ijj} = c^\mu_j$, $\bar{X}^I \in \Lambda_0$, $\bar{X}^I_{ijj} = \phi^I_i$, then $[C^\mu, \bar{X}^I, Y]_{ijk} = (\phi^I_{jk} c^\mu_i + \phi^I_{ki} c^\mu_j + \phi^I_{ij} c^\mu_k) Y_{ijk}$, $\forall\, Y \in \Lambda$. This is just the equation proposed in (2.95). Besides, in the previous discussion, C^μ is given by N vectors $\{c^\mu_1 \cdots c^\mu_N\}$, which is consistent with the cubic matrices and the Nambu algebra, for both of which, $dim\bar{\Lambda} \sim N$.

3. MOMENTUM MODES OF $M5$-BRANES IN A $2D$ SPACE*

We study $M5$ branes by considering the selfdual strings with the orientation covering a plane. With the internal oscillation frozen, each selfdual string gives a $5d$ SYM field. All selfdual strings together give a $6d$ field with 5 scalars, 3 gauge degrees of freedom and 8 fermionic degrees of freedom in adjoint representation of $U(N)$. Selfdual strings with the same orientation have the SYM-type interaction. For selfdual strings with the different orientations, which could also be taken as the unparallel momentum modes of the $6d$ field on that plane or the (p, q) (r, s) strings on $D3$ with $(p, q) \neq (r, s)$, the $[i, j] + [j, k] \to [i, k]$ relation is not valid, so the coupling cannot be written in terms of the standard $N \times N$ matrix multiplication. 3-string junction, which is the bound state of the unparallel $[i, j]$ $[j, k]$ selfdual strings, may play a role here.

The effective theory on $M5$ branes is special in that the basic excitations, the selfdual strings, are $1d$ objects other than the $0d$ objects, like those on D branes or $M2$ branes [1, 3]. If the selfdual strings can be closed and can shrink to point like the fundamental strings, then we will still have a theory with the semi-point-like excitations. For selfdual strings without the charge, this is indeed the case. In abelian theory, the quantization of the point-like $M2$ confined to $M5$ brane gives the $(2, 0)$ tensor multiplet [12, 10, 11]. Moreover, the basic excitations on $(2, 0)$ little string theory [62] living on N coincident type IIA $NS5$ branes are closed fundamental strings, which are also the closed selfdual strings coming from $M2$ wrapping the M theory circle intersecting $NS5$ along a closed curve. There is no evidence showing

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that for selfdual strings carrying charge, the situation is the same. On $D4$ branes, the $[i, j]$ monopole string can carry the $D0$ charge. The closed $[i, j]$ monopole string with the vanishing length carrying $D0$ will appear as the point-like instanton with charge $[i, j]$. However, the classical instanton solution with charge $[i, j]$ is associated with the $[i, j]$ monopole string extending along a straight line, while the point-like[1] 1/2 BPS instanton solutions are always chargeless. The closed monopole string with no $D0$ charge is the selfdual string with the winding number and the momentum both zero along x_5, which will give a $5d$ massless SYM field in adjoint representation of $U(N)$, in addition to the original $5d$ SYM field coming from the selfdual strings winding x_5 once. The SYM field like this is always massless even if the $D4$ branes are separated from each other, so it will dominate at the Coulomb branch. However, on $D4$, no such field exists. We do not get the clue for the existence of the closed charged selfdual strings. Actually, when the charged selfdual string becomes curved, different parts of it may exert force to each other, so it cannot vibrate freely and cannot be closed as the chargeless strings do.

Then we have to incorporate the $1d$ object in a $6d$ field theory. In this note, we will take the $[i, j]$ tensionless selfdual string extending along, for example, the x_5 direction, as the point-like $6d$ excitation, which is in the position eigenstate in 1234 space but in the $P_5 = 0$ momentum eigenstate in x_5. If so, selfdual strings extending along the same direction cannot give the complete Hilbert space for the 6d particle. To get the full Hilbert space, we need to consider selfdual strings with the orientations covering all directions in a plane. The superposition of the selfdual strings with orientations covering a plane can give the 6d point-like excitations localized in 12345 space, but it seems that somehow, the position representation is not the suitable one, since it is the $[i, j]$ selfdual strings other than the $[i, j]$ point-like excitations that naturally

[1] By point-like, we mean the instanton solution is localized in R^4, centered around a point.

exist. One plane is already enough to define a $6d$ field theory, so different planes may give the dual versions for the same $6d$ theory. This is quite similar with the $\mathcal{N}=4$ SYM theory, for which one (p,q) string defines a $4d$ theory, while the rest (p,q) strings give the dual $4d$ theories.

Theories with the line-like excitations are intrinsically different from those with the point-like excitations. If the excitations are line-like, a reduction on x_5 will give the selfdual strings extending along x_5, while a further reduction on x_4 will make $P_4=0$. The first reduction selects a particular selfdual string; the second one is just the normal reduction in local field theories. With 4 and 5 switched, we will get the selfdual strings extending along x_4 with $P_5=0$, which is S-dual to the selfdual strings extending along x_5 with $P_4=0$. On the other hand, if the excitations are point-like, both sequences will give the point-like selfdual strings with $P_4=P_5=0$. In [63, 64], Witten has shown that due to the conformal symmetry, the 45 and 54 reductions for the $6d$ $(2,0)$ theory will give two S-dual $4d$ SYM theories other than one $4d$ theory, which strongly indicates that the basic excitations on $M5$ cannot be point-like.

Recall that in [23], the equations of motion for the 3-algebra valued $(2,0)$ tensor multiplet contain a constant vector field C_μ. A given C_μ will reduce the dynamics from $6d$ to $5d$. However, if C_μ covers all directions in a plane, we will get a set of θ-parameterized $5d$ SYM theories[2], which is equivalent to a $6d$ theory with 5 scalars, 3 gauge degrees of freedom and 8 fermionic degrees of freedom. Each $5d$ SYM theory has the $5d$ vector multiplet in adjoint representation of $U(N)$, arising from the quantization of the open $[i,j]$ $M2$ intersecting $M5$ along the $C_\mu(\theta)$ direction. The oscillation along $C_\mu(\theta)$ is frozen, so the spectrum is the same as that from

[2]Here and in the following, by $5d$ SYM theory, we only mean the $6d$ $C_\mu(\theta)$-translation invariant fields have the SYM-type coupling. It's not the genuine $5d$ SYM theory. A single $5d$ SYM theory already contains the complete KK modes as the nonpertubative states [65, 58].

the quantization of the open string. For the given $6d$ $(2,0)$ tensor multiplet field configuration, each $5d$ SYM field comes from the reduction of the $6d$ field along $C_\mu(\theta)$. This is actually a special kind of KK compactification, using the polar coordinate other than the rectangular coordinate.

Suppose the selfdual string orientations are restricted in 45 plane, then the common eigenstates of $[\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{P}_4, \hat{P}_5]$ can be selected as the bases to generate the Hilbert space. In this respect, it is convenient to consider the KK mode of the $6d$ $(2,0)$ theory with x_4 and x_5 compactified to circles with the radii R_4 and R_5. $M5$ branes with the longitudinal x_4 and x_5 compactified is dual to the $D3$ branes with the transverse x'^{45} compactified. The vacuum expectation values of the 2-form field on $x_4 \times x_5$ is converted to the transverse positions of the $D3$ branes on x'^{45}. The duality differs from the T-duality in that two longitudinal dimensions are converted to one transverse dimension so that the total dimensions are reduced from 11 to 10. The $(n/R_4, m/R_5)$ momentum mode of the $6d$ theory is dual to the (p,q) string winding x'^{45} k times, with $n = kp$, $m = -kq$, p and q are co-prime. The $[i,j]$ (p,q) strings form the adjoint representation of $U(N)$. Correspondingly, the momentum modes as well as the original $6d$ field are also in the $U(N)$ adjoint representation.

The $(n/R_4, m/R_5)$ momentum modes are in $4d$ vector multiplet V_4, which, when combine together, give the $6d$ tensor multiplet T_6 in $U(N)$ adjoint representation. Although the $6d$ tensor multiplet is in the adjoint representation, the coupling involving more than two fields cannot be realized as the standard matrix multiplication, as the SYM-type couplings do. SYM coupling is obtained by studying the scattering amplitude of the open strings ending on D branes. $[i,j] + [j,k] \to [i,k]$, so the $L^i_j M^j_k N^k_i$ type coupling is possible. On the other hand, for $M5$, we need to consider the scattering of the selfdual strings parallel to a given plane. Selfdual strings with the same orientation still have the SYM-coupling, while for selfdual strings with different orientations, the $[i,j]+[j,k] \to [i,k]$ relation is not valid, so the $[i,j]$ $(n/R_4, m/R_5)$ mode, the $[j,k]$ $(k/R_4, l/R_5)$ mode, and the $[k,i]$ $(-(n+k)/R_4, -(m+l)/R_5)$ mode of the $6d$ field cannot couple unless $nl = mk$, in which case all of them belong to the same $5d$ SYM theory with $\tan\theta = -\frac{nR_5}{mR_4}$. The difference between the SYM theory and the effective theory on $M5$'s is rooted in the fact that the boundary of the open string is the point, while the boundary of the open $M2$ is the straight line.

The $(n/R_4, m/R_5)$ momentum mode of the $6d$ SYM theory on $D5$ is dual to the open $F1$ ending on $D3$ with the winding number (n,m) around $x'_4 \times x'_5$. From the scattering amplitude of the massive winding open strings and the massless open strings on $D3$, one can reconstruct the original $6d$ SYM theory. Similarly, for $M5$, we need to consider the interaction of the open (p,q) strings ending on $D3$ winding x'^{45} k times for all co-prime (p,q) and all nonnegative k, or in other words, the interaction for all of the monopoles and dyons in $\mathcal{N}=4$ SYM theory. When the scalar fields on $M5$ branes get the vacuum expectation value, $D3$ branes will be separated in the rest $5d$ transverse space, and then the 3-string junctions [45, 66], which are also the bound states of the $[i,j]$ $[j,k]$ selfdual strings each carrying the transverse momentum in 45 plane, can be formed. The 3-string junction is characterized by three vectors (r_4, r_5), (s_4, s_5) and (t_4, t_5) in 45 plane, which may couple with the $(n/R_4, m/R_5)$ momentum modes as long as $(n,m) \propto (r_4, -r_5)$, or $(n,m) \propto (s_4, -s_5)$ or $(n,m) \propto (t_4, -t_5)$. The quantization of the 3-string junction with the lowest spin content gives the 1/4 BPS multiplet $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ with 2^6 states [67], which, when lifted to $6d$, becomes the $(2,1)$ multiplet with 2^7 states, among which, half are tri-fundamental and half are tri-anti-fundamental representation of $U(N)$. $[i,l]+[l,j,k] \to [i,j,k]$, $[j,m]+[i,m,k] \to [i,j,k]$, $[k,n]+[i,j,n] \to [i,j,k]$, so we may have couplings like $V^i_l T_{ijk} T^{ljk}$ or $V^i_l T_{ijk} T^{ljn} V^k_n$. [3]

[3] In [68], the scattering amplitude involving two charged $5d$ KK modes and one $5d$ zero mode

However, with the given scalar vacuum expectation value $\vec{v}_i$ on $M5$, the bound states of the $[i,j]$ $\vec{v}_{ij}$ $(n/R_4, m/R_5)$ momentum mode and the $[j,k]$ $\vec{v}_{jk}$ $(k/R_4, l/R_5)$ momentum mode exist only under the certain condition $h(\vec{v}_{ij}, \vec{v}_{jk}, n/R_4, m/R_5, k/R_4, l/R_5) > 0$ with $h = 0$ specifying the marginal stability curve [69]. Especially, when $\vec{v}_{ij} = 0$, $\forall\, i,j$, except for $n = m = 0$ or $k = l = 0$, which is at the curve of the marginal stability, the rest bound states do not exist. The bound state of the $[i,j]$ $(0,0)$ momentum mode and the $[j,k]$ $(n/R_4, m/R_5)$ momentum mode could be taken as the tensionless selfdual string carrying the longitudinal momentum. It is unclear whether it is the bound state or just two separate states.

When compactified on x_5, the $[i,j]$ selfdual string extending along x_5 becomes the $[i,j]$ $F1$ localized in 1234 space. For the rest selfdual strings, to get the definite P_5 momentum, they must carry the definite transverse momentum thus are projected to the bound states of the $[i,j]$ $F1$ and the $[i,j]$ monopole string, each carrying the suitable P_4 and P_5 transverse momentum, localized in 123 space. The 1/2 BPS solutions for the BPS equations in $5d$ SYM theory match well with the above states, except for instantons, which is also localized in 1234 space. Similarly, the 3-string junctions in $6d$, when projected to $5d$, become the bound states of the $[i,j]$ $F1$ and the $[j,k]$ monopole string with P_4 and P_5 transverse momentum, localized in 123 space. Except for the dyonic instantons, the generic 1/4 BPS solutions in $5d$ SYM theory involving no more than three $D4$'s have a one-to-one correspondence with these states.

The rest of this chapter is organised as follows: In section 2, we discuss the longitudinal momentum mode on branes with special emphasis on the point-like

for $M5$ branes compactified on S^1 is considered. The charged KK mode is the 1/4 BPS dyonic instanton in $5d$ massive $(2,1)$ multiplet, while the zero mode is in $5d$ massless vector multiplet. The incorporation of the spin-3/2 particles of the $(2,1)$ multiplet into the theory requires a novel fermionic symmetry.

charged 1/2 BPS instantons on $D4$ branes. In section 3, we consider the selfdual strings on $M5$ branes with the orientation covering the 45 plane, or equivalently, the KK momentum mode of the $6d$ theory upon the compactification on $x_4 \times x_5$. In section 4, we study the interaction of the $6d$ $(2, 0)$ theory by considering its KK mode on $x_4 \times x_5$. In section 5, we consider various momentum-carrying BPS states in $5d$ SYM theory, especially, the monopole string carrying the longitudinal momentum and the string carrying $D0$ charge. In section 6, we discuss the states living at the triple intersection of $M5$ branes. The discussion is in section 7.

3.1 The longitudinal momentum mode on branes

For a Dp brane with the transverse dimension x_{p+1} compactified to a circle S^1, the brane may locate at a particular point in S^1 or have the definite momentum along S^1. In the former situation, after the T-duality transformation along x_{p+1}, D_p becomes D_{p+1} with the gauge field A_{p+1} getting the vacuum expectation value. In the latter case, the T-duality transformation converts D_p into D_{p+1} with the definite electric flux $F_{0(p+1)}$. D_{p+1} cannot have the definite A_{p+1} and $F_{0(p+1)}$ simultaneously, just as D_p cannot have the definite x_{p+1} and P_{p+1} at the same time. In M theory, if x_5 is compactified to S^1, $M2$ transverse to x_5 may either locate at a particular point in S^1 or have the definite P_5 momentum. $M2$ with zero P_5 momentum is $D2$ in type IIA string theory. $M2$ with nonzero P_5 momentum is the $D2$-$D0$ bound state. The transverse velocity should be the same everywhere on $D2$ so the $D0$ charges are uniformly distributed over $D2$. If the masses of the $D2$ and $D0$ are m_2 and m_0 respectively, the energy of the $D2$-$D0$ bound state is $\sqrt{m_2^2 + m_0^2}$, in contrast to the energy of the $D4$-$D0$ bound state, which is $m_4 + m_0$.

For Dp brane with the longitudinal dimension x_p compactified to a circle S^1, D_p may carry momentum along x_p. Under the T-duality transformation in p direction, we get the D_{p-1}-$F1$ bound state with $F1$ ending on D_{p-1} winding the transverse circle x'^p. If the i_{th} and the j_{th} D_{p-1} branes are separated along another transverse dimension x'^{p+1}, the closed $F1$ becomes open, which, under the T-duality transformation along x'^p, gives the open $[i,j]$ string ending on Dp branes carrying the P_p momentum. So, more precisely, the P_p longitudinal momentum of the D_p brane is the P_p transverse momentum of the open strings living on them. Consider the i_{th} and the j_{th} D_p branes with the $[i,j]$ string orthogonally connecting them carrying momentum P_p, the total energy is $m_p + \sqrt{T_{F1}^2|\vec{v}_i - \vec{v}_j|^2 + P_p^2}$, where m_p and $T_{F1}|\vec{v}_i - \vec{v}_j|$ are masses of D_p and $[i,j]$ string respectively. When $\vec{v}_i = \vec{v}_j$, the energy reduces to $m_p + |P_p|$, corresponding to the D_p branes carrying the $[i,j]$ P_p longitudinal momentum. The low energy effective action on N coincident Dp branes is the $U(N)$ SYM theory. When compactified on x_p, one may get an infinite tower of KK modes still in the adjoint representation of $U(N)$. We have seen that the P_p momentum carries charge, thus is indeed in the adjoint representation.

For $M2$ branes, two $M2$ branes orthogonally intersecting at a point may form the threshold bound state, so the transverse momentum of one $M2$ gives the longitudinal momentum of the other, as long as the two can keep intersecting at one point. Similarly, it is natural to expect that for $M5$ branes, the P_k longitudinal momentum may actually be the P_k transverse momentum of the selfdual strings living in them. Especially, for $5d$ SYM theory, the P_5 momentum may just come from the monopole strings living in $D4$. However, P_5 like this is distributed in a straight line other than localized at a point. Localized instantons do exist, which are $D0$ branes resolved in $D4$. The line-like P_5 momentum carried by selfdual strings has the $[i,j]$ charge, while the point-like P_5 momentum corresponding to $D0$ branes is chargeless. It is interesting to see the relation between these two kinds of momentum modes on $M5$, and especially, whether it is possible to obtain the point-like 1/2 BPS momentum mode carrying charge.

One may want to consider the closed selfdual strings, which, with the length shrinking to zero, may carry the point-like momentum. However, the selfdual strings carrying charge must extend along a straight line. We cannot get the closed charged selfdual strings unless the worldvolume of the $M5$ branes has the nontrivial 1-cycle. Consider the $[i, j]$ selfdual string segment extending along ABC, where A B C are three points in the worldvolume of $M5$. Suppose $AB \perp BC$, then the AB, BC strings are actually the same as the $[i, j]$ $F1$ and the $[i, j]$ $D1$. The configuration like this is not BPS, so $F1$ and $D1$ may exert force to each other. The $[i, j]$ $F1$-$D1$ bound state is not at the threshold and has the mass $|\vec{v}_{ij}|\sqrt{|AB|^2 + |BC|^2}$ due to the binding energy. So we are actually talking about the $[i, j]$ selfdual string segment extending along AC. The $[i, j]$ selfdual string cannot vibrate freely, because different parts may exert force to each other.

On the other hand, the chargeless selfdual strings do not have this problem. They can be closed and may carry the point-like momentum. One such example is the $[i, i]$ selfdual string. On the i_{th} $M5$ brane, we have the zero length $[i, i]$ closed selfdual string, or in other words, the collapsed $M2$ brane, the quantization of which gives the expected $U(1)$ $(2, 0)$ tensor multiplet [12, 10, 11]. When compactified on x_5, the point-like $D2$ with P_5 momentum becomes the $D0$ confined to the i_{th} $D4$. Another example is the little string theory [10, 11, 14, 8, 9]. Consider N coincident type IIA $NS5$ branes with the longitudinal dimension x_5 compactified to a circle with the radius R_5. The 11_{th} dimension is x_{10} which is compactified with the radius R_{10}. There are closed fundamental strings with tension $2\pi R_{10} T_{M2}$ living in $NS5$, which are closed $M2$'s wrapping x_{10} intersecting $M5$ along a closed curve. After a series of duality transformations, the momentum mode (carried by the type IIA string) along x_5 is converted to the $D0$ branes living in $D4$ branes with the compactified transverse dimension x'_5. If the original type IIA closed string has the finite length, we will get a closed $D2$ wrapping x'_5 intersecting $D4$ along a closed curve and carrying the uniformly distributed $D0$ charge. The $D0$ branes are obtained when the size of the $D2$ brane carrying them shrinks to zero. Actually, the type IIA $NS5$ brane picture and the $D4$ brane picture are S-dual to each other with 5 and 10 switched. On type IIA $NS5$ branes, the P_5 momentum is carried by the closed string. When the string shrinks to a point, we simply take it as a momentum mode without the string involved. Similarly, on $D4$ branes, we may have closed $D2$ carrying P_5 momentum. With the closed $D2$ shrinking to a point, we are left with the $D0$ brane/P_5 momentum.

On a single $NS5$ brane, purely P_5 momentum is carried by strings that do not wind x_5, or alternatively, $M2$'s that do not wrap x_5. Correspondingly, on a single $D4$ brane, the purely P_5 momentum mode should be carried by the closed $D2$ branes other than strings. The complete KK modes on $NS5$ branes upon the compactification on x_5 are characterized by (m, n), where m and n are the winding number and the momentum mode of the string along x_5 respectively. The (m, n) mode has the mass $2\pi m T_{F1} R_5 + n/R_5$. $(m, 0)$ mode, $(0, n)$ mode and (m, n) mode are in the $5d$ $(1, 1)$, $(2, 0)$ and $(2, 1)$ multiplets preserving 1/2, 1/2, and 1/4 supersymmetries respectively [70]. For the dyonic strings in [58], with x_6 compactified to a circle, the bound state of the $[1, 2]$ and $[2, 1]$ dyonic strings carrying n instanton number is just be the $(1, n)$ mode here. The generic (m, n) mode is obtained by the quantization of the type IIA strings. Especially, with $N_L = 0$, the level-matching condition requires $N_R = mn$, so the oscillation mode along the string must be turned on [70, 71]. This is easy to understand. For type IIA string wrapping x_5, P_5 can only come from the internal oscillation since there is no transverse momentum along x_5. On the other hand, if the $[i, j]$ selfdual strings wrapping x_5 cannot oscillate, the P_5 momentum carried by it can only come from selfdual strings extending in 1234 space. The (m, n) mode in this case is actually the threshold bound state of the $(m, 0)$ mode and the $(0, n)$ mode. The former is associated with the selfdual string extending along x_5, while the latter is given by the tensionless selfdual strings extending along 1234 space carrying the P_5 momentum. It is possible for the $[i, j]$ $(m, 0)$ mode and the $[j, k]$ $(0, n)$ mode to form the threshold bound state with ijk indices, which we will discuss later.

Now, let us consider the relation between the point-like charged P_5 momentum mode and the line-like charged P_5 momentum mode. For N coincident $D4$ branes with the longitudinal dimension x_4 compactified, the T-duality transformation along x_4 converts the $D4$-$D0$ bound state into the $D3$-$D1$ bound state with $D1$ winding x'^4. More precisely, $D1$ carrying the definite electric flux, or equivalently, $D1$-$F1$ bound state, corresponds to $D0$ carrying the definite P_4 momentum, while $D1$ with the definite A_4 field corresponds to $D0$ with the exact x_4 position. We may take the $(n, 1)$ strings in $D3$ as the bases, the superposition of which gives $D1$ with the definite A_4, which is also the $D0$ located at a definite point in $D4$. Moreover, since $D1$ ending on $D3$'s can also carry charge, the $[i, j]$ $D1$ wrapping x'^4 is dual to the $[i, j]$ $D2$ wrapping x_4 with the $D0$ charge spreading over $\vec{v}_{ij} \times x_4$. When $\vec{v}_{ij} = 0$, we are left with the tensionless $[i, j]$ monopole string winding x_4 carrying the uniformly distributed $D0$ charge, which could also be taken as the $[i, j]$ $D0$ with the zero momentum along x_4. Similarly, the $[i, j]$ $(n, 1)$ string is dual to the tensionless $[i, j]$ monopole string winding x_4 carrying the uniformly distributed $D0$ charge and the massless $[i, j]$ string carrying P_4 momentum, which could be simply taken as the $[i, j]$ $D0$ carrying P_4. Still, the superposition of the $[i, j]$ $(n, 1)$ strings gives the $[i, j]$ $D0$ located at a definite point in $D4$. The instanton solutions describing the $[i, j]$ $D0$ with the definite P_4 momentum have the translation invariance along x_4, involving both magnetic and the electric fields. These states compose the complete spectrum for the charged $D0$ living in $D4$, while the localized charged $D0$ is the superposition of them.

The above discussion can be directly extended to $M5$ branes. Consider $M5$ branes with x_4 and x_5 compactified with the radii R_4 and R_5, $B_{45} = \frac{k}{2\pi R_4 R_5}$. Tensionless selfdual string winding x_4 and x_5 m and n times will carry the transverse momentum, which can be described by the wave function

$$\frac{1}{4\pi^2 R_4 R_5} \exp\left\{ik(-\frac{nx_4}{R_4} + \frac{mx_5}{R_5})\right\} \delta(x_1 - X_1)\delta(x_2 - X_2)\delta(x_3 - X_3). \tag{3.1}$$

When $k = 1$, the P_5 momentum localized in x_4 has the wave function

$$\frac{1}{2\pi R_5} \exp\left\{\frac{imx_5}{R_5}\right\} \delta(x_1 - X_1)\delta(x_2 - X_2)\delta(x_3 - X_3)\delta(x_4 - X_4), \tag{3.2}$$

which is the superposition of (3.1) with all $n \in \mathbf{Z}$. In this respect, at least for $M5$ with at least two dimensions compactified, the longitudinal momentum is still given by the basic excitations, which are the selfdual strings here. The only difference is that the selfdual string is the one dimensional object, so the transverse momentum carried by it will appear as the two dimensional wave other than the one dimensional wave. The momentum mode carried by the particles is the one dimensional wave. To get the complete spectrum, we need the particles with the location covering x_4, or the selfdual strings with the orientation covering the 45 plane. $\{\delta(x_4 - X_4)|X_4 \in [0, 2\pi R_4)\}$ and $\left\{\frac{1}{2\pi R_4} e^{\frac{inx_4}{R_4}} |n \in \mathbf{Z}\right\}$ are different bases for the same Hilbert space.

For $M5$ branes with x_3 x_4 x_5 compactified to circles with the radii R_3 R_4 R_5, if $B_{35} = \frac{1}{2\pi R_3 R_5}$, $B_{45} = \frac{1}{2\pi R_4 R_5}$, the tensionless selfdual string localized in x_4 winding

R_3 R_5 n l times could be described by the wave function

$$\frac{1}{4\pi^2 R_3 R_5} \exp\left\{ i(-\frac{l x_3}{R_3} + \frac{n x_5}{R_5}) \right\} \delta(x_1 - X_1)\delta(x_2 - X_2)\delta(x_4 - X_4), \qquad (3.3)$$

while the tensionless selfdual string localized in x_3 winding R_4 R_5 n m times could be described by the wave function

$$\frac{1}{4\pi^2 R_4 R_5} \exp\left\{ i(-\frac{m x_4}{R_4} + \frac{n x_5}{R_5}) \right\} \delta(x_1 - X_1)\delta(x_2 - X_2)\delta(x_3 - X_3). \qquad (3.4)$$

Now there are two sets of bases for the P_5 momentum mode. One may wonder whether the superposition of either of them can give the same $\exp\left\{\frac{i n x_5}{R_5}\right\}\delta(x_1 - X_1)\delta(x_2 - X_2)\delta(x_3 - X_3)\delta(x_4 - X_4)$.

This is indeed the case. Consider the coincident $D4$ branes with the longitudinal x_3 and x_4 compactified to circles. Do the T-duality transformations along x_3 and x_4 successively. The $[i, j]$ monopole string wrapping x_3, localized in x_4, carrying one unit of $D0$ charge and the $[i, j]$ longitudinal momentum P_3 becomes $D2$ wrapping x_3' and x_4', intersecting the i_{th} and the j_{th} $D2$ branes at one point, carrying the definite F_{03} and A_4. The $[i, j]$ monopole string wrapping x_4, localized in x_3, carrying one unit of $D0$ charge and the $[i, j]$ longitudinal momentum P_4 becomes $D2$ carrying the definite F_{04} and A_3. The superposition of $D2$ states with either the definite A_4 and all possible F_{03} or the definite A_3 and all possible F_{04} will give the $D2$ state with the definite A_3 and A_4, which is dual to the $D0$ localized in x_3 and x_4. So, the $[i, j]$ tensionless selfdual strings wrapping x_4 carrying the given P_5 momentum and all possible P_4 momentum compose the bases which is equivalent to the $[i, j]$ tensionless selfdual strings wrapping x_3 carrying the same P_5 momentum and all possible P_3 momentum. The superposition of either of them may give the $[i, j]$ $D0$ located at a definite point in $x_3 \times x_4$. Notice that for this equivalence to be valid, $\vec{v}_{ij} = 0$ is necessary, otherwise the T-dual $[i, j]$ $D2$ brane will be a tube wrapping either x_3 or x_4 depending on which circle the selfdual string wraps. So, for coincident $M5/D4$ branes, to get the complete spectrum for P_5 momentum, we only need to select an arbitrary selfdual string extending in 1234 space, carrying the given P_5 momentum and all possible longitudinal momentum. The generated Hilbert space is the same no matter which selfdual string is selected.

On $D4$ branes, the $[i, j]$ P_5 momentum can only be carried by the $[i, j]$ selfdual strings. There is no classical solution for the point-like instanton with the charge $[i, j]$. However, we do have the solution for the chargeless point-like instantons, which may consist of N instanton partons with charge $[1, 2]$, $\cdots$, $[N-1, N]$, $[N, 1]$, while the size ρ is the parameter characterizing the distance between the instanton partons [72, 73]. Similarly, for type IIA $NS5$ branes with x_5 compactified, the P_5 momentum is carried by the point-like closed strings, which are also composed by the $[1, 2]$, $\cdots$, $[N-1, N]$, $[N, 1]$ closed selfdual strings from M theory's point of view. It is difficult to get a single closed selfdual string with charge $[i, j]$. However, if one longitudinal dimension of $D4$ is compactified, an instanton on $D4$ branes will be dual to a D-string on $D3$ branes winding the transverse circle one time. A closed D-string is composed by the $[1, 2]$, $\cdots$, $[N-1, N]$, $[N, 1]$ D-string segments [72, 73]. The $[i, j]$ D-string segment can exist independently, because it is the $[i, j]$ monopole string extending along the compactified longitudinal dimension carrying the transverse P_5 momentum. Similarly, for the P_4 longitudinal momentum mode on $D4$ branes, we have the $[i, j]$ P_4 mode carried by the $[i, j]$ open string, which is the $[i, j]$ selfdual string winding x_5 one time, carrying the transverse P_4 momentum, and so can also exist separately. The $[1, 2]$, $\cdots$, $[N-1, N]$, $[N, 1]$ P_4 modes can combine together to give a chargeless P_4 mode as well, but it is not so necessary here.

For N coincident $M5$ branes with the compactified x_5, the $6d$ $(2,0)$ tensor multiplet could be decomposed into the zero mode and an infinite tower of KK modes. The zero mode is the $5d$ vector multiplet in adjoint representation of $U(N)$. The KK modes are in the $5d$ $(2,0)$ tensor multiplet. The $U(1)^N$ part of the tensor multiplet is easy to construct, which is related with the $D0$ brane (point-like $M2$ carrying P_5) confined to each $M5$. There is no similar point-like $[i,j]$ P_5 mode (the $[i,j]$ $D0$ brane), which is supposed to give the nonabelian part of the $5d$ tensor multiplet. We do have the line-like $[i,j]$ P_5 mode, which is in the vector other than the tensor multiplet. This is expected. The line-like P_5 mode extending along x_4 could be taken as the point-like P_5 mode with $P_4 = 0$. So we actually truncate the field content of the $5d$ $(2,0)$ multiplet to its zero mode along x_4, which is in the $4d$ vector multiplet. To recover the $5d$ $(2,0)$ tensor multiplet, all momentum along x_4 should be included, and so we should consider all tensionless strings extending in 45 plane, carrying the transverse momentum (P_4, P_5), with the same P_5 but all P_4. In this way, four moduli of the point-like P_5 mode is recovered, with one position moduli replaced by the momentum moduli. $5d$ massive tensor multiplet is decomposed into the sum of the $4d$ KK modes in vector multiplet, forming the $U(N)$ adjoint representation.

In the language of the $4d$ SYM theory, the P_5 momentum carried by the $[i,j]$ tensionless selfdual string is actually the $[i,j]$ $(0,1)$ string with mass P_5, whose moduli space is $\mathbf{R}^3 \times S^1$, the same as the center of mass part of the moduli space of the instanton in $\mathbf{R}^3 \times S^1$ [74]. The momentum along S^1 is the electric charge. The $(0,1)$ string carrying n momentum mode along S^1 is the $(n,1)$ string. In $S^1 \to \infty$ limit, the quantization of the instanton in $\mathbf{R}^4$ gives the $5d$ massive $(2,0)$ multiplet. Correspondingly, the quantization of the $(n,1)$ strings in $\mathbf{R}^3$ for all $n \in Z$ should give the same multiplet.

Consider $B_{\mu\nu}$ in $5d$ massive $(2,0)$ multiplet with mass $1/R_5$. $B_{\mu\nu}$ satisfies the selfduality condition

$$B_{\mu\nu} = -\frac{iR_5}{2}\epsilon_{\mu\nu\lambda\rho\sigma}\partial^\lambda B^{\rho\sigma} \tag{3.5}$$

and the equation of motion

$$\partial^\lambda\partial_\lambda B_{\mu\nu} + \frac{1}{R_5^2}B_{\mu\nu} = 0, \tag{3.6}$$

where $\mu, \nu, \lambda, \rho, \sigma = 0, 1, 2, 3, 4$ [58]. Do a further compactification on x_4,

$$B_{\mu\nu} = \sum_k e^{ikx_4/R_4} B^{(k)}_{\mu\nu}. \tag{3.7}$$

Due to (3.5), for $i, j = 0, 1, 2, 3$ and $k \in Z$, $B^{(k)}_{ij}$ could be expressed in terms of $B^{(k)}_{i4}$ thus could be dropped. We are left with a tower of the $4d$ massive vector field $A^{(k)}_i = B^{(k)}_{i4}$ satisfying the constraint

$$\partial^i A^{(k)}_i = 0 \tag{3.8}$$

as well as the equation of motion

$$\partial_j\partial^j A^{(k)}_i + (\frac{1}{R_5^2} + \frac{k^2}{R_4^2})A^{(k)}_i = 0. \tag{3.9}$$

Each $A^{(k)}_i$ carries 3 degrees of freedom, the same as $B^{(k)}_{ij}$. In 45 plane, the $(n, 1)$ string carries the momentum $(n/R_4, 1/R_5)$ thus gives the $4d$ vector multiplet $A^{(n)}_i$. All of the $A^{(n)}_i$ are on the equal footing, which is consistent with the S-duality. To account for the P_5 momentum m/R_5 with $m > 1$, we need (n, m) strings, so altogether, all (m, n) strings should be included to give the complete $6d$ dynamics. Under the compactification on x_4 and x_5, the $6d$ field $B_{\alpha\beta}$ with $\alpha, \beta = 0, \cdots, 5$ is decomposed into the $4d$ KK modes $(n/R_4, m/R_5)$ corresponding to the (m,n) string. Each KK mode gives a $4d$ massive vector field $A_i^{(n,m)}$, for which, the constraint and the equation of motion could be obtained by replacing R_5 and k in (3.9) by R_5/m and n.

Extending the discussion to the nonabelian case is a little difficult, since we don't know the equations for the nonabelian tensor field. However, we do know that the $(n/R_4, 0)$ mode, which is the KK mode of the $5d$ massless SYM field, is in $4d$ adjoint massive vector multiplet. The rest $(n/R_4, m/R_5)$ modes are related with $(n/R_4, 0)$ via the S-duality, so they should also form the $4d$ adjoint massive vector multiplet. The whole KK tower of the $4d$ vector multiplet together may give the $6d$ $(2,0)$ tensor multiplet in adjoint representation of $U(N)$.

3.2 Selfdual strings with the orientation covering a plane and the $M5$-$D3$ duality

In previous discussion, we have seen that a tensionless selfdual string with the transverse position (X_1, X_2, X_3, X_4) can be taken as a $6d$ massless particle with the wave function $\delta(x_1-X_1)\delta(x_2-X_2)\delta(x_3-X_3)\delta(x_4-X_4)$. In x_5 direction, it is a plane wave with the zero momentum. Another selfdual string with the transverse position (X_1, X_2, X_3, X_5) gives the wave function $\delta(x_1-X_1)\delta(x_2-X_2)\delta(x_3-X_3)\delta(x_5-X_5)$. In 45 plane, the complete spectrum is $\left\{e^{i(P_4x_4+P_5x_5)} | \forall\, P_4, P_5\right\}$, so selfdual strings extending along all possible directions in 45 plane should be included to give the complete spectrum for a single $6d$ particle. The common eigenstates of $\hat{X}_1$ $\hat{X}_2$ $\hat{X}_3$ $\hat{P}_4$ $\hat{P}_5$,

$$\Lambda = \left\{\delta(x_1-X_1)\delta(x_2-X_2)\delta(x_3-X_3)e^{iP_4x_4}e^{iP_5x_5} | \forall\, X_1, X_2, X_3, P_4, P_5\right\} \qquad (3.10)$$

may be the suitable bases, the superposition of which can give a $6d$ particle localized in $(X_1, X_2, X_3, X_4, X_5)$. Although the position eigenstates can also be obtained, the basic excitations are $[i,j]$ selfdual strings other than the $[i,j]$ particles. Since it is (3.10) other than the position eigenstates that is naturally realized, the KK modes in this theory may tell us more than the KK modes in theories with point-like excitations.

Until now, our discussion is only restricted to coincident $M5$ branes, for which, the (P_4, P_5) momentum, even with the charge $[i, j]$, always takes the value $(n/R_4, m/R_5)$. The situation will be different if $\vec{v}_{ij} \neq 0$, so there must be something carrying the momentum. The natural candidate carrying the (P_4, P_5) momentum is the selfdual string extending along the 45 plane. Consider the $[i, j]$ selfdual strings with the length and the orientation characterized by the vector (qR_4, pR_5) in 45 plane. p and q are co-prime, so the selfdual string only winds $x_4 \times x_5$ once. In 123 space, the string is localized at a point. The Wilson surface in $x_4 \times x_5$ is trivial. Nevertheless, each $[i, j]$ string can still effectively pick up the background 2-form field $B_{45} = k/(2\pi R_4 R_5)$, $\forall\, k \in \mathbf{N}$. $\forall\, m, n \in \mathbf{Z}$, $\exists\, k, p, q$, $m = kq$, $n = kp$. In 45 plane, the $[i, j]$ string will get the definite transverse momentum $(n/R_4, -m/R_5)$, thus could be taken as the plane wave $e^{i(\frac{nx_4}{R_4} - \frac{mx_5}{R_5})}$. If the momentum in 123 space is (P_1, P_2, P_3), the energy will be

$$E = \sqrt{P_1^2 + P_2^2 + P_3^2 + \frac{n^2}{R_4^2} + \frac{m^2}{R_5^2}}, \tag{3.11}$$

which is the energy of a $6d$ massless particle. When $\vec{v}_{ij} \neq 0$, the $[i, j]$ selfdual string will have the rest mass $2\pi|\vec{v}_{ij}|\sqrt{q^2 R_4^2 + p^2 R_5^2}$, so the energy should be modified to

$$E = \sqrt{P_1^2 + P_2^2 + P_3^2 + \frac{n^2}{R_4^2} + \frac{m^2}{R_5^2} + 4\pi^2|\vec{v}_{ij}|^2(q^2 R_4^2 + p^2 R_5^2)}, \tag{3.12}$$

which is the energy of a $6d$ massive particle.

Notice that there is an ambiguity for the mass of the zero mode in $4d$. With $k = 0$, any (qR_4, pR_5) string can be the $4d$ zero mode with mass $2\pi|\vec{v}_{ij}|\sqrt{q^2 R_4^2 + p^2 R_5^2}$.

However, the zero mode is unique. In the dual $D3$ picture, there are unwrapped $[i, j]$ (p, q) strings with the length $|\vec{v}_{ij}|$. One may choose one possible $(qR_4, pR_5)/(p, q)$ as the zero mode. A particular S-frame is selected in this way, while in other S-frames, all (qR_4, pR_5) can get the chance to act as the zero mode. For the point-like excitations, the $5d$ momentum can uniquely fix the state; however, for the line-like excitations, with the given $5d$ momentum, the selfdual string orientations can still vary in a $4d$ space orthogonal to the momentum. If the selfdual string orientations are restricted to a plane, a one-to-one correspondence between the momentum and state may be realized except for the momentums orthogonal to that plane. So, the fixing of the S-frame is necessary. As is shown in later discussions, the $M5$-$D3$ duality also intrinsically involves the selection of the S-frame.

For $M5$ compactified on x_5, the P_5 zero mode should be the state with the zero momentum along x_5, localized in 1234 space. Selfdual string extending along x_5 is the only one meeting the requirement, and so, in Coulomb branch, the mass of the W-bosons is given by $2\pi|\vec{v}_{ij}|R_5$ without the ambiguity. Notice that there is a distinction between the little string theory and the theory on coincident $M5$ branes. For little string theory with x_5 compactified, there are momentum mode and the winding mode. The momentum mode is carried by the closed string. Although the string has the finite tension, the mass of the momentum mode is still m/R_5, because the string without winding x_5 can shrink to point thus has the zero mass and has no contribution to the energy. On the other hand, for $6d$ $(2, 0)$ theory with x_5 compactified, the zero mode is the $5d$ SYM field. In Coulomb branch, the mass of the $[i, j]$ zero mode is $2\pi|\vec{v}_{ij}|R_5$ other than 0, since the zero mode is actually the selfdual string winding x_5 once. There is no way to get rid of the lowest winding mode, because we do not have the closed charged selfdual string, while the straight selfdual string localized in 1234 space must extend along x_5.

The direct study of the $6d$ $(2,0)$ theory is difficult. The x_5 compactification will give the $5d$ massive tensor multiplet, which is also not quite accessible. The $4d$ KK modes upon the compactification on $x_4 \times x_5$ are relatively easy to study. Moreover, the previous discussion indicates that (3.10) might be the suitable bases to consider the $6d$ theory, so in the following, we will focus on the $4d$ KK modes arising from the $6d$ theory.

Actually, $M5$ with x_4 and x_5 compactified in $11d$ spacetime is dual to $D3$ with one transverse dimension compactified in $10d$ spacetime. Consider N coincident type IIB $NS5$ branes with x_4 and x_5 compactified to circles with the radii R_4 and R_5. The vacuum expectation values of the gauge fields along x_4 and x_5 are A_{4i} and A_{5i} respectively. The T-duality transformation along x_5 gives N coincident type IIA $NS5$ branes with x_4' and x_5' compactified to circles with the radii R_4 and R_5'. $R_5' = 1/(4\pi^2 T_{M2} R_{10} R_5)$, where R_{10} the radius of x_{10}', the M theory dimension. The positions of the type IIA $NS5$ branes on x_{10}' are $A_{5i}/(2\pi T_{M2} R_5')$. The 2-form field B on type IIA $NS5$ branes also gets the vacuum expectation value $B_{45i} = A_{4i}/(2\pi R_5')$.[4] Under the S-duality transformation, type IIB $NS5$ branes become the $D5$ branes, which, under the T-duality transformation along x_5, turn into the $D4$ branes with the transverse dimension x_5'' compactified to a circle with the radius R_5''. $R_5'' = 1/(4\pi^2 T_{M2} R_{10}'' R_5)$, where R_{10}'' is the radius of the new M theory circle. The positions of the $D4$ branes on x_5'' are $A_{5i}/(2\pi T_{M2} R_{10}'')$. Another T-duality transformation along x_4'' takes $D4$ into $D3$ branes with the transverse dimension x_4''' and x_5''' compactified to circles with the radii $1/(4\pi^2 T_{M2} R_{10}'' R_4)$ and $1/(4\pi^2 T_{M2} R_{10}'' R_5)$. The positions of

[4] If the vacuum expectation values $A_{\mu i}$ for $\mu = 1,2,3$ on type IIB $NS5$ branes are also turned on, on type IIA $NS5$ branes, there will be $B_{\mu 5i} = A_{\mu i}/(2\pi R_5')$. So, among the five gauge fields on type IIB $NS5$ branes, one, for example, A_{5i}, is converted to the position along the M theory circle, while the rest four turn into $B_{\mu 5i}$. The four scalar fields on type IIB $NS5$ branes become four scalar fields on type IIA $NS5$ branes. The $6d$ $(1,1)$ theory has $4+4$ bosonic degrees of freedom, while the $6d$ $(2,0)$ theory has $3+5$ bosonic degrees of freedom. The T-duality transformation converts one gauge field into the scalar field.

the $D3$ branes on x_4''' and x_5''' are $[A_{4i}/(2\pi T_{M2}R''_{10}), A_{5i}/(2\pi T_{M2}R''_{10})]$.

Until now, we only used the T-duality and S-duality for type IIA and type IIB string theories with no M theory involved. Let us use the new symbols to give a summary. Consider N coincident type IIA $NS5$ branes with x_4, x_5 and the M theory dimension x_{10} compactified to circles with the radii R_4, R_5 and R_{10}. Suppose the positions of the $NS5$ branes on x_{10} are X_i^{10}, the vacuum expectation values of the 2-form field on $x_4 \times x_5$ are B_{45i}. After a series of duality transformations, we get N $D3$ branes with the transverse dimension x'_{10} and x'_{45} compactified to circles with the radii $R'_{10} = R_{10}R_5/\tilde{R}_{10}$ and $R'_{45} = 1/(4\pi^2 T_{M2}R_4\tilde{R}_{10})$. The positions of the $D3$ branes on $x'_{10} \times x'_{45}$ are $[X_i^{10}R_5/\tilde{R}_{10}, B_{45i}R_5/(T_{M2}\tilde{R}_{10})]$. $T_{F1} = 2\pi T_{M2}\tilde{R}_{10}$, $T_{D1} = 2\pi T_{M2}\tilde{R}_{10}R_4/R_5$. The duality transformations also take the $NS5$ branes into N $D4$ branes with the transverse dimension x''_5 compactified to circle with the radius $R''_5 = R_{10}R_5/\tilde{R}_{10}$. The positions of the $D4$ branes on x''_5 are $X_i^{10}R_5/\tilde{R}_{10}$. The $NS5_A - D4$ and $NS5_A - D3$ duality shows that the both sides have the same degrees of freedom. Especially, the four vector fields on $D4$ as well as the three vector fields and one scalar field on $D3$ are dual to the four 2-form fields $B_{\mu 5}$ on $NS5$. The rest 2-form fields on $NS5$ have no counterpart thus could be dropped. This is consistent with the self-duality condition on $NS5$. Especially, for $NS5$ compactified on $x_4 \times x_5$,

$$B_{\mu\nu}(x_4, x_5, \vec{x}) = \frac{1}{2\pi\sqrt{R_4R_5}} \sum_{n,m} e^{i(nx_4/R_4 + mx_5/R_5)} B_{\mu\nu}^{(n,m)}(\vec{x}). \tag{3.13}$$

The zero mode has no winding number around x'^{45}. $B_{45}^{(0,0)}(\vec{x}) \to X^{45(0,0)}(\vec{x})$, $B_{i5}^{(0,0)}(\vec{x}) \to A_i^{(0,0)}(\vec{x})$, where $i = 1, 2, 3$. The rest $B_{\mu\nu}^{(0,0)}$ could be neglected. The bosonic degrees of freedom are $6 + 2 = 8$. The higher mode has the nonzero x'^{45} winding number, so there is no $X^{45(n,m)}$ for $m, n \neq 0$. $B_{i5}^{(n,m)}(\vec{x}) \to A_i^{(n,m)}(\vec{x})$, where $i = 1, 2, 3$. $B_{45}^{(n,m)}(\vec{x})$ together with $A_i^{(n,m)}(\vec{x})$ gives $4 - 1 = 3$ gauge degrees of freedom, so the total bosonic degrees of freedom are still $5+3=8$.

In $R_{10} \to \infty$ limit, $NS5$ branes become $M5$ branes, while on $D3$ and $D4$ branes side, x'_{10} and x''_5 are decompactified. So, $M5$ branes with x_4 and x_5 compactified is dual to the $D3$ branes with the transverse x'^{45} compactified; $M5$ branes with x_5 compactified is dual to $D4$ branes. For T-duality, D_p with the longitudinal x_p compactified is dual to D_{p-1} with the transverse x'^p compactified, with A_p converted to X^p, the momentum mode along x_p transformed to the winding mode along x'^p. For $M5$, we have $B_{\mu\nu}$ instead of A_μ, so two longitudinal dimensions x_4 x_5 are transformed to one transverse dimension x'^{45}, while the (P_4, P_5) momentum modes become the winding modes of the (p,q) strings along x'^{45} for all co-prime p q.

For $D3$, $R'_{45} = 1/(2\pi R_4 T_{F1}) = 1/(2\pi R_5 T_{D1})$. The five transverse dimensions of $M5$, x^I, are dual to the rest five transverse dimensions of $D3$, x'^I. $I = 6 \cdots 10$. If the B_{45} on $M5$ branes gets the vacuum expectation value B_{45i}, $D3$ branes will be separated along x'^{45} with the transverse positions

$$X_i'^{45} = 2\pi R_5 B_{45i}/T_{F1} = 2\pi R_4 B_{45i}/T_{D1}. \tag{3.14}$$

If the scalar fields on $M5$ branes get the vacuum expectation value Φ_i^I, $D3$ branes will be separated along x'^I with the transverse positions

$$X_i'^I = 2\pi R_5 \Phi_i^I/T_{F1} = 2\pi R_4 \Phi_i^I/T_{D1}. \tag{3.15}$$

If the $B_{\mu 5}$ for $\mu = 1,2,3$ on $M5$ branes gets the vacuum expectation value $B_{\mu 5i}$, the gauge field $A_{\mu i}$ on $D3$ branes will get the vacuum expectation value

$$A_{\mu i} = 2\pi R_5 B_{\mu 5i}. \tag{3.16}$$

(3.16) indicates that the $(0,0)$ mode of $M5$ is dual to the $(1,0)$ string on $D3$ with the winding number 0. A particular S-frame is selected.

For $D4$, if the five scalar fields on $M5$ branes get the vacuum expectation value Φ_i^I, $D4$ branes will be separated along x''^I with the transverse positions

$$X_i''^I = R_5\Phi_i^I/(T_{M2}\tilde{R}_{10}). \tag{3.17}$$

If the $B_{\mu 5}$ for $\mu = 1,2,3,4$ on $M5$ branes gets the vacuum expectation value $B_{\mu 5i}$, the gauge field $A_{\mu i}$ on $D4$ branes will get the vacuum expectation value

$$A_{\mu i} = 2\pi R_5 B_{\mu 5i}. \tag{3.18}$$

There's no other Bosonic fields on $D4$ branes, so the rest 2-form fields on $M5$ branes can be neglected. $D4$ is equivalent to $M5$ compactified on x_5 other than $M5$ reduced along x_5.

Let us concentrate on the $M5 - D3$ duality. The $[i,j]$ (qR_4, pR_5) selfdual string on $M5$ is dual to the $[i,j]$ (p,q) string on $D3$. If x_4 and x_5 are compact, x'_{45} will also be compact, so even if $B_{45} = 0$, the covering space of x'_{45} will still have N coincident $D3$ branes distributed with the period $2\pi R'_{45}$. $\forall\, i,j$, we have the $[i,j]$ (p,q) string connecting the i_{th} and the j_{th} $D3$ branes with the length $2\pi k R'_{45}$, corresponding to the $[i,j]$ (qR_4, pR_5) selfdual string coupling with the 2-form field $B_{45} = k/(2\pi R_4 R_5)$, getting the momentum $(kp/R_4, -kq/R_5)$. If the other transverse fields on $D3$ also get the vacuum expectation value, the mass of the $[i,j]$ (p,q) string will be

$$M = 2\pi\sqrt{q^2R_4^2 + p^2R_5^2}\sqrt{(\frac{k}{2\pi R_4 R_5})^2 + |\vec{\Phi}_{ij}|^2}, \tag{3.19}$$

which is the same as the energy of the $[i,j]$ (qR_4, pR_5) selfdual string.

The $(kp/R_4, -kq/R_5)$ momentum mode is dual to the $[i,j]$ (p,q) string winding x'^{45} k times. $D3$ with one transverse dimension compactified is T-dual to $D4$ with one longitudinal dimension compactified, with the winding mode converted to the momentum mode. So, the (p,q) string with all possible winding numbers gives a $5d$ SYM theory whose basic excitations are (p,q) strings. In this way, the $4d$ KK modes are equivalent to a series of $5d$ SYM fields labeled by (p,q) with p and q co-prime. The (p,q) $5d$ SYM fields, when lifted to $6d$, are translation invariant along the (qR_4, pR_5) direction. They are the fields related with the (qR_4, pR_5) selfdual strings. With all (p,q) included, the selfdual string orientation then covers the whole 45 plane. In $R_4, R_5 \to \infty$ limit, the selfdual string orientation can be represented by the angle θ. For every $\theta \in [0,\pi)$, there is a corresponding $5d$ SYM theory. Fields related with the same θ have the standard SYM type coupling. The coupling between the SYM fields related with the different θ will be discussed later.

One can make a comparison between the $6d$ $(2,0)$ theory and the $6d$ SYM theory living in $D5$ branes. $D5$ with the longitudinal dimension x_4 and x_5 compactified is T-dual to $D3$ with the transverse dimension x'_4 and x'_5 compactified. The $[i,j]$ $(n/R_4, m/R_5)$ momentum mode is dual to the $[i,j]$ $F1$ winding x'_4 and x'_5 n and m times. With m fixed, the winding modes along x'_5 give the $5d$ massive SYM theory, which, with all of m included, becomes the $6d$ SYM theory. On the other hand, for the $6d$ $(2,0)$ theory living in $M5$ branes, when $m=0$, the $[i,j]$ $(n/R_4, 0)$ momentum mode is dual to the $[i,j]$ $(1,0)$ string winding x'^{45} n times, from which, one can indeed get a $5d$ SYM theory. However, when $m=1$, the $(n/R_4, 1/R_5)$ mode is dual to the $(n,1)$ string winding x'^{45} one time. Such strings, when combine together, will give the $5d$ massive tensor multiplet rather than the $5d$ massive vector multiplet. Since the $(n,1)$ strings are also in the adjoint representation of $U(N)$, the $5d$ massive tensor multiplet will also form the $U(N)$ adjoint representation. Even though, the nonabelian interaction for them is different from the vector multiplet.

When $\vec{v}_i = 0$, all $D3$ branes are separated along a straight line, so the possible BPS states are still the original 1/2 BPS states. To get the new states, the vacuum expectation values of the five scalar fields on $M5$ branes must be turned on. In $D3$ brane picture, $D3$'s then appear as N arbitrary points in the $5d$ transverse space orthogonal to x'^{45}. The only possible new BPS states are 1/4 BPS 3-string junctions, which are also the bound states of the $[i,j]$ (p,q) string and the $[j,k]$ (r,s) string. On $M5$ side, the 3-string junction is the bound state of the $[i,j]$ $[j,k]$ selfdual strings each carrying the transverse momentum $(kp/R_4, -kq/R_5)$ and $(hr/R_4, -hs/R_5)$.

We are interested with the N coincident $M5$ branes, since in that case, the states can be massless in six dimensional sense thus will contribute to the entropy. We have seen that the bound states of the momentum mode cannot give the new degrees of freedom, but we haven't considered the bound states of the momentum mode and the zero mode, which, for example, can be taken as the tensionless $[i,j]$ $(0, R_5)$ string. In $D3$ brane picture, that is the massless $[i,j]$ $(1,0)$ string, which may form the 1/4 BPS threshold bound state with any $[j,k]$ (p,q) strings with the length $2\pi k R'_{45}$. In $6d$, they are the $[i,j]$ $(0, R_5)$ and the $[j,k]$ (qR_4, pR_5) tensionless selfdual strings located at the same point in 123 space. The former has the zero momentum in 45 plane, while the later carries the transverse momentum $(kp/R_4, -kq/R_5)$. It is unclear whether they may form the threshold bound state or not, but there is no other possibilities left. In $4d$ SYM theory, these states are massive unless $k = 0$ but in $6d$, they are massless.

We can also consider the bound state of the zero mode and momentum mode when the zero mode has the finite mass, for example, the $[i,j]$ $(R_4, 0)$ selfdual string with tension $|\vec{v}_{ij}|$ and the momentum mode $(kp/R_4, -kq/R_5)$ carried by the tensionless $[j,k]$ (qR_4, pR_5) selfdual string. When $q = 0$, the bound state is at the threshold, which is the the $[i, j]$ selfdual string carrying the $[j, k]$ longitudinal momentum. Notice that the $(0, R_5)$ selfdual string appears as the zero momentum plane wave in x_5 direction. If x_5 is taken as the M theory direction, then in type IIA, selfdual string extending along x_4 is just the monopole string on $D4$, which, of course, also has the zero momentum along x_5. The above bound state is the $[i, j]$ monopole string carrying the longitudinal momentum offered by the $[j, k]$ (massless) open string. On the other hand, the selfdual string localized in x_5 should carry the point-like P_4 momentum, which, however, only naturally exists in chargeless situation.

3.3 The interaction of the $6d$ $(2, 0)$ theory seen from its KK modes on $x_4 \times x_5$

Recall that in [23], the equations of motion for the 3-algebra valued $(2, 0)$ tensor multiplet involve a constant vector field C_μ, giving a direction along which all of the fields are required to be translation invariant. The theory with the fixed C_μ describes the selfdual string extending along it. The selfdual string has the zero momentum along C_μ but may get the arbitrary momentum along the four transverse dimensions, so the theory describing it is just the $5d$ $U(N)$ SYM theory, which is the reduction of the $6d$ $(2, 0)$ theory along C_μ. To recover the full $6d$ theory, we need the selfdual strings with the orientations covering all directions in a plane, which, for definiteness, is taken as the 45 plane. Correspondingly, C_μ is replaced by $C_\mu(\theta) = \cos\theta\delta^4_\mu + \sin\theta\delta^5_\mu$, while the original fields $f(x_\mu)$ now become $f(\theta, x_\mu)$. $\mu = 0 \cdots 5$. With the extra dimension θ added, we get a $7d$ theory, which, under the constraint $C_\mu(\theta)\partial^\mu f(\theta, x_\mu) = 0$, is a $6d$ theory again.

Suppose the $U(N)$ $6d$ $(2, 0)$ tensor multiplet field configuration is given. For simplicity, consider the scalar fields $X^I(x_m, x_4, x_5)$, where $m = 0, 1, 2, 3$, $I = 6, 7, 8, 9, 10$.

$$\int dx_\theta \; X^I(x_m, \cos\theta x_\theta + \sin\theta y_\theta, -\sin\theta x_\theta + \cos\theta y_\theta) = \Phi^I(x_m, \theta, y_\theta) \tag{3.20}$$

is the scalar field in the $5d$ SYM theory related with θ. X^I and Φ^I have the scaling dimensions 2 and 1 respectively. Φ^I is the zero mode of X^I along $C_\mu(\theta)$. Also, notice that

$$\int dx_\theta dy_\theta\, X^I(x_m, \cos\theta x_\theta + \sin\theta y_\theta, -\sin\theta x_\theta + \cos\theta y_\theta) = \phi^I(x_m) \tag{3.21}$$

is independent of θ. ϕ^I is the zero mode of $\Phi^I(x_m, \theta, y_\theta)$ in the $4d$ spacetime. All of the $5d$ SYM theories share the same zero mode in $4d$, because the $6d$ theory has the unique zero mode in $4d$. The vector field A and the spinor field η with the scaling dimensions 1 and 3/2 in $5d$ SYM theory could be constructed in the similar way from the $6d$ 2-form field B and the spinor field Ψ with the scaling dimensions 2 and 5/2. Since the integration is carried out along a particular direction, more precisely, the original scalar fields, 2-form field, and the spinor field are converted into the vector fields, vector field, and the spinor-vector field respectively.

One may also want to reconstruct X^I from Φ^I.

$$X^I(x_m, x_4, x_5) = \int dp_4 dp_5\, e^{i(p_4x_4+p_5x_5)}\phi^I_{(p_4,p_5)}(x_m). \tag{3.22}$$

If x_4 and x_5 are compact, $p_4 = k\bar{p}_4/R_4$, $p_5 = -k\bar{p}_5/R_5$, $(\bar{p}_4, \bar{p}_5)$ is the co-prime pair,

$$X^I(x_m, x_4, x_5) = \sum_{(\bar{p}_4,\bar{p}_5)} \sum_k e^{ik(\frac{\bar{p}_4x_4}{R_4} - \frac{\bar{p}_5x_5}{R_5})}\phi^I_{(\bar{p}_4,\bar{p}_5;k)}(x_m) = \sum_{(\bar{p}_4,\bar{p}_5)} \Phi^I(x_m, \bar{p}_4R_5x_4 - \bar{p}_5R_4x_5). \tag{3.23}$$

$\Phi^I(x_m, \bar{p}_4R_5x_4 - \bar{p}_5R_4x_5)$ is the discrete version of Φ^I. In continuous limit,

$$\begin{aligned} X^I(x_m, x_4, x_5) &= \int d\theta dp_\theta p_\theta\, e^{ip_\theta(-\sin\theta x_4 + \cos\theta x_5)}\phi^I_{(\theta;p_\theta)}(x_m) \\ &= \int d\theta\, \tilde{\Phi}^I(x_m, -\sin\theta x_4 + \cos\theta x_5). \end{aligned} \tag{3.24}$$

However, $\tilde{\Phi}^I$ is not the Φ^I in (3.20). The latter is

$$\Phi^I(x_m, -\sin\theta x_4 + \cos\theta x_5) = \int dp_\theta \; e^{ip_\theta(-\sin\theta x_4 + \cos\theta x_5)} \phi^I_{(\theta;p_\theta)}(x_m) \tag{3.25}$$

with p_θ left out in the integral. X^I is only the direct superposition of $\tilde{\Phi}^I(\theta)$, which is not the zero mode in $C_\mu(\theta)$ direction. Nevertheless, $\left\{\tilde{\Phi}^I(\theta) \mid \forall\theta \in [0,\pi)\right\}$ and $\left\{\Phi^I(\theta) \mid \forall\theta \in [0,\pi)\right\}$ are equivalent bases.

We now have a series of θ-parameterized $5d$ $U(N)$ SYM theories, which is effectively a $6d$ theory with 5 scalars, 3 gauge degrees of freedom and 8 fermionic degrees of freedom. It may at least exhaust the $1/2$ BPS field content of the $6d$ $(2,0)$ theory. The next problem is the interaction. Fields belong to the same $5d$ SYM theory have the standard SYM coupling among themselves. It is also necessary to consider the couplings involving fields in different $5d$ SYM theories. Actually, the $6d$ SYM theory could also be decomposed in this way, while the local interactions in the original $6d$ theory induce the couplings among the $5d$ theories labeled by different θ.

To see this coupling more explicitly, we'd better decompose the $6d$ fields into the $4d$ KK modes. For scalars, the decomposition is as that in (3.22). Similarly, for the $6d$ SYM fields such as the scalars $Y^L(x_m, x_4, x_5)$, $L = 6, 7, 8, 9$, we also have

$$Y^L(x_m, x_4, x_5) = \int dp_4 dp_5 \; e^{i(p_4 x_4 + p_5 x_5)} \varphi^L_{(p_4,p_5)}(x_m). \tag{3.26}$$

The two-field coupling $Y^L Y^{L'}$ gives

$$\int dx_4 dx_5 \; Y^L_{ij} Y^{L'}_{ji} = \int dp_4 dp_5 \; \varphi^L_{(p_4,p_5)ij}(x_m) \varphi^{L'}_{(-p_4,-p_5)ji}(x_m), \tag{3.27}$$

the three-field coupling $Y^L Y^{L'} Y^{L''}$ gives

$$\int dx_4 dx_5 Y^L_{ij} Y^{L'}_{jk} Y^{L''}_{ki} = \int dp_4 dp_5 dq_4 dq_5 \varphi^L_{(p_4,p_5)ij}(x_m) \varphi^{L'}_{(q_4,q_5)jk}(x_m) \varphi^{L''}_{(-p_4-q_4,-p_5-q_5)ki}(x_m), \tag{3.28}$$

and similarly for the n-field coupling. In the dual $D3$ brane picture, $\varphi^L_{(p_4,p_5)ij}(x_m)$ corresponds to the F-string connecting the i_{th} and the j_{th} $D3$ branes represented by the vector (p_4, p_5) in transverse space. The above coupling is possible because the bound state of the $[i,j]$ (p_4,p_5) F-string and the $[j,k]$ (q_4,q_5) F-string is the $[i,k]$ (p_4+q_4, p_5+q_5) F-string. The conclusion also holds in Coulomb branch. $\varphi^L_{(p_4,p_5)ij}(x_m)$ then corresponds to the F-string represented by the vector $(p_4, p_5, \vec{v}_{ij})$ in transverse space.

$$(p_4, p_5, \vec{v}_{ij}) + (q_4, q_5, \vec{v}_{jk}) = (p_4 + q_4, p_5 + q_5, \vec{v}_{ik}). \tag{3.29}$$

On the other hand, for fields in tensor multiplet, such as X^I, the two-field coupling $X^I X^{I'}$ is indeed

$$\int dx_4 dx_5 \, X^I_{ij} X^{I'}_{ji} = \int dp_4 dp_5 \, \phi^I_{(p_4,p_5)ij}(x_m) \phi^{I'}_{(-p_4,-p_5)ji}(x_m), \tag{3.30}$$

but the three-field coupling and the n-field coupling cannot take the similar form as (3.28). On $D3$ branes, $\phi^I_{(p_4,p_5)ij}(x_m)$ corresponds to the $[i,j]$ (p_4,p_5) string[5]. When $(p_4,p_5) \propto (q_4,q_5)$, the bound state of the $[i,j]$ (p_4,p_5) string and the $[j,k]$ (q_4,q_5) string is still the $[i,k]$ (p_4+q_4, p_5+q_5) string, so, the coupling like (3.28) is possible. $\phi^I_{(p_4,p_5)ij}(x_m)$ and $\phi^{I'}_{(q_4,q_5)jk}(x_m)$ belong to the same $5d$ SYM theory with $\theta = -\arctan(p_4/p_5)$. However, for unparallel (p_4,p_5) and (q_4,q_5), the bound state will be the 3-string junction other than the single string[6]. The similar problem also

[5] More accurately, it is the $[i,j]$ $(\bar{p}_4, \bar{p}_5)$ string winding x'^{45} k times. $p_4 = k\bar{p}_4/R_4$, $p_5 = -k\bar{p}_5/R_5$, $\bar{p}_4$ and $\bar{p}_5$ are co-prime. For simplicity, we just denote it by (p_4,p_5).

[6] The bound state exists only when the $[i,j]$ $[j,k]$ strings have the suitable mass. Here, we just

exists for the $5d$ massive tensor multiplet. The KK modes in $4d$ could be represented by the $[i,j]$ (p_4,p_5) strings with the fixed p_5 but all possible p_4. Obviously, (p_4,p_5) and (p_4',p_5) are not parallel unless $p_4=p_4'$. To summarize, if we concentrate on a single kind of the selfdual strings, the theory will be the $5d$ SYM theory; if we consider the selfdual strings with the different orientations, the theory will involve the tensor multiplet, for which, the interaction is not the standard SYM type.

Then the problem reduces to the coupling between $\phi_{(p_4,p_5)ij}(x_m)$ and $\phi'_{(q_4,q_5)jk}(x_m)$ for the unparallel (p_4,p_5) and (q_4,q_5). The bound state of the $[i,j]$ (p_4,p_5) string and the $[j,k]$ (q_4,q_5) string is the 3-string junction other than the traditional $[i,k]$ (P_4,P_5) string. Unlike the $6d$ SYM theory, we now get more states and should also quantize them. A given 3-string junction is characterized by the charge vector $\mathbf{v}_e=(r_4,s_4,t_4)$ and $\mathbf{v}_m=(r_5,s_5,t_5)$, for which, no common divisor exists. r s t are related with the i j k branes, while the rest $N-3$ branes are neglected.

$$r_4+s_4+t_4=r_5+s_5+t_5=0. \tag{3.31}$$

In x^{45}, $v^{45}_{ij}=2\pi kR'_{45}$, $v^{45}_{jk}=2\pi hR'_{45}$. In transverse space, we may also have $\vec{v}_{ij}$ and $\vec{v}_{jk}$, which will make the string junction massive. The total momentum of the 3-string junction is

$$(P_4,P_5)=(\frac{kr_4-ht_4}{R_4},\frac{-kr_5+ht_5}{R_5})=(p_4+q_4,p_5+q_5). \tag{3.32}$$

where $(p_4,p_5)=(kr_4/R_4,-kr_5/R_5)=(k\bar{p}_4/R_4,-k\bar{p}_5/R_5)$, $(q_4,q_5)=(-ht_4/R_4,$ $ht_5/R_5)=(h\bar{q}_4/R_4,-h\bar{q}_5/R_5)$. $\bar{p}_4$ and $\bar{p}_5$, $\bar{q}_4$ and $\bar{q}_5$ are not necessarily co-prime now. For $(\bar{p}_4,\bar{p}_5)\propto(\bar{q}_4,\bar{q}_5)$, $(P_4,P_5)\propto(\bar{q}_4,\bar{q}_5)$, while for the unparallel $(\bar{p}_4,\bar{p}_5)$

assume so.

and $(\bar{q}_4, \bar{q}_5)$, k and h may generate two dimensional momentum. Especially, if the $SL(2, \mathbf{Z})$ invariant intersection number [69] $I = t_5 r_4 - t_4 r_5 = \pm 1$, (P_4, P_5) can cover all of $(n/R_4, m/R_5)$; otherwise, it can only cover $(nI/R_4, mI/R_5)$. We will use $(\bar{p}_4, \bar{p}_5)$, $(\bar{q}_4, \bar{q}_5)$ and (P_4, P_5) to denote the 3-string junction. When $R_4, R_5 \to \infty$, the 3-string junction is characterized by the $2d$ vectors $\vec{C}(\theta_1)$, $\vec{C}(\theta_2)$ and (P_4, P_5). $\vec{C}(\theta_3) = -\vec{C}(\theta_1) - \vec{C}(\theta_2)$, $(P_4, P_5) = \hat{\theta}_1 p_{\theta_1} - \hat{\theta}_2 p_{\theta_2}$, where $\hat{\theta}_1$ and $\hat{\theta}_2$ are unit vectors. For the given $\vec{v}_{ij}$ and $\vec{v}_{jk}$, the 3-string junction exists only when (P_4, P_5) satisfies some particular condition

$$f_{(\bar{p}_4, \bar{p}_5, \bar{q}_4, \bar{q}_5, \vec{v}_{ij}, \vec{v}_{jk})}(P_4, P_5) > 0, \tag{3.33}$$

so the 3-string junction cannot have the arbitrary momentum in 45 plane. The selfdual string in 45 plane can only carry the $1d$ transverse momentum. Here, the bound state of two unparallel selfdual strings can carry the $2d$ momentum, but this momentum cannot cover the $2d$ space.

Fields arising from the quantization of the 3-string junctions can then be denoted by $\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}(x_m)$ or $\phi_{(\theta_1,\theta_2,\theta_3;P_4,P_5)ijk}(x_m)$ in decompactification limit. $\vec{r}+\vec{s}+\vec{t} = 0$. The corresponding $6d$ field is

$$X_{(\vec{r},\vec{s},\vec{t})ijk}(x_m, x_4, x_5) = \sum_{P_4, P_5} e^{i(P_4 x_4 + P_5 x_5)} \phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}(x_m), \tag{3.34}$$

or

$$X_{(\theta_1,\theta_2,\theta_3)ijk}(x_m, x_4, x_5) = \int dP_4 dP_5 \; e^{i(P_4 x_4 + P_5 x_5)} \phi_{(\theta_1,\theta_2,\theta_3;P_4,P_5)ijk}(x_m). \tag{3.35}$$

In polar coordinate, (3.35) could also be written as

$$
\begin{aligned}
& X_{(\theta_1,\theta_2,\theta_3)ijk}(x_m,\theta,\rho) \\
= \; & \sin(\theta_2-\theta_1)\int dp_{\theta_1}dp_{\theta_2}\; e^{i\rho[\sin(\theta-\theta_1)p_{\theta_1}-\sin(\theta-\theta_2)p_{\theta_2}]}\phi_{(\theta_1,\theta_2,\theta_3;p_{\theta_1},p_{\theta_2})ijk}(x_m). \qquad (3.36)
\end{aligned}
$$

In terms of k and h, (3.34) becomes

$$
X_{(\vec{r},\vec{s},\vec{t})ijk}(x_m,x_4,x_5)=\sum_{k,h} e^{i[k(\frac{r_4x_4}{R_4}-\frac{r_5x_5}{R_5})-h(\frac{t_4x_4}{R_4}-\frac{t_5x_5}{R_5})]}\phi_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m). \qquad (3.37)
$$

In (3.34)-(3.37), (P_4,P_5) is in the range specified by (3.33).

Similarly, we can also consider fields related with string webs, if there is only one scalar field gets the vacuum expectation value. For simplicity, consider the $4d$ $(n/R_4,1/R_5)$ momentum modes, or alternatively, the $(n,1)$ strings with $n\in\mathbf{Z}$. One may take the $[1,2]$, $\cdots$, $[N-1,N]$ strings as the simple modes, from which, the $5d$ field can be constructed as

$$
Z^{(1)}_{i,i+1}(x_m,x_4)=\int dp_4\; e^{ip_4x_4}\phi_{(p_4)i,i+1}(x_m), \qquad (3.38)
$$

The $5d$ field is in the $(2,0)$ tensor multiplet, but we only consider the scalar here. The bound state of the $[i,i+1]$, $\cdots$, $[i+k-1,i+k]$ strings has the charge vector

$$
\mathbf{v}_e=(0,\cdots,0,n_i,n_{i+1}-n_i,\cdots,-n_{i+k-1},0,\cdots,0), \qquad (3.39)
$$

$$
\mathbf{v}_m=(0,\cdots,0,1,0,\cdots,0,-1,0,\cdots,0) \qquad (3.40)
$$

and the total momentum $(\sum_{a=0}^{k-1}n_{i+a}/R_4,k/R_5)$. When $n_i=\cdots=n_{i+k-1}$, the bound state reduces to the $[i,i+k]$ $(\sum_{a=0}^{k-1}n_{i+a}/R_4,k/R_5)$ mode. The corresponding $5d$ field is

$$Z^{(k)}_{(n_i,\cdots,n_{i+k-1})i,\cdots,i+k}(x_m,x_4) = \int dp_4\, e^{ip_4x_4}\phi_{(n_i,\cdots,n_{i+k-1};p_4)i,\cdots,i+k}(x_m). \tag{3.41}$$

It seems that the quantization of the string webs gives an infinite number of fields with the infinite degrees of freedom. However, these fields can be neatly reorganized to give the $(N^3-N)/6$ degrees of freedom. For example, in (3.41), the field is actually $Z_{p_{4\,i},\cdots,p_{4\,i+k-1}}(x_m)$, which is equivalent to $Z(x_m,x_{4\,i},\cdots,x_{4\,i+k-1}) \sim Z(x_m,x_{4\,i})\cdots Z(x_m,x_{4\,i+k-1})$ thus has k degrees of freedom. Altogether,

$$\sum_{k=1}^{N-1} k(N-k) = \frac{N^3-N}{6}. \tag{3.42}$$

Recall that in previous discussion, the N^3 degrees of freedom comes from the $[i,j]$ $[j,k]$ string bound state. These two are actually equivalent. Here, we take $N-1$ simple modes as the basic degrees of freedom. We can also take $[i,j]$ modes with $i<j$ fundamental, then each $[i,j]$ $[j,k]$ bound state gives one degrees of freedom[7], which, together with the $[i,j]$ modes, becomes

$$C_N^3 + C_N^2 = \frac{N^3-N}{6}. \tag{3.43}$$

However, C_N^2 out of $(N^3-N)/6$ comes from the $[i,j]$ selfdual strings, while the rest C_N^3 is not enough to produce the anomaly on $M5$ branes [75, 76]. To give the right anomaly coefficient, we need the $[i,j]$ $[j,k]$ selfdual string bound state including the situation when two of i j k are the same. The total number of the bound states is N^3-N.

With the fields related with strings as well as the string junctions, we can consider the possible couplings among them. First, the bound state of the $[i,j]$ (r_4,r_5) string

[7]It is one other than two because the rest one has already been counted by the $[i,j]$ modes.

and the $[j,k]$ $(-t_4,-t_5)$ string, or the $[j,k]$ (s_4,s_5) string and the $[k,i]$ $(-r_4,-r_5)$ string, or the $[k,i]$ (t_4,t_5) string and the $[i,j]$ $(-s_4,-s_5)$ string is the $[i,j,k]$ 3-string junction with $\mathbf{v}_e = (r_4, s_4, t_4)$ and $\mathbf{v}_m = (r_5, s_5, t_5)$. The momentum of the 3-string junction is the sum of the two individual strings. $2+2 \to 3$.

$$\phi_{(r_4,r_5;k)ij}(x_m)\phi'_{(-t_4,-t_5;h)jk}(x_m) \sim \phi''_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m), \tag{3.44}$$

$$\phi_{(s_4,s_5;h)jk}(x_m)\phi'_{(-r_4,-r_5;-k-h)ki}(x_m) \sim \phi''_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m), \tag{3.45}$$

$$\phi_{(t_4,t_5;-k-h)ki}(x_m)\phi'_{(-s_4,-s_5;k)ij}(x_m) \sim \phi''_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m), \tag{3.46}$$

which could be derived from the coupling

$$\int dx_4 dx_5 \; X_{(r_4,r_5)ij}(x_m,x_4,x_5)X'_{(-t_4,-t_5)jk}(x_m,x_4,x_5)X''_{(\vec{r},\vec{s},\vec{t})ijk}(x_m,x_4,x_5), \tag{3.47}$$

with

$$X_{(r_4,r_5)ij}(x_m,x_4,x_5) = \sum_k \; e^{ik(\frac{r_4x_4}{R_4}-\frac{r_5x_5}{R_5})}\phi_{(r_4,r_5;k)ij}(x_m), \tag{3.48}$$

$$X'_{(-t_4,-t_5)jk}(x_m,x_4,x_5) = \sum_h \; e^{-ih(\frac{t_4x_4}{R_4}-\frac{t_5x_5}{R_5})}\phi_{(-t_4,-t_5;h)jk}(x_m). \tag{3.49}$$

In polar coordinate, the coupling is

$$\int \rho d\rho d\theta \; X_{(\theta_1)ij}(x_m,\theta,\rho)X'_{(\theta_2+\pi)jk}(x_m,\theta,\rho)X''_{(\theta_1,\theta_2,\theta_3)ijk}(x_m,\theta,\rho), \tag{3.50}$$

where

$$X_{(\theta_1)ij}(x_m,\theta,\rho) = \int dp_{\theta_1} \; e^{ip_{\theta_1}\rho\sin(\theta-\theta_1)}\phi_{(\theta_1;p_{\theta_1})ij}(x_m), \tag{3.51}$$

$$X_{(\theta_2+\pi)jk}(x_m,\theta,\rho) = \int dp_{\theta_2} \; e^{-ip_{\theta_2}\rho\sin(\theta-\theta_2)}\phi_{(\theta_2+\pi;p_{\theta_2})jk}(x_m). \tag{3.52}$$

Now, consider the bound state of $\phi_{(u_4,u_5;g)li}(x_m)\ /\phi_{(u_4,u_5;-g)il}(x_m)$ and $\phi'_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m)$.

If $(u_4, u_5) = (r_4, r_5)$ or $(u_4, u_5) = (-r_4, -r_5)$, the bound state will still be the 3-string junction $\phi''_{(\vec{r},\vec{s},\vec{t};k+g,h)ljk}(x_m)$, $2+3 \to 3$; otherwise, it is a 4-string junction, $2+3 \to 4$. The situation is similar if i is replaced by j or k. The $2+3 \to 3$ type relation may give the couplings like

$$X_{li}X'_{ijk}X''_{ljk}, \quad X_{il}X'_{ijk}X''_{ljk}, \quad X_{li}X'_{ijk}X''_{kn}X'''_{ljn} \tag{3.53}$$

and so on.

The $2+2 \to 2$, $2+3 \to 3$ couplings could be realized as the matrix multiplication. Moreover, they can also be visualized as the junction of two 2-boundary-$M2$'s and the junction of one 2-boundary-$M2$ and one 3-boundary-$M2$ respectively. Therefore, they are more reasonable than the couplings like $2+2 \to 3$ and $2+3 \to 4$.

We now have two sets of fields $f_{(r_4,r_5)ij}(x_m, x_4, x_5)$ and $f_{(\vec{r},\vec{s},\vec{t})ijk}(x_m, x_4, x_5)$, or alternatively, $f_{(\theta)ij}(x_m, \alpha, \rho)$ and $f_{(\theta_1,\theta_2,\theta_3)ijk}(x_m, \alpha, \rho)$. $f_{(r_4,r_5)ij}(x_m, x_4, x_5)$ is translation invariant along the (r_5R_4, r_4R_5) direction, so it is the previous discussed field satisfying the constraint $C_\mu(\theta)\partial^\mu f_{(\theta)ij}(x_m, \alpha, \rho) = 0$. Conversely, $f_{(\theta_1,\theta_2,\theta_3)ijk}(x_m, \alpha, \rho)$ is a $6d$ field without the constraint[8]. $f_{(\theta)ij}$ with all θ included is equivalent to a $6d$ field. $f_{(\theta_1,\theta_2,\theta_3)ijk}$ with all θ_1 θ_2 θ_3 included is equivalent to a $9d$ field, since it is related with the bound state. $f_{(\theta_1)}$, $f_{(\theta_2)}$ and $f_{(\theta_3)}$ may couple with each other through $f_{(\theta_1,\theta_2,\theta_3)}$.

$f_{(\theta)ij}$ is a vector multiplet composed by the scalars $\Phi^I_{(\theta)}$, the vector $A_{(\theta)\mu}$ and the spinor $\eta_{(\theta)}$ with the scaling dimensions 1, 1 and 3/2 respectively, coming from the $C_\mu(\theta)$ direction integration of the scalars X^I, the 2-form $B_{\mu\nu}$ and the spinor Ψ with the scaling dimensions 2, 2 and 5/2. As a $6d$ vector, $C^\mu(\theta)A_{(\theta)\mu} = 0$.

The field content of $f_{(\theta_1,\theta_2,\theta_3)ijk}$ can be reconstructed from the $4d$ KK mode.

[8](3.33) gives a restriction on the range of k and h in (3.37). Especially, if $k = 0$ or $h = 0$, $f_{(\theta_1,\theta_2,\theta_3)ijk}$ is also translation invariant along one direction, as we will see later.

The KK compactification of $f_{(\theta_1,\theta_2,\theta_3)ijk}$ on $x_4 \times x_5$ gives the $4d$ field $\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}$, which, in $4d$ SYM theory, is related to the 3-string junction with the charge vector $\mathbf{v}_e = (r_4, s_4, t_4)$ and $\mathbf{v}_m = (r_5, s_5, t_5)$, having the total mass P_5 and the total electric charge P_4. The multiplet structure of $\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}$ is $V_4 \otimes V_{in}$, where V_4 is the vector supermultiplet coming from the free center-of-mass part, V_{in} is the internal part determined by $\mathbf{v}_e$ and $\mathbf{v}_m$. For $\mathbf{v}_e = (r_4, s_4, t_4)$, $\mathbf{v}_m = (1, 0, -1)$, $V_{in} = [|s_4|/2] \oplus [|s_4|/2 - 1/2] \oplus [|s_4|/2 - 1/2] \oplus [|s_4|/2 - 1]$, giving a total of $4|s_4|$ states [67]. As the string web, it has $E_{ext} = 3$ external points and $F_{int} = |s_4|$ internal points [69]. If $\phi_{(\vec{r},\vec{s},\vec{t};P_4,P_5)ijk}$ is lifted into the $6d$ field $f_{(\vec{r},\vec{s},\vec{t})ijk}$, $V_4 \otimes V_{in}$ will become $T_6 \otimes V_{in}$, with T_6 the tensor supermultiplet from the center-of-mass part. So, $f_{(\vec{r},\vec{s},\vec{t})ijk}$ at least contains a tensor multiplet factor.

It is difficult to determine V_{in} in decompactification limit. The simplest possibility is $|s_4| = 1$ with one internal point, and then $V_{in} = [1/2] \oplus [0] \oplus [0]$. Recall that for 1/2 BPS states with the degeneracy of 2^4, we have the $6d$ $(2, 0)$ tensor multiplet T_6, whose KK modes along x_5 are the $5d$ massless vector multiplet V_5 and the $5d$ massive $(2, 0)$ tensor multiplets T_5. The KK modes of V_5 and T_5 on x_4 are the $4d$ vector multiplets V_4. The massless limit of the $5d$ massive tensor multiplet is the $5d$ massless vector multiplet. For 1/4 BPS states, in $4d$, we get $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$. The $V_4 \otimes [1/2]$ part gives

j	3/2	1	1/2	0	-1/2	-1	-3/2
Degeneracy	1	4	7	8	7	4	1

which, when combines with the rest two V_4, could be organized into 1 spin-3/2 fermion, 6 vectors, 14 spin-1/2 fermions and 14 scalars, forming the massive representation of the $4d$ $\mathcal{N} = 4$ superalgebra with 2^6 states. In massless limit, the bosonic part of $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ is composed by 6 vectors and 20 scalars, with each vector containing two degrees of freedom. V_4, when lifted to $5d$ with $P_5 = 0$ or $P_5 \neq 0$, becomes V_5 or T_5. The lifted $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ could be naively denoted by $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ and $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$, which are all complex now. Actually, one $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ only gives the 3-string junction with one possible orientation; if the other orientation is taken into account, we will also get $2^6 \times 2 = 2^7$ states. The field content of $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ could be organized into 1 spin-3/2 fermion, 6 vectors, 13 spin-1/2 fermions and 14 scalars, while the field content of $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$ could be organized into 2 selfdual tensors, 1 spin-3/2 fermion, 4 vectors, 13 spin-1/2 fermions and 10 scalars. $T_6 \otimes ([1/2] \oplus [0] \oplus [0])$ and $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$ have the same field content, forming the $6d$ massless $(2, 1)$ multiplet and the $5d$ massive $(2, 1)$ multiplet respectively. The $5d$ massive selfdual tensors and the $5d$ massive vectors, containing 3 and 4 degrees of freedom, become the $6d$ massless selfdual tensors and the $6d$ massless vectors, still with 3 and 4 degrees of freedom. $T_6 \otimes ([1/2] \oplus [0] \oplus [0])$ compactified on x_5 gives $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ and $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$, which, when further compactified on x_4, becomes $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$. Just as V_5 is the massless limit of T_5, $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ could also be taken as the massless limit of $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$. The 2 massive $5d$ selfdual tensors become 2 massless $5d$ vectors, while the 4 massive $5d$ vectors become 4 massless $5d$ vectors plus 4 scalars. $T_6 \otimes ([1/2] \oplus [0] \oplus [0])$, $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$ and $V_5 \otimes ([1/2] \oplus [0] \oplus [0])$ are all complex, so the total states for each are 2^7 other than 2^6. Each multiplet will form the $N \times N \times N$ or $\bar{N} \times \bar{N} \times \bar{N}$ representation of $U(N)$, so they cannot be real, as the fields in adjoint representation do.

$f_{(\theta_1,\theta_2,\theta_3)ijk}$ may be a $(2, 1)$ multiplet composed by the scalars $X_{(\theta_1,\theta_2,\theta_3)}$, the vectors $V_{(\theta_1,\theta_2,\theta_3)}$, the 2-forms $B_{(\theta_1,\theta_2,\theta_3)}$, the spin-1/2 fermions $\Psi_{(\theta_1,\theta_2,\theta_3)}$ and the spin-3/2 fermions $\eta_{(\theta_1,\theta_2,\theta_3)}$. In principle, the $6d$ $(2, 0)$ theory can only contain the $(2, 0)$ tensor multiplet, but now, the $(2,1)$ multiplet is also added. There will be the couplings between the $(2,1)$ multiplet and the vector multiplet arising from the reduction of the tensor multiplet along a particular direction. The incorporation of the $(2,1)$ multiplet into the scattering amplitude is also discussed in [68] for $M5$ compactified on S^1. It was shown that the $BB'A$ coupling is one of the possibilities. B and B' are the 2-forms in $5d$ massive $(2,1)$ multiplet, while A is the zero mode vector in $5d$. In the following, we will only discuss X, B and Ψ with the scaling dimensions 2, 2, 5/2 respectively. Of course, X, B and Ψ are not necessarily the R-symmetry singlet, but the indices in transverse dimensions are neglected for simplicity.

Let us consider the possible dimension six couplings for these fields. For two-field couplings, there are

$$\begin{aligned}&\partial X\partial X,\ \ \partial X_{(\theta_1,\theta_2,\theta_3)}\partial X^*_{(\theta_1,\theta_2,\theta_3)},\ \ \bar{\Psi}\partial\Psi,\ \ \bar{\Psi}_{(\theta_1,\theta_2,\theta_3)}\partial\Psi_{(\theta_1,\theta_2,\theta_3)},\\&\partial B\partial B,\ \ \partial B_{(\theta_1,\theta_2,\theta_3)}\partial B^*_{(\theta_1,\theta_2,\theta_3)}.\end{aligned}\tag{3.54}$$

X Ψ B compose a $6d$ $(2,0)$ tensor multiplet in adjoint representation of $U(N)$, which is equivalent to $\Phi_{(\theta)}$ $A_{(\theta)}$ $\eta_{(\theta)}$ with all θ included. We do not have terms like $X_{ij}X'_{jk}X''_{ki}$, but the two-field couplings like $X_{ij}X'_{ji}$ are allowed. The tensor multiplet representation works well in free theory. The possible three-field couplings are

$$\begin{aligned}&A_{\theta_a}X_{(\theta_1,\theta_2,\theta_3)}\partial X^*_{(\theta_1,\theta_2,\theta_3)},\qquad A_{\theta_a}\bar{\Psi}_{(\theta_1,\theta_2,\theta_3)}\Psi_{(\theta_1,\theta_2,\theta_3)},\\&\Phi_{\theta_a}\bar{\Psi}_{(\theta_1,\theta_2,\theta_3)}\Psi_{(\theta_1,\theta_2,\theta_3)},\qquad A_{\theta_a}B_{(\theta_1,\theta_2,\theta_3)}\partial B^*_{(\theta_1,\theta_2,\theta_3)},\end{aligned}\tag{3.55}$$

where $a=1,2,3$. The possible four-field couplings are

$$A_{\theta_a}X_{(\theta_1,\theta_2,\theta_3)}A_{\theta_b}X^*_{(\theta_1,\theta_2,\theta_3)},\qquad \Phi_{\theta_a}X_{(\theta_1,\theta_2,\theta_3)}\Phi_{\theta_b}X^*_{(\theta_1,\theta_2,\theta_3)},$$

$$A_{\theta_a} B_{(\theta_1,\theta_2,\theta_3)} A_{\theta_b} B^*_{(\theta_1,\theta_2,\theta_3)}, \qquad \Phi_{\theta_a} B_{(\theta_1,\theta_2,\theta_3)} \Phi_{\theta_b} B^*_{(\theta_1,\theta_2,\theta_3)}, \tag{3.56}$$

with $a, b = 1, 2, 3$. Based on the above couplings, the nonabelian generalization of $H_{\mu\nu\lambda}$ can then be defined as

$$H_{ijk} = dB_{ijk} + A_i^l \wedge B_{ljk} + A_j^m \wedge B_{imk} + A_k^n \wedge B_{ijn}, \tag{3.57}$$

with $H \sim H_{(\theta_1,\theta_2,\theta_3)\mu\nu\lambda}$, $A_i^l \sim A^l_{(\theta_1)\mu i}$, $A_j^m \sim A^m_{(\theta_2)\mu j}$, $A_k^n \sim A^n_{(\theta_3)\mu k}$, $B \sim B_{(\theta_1,\theta_2,\theta_3)\mu\nu}$.

Fermions may get mass through the Yukawa coupling $\Phi_{\theta_a} \bar{\Psi}_{(\theta_1,\theta_2,\theta_3)} \Psi_{(\theta_1,\theta_2,\theta_3)}$. In order to compare with the 3-string junctions in $4d$ SYM theory, we will use $(\vec{r}, \vec{s}, \vec{t})$ instead of $(\theta_1, \theta_2, \theta_3)$. Consider $\Psi_{(\vec{r},\vec{s},\vec{t})ijk}$ and $\Phi^I_{(\vec{u};lm)\mu}$. The vacuum expectation value of X^I is $\bar{X}^I_{lm} = v^I_m \delta_{lm}$, then the induced vacuum expectation value for Φ^I_μ is $\bar{\Phi}^I_{(\vec{u};lm)\mu} = \tilde{u}_\mu v^I_m \delta_{lm}$. Similar with the equation for fermions in [23],

$$\Gamma^\mu D_\mu \Psi_A + X^I_C C^\nu_B \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A = 0, \tag{3.58}$$

we may have

$$\begin{aligned} & i\Gamma^0\Gamma^\mu\Gamma_I[\bar{\Phi}^I_{(\vec{r};il)\mu}\Psi_{(\vec{r},\vec{s},\vec{t})ljk} + \bar{\Phi}^I_{(\vec{s};jl)\mu}\Psi_{(\vec{r},\vec{s},\vec{t})ilk} + \bar{\Phi}^I_{(\vec{t};kl)\mu}\Psi_{(\vec{r},\vec{s},\vec{t})ijl}] \\ = \; & i\Gamma^0\Gamma^\mu\Gamma_I[\tilde{r}_\mu v^I_i \delta_{il}\Psi_{(\vec{r},\vec{s},\vec{t})ljk} + \tilde{s}_\mu v^I_j \delta_{jl}\Psi_{(\vec{r},\vec{s},\vec{t})ilk} + \tilde{t}_\mu v^I_k \delta_{kl}\Psi_{(\vec{r},\vec{s},\vec{t})ijl}] \\ = \; & i\Gamma^0\Gamma^\mu\Gamma_I(\tilde{r}_\mu v^I_i + \tilde{s}_\mu v^I_j + \tilde{t}_\mu v^I_k)\Psi_{(\vec{r},\vec{s},\vec{t})ijk} = M\Psi_{(\vec{r},\vec{s},\vec{t})ijk}, \end{aligned} \tag{3.59}$$

where in the last step, we assume $\Psi_{lmn} = 0$ for $l, m, n \neq i, j, k$ so that Ψ is a generator with the index $[i, j, k]$.

$$M = i\Gamma^0\Gamma^\mu\Gamma_I(\tilde{r}_\mu v^I_{ij} - \tilde{t}_\mu v^I_{jk}) = i\Gamma^0\Gamma^\mu\Gamma_I(\tilde{s}_\mu v^I_{jk} - \tilde{r}_\mu v^I_{ki}) = i\Gamma^0\Gamma^\mu\Gamma_I(\tilde{t}_\mu v^I_{ki} - \tilde{s}_\mu v^I_{ij}). \tag{3.60}$$

The $[i,j,k]$ $(\vec{r},\vec{s},\vec{t})$ string is the bound state of the $[i,j]$ $(\vec{r})$ and $[j,k]$ $(-\vec{t})$ strings or the $[j,k]$ $(\vec{s})$ and $[k,i]$ $(-\vec{r})$ strings or the $[k,i]$ $(\vec{t})$ and $[i,j]$ $(-\vec{s})$ strings. In (3.60), the mass of the bound state is expressed in terms of the component strings. $\tilde{r}_4 = 2\pi r_5 R_4$, $\tilde{r}_5 = 2\pi r_4 R_5$, $\tilde{r}_\mu = 0$, for $\mu = 0,1,2,3$, so

$$i\Gamma^0\Gamma^\mu\Gamma_I\tilde{r}_\mu v_i^I = i\Gamma^0\Gamma^4\Gamma_I 2\pi r_5 R_4 v_i^I + i\Gamma^0\Gamma^5\Gamma_I 2\pi r_4 R_5 v_i^I, \tag{3.61}$$

and similarly for $\tilde{s}_\mu$ and $\tilde{t}_\mu$. As a result,

$$\begin{aligned} M &= i\Gamma^0\Gamma^4\Gamma_I(\tilde{r}_4 v_i^I + \tilde{s}_4 v_j^I + \tilde{t}_4 v_k^I) + i\Gamma^0\Gamma^5\Gamma_I(\tilde{r}_5 v_i^I + \tilde{s}_5 v_j^I + \tilde{t}_5 v_k^I) \\ &= i\Gamma^0\Gamma^4\Gamma_I Q_M^I + i\Gamma^0\Gamma^5\Gamma_I Q_E^I. \end{aligned} \tag{3.62}$$

where Q_E^I and Q_M^I are the electric and the magnetic charge vectors in $4d$ SYM theory.

$$M^2 = |\vec{Q}_E|^2 + |\vec{Q}_M|^2 + \Gamma_I\Gamma_J\Gamma^4\Gamma^5(Q_M^I Q_E^J - Q_M^J Q_E^I). \tag{3.63}$$

The third term is a matrix, nevertheless,

$$\sqrt{[\Gamma_I\Gamma_J\Gamma^4\Gamma^5(Q_M^I Q_E^J - Q_M^J Q_E^I)]^2} = 2|\vec{Q}_E \times \vec{Q}_M| \tag{3.64}$$

The above result can be compared with the mass of the 3-string junctions in $4d$ SYM theory, which is

$$Z_+^2 = |\vec{Q}_E|^2 + |\vec{Q}_M|^2 + 2|\vec{Q}_E \times \vec{Q}_M|. \tag{3.65}$$

The mass term together with $i\Psi^+\Gamma_\mu\partial^\mu\Psi$ gives the energy

$$E = \Gamma_0\Gamma_\mu p^\mu + i\Gamma^0\Gamma^4\Gamma_I Q_M^I + i\Gamma^0\Gamma^5\Gamma_I Q_E^I, \tag{3.66}$$

where $\mu = 1,2,3,4,5$, $\Gamma^+_\mu = -\Gamma_\mu$.

$$E^2 = |\vec{p}|^2 + |\vec{Q}_E|^2 + |\vec{Q}_M|^2 + \Gamma_I\Gamma_J\Gamma^4\Gamma^5(Q^I_M Q^J_E - Q^J_M Q^I_E) + 2i\Gamma_I(Q^I_M p^4 + Q^I_E p^5). \quad (3.67)$$

(3.67) can be rewritten as

$$\begin{aligned} E^2 &= p_a p_a + (Q^{45}_E Q^{45}_E + Q^{45}_M Q^{45}_M) + (Q^I_E Q^I_E + Q^I_M Q^I_M) \\ &+ \Gamma_I\Gamma_J\Gamma^4\Gamma^5(Q^I_M Q^J_E - Q^J_M Q^I_E) + 2i\Gamma_I(Q^{45}_E Q^I_M - Q^I_E Q^{45}_M), \end{aligned} \quad (3.68)$$

where $a = 1,2,3$. $Q^{45}_E = p^4$, $Q^{45}_M = -p^5$. p_4 and p_5 enter the energy formula as another charge vector Q^{45}_E and Q^{45}_M. p^1, p^2 and p^3 appear as the normal transverse momentum. In $4d$ SYM theory, with v^{45}_i and v^I_i turned on, the energy of the 3-string junction carrying the transverse momentum (p^1, p^2, p^3) is consistent with (3.68). The above result can be compared with the $6d$ SYM theory, for which,

$$E = \Gamma_0\Gamma_\mu p^\mu + \Gamma_0\Gamma_I v^I_{ij}, \quad (3.69)$$

so

$$E^2 = p_\mu p_\mu + v^I_{ij} v^I_{ij}, \quad (3.70)$$

which is the energy of a particle with the rest mass $\sqrt{v^I_{ij} v^I_{ij}}$ carrying the $5d$ momentum p^μ. Now, we have different Dirac operator, giving rise to a dispersion relation different from the standard $\sqrt{m^2 + p^2}$ type. $\sqrt{m^2 + p^2}$ is the dispersion relation for a Lorentz invariant theory. The 3-string junctions breaks the $SO(5,1)$ symmetry into $SO(3,1)$.

Obviously, $f^i_{(\theta)j}$ and $f_{(\theta_1,\theta_2,\theta_3)ijk}$ are in the adjoint and the $N \times N \times N$ representations of $SU(N)$ respectively.

$$f^{i}_{(\theta)j} \to U^{i}_{l} f^{l}_{(\theta)n} U^{+n}_{j}, \quad f_{(\theta_1,\theta_2,\theta_3)ijk} \to U^{l}_{i} U^{m}_{j} U^{n}_{k} f_{(\theta_1,\theta_2,\theta_3)lmn}, \quad \forall\, U \in SU(N). \quad (3.71)$$

A_{N-1} $6d$ $(2,0)$ theory compactified on a Riemann surface with the genus $g > 1$ could be decomposed into the T_N part and the I_N part. Each T_N part has the $SU(N)^3$ symmetry, while each I_N part gives a $SU(N)$ gauge group [47, 48]. Still, there are two sets of fields with the index $[i,j,k]$ and $[i,j]$ which may couple with each other, quite like what we have discussed above. This is not accidental. The 3-string junction on $D3$, when lifted to M theory, corresponds to $M2$ with three boundaries, which may be denoted by $M2(C_1, C_2, C_3)$, with $C_1 \sim \vec{r}$, $C_2 \sim \vec{s}$, $C_3 \sim \vec{t}$ [77]. $M2$ with two boundaries is $M2(C)$. $M2(C_1, C_2, C_3)$ and $M2(C)$ may couple at the boundary as long as $C = C_1$, or $C = C_2$, or $C = C_3$, while the product is still $M2(C_1, C_2, C_3)$. Likewise, the T_N part of the Riemann surface offers the nontrivial 1-cycles $[C_1]$, $[C_2]$, $[C_3]$ for $M2$ to end. $[C_1] + [C_2] + [C_3] = 0$. Each $M2([C_1], [C_2], [C_3])$ can only couple with the adjacent $M2([C_1])$, $M2([C_2])$, and $M2([C_3])$. The boundary coupling is the standard $\sum_i L_{a\cdots bi} M_{ic\cdots d} = N_{a\cdots bc\cdots d}$ type, indicating that $M2(C_1, C_2, C_3)$ should be in the $N \times N \times N$ or $\bar{N} \times \bar{N} \times \bar{N}$ representation. The Σ_g theory has $3(g-1)$ $SU(N)$ gauge groups associated with each 1-cycle. Fields related with the different $SU(N)$ groups cannot couple to each other directly. Similarly, the $6d$ $(2,0)$ theory may contain a series of $SU(N)$ groups associated with the selfdual strings labeled by θ. Still, fields related with different θ do not have the direct coupling. The situation is different for the $6d$ SYM theory, in which, there is only one gauge group. Even if the $6d$ SYM theory is compactified on a Riemann surface with $g > 1$, there is still only one gauge group. The reason is that the basic excitations on $M5$ is line-like, while the basic excitations on $D5$ is point-like.

In above discussion, we didn't pay too much attention to the condition (3.33). For the given v_i^I and $(\vec{r},\vec{s},\vec{t})$, the allowed (P_4,P_5) are not arbitrary. Especially, when $v_{ij}^I=0$, $\forall\, i,j$, no (P_4,P_5) can satisfy (3.33). Nevertheless, when $P_4=0$ or $P_5=0$, the equality can be saturated, while the bound states are at the threshold or just decay. If they do not decay, then (3.36) and (3.37) should be replace by

$$X_{(\theta_1,\theta_2,\theta_3)ijk}(x_m,\sin(\theta-\theta_1)\rho)=\sin(\theta_2-\theta_1)\int dp_{\theta_1}\; e^{ip_{\theta_1}\rho\sin(\theta-\theta_1)}\phi_{(\theta_1,\theta_2,\theta_3;p_{\theta_1},0)ijk}(x_m) \tag{3.72}$$

and

$$X_{(\vec{r},\vec{s},\vec{t})ijk}(x_m,r_4x_4R_5-r_5x_5R_4)=\sum_k\; e^{ik(\frac{r_4x_4}{R_4}-\frac{r_5x_5}{R_5})}\phi_{(\vec{r},\vec{s},\vec{t};k,0)ijk}(x_m). \tag{3.73}$$

(3.72) and (3.73) are translation invariant along the θ_1 direction and the (r_5R_4,r_4R_5) direction respectively. They are the zero mode of the original $6d$ field (3.36) and (3.37) along the θ_1 and the (r_5R_4,r_4R_5) directions. $\phi_{(\vec{r},\vec{s},\vec{t};k,0)ijk}(x_m)$ is the $4d$ 1/4 BPS field in $V_4\otimes V_{in}$ multiplet. Summing over all possible k will give a $5d$ field in $V_5\otimes V_{in}$ multiplet. On the other hand, summing over $\phi_{(\vec{r},\vec{s},\vec{t};k,h)ijk}(x_m)$ with all possible k but the fixed nonzero h will give a $5d$ field in $T_5\otimes V_{in}$ multiplet. (3.72) and (3.73) are in the $V_5\otimes V_{in}$ multiplet. They are actually the bound state of the $[i,j]$ (r_4,r_5) string with momentum $(kr_4/R_4,-kr_5/R_5)$ and the $[j,k]$ $(-t_4,-t_5)$ string with momentum $(0,0)$. As is mentioned before, the $4d$ $(0,0)$ mode of the $6d$ field is unique, so $(-t_4,-t_5)$ should be fixed, while the field in (3.72) and (3.73) could simply be denoted by $X_{(\theta_1)ijk}(x_m,\sin(\theta-\theta_1)\rho)$ and $X_{(\vec{r})ijk}(x_m,r_4x_4R_5-r_5x_5R_4)$. Although $X_{(\theta_1)ijk}$ or $X_{(\vec{r})ijk}$ is a $5d$ field, with all θ_1 or (r_4,r_5) included, the $6d$ field can be recovered again.

$$\phi_{(\vec{r};k)ijk}(x_m)\phi'_{(\vec{r};g)li}(x_m)\sim\phi''_{(\vec{r};k+g)ljk}(x_m). \tag{3.74}$$

$X_{(\vec{r})ijk}$ or $X_{(\theta_1)ijk}$ can only couple with $X_{(\vec{r})li}$ or $X_{(\theta_1)li}$. Both of them are translation invariant along the same direction, so the coupling is still 5 dimensional. Now, we have the $7d$ fields $f_{ij}(\theta, x_\mu)$ together with $g_{ijk}(\theta, x_\mu)$ subject to the constraints $C_\mu(\theta)\partial^\mu f_{ij}(\theta, x_\mu) = 0$ and $C_\mu(\theta)\partial^\mu g_{ijk}(\theta, x_\mu) = 0$. Fields related with different θ cannot couple with each other. It must be admitted that such scenario is not quite interesting.

3.4 The momentum-carrying BPS states in $5d$ SYM theory

Until now, all of the discussions are carried out in $6d$ theory's framework. In $6d$ theory, the KK modes are fields thus are treated perturbatively. When we say $5d$ SYM theory, we only mean the KK modes of the $6d$ fields have the SYM type coupling. In this section, we will turn to the genuine $5d$ SYM theory.

The $6d$ tensor multiplet field compactified on x_5 gives the $5d$ massless vector multiplet field and a tower of $5d$ massive tensor multiplet fields, while the $6d$ tensor multiplet field compactified on $x_4 \times x_5$ gives the $4d$ massless vector multiplet field and a tower of $4d$ massive vector multiplet fields. The $5d$ SYM theory contains the perturbative $5d$ massless SYM field and a tower of instantons, the quantization of which gives the $5d$ massive tensor multiplet. The $5d$ SYM theory with x_4 compactified also contains the BPS states, the quantization of which gives the $4d$ vector multiplets. All KK modes could be realized in $5d$ SYM theory as the nonpertubative states. It is expected that the $5d$ SYM theory may give another definition for the $6d$ $(2, 0)$ theory [65, 58].

The field configurations in SYM theories are classified by boundary conditions. For $5d$ SYM theory, the boundary configurations are characterized by $\Pi_3(SU(N)) \cong \mathbf{Z}$ with $k \in \mathbf{Z}$ the winding number. The sector with the particular k corresponds to the KK mode with $P_5 = k/R_5$. The energy is bounded by

$$E \geq |k|/R_5. \tag{3.75}$$

The equality holds for configurations representing the localized k/R_5 mode which have the zero average momentum in 1234 space. The path integral covers all these configurations, so the $5d$ SYM theory will unavoidably carry the $6d$ information and is intrinsically a $6d$ theory. $5d$ theory can be obtained only by an artificial truncation. Since the configuration only carries the chargeless P_5 momentum, there might be some kind of confinement happen.

In the following, we will discuss the generic BPS states in $5d$ SYM theory. The (P_4, P_5) modes will appear as the BPS states in $4d$ SYM theory only when one transverse dimension is compactified and so the extra degrees of freedom are added. For $5d$ SYM theory, KK modes are intrinsically included.

The field content of the $5d$ $\mathcal{N} = 2$ $U(N)$ SYM theory consists of a vector A_μ with $\mu = 0, 1, 2, 3, 4$, five scalars X^I with $I = 6, 7, 8, 9, 10$ and fermions Ψ. x_5 is the extra dimension associated with M-theory. The action is

$$\begin{aligned} S &= -\frac{1}{g_{YM}^2} \int d^5x \, \mathrm{tr}\Big(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu X^I D^\mu X^I - \frac{i}{2} \bar{\Psi} \Gamma^\mu D_\mu \Psi \\ &\qquad + \frac{1}{2} \bar{\Psi} \Gamma^5 \Gamma^I [X^I, \Psi] - \frac{1}{4} \sum_{I,J} [X^I, X^J]^2 \Big) \,, \end{aligned} \tag{3.76}$$

where $D_\mu X^I = \partial_\mu X^I - i[A_\mu, X^I]$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. For time-independent bosonic solutions with a single non-vanishing scalar field X^6, the associated energy is

$$E = \frac{1}{g_{YM}^2} \int d^4x \, \mathrm{tr}\Big[\frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} F_{0i} F_{0i} + \frac{1}{2} D_i X^6 D_i X^6 \Big], \tag{3.77}$$

where $i = 1, 2, 3, 4$. For an arbitrary vector C_i with $|C| = 1$, E could be rewritten as

$$\begin{aligned} E &= \frac{1}{g_{YM}^2} \int d^4x \, \mathrm{tr}\Big[\frac{1}{2}(F_{0i} - \sin\theta C_k F_{ik} + \cos\theta D_i X^6)^2 \\ &\quad + \frac{1}{2}(\frac{1}{2}C_k \varepsilon_{ilmk} F_{lm} \pm \cos\theta C_k F_{ik} \pm \sin\theta D_i X^6)^2 \\ &\quad + \sin\theta \, (F_{0i} C_k F_{ik} \mp \frac{1}{2} C_k \varepsilon_{ilmk} F_{lm} D_i X^6) \\ &\quad + \cos\theta \, (\mp \frac{1}{8} \varepsilon_{iklm} F_{ik} F_{lm} - F_{0i} D_i X^6)\Big] \, . \end{aligned} \tag{3.78}$$

Note that

$$\begin{aligned} P_k &= -\frac{1}{g_{YM}^2} \int d^4x \, \mathrm{tr}(F_{0i} F_{ik}), \quad Q_{Mk} = Z_k^6 = -\frac{1}{2g_{YM}^2} \int d^4x \, \mathrm{tr}(\varepsilon_{iklm} F_{lm} D_i X^6), \\ P_5 &= -\frac{1}{8g_{YM}^2} \int d^4x \, \mathrm{tr}(\varepsilon_{iklm} F_{lm} F_{ik}), \quad Q_E = Z_5^6 = \frac{1}{g_{YM}^2} \int d^4x \, \mathrm{tr}(F_{0i} D_i X^6). \end{aligned} \tag{3.79}$$

So

$$E \geq \sin\theta C_k(-P_k \pm Q_{Mk}) + \cos\theta(\pm P_5 - Q_E) \geq Max(Z_+, Z_-), \tag{3.80}$$

where

$$Z_\pm = \Big[(C_k P_k \pm C_k Q_{Mk})^2 + (P_5 \pm Q_E)^2\Big]^{\frac{1}{2}}. \tag{3.81}$$

If $Z_+ \geq Z_-$, $E = Z_+$ for

$$F_{0i} = \sin\theta C_k F_{ik} - \cos\theta D_i X^6, \tag{3.82}$$

$$\frac{1}{2} C_k \varepsilon_{ilmk} F_{lm} = \cos\theta C_k F_{ik} + \sin\theta D_i X^6. \tag{3.83}$$

If $Z_+ \leq Z_-$, $E = Z_-$ for

$$F_{0i} = \sin\theta C_k F_{ik} - \cos\theta D_i X^6, \tag{3.84}$$

$$\frac{1}{2}C_k\varepsilon_{ilmk}F_{lm} = -\cos\theta C_k F_{ik} - \sin\theta D_i X^6. \tag{3.85}$$

In both cases,

$$E = \frac{1}{g_{YM}^2}\int d^4x \ \mathrm{tr}\Big[(C_k F_{ik})^2 + (D_i X^6)^2\Big]. \tag{3.86}$$

Moreover, if $\theta \neq 0$, from (3.82-3.85), we also have $C_i D_i X^6 = C_i F_{0i} = 0$. For simplicity, in the following, we will only consider the case with $Z_+ \geq Z_-$. The situation with $Z_+ \leq Z_-$ is similar.

Without loss of generity, let $C_k = \delta_k^4$, then (3.82) and (3.83) become

$$F_{0i} = \sin\theta F_{i4} - \cos\theta D_i X^6, \tag{3.87}$$

$$\frac{1}{2}\varepsilon_{ilm4}F_{lm} = \cos\theta F_{i4} + \sin\theta D_i X^6. \tag{3.88}$$

$$E = \Big[(P_4 + Q_{M4})^2 + (P_5 + Q_E)^2\Big]^{\frac{1}{2}}. \tag{3.89}$$

For $\theta \neq 0$, $F_{04} = D_4 X^6 = 0$. When $\theta = 0$, (3.87)-(3.89) reduce to

$$F_{0i} = -D_i X^6, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = F_{i4}. \tag{3.90}$$

$$E = |P_5 + Q_E|. \tag{3.91}$$

These are the equations for the dyonic instantons discussed in [58]. $F_{04} = D_4 X^6 = 0$ is not necessary. If is imposed, the original $SO(4)$ symmetry will be broken to $SO(3)$. $P_4 \neq 0$, $Q_{M4} \neq 0$, but $P_4 + Q_{M4} = 0$. When $\theta = \pi/2$,

$$F_{0i} = F_{i4}, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = D_i X^6. \tag{3.92}$$

$$E = |P_4 + Q_{M4}|. \tag{3.93}$$

The solution describes the monopole string extending along the x_4 direction, carrying momentum P_4. $P_5 \neq 0$, $Q_E \neq 0$, but $P_5 + Q_E = 0$.

For the time-independent bosonic solutions with $C_k = \delta_k^4$, the supersymmetry transformation becomes

$$\begin{aligned}
\delta_\epsilon \Psi &= \frac{1}{2} F_{\mu\nu} \Gamma^{\mu\nu} \Gamma_5 \epsilon + D_\mu X^6 \Gamma^\mu \Gamma^6 \epsilon \\
&= D_a X^6 \Gamma_a (\Gamma^6 + \cos\theta \Gamma_{05} + \sin\theta \Gamma_{123} \Gamma_5) \epsilon \\
&\quad + F_{a4} \Gamma_a (\Gamma_{45} - \sin\theta \Gamma_{05} + \cos\theta \Gamma_{123} \Gamma_5) \epsilon,
\end{aligned} \tag{3.94}$$

where $a = 1, 2, 3$. $D_4 X^6 = F_{04} = 0$ is imposed. $\delta_\epsilon \Psi = 0$, ϵ should satisfy

$$(1 + \cos\theta \Gamma_{056} - \sin\theta \Gamma_{046}) \epsilon = 0, \tag{3.95}$$

$$(1 + \sin\theta \Gamma_{04} - \cos\theta \Gamma_{05}) \epsilon = 0, \tag{3.96}$$

in which $\Gamma_{012345} \epsilon = \epsilon$ is used. The solution is 1/4 BPS. For $\theta = 0$, we have $\epsilon = -\Gamma_{056} \epsilon = \Gamma_{05} \epsilon$, which are the supersymmetries preserved by dyonic instantons [58]. For $\theta = \pi/2$, $\epsilon = \Gamma_{046} \epsilon = -\Gamma_{04} \epsilon$, which are the supersymmetries preserved by the monopole strings extending along x_4 carrying momentum P_4. If $D_a X^6$ and F_{a4} are not independent, for example, $D_a X^6 = \sin\theta D_a \Phi$ and $F_{a4} = \cos\theta D_a \Phi$ as that in [58], (3.94) will reduce to

$$\begin{aligned}
\delta_\epsilon \Psi &= D_a \Phi \Gamma_a (\sin\theta \Gamma^6 + \cos\theta \Gamma_{45} - \Gamma_{04}) \epsilon \\
&= D_a \Phi \Gamma_a \Gamma_{04} (\sin\theta \Gamma_{04} \Gamma^6 + \cos\theta \Gamma_{05} - 1) \epsilon = 0.
\end{aligned} \tag{3.97}$$

The solution becomes 1/2 BPS. Moreover, for this state, $F_{0i} = 0$, so $P_k = Q_E = 0$, $E = \sqrt{Q_{M4}^2 + P_5^2}$. It may describe the monopole string extending along x_4 carrying the uniformly distributed $D0$ charge. Conversely, if $F_{a4} = \sin\theta D_a\Phi$, $D_a X^6 = -\cos\theta D_a\Phi$, $F_{ab} = 0$, $E = \sqrt{Q_E^2 + P_4^2}$. The solution describes the $F1$ string carrying P_4 momentum, which is also 1/2 BPS.

Another special kind of 1/2 BPS states have $F_{i4} = 0$ or $X^6 = 0$. When $X^6 = \theta = 0$, we get the instanton equation

$$F_{0i} = 0, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = F_{i4}, \tag{3.98}$$

the solution of which describes the $D0$ branes revolved in $D4$ branes. $E = |P_5|$. The quantization of the instanton state gives the $5d$ massive $(2,0)$ tensor multiplet T_5 without charge. When $\theta \neq 0$,

$$F_{0i} = \sin\theta F_{i4}, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = \cos\theta F_{i4}. \tag{3.99}$$

The $SO(4)$ symmetry is broken to $SO(3)$. Therefore, we may look for solutions which are translation invariant along x_4. $E = \sqrt{P_4^2 + P_5^2}$. The solution describes the $D0$ branes localized in R^3 carrying momentum P_4, which, in $D3$ picture, is the (p,q) strings winding x'^4. The quantization gives the $4d$ massive vector multiplet V_4 that is also the KK mode of the $5d$ massive tensor multiplet T_5. The original four position moduli of the instantons become the three position moduli plus one momentum moduli. $\tan\theta = P_4/P_5$. The (p,q) string can be open or closed, thus carries the $[i,j]$ charge or not, so is the corresponding $4d$ vector multiplet. On the other hand, if $F_{i4} = 0$, $\theta = 0$, the equations will be

$$F_{0i} = -D_i X^6, \qquad \frac{1}{2}\varepsilon_{ilm4}F_{lm} = 0, \tag{3.100}$$

whose solutions are $[i,j]$ $F1$ strings, the quantization of which gives the $5d$ vector multiplet V_5. $E = |Q_E|$. When $\theta \neq 0$,

$$F_{0i} = -\cos\theta D_i X^6, \qquad \frac{1}{2}\varepsilon_{ilm4} F_{lm} = \sin\theta D_i X^6. \tag{3.101}$$

The solution describes the bound state of the $[i,j]$ $F1$ and the $[i,j]$ monopole string extending along x_4, whose quantization also gives the $4d$ vector multiplet V_4. $E = \sqrt{Q_E^2 + Q_{M4}^2}$. $\tan\theta = Q_{M4}/Q_E$. In this case, θ is just the previously mentioned label for the selfdual strings parallel to the 45 plane. A reduction along x_5 is made to get the states with $P_5 = 0$. Selfdual strings extending along x_5 already have $P_5 = 0$ and is projected to a point in $5d$. The rest selfdual strings are projected to a straight line extending along x_4, which is the bound state of the $[i,j]$ $F1$ and the $[i,j]$ monopole string. $F1$ has the definite momentum $P_4 = 0$, while the monopole string carries no $D0$ charge, so the bound state is the zero mode of the $6d$ theory on $x_4 \times x_5$, which should be unique, but is now degenerate.

For 1/4 BPS state, when $\theta = 0$, we get (3.90), whose solution is the dyonic instanton, the quantization of which gives the $5d$ massive $(2,1)$ multiplet with 2^6 complex states composed by 1 spin-3/2 fermion, 13 spin-1/2 fermions, 2 selfdual tensors, 4 vectors and 10 scalars [58], which is actually the previously mentioned $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$. When $\theta \neq 0$, the equations are (3.87) and (3.88). The solution corresponds to the bound state of the string and the monopole string, carrying the P_4 P_5 transverse momentum respectively. The string and the monopole string carry the different charge, for example, $[i,j]$ and $[j,k]$. The quantization gives the $4d$ $V_4 \otimes ([1/2] \oplus [0] \oplus [0])$ multiplet with 2^6 real states composed by 1 spin-3/2 fermion, 14 spin-1/2 fermions, 6 vectors and 14 scalars, which is the massive KK mode of $T_5 \otimes ([1/2] \oplus [0] \oplus [0])$. Notice that for the $F1$-$D0$ bound state, $D0$ is chargeless, so the corresponding multiplet can only carry the $[i,j]$ charge. On the other hand, for the $F1$-$D2$ bound state with the transverse momentum involved, $F1$ and $D2$ may carry the $[i,j]$ and $[j,k]$ charges, and so the corresponding multiplet may have the index $[i,j,k]$. Just as the 1/2 BPS case, $D0$ in momentum other than position eigenstate of x_4 can carry charge.

It is convenient to work in $D3$ picture. With x_4 compactified, under the T-duality transformation along x_4, $A_4 \to X^4$. Let $F_{0a} = E_a$, $\frac{1}{2}\epsilon_{abc4}F_{bc} = B_a$, (3.87) and (3.88) could be rewritten as

$$E_a = \sin\theta D_a X^4 - \cos\theta D_a X^6, \tag{3.102}$$

$$B_a = \cos\theta D_a X^4 + \sin\theta D_a X^6, \tag{3.103}$$

which are the standard BPS equations for the $\mathcal{N}=4$ SYM theory with two scalar fields X^4 and X^6 turned on. In the language of the $\mathcal{N}=4$ $SU(N)$ SYM theory,

$$\int dS_a\, E_a = e\mathbf{p}\cdot\mathbf{H}, \quad \int dS_a\, B_a = \frac{4\pi}{e}\mathbf{q}\cdot\mathbf{H}, \tag{3.104}$$

where the vectors $\mathbf{p}$ and $\mathbf{q}$ are the electric and the magnetic charges respectively. $\mathbf{H}$ generates the Cartan subalgbra of $SU(N)$.

$$\mathbf{p}\cdot\mathbf{H} = \mathrm{diag}(p_1, p_2, \cdots, p_N), \quad \mathbf{q}\cdot\mathbf{H} = \mathrm{diag}(q_1, q_2, \cdots, q_N), \tag{3.105}$$

$\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i = 0$. Suppose $\left\langle X^I \right\rangle = \mathbf{v}^I \cdot \mathbf{H} = \mathrm{diag}(v_1^I, v_2^I, \cdots, v_N^I)$, $v_1^6 \geq v_2^6 \geq \cdots \geq v_N^6$,

$$\mathcal{Q}_E^4 = \int d^3x\, \partial_a \mathrm{tr}\Big[X^4 E_a\Big] = e\mathbf{p}\cdot\mathbf{v}^4 = e\sum_{i=1}^{N} p_i v_i^4 \sim -P_4, \tag{3.106}$$

$$\mathcal{Q}_E^6 = \int d^3x\, \partial_a \mathrm{tr}\Big[X^6 E_a\Big] = e\mathbf{p}\cdot\mathbf{v}^6 = e\sum_{i=1}^{N} p_i v_i^6 \sim Q_E, \tag{3.107}$$

$$\mathcal{Q}_M^4 = \int d^3x \, \partial_a \mathrm{tr}\Big[X^4 B_a\Big] = \frac{4\pi}{e}\mathbf{q}\cdot\mathbf{v}^4 = \frac{4\pi}{e}\sum_{i=1}^N q_i v_i^4 \sim -P_5, \tag{3.108}$$

$$\mathcal{Q}_M^6 = \int d^3x \, \partial_a \mathrm{tr}\Big[X^6 B_a\Big] = \frac{4\pi}{e}\mathbf{q}\cdot\mathbf{v}^6 = \frac{4\pi}{e}\sum_{i=1}^N q_i v_i^6 \sim -Q_{M4}. \tag{3.109}$$

The energy becomes[9]

$$E = \sqrt{(\mathcal{Q}_E^4 + \mathcal{Q}_M^6)^2 + (\mathcal{Q}_E^6 - \mathcal{Q}_M^4)^2}. \tag{3.110}$$

$x^4 \sim x^4 + 2\pi nR$. The transverse position of the i_{th} $D3$ brane in 4-6 plane could be denoted by (v_i^4, v_i^6), where $v_i^I \in (-\infty, +\infty)$.

The generic 1/2 BPS state is the (p,q) string connecting the i j $D3$ branes with the mass

$$E = \sqrt{[ep(v_i^6 - v_j^6) - \frac{4\pi}{e}q(v_i^4 - v_j^4)]^2 + [\frac{4\pi}{e}q(v_i^6 - v_j^6) + ep(v_i^4 - v_j^4)]^2}. \tag{3.111}$$

Especially, if $e^2p(v_i^6 - v_j^6) = 4\pi q(v_i^4 - v_j^4)$, the state will reduce to a $[j,i]$ $D2$ brane carrying $[j,i]$ P_4 momentum, while if $4\pi q(v_i^6 - v_j^6) = -e^2p(v_i^4 - v_j^4)$, the state will become a $[i,j]$ string with $[j,i]$ $D0$ charge. Notice that in this case, the P_4 momentum and the $D0$ charge spread uniformly over the $D2$ branes and the strings.

The simplest 1/4 BPS state is the 3-string junction with i j k representing three distinct $D3$ branes with coordinates (v_i^4, v_i^6), (v_j^4, v_j^6), (v_k^4, v_k^6). With the charge vector $\mathbf{v}_e = (1, 0, -1)$, $\mathbf{v}_m = (0, 1, -1)$, the mass is

$$E = \sqrt{[e(v_i^6 - v_k^6) - \frac{4\pi}{e}(v_j^4 - v_k^4)]^2 + [\frac{4\pi}{e}(v_j^6 - v_k^6) + e(v_i^4 - v_k^4)]^2}. \tag{3.112}$$

The corresponding state on $D4$ is a $[i,k]$ string with $v_k^4 - v_j^4$ $D0$ charge and a $[k,j]$

[9] For (3.110) to be valid, for the given $\mathbf{v}^I$, $\mathbf{p}$ $\mathbf{q}$ should be selected so that $Z_+ \geq Z_-$, otherwise $E = \sqrt{(\mathcal{Q}_E^4 - \mathcal{Q}_M^6)^2 + (\mathcal{Q}_E^6 + \mathcal{Q}_M^4)^2}$.

$D2$ brane with $v_k^4 - v_i^4$ P_4 momentum. Especially, when $e^2(v_i^6 - v_k^6) = 4\pi(v_j^4 - v_k^4)$, the state reduces to the $[k, j]$ $D2$ brane carrying $[k, i]$ P_4 momentum, while when $4\pi(v_j^6 - v_k^6) = e^2(v_k^4 - v_i^4)$, the state reduces to the $[i, k]$ string carrying $[k, j]$ $D0$ charge. The $[i, k]$ string and the $[k, j]$ $D2$ brane are parallel, so the bound state does not exist, nevertheless, with suitable amount of P_4 momentum and the $D0$ charge, the bound state may form. With the given A_4, the $[i, j]$ string ($D2$ brane) can only carry the $[i, j]$ P_4 momentum ($D0$ charge). However, they can carry the $[j, k]$ or $[i, k]$ $D0$ charge (P_4 momentum), which is actually the transverse momentum of the $[j, k]$ or $[i, k]$ $D2$ brane (string).

Now, consider string webs with more external legs. For $i \geq k \geq l \geq j$, the $[i, j]$ $D2$ brane (string) $[k, l]$ string ($D2$ brane) bound states do not exist. The bound state may exist if the $[k, l]$ string ($D2$ brane) carries the appropriate P_4 momentum ($D0$ charge). In general, the charge vector can be taken as $p_i = 1$, $p_j = -1$, $p_a = 0$ for $a \neq i, j$, $q_a = 0$ for $a > i$ or $a < j$.

$$\begin{aligned} P_4 &= e(v_j^4 - v_i^4), \quad Q_E = e(v_i^6 - v_j^6), \\ P_5 &= -\frac{4\pi}{e}\sum_{m=j}^{i} q_m v_m^4 = \frac{4\pi}{e}\sum_{m=j}^{i} r_m (v_{m+1}^4 - v_m^4), \\ Q_{M4} &= -\frac{4\pi}{e}\sum_{m=j}^{i} q_m v_m^6 = \frac{4\pi}{e}\sum_{m=j}^{i} r_m (v_{m+1}^6 - v_m^6). \end{aligned} \tag{3.113}$$

This is the bound state of the $[i, j]$ string and r_m $[m+1, m]$ $D2$ branes each carrying $v_{m+1}^4 - v_m^4$ $[m+1, m]$ $D0$ charge. Especially, if the $[i, j]$ string carries the transverse momentum P_4 so that $P_4 + Q_{M4} = 0$, the state will reduce to the $[i, j]$ string with r_m $v_{m+1}^4 - v_m^4$ $[m+1, m]$ $D0$ charge. Conversely, one may let $q_i = 1$, $q_j = -1$, $q_a = 0$

for $a \neq i, j$, $p_a = 0$ for $a > i$ or $a < j$.

$$
\begin{aligned}
P_5 &= \frac{4\pi}{e}(v_j^4 - v_i^4), \quad Q_{M4} = \frac{4\pi}{e}(v_j^6 - v_i^6) \\
P_4 &= -e\sum_{m=j}^{i} p_m v_m^4 = e\sum_{m=j}^{i} s_m(v_{m+1}^4 - v_m^4), \\
Q_E &= e\sum_{m=j}^{i} p_m v_m^6 = -e\sum_{m=j}^{i} s_m(v_{m+1}^6 - v_m^6).
\end{aligned} \tag{3.114}
$$

The corresponding state is the bound state of the $[j,i]$ $D2$ brane and s_m $[m, m+1]$ strings each carrying the $v_{m+1}^4 - v_m^4$ P_4 momentum. When $P_5 + Q_E = 0$, the state becomes $[j,i]$ $D2$ brane carrying s_m $v_{m+1}^4 - v_m^4$ $[m+1, m]$ P_4 momentum.

We can give a more precise description for these longitudinal-momentum-carrying states. For example, for string web in Fig. 3.1, suppose the strings extending in x_4 x_6 directions are $(1,0)$ and $(0,1)$ strings, while the rest ones are $(1,1)$ strings, then the state could be taken as the $[i,n]$ $D2$ extending along $x_4 \times x_6$, for which, the $[i,j]$ $[k.l]$ $[m,n]$ $D2$ carry the zero P_4 momentum, the $[j,k]$ $[l,m]$ $D2$ have the uniformly distributed $T_{F1}|v_{ab}^4|$, $T_{F1}|v_{cd}^4|$, P_4 momentum, while the rest $T_{F1}|v_{ja}^4|$, $T_{F1}|v_{bk}^4|$, $T_{F1}|v_{lc}^4|$, $T_{F1}|v_{dm}^4|$, P_4 momentums are localized on the j_{th}, k_{th}, l_{th}, m_{th} $D4$ branes.

$[i,j]$ $D2$ $(F1)$ is composed by $[i,i+1] \cdots [j-1,j]$ $D2$'s $(F1$'s). Each $[a,a+1]$ $D2$ $(F1)$ must have the same transverse velocity, otherwise, the bound state cannot be formed. On the other hand, the longitudinal momentums along x_4 (x_5) on each $[a,a+1]$ $D2$ $(F1)$ are independent, so the degrees of freedom on the $[i,j]$ $D2$ $(F1)$ is $j-i$. Altogether, there are $N(N-1)/2$ $[i,j]$ $D2$ $(F1)$, therefore, the total number of degrees of freedom is $(N^3-N)/6$. The N^3 scaling comes from the longitudinal momentum. Both transverse momentum and the longitudinal momentum carry the charge. The $[i,j]$ $D2$ $(F1)$ can only carry the $[i,j]$ transverse momentum but the $[k,l]$ longitudinal momentum for any $i \leq k < l \leq j$. The index calculation in [78]

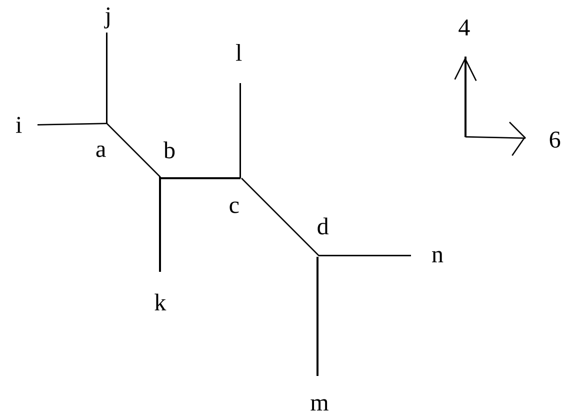

Figure 3.1: The $ijklmn$ string web

also showed the $(N^3 - N)/6$ degrees of freedom for the longitudinal momentum mode on open $D2$'s connecting $D4$'s.

If there are more than one scalar fields get the vacuum expectation value, the string webs with more than three external points may not be BPS. For 3-string junctions $i\ j\ k$, one can always choose two particular orthogonal transverse directions, in terms of which, the coordinates are $(v_i^4, \alpha_i, \beta_i)$, $(v_j^4, \alpha_j, \beta_j)$, $(v_k^4, \alpha_k, \beta_k)$ and the mass formula is

$$E = \sqrt{[|\alpha_{jk}| + (\beta_{ik}^2 + v_{ik}^{4\,2})^{\frac{1}{2}}]^2 + [|\alpha_{ik}| - (\beta_{jk}^2 + v_{jk}^{4\,2})^{\frac{1}{2}}]^2}. \tag{3.115}$$

When $i\ j\ k$ are located in a straight line, $\beta_{i,j,k} = 0$, (3.115) reduces to the previous equation.

3.5 The degrees of freedom at the triple intersection of $M5$ branes

In this section, we will consider the the triple intersecting configuration of the $M5$ branes $5\perp5\perp5$. Suppose there are N_1 $M5$ branes extending in 0 1 2 3 4 5 direction,

N_2 $M5$ branes extending in 0 1 2 3 6 7 direction, and N_3 $M5$ branes extending in 0 1 4 5 6 7 direction (see Fig. 3.2). The common transverse space is x_8 x_9 x_{10},

	0	1	2	3	4	5	6	7	8	9	10
M5	*	*	*	*	*	*					
M5	*	*	*	*			*	*			
M5	*	*			*	*	*	*			

Figure 3.2: The $M5$ $M5$ $M5$ configuration

while the common longitudinal spacetime is x_0 x_1 with x_0 the time direction. If $x_8 = x_9 = x_{10}$, the $N_1 + N_2 + N_3$ $M5$ branes will have $N_1N_2N_3$ triple intersections no matter whether each bunch of $M5$ branes are coincident or not. The black hole entropy calculation shows that there are $N_1N_2N_3$ degrees of freedom at the triple intersections, so each intersection will offer one degree of freedom [79]. The situation can be compared with the $4\perp4$ configuration for N_1 and N_2 intersecting $D4$ branes with N_1N_2 $3d$ intersections. There are $U(1)\times U(1)$ massless hypermultiplets living at each intersection producing the N_1N_2 entropy. So, we may expect that similarly the triple intersection will also capture some nonabelian features of $M5$.

Consider one intersection and label the three $M5$ branes by i, j, k. In the most generic case, i j k $M5$ branes appear as three points v_i^I v_j^I v_k^I in $x_8\times x_9\times x_{10}$ transverse space. Still, we want to compactify two longitudinal dimensions of $M5$ branes to simplify the problem. There are two distinct possibilities: $x_2\times x_4$ and

$x_2 \times x_1$. M theory compactified on x_2 gives the type IIA string theory, with the i j k $M5$'s becoming the $D4$ $D4$ $NS5$. The triple intersection of the $D4$ $D4$ $NS5$ branes still have the $N_1N_2N_3$ entropy, so the KK mode along x_2 can be safely dropped[10].

	0	1	2	3	4	5	6	7	8	9	10
D3	*	*		*		*					
D5	*	*		*	*		*	*			
NS5	*	*			*	*	*	*			

Figure 3.3: The $D3$ $D5$ $NS5$ configuration

Then compactify on x_4 with the radius R_4 and do a T-duality transformation, we get $D3$ $D5$ $NS5$ (see Fig. 3.3). The state carrying $[i,j,k]$ index is the 3-string junction with (p,q), $(p,0)$, $(0,q)$ strings ending on $D3$ $D5$ $NS5$, which will become massless when $v_i^I = v_j^I = v_k^I$. This is the scenario discussed in [43]. The 3-string junction is the point-like particle in $x_0 \times x_1$, so they may give the field $f^{ijk}(x_0, x_1)$ localized at the intersection. In M theory with $x_2 \times x_4$ compactified to T^2, the 3-string junction is lifted to a $M2$ embedded along a holomorphic curve in $x_2 \times x_4 \times x_8 \times x_9 \times x_{10}$, ending on the three $M5$'s along (pR_2, qR_4), pR_2, qR_4 [77]. Still, the problem is that when the three $M5$ branes intersect, the 3-string junction is at the threshold and may decay into the component strings. If they do decay, then at the triple intersection,

[10]The P_2 momentum may have the relevance with the $c = 6$ central charge. With one $M5$ fixed, there are 4 moduli to characterize the $5\perp5\perp5$ intersection, while for $D4$ $D4$ $NS5$, only 3 moduli are left, since the motion along x_2 is neglected.

there will be no BPS state related with all three branes.

	0	1	2	3	4	5	6	7	8	9	10
D3	*			*	*	*					
D3	*			*			*	*			
NS5	*	*			*	*	*	*			

Figure 3.4: The $D3$ $D3$ $NS5$ configuration

The other possibility is to compactify on x_1 with the radius R_1 and also do a T-duality transformation. We get $D3$ $D3$ $NS5$ (see Fig. 3.4). In $x_8 \times x_9 \times x_{10}$, no string junction can be formed. We may consider the 3-string junction in, for example, $x_1 \times x_8$ plane. The KK mode along x_1 cannot be dropped. Actually, in the T-dual picture, x_1 is a circle with the radius $\frac{1}{T_{F1}R_1}$, so $D3$ and $D3$ will be separated with the distance $\frac{m}{T_{F1}R_1}$ in the covering space.

In $x_1 \times x_8$ plane, $NS5$ is a straight line locating in v_k^8, while the i j $D3$'s appear as two points with coordinates (v_i^1, v_i^8), (v_j^1, v_j^8), $v_{ij}^1 = \frac{m}{T_{F1}R_1}$. The simplest 3-string junctions are given in Fig. 3.5. In Fig. 3.5 (A) and (C), the io oj ok strings carry the charge (p,q) $(p,0)$ $(0,q)$, $\tan\angle ioj = -\frac{qR_1}{pR_2}$. In Fig. 3.5 (B), the io oo' $o'j$ strings carry the charge (p,q) $(p,0)$ (p,q), $\tan\angle ioo' = \tan\angle oo'j = -\frac{qR_1}{pR_2}$. Actually, there are also $(0,q)$ oa string and the $(0,q)$ bo' string ending on $NS5$ with the zero length. The string junctions like this always exist as long as the suitable charge vectors are taken. The mass of the string junctions in (A) and (C) is $qT_{D1}|v_{ik}^8| + pT_{F1}|v_{ij}^1|$. The

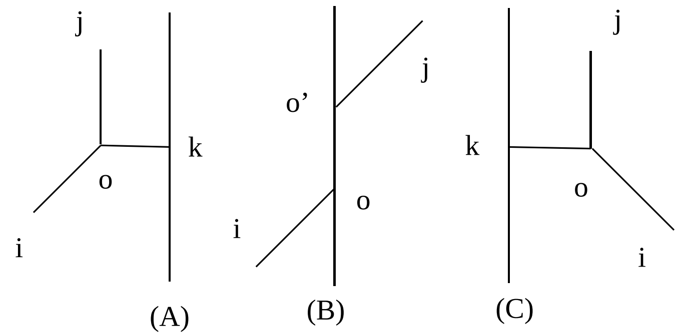

Figure 3.5: The 3-string junction in 18 plane

mass of the string junctions in (B) is $qT_{D1}|v_{ij}^8| + pT_{F1}|v_{ij}^1|$. In the T-dual picture, Fig. 3.5 (A) corresponds to q $[i,k]$ monopole strings with tension $T_{M2}|v_{ik}^8|$ wrapping x_1 carrying the $[i,j]$ longitudinal momentum $P_1 = pm/R_1$. The situation is similar for Fig. 3.5 (C). Fig. 3.5 (B) corresponds to q $[i,j]$ monopole strings with tension $T_{M2}|v_{ij}^8|$ wrapping x_1 carrying the $[i,j]$ longitudinal momentum $P_1 = pm/R_1$.

When $v_i^8 = v_j^8 = v_k^8$, Fig. 3.5 (A) and (C) reduce to the $[i,k]$ tensionless monopole strings wrapping x_1 carrying the $[i,j]$ longitudinal momentum $P_1 = m/R_1$, which is offered by the potentially existing $[i,j]$ massless string. In Fig. 3.5 (B), with *oa* oo' or oo' bo' kept, the state becomes the $[i,k]$ or $[k,j]$ tensionless monopole string wrapping x_1 carrying the $[i,j]$ longitudinal momentum $P_1 = m/R_1$ offered by the $[i,j]$ massless string. Monopole string wrapping x_1 carrying P_1 momentum corresponds to the KK mode of $f^{ijk}(x_0,x_1)$ along x_1. Again, at the intersection, the state may decay into the monopole string and the string, and then there will be no BPS state relevant to all three branes.

At the 5⊥5⊥5 intersection of three $NS5_A$'s or $NS5_B$'s or $D5$'s, there are type IIA strings, or type IIB strings, or D-strings living at the intersection. The F-string or D-string has the $4d$ transverse monition thus may produce the $c = 6$ central charge.

The oscillation of the F-string or D-string gives the P_1 momentum. The problem is that neither F-string nor D-string carries charge, so it is difficult to explain their relation with the three intersecting branes. Moreover, this picture is different from the pair-wise D brane intersection case, for which, the state at the intersection is the $[i, j]$ open string.

4. THE EFFECTIVE THEORY FOR KK MODES

In this chapter, we will explicitly discuss the 4d fields arising from the quantization of the 3-string junctions as well as their 5d and the 6d lifts. We will show that the non-chiral part of the fields are suppressed in 6d, while the chiral part of the fields are exactly $4N^3$ copies of the (2, 0) tensor multiplet. On the other hand, the scale anomaly of the 6d (2, 0) theory in large N limit is indeed equal to the scale anomaly of $4N^3$ copies of the (2, 0) tensor multiplet. In 4d, the interaction between the 3-string junctions is mediated by the N=4 vector multiplet arising from the quantization of the open strings. We will try to construct the 4d Lagrangian describing the coupling between the 3-string junctions and the open strings.

4.1 The field content of the 3-string junction in 4d and its 5d and 6d lifts

On N parallel $D3$ branes, various string webs can be constructed. Let F_{int} and E_{ext} represent the numbers of the internal faces and the external legs of the string webs respectively. When $F_{int} = 0$, $E_{ext} = n$, the quantization of the string webs gives the supermultiplet with the highest spin $n/2$ [69]. For $n = 1$, we get the hypermultiplet in the fundamental representation of $U(N)$; for $n = 3$, we get the N=6 multiplet in the tri-fundamental representation of $U(N)$. The gauge interaction is mediated by the open strings with $n = 2$, the quantization of which gives the N=4 vector multiplet in the adjoint representation of $U(N)$. With the co-prime p and q fixed, the low energy effective theory describing the interaction of the (p, q) open strings is the $N = 4$ SYM theory with the dimensionless coupling constant

$$g(p,q) = \frac{|p - q\tau|^2}{\Im\tau}. \tag{4.1}$$

Hypermultiplet and the N=6 multiplet can also be added as the matter.

D3 with the transverse x_{45} compactified has the R-symmetry $SO(5)$. The existence of the 3-string junction will also need the D3 branes to be separated along another transverse direction, say x_6, which will then reduce the R-symmetry to $SO(4) \sim SU(2) \times SU(2)$. Quantization of the 3-string junctions gives the massive $N = 6$ multiplet in $4d$, whose representation under the little group and the R-symmetry group $SO(3) \times SU(2) \times SU(2)$ is

field	$SO(3) \times SU(2) \times SU(2)$
S_μ	$(4; 1, 1)$
V_μ	$(3; 1, 2) \oplus 2 \otimes (3; 2, 1)$
ζ	$3 \otimes (2; 1, 1) \oplus 2 \otimes (2; 2, 2) \oplus (2; 3, 1)$
φ	$(1; 3, 2) \oplus 2 \otimes (1; 2, 1) \oplus 2 \otimes (1; 1, 2)$

The 12 real supercharges have the representation

supercharge	$SO(3) \times SU(2) \times SU(2)$
Q_1	$(2; 1, 2)$
Q_2	$(2; 2, 1)$
Q_3	$(2; 2, 1)$

When the $SO(5)$ R-symmetry is recovered, the representations of the fields and the supercharges under $SO(3) \times SO(5) \times SU(2)$ are

field	$SO(3) \times SO(5) \times SU(2)$
S_μ	$(4; 1, 1)$
V_μ	$(3; 1, 2) \oplus (3; 4, 1)$
ζ	$(2; 1, 1) \oplus (2; 4, 2) \oplus (2; 5, 1)$
φ	$(1; 5, 2) \oplus (1; 4, 1)$

supercharge	$SO(3) \times SO(5) \times SU(2)$
Q^{α_1}	$(2;1,2)$
Q_γ	$(2;4,1)$

There are altogether 64 real states arising from the quantization of the 3-string junction $[i,j,k]$ with the give charge vector $(p,q,-p-q)$. The charge conjugate is the 3-string junction $[i,j,k]$ with the charge vector $(-p,-q,p+q)$. So finally, we get the CPT self conjugate multiplet with 64 complex states.

The decomposition of the N=4 vector multiplet under $SO(3) \times SU(2) \times SU(2)$ is

field	$SO(3) \times SU(2) \times SU(2)$
A_μ	$(3;1,1)$
ξ	$(2;1,2) \oplus (2;2,1)$
X	$(1;2,2) \oplus (1;1,1)$

The 8 real supercharges have the representation

supercharge	$SO(3) \times SU(2) \times SU(2)$
Q_1	$(2;1,2)$
Q_2	$(2;2,1)$

When the $SO(5)$ R-symmetry is recovered, the representations of the fields and the supercharges under $SO(3) \times SO(5) \times SU(2)$ are

field	$SO(3) \times SO(5) \times SU(2)$
A_μ	$(3;1,1)$
ξ	$(2;4,1)$
X	$(1;5,1)$

supercharge	$SO(3) \times SO(5) \times SU(2)$
Q_γ	$(2; 4, 1)$

They are all singlet under the second $SU(2)$ of the R-symmetry group. There are 16 real states arising from the quantization of the $[i, j]$ open string with the charge vector $(p, -p)$. The charge conjugate is the $[i, j]$ $(-p, p)$, or equivalently, the $[j, i]$ $(p, -p)$ string. So, no extra states need to be added. We get the CPT self conjugate multiplet with 16 real states.

When lifted to 5d and 6d the little group and the R-symmetry group becomes $SU(2) \times SU(2) \times SU(2) \times SU(2)$, under which, the fields and the supercharges have the representation [58, 68]

field	$SU(2) \times SU(2) \times SU(2) \times SU(2)$
s_μ	$(3, 2; 1, 1)$
a_μ	$(2, 2; 1, 2) \oplus (2, 2; 2, 1)$
λ	$(1, 2; 1, 1) \oplus (1, 2; 2, 2)$
$b_{\mu\nu}$	$(3, 1; 2, 1)$
χ	$(2, 1; 1, 1) \oplus (2, 1; 3, 1) \oplus (2, 1; 2, 2)$
Φ	$(1, 1; 1, 2) \oplus (1, 1; 3, 2) \oplus (1, 1; 2, 1)$

field	$SU(2) \times SU(2) \times SU(2) \times SU(2)$
A_μ	$(3, 1; 1, 1)$
ξ	$(2, 1; 1, 2) \oplus (2, 1; 2, 1)$
X	$(1, 1; 2, 2) \oplus (1, 1; 1, 1)$

supercharge	$SU(2)\times SU(2)\times SU(2)\times SU(2)$
Q^{α_1}	$(1,2;2,1)$
Q^{α_2}	$(2,1;2,1)$
Q_β	$(2,1;1,2)$

Notice that under the second $SU(2)$ of the $SU(2)\times SU(2)$ little group, $(b_{\mu\nu},\chi,\Phi)$, (A_μ,ξ,X) and (Q^{α_2},Q_β) are all singlet, while (s_μ,a_μ,λ) and Q^{α_1} are doublet. As a result, the supersymmetry transformation generated by (Q^{α_2},Q_β) will keep $(b_{\mu\nu},\chi,\Phi)$ (s_μ,a_μ,λ) and (A_μ,ξ,X) separate, while the supersymmetry transformation generated by Q^{α_1} will transform $(b_{\mu\nu},\chi,\Phi)$ and (s_μ,a_μ,λ) into each other. $(b_{\mu\nu},\chi,\Phi)$ is the chiral sector of the (2, 1) multiplet $(b_{\mu\nu},\chi,\Phi,s_\mu,a_\mu,\lambda)$.

When the $SO(5)$ R-symmetry is recovered, the representations of the fields and the supercharges under $SU(2)\times SU(2)\times SO(5)\times SU(2)$ become

field	$SU(2)\times SU(2)\times SO(5)\times SU(2)$
s_μ	$(3,2;1,1)$
a_μ	$(2,2;4,1)$
λ	$(1,2;5,1)$
$b_{\mu\nu}$	$(3,1;1,2)$
χ	$(2,1;4,2)$
Φ	$(1,1;5,2)$

field	$SU(2)\times SU(2)\times SO(5)\times SU(2)$
A_μ	$(3,1;1,1)$
ξ	$(2,1;4,1)$
X	$(1,1;5,1)$

supercharge	$SU(2) \times SU(2) \times SO(5) \times SU(2)$
Q^{α_1}	$(1, 2; 1, 2)$
Q_γ	$(2, 1; 4, 1)$

Let $SU(2)_1$ and $SU(2)_2$ represent the first and the second $SU(2)$ of the little group respectively, $SU(2)_R$ represent the $SU(2)$ in R-symmetry group, then $(b_{\mu\nu}, \chi, \Phi)$, (s_μ, a_μ, λ) and (A_μ, ξ, X) all form the $[(3, 1), (2, 4), (1, 5)]$ representation of $SU(2)_1 \times SO(5)$. With respect to $SU(2)_2 \times SU(2)_R$, $(b_{\mu\nu}, \chi, \Phi)$, (s_μ, a_μ, λ) and (A_μ, ξ, X) are in the $(1, 2)$, $(2, 1)$ and $(1, 1)$ representations respectively. Q_γ is the standard (2, 0) supercharges in the 6d (2, 0) theory, and is in the $(2, 4)$ representation of $SU(2)_1 \times SO(5)$, $(1, 1)$ representation of $SU(2)_2 \times SU(2)_R$. Therefore, the way Q_γ acting on $(b_{\mu\nu}, \chi, \Phi)$, (s_μ, a_μ, λ) and (A_μ, ξ, X) is exactly the same as the action of the (2, 0) supercharge on the (2, 0) tensor multiplet, or the N=4 supercharge on the massive N=4 vector multiplet when reduced to 4d. $(b_{\mu\nu}, \chi, \Phi)$, (s_μ, a_μ, λ) and (A_μ, ξ, X) transform separately. Namely,

$$b_{\mu\nu} \overset{Q_\gamma}{\leftrightarrow} \chi \overset{Q_\gamma}{\leftrightarrow} \Phi, \qquad s_\mu \overset{Q_\gamma}{\leftrightarrow} a_\mu \overset{Q_\gamma}{\leftrightarrow} \lambda, \qquad A_\mu \overset{Q_\gamma}{\leftrightarrow} \xi \overset{Q_\gamma}{\leftrightarrow} X. \tag{4.2}$$

On the other hand, Q^{α_1} is in the $(1, 1)$ representation of $SU(2)_1 \times SO(5)$, $(2, 2)$ representation of $SU(2)_2 \times SU(2)_R$, so it can only make the

$$s_\mu \overset{Q^{\alpha_1}}{\leftrightarrow} b_{\mu\nu}, \qquad a_\mu \overset{Q^{\alpha_1}}{\leftrightarrow} \chi, \qquad \lambda \overset{Q^{\alpha_1}}{\leftrightarrow} \Phi \tag{4.3}$$

transformations.

4.2 The conformal anomaly of the 6d (2, 0) theory and the (2, 1) multiplet

The conformal anomaly of the 6d (2, 0) theory and the (2, 1) multiplet

We may determine the 5d and 6d Weyl weight of the (2, 1) multiplet $(b_{\mu\nu}, \chi, \Phi, s_\mu,$

$a_\mu, \lambda)$. $b_{\mu\nu}$ has the dimension 2, a_μ has the dimension 1, so if η^{α_1} and η_γ are the supersymmetry transformation parameters of Q^{α_1} and Q_γ respectively, s_μ, η^{α_1} and η_γ will have the dimension 3/2, 1/2 and −1/2 respectively. The Weyl weight of the whole multiplet is then

field	s_μ	a_μ	λ	$b_{\mu\nu}$	χ	ϕ
weight	3/2	1	3/2	2	5/2	2

On the other hand, when reduced to 4d, the (2, 1) multiplet becomes the N=6 multiplet (S, V, ζ, φ), for which, the Weyl weight is

field	S_μ	V_μ	ζ	φ
weight	3/2	1	3/2	1

In 4d, N=6 multiplet is the conformal field, however, in 6d, only the $(b_{\mu\nu}, \chi, \Phi)$ sector in the (2, 1) multiplet is conformal.

The kinetic terms for the (2, 1) multiplet in 6d can then be written as

$$\begin{aligned} S_k^6 &= \int d^6x \ [db^+ \wedge *db + \frac{i}{2}\bar{\chi}\gamma^\mu\partial_\mu\chi + \frac{1}{2}\partial^\mu\phi^+\partial_\mu\phi] \\ &+ \frac{1}{R_4^2}\,[\frac{i}{2}\bar{s}_\mu\gamma^{\mu\nu\rho}\partial_\nu s_\rho + da^+ \wedge *da + \frac{i}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda]. \end{aligned} \tag{4.4}$$

All fields are complex and are in the tri-fundamental representation of $U(N)$. The reduction along x_5 gives

$$\begin{aligned} S_k^5 &= \int d^5x \ R_5[db^+ \wedge *db + \frac{i}{2}\bar{\chi}\gamma^\mu\partial_\mu\chi + \frac{1}{2}\partial^\mu\phi^+\partial_\mu\phi] \\ &+ \frac{R_5}{R_4^2}\,[\frac{i}{2}\bar{s}_\mu\gamma^{\mu\nu\rho}\partial_\nu s_\rho + da^+ \wedge *da + \frac{i}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda]. \end{aligned} \tag{4.5}$$

Reducing along x_4 further, b, χ, ϕ will absorb R_4 to give the vector $\tilde{a}$, spinor $\tilde{\chi}$, and scalar $\tilde{\phi}$ with the dimension 1, 3/2, 1, respectively, i.e. $R_4 b \sim \tilde{a}$, $R_4\chi \sim \tilde{\chi}$, $R_4\phi \sim \tilde{\phi}$.

The 4d action is then

$$S_k^4 = \frac{R_5}{R_4}\int d^4x\ [d\tilde{a}^+\wedge *d\tilde{a}+\frac{i}{2}\bar{\tilde{\chi}}\gamma^\mu\partial_\mu\tilde{\chi}+\frac{1}{2}\partial^\mu\tilde{\phi}^+\partial_\mu\tilde{\phi}+\frac{i}{2}\bar{s}_\mu\gamma^{\mu\nu\rho}\partial_\nu s_\rho+da^+\wedge *da+\frac{i}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda] \tag{4.6}$$

with no dimensional coupling constant. (4.6) is the 4d free action for fields arising from the quantization of the 3-string junction, which is our starting point. (4.4) and (4.5) take the present form due to the requirement that the dimensional reduction to 4d should give (4.6). In the decompactification limit, $R_4, R_5 \to \infty$, the non-conformal sector (s_μ, a_μ, λ) in (4.4) vanishes since $(b_{\mu\nu}, \chi, \Phi)$ and (s_μ, a_μ, λ) are assumed to have the comparable magnitude.

$$S_k^6 = \int d^6x\ [db^+ \wedge *db + \frac{i}{2}\bar{\chi}\gamma^\mu\partial_\mu\chi + \frac{1}{2}\partial^\mu\phi^+\partial_\mu\phi]. \tag{4.7}$$

With the 6d kinetic terms given, it is straightforward to calculate the free field conformal anomaly of the $(2,1)$ multiplet. If the free action for a field f in the curved background is $f\Delta_f f$, the partition function and the effective action will then be

$$Z = \int\ df\ e^{-\frac{1}{2}\int d^6x\ f\Delta_f f}, \quad \Gamma = \frac{1}{2}\ln\det\Delta_f. \tag{4.8}$$

The heat kernel expansion is [80]

$$(\mathrm{Tr}e^{-s\Delta_f})_{s\to 0} \sim \sum_{p=0}^{6} s^{\frac{1}{2}(p-6)}\int\ d^6x\ \sqrt{g}\ b_p. \tag{4.9}$$

The logarithmical divergence comes from b_6, which is the conformal anomaly. $b_6 = b_6(\Delta_f)$ is determined by the type of the field.

Go back to (4.4), in the $R_4 \to \infty$ limit, (s_μ, a_μ, λ) do not have the contribution to the conformal anomaly. According to (4.1), $(b_{\mu\nu}, \chi, \phi)$ contains 10 scalars, 2 selfdual tensors, 8 chiral spinors, all complex and are in the tri-fundamental representation of $U(N)$, so (4.4) is just the free action for $4N^3$ copies of the $(2,0)$ tensor multiplet. The conformal anomaly of one copy of $(2,0)$ tensor multiplet is [81]

$$b_6^{tens} = \frac{1}{(4\pi)^3 7!}(-\frac{245}{8}E_6 - 1680 I_1 - 420 I_2 + 140 I_3 + \nabla_i J^i), \tag{4.10}$$

so the conformal anomaly of $(b_{\mu\nu}, \chi, \phi)$ is just

$$b_6^{(b_{\mu\nu},\chi,\phi)} = \frac{4N^3}{(4\pi)^3 7!}(-\frac{245}{8}E_6 - 1680 I_1 - 420 I_2 + 140 I_3 + \nabla_i J^i). \tag{4.11}$$

The type-D anomaly $\nabla_i J^i$ is not written explicitly, since it is scheme-dependent. AdS/CFT predicts that in large N limit, the conformal anomaly of the interacting 6d $(2,0)$ theory should be

$$A_{(2,0)} = \frac{4N^3}{(4\pi)^3 7!}(-\frac{35}{2}E_6 - 1680 I_1 - 420 I_2 + 140 I_3 + \nabla_i J^i). \tag{4.12}$$

Compared with (4.11), the coefficients in front of E_6, the type-A anomaly, are different, while the coefficients for type-B anomalies are the same. In 6d, the coefficients of the type-B anomalies are related with the 2 and 3 point functions thus is un-renormalized. On the other hand, the coefficient of the type-A anomaly is related with some 4 point functions, so its value may differ in free theory and the interacting theory [81]. The situation is different for 4d $N = 4$ SYM theory, in which, the type-B and type-A anomalies are related with the 2 and 3 point functions respectively thus are both un-renormalized. As a result, b_4 arising from N^2 copies of the free $N = 4$ vector multiplet exactly gives the conformal anomaly of the full 4d interacting theory.

The (s_μ, a_μ, λ) sector is suppressed in 6d. The $(b_{\mu\nu}, \chi, \Phi)$ sector then form the representation of the $(2,0)$ supersymmetry. Q^{α_1}, the $(0,1)$ part of $(2,1)$ can be neglected in 6d, while the remaining Q_γ, the $(2,0)$ part of the $(2,1)$, is the surviving supersymmetry in 6d, so the 6d theory is indeed chiral.

Aside from $(b_{\mu\nu}, \chi, \Phi)$ and (s_μ, a_μ, λ), in $6d$, we also have $(B_{\mu\nu}, \Psi, Y)$ in adjoint representation of $U(N)$, arising from the selfdual string. They are the 6d lift of the 5d vector multiplet (A_μ, ξ, X).

$$A_\mu \sim R_\theta B_{\mu\nu}, \qquad \xi \sim R_\theta \Psi, \qquad X \sim R_\theta Y. \tag{4.13}$$

The kinetic term for $(B_{\mu\nu}, \Psi, Y)$ is

$$s_k^6 = \int d^6x\, [dB \wedge *dB + \frac{i}{2}\bar{\Psi}\gamma^\mu \partial_\mu \Psi + \frac{1}{2}\partial^\mu Y \partial_\mu Y]. \tag{4.14}$$

The interaction terms take the form of $\bar{\chi}\gamma_I X^I \chi$, so (A_μ, ξ, X) and $(b_{\mu\nu}, \chi, \Phi)$ are of the comparable magnitude. When $R_\theta \to \infty$, $s_k^6 \to 0$. $(B_{\mu\nu}, \Psi, Y)$ do not have the contribution to the conformal anomaly.

4.3 The coupling between the N=6 multiplet and the N=4 vector multiplet

The coupling between the N=6 multiplet and the N=4 vector multiplet Now we need to consider the coupling between the N=6 or (2, 1) multiplet and the N=4 vector multiplet. The first subtlety is that the N=6 multiplet is the $N \times N \times N$ matrix, while the N=4 multiplet is the $N \times N$ matrix. It is necessary to define the multiplication between these two kinds of matrices. There are two kinds of multiplications. Let U^i_j be the unitary $N \times N$ matrix, V^i_j the adjoint $N \times N$ matrix, then

$$(Uf)^{ijk} \equiv U^i_l U^j_m U^k_n f^{lmn}, \tag{4.15}$$

$$(Vf)^{ijk} \equiv V^i_l f^{ljk} + V^j_m f^{imk} + V^k_n f^{ijn}. \tag{4.16}$$

For the 3-string junction with the charge vector (k_1, k_2, k_3), let $f^{ijk}_{(k_1,k_2,k_3)}$ represent the field in $N = 6$ multiplet, the covariant derivative can then be defined as

$$\begin{aligned} D_\mu f^{ijk}_{(k_1,k_2,k_3)} &= \partial_\mu f^{ijk}_{(k_1,k_2,k_3)} - i(A_\mu f)^{ijk}_{(k_1,k_2,k_3)} \qquad (4.17) \\ &= \partial_\mu f^{ijk}_{(k_1,k_2,k_3)} - iA^{i}_{(k_1)\mu\, l} f^{ljk}_{(k_1,k_2,k_3)} - iA^{j}_{(k_2)\mu\, m} f^{imk}_{(k_1,k_2,k_3)} - iA^{k}_{(k_3)\mu\, n} f^{ijn}_{(k_1,k_2,k_3)}, \end{aligned}$$

where $A_{(k_1)}$, $A_{(k_2)}$, $A_{(k_3)}$ are gauge fields for open strings with the charge vector k_1, k_2, k_3 respectively. Under the local $U(N)$ gauge transformation,

$$f^{ijk}_{(k_1,k_2,k_3)} \to U^i_l U^j_m U^k_n f^{lmn}_{(k_1,k_2,k_3)}, \qquad A^i_{(k_a)\mu\, j} \to U^i_m U^{+n}_j A^m_{(k_a)\mu\, n} + iU^i_l \partial_\mu U^{+l}_j, \qquad (4.18)$$

$$D_\mu f^{ijk}_{(k_1,k_2,k_3)} \to U^i_l U^j_m U^k_n [D_\mu f_{(k_1,k_2,k_3)}]^{lmn}. \qquad (4.19)$$

For K, L, M in anti-tri-fundamental, adjoint and tri-fundamental representation of $U(N)$,

$$LM \equiv (LM)^{abc} = L^a_p M^{pbc} + L^b_q M^{aqc} + L^c_r M^{abr}, \qquad (4.20)$$

$$LK \equiv (LK)_{abc} = L^p_a K_{pbc} + L^q_b K_{aqc} + L^r_c K_{abr}, \qquad (4.21)$$

$$\begin{aligned} K(LM) &= K_{abc}(LM)^{abc} = K_{abc} L^a_p M^{pbc} + K_{abc} L^b_q M^{aqc} + K_{abc} L^c_r M^{abr} \qquad (4.22) \\ &= (LK)_{abc} M^{abc} = (KL)M = KLM. \end{aligned}$$

$$L_1(L_2 M) - L_2(L_1 M) = [L_1, L_2]M, \qquad (4.23)$$

$$L_1(L_2 K) - L_2(L_1 K) = [L_1, L_2]K. \qquad (4.24)$$

As a result,

$$D_\mu(LM) = (D_\mu L)M + L(D_\mu M), \qquad (4.25)$$

where $D_\mu L = \partial_\mu L - i[A_\mu, L]$ is the ordinary covariant derivative for fields in adjoint representation.

$$\begin{aligned} & [D_\mu, D_\nu] f^{ijk}_{(k_1,k_2,k_3)} \\ = & -\frac{i}{3}[F^i_{(k_1)\mu\nu\ l} f^{ljk}_{(k_1,k_2,k_3)} + F^j_{(k_2)\mu\nu\ m} f^{imk}_{(k_1,k_2,k_3)} + F^k_{(k_3)\mu\nu\ n} f^{ijn}_{(k_1,k_2,k_3)}] \\ = & -\frac{i}{3}(F_{\mu\nu} f)^{ijk}_{(k_1,k_2,k_3)}, \end{aligned} \tag{4.26}$$

where $F_{(k_a)\mu\nu}$ is the field strength of $A_{(k_a)\mu}$. Although $L_1(L_2M) \neq (L_1L_2)M$, fields in tri-fundamental representation have the enough algebra properties so that during the calculation, we can simply take them as the fields in fundamental representation.

To see it more explicitly, consider the coupling between the N=1 hypermultiplet in tri-fundamental representation and the N=1 vector multiplet in adjoint representation. The superfield for hypermultiplet is Φ^{ijk}, while the superfield for vector multiplet is V^i_j. Under the local $U(N)$ gauge transformation,

$$\Phi^{lmn} \to (e^{-i\Lambda})^l_p (e^{-i\Lambda})^m_q (e^{-i\Lambda})^n_r \Phi^{pqr}, \quad \Phi^+_{lmn} \to (e^{i\Lambda^+})^p_l (e^{i\Lambda^+})^q_m (e^{i\Lambda^+})^r_n \Phi^+_{pqr}, \tag{4.27}$$

$$(e^V)^a_b \to (e^{-i\Lambda^+})^a_c (e^V)^c_d (e^{i\Lambda})^d_b, \tag{4.28}$$

where Λ and Λ^+ are chiral and anti-chiral superfield in adjoint representation respectively.

$$\Phi^+ e^V \Phi = \Phi^+_{ijk} (e^V)^i_l (e^V)^j_m (e^V)^k_n \Phi^{lmn} \to \Phi^+ e^V \Phi \tag{4.29}$$

is then gauge invariant. In WZ gauge,

$$e^V = 1 + V + \frac{1}{2} V^2. \tag{4.30}$$

Plug (4.30) into $\Phi^+ e^V \Phi$, the D-term is

$$\Phi^+ e^V \Phi|_{\theta\theta\bar{\theta}\bar{\theta}} = [\Phi^+\Phi + \Phi^+ V\Phi + \frac{1}{2}\Phi^+ VV\Phi]|_{\theta\theta\bar{\theta}\bar{\theta}}. \tag{4.31}$$

This is almost the same as the situation when Φ is in fundamental representation. The only difference is that the way V acting on Φ is as that in (4.16). Also,

$$\Phi^+ VV\Phi = \Phi^+ V(V\Phi) = (\Phi^+ V)(V\Phi) \neq \Phi^+ V^2 \Phi. \tag{4.32}$$

Now, we may try to write down the interacting Lagrangian for the N=6 multiplet. For simplicity, we will consider the case when the $SO(5)$ symmetry is recovered, that is, the R-symmetry group is $SO(5)\times SU(2)$. N=6 multiplet is the 4d KK mode of the 5d or 6d (2, 1) multiplet. The reduction of $(b_{\mu\nu}, \chi, \Phi)$ gives $(a_\mu^{\alpha_1}, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$, while the reduction of (s_μ, a_μ, λ) gives $(s_\mu, a_\mu^\gamma, \lambda^I, \psi, \varphi^\gamma)$. For the sake of the explicitness, the R-symmetry index is also added. $\alpha_1 = 1, 2$, $\gamma = 1, 2, 3, 4$, $I = 1, 2, 3, 4, 5$. The N=4 vector massive multiplet is then (A_μ, ξ^γ, X^I).

field	$SO(3)\times SO(5)\times SU(2)$
A_μ	$(3; 1, 1)$
ξ^γ	$(2; 4, 1)$
X^I	$(1; 5, 1)$

field	$SO(3)\times SO(5)\times SU(2)$
$a_\mu^{\alpha_1}$	$(3; 1, 2)$
$\chi^{\gamma\,\alpha_1}$	$(2; 4, 2)$
$\Phi^{I\,\alpha_1}$	$(1; 5, 2)$

field	$SO(3)\times SO(5)\times SU(2)$
s_μ	$(4;1,1)$
a_μ^γ	$(3;4,1)$
λ^I	$(2;5,1)$
ψ	$(2;1,1)$
φ^γ	$(1;4,1)$

Under the reduction, $s_\mu \to s_\mu + \psi$, $a_\mu \to a_\mu^\gamma + \varphi^\gamma$.

The massive fields $(a_\mu^{\alpha_1}, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$ in $4d$ are the KK modes of the $6d$ massless fields $(\tilde{b}_{mn}^{\alpha_1}, \tilde{\chi}^{\gamma\,\alpha_1}, \tilde{\Phi}^{I\,\alpha_1})$, where $m = 0, \cdots, 5$. The $6d$ kinetic terms for $(\tilde{b}_{mn}^{\alpha_1}, \tilde{\chi}^{\gamma\,\alpha_1}, \tilde{\Phi}^{I\,\alpha_1})$ can be written as

$$\tilde{L}_k = \frac{1}{24}\tilde{h}_{lmn}^{\alpha_1 +}\tilde{h}^{lmn\,\alpha_1} + \frac{i}{2}\bar{\tilde{\chi}}^{\gamma\,\alpha_1}\gamma^m\partial_m\tilde{\chi}^{\gamma\,\alpha_1} + \frac{1}{2}\partial^m\tilde{\Phi}^{I\,\alpha_1 +}\partial_m\tilde{\Phi}^{I\,\alpha_1}, \tag{4.33}$$

where $\tilde{h}_{lmn}^{\alpha_1} = 3\partial_{[l}\tilde{b}_{mn]}^{\alpha_1}$. (4.33) is invariant under the supersymmetry transformation

$$\begin{aligned}
\delta\tilde{b}_{mn}^{\alpha_1} &= i\bar{\eta}^\gamma\gamma_{mn}\tilde{\chi}^{\gamma\,\alpha_1}, \\
\delta\tilde{\chi}^{\gamma\,\alpha_1} &= \frac{1}{12}\gamma^{lmn}\tilde{h}_{lmn}^{\alpha_1}\eta^\gamma + \gamma_m\gamma_I\eta^\gamma\partial^m\tilde{\Phi}^{I\,\alpha_1}, \\
\delta\tilde{\Phi}^{I\,\alpha_1} &= -i\bar{\eta}^\gamma\gamma^I\tilde{\chi}^{\gamma\,\alpha_1}.
\end{aligned} \tag{4.34}$$

(4.34) is closed up to a gauge transformation for $\tilde{b}_{mn}^{\alpha_1}$.

$$\delta_{[2}\delta_{1]}\tilde{b}_{mn}^{\alpha_1} = \cdots + 2i\bar{\eta}_2^\gamma(\gamma_n\partial_m\tilde{\Phi}^{I\,\alpha_1} - \gamma_m\partial_n\tilde{\Phi}^{I\,\alpha_1})\gamma_I\eta_1^\gamma. \tag{4.35}$$

The compactification on x^5 gives the $5d$ KK modes with the momentum p_5.

$\partial_5 f = ip_5 f$, so (4.33) can be rewritten as

$$\tilde{L}_k = \frac{1}{24} h^{\alpha_1 +}_{ijk} h^{ijk\,\alpha_1} + \frac{p_5^2}{8} b^{ij\,\alpha_1 +} b^{\alpha_1}_{ij} + \frac{i}{2} \bar{\tilde{\chi}}^{\gamma\,\alpha_1} \gamma^i \partial_i \tilde{\chi}^{\gamma\,\alpha_1} - \frac{p_5}{2} \bar{\tilde{\chi}}^{\gamma\,\alpha_1} \gamma^5 \tilde{\chi}^{\gamma\,\alpha_1} + \frac{1}{2} \partial^i \tilde{\Phi}^{I\,\alpha_1 +} \partial_i \tilde{\Phi}^{I\,\alpha_1} + \frac{p_5^2}{2} \tilde{\Phi}^{I\,\alpha_1 +} \tilde{\Phi}^{I\,\alpha_1}, \tag{4.36}$$

where

$$b^{\alpha_1}_{ij} = \tilde{b}^{\alpha_1}_{ij} + \frac{i}{p_5} \partial_i \tilde{b}^{\alpha_1}_{5j} - \frac{i}{p_5} \partial_j \tilde{b}^{\alpha_1}_{5i} = -\frac{i}{p_5} \tilde{h}^{\alpha_1}_{5ij} \tag{4.37}$$

is gauge invariant. $i, j, k = 0, \cdots, 4$, $h^{\alpha_1}_{ijk} = 3\partial_{[i} b^{\alpha_1}_{jk]}$. The selfduality condition becomes

$$-6ip_5 b^{\alpha_1}_{ij} = \epsilon_{ijklm} h^{klm\,\alpha_1}, \tag{4.38}$$

or equivalently

$$-2ip_5 b^{\alpha_1}_{ij} = \epsilon_{ijklm} \partial^k b^{lm\,\alpha_1}, \tag{4.39}$$

which indicates the equation

$$\partial^i h^{\alpha_1}_{ijk} - p_5^2 b^{\alpha_1}_{jk} = 0. \tag{4.40}$$

$\partial^j b^{\alpha_1}_{jk} = 0$, so (4.40) is equivalent to

$$\partial^2 b^{\alpha_1}_{jk} - p_5^2 b^{\alpha_1}_{jk} = 0. \tag{4.41}$$

(4.36) can be corrected into

$$\tilde{L}_k = \frac{1}{8} \partial^i b^{jk\,\alpha_1 +} h^{\alpha_1}_{ijk} + \frac{p_5^2}{8} b^{\alpha_1 +}_{ij} b^{ij\,\alpha_1} + \frac{i}{2} \bar{\tilde{\chi}}^{\gamma\,\alpha_1} \gamma^i \partial_i \tilde{\chi}^{\gamma\,\alpha_1} - \frac{p_5}{2} \bar{\tilde{\chi}}^{\gamma\,\alpha_1} \gamma^5 \tilde{\chi}^{\gamma\,\alpha_1} + \frac{1}{2} \partial^i \tilde{\Phi}^{I\,\alpha_1 +} \partial_i \tilde{\Phi}^{I\,\alpha_1} + \frac{p_5^2}{2} \tilde{\Phi}^{I\,\alpha_1 +} \tilde{\Phi}^{I\,\alpha_1}. \tag{4.42}$$

The supersymmetry transformation becomes

$$\begin{aligned}
\delta b_{ij}^{\alpha_1} &= i\bar{\eta}^{\gamma}\gamma_{ij}\tilde{\chi}^{\gamma\,\alpha_1} - \frac{1}{p_5}\bar{\eta}^{\gamma}\gamma_5(\gamma_j\partial_i\tilde{\chi}^{\gamma\,\alpha_1} - \gamma_i\partial_j\tilde{\chi}^{\gamma\,\alpha_1}), \\
\delta\tilde{\chi}^{\gamma\,\alpha_1} &= \frac{1}{12}\gamma^{ijk}h_{ijk}^{\alpha_1}\eta^{\gamma} + \frac{ip_5}{4}\gamma^5\gamma^{jk}b_{jk}^{\alpha_1}\eta^{\gamma} + \gamma_i\gamma_I\eta^{\gamma}\partial^i\tilde{\Phi}^{I\,\alpha_1} + ip_5\gamma_5\gamma_I\eta^{\gamma}\tilde{\Phi}^{I\,\alpha_1}, \\
\delta\tilde{\Phi}^{I\,\alpha_1} &= -i\bar{\eta}^{\gamma}\gamma^I\tilde{\chi}^{\gamma\,\alpha_1}.
\end{aligned} \tag{4.43}$$

(4.43) is closed and also leaves (4.42) invariant. Also, the supersymmetry transformation does not does not give $b_{ij}^{\alpha_1}$ a gauge transformation any more.

The further compactification on x^4 gives the $4d$ KK modes with the momentum (p_4, p_5), $\partial_4 f = ip_4 f$, $\partial_5 f = ip_5 f$. From (4.38), $b_{\mu\nu}^{\alpha_1}$ can be solved in terms of $b_{4\mu}^{\alpha_1}$, i.e.

$$b_{\mu\nu}^{\alpha_1} = \frac{i}{p_5^2 - p_4^2}[p^4(\partial_\mu b_{4\nu}^{\alpha_1} - \partial_\nu b_{4\mu}^{\alpha_1}) - \frac{p^5}{2}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho b^{4\sigma\,\alpha_1} - \partial^\sigma b^{4\rho\,\alpha_1})], \tag{4.44}$$

where $\mu, \nu, \rho, \sigma = 0, \cdots, 3$. So, it is enough to consider the action with $b_{4\mu}^{\alpha_1}$. Let

$$R_4 b_{4\mu}^{\alpha_1} = a_\mu^{\alpha_1}, \quad R_4\tilde{\chi}^{\gamma\,\alpha_1} = \chi^{\gamma\,\alpha_1}, \quad R_4\tilde{\Phi}^{I\,\alpha_1} = \Phi^{I\,\alpha_1}, \tag{4.45}$$

consider the $4d$ action

$$\begin{aligned}
R_4^2 L_k^1 \;\; = \;\; & -\frac{1}{4}a^{\mu\nu\,\alpha_1 +}a_{\mu\nu}^{\alpha_1} - \frac{p_4^2 + p_5^2}{2}a_\mu^{\alpha_1 +}a^{\mu\,\alpha_1} \\
& - \frac{i}{2}\bar{\chi}^{\gamma\,\alpha_1}\gamma^\mu\partial_\mu\chi^{\gamma\,\alpha_1} + \frac{1}{2}\bar{\chi}^{\gamma\,\alpha_1}(p_4\gamma^4 + p_5\gamma^5)\chi^{\gamma\,\alpha_1} \\
& - \frac{1}{2}\partial^\mu\Phi^{I\,\alpha_1 +}\partial_\mu\Phi^{I\,\alpha_1} - \frac{p_4^2 + p_5^2}{2}\Phi^{I\,\alpha_1 +}\Phi^{I\,\alpha_1},
\end{aligned} \tag{4.46}$$

and

$$\delta a_\mu^{\alpha_1} = i\bar{\eta}^{\gamma}[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2 + p_5^2)^{1/2}}]\gamma_\mu\chi^{\gamma\,\alpha_1} + \frac{1}{(p_4^2 + p_5^2)^{1/2}}\bar{\eta}^{\gamma}\gamma_5\gamma_4\partial_\mu\chi^{\gamma\,\alpha_1}, \tag{4.47}$$

$$\delta\chi^{\gamma\,\alpha_1} = \frac{1}{2}[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma^{\mu\nu}\eta^\gamma a^{\alpha_1}_{\mu\nu} - i(p_4^2+p_5^2)^{1/2}\gamma^5\gamma^4\gamma^\mu a^{\alpha_1}_\mu\eta^\gamma$$
$$+ \gamma_\mu\gamma_I\eta^\gamma\partial^\mu\Phi^{I\,\alpha_1} + i(p_4\gamma^4+p_5\gamma^5)\gamma_I\eta^\gamma\Phi^{I\,\alpha_1},$$
$$\delta\Phi^{I\,\alpha_1} = -i\bar{\eta}^\gamma\gamma^I\chi^{\gamma\,\alpha_1}.$$

(4.46) is invariant under the supersymmetry transformation (4.47), which is the (2, 0) supersymmetry transformation generated by Q_γ. (4.47) is closed. $[\delta_1,\delta_2]$ does not generate the gauge transformation for $a^{\alpha_1}_\mu$, which is massive.

(4.46) is the free kinetic term for $(a^{\alpha_1}_\mu, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$. Similarly, the free kinetic term for $(s_\mu, a^\gamma_\mu, \lambda^I, \psi, \varphi^\gamma)$ can be taken as

$$L^2_k = -\frac{i}{2}\bar{s}_\mu\gamma^{\mu\nu\rho}\partial_\nu s_\rho - \frac{1}{4}a^{\mu\nu\,\gamma+}a^\gamma_{\mu\nu} - \frac{i}{2}\bar{\lambda}^I\gamma^\mu\partial_\mu\lambda^I - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi$$
$$- \frac{1}{2}\partial^\mu\varphi^{\gamma+}\partial_\mu\varphi^\gamma - \frac{1}{2}\bar{s}_\mu\gamma^{\mu\nu}(p_4\gamma^4+p_5\gamma^5)s_\nu - \frac{p_4^2+p_5^2}{2}a^{\gamma+}_\mu a^{\mu\,\gamma}$$
$$+ \frac{1}{2}\bar{\lambda}^I(p_4\gamma^4+p_5\gamma^5)\lambda^I + \frac{1}{2}\bar{\psi}(p_4\gamma^4+p_5\gamma^5)\psi - \frac{p_4^2+p_5^2}{2}\varphi^{\gamma+}\varphi^\gamma. \quad (4.48)$$

Under the (2, 0) supersymmetry transformation generated by Q_γ,

$$\delta s_\mu = \frac{i}{4}(\gamma^{\nu\rho}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\nu\rho})a^\gamma_{\nu\rho}\eta^\gamma - \frac{1}{6}\partial_\mu(4i\gamma^\nu a^\gamma_\nu\eta^\gamma \quad (4.49)$$
$$+ \frac{p_4\gamma^4+p_5\gamma^5}{p_4^2+p_5^2}\gamma^{\nu\rho}a^\gamma_{\nu\rho}\eta^\gamma) - \frac{2(p^4\gamma_4+p^5\gamma_5)}{3}(a^\gamma_\mu\eta^\gamma - \frac{1}{2}\gamma_{\mu\nu}a^{\nu\,\gamma}\eta^\gamma),$$
$$\delta a^\gamma_\mu = \bar{\eta}^\gamma s_\mu - i\bar{\eta}^\gamma[\frac{p^5\gamma_4-p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma_\mu\gamma_I\lambda^I - \frac{1}{(p_4^2+p_5^2)^{1/2}}\bar{\eta}^\gamma\gamma_5\gamma_4\gamma_I\partial_\mu\lambda^I,$$
$$\delta\lambda^I = \frac{1}{2}\gamma^I[\frac{p^5\gamma_4-p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma^{\mu\nu}\eta^\gamma a^\gamma_{\mu\nu} - i(p_4^2+p_5^2)^{1/2}\gamma^I\gamma^5\gamma^4\gamma^\mu a^\gamma_\mu\eta^\gamma,$$
$$\delta\psi = -i\gamma_\mu\eta^\gamma\partial^\mu\varphi^\gamma + (p^4\gamma_4+p^5\gamma_5)\eta^\gamma\varphi^\gamma,$$
$$\delta\varphi^\gamma = \bar{\eta}^\gamma\psi.$$

(4.48) is invariant.

The (0, 1) supersymmetry transformation generated by Q^{α_1} is

$$\begin{aligned}
\delta s_\mu &= \frac{i}{4}(\gamma^{\nu\rho}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\nu\rho})a^{\alpha_1}_{\nu\rho}\eta^{\alpha_1} - \frac{1}{6}\partial_\mu(4i\gamma^\nu a^{\alpha_1}_\nu\eta^{\alpha_1} \\
&\quad + \frac{p_5\gamma^4 + p_5\gamma^5}{p_4^2+p_5^2}\gamma^{\nu\rho}a^{\alpha_1}_{\nu\rho}\eta^{\alpha_1}) + \frac{2(p^4\gamma_4 + p^5\gamma_5)}{3}(a^{\alpha_1}_\mu\eta^{\alpha_1} - \frac{1}{2}\gamma_{\mu\nu}a^{\nu\,\alpha_1}\eta^{\alpha_1}), \\
\delta a^{\alpha_1}_\mu &= \bar\eta^{\alpha_1}s_\mu + i\bar\eta^{\alpha_1}[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma_\mu\psi - \frac{1}{(p_4^2+p_5^2)^{1/2}}\bar\eta^{\alpha_1}\gamma_5\gamma_4\partial_\mu\psi, \\
\delta\chi^{\gamma\,\alpha_1} &= \frac{1}{2}[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma^{\mu\nu}\eta^{\alpha_1}a^\gamma_{\mu\nu} - i(p_4^2+p_5^2)^{1/2}\gamma^5\gamma^4\gamma^\mu a^\gamma_\mu\eta^{\alpha_1} \\
&\quad - i\gamma_\mu\eta^{\alpha_1}\partial^\mu\varphi^\gamma + (p^4\gamma_4 + p^5\gamma_5)\eta^{\alpha_1}\varphi^\gamma, \\
\delta\Phi^{I\,\alpha_1} &= \bar\eta^{\alpha_1}\lambda^I, \\
\delta a^\gamma_\mu &= i\bar\eta^{\alpha_1}[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma_\mu\chi^{\gamma\,\alpha_1} - \frac{1}{(p_4^2+p_5^2)^{1/2}}\bar\eta^{\alpha_1}\gamma_5\gamma_4\partial_\mu\chi^{\gamma\,\alpha_1}, \\
\delta\lambda^I &= \gamma_\mu\eta^{\alpha_1}\partial^\mu\Phi^{I\,\alpha_1} + i(p^4\gamma_4 + p_5\gamma_5)\eta^{\alpha_1}\Phi^{I\,\alpha_1}, \\
\delta\psi &= \frac{1}{2}[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma^{\mu\nu}\eta^{\alpha_1}a^{\alpha_1}_{\mu\nu} - i(p_4^2+p_5^2)^{1/2}\gamma^5\gamma^4\gamma^\mu a^{\alpha_1}_\mu\eta^{\alpha_1}, \\
\delta\varphi^\gamma &= \bar\eta^{\alpha_1}\chi^{\gamma\,\alpha_1}.
\end{aligned} \tag{4.50}$$

The interaction is mediated by the N=4 multiplet (A_μ, ξ^γ, X^I). (A_μ, ξ^γ, X^I) is the singlet of the $SU(2)_R$, so the possible couplings are

$$(s_\mu, a^\gamma_\mu, \lambda^I, \psi, \varphi^\gamma) - (A_\mu, \xi^\gamma, X^I) \quad \text{and} \quad (a^{\alpha_1}_\mu, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1}) - (A_\mu, \xi^\gamma, X^I). \tag{4.51}$$

(s_μ, a_μ, λ) is suppressed in $6d$, therefore, the $4d$ fields of interest are $(a^{\alpha_1}_\mu, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$ and (A_μ, ξ^γ, X^I), while the relevant $4d$ coupling is $(a^{\alpha_1}_\mu, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1}) - (A_\mu, \xi^\gamma, X^I)$.

With ∂_μ replaced by D_μ, the kinetic terms become

$$\begin{aligned}
L_k &= -\frac{1}{4}a^{\mu\nu\,\alpha_1+}a^{\alpha_1}_{\mu\nu} - \frac{p_4^2+p_5^2}{2}a^{\alpha_1+}_\mu a^{\mu\,\alpha_1} \\
&\quad - \frac{i}{2}\bar\chi^{\gamma\,\alpha_1}\gamma^\mu D_\mu\chi^{\gamma\,\alpha_1} + \frac{1}{2}\bar\chi^{\gamma\,\alpha_1}(p_4\gamma^4 + p_5\gamma^5)\chi^{\gamma\,\alpha_1}
\end{aligned} \tag{4.52}$$

$$-\frac{1}{2}D^\mu\Phi^{I\,\alpha_1+}D_\mu\Phi^{I\,\alpha_1}-\frac{p_4^2+p_5^2}{2}\Phi^{I\,\alpha_1+}\Phi^{I\,\alpha_1},$$

We may add the interaction terms

$$\begin{aligned} L_i \;=\; & \frac{1}{6}\bar{\chi}^{\gamma\,\alpha_1}\gamma^I\xi^\gamma\Phi^{I\,\alpha_1}-\frac{1}{6}\bar{\chi}^{\gamma\,\alpha_1}[\frac{p^5\gamma_4-p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma^\mu\xi^\gamma a_\mu^{\alpha_1} \\ & -\frac{i}{6}a^{\mu\,\alpha_1+}F_{\mu\nu}a^{\nu\,\alpha_1}-\frac{1}{2}\bar{\chi}^{\gamma\,\alpha_1}\gamma_I X^I\chi^{\gamma\,\alpha_1}-\frac{1}{2}a_\mu^{\alpha_1+}X^I X_I a^{\mu\,\alpha_1} \\ & -\Phi^{I\,\alpha_1+}[X_I,X_J]\Phi^{J\,\alpha_1}-\frac{1}{2}\Phi^{I\,\alpha_1+}X^J X_J\Phi_I^{\alpha_1}. \end{aligned} \tag{4.53}$$

The supersymmetry transformation of $(a_\mu^{\alpha_1},\chi^{\gamma\,\alpha_1},\Phi^{I\,\alpha_1})$ is then modified into

$$\begin{aligned} \delta a_\mu^{\alpha_1} &= i\bar{\eta}^\gamma[\frac{p^5\gamma_4-p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma_\mu\chi^{\gamma\,\alpha_1}+\frac{1}{(p_4^2+p_5^2)^{1/2}}\bar{\eta}^\gamma\gamma_5\gamma_4 D_\mu\chi^{\gamma\,\alpha_1}, \\ \delta\chi^{\gamma\,\alpha_1} &= \frac{1}{2}[\frac{p^5\gamma_4-p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma^{\mu\nu}\eta^\gamma a_{\mu\nu}^{\alpha_1}-i(p_4^2+p_5^2)^{1/2}\gamma^5\gamma^4\gamma^\mu a_\mu^{\alpha_1}\eta^\gamma \\ & \quad -i[\frac{p^5\gamma_4-p^4\gamma_5}{(p_4^2+p_5^2)^{1/2}}]\gamma_I\gamma^\mu\eta^\gamma X^I a_\mu^{\alpha_1} \\ & \quad +\gamma_\mu\gamma_I\eta^\gamma D^\mu\Phi^{I\,\alpha_1}+i(p^4\gamma_4+p^5\gamma_5)\gamma_I\eta^\gamma\Phi^{I\,\alpha_1}-i\gamma^I\gamma^J\eta^\gamma X_I\Phi_J^{\alpha_1}, \\ \delta\Phi^{I\,\alpha_1} &= -i\bar{\eta}^\gamma\gamma^I\chi^{\gamma\,\alpha_1}. \end{aligned} \tag{4.54}$$

For (A_μ,ξ^γ,X^I), we have

$$\begin{aligned} \delta A_\mu &= -\frac{i}{3}\bar{\eta}^\gamma\gamma_\mu\xi^\gamma \\ \delta\xi^\gamma &= \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\eta^\gamma+3D_\mu X^I\gamma^\mu\gamma_I\eta^\gamma-3iX^IX^J\gamma_{IJ}\eta^\gamma, \\ \delta X^I &= -\frac{i}{3}\bar{\eta}^\gamma\gamma^I\xi^\gamma. \end{aligned} \tag{4.55}$$

$L=L_k+L_i$ is not completely invariant under the supersymmetry transformation (4.54) and (4.55). Some additional terms are to be added to make the susy completion.

If we are only interested in the $6d$ dynamics, then this is all we need to consider, since (s_μ, a_μ, λ) is suppressed in $6d$, $(s_\mu, a^\gamma_\mu, \lambda^I, \psi, \varphi^\gamma)$ and $(a^{\alpha_1}_\mu, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$ are decoupled. Nevertheless, in $4d$, we can still write down an interacting Lagrangian describing the coupling between $(s_\mu, a^\gamma_\mu, \lambda^I, \psi, \varphi^\gamma)$ and (A_μ, ξ^γ, X^I). With ∂_μ replaced by D_μ, the kinetic terms (4.48) becomes

$$
\begin{aligned}
L^2_k &= \frac{i}{2}\bar{s}_\mu\gamma^{\mu\nu\rho}D_\nu s_\rho - \frac{1}{4}a^{\mu\nu\,\gamma+}a^\gamma_{\mu\nu} + \frac{i}{2}\bar{\lambda}^I\gamma^\mu D_\mu\lambda^I + \frac{i}{2}\bar{\psi}\gamma^\mu D_\mu\psi + \frac{1}{2}D^\mu\varphi^{\gamma+}D_\mu\varphi^\gamma \\
&\quad + \frac{1}{2}\bar{s}_\mu\gamma^{\mu\nu}(p_4\gamma^4 + p_5\gamma^5)s_\nu + \frac{p_4^2 + p_5^2}{2}a^{\gamma+}_\mu a^{\mu\,\gamma} \\
&\quad - \frac{1}{2}\bar{\lambda}^I(p_4\gamma^4 + p_5\gamma^5)\lambda^I - \frac{1}{2}\bar{\psi}(p_4\gamma^4 + p_5\gamma^5)\psi + \frac{p_4^2 + p_5^2}{2}\varphi^{\gamma+}\varphi^\gamma,
\end{aligned}
$$

with the interaction

$$
\begin{aligned}
L^2_i &= \frac{i}{6}\bar{s}_\mu\gamma^{\mu\nu}\xi^\gamma a^\gamma_\nu + \frac{i}{6}\bar{\psi}\xi^\gamma\varphi^\gamma + \frac{1}{6}\bar{\lambda}^I\gamma_I[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2 + p_5^2)^{1/2}}]\gamma^\mu\xi^\gamma a^\gamma_\mu - \frac{i}{6}a^{\mu\,\gamma+}F_{\mu\nu}a^{\nu\,\gamma} \\
&\quad + \frac{1}{2}\bar{s}_\mu\gamma^{\mu\nu}\gamma_I X^I s_\nu + \frac{1}{2}\bar{\psi}\gamma_I X^I\psi + \frac{1}{2}\bar{\lambda}^I\gamma_J X^J\lambda^I \\
&\quad - \frac{1}{2}a^{\gamma+}_\mu X^I X_I a^{\mu\,\gamma} + \frac{1}{2}\varphi^{\gamma+}X_I X^I\varphi^\gamma.
\end{aligned}
$$

The $(2,0)$ supersymmetry transformation is corrected into

$$
\begin{aligned}
\delta s_\mu &= \frac{i}{4}(\gamma^{\nu\rho}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\nu\rho})a^\gamma_{\nu\rho}\eta^\gamma - \frac{1}{6}D_\mu(4i\gamma^\nu a^\gamma_\nu\eta^\gamma + \frac{p_5\gamma^4 + p_5\gamma^5}{p_4^2 + p_5^2}\gamma^{\nu\rho}a^\gamma_{\nu\rho}\eta^\gamma) \\
&\quad + \frac{2(p^4\gamma_4 + p^5\gamma_5)}{3}(a^\gamma_\mu\eta^\gamma - \frac{1}{2}\gamma_{\mu\nu}a^{\nu\,\gamma}\eta^\gamma) + \gamma^I X_I\eta^\gamma a^\gamma_\mu, \\
\delta a^\gamma_\mu &= \bar{\eta}^\gamma s_\mu - i\bar{\eta}^\gamma[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2 + p_5^2)^{1/2}}]\gamma_\mu\gamma_I\lambda^I + \frac{1}{(p_4^2 + p_5^2)^{1/2}}\bar{\eta}^\gamma\gamma_5\gamma_4\gamma_I D_\mu\lambda^I, \\
\delta\lambda^I &= \frac{1}{2}\gamma^I[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2 + p_5^2)^{1/2}}]\gamma^{\mu\nu}\eta^\gamma a^\gamma_{\mu\nu} - i(p_4^2 + p_5^2)^{1/2}\gamma^I\gamma^5\gamma^4\gamma^\mu a^\gamma_\mu\eta^\gamma \\
&\quad + i\gamma^{IJ}[\frac{p^5\gamma_4 - p^4\gamma_5}{(p_4^2 + p_5^2)^{1/2}}]\gamma^\mu\eta^\gamma X_J a^{\alpha_1}_\mu, \\
\delta\psi &= -i\gamma_\mu\eta^\gamma D^\mu\varphi^\gamma + (p^4\gamma_4 + p^5\gamma_5)\eta^\gamma\varphi^\gamma - \gamma^I X_I\eta^\gamma\varphi^\gamma,
\end{aligned}
$$

$$\delta\varphi^{\gamma} = \bar{\eta}^{\gamma}\psi.$$

Still, both $L^2 = L_k^2 + L_i^2$ and the above transformation are to be completed.

Up to present, we only considered the Lagrangian for $(s_\mu, a_\mu^\gamma, \lambda^I)$, $(b_{\mu\nu}^{\alpha_1}, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$, (A_μ, ξ^γ, X^I), when the $SO(5)$ R-symmetry is recovered. If the R-symmetry is $SO(4)$, the $6d/5d$ field content are $(s_\mu, a_{\mu\,\beta}, a_\mu^{\alpha_2}, \psi, \psi_\beta^{\alpha_2})$ and $(b_{\mu\nu}^{\alpha_1}, \psi^{\alpha_1,\alpha_2}, \psi_\beta^{\alpha_1}, \Phi_\beta^{\alpha_1,\alpha_2}, \Phi^{\alpha_1})$, which, when compactified to $4d$, becomes

$$(s_\mu, a_{\mu\,\beta}, a_\mu^{\alpha_2}, \psi, \psi_\beta^{\alpha_2}) \longrightarrow (s_\mu, \rho, a_{\mu\,\beta}, a_\mu^{\alpha_2}, \varphi_\beta, \varphi^{\alpha_2}, \psi, \psi_\beta^{\alpha_2}), \tag{4.56}$$

$$(b_{\mu\nu}^{\alpha_1}, \psi^{\alpha_1,\alpha_2}, \psi_\beta^{\alpha_1}, \Phi_\beta^{\alpha_1,\alpha_2}, \Phi^{\alpha_1}) \longrightarrow (a_\mu^{\alpha_1}, \psi^{\alpha_1,\alpha_2}, \psi_\beta^{\alpha_1}, \Phi_\beta^{\alpha_1,\alpha_2}, \Phi^{\alpha_1}). \tag{4.57}$$

$\alpha_1, \alpha_2, \beta = 1, 2$. (A_μ, ξ^γ, X^I) could be written as $(A_\mu, \xi^{\alpha_2}, \xi_\beta, X_\beta^{\alpha_2}, X)$.

L_k and L_k^2 then become

$$\begin{aligned} L_k \;=\; & -\frac{1}{4}a^{\mu\nu\,\alpha_1+}a_{\mu\nu}^{\alpha_1} + \frac{p_4^2+p_5^2}{2}a_\mu^{\alpha_1+}a^{\mu\,\alpha_1} \\ & + \frac{i}{2}\bar{\psi}^{\alpha_1,\alpha_2}\gamma^\mu D_\mu\psi^{\alpha_1,\alpha_2} - \frac{1}{2}\bar{\psi}^{\alpha_1,\alpha_2}(p_4\gamma^4 + p_5\gamma^5)\psi^{\alpha_1,\alpha_2} \\ & + \frac{i}{2}\bar{\psi}_\beta^{\alpha_1}\gamma^\mu D_\mu\psi_\beta^{\alpha_1} - \frac{1}{2}\bar{\psi}_\beta^{\alpha_1}(p_4\gamma^4 + p_5\gamma^5)\psi_\beta^{\alpha_1} \\ & + \frac{1}{2}D^\mu\Phi_\beta^{\alpha_1,\alpha_2+}D_\mu\Phi_\beta^{\alpha_1,\alpha_2} + \frac{p_4^2+p_5^2}{2}\Phi_\beta^{\alpha_1,\alpha_2+}\Phi_\beta^{\alpha_1,\alpha_2} \\ & + \frac{1}{2}D^\mu\Phi^{\alpha_1+}D_\mu\Phi^{\alpha_1} + \frac{p_4^2+p_5^2}{2}\Phi^{\alpha_1+}\Phi^{\alpha_1}, \end{aligned} \tag{4.58}$$

$$\begin{aligned} L_k^2 \;=\; & \frac{i}{2}\bar{s}_\mu\gamma^{\mu\nu\rho}D_\nu s_\rho - \frac{1}{4}a^{\mu\nu\,\alpha_2+}a_{\mu\nu}^{\alpha_2} - \frac{1}{4}a_\beta^{\mu\nu+}a_{\mu\nu\,\beta} \\ & + \frac{i}{2}\bar{\psi}_\beta^{\alpha_2}\gamma^\mu D_\mu\psi_\beta^{\alpha_2} + \frac{i}{2}\bar{\psi}\gamma^\mu D_\mu\psi + \frac{i}{2}\bar{\rho}\gamma^\mu D_\mu\rho \\ & + \frac{1}{2}D^\mu\varphi^{\alpha_2+}D_\mu\varphi^{\alpha_2} + \frac{1}{2}D^\mu\varphi_\beta^+D_\mu\varphi_\beta \\ & + \frac{1}{2}\bar{s}_\mu\gamma^{\mu\nu}(p_4\gamma^4 + p_5\gamma^5)s_\nu + \frac{p_4^2+p_5^2}{2}(a^{\mu\,\alpha_2+}a_\mu^{\alpha_2} + a_\beta^{\mu+}a_{\mu\,\beta}) \end{aligned} \tag{4.59}$$

$$
\begin{aligned}
&-\frac{1}{2}\bar{\psi}^{\alpha_2}_{\beta}(p_4\gamma^4+p_5\gamma^5)\psi^{\alpha_2}_{\beta}-\frac{1}{2}\bar{\psi}(p_4\gamma^4+p_5\gamma^5)\psi-\frac{1}{2}\bar{\rho}(p_4\gamma^4+p_5\gamma^5)\rho \\
&+\frac{p_4^2+p_5^2}{2}(\varphi^{\alpha_2+}\varphi^{\alpha_2}+\varphi^{+}_{\beta}\varphi_{\beta}),
\end{aligned}
$$

5. CONCLUSIONS*

5.1 Hopf-Wess-Zumino term in the effective action of the $6d$ $(2,0)$ field theory revisited

We discussed the WZ term in the low energy effective action of the $6d$ $(2,0)$ field theory in the generic Coulomb branch. As a topological term, WZ term does not depend on the metric nor the coupling, so it is protected. For such terms, the supergravity calculation and the 1-loop calculation in field theory will give the same result. There is no available $6d$ $(2,0)$ field theory at present, so we will first calculate the WZ term on supergravity side. We then show that the obtained WZ term could indeed compensate the anomaly deficit, indicating that it is the desired term required by the anomaly matching condition.

For SYM theory in a generic Coulomb branch, each WZ term involves one root $e_i - e_j$, which is consistent with the fact that the supergravity interaction is produced by the integrating out of massive strings connecting the i_{th} and the j_{th} D branes. On the other hand, for M5 branes, each WZ term involves two roots $e_i - e_j$ and $e_k - e_j$. One may expect that such kind of triple interaction may be generated by the integrating out of the massive objects carrying (i, j, k) indices. A natural candidate is the string junctions with tension $(|\phi_{ij}|, |\phi_{jk}|, |\phi_{ki}|)$ proposed in [20, 21].

$6d$ $(2,0)$ theory should be a theory for strings. These strings are strongly-coupled. In [23], the equations of motion for the 3-algebra valued $(2,0)$ tensor multiplet were obtained. Later, in [31], it was shown that these equations have the natural in-

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terpretation as the supersymmetric gauge field equations in loop space, while the auxiliary field C^μ is the vector tangent to the loop, or in other words, the selfdual string. 3-algebra and the string structure are two appealing features for a theory on M5 branes, so we take the equations in [23] as the starting point to derive the WZ term. To relate the WZ term obtained from the field theory to the WZ term on supergravity side, we made special assumptions on fermion mass, which could be determined only after the specification of the 3-algebra. It may be possible to get the $H_3 \wedge A_3$ part of the WZ term, but for $A_3 \wedge F_4$, some indefiniteness exists and we can at most make the both sides have the similar structure. Moreover, it seems that $H_3 \wedge A_3$ requires the $5d$ momentum while $A_3 \wedge F_4$ indicates the $6d$ dynamics.

Finally, we discussed the representation of the 3-algebra. If the fundamental degrees of freedom carry three indices, the natural algebra structure for them is the 3-algrbra with N^3 scaling elements. An obvious problem is there is no finite dimensional Euclidean 3-algrbra satisfying the fundamental identity except for A_4. We show that the fundamental identity in [23] could be relaxed to a weaker version, but even though, it is still difficult to get a satisfactory representation.

5.2 Momentum modes of $M5$-branes in a $2d$ space

We considered the momentum modes of the $M5$ branes on a plane, which are the transverse momentum of the selfdual strings parallel to that plane. Different from the D branes, on which, the momentum modes are carried by the same kind of point-like excitations, here, the unparallel momentum modes are carried by selfdual strings with the different orientations. Selfdual strings with the same orientation gives a $5d$ SYM theory with the field configurations taking the zero Pontryagin number. The original $6d$ $(2,0)$ tensor multiplet field is then decomposed into a series of θ-parameterized $5d$ $U(N)$ SYM fields, among which, fields labeled by the same θ have the standard

SYM-type interaction. Fields labeled by different θ are associated with the selfdual strings with the different orientations. As a result, the $[i,j]+[j,k] \to [i,k]$ relation is not valid and the coupling cannot be realized as the standard matrix multiplication.

Since the bound state of the $[i,j]$ θ_1 selfdual string and the $[j,k]$ θ_2 selfdual string is not some $[i,k]$ selfdual string but the 3-string junction, we may also include the string junction into the theory. Each 3-string junction is characterized by $(\theta_1, \theta_2, \theta_3)$, forming the tri-fundamental or anti-tri-fundamental representation of $U(N)$, and may couple with the θ_1 θ_2 θ_3 selfdual strings in adjoint representation of $U(N)$. $[i,l]+[l,j,k] \to [i,j,k]$, $[j,m]+[i,m,k] \to [i,j,k]$, $[k,n]+[i,j,n] \to [i,j,k]$.

The quantization of the 3-string junction will give the higher-spin multiplet, for which the simplest one is the $(2,1)$ multiplet with the highest spin $3/2$. It is unclear whether the introducing of the 3-string junction will solve the problem or bring more problems, since at the beginning, we only want to get a theory for the $(2,0)$ tensor multiplet. The incorporation of the $5d$ massive $(2,1)$ multiplet into the $6d$ $(2,0)$ theory compactified on S^1 was also discussed in [68], where it was suggested that the algebraic structure of the $6d$ $(2,0)$ theory may have a fermionic symmetry in addition to the self-dual tensor gauge symmetry.

Each 3-string junction carries three indices, so they may offer the N^3 degrees of freedom on N $M5$ branes. However, the existing of the 3-string junction is severely restricted by the marginal stability curve, outside of which, the string junction may decay into the strings. For the given vacuum expectation values of the scalar fields on $M5$, the momentum of the string junction on that plane cannot be arbitrary. Especially, on coincident $M5$ branes, the 3-string junctions are at best at the marginal stability curve, so it is quite likely that they may decay.

Maybe it is easier consider the problem in the dual $D3$ picture. For $D3$ with the transverse dimension x'^{45} compactified, the winding mode of the (p,q) strings is dual to the $(n/R_4, m/R_5)$ momentum mode on $M5$. For the give p and q, open (p,q) strings with the arbitrary winding numbers have the SYM interaction. Then the questions are whether the open (p,q) (r,s) strings can interact or not, if can, in which way, what is the situation when $D3$ branes are coincident.

Among all selfdual strings, only those parallel to a given plane are taken as the perturbative degrees of freedom; nevertheless, different planes give the dual theories. One may compare the $6d$ theory with the $5d$ and $4d$ theories coming from the reductions on x_5 and $x_4 \times x_5$. Obviously, selfdual string extending along x_5 is the only candidate to define the $5d$ theory. However, for $4d$ theory, any selfdual string parallel to the 45 plane, carrying zero transverse momentum along it can act as the perturbative degrees of freedom. Only one is selected to give the $4d$ field, while the rest ones define the dual theories. Similarly, for $6d$ theory, selfdual strings parallel to a specific plane can be taken to give the $6d$ field, while the other planes give the dual versions. $M5$ on $S_1 \times S_2 \times S_3 \times S_4 \times S_5$ is dual to $D3$ on $S_i \times S_j \times S_k$ with a transverse S'^{lm}, where $\{1,2,3,4,5\} = \{i,j,k,l,m\}$. The (p,q) string ending on $D3$ winding S'^{lm} is dual to the selfdual string extending in (qR_l, pR_m) direction, carrying transverse momentum in $x_l \times x_m$, localized in $x_i \times x_j \times x_k$. There are 10 possible dual theories, corresponding to choosing the selfdual strings parallel to 10 different $2d$ subspaces.

$M5$ on T^5 is $SL(5,Z)$ invariant. However, the $6d$ theory on $M5$ does not have the explicit $SL(5,Z)$ invariance. The $SL(5,Z)$ U-duality transformation, or the $SO(5)$ U-duality transformation in R^5, is not a simple differorphism transformation but is also accompanied by a reallocation of the perturbative and the nonpertubative degrees of freedom. The U-dual $6d$ theories are equivalent, so the $SL(5,Z)$ transformation is just like a change of the gauge. This is similar with the $D3$. Although $D3$ is S-duality invariant, the $4d$ theory on $D3$ does not have the $SL(2,Z)$ invariance.

The nonpertubative $SL(2,Z)$ transformation gives the equivalent $4d$ theories.

5.3 The effective theory for KK modes

We discussed the 4d fields arising from the quantization of the 3-string junctions as well as their 5d and the 6d lifts. In 4d, we get the N=6 multiplet in tri-fundamental representation of $U(N)$, which, when lifted to 5d and 6d, becomes the (2, 1) multiplet. The (2, 1) multiplet is composed by two sets of fields $(b_{\mu\nu}, \chi, \Phi)$ and (s_μ, a_μ, λ), while the (2, 1) supercharge can be decomposed into the (2, 0) part Q^γ and the (0, 1) part Q^α. The action of the (2, 1) supercharge on (2, 1) multiplet can be summarized as

$$b_{\mu\nu} \overset{Q^\gamma}{\leftrightarrow} \chi \overset{Q^\gamma}{\leftrightarrow} \Phi, \qquad s_\mu \overset{Q^\gamma}{\leftrightarrow} a_\mu \overset{Q^\gamma}{\leftrightarrow} \lambda, \tag{5.1}$$

and

$$s_\mu \overset{Q^\alpha}{\leftrightarrow} b_{\mu\nu}, \qquad a_\mu \overset{Q^\alpha}{\leftrightarrow} \chi, \qquad \lambda \overset{Q^\alpha}{\leftrightarrow} \Phi, \tag{5.2}$$

from which, we can determine the Weyl weight

field	s_μ	a_μ	λ	$b_{\mu\nu}$	χ	ϕ
weight	3/2	1	3/2	2	5/2	2

$(b_{\mu\nu}, \chi, \Phi)$ has the Weyl weight $(2, 5/2, 2)$, thus is the conformal sector in $6d$. On the other hand, (s_μ, a_μ, λ) is not conformal in $6d$. Therefore, (s_μ, a_μ, λ) is suppressed in $6d$. It is $(b_{\mu\nu}, \chi, \Phi)$ that will have the contribution to the conformal anomaly. $(b_{\mu\nu}, \chi, \Phi)$ is $4N^3$ copies of the (2, 0) tensor multiplet, so the scale (type-B) anomaly of the (2, 1) multiplet in 6d is $4N^3 b_{6B}$, which is consistent with the AdS/CFT prediction for the scale anomaly of the 6d (2, 0) theory in large N limit.

When compactified to 4d, $(b_{\mu\nu}, \chi, \Phi)$ will become $(a_\mu^{\alpha_1}, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$ if the R-symmetry is $SO(5)$, $(a_\mu^{\alpha_1}, \psi^{\alpha_1,\alpha_2}, \psi_\beta^{\alpha_1}, \Phi_\beta^{\alpha_1,\alpha_2}, \Phi^{\alpha_1})$ if the R-symmetry is $SO(4)$; $(s_\mu, a_\mu,$ $\lambda)$ will become $(s_\mu, a_\mu^\gamma, \lambda^I, \psi, \varphi^\gamma)$ if the R-symmetry is $SO(5)$, $(s_\mu, \rho, a_{\mu\,\beta}, a_\mu^{\alpha_2}, \varphi_\beta, \varphi^{\alpha_2},$

$\psi, \psi_\beta^{\alpha_2}$) if the R-symmetry is $SO(4)$. The interaction is mediated by the N=4 multiplet (A_μ, ξ^γ, X^I). It is (A_μ, ξ^γ, X^I) $-(a_\mu^{\alpha_1}, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$ coupling and $(A_\mu, \xi^\gamma, X^I) - (s_\mu, a_\mu^\gamma, \lambda^I, \psi, \varphi^\gamma)$ coupling that is possible. We constructed the 4d Lagrangian with $(A_\mu, \xi^\gamma, X^I) - (a_\mu^{\alpha_1}, \chi^{\gamma\,\alpha_1}, \Phi^{I\,\alpha_1})$ coupling and $(A_\mu, \xi^\gamma, X^I) - (s_\mu, a_\mu^\gamma, \lambda^I, \psi, \varphi^\gamma)$ coupling and the corresponding N=4 susy transformation. The Lagrangian is not entirely invariant under the N=4 supersymmetry. Some additional terms are to be added.

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APPENDIX A

THE ON-SHELL ACTION FOR BRANE-GRAVITY COUPLED SYSTEM*

Consider the $(d-1)$-brane couples with the supergravity fields. The action is [82]

$$S = S_{brane} + S_{gravity} \tag{A.1}$$

$$\begin{aligned} S_{brane} &= T_d \int d^d\xi [-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^M \partial_j X^N g_{MN} e^{\alpha(d)\phi/d} + \frac{d-2}{2}\sqrt{-\gamma} \\ &\quad -\frac{1}{d!}\epsilon^{i_1 i_2 \cdots i_d}\partial_{i_1} X^{M_1}\partial_{i_2} X^{M_2}\cdots\partial_{i_d} X^{M_d} A_{M_1 M_2 \cdots M_d}] \\ &= S_1 + S_2 + S_3 \end{aligned} \tag{A.2}$$

$$S_{gravity} = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(d+1)!}e^{-\alpha(d)\phi}F^2_{d+1}] \tag{A.3}$$

The action of this form is valid for both electric and magnetic branes, while for magnetic branes, just let $\alpha(d) \to -\alpha(d)$. Variation with respect to g_{MN}, $A_{M_1 M_2 \cdots M_d}$, and γ_{ij} gives

$$\begin{aligned} T^{MN}_{brane} &= -T_d \int d^d\xi \sqrt{-\gamma}\gamma^{ij}\partial_i X^M \partial_j X^N e^{\alpha(d)\phi/d}\frac{\delta^D(x-X)}{\sqrt{-g}} \\ &= \frac{1}{\kappa^2}[R^{MN} - \frac{1}{2}g^{MN}R - \frac{1}{2}(\partial^M\phi\partial^N\phi - \frac{1}{2}g^{MN}(\partial\phi)^2) \\ &\quad -\frac{1}{2d!}(F^M{}_{M_1\cdots M_d}F^{NM_1\cdots M_d} - \frac{1}{2(d+1)}g^{MN}F^2)e^{-\alpha(d)\phi}] \end{aligned} \tag{A.4}$$

---

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$$
\begin{aligned}
J^{M_1\cdots M_d}_{brane} &= T_d \int d^d\xi\, \epsilon^{i_1 i_2\cdots i_d}\partial_{i_1}X^{M_1}\partial_{i_2}X^{M_2}\cdots\partial_{i_d}X^{M_d}\frac{\delta^D(x-X)}{\sqrt{-g}} \\
&= \frac{1}{2\kappa^2\sqrt{-g}}\partial_M(\sqrt{-g}e^{-\alpha(d)\phi}F^{MM_1\cdots M_d})
\end{aligned} \tag{A.5}
$$

and

$$
\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN} e^{\alpha(d)\phi/d}. \tag{A.6}
$$

Then

$$
\begin{aligned}
S_1 &= \frac{1}{2}\int d^D x\sqrt{-g}\, T^{MN}_{brane}\, g_{MN} \\
&= \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}[(1-\frac{D}{2})(R-\frac{1}{2}(\partial\phi)^2) - \frac{1}{2d!}(1-\frac{D}{2(d+1)})F^2 e^{-\alpha(d)\phi}]
\end{aligned} \tag{A.7}
$$

Plug (A.6) into (A.2), we get

$$
S_1 + S_2 = \frac{2}{d}S_1 \tag{A.8}
$$

From (A.5),

$$
\begin{aligned}
S_3 &= -\frac{1}{d!}\int d^D x\sqrt{-g}\, J^{M_1\cdots M_d}_{brane} A_{M_1 M_2\cdots M_d} \\
&= \frac{1}{2\kappa^2}\int d^D x \frac{1}{(d+1)!}\sqrt{-g}e^{-\alpha(d)\phi}F^2 \\
&\quad -\frac{1}{d!}\partial_M(\sqrt{-g}e^{-\alpha(d)\phi}A_{M_1 M_2\cdots M_d}F^{MM_1\cdots M_d})
\end{aligned} \tag{A.9}
$$

As a result,

$$
\begin{aligned}
S_{brane} &= S_1 + S_2 + S_3 \\
&= \frac{2-D}{d}\frac{1}{2\kappa^2}\int d^D x\sqrt{-g}[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(d+1)!}e^{-\alpha(d)\phi}F^2_{d+1}] \\
&\quad -\frac{1}{2\kappa^2}\int d^D x\frac{1}{d!}\partial_M(\sqrt{-g}e^{-\alpha(d)\phi}A_{M_1 M_2\cdots M_d}F^{MM_1\cdots M_d}) \\
&= \frac{2-D}{d}S_{gravity} + S_{boundary}
\end{aligned} \tag{A.10}
$$

For $d < 3$, or equivalently, for purely electric branes, $S_{boundary} = 0$. If M_D has no boundary, $S_{boundary}$ could also be dropped. Then for the given brane configuration, if S is on-shell with respect to supergravity, we have

$$S_{brane} : S_{gravity} : S = (D-2) : (-d) : (D-2-d), \tag{A.11}$$

where

$$\begin{aligned} S_{brane} &= T_d \int d^d\xi[-\sqrt{-\det(\partial_i X^M \partial_j X^N g_{MN} e^{\alpha(d)\phi/d})} \\ &\quad -\frac{1}{d!}\epsilon^{i_1 i_2 \cdots i_d}\partial_{i_1} X^{M_1}\partial_{i_2} X^{M_2}\cdots\partial_{i_d} X^{M_d} A_{M_1 M_2 \cdots M_d}] \end{aligned} \tag{A.12}$$

The extension to multi-brane configurations is straightforward, and (A.11) still holds. When the dimensions of the branes are different, the exact proportional relation is not valid anymore. Besides, when F_{ij} does not vanish, i.e. the $(p-1)$-brane carries $p-1-2n$ charge, (A.11) does not hold. Naively, when $D = 11$, $d = 6$, neglecting the boundary term, $S_{brane} : S_{gravity} : S = 3 : (-2) : 1$[1]. However, S here is not exactly the action for M5 branes coupling with supergravity. In the following, we will use (2.1) as the action to get the same conclusion.

Now, consider

$$S = S_g + S_{M5} \tag{A.13}$$

[1]Note that for D3, M2, M5 branes, S_{brane}/S equals to 2, 3/2, and 3, while the degrees of freedom on these branes scale as N^2, $N^{3/2}$, and N^3. This is not the coincidence. Suppose the degrees of freedom on N branes scale as N^α. Also suppose that in S_{brane}, there is a term T has the N^α scaling. Consider $N+1$ branes with large N, when the symmetry is broken from $SU(N+1)$ to $SU(N)\times U(1)$, $(N+1)^\alpha - N^\alpha \sim \alpha N^{\alpha-1}$ number of T will enter into S_{brane}. On the other hand, the effective action of the system could be approximated as the action of a single brane on the background generated by the rest N branes. In S_{eff}, one may get $N^{\alpha-1}T$. Obviously, T in S_{brane} and T in S_{eff} differ by a α factor. For $D3$, T is $\int F_5$, for $M5$, T is $\int A_3 \wedge F_4$, while for $M2$, T is obscure.

with

$$S_g = \frac{1}{2\kappa^2}\int_{M_{11}} *R - \frac{1}{2} * \hat{F}_4 \wedge \hat{F}_4 - \frac{1}{6}F_4 \wedge F_4 \wedge A_3 \tag{A.14}$$

$$\begin{aligned} S_{M5} &= -T_5 \int_{W_6} d^6\xi \sqrt{-\det(g_{\mu\nu} + (i_{v_1}\tilde{*}h_3)_{\mu\nu})} + \frac{1}{2}v_1 \wedge h_3 \wedge \tilde{*}(v_1 \wedge \tilde{*}h_3) \\ &+ \frac{T_5}{2}\int_{W_6} db_2 \wedge A_3 + \frac{T_5}{2}\int_{W_7} A_3 \wedge F_4 \end{aligned} \tag{A.15}$$

The field equations are [32]

$$T^{MN}_{M5} = \frac{1}{\kappa^2}\left[R^{MN} - \frac{1}{2}g^{MN}R - \frac{1}{12}(\hat{F}^M_4{}_{PQL}\hat{F}_4^{NPQL} - \frac{1}{8}g^{MN}\hat{F}_4^2)\right] \tag{A.16}$$

$$d * \hat{F}_4 + \frac{1}{2}F_4 \wedge F_4 = -2\kappa^2 T_5(-A_3 \wedge *J_6 + F_4 \wedge *G_7) \tag{A.17}$$

$$d\hat{F}_4 = 2\kappa^2 T_5 * J_6, \tag{A.18}$$

The vacuum expectation value of b_2 are taken to be zero, otherwise, (A.11) does not hold. From (A.16),

$$\frac{1}{2}\int_{W_6} d^6\xi\sqrt{-g}\,T^{MN}_{M5}\,g_{MN} = \frac{1}{2\kappa^2}\int_{M_{11}} -\frac{9}{2} * R + \frac{3}{4} * \hat{F}_4 \wedge \hat{F}_4 \tag{A.19}$$

From (A.17),

$$\frac{T_5}{2}\int_{W_7} A_3 \wedge F_4 = \frac{1}{2\kappa^2}\int_{M_{11}} \frac{1}{2} * \hat{F}_4 \wedge F_4 + \frac{1}{4}F_4 \wedge F_4 \wedge A_3 - \frac{1}{4\kappa^2}\int_{\partial M_{11}} A_3 \wedge *\hat{F}_4 \tag{A.20}$$

We will still use the general relation (A.8) for M5 branes. The Nambu-Goto action for M5 branes is more involved than that for D branes or M2 branes, so the correction may exist, but that will not bring too many problems, since our main concern is the

WZ term.

$$S_{M5} = \frac{1}{2\kappa^2}\int_{M_{11}} -\frac{3}{2}*R+\frac{3}{4}*\hat{F}_4\wedge\hat{F}_4+\frac{1}{4}F_4\wedge F_4\wedge A_3+\frac{T_5}{2}\int_{W_7}*\hat{F}_4-\frac{1}{4\kappa^2}\int_{\partial M_{11}} A_3\wedge*\hat{F}_4 \tag{A.21}$$

$$S = \frac{1}{2\kappa^2}\int_{M_{11}} -\frac{1}{2}*R+\frac{1}{4}*\hat{F}_4\wedge\hat{F}_4+\frac{1}{12}F_4\wedge F_4\wedge A_3+\frac{T_5}{2}\int_{W_7}*\hat{F}_4-\frac{1}{4\kappa^2}\int_{\partial M_{11}} A_3\wedge*\hat{F}_4 \tag{A.22}$$

Because of the last two terms, the exact proportional relation does not hold. This is the general solution. Now, consider the multi-center M5 brane solutions and the corresponding WZ term. In this case, $d*\hat{F}_4 = 0$, (A.20) reduces to[2]

$$\frac{T_5}{2}\int_{W_7} A_3\wedge F_4 = \frac{1}{2\kappa^2}\int_{M_{11}}\frac{1}{4}F_4\wedge F_4\wedge A_3 \tag{A.23}$$

In S_g, we have $-\frac{1}{6}F_4\wedge F_4\wedge A_3$, so altogether,

$$\Gamma_{WZ} = \frac{1}{2\kappa^2}\int_{M_{11}}\frac{1}{12}F_4\wedge F_4\wedge A_3 \tag{A.24}$$

A factor of 1/3 is involved.

Similarly, for D3 branes, one may expect that the WZ term should be one half of the corresponding term in the D3 brane action. This is indeed the case, and it is just this rescaled term that is obtained from the 1-loop integration and compensates the anomaly deficit.

[2] Plug (2.2) into (A.23), since $\hat{F}_4\wedge\hat{F}_4 = 0 = *G_7\wedge*G_7$, (A.23) becomes an identity. However, the right-hand side of (A.23) cannot be used for anomaly counting directly.

APPENDIX B

THE CONSTRAINTS ON THE STRUCTURE CONSTANT FOR THE CLOSURE OF THE SUPERSYMMETRY

In [23], the supersymmetry transformation of the tensor multiplets together with the gauge field A_μ and the vector field C_μ is discussed. We will focus on the resulted conditions that should be imposed on the structure constant.

The closure on C^μ_A gives the constraint $C^\mu_B C^\nu_C f^{CDB}{}_A = 0$. For the closure on $A^B_{\mu A}$, we have

$$\begin{aligned}[][\delta_1, \delta_2] A^B_{\mu\,A} &= -2i(\epsilon_2 \Gamma_\lambda \Gamma^I \epsilon_1) C^\lambda_C D_\mu X^I_D h^{CDB}{}_A - v^\nu C^\lambda_C H_{\mu\nu\lambda D} h^{CDB}{}_A \\ &\quad -i(\bar\epsilon_2 \Gamma_{\mu\nu\lambda} \Gamma^{IJ} \epsilon_1) C^\nu_C C^\lambda_G X^I_E X^J_F f^{EFG}{}_D h^{CDB}{}_A \\ &\quad +2i(\epsilon_2 \Gamma_\mu \Gamma^I \epsilon_1) C^\lambda_C D_\lambda X^I_D h^{CDB}{}_A \\ &= v^\nu \tilde F_{\mu\nu}{}^B{}_A + D_\mu \tilde\Lambda^B{}_A\,. \end{aligned} \tag{B.1}$$

The second line must vanish. $\Gamma_{\mu\nu\lambda}$ is antisymmetric with respect to ν and λ, so

$$(C^\nu_C C^\lambda_G - C^\lambda_C C^\nu_G) f^{EFG}{}_D h^{CDB}{}_A = (C^\nu_C C^\lambda_G - C^\lambda_C C^\nu_G) f^{EFG}{}_D f^{DBC}{}_A = 0. \tag{B.2}$$

For the closure on $H_{\mu\nu\lambda A}$,

$$\begin{aligned}[][\delta_1, \delta_2] H_{\mu\nu\lambda\,A} &= v^\rho D_\rho H_{\mu\nu\lambda\,A} - 2i(\bar\epsilon_2 \Gamma_\rho \Gamma^I \epsilon_1) C^\rho_C X^I_D g^{DBC}{}_A H_{\mu\nu\lambda\,B} \\ &\quad -6i(\bar\epsilon_2 \Gamma_{[\mu} \Gamma^I \epsilon_1)\Big(\tilde F_{\nu\lambda]}{}^C{}_A - C^\rho_B H_{\nu\lambda]\rho\,D} g^{CDB}{}_A\Big) X^I_C \\ &\quad -6i(\bar\epsilon_2 \Gamma_{\rho[\mu\nu} \Gamma^{IJ} \epsilon_1) C^\rho_B X^I_C D_{\lambda]} X^J_D (f^{CDB}{}_A - g^{CDB}{}_A) \\ &\quad -\frac{3i}{8}(\bar\epsilon_2 \Gamma^\sigma \Gamma^J \epsilon_1)(\bar\Psi_C \Gamma_{\mu\nu\lambda\rho\sigma} \Gamma^J \Psi_D) C^\rho_B (h^{DBC}{}_A - g^{CDB}{}_A) \\ &\quad +2i(\bar\epsilon_2 \Gamma^\tau \Gamma^K \epsilon_1)\epsilon_{\mu\nu\lambda\rho\sigma\tau} C^\rho_B C^\sigma_E X^I_C X^I_F X^K_G g^{D[B|C}{}_A f^{FG|E]}{}_D \end{aligned}$$

$$
\begin{aligned}
&+i(\bar{\epsilon}_2\Gamma_{\mu\nu\lambda}\Gamma_{LM}\epsilon_1)\epsilon^{IJKLM}C^{\kappa}_{B}C_{\kappa\ E}X^I_C X^J_F X^K_G g^{DB[C}{}_{A}f^{FG]E}{}_{D}\\
&+3i(\bar{\epsilon}_2\Gamma_{\rho[\mu\nu}\Gamma_{LM}\epsilon_1)\epsilon^{IJKLM}C^{\rho}_{B}C_{\lambda]\ E}X^I_C X^J_F X^K_G g^{DB[C}{}_{A}f^{FG]E}{}_{D}\\
&+v^{\rho}\Big(4D_{[\mu}H_{\nu\lambda\rho]\ A}+\epsilon_{\mu\nu\lambda\rho\sigma\tau}C^{\sigma}_{B}X^I_C D^{\tau}X^I_D g^{CDB}{}_{A}\\
&\qquad\qquad +\frac{i}{2}\epsilon_{\mu\nu\lambda\rho\sigma\tau}C^{\sigma}_{B}\bar{\Psi}_C\Gamma^{\tau}\Psi_D g^{CDB}{}_{A}\Big)\\
=\ & v^{\rho}D_{\rho}H_{\mu\nu\lambda\ A}+\tilde{\Lambda}^{B}{}_{A}H_{\mu\nu\lambda\ B}\ ,
\end{aligned}
\tag{B.3}
$$

The fifth, sixth, and seventh lines must vanish. The fifth line equals to zero because of (B.2). The sixth and the seventh lines vanish if

$$
C^{\mu}_{B}C^{\nu}_{E}f^{DB[C}{}_{A}f^{FG]E}{}_{D}=0. \tag{B.4}
$$

Finally, the invariance of the inner product under the gauge transformation gives the constraint

$$
C^{\mu}_{D}(f^{DCA}{}_{E}h^{EB}+f^{DCB}{}_{E}h^{AE})=0, \tag{B.5}
$$

where $\left\langle t^A|t^B\right\rangle = h^{AB}$. $f^{ABC}{}_{D}=f^{[ABC]}{}_{D}$.

The above constraints could be explicitly written as

$$
[C^{\mu},C^{\nu},*]=0, \tag{B.6}
$$

$$
\begin{aligned}
[C^{\mu},A,[C^{\nu},B,Y]]-[C^{\nu},B,[C^{\mu},A,Y]] &= [[C^{\mu},A,C^{\nu}],B,Y]+[C^{\nu},[C^{\mu},A,B],Y]\\
&= [C^{\nu},[C^{\mu},A,B],Y]\\
&= [C^{\mu},[C^{\nu},A,B],Y].
\end{aligned}
\tag{B.7}
$$

The fundamental identity should be satisfied for gauge transformations generated by $[C,A,*]$, for which, it reduces to the Jacobi identity.