A STUDY OF QUINTESSENTIAL INFLATION

A Senior Scholars Thesis

by

CHRISTIAN DANIEL FREEMAN

Submitted to Honors and Undergraduate Research
Texas A&M University
in partial fulfillment of the requirements for the designation as

UNDERGRADUATE RESEARCH SCHOLAR

May 2012

Major: Physics
Mathematics

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ABSTRACT

A Study of Quintessential Inflation. (May 2012)

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I present an analysis of a class of joint Inflation and Dark Energy models called quintessential inflation models. Such models are motivated by the common mathematical formalism describing the early expansion of the universe, i.e. inflation, and the late time accelerated expansion of space, i.e. quintessence. In particular, I examine the historical motivation and theoretical underpinnings of the models. I also consider the feasibility of such models and the ability of a subset of the proposed models in the literature to withstand the increasingly stringent conditions imposed by observational tests such as, but not limited to, the Wilkinson Microwave Anisotropy Probe (WMAP) and the Supernova Legacy Survey. While able to withstand observational constraints, the models considered offer little, if any predictive power, and it is too early to be able to meaningfully distinguish quintessential inflation models from other theoretical competitors.

ACKNOWLEDGMENTS

I'd like to thank Dr. Bhaskar Dutta for his continuing support and guidance throughout this product. His ability to dissolve my woes with a few short words on many an occasion has made this project immensely more enjoyable.

I'd also like to thank Sean Downes for many stimulating conversations, and reassuring me that what I'm trying to do is not, in fact, impossible.

NOMENCLATURE

$a(t)$	Scale factor
H	Hubble factor
Ω_i	Energy density fraction of critical density
ρ_c	Critical density
z	Redshift
Q.I.	Quintessential inflation
n_s	Spectral index

TABLE OF CONTENTS

Page

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Quintessential inflation models are the brainchild of a thrust of theoretical results spanning the last thirty years. I will present a brief outline of those results to ground my analysis, as well as a sketch of the prototypical quintessential inflation toy model discovered in the late 90s.

An intuitive approach to scalar fields

The common thread tying quintessence to inflation will be the careful use of a scalar field. An intuitive, although not all-encompassing example of something like a scalar field is a finite one dimensional spring-mass system as in Figure 1 [1]:

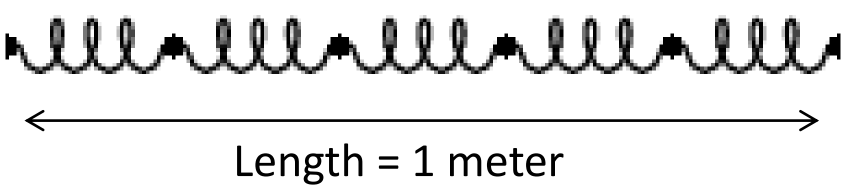

FIG. 1. A finite chain of springs linking masses.

The Lagrangian of the system can be written[1]:

$$L = \sum_{r=1}^{N/\epsilon} \frac{\epsilon}{2}\left(\frac{m_r}{\epsilon}\ (\dot{x}_r)^2 - k\,\frac{(x_r - x_{r+1})^2}{\epsilon}\right) \qquad (1)$$

This thesis follows the style of Physical Review D.

Where this is just a sum over the kinetic energies of each mass (the first term) minus the potential energy of each spring-mass unit with respect to its immediate neighbor (the second term). The ϵ term represents the size of an individual spring, so taking the limit as ϵ approaches zero is akin to linking infinitely many springs together on the same finite interval. In the limit $\epsilon \to 0$, we may write the equivalent expression[1]:

$$L = \int dx \frac{1}{2} \left[\mu \dot{\phi}(x)^2 - Y \left(\frac{\partial \phi(x)}{\partial x} \right)^2 \right] \tag{2}$$

The limiting procedure has replaced the sum with an integral, our previous m/ϵ term has become a "mass per unit length" μ, and our previous k/ϵ term can be interpreted as a sort of Young's Modulus Y. Also, we have replaced the coordinate x_k with the function $\phi(x)$, because we have moved from a system of discrete springs to a continuum of spring-like points. A standard application of the Euler-Lagrange equations yields:

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\mu}{Y} \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{3}$$

Which is the wave equation from classical physics. Kaku notes that this result is not unexpected, as it is a statement of the intuitive notion that waves can propagate on systems of springs. The theoretical weight of this result is that this system represents an infinite number of degrees of freedom—that is, each point along our spring-mass system has its own associated kinetic energy and potential energy, because each point along the strip really represents an infinitesimal mass-spring unit. This system can be interpreted as a sort of scalar field defined along some finite strip of length 1 meter. The analogy breaks down in that real scalar fields are not so physically tangible. The kinetic and potential energy of scalar fields exists at points in space more how the Electric and Magnetic field exists at points in space—essentially as an extra degree of freedom for energy. We will recover these ideas with quintessence and inflation models, wherein the scalar fields permeate all of space. We will also state the precise lagrangian density for the scalar fields under investigation, as well as the relevant equation of motion, which differ qualitatively from the wave equation shown here.

Inflation and the horizon problem

The horizon problem plagued Big Bang cosmology for much of the 1970s until theorists advanced several competing explanations, among which one particularly successful solution is inflation. The precise problem is nuanced, but the simplest statement involves simply looking at the night sky[2]. If one were to go outside and measure the cosmic microwave background radiation coming from some direction, one would get a "temperature" of approximately 2.75 Kelvin. If one were to whirl themselves about and measure the same cosmic microwave background radiation from another direction, one would get essentially the same reading. This radiation, or, these photons correspond to an event that happened approximately 300,000 years after the Big Bang, sometimes referred to as recombination, but more accurately called decoupling[3]. Recombination refers to the period during the early universe when the universe cooled just enough to allow hydrogen to form stably, producing a plethora of photons in the process. Decoupling refers to the time shortly thereafter when the density of the universe was low enough that the photons produced during recombination were no longer interacting with the surrounding soup of hydrogen nuclei.

The problem arises when we consider how these photons coming from completely different directions in the night sky could have the same temperature. For this, we will refer to the notion of a *horizon*, or roughly, the scale at which we expect objects to be within causal contact. Because the speed of light is finite, and the speed of light governs the speed at which different regions can communicate, if two points in space are sufficiently far apart, they may be causally disconnected—that is, too far apart to ever communicate with one another within the lifetime of the universe. We'll consider a handful of different distance measures that, but a particularly important one is the horizon. The size of the horizon formalizes this idea of the distance at which objects are no longer in causal contact. One can derive an expression relating the horizon size to the entropy contained within the volume bounded by the horizon[4]:

$$S_{HOR} = s\frac{4\pi}{3}t^3 \approx \begin{cases} 0.05 g_*^{-\frac{1}{2}}\left(\frac{m_{pl}}{T}\right)^3 & for\ t < t_{eq} \\ 3 * 10^{87}(\Omega_0 h^2)^{-\frac{3}{2}}(1+z)^{-\frac{3}{2}} & for\ t > t_{eq} \end{cases} \quad (4)$$

Using an entropy of approximately 10^{83}, which is a fairly close approximation of the entropy of the universe during recombination, yields $\approx 10^5$ separate causally disconnected regions. More visually, this implies that about .8° of arc in the night sky is the largest distance scale across which we expect there to have been causal contact since the big bang, and thus uniformity. But we see uniformity across the entire night sky, down to approximately 5 decimal places. Even worse, considering Big Bang nucleosynthesis arguments requires uniformity in 10^{25} causally disconnected patches to reproduce the uniform distribution of elements we see today[4].

One could always posit that the universe just happened to have remarkable agreement on all scales due to its initial conditions—but this requires a miraculous amount of fine tuning. In 1980, Alan Guth suggested a more amenable approach that he called *inflation*. The idea attacks the problem above directly: it provides a mechanism for the universe to go from a small, causally connected patch, as the universe was in the brief fractions of a second after the big bang, to a vast expanse of space with a large degree of uniformity throughout.

The mechanism of inflation

We will make extensive use of the quantity $a(t)$ or sometimes a, which represents the relative distance between objects at different times throughout the age of the universe. This quantity is called the *scale factor* in the literature. The quantity $\frac{\dot{a}}{a}$ or commonly H, known as the *Hubble factor* in the literature, represents the growth rate of the universe. The quantity $1/aH$, or the *comoving Hubble radius*, represents the maximum possible distance a particle can travel during the amount of time in which the scale factor approximately doubles. This can be thought of as another qualitative measure of the distance at which particles will remain in causal contact, like the horizon.

For the space we see now to have been in causal contact at some point in the past (so that it could have been in thermal equilibrium, and have produced the cosmic microwave background spectrum that we see today), the commoving Hubble radius must have decreased dramatically in the past. We can write this decrease in the comoving Hubble radius as[3]:

$$\frac{d}{dt}[aH] = \frac{d}{dt}\left[a\frac{da/dt}{a}\right] = \frac{d^2a}{dt^2} > 0 \qquad (5)$$

That is, the scale factor must increase at an accelerated rate. The standard argument lets the scale factor grow exponentially as:

$$a(t) = a_e e^{H(t-t_e)} \quad for\ t < t_e \qquad (6)$$

A handful[3] of theoretical considerations, which are detailed in Chapter three, lead to this period of rapid expansion lasting long enough to cause the scale factor and thus the volume of the universe to increase by roughly 60 orders of magnitude, or 60 *e-foldings*.

Inflation as a scalar field

To reconcile the need for the universe to expand by 60 orders of magnitude, we must turn to Einstein's field equations to see how such a large amount of inflation could be physically realized. Einstein's field equations dictate how different forms of energy affect the geometry of spacetime. The zeroth order equations can be written[3]:

$$\frac{d^2a/dt^2}{a} = -\frac{4\pi G}{3}(\rho + 3P) \qquad (7)$$

Combining this with our inequality from (5), we have the well known statement[3]:

$$P < -\frac{\rho}{3} \tag{8}$$

That is, the required type of energy to drive inflation must have a negative pressure. This is strange because no other known type of energy has this exotic property, but it is a necessary condition for inflation to occur. With this result in mind, the simplest general scalar field equation can be written[3]:

$$\mathrm{L} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{9}$$

Notice we have the familiar $\dot{\phi}^2$ term as in our spring-mass system example, which represents the kinetic energy of the field. The potential term, however, is left unspecified intentionally. Einstein's field equations offer a straightforward way to extract the pressure and energy density of an object governed by such a Lagrangian Density. The Stress-Energy tensor for $\phi(t)$ reads[3]:

$$T^{\alpha}_{\beta} = g^{\alpha\nu}\, \frac{\partial\phi}{\partial x^{\nu}}\frac{\partial\phi}{\partial x^{\beta}} - g^{\alpha}_{\beta}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right] \tag{10}$$

Combining (9) and (10), and taking the greek and latin indices separately, we have[3]:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{11}$$

$$\mathrm{P} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{12}$$

Equation (12) implies that to achieve our negative pressure condition, the potential energy has to dominate the kinetic energy of our field (this is a bit of a simplification, as there are models that produce inflation without this condition). The details of determining the form of $V(\phi)$ that produce a sort of inflation compatible with observation comprise the bulk of inflationary research today. These potentials and a more precise notion of compatibility will be discussed in detail in Chapter 2.

Quintessence and the budget problem

For this section, we'll turn our attention to a quantity called the *luminosity distance*. In generality, this quantity can be written as a function of redshift[5]:

$$d_L(z) = \frac{1+z}{H_0}\int_{1/(1+z)}^{1} \frac{da}{x^2\sqrt{\Omega_\Lambda + \Omega_M a^{-3} + \Omega_R a^{-4}}} \tag{13}$$

Where the Ω_is represent the different energy density fractions of the universe, so $\Omega_\Lambda + \Omega_M + \Omega_R = 1$. M represents matter, R represents radiation, and Λ represents some energy density associated with the vacuum. The different factors of a^k accompanying the different terms follow from general relativistic constraints on how the different forms of energy density affect spacetime.

A less opaque statement of the luminosity distance reads[3]:

$$d_L(z) = \sqrt{\frac{L}{4\pi F}} \tag{14}$$

Essentially, the luminosity distance goes as the square root of the luminosity of an object divided by its flux. In the late 90s, a curious observation [6] was made of a handful of type 1a supernovae. Astronomers were interested in these objects because the luminosity of these types of supernovae are well understood, so by measuring their flux, astronomers could obtain a good measure of the luminosity distance. Effectively, they could calculate the luminosity distance using equation (14), and then use this value to see what the different energy density fractions of the universe had to be to reproduce their measured value by using equation (13).

We have, in a sense, given away the punch line by including a term of the form Ω_Λ. The astronomers realized that there was no consistent allocation of energy densities that reproduced their value for the luminosity distance to various supernovae without attributing a sizeable fraction of the energy density of the universe to some previously unobserved form of vacuum energy density—i.e. *dark energy*.

For historical[7] and aesthetic reasons, this energy density was initially thought to be constant, hence the name Cosmological Constant. More recent results[8] suggest that this energy density may not have always been constant, ushering forth the idea of *quintessence*, or a scalar field that gives rise to this large fraction of energy density today.

The necessity of dark energy

In stating equation (13), we have glazed over a great deal of arguments from General Relativity. More generally, we could write[5]:

$$d_L(z) = \frac{1+z}{H_0}\int_{1/(1+z)}^{1} \frac{da}{x^2\sqrt{\sum_i \Omega_i^{3(1+w_i)}}} \qquad (15)$$

Where the sum in the denominator runs over all types of energy density, and the quantities w_i represent the ratio of the pressure to the energy density of the different contributing types of energy. Using $w_M = 0$, $w_R = 1/3$, and $w_\Lambda = -1$ reproduces equation (13). These values for matter and radiation follow from classical theory, but we do not know a priori the w_Λ parameter. Leaving it as a free variable, the ESSENCE (Equation of State: Supernovae trace Cosmic Expansion) group came up with a value of $w_\Lambda = -1.09 \pm 0.09(stat, 1\sigma) \pm 0.13(syst)$[9] as a best fit to the data from the ongoing Supernova Legacy Survey (assuming w_Λ is time independent, as in the case of a Cosmological Constant). Comparing this with equation (8):

$$w_\Lambda = \frac{P_\Lambda}{\rho_\Lambda} \approx -1 \quad versus \quad w_{Inflation} \lesssim -\frac{1}{3} \tag{16}$$

Where the right hand side follows from rearrangement of (8). The prominent feature here is that both forms of energy have a decidedly negative w parameter, and both forms of energy give rise to the accelerated expansion of space. If we posit that dark energy actually exists as some sort of scalar field, we may invoke all of the arguments we used for the scalar field governing inflation (i.e. equations (9) through (12)). Once again, the problem reduces to finding the specific form of $V(\phi)$ for the quintessence field that reproduces the desired behavior.

Invoking the use of a scalar field might seem strange here, considering no scalar fields are actually known to exist. Modern physics does, however, make extensive use of other, more complicated field theoretic objects, as in quantum field theory, which are more epistemically accessible, and arguably more esoteric.

Quintessential inflation

A deeper question arises: can we find some very special form of $V(\phi)$ which fits all of the requirements of both inflation and dark energy? Peebles and Vilenkin did just this for the first time in 1999 with the following scalar field potential energy function[10]:

$$V(\phi) = \lambda(\phi^4 + M^4) \; for \; \phi < 0 \tag{17}$$

$$V(\phi) = \frac{\lambda M^8}{\phi^4 + M^4} \quad for \; \phi \geq 0$$

Where λ is some dimensionless constant, and M sets the energy scale. Those familiar with the literature will recognize this as a chaotic inflation potential for $-\phi \gg M$ essentially glued to a quintessence potential for $\phi \gg M$. While a completely ad hoc construction meant only to be a proof of concept, this model remains useful. This model is able to account for nontrivial cosmological processes like reheating, matter domination, and radiation domination between the epochs of inflation and dark energy domination. The largest problem with the model is that it has essentially no physical motivation[11]. No known process can account for a potential of the form in equation (17). As this model is now nearing 13 years old, several other complete classes of Quintessential Inflation models have arisen, as well as claims that quintessential inflation might not even be physically realizable, in principle [12]. We will examine these ideas in detail in the subsequent chapters.

CHAPTER II

METHODS

Many quintessential inflation models were proposed in the literature in the past decade since Peebles' and Vilenkin's work. This is largely due to the vast number of different inflation theories and similarly large number of different quintessence theories. Furthermore, there's no scholarly consensus on the specifics of the bridging process of reheating—a thesis unto itself. For the purposes of model building, this chapter will examine the most popular and most reasonable components that could make up a successful quintessential inflation theory. The components outlined herein are not at all meant to be exhaustive, but are meant to represent a coarse classification of differing Q.I. theories, and the theoretical commitments therein.

Choices for inflation models

The first such theoretical commitment is the specific form of the potential component of the inflation scalar field. The literature broadly divides these into (among other categories) high and low energy scale potentials, the simplest examples of which correspond to new inflation/chaotic inflation theories and inflection point inflation theories, respectively.

Chaotic inflation and model independent constraints

An inflationary potential that captures most of the relevant features of chaotic inflation looks something like [5]:

$$V(\phi) = \frac{1}{2}\mathrm{m}^2\phi^2 \tag{18}$$

The reason these sorts of potentials are considered "high energy" is that the starting value of the inflaton field is generally very high, much greater than of $10^{19}GeV$[13]. Now, from our previous arguments in chapter one, we know that this potential term must dominate the kinetic term. That is, equation (16) must hold. This allows us to apply the Euler-Lagrange equations and drop terms of inconsequential order[5].

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = \ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \tag{19}$$

The usual classical interpretation of this equation suggests something like a ball, ϕ, rolling down a hill described by the function V, as in Figure 2. The middle term, $3H\dot{\phi}$, is often called the Hubble Friction, and can just be thought of as a damping term that drains scalar field energy and dumps it into the expansion of space. This rapid expansion of space can be made more concrete in terms of the number of e-foldings (as hinted at in equation (6)) in the form of the first compatibility condition for our inflaton field[20]:

$$N(\phi) = \int_{\phi}^{\phi_e} \frac{Hd\phi}{\dot{\phi}} \lesssim 65 \tag{20}$$

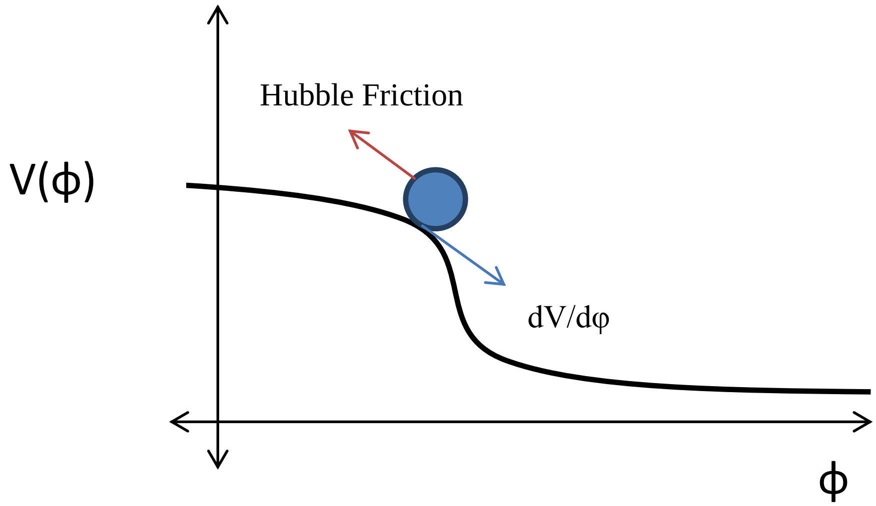

FIG. 2. Scalar field idealized as a ball. The “ball” rolls down the hill, responding to the shape of the hill (the potential), and Hubble Friction.

Note also that, for this specific potential (which has yet to be ruled out by observation) the form of the differential equation exactly mirrors that of a spring-mass system with some damping term.

For the rest of cosmic history to occur, as we understand it, the huge amount of energy in this scalar field has to be transferred to other types of energy in a process called *reheating*. While I’ve yet to discuss other inflationary models, reheating must occur in all models. Generally, this transfer occurs via coupling of the inflation field to some other scalar field, call it χ. As the inflation field has some velocity when it finally does reach the minimum, it proceeds to oscillate about the minimum of the potential specified in (18). As it oscillates, it dumps energy into χ, and the χ field is responsible for the subsequent radiation-dominated era of our cosmic history.

That we know reheating occurred suggests the second such compatibility condition mentioned in Chapter 1. The inflaton field must perform a so called "graceful exit"[14]. If the velocity of the field isn't properly tuned as it approaches the minimum, reheating may never occur, or the field could just completely overshoot the minimum. The constraint arises in two direct forms: first, this imposes model dependent upper and lower bounds for the initial value of the field. Second, if the shape of the potential doesn't satisfy the "slow roll" constraints, inflation will never happen, even in principle. The "slow roll" parameters are usually defined as follows [15]:

$$\epsilon \equiv \frac{m_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \qquad (21)$$

$$\eta \equiv \frac{m_{pl}^2}{8\pi}\left(\frac{V''}{V}\right)$$

These are qualitatively the slope and curvature of the potential function. For inflation to occur, it can be shown that both of these must be much less than 1 in the neighborhood of the scalar field value during inflation. Furthermore, when η is $O(1)$, inflation ends [20].

Calculations have also been done that strongly suggest quantum fluctuations in the inflaton field directly seed the large scale structure of the universe as we see it today[5]. These fluctuations can also give rise to small deviations in scale invariance of the cosmic microwave background spectrum[16], which admits another semi-model independent compatibility condition.

Before discussing that compatibility condition, we need a more precise notion of what scale invariance even means. By examining the cosmic microwave background, we can express the power of the spectrum as a function of the scale of the features in that spectrum via a sort of Fourier decomposition. A completely scale invariant power spectrum would be flat—that is, there would be an equal contribution to the overall power of the observed spectrum from every length scale—usually called Gaussian at all length scales. Inflationary arguments suggest a power spectrum of the form[17]:

$$P_s(k) \propto k^{n_s-1} \tag{22}$$

Where k is the Fourier coordinate (the length scale), and n_s is the so called *spectral index*. A careful quantum field theoretic treatment of the density perturbations introduced by many inflationary theories, including our simple quadratic chaotic potential in equation (18), suggest a spectral index of nearly 1[18]. This bodes well, because we observe that the CMB's spectrum is very close to scale invariant, with an observed spectral index of $.963 \pm .012$ (note that if the exponent is 0, the power spectrum is flat, that is, $P_s(k) \propto 1$). Doubly fortuitous is the recent result that inflation models suggest very slight nongaussianity, which currently agrees with observation[19].

On general inflation-phenomenological grounds, we can also express the spectral index as (as long as slow roll conditions are satisfied) and the power spectrum [20]:

$$n_s - 1 \approx -3M_{pl}^2\left(\frac{V'}{V}\right)^2 + 2M_{pl}^2\left(\frac{V''}{V}\right) \tag{23}$$

$$\boxed{n_s = 1 + 2\eta(\phi_{cmb}) - 6\epsilon(\phi_{cmb})}$$

$$\boxed{P_R = \frac{1}{24\pi^2 M_{pl}^4}\frac{V_0}{\epsilon_{cmb}}}$$

Where M_{pl} is the reduced Planck mass equal to approximately $2.4 * 10^{18} GeV$ (note that this equation is intimately related to the slow roll conditions from equation (21), as they are used in deriving it). The cmb subscript indicates the value of ϕ to use to reproduce the perturbations we observe today. The best value for the latter amplitudes is $2.43 \pm .11 * 10^{-9}$ [20]. These formulae give us an easy way to calculate the spectral index and power spectrum compatibility conditions.

Inflection point inflation and the narrowing parameter space

The picture of inflation so far presented has focused on the high energy regime of the total parameter space. I'd like to contrast this with a different approach both for didactic purpose, as well as to demonstrate how the compatibility conditions shift under different initial assumptions about the inflaton field. The better we understand the landscape of available inflations model, the better we can ground our eventual Q.I. model.

Inflection point inflation models are characterized by two things—their much lower energy scale, and the existence of an inflection point in the particular potential used, where an inflection point is just some point on the potential satisfying $\frac{d^2V}{d\phi^2} = 0$. A fairly standard potential specification with a reasonable theoretical motivation could look something like [20]:

$$V(\phi) = A\phi^2 - C\phi^3 + B\phi^4 \tag{24}$$

Where the C can be expressed as a specific function of A and B and can be tuned to move the point of inflection around, effectively specifying some two variable parameterization of the potential. A more general treatment of the theory of inflection points is detailed in [21]. Where before we were concerned with upper and lower bounds on the initial value of the field, now all we require is that the starting field value be close to the inflection point. Notice also that the η slow roll condition in equation (21) is trivially satisfied by the inflection point requirement.

The dynamical picture here is much the same as it was before. Our inflaton field slowly rolls down the potential until sufficiently many e-foldings have occurred to fix the horizon problem, and reheating sets in via coupling to some χ field. The main point of contention with these models is the inescapable need for fine tuning to get the desired behavior during inflation (which is qualitatively "worse" for inflection point models than it is for chaotic potentials). Some[20] have suggested adding an additional scalar field during inflation to ameliorate some of the fine-tuning issues, but while it works, it is not in the spirit of a quintessential inflation model—we seek a single dynamical explanation, and introducing extra fields defeats the purpose of the exercise. Fine tuning can also be reduced by raising the scale of inflation[20], but this also somewhat defeats the purpose of resorting to a low energy scale argument in the first place.

Tracker quintessence models

Just like in inflation, there are a veritable zoo of potentials that have been proposed[22][25] in the literature for which scalar field *could* describe the phenomenon of the late time accelerated expansion of space. In the spirit of choosing models that are as free from fine tuning as possible, I'll stick to "tracker" solutions[5], which have the nice property that vast regions of the parameter space of solutions all evolve to approximately the same final value. Once again, I'll examine some compatibility conditions for what makes a "good" quintessence potential.

Constraining a potential

When selecting quintessence potentials, the chief parameter of study is the equation of state or w_Λ parameter introduced in equation (16), reproduced here, making use of equations (11) and (12):

$$w_{\Lambda\,\mathrm{today}} = \frac{P_\Lambda}{\rho_\Lambda} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \approx -1 \tag{25}$$

So the potential we seek actually hides within the w_Λ parameter. Further, it is common in the literature to write down a simple parameterization of the w_Λ parameter, as current and future data, through 2020, will only constrain, at best, a two parameter model.[23] The popular choice seems to be[24]:

$$w_\Lambda = w_0 + w_a \frac{z}{1+z} \tag{26}$$

Making this parameterization is not necessary, though, and it really depends on whether or not we're trying to fit w_Λ directly, or $V(\phi)$.

The standard model-building procedure once again is to pick a potential and check the behavior of that potential versus observational constraints—that is, the compatibility conditions. First, we'd like to reduce the parameter space, so we will impose so conditions on the shape of the potential. As I mentioned tracker solutions, all tracker solutions must satisfy the relation[25]:

$$\frac{V''V}{V'} \geq 1 \tag{27}$$

We will also impose the additional conditions of [22] that the quintessence field is not allowed to "turn around" as it rolls down the potential—that is $\frac{dV}{d\phi}$ is not allowed to change sign—and the field must only roll down the potential. This modestly constrains our parameter space, and many of our previously considered inflaton potentials fortuitously satisfy the first of these relations.

The w_Λ parameter falls naturally out of the luminosity distance which we already discussed in equation 15, but that parameter was written assuming a constant equation of state. We will also be concerned with a more readily measurable quantity, directly related to the luminosity distance, called the angular diameter distance, which, qualitatively, tells us the ratio of an objects apparent size (as some number of degrees subtended in the sky) to its actual size[24]:

$$D_A(z) = \frac{1}{1+z}\int_0^z \frac{dz'}{H(z')} \tag{28}$$

As well as the $D_V(z)$ parameter of Eisenstein et. al which comes from Baryon Acoustic Oscillation measurements[24]:

$$D_V(z) = \left((1+z)^2 D_A^2(z)\frac{z}{H(z)}\right)^{\frac{1}{3}} \tag{29}$$

We will use these shortly.

These equations make use of the Friedmann equation's formula for the Hubble parameter, which we need to rewrite for a now dynamical w_Λ[24]:

$$H^2(z) = H_0^2\,[\Omega_m\,(1+z)^3 + \,\Omega_\Lambda(\mathrm{z})] \tag{30}$$

$$\Omega_\Lambda(z) = \frac{\rho_\Lambda}{\rho_c} = (1-\Omega_m)\exp\left(3\int_0^z [1+w_\Lambda(z')]\frac{dz'}{1+z'}\right)$$

The dynamical nature of the equation of state gives rise to the integral expression for Ω_Λ in equation (30) above. We'd also like to have expressions for both the potential and the scalar field itself as a function of the w_Λ parameter, so we can rewrite the equation of state[24]:

$$\phi(z) - \phi_0 = \pm\int_0^z \frac{\sqrt{[1+w_\Lambda(z')]\rho_c\Omega_\Lambda(z')}}{H(z')}\frac{dz'}{1+z'} \tag{31}$$

$$V(z) = [1-w_\Lambda(z)]\,\frac{\rho_c\Omega_\Lambda(z')}{2}$$

Where these directly follow from the definition of the equation of state and its time-integral (where we change variables from t to z). These equations can of course be inverted to give expressions for $w_\Lambda(z)$ in terms of the potential and field itself.

With these preliminary results in hand, we can dive into the actual constraints from the WMAP and BAO data[24]:

$$R \equiv \sqrt{\Omega_m H_0^2} \int_0^{1090.04} \frac{dz'}{H(z')} = 1.710 \pm .019 \tag{32}$$

$$D_V(.35)/D_V(.2) = 1.812 \pm .06 \tag{33}$$

The first of these is the "cosmic shift parameter" of Bond et. al, and the second is the aforementioned D_V parameter—a sort of volumetric angular diameter distance.

Our last condition is derived from Big Bang Nucleosynthesis arguments, and simply requires that Ω_Λ during the radiation dominated era is approximately less than or equal to .05[5]. With these two constraints, we're essentially a numerical integration away from constraining any quintessence model.

Picking a potential—freezing and thawing

The dynamical behavior of the field for quintessence models can be broadly divided in two[12]: models where the field has only recently begun rolling down the potential—thawing models—and fields that are gradually slowing down over time—freezing models. Which class our quintessence potential happens to fall under depends on the details of the inflaton field, and chapter three will detail my efforts at determining how the initial conditions of the inflaton field affect the dynamical behavior at late times. A similar demarcation could be made between models that track and models that don't track, as in[12], but again, we're only focusing on tracking solutions to simply the picture. The details are intentionally sparse here, as the details of reheating are also important for determining how late time dynamics of the Q.I. field, but those details are a bit outside the scope of this paper.

Marrying inflation and quintessence

With general procedures outlined for tailoring any given inflation or quintessence model to observation, the question remains—how do we join them together? We saw a hint of this with Peebles' and Vilenkin's model at the end of chapter one, where they simply massaged the boundary conditions of two completely different scalar field models together into a single model—a bit of a haphazard gluing process. We could also stipulate any number of nontrivial couplings between the Q.I. field and the matter or radiation fields to get the desired behavior, as in [11]. There is not really any general procedure to go about because we do not yet know enough about the nature of the quintessence and inflation fields themselves.

Efforts have been done to constrain the possibility of quintessential inflation, as in [12], but the results are highly model dependent, just as they will be for my studies. The eventual goal is to attack the problem in slightly more generality, but with too much generality, the problem becomes untenable. Again, chapter three will detail some modest results concerning the types of models outlined herein.

CHAPTER III

RESULTS

This chapter will serve as a sort of work flow for the theoretical tools developed in chapter two. As the space of possible potentials is vast, I will focus on the relaxed problem of whether or not an inflection point inflation potential can be massaged into a quintessence potential. That is, I will examine the parameter space of potentials of the form:

$$V(\phi) = A\phi^2 - C\phi^3 + B\phi^4 \tag{34}$$

This chapter will concern the inflation side of the potential, and chapter four will be devoted to a discussion of Quintessential Inflation models in a broader context with respect to the results herein, as well as results elsewhere.

Constraining the inflationary era

Reference [20] thoroughly examines the inflationary constraints of such a parameterization, essentially by using the constraints listed here as equations (20) and (23). We proceed in slightly more generality, as [20] is motivated by string theoretic arguments, where we are simply outlining a strategy. For potentials of this shape, we can also examine some characteristic behaviors of chaotic-like models as well.

We will first pursue the e-foldings constraint. Equation (30) admits two inflection points:

$$\phi = \frac{3C - \sqrt{3}\sqrt{-8AB + 3C^2}}{12B} \tag{35}$$

$$\phi = \frac{3C + \sqrt{3}\sqrt{-8AB + 3C^2}}{12B}$$

We will consider the bottom inflection point first to point out some features of chaotic models, and then move to the regime for inflection point models (the top value).

For example, if we set A=.256, B=.008, and C=.626 in equation (34), we have the potential:

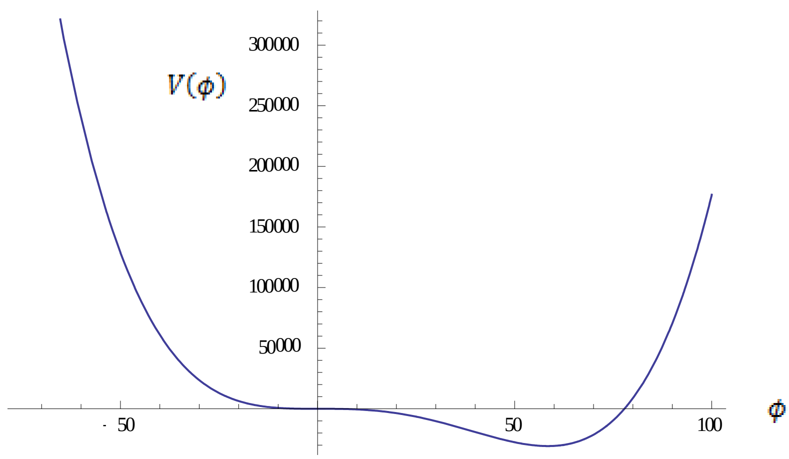

FIG. 3. Example ad-hoc inflection point potential.

Suppose we specify that the field starts slightly to the right of the larger inflection point in Figure 3 (approximate 38.9 units along the x-axis of this graph via equation (35)). We would like to know the behavior of the field. Equation (19) gives us the equations of motion, so after a quick numerical integration, we have (where we have abandoned units for simplicity):

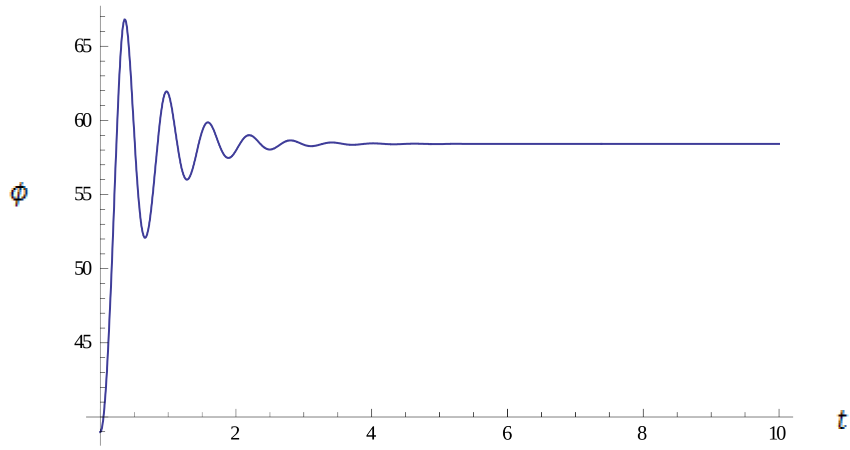

FIG. 4. Inflation field ringing about the global minimum. The y axis in this figure is the x axis in Figure 3, and the x axis here is "time".

As we expected, Figure 4 depicts the inflaton rolling back and forth about the global minimum. We know that inflation ends when $|\eta| \sim 1$. Plotting $\eta(t)$:

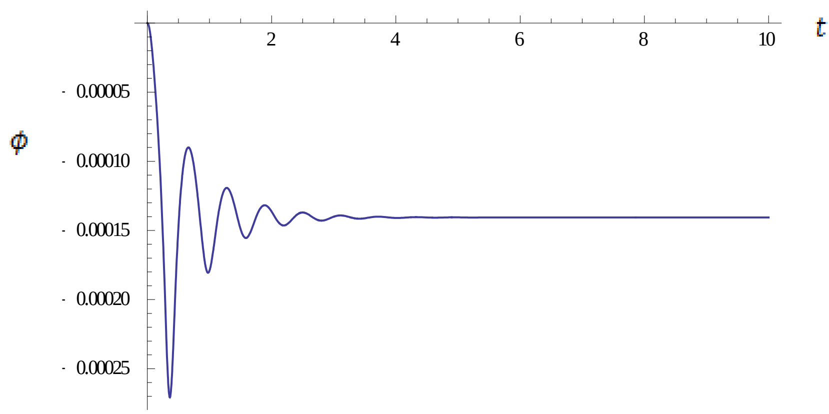

FIG. 5. η parameter ringing about its minimum value. Note the smallness.

Figure 5 shows that η levels out much too soon on this model. After a numerical integration for the e foldings, we calculate that, for these coefficients, only about 1 e-folding occurs—a common failure for poorly chosen model coefficients.

A slightly more judicious choice of coefficients A,B, and C → (.07,.0038,.053) yields a largely similar shape:

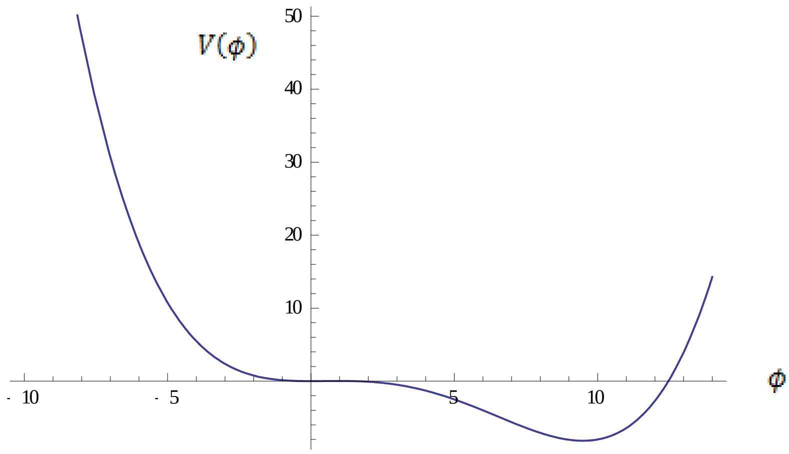

FIG. 6. Another example ad-hoc inflection point potential.

But now we see an overdamped convergence to the global minimum:

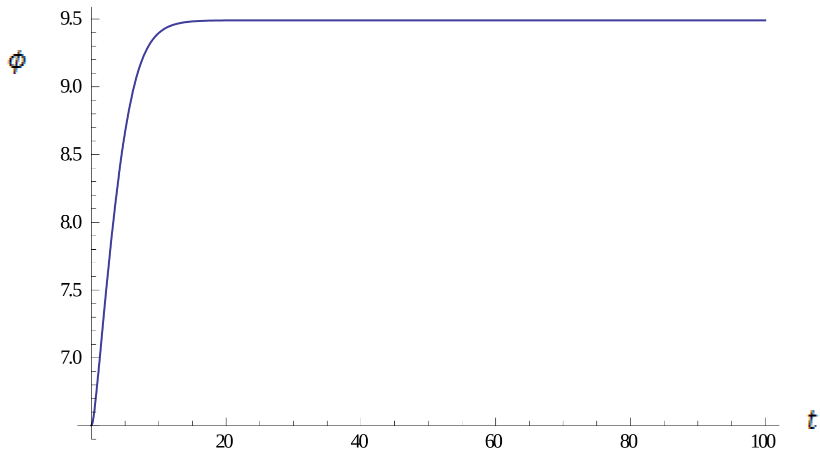

FIG. 7. Inflation field converging to the global minimum. Intuitively looks like an overdamped oscillator, or a ball rolling down a hill with a large amount of friction.

And a quick calculation of the number of e-foldings for the potential in Figure 6 yields about 55—a realistic value. Note that both sorts of dynamical behaviors can result in “valid” inflationary models, it just so happened that my overdamped trajectory, depicted in Figure 7, better satisfied the constraints.

This is essentially the game we play. We select coefficients, we calculate the equations of motion, and we determine the number of e-foldings. Those models that satisfy the e-foldings condition are kept, and those that don’t are rejected. Models that make the first round pass are further scrutinized with additional constraints. In an exhaustive analysis, further care must be taken with units, as I have somewhat haphazardly set lingering constants equal to 1 to illustrate the general concept as well as the dynamical behaviors involved. Furthermore, the vacuum expectation value of the field is generally fine-tuned to ≈ 0, which I have neglected to do here, but will account for the in following analysis. For completeness, it should be noted that the value of the Hubble constant during inflation can be written[20]:

$$H_{inf} \approx \frac{V_0^{\frac{1}{2}}}{\sqrt{3}M_{pl}} \tag{36}$$

Where V_0 is the value of the potential at the inflection point (H_{inf} has been $\approx O(1)$ for my previous analysis, in Planck units). In absolute generality, the Hubble constant of course depends on the field value via the Friedmann equation:

$$H = \sqrt{\frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)} \tag{37}$$

The slow roll constraint accounts for the lack of a $\dot{\phi}$ term in equation (36).

As a general comment, it's often useful to reparameterize the potential in terms of something more tangible than the leading coefficients of the polynomial. For example, we could reparameterize to a set of three controls where we directly control the width of the potential, the separation of the inflection point and the global minimum, and the separation of the inflection point and zero. While this makes the controls intuitively clear, such a set is not *mathematically* clear, so we will proceed differently.

Before we proceed to the power spectrum analysis, there are a couple more salient dynamical features of this model. Potentials of this shape admit two inflection points, so it is natural to ask about the trajectories starting on the far left side. In fact, inflection point inflation models are primarily concerned with trajectories beginning in the blue region in Figure 8:

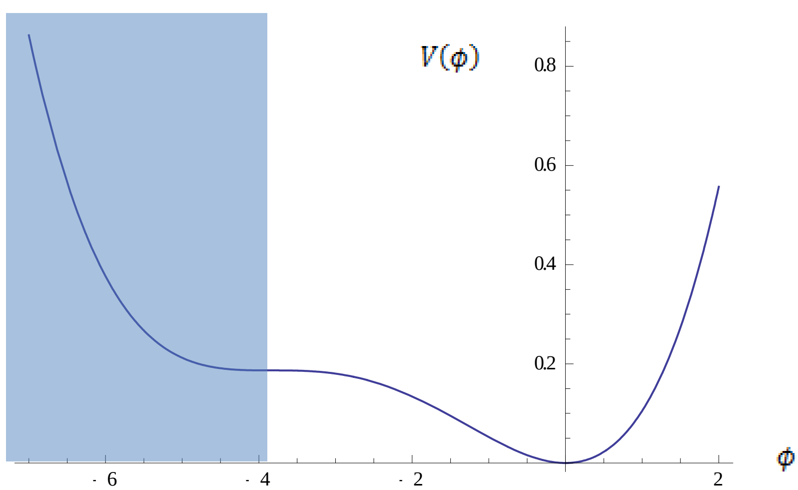

FIG. 8. Sample field with region of interest shaded.

Inflection point potentials have the interesting property that, for a large range of initial conditions, the field is attracted to the inflection point—that is, the inflection point is an attractor. For this above parameter set $(A, B, C \rightarrow\ (.39^2),\ (-.176 * 39),\ (.76^2))$, starting the field anywhere between the leftmost inflection point and ≈-10 causes the field to end on the that leftmost inflection point—not the global minimum—in Figure 9:

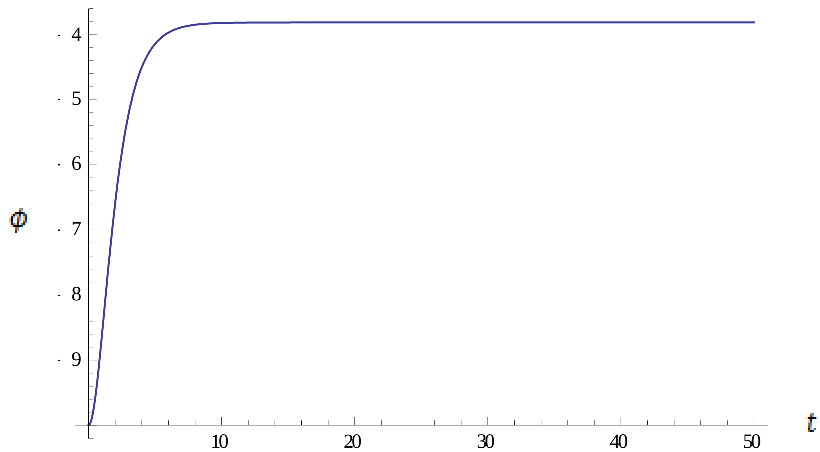

FIG. 9. Inflation field converging to local minimum.

To beat this attractor property, and to allow inflation to actually end, the field has to be perturbed slightly at the inflection point. With the addition of this perturbation, and starting slightly above this starting value, brings the field down:

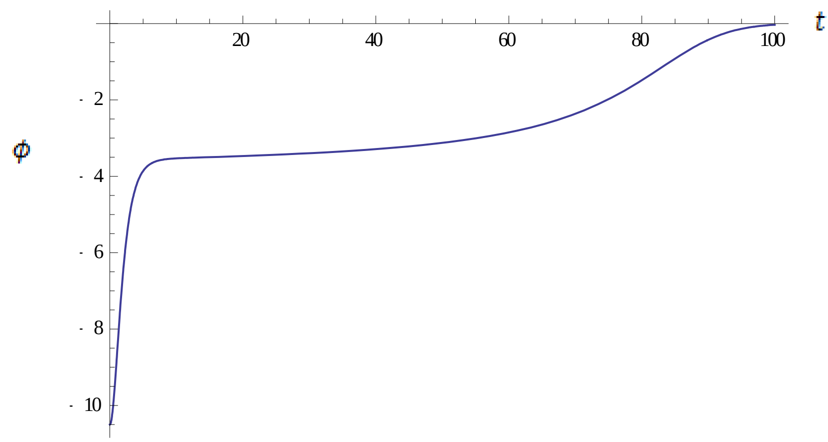

FIG. 10. Inflation field marginally overshoots the local minimum. The field eventually converges to global minimum.

The potential shape in Figure 10 suggests an interesting possibility for those solutions which are not overdamped. Ringing, if properly tuned, can eventually converge to either the global, or local minimum:

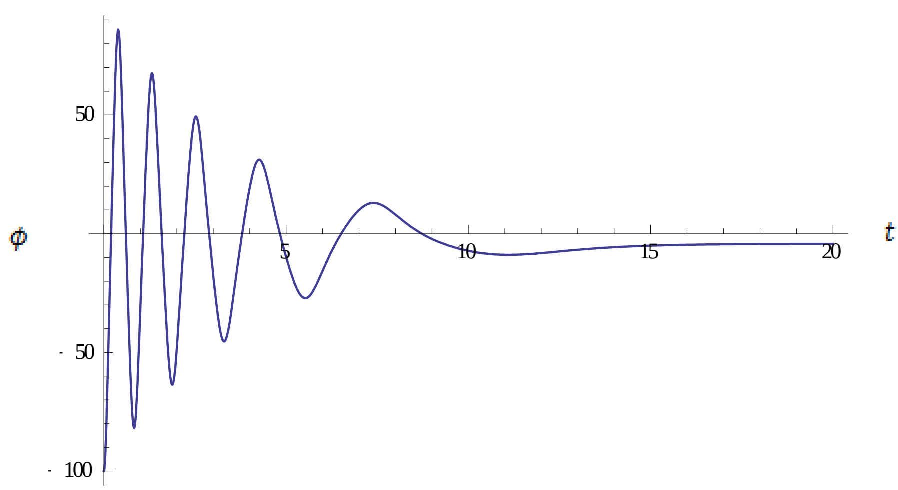

FIG. 11. Ringing behavior getting caught on the local minimum.

Convergence to the local minimum means that inflation never ends, like in Figure 11. These fine tuning issues will be revisited in chapter four. In particular, the extra degree of freedom required to tune the perturbation has been a prime criticism of inflection point inflation models. For the power spectrum and spectral index analysis I will adapt the parameterization of [20] to clarify the analysis of the parameter space. That is, I will write:

$$V(\phi) = \frac{m^2}{2}\phi^2 - \frac{2hm}{3\sqrt{2}}\phi^3 + \frac{h^2}{12}\phi^4 - \lambda\phi \qquad (37)$$

Where λ is small. This parameterization offers a particularly simple form of the coordinates of the inflection and critical points. For this analysis, we must consider equation (23). Notice in the two sample cases above, the field rolls to a minimum where V' is trivially zero, hence η dominates ϵ for the spectral index. The amplitudes, however, are independent of η, so combining these results, we essentially have constraints on η and ϵ individually. Further, the time at which these parameters are evaluated is critical, as after inflation has ended, these quantities are, in a sense, frozen out. Figure 12 details these different regimes:

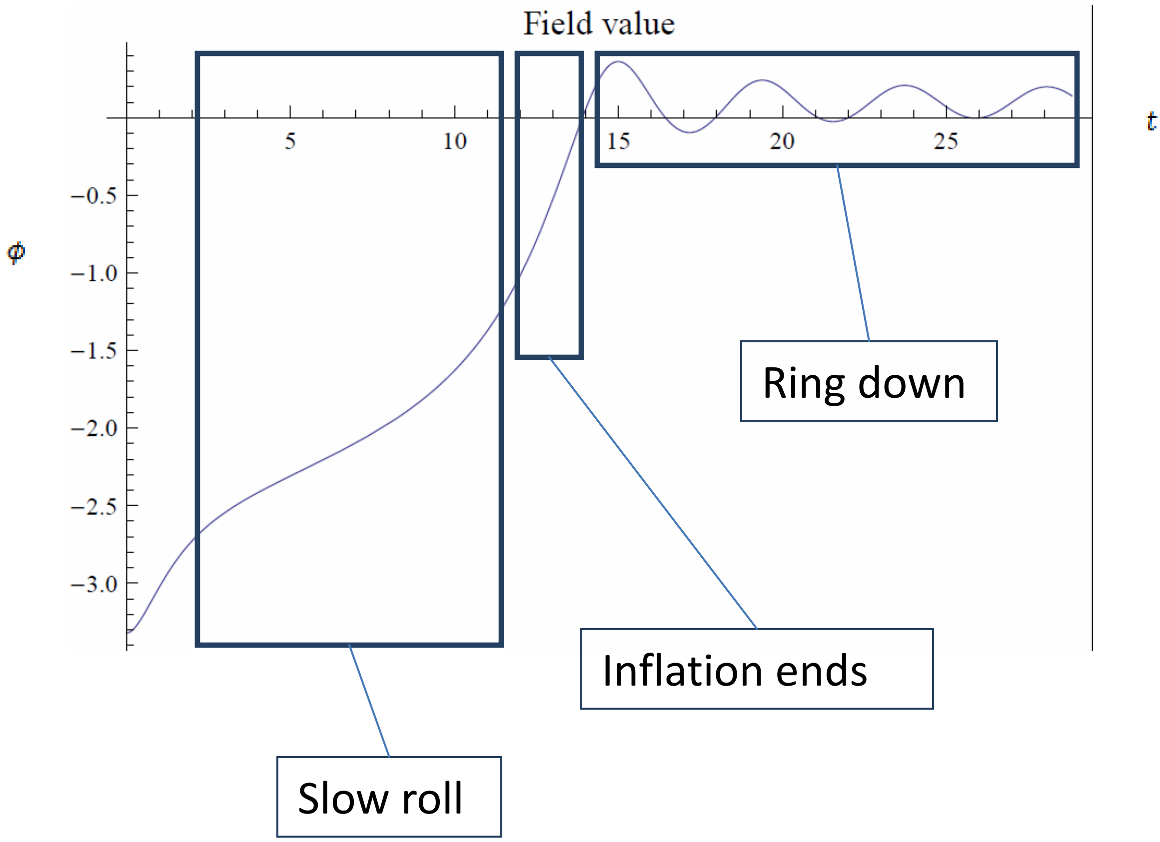

FIG. 12. Inflationary epochs for an example field. $m, h, \lambda \rightarrow -1.35, 1, .2$

Just to emphasize that the e-foldings are actually frozen out, Figure 13 depicts a plot of number of e-foldings versus time for the above model:

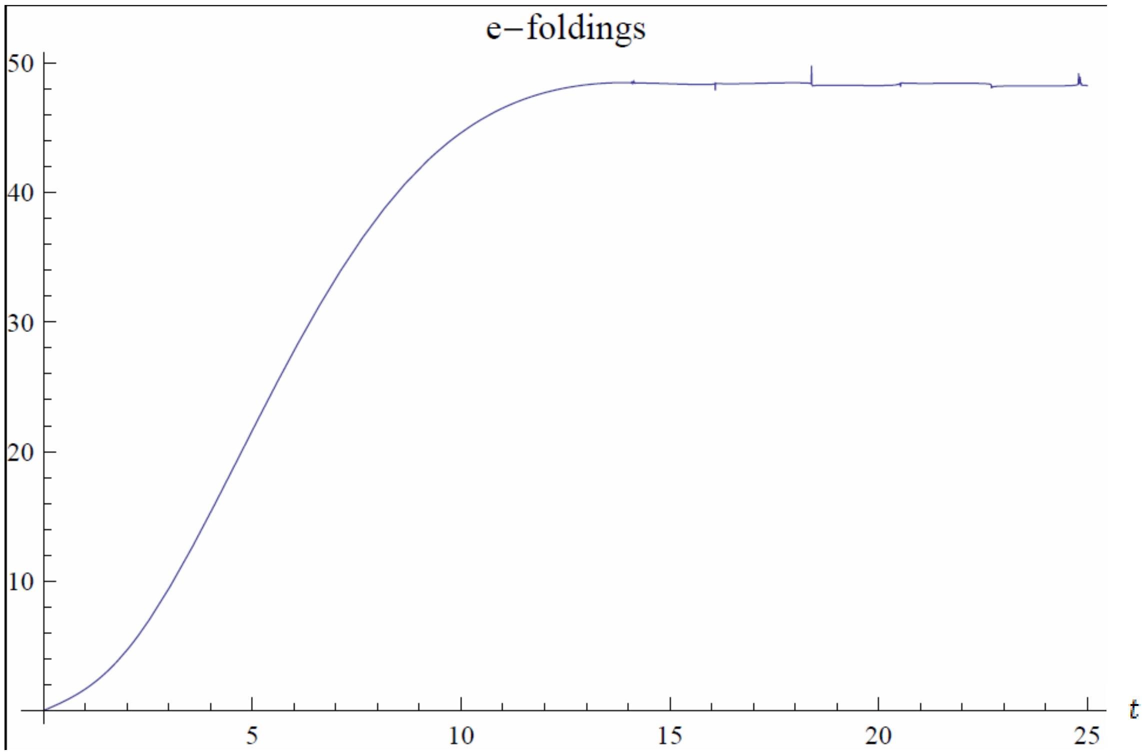

FIG. 13. Number of e-foldings versus time. The artifacts past t≈14 are due to errors in the precision of the numerical integration.

This suggests that the evaluation time for the spectral index and the power spectrum amplitudes should be around the time when the e-foldings level out. Plotting the spectral index (i.e. $1 + 2\eta(\phi)$) and power spectrum together for the sample model above, we see that the region satisfying the constraint for the spectral index is not quite the same region satisfying our constraint on the power spectrum.

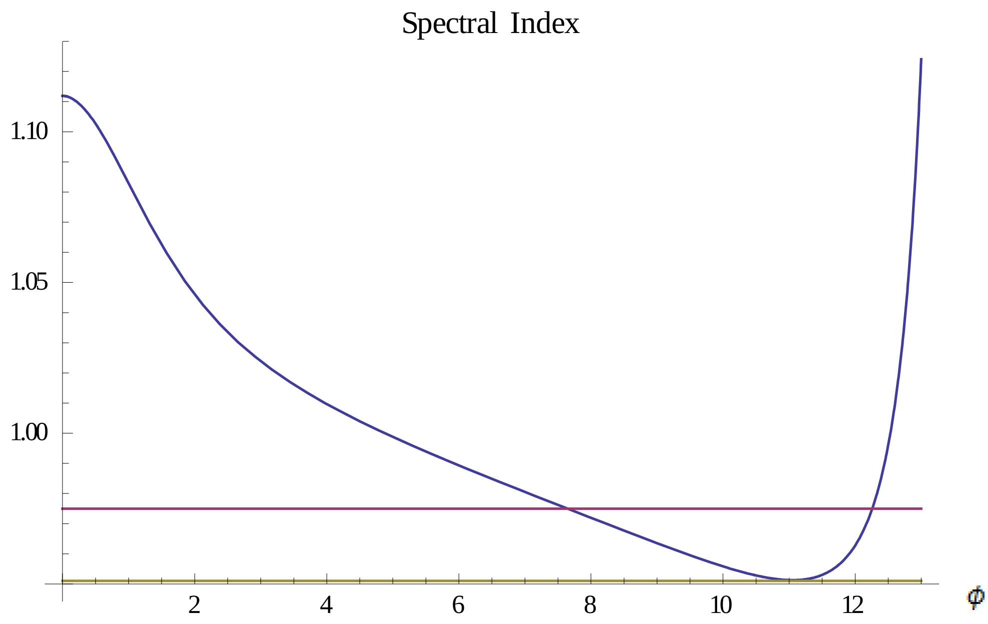

FIG. 14. The spectral index for different values of ϕ. The x-axis is field coordinate, and the y axis is the spectral index. Region below horizontal line satisfies constraint.

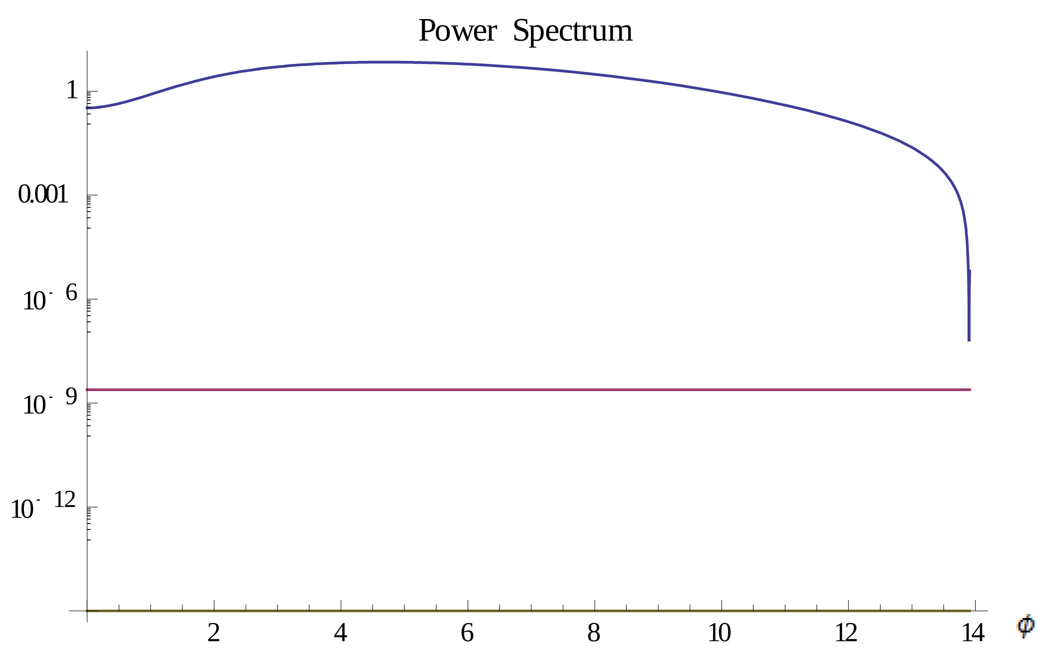

FIG. 15. The power spectrum for different values of ϕ. The x-axis is the field coordinate, and the y-axis is the power spectrum amplitude. All amplitudes too large.

Note that, for this model, the spectral index in Figure 14 lands neatly into the required range, but there's no corresponding region for the power spectrum amplitudes in Figure 15. Once again, this is the game we must play. We must vary the collection of coefficients, the field starting value, and the linear perturbation so as to match our 3 indicated constraints. As [20] already performed this analysis, we reproduce it here in Figure 16:

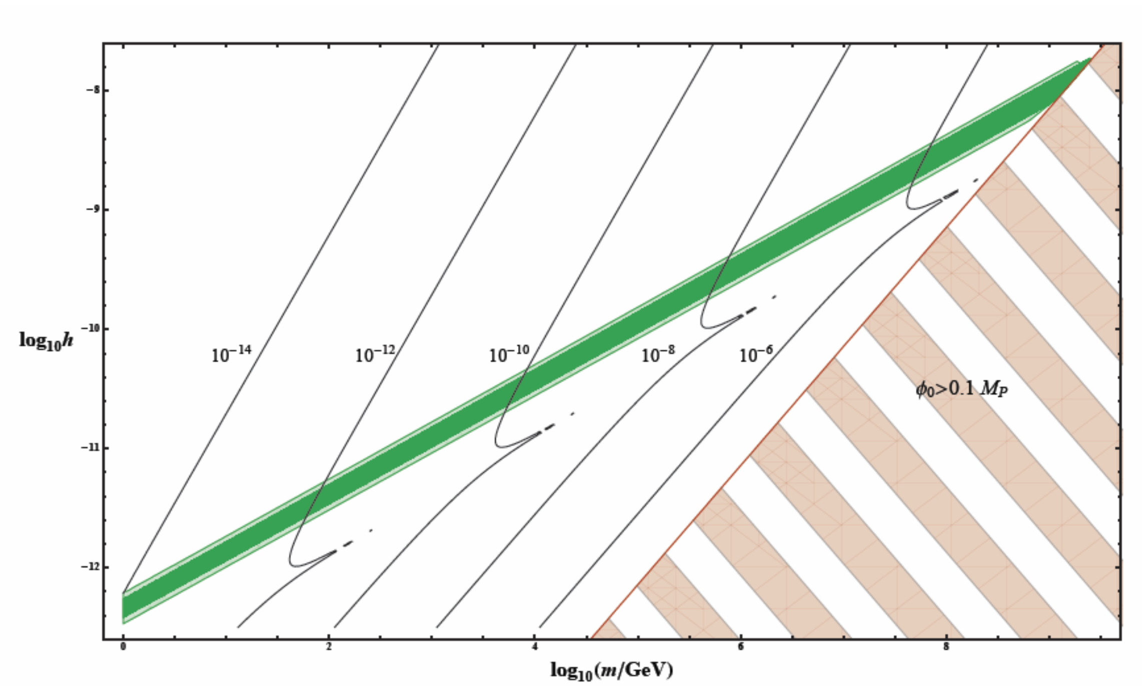

FIG. 16. Phase space analysis of the inflection point potential presented in [20]. The green band corresponds to combinations of coefficients that satisfy all the parameters I have described. Further, the crossed out region is rejected by nature of the resulting inflationary mechanism being too large of an energy scale.

This leaves a large swath of parameter space available to tease into a quintessence potential, to be discussed in chapter four.

CHAPTER IV

SUMMARY AND CONCLUSIONS

Ideally, as observational constraints become increasingly stringent, bands of allowed parameter space, like in Figure 16, will continue to shrink. But continually being in a state of not-yet-excluded is an epistemically precarious place to be. The dream would, of course, be to construct a quintessential inflationary potential with some fundamental physical motivation that matched the constraints outlined within this paper as well as elsewhere.

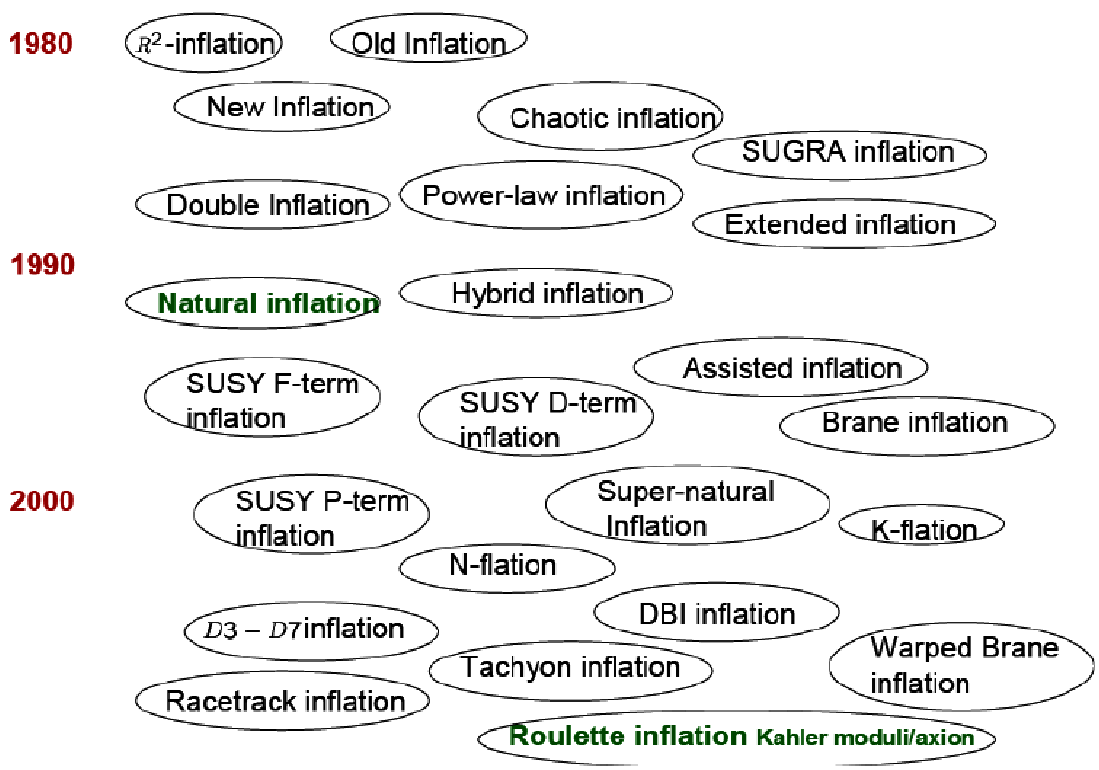

FIG. 17. Competing theories of inflation ordered from the oldest to newest. Image from [26].

Unfortunately, nature is not so simple. Figure 17 details the history of proposed inflationary mechanisms. Most of them are not yet ruled out observationally. Some are qualitatively similar to the mechanisms I have discussed, particularly chaotic-like and inflection-point-like models. Some use completely different mechanisms. This is not to mention the different, competing models the process of reheating, and the different, competing models for the entirely separate problem of dark energy like Chaplygin gas models, holographic models, and quintessence models—the last of which I briefly discussed.

This places the model-builder in an awkward situation. Speculative model building can, and has, in my opinion, answered the question “Can a quintessential inflation model be constructed?”. Much less compelling are the proposed answers to the question, “Are quintessential inflation models *reasonable*?”. Constructability and reasonability are massively different. Ptolemaic epicycles are constructible. They are even accurate, in a twisted sense. But they are not reasonable, nor are they predictive of the underlying mechanisms of gravity.

In my research, much of the actual model building was stalled by the sneaking suspicion that my efforts were for naught. Suppose we set aside the vast number of competing explanations for the processes of inflation, reheating, and dark energy. Suppose that some data suggests one scalar field must be responsible for inflation and dark energy (which no data does). The proposed models still require miraculous amounts of fine tuning to satisfy all of the compatibility conditions listed above—the linear perturbation to the field in equation (37) is a particularly obvious example of this. Reference [14] in particular was able to actually exclude a certain class of Q.I. models, but only for the narrow window in theory-space comprised of Braneworld inflaton fields. The eager model builder need only tap-dance to another, competing mechanism for inflation to hide within the safety of not-yet-exclusion. This is not at all satisfying.

I was not able to probe the space of quintessence models to my satisfaction, but it is worthwhile to point out that a simple cosmological constant as an explanation for dark energy has not yet been ruled out. I would argue that such models trump Q.I. models in both aesthetic and real simplicity, as would Occam's Razor. The onus, then, really is on the model builder to properly motivate the model, which is something that I was not able to do for my own model, nor elsewhere. In my literature survey, I was not able to find any sort of predictive power unique to Q.I. models.

This criticism should not be confused with a criticism of the study of inflation or quintessence individually, nor a suggestion that there is no predictive power to such models, in principle. In fact, it is in large part because of the difficulty of the individual problems that tying the models together seems premature. A recent result [27] suggests that parts of the process of inflation may always be a black box, which may mean that we will never have access to information before a certain point in cosmic history. The authors discovered a phase transition in inflection point models that essentially blurs our ability to meaningfully distinguish between models before a certain period—that is, many different initial conditions of inflection point type models all satisfy the compatibility conditions I have outlined in the same way. In one interpretation, this makes it "easier" to craft a Q.I. model, as the space of allowed initial values increases; however, this does not seem like an honest victory for the model builder. Having more room in parameter space in which to construct a theory is all well and good, but vastly more satisfying would be to be able to construct a model that had more to say for itself than that it satisfied constraints.

Quintessential inflation, then, is a curiosity. It bears striking similarity to the problem of Einstein's cosmological constant. It is motivated by aesthetics—we do not have a very good reason to believe that inflation and dark energy are mediated by the one same field, but that fields can be constructed to solve both problems is interesting. Unfortunately, the process of contorting the fields into the required forms leaves potential shapes with no known physical motivation—again, much like Einstein's ad hoc addition of a constant so that the universe may be static. Perhaps, in a hundred years, we will have a good reason to think one field governs these problems, and the next generation of physicists will see that yet another century old curiosity turned out to be on the right track, if a bit off the mark.

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