ON PARTIAL AND GENERIC UNIQUENESS OF BLOCK TERM TENSOR DECOMPOSITIONS IN SIGNAL PROCESSING

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ABSTRACT

In this dissertation, we study the partial and generic uniqueness of block term tensor decompositions in signal processing. We present several conditions for generic uniqueness of tensor decompositions of multilinear rank $(1,\ L_1,\ L_1), \ldots, (1,\ L_R,\ L_R)$ terms. Our proof is based on algebraic geometric methods. Mathematical preliminaries for this dissertation are multilinear algebra, and classical algebraic geometry.

In geometric language, we prove that the joins of relevant subspace varieties are not tangentially weakly defective. We also give conditions for partial uniqueness of block term tensor decompositions by proving that the joins of relevant subspace varieties are not defective. The main result is the following. For a tensor Y belong to the tensor product of three complex vector spaces of dimensions I, J, K, we assume that $L_1, L_2, \ldots, L_R$ is from small to large, K is bigger or equal to J, and J is strictly bigger than L_R. If the dimension of ambient space is strictly less than IJK, then for general tensors among those admitting block term tensor decomposition, the block term tensor decomposition is partially unique under the condition that the binomial coefficient indexed by J and L_R is bigger or equal to R, and I is bigger or equal to 2; it has infinitely many expressions under the condition IJK is strictly less than the sum from L_1^2 to L_R^2; it is essentially unique under any of the following there conditions: (i) I is bigger or equal to 2, J, K is bigger or equal to the sum from L_1 to L_R (ii) R is 2, I is bigger or equal to 2 (iii) I is bigger or equal to R, K is bigger or equal to the sum from L_1 to L_R, J is bigger or equal to $2L_R$, the binomial coefficient indexed by J and L_R is bigger or equal to R.

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TABLE OF CONTENTS

1. INTRODUCTION AND BACKGROUND

1.1 The history and background of tensor decomposition

Tensor decomposition [7, 18] arises in numerous application areas: locating the area causing epileptic seizures in a brain, determining the compounds in a solution using fluorescence spectroscopy, and data mining, to name a few. In each case, researchers compile data into a multi-way array and isolate essential features of the data by decomposing the corresponding tensor as a sum of rank one tensors.

A *tensor decomposition* (see Chapter 11 of [21]) is the expression of a tensor as a linear combination of other tensors (presumably of lower rank). Concerning the uniqueness, imagine a doctor wants to perform surgery in the brain of a patient. The signal processing tool he is using computes the location of the pathological cerebral source with the help of tensor decompositions. If there are infinitely many solutions, the result cannot be exploited. On the other hand, if there are two or three solutions, the doctor may be able to select the the only one that is plausible. In medicine, other applications exist in the frame of Brain-Computer Interface for handicapped persons. In telecommunications, similar problems arise: receivers need to eliminate interference to improve on performance [15]. If tensor tools are able to compute finitely many directions of arrival of interferences, one can construct notch filters match to each of these directions. If there are infinitely many, nothing can be done. Many examples of this kind can be put forward, e.g. in data analysis [19].

Recently, De Lathauwer [12, 9, 10, 13] introduced the concept of block term tensor decompositions, because it is natural for certain source separation problems in signal processing, and often has better uniqueness properties than decompositions by tensor rank. Data matrices (see Chapter 1 of [19]) in signal processing are often noisy versions of low-rank matrices. Many applications essentially rely on the estimation of the true column space. In cases where the data matrix is not just a perturbed version of a low-rank matrix, it may still be interesting to know the subspaces of row and column space that contribute most to the data matrix. This is related to Principal Component Analysis [23]. Sometimes the goal is just a reduction of computational complexity: large problems are reduced to a more practical size by approximating the given data matrix by a matrix of low rank (the given matrix is compressed to a matrix of size $(R \times R)$, where R is the rank). Higher-order variants often lead to the approximation of a given higher-order tensor by a tensor of low multilinear rank. We refer the reader to [14] and [11] for the background and applications of block term decomposition in blind source separation. Therefore, the study of the uniqueness property of this kind of tensor decompositions is of interest.

Throughout this paper, for basic definitions, notation and results, we follow [21], which is addressed to both the numerical and the algebraic geometrical research communities.

First, we recall the celebrated theorem of Kruskal about the uniqueness of tensor decomposition in rank-1 tensors.

Definition 1.1.1. (see Section 1.1 of [4]) Let A, B, C three complex vector spaces, of dimensions a, b, c, respectively. A tensor $t \in A \otimes B \otimes C$ is said to have *rank* k if there is a decomposition

$$t = \sum_{i=1}^{k} u_i \otimes v_i \otimes w_i$$

with $u_i \in A, v_i \in B, w_i \in C$ and the number of summands k is minimal. Such a decomposition is said to be *unique* if for any other expression

$$t = \sum_{i=1}^{k} u_i' \otimes v_i' \otimes w_i'$$

there is a permutation σ of $\{1, \dots, r\}$ such that

$$u_i \otimes v_i \otimes w_i = u'_{\sigma(i)} \otimes v'_{\sigma(i)} \otimes w'_{\sigma(i)} \quad \forall i = 1, \dots, k.$$

An interesting property of higher-order tensors is that their decompositions are often unique, whereas matrix decompositions are not, for the tensor product of two vector spaces, an expression as a sum of r elements is never unique unless $r = 1$(see Chapter 12 Section 5 of [21]). Thus an obvious necessary condition for uniqueness is that we cannot be reduced to a two factor situation. For example, an expression of the form

$$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3 + \cdots + a_r \otimes b_r \otimes c_r$$

where each of the sets $\{a_i\}, \{b_j\}, \{c_k\}$ are linearly independent is not unique because of the first two terms.

Kruskal's theorem regarding uniqueness of expressions for tensors is well known, and we phrase it in geometric language.

Definition 1.1.2. (see Definition 11.3.2.1 of [21]) Let $\mathcal{S} = \{x_1; \dots; x_p\} \subset \mathbb{P}W$ be a set of points. We say the points of $\mathcal{S}$ are in 2-general linear position if no two points coincide, they are in 3-general linear position if no three lie on a line and more generally they are in *r-general linear position* if no $r-1$ of them lie in a $\mathbb{P}^{r-2}$. We let the *Kruskal rank* of $\mathcal{S}$, $k_{\mathcal{S}}$, be the maximum number r such that the points of $\mathcal{S}$ are in r-general linear position.

If one chooses a basis for W so that the points of $\mathcal{S}$ can be written as columns of a matrix (well defined up to rescaling columns), then $k_{\mathcal{S}}$ will be the maximum number r such that all subsets of r column vectors of the corresponding matrix are linearly independent. (This was Kruskal's original definition.) The following result is well known.

Theorem 1.1.3. *(Kruskal, [20]) Let $T \in A \otimes B \otimes C$. Say T admits an expression $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$. Let $\mathcal{S}_A = \{[u_i]\}$, $\mathcal{S}_B = \{[v_i]\}$, $\mathcal{S}_C = \{[w_i]\}$. If*

$$r \leq \frac{1}{2}(k_{\mathcal{S}_A} + k_{\mathcal{S}_B} + k_{\mathcal{S}_C}) - 1$$

then T has rank r and its expression as a rank r tensor is essentially unique.

In [4], the authors introduce an inductive method for the study of the uniqueness of decompositions of tensors, by means of tensors of rank 1. The method is based on the geometric notion of *weak defectivity*. For three-dimensional tensors of type (a, b, c), $a \leq b \leq c$, their method proves that the decomposition is unique for general tensors of rank k, as soon as $k \leq (a+1)(b+1)/16$. This improves considerably the known range for uniqueness.

Theorem 1.1.4. *(see Theorem 1.1 in [4]) Let $a \leq b \leq c$. Let α, β be maximal such that $2^\alpha \leq a$ and $2^\beta \leq b$. The general tensor $t \in A \otimes B \otimes C$ of rank k has a unique decomposition, if $k \leq 2^{\alpha+\beta-2}$.*

The results of my thesis, mainly concerning the uniqueness property, extend the range of applicability of block term tensor decompositions. To present these results, we first recall the definition of a block term decomposition of a tensor.

Definition 1.1.5. (see Definition 2.1 in [14]) A *block term tensor decomposition* of a tensor $Y \in A \otimes B \otimes C \cong \mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K$ in a sum of multilinear rank $(1,\ L_1,\ L_1), \ldots, (1,\ L_R,\ L_R)$ terms, is a decomposition of Y of the form

$$Y = \sum_{r=1}^{R} a_r \otimes X_r, \tag{1.1.1}$$

in which $a_r \in \mathbb{C}^I$, and $X_r \in \mathbb{C}^J \otimes \mathbb{C}^K$ is of rank L_r. (Each term consists of the outer product of a vector and a rank-L_r matrix.)

In (1.1.1), one can permute the r-th and r'-th term when $L_r = L_{r'}$. Also one can scale X_r, provided a_r is counter scaled as well. The decomposition is said to be *essentially unique* when it is subject only to these trivial identifications. The decomposition is said to be *partially unique* when it has finite number of representations modulo trivial identifications.

1.2 Statements of main results

The main result in this thesis is the following.

Theorem 1.2.1. *Assume that* $L_1 \leq L_2 \leq \ldots \leq L_R,\ K \geq J > L_R$. *If*

$$\sum_{1 \leq r \leq R} (J \cdot L_r + L_r \cdot (K - L_r) + I - 1) < IJK, \tag{1.2.1}$$

then for general *tensors among those admitting block term tensor decomposition as in* (1.1.1)*, the block term tensor decomposition*

(ı) *is partially unique under the condition*

$$(A)\ \binom{J}{L_R} \geq R,\ I \geq 2,$$

(ıı) *has infinitely many expressions under the condition*

$$(B)\ IJK < L_1^2 + \cdots + L_R^2,$$

($\imath\imath\imath$) *is essentially unique under the following conditions:*

$$(C)\ I \geq 2,\ \ J, K \geq \sum_{r=1}^{R} L_r,$$

$$(D)\ R = 2,\ \ I \geq 2,$$

$$(E)\ I \geq R,\ \ K \geq \sum_{r=1}^{R} L_r,\ \ J \geq 2L_R, \binom{J}{L_R} \geq R.$$

Remark. Here the meaning of *general* is that the set of tensors which do not have the respective uniqueness property is included in a proper subvariety (see Remark 2.2). Relation (D) in ($\imath\imath\imath$) also appears as Theorem 2.2 in [14] in a different context. The other relations are new.

Our proof is based on algebraic geometric methods presented in the next section. Also, the hypothesis (1.2.1) is actually not restrictive (see the beginning of Section 4). And if $\sum_{1 \leq r \leq R}(J \cdot L_r + L_r \cdot (K - L_r) + I - 1) > IJK$, then (1.1.1) has infinitely many expressions.

Using similar methods, we also establish results valid for:

($\imath$) any tensors and any multilinear rank (see Proposition 2.3.3),

($\imath\imath$) any 3-tensors of any multilinear rank (see Proposition 2.3.4).

1.3 Notations

1. As in [21], for a finite dimensional complex vector space V, $\mathbb{P}V$ denotes the projective space associated to V, π denotes the projection of $V\backslash\{0\}$ onto $\mathbb{P}V$; for a variety $X \subset \mathbb{P}V$, $\hat{X} \subset V$ denotes its inverse image under the projection π, which is the (affine) cone over X in V, and for $x \in X$, $[x]$ denotes $\pi(x)$.

2. Let S be a subset of $\mathbb{P}V$, then the span $\langle S\rangle$ is by definition the image of π on the usual vector span of $\hat{S}$ in V.

3. The Zariski closure of S in $\mathbb{P}V$ will be denoted by $\bar{S}$.

4. When we will need to specify the the elements of S and their linear span, we use the notation $\{s_1, s_2, \cdots\}$ and $\langle s_1, s_2, \cdots\rangle$ respectively.

5. For $x \in \hat{X}$, $\hat{T}_{[x]}X := \hat{T}_x\hat{X}$ is the affine tangent space to X at $[x]$.

6. For a vector space V, its dual space is denoted V^*. If $A \subset V$ is a subspace, $A^\perp \subset V^*$ is its annihilator, namely the space consisting of those $f \in V^*$ satisfying $f(a) = 0,$ for all $a \in A$.

7. $\hat{T}_x^\perp X := (\hat{T}_x X)^\perp$ is the affine conormal space of X at x.

8. $\mathfrak{S}_n$ is the symmetric group on n elements. Given $\sigma \in \mathfrak{S}_n$, we can express σ as disjoint product of cycles, and we can denote the conjugacy class of σ by $(1^{i_1}2^{i_2}\ldots n^{i_n})$, meaning that σ is a disjoint product of i_1 1-cycles, i_2 2-cycles,$\ldots$, i_n n-cycles. Sometimes we might use $(k_1, k_2, ..., k_p)$ (where $n \geq k_1 \geq k_2 \geq ... \geq k_p \geq 1$ and $\sum_{i=1}^p k_i = n$) to indicate the cycle type of σ. This notation means that σ contains a k_1-cycle, a k_2-cycle, $\ldots$ and a k_p-cycle.

2. PRELIMINARIES

2.1 Multilinear algebra

Definition 2.1.1. Let U and V be complex vector spaces and let U^* and V^* be dual vector spaces of U and V. We define $U^* \otimes V^*$ to be the set of all bilinear functions $f : U \times V \mapsto \mathbb{C}$.

Let A_j, $1 \leq j \leq n$, be finite dimensional complex vector spaces. The elements of $A_1 \otimes \ldots \otimes A_n$ are called n-tensors. When no confusion can occur, they are simply called tensors. For tensors there are several different notions of rank that we review below.

For $\beta_j \in A_j^*$, $1 \leq j \leq n$, where A_j^* is the dual space of A_j, let an element $\beta_1 \otimes \cdots \otimes \beta_n$ denote the unique element in $A_1^* \otimes \cdots \otimes A_n^*$ determined by the condition

$$(\beta_1 \otimes \cdots \otimes \beta_n) \vdash (v_1, \ldots, v_n) := \beta_1(v_1) \cdots \beta_n(v_n), \ v_j \in A_j. \tag{2.1.1}$$

We introduce the symbol $\vdash$, which will be used in Chapter 4 and Chapter 5. An element in $A_1^* \otimes \cdots \otimes A_n^*$ is said to have *rank* one if it can be written as $\beta_1 \otimes \cdots \otimes \beta_n$, where $\beta_j \in A_j^*$, $1 \leq j \leq n$. Using the obvious reflexity $(A_1^* \otimes \cdots \otimes A_n^*)^* = A_1 \otimes \cdots \otimes A_n$, a rank one tensor in $A_1 \otimes \cdots \otimes A_n$ is defined similarly.

Definition 2.1.2. The *rank* of a tensor $T \in A_1 \otimes \cdots \otimes A_n$, denoted $\mathbf{R}(T)$, to be the minimal number r such that

$$T = \xi_1 + \cdots + \xi_r, \tag{2.1.2}$$

with each ξ_j of rank one.

Remark. (see Theorem 3.1.1.1 in [21]) Let A, B, C be finite dimensional complex vector spaces and $T \in A \otimes B \otimes C$, then $\mathbf{R}(T)$ equals the number of rank one matrices needed to span $T(A^*) \subset B \otimes C$.

Definition 2.1.3. When studying tensors in $A_1 \otimes \cdots \otimes A_n$, it is convenient to introduce the notation $A_{\hat{j}} := A_1 \otimes \cdots A_{j-1} \otimes A_{j+1} \otimes A_n$. Also, given $T \in A_1 \otimes \cdots \otimes A_n$, it canonically defines a linear map $A_j^* \to A_{\hat{j}}$ for all $j \in \{1, \ldots, n\}$. The image of this map will be denoted by $T(A_j^*) \subset A_{\hat{j}}$ and the image of the transpose will be denoted by $T^t(A_{\hat{j}}^*) \subset A_j$.

Definition 2.1.4. Let V be a finite dimensional complex vector space. Define the d-th symmetric power $S^d V$ of V to be the linear space spanned by elements of the form

$$v_1 \circ \cdots \circ v_n := \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

$v_1 \circ \cdots \circ v_n$ is called the symmetric product of $v_1, \ldots, v_n$.

Remark. Suppose V has dimension n, then $S^d V$ is a complex vector space of dimension $\binom{n+d-1}{d}$.

We define $S^d V^*$ as the space of symmetric d-linear forms on V. We can also identify $S^d V^*$ as the space of homogeneous polynomials in degree d on V, since we have the polarization of any homogeneous polynomial. Let Q be a homogeneous polynomial in degree d on V, then the polarization $\overline{(Q)}$ of Q is defined as a d-linear form:

$$\overline{(Q)}(x_1, ..., x_d) = \frac{1}{d!} \sum_{I \subset [k], I \neq \emptyset} (-1)^{d-|I|} Q(\sum_{i \in I} x_i),$$

where $[d] = \{1, \ldots, d\}$ and $x_1, \ldots, x_d$ are elements in V.

Definition 2.1.5. Let V be a finite dimensional complex vector space and d a nonnegative integer. Define the d-th alternating d-tensors $\bigwedge^d V$ of V to be the linear space spanned by elements of the form

$$v_1 \wedge \cdots \wedge v_d := \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

$v_1 \wedge \cdots \wedge v_d$ is called the exterior product of $v_1, \ldots, v_d$.

Definition 2.1.6. The *multilinear rank* of $T \in A_1 \otimes \cdots \otimes A_n$ is the n-tuple of natural numbers

$$\mathbf{R}_{multlin}(T) := (\dim T(A_1^*), \ldots, \dim T(A_j^*), \ldots, \dim T(A_n^*)). \tag{2.1.3}$$

The number $\dim T(A_j^*)$ is called *the mode j rank* of T.

Remark. Observe that for a matrix (i.e., the case $n = 2$), the rank, the mode-1 rank, and mode-2 rank are all equal.

Remark. (see Page 34 Exercise 2.4.2.6 in [21]) If $T \in A_1 \otimes \cdots \otimes A_n$, then the multilinear rank $(b_1, \ldots, b_n)$ of T satisfies $b_i \leq \min\ (a_i, \prod_{j \neq i} a_j)$ and equality holds for general tensors.

Example 2.1.7. *Let $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ to be*

$$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1,$$

where a_1, a_2 are linear independent, as well as b_1, b_2 and c_1, c_2, one can show by simple calculation

$$\mathbf{R}(T) = 3, \ \mathbf{R}_{multlin}(T) = (2, 2, 2).$$

And T can be approximated as closely as one likes by tensors of rank two, as consider:

$$T(\varepsilon) = \frac{1}{\varepsilon}[(\varepsilon - 1)a_1 \otimes b_1 \otimes c_1 + (a_1 + \varepsilon a_2) \otimes (b_1 + \varepsilon b_2) \otimes (c_1 + \varepsilon c_2)].$$

Intuitively, $T(\varepsilon)$ is a point on the line spanned by the two tensors $a_1 \otimes b_1 \otimes c_1$ and $z(\varepsilon) := (a_1 + \varepsilon a_2) \otimes (b_1 + \varepsilon b_2) \otimes (c_1 + \varepsilon c_2)$ inside the set of rank one tensors. Draw $z(\varepsilon)$ as a curve, for $\varepsilon > 0$, $T(\varepsilon)$ is a point on the secant line through $z(0)$ and $z(\varepsilon)$, and in the limit, one obtains a point on the tangent line to $z(0) = a_1 \otimes b_1 \otimes c_1$.

Definition 2.1.8. A tensor T has *border rank* r if it is a limit of tensors of rank r but is not a limit of tensors of rank s for any $s < r$.

Remark. Some researchers [19, 11, 23] like to picture tensors given in bases in terms of slices. Let A have basis $a_1, \ldots, a_\alpha$, where $\alpha = \dim A$ and similarly for B, C, let $T \in A \otimes B \otimes C$, so in bases $T = T_{i,j,k} a_i \otimes b_j \otimes c_k$. Then one forms an $\alpha \times \beta \times \gamma$ rectangular solid table whose entries are the T_{ijk}. This solid is then decomposed into modes or slices, e.g., consider T as a collection of a matrices of size $\beta \times \gamma$: $(T_1, j, k), \ldots, (T_\alpha, j, k)$, which might be referred to as horizontal slices, or a collection of β matrices $(T_{i,1,k}), \ldots, (T_{i,\beta,k})$ called lateral slices, or a collection of γ matrices called frontal slices. When two indices are fixed, the resulting vector in the third space is called a fiber.

2.2 Basic algebraic geometric methods

Let V be a finite dimensional complex vector space of dimension v. The projective space $\mathbb{P}(V) = \mathbb{P}^{v-1}$ associated to V is the set of lines through the origin in V. Its precise definition is as follows:

Definition 2.2.1. Projective space $\mathbb{P}(V)$ is the set whose points $[v] \in \mathbb{P}(V)$ are equivalence classes of nonzero elements $v \in V$, where $[v] = [w]$ if and only if there exists a nonzero $\lambda \in \mathbb{C}$ such that $w = \lambda v$.

Let $\pi : V \backslash \{0\} \to \mathbb{P}V$ denote the projection. As a quotient of $V \backslash \{0\}$, projective space inherits aspects of the linear structure on V. When $U \subset V$ is a linear subspace one also says that $\mathbb{P}U \subset \mathbb{P}V$ is a linear subspace. Just as in affine space, given any two distinct points $x, y \in \mathbb{P}\ V$, there exists a unique line $\mathbb{P}^1_{xy}$ containing them. A line in $\mathbb{P}V$ is the image under π of a 2-plane through the origin in V . An essential property of $\mathbb{P}^2$ is that any two distinct lines will intersect in a point.

Definition 2.2.2. For a subset $Z \subset \mathbb{P}\ V$, let $\hat{Z} := \pi^{-1}(Z)$ denote the affine cone over Z. The image of an affine cone $\mathcal{C}$ in projective space is called its projectivization, and we often write $\mathbb{P}\mathcal{C}$ for $\pi(\mathcal{C})$.

An algebraic variety is the image under $\pi : V \backslash \{0\} \to \mathbb{P}V$ of the set of common zeros of a collection of homogeneous polynomials on V.

Let $S^\bullet V = \bigoplus_{k \geq 0} S^k V$. The ideal $I(X) \in S^\bullet V$ of a variety $X \subset \mathbb{P}V$ is the set of all polynomials vanishing on X. Define $\overline{X}$ to be the set of common zeros of $I(X)$. $\overline{X}$ is called the Zariski closure of X.

Definition 2.2.3. A variety $X \subset \mathbb{P}V$ is said to be reducible if there exist varieties $Y, Z \subsetneq X$ such that $X = Y \cup Z$. Equivalently, X is reducible if there exists nontrivial ideals $I(Y), I(Z)$, $I(Y), I(Z) \supsetneq I(X)$, such that $I(X) = I(Y) \cap I(Z)$, and otherwise X is said to be irreducible.

Definition 2.2.4. Let $I(X)$ be the ideal of a variety X and let $\{f_1, \ldots, f_r\}$ be a set of generators of $I(X)$. Then the common zero set of $(n-r) \times (n-r)$ minors of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ is called the singular locus of X. Any point in X that is not in the singular locus is called a non-singular point or a smooth point.

Remark. The definition of the singular locus depends on the choices of generators of the ideal, but it turns out that different choices of generators give the same singular locus, see Chapter 1 of [16].

Example 2.2.5. *n-factor Segre variety (Definition* 4.3.5.1 *of [21]).*

Let A_j be vector spaces, let $V = A_1 \otimes \cdots \otimes A_n$. The classical n-factor Segre variety is the image of the map

$$Seg : \mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n \to \mathbb{P}V$$
$$([v_1], \ldots, [v_n]) \to [v_1 \otimes \cdots \otimes v_n].$$

This map is called the Segre embedding of a product of projective spaces. It is easy to see that Seg is well defined and is a differentiable mapping.

Definition 2.2.6. (See Definition 1 in [22]) *Subspace varieties*, denoted by $Sub_{k_1,\ldots,k_n}(A_1 \otimes \cdots \otimes A_n) \in \mathbb{P}(A_1 \otimes \ldots \otimes A_n)$ are defined as

$$Sub_{k_1,\ldots,k_n}(A_1 \otimes \cdots \otimes A_n)$$
$$:= \{[T] \in \mathbb{P}(A_1 \otimes \cdots \otimes A_n) \mid \forall i \ \exists A_i' \subset A_i, \ \dim A_i' = k_i, \ T \in A_1' \otimes \cdots \otimes A_n'\}$$
$$:= \{[T] \in \mathbb{P}(A_1 \otimes \cdots \otimes A_n) \mid \dim T(A_i^*) \le k_i\}.$$

Remark. The multilinear rank of a tensor $[T] \in \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$ is the minimum

$(k_1, \ldots, k_n)$ such that $[T] \in Sub_{k_1,\cdots,k_n}(A_1 \otimes \cdots \otimes A_n)$. The general elements (if they exist) in $Sub_{k_1,\ldots,k_n}(A_1 \otimes \cdots \otimes A_n)$ are of multilinear rank-$(k_1, \ldots, k_n)$.

Terracini [4, 5] introduced an algebraic geometric criterion of uniqueness of tensor decomposition. We will use a Corollary of Terracini's lemma, which appears as Proposition 2.4 in [4] for secant varieties.

Definition 2.2.7. If $X_i,\ i = 1, \ldots, k,\ k \leq n$ are projective algebraic varieties of $\mathbb{P}^n = \mathbb{P}V, V = \mathbb{C}^{n+1}$, then the *join* of $X_1, \ldots, X_k$ is

$$\mathbf{J}(X_1, \ldots, X_k) := \overline{\cup\{\langle [P_1], \ldots, [P_k] \rangle | P_i \in \hat{X}_i,\ 1 \leq i \leq k\}},$$

where P_i, $i = 1, \ldots, k$, are linearly independent vectors in V. If $X_1 = \cdots = X_k = X$, then we write $\mathbf{J}(X_1, \ldots, X_k) = \sigma_k(X)$ and we call this the k-th secant variety to X.

In our considerations, the following fact will play an essential role.

Remark. There is a normal form for a point $[p]$ of $\sigma_L(\mathbb{P}B \times \mathbb{P}C)$ (see Proposition 5.3.0.5 in [21] and also Chapter 11 in [21]), which is of the form

$$p = b_1 \otimes c_1 + \cdots + b_L \otimes c_L.$$

and we may assume that all the $b_i, 1 \leq i \leq L$ are linearly independent in B as well as all the $c_i, 1 \leq i \leq L$ in C(otherwise one would have $[p] \in \sigma_{L-1}(\mathbb{P}B \times \mathbb{P}C)$), then a general element $[\varphi] \in Sub_{1,L,L}(\mathbb{P}A \otimes \mathbb{P}B \times \mathbb{P}C)$ is of the form

$$\varphi = a_1 \otimes (b_1 \otimes c_1 + \cdots + b_L \otimes c_L),$$

where a_1 is a nonzero vector in A.

Definition 2.2.8. Define the tangent space to a point x of a subset M of a vector space V, $\hat{T}_x M \subset V$, to be the span of all vectors in V obtained as the derivative $\alpha'(0)$ of a smooth analytic parametrized curve $\alpha : \mathbb{C} \mapsto M$ with $\alpha(0) = x$ considered as a vector in V based at x. If M is a cone through the origin in V, minus the vertex, then dim $\hat{T}_x M$ is constant along rays of the cone.

Definition 2.2.9. For a variety $X \subset \mathbb{P}V$, and $x \in X$, define the affine tangent space to X at x, $\hat{T}_x X := \hat{T}_{\bar{x}} \hat{X}$, where $\bar{x} \in \hat{x}$. If dim $\hat{T}_x X$ is locally constant near x, we say x is a smooth point of X. Let X_{smooth} denote the set of smooth points of X. Otherwise, one says that x is a singular point of X. Let $X_{sing} = X \backslash X_{smooth}$ denote the singular points of X. If $X_{sing} = \emptyset$, one says X is singular.

Example 2.2.10. *The affine tangent space of the two factor Segre variety* $Seg(\mathbb{P}A \times \mathbb{P}B)$ *at* $[a \otimes b]$.

Any curve in $Seg(\mathbb{P}A \times \mathbb{P}B)$ is of the form $[a(t) \otimes b(t)]$ for curves $a(t) \subset A, b(t) \subset B$, where $a(0) = a, b(0) = b$.

Differentiating

$$(a(t) \otimes b(t))'|_{t=0} = a'(0) \otimes b(0) + a(0) \otimes b'(0)$$

shows that

$$\hat{T}_{[a \otimes b]} Seg(\mathbb{P}A \times \mathbb{P}B) = A \otimes b + a \otimes B,$$

where the sum is not direct and the intersection is $\langle a \otimes b \rangle$.

Throughout up to the end of this section, $X_1, \ldots, X_k$ will be as in the above definition.

Theorem 2.2.11. *(Lemma* 5.3.0.2 *in [21], a modern version of* Terracini's Lemma *in [30])*

Let $P_i \in \hat{X}_i$ be a general point of $\hat{X}_i$, for each $i = 1, \ldots, k$, then for $[P] := [P_1 + \cdots + P_k]$,

$$\hat{T}_{[P]}\mathbf{J}(X_1, \ldots, X_k) = \hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k. \tag{2.2.1}$$

This result reduces the determination of the dimension of the join $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), \ldots, Sub_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ to be the calculation of the dimensions of the tangent spaces at general points of our varieties $Sub_{1,L_r,L_r}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)), 1 \leq r \leq R$.

Definition 2.2.12. The *expected dimension* of $\mathbf{J}(X_1, \ldots, X_k)$ is $\min\{d_1 + \cdots + d_k + k - 1,\ n\}$, where $d_i = \dim X_i$, $1 \leq i \leq k$. The *defect* of $\mathbf{J}(X_1, \ldots, X_k)$ is

$$\delta(\mathbf{J}(X_1, \ldots, X_k)) = d_1 + \cdots + d_k + k - 1 - \dim\, \mathbf{J}(X_1, \ldots, X_k).$$

When the defect of $\mathbf{J}(X_1, \ldots, X_k)$ is positive, we say that $\mathbf{J}(X_1, \ldots, X_k)$ is *defective.*

Remark. By the upper semicontinuity of dimension of tangent space (see Exercise II.3.22 of [16]), if for one particular set of general points $\{P_1, \cdots, P_k\}$, $\hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k$ has the expected dimension, then $\mathbf{J}(X_1, \ldots, X_k)$ is not defective.

Corollary 2.2.13. *If $\boldsymbol{J}(X_1, \ldots, X_k)$ is not defective, then general points $[\varphi]$ on $\boldsymbol{J}(X_1, \ldots, X_k)$ have a finite number of decompositions*

$$\varphi = P_1 + \cdots + P_k,$$

with $[P_i] \in X_i$, $1 \leq i \leq k$. Moreover, if $\boldsymbol{J}(X_1, \ldots, X_k)$ is defective, then all $[\varphi] \in \boldsymbol{J}(X_1, \ldots, X_k)$ have infinitely-many decompositions.

Proof. From (2.2.1), we have

$$\begin{aligned}\dim\ \mathbf{J}(X_1,\ldots,X_k) &= \dim\ \hat{T}_P\mathbf{J}(X_1,\ldots,X_k)-1\\ &= \dim\ \langle \hat{T}_{P_1}X_1,\ldots,\hat{T}_{P_k}X_k\rangle - 1.\end{aligned}$$

Note that $\dim\ \mathbf{J}(X_1,\ldots,X_k)$ is the expected value precisely when these different tangent spaces do not intersect, which is $\min\ \{d_1+\cdots+d_k+k-1,\ N\}$.

If $\mathbf{J}(X_1,\ldots,X_k)$ is not defective, assume there were an infinite number of points, there would have to be a curve's worth (as X is compact algebraic), in which case the join would have to be degenerate. □

Remark. Geometrically, the expected dimension of $J(Y,Z)$ is $\min\ \{\dim Y+\dim Z+1, \dim \mathbb{P}V\}$ because a point $x\in J(Y,Z)$ is obtained by picking a point of Y, a point of Z, and a point on the line joining the two points. This expectation fails if and only if every point of $J(Y,Z)$ lies on a one-parameter family of lines intersecting Y and Z, as when this happens one can vary the points on Y and Z used to form the secant line without varying the point x. Similarly, the expected dimension of $\sigma_r(Y)$ is $r(\dim\ Y)+r-1$ which fails if and only if every point of $\sigma_r(Y)$ lies on a curve of secant $\mathbb{P}^{r-1}$s to Y.

Example 2.2.14. $\sigma_2(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)=\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$.

Proof. Let $X = Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$, where $A=\langle a_1,a_2\rangle,\ B=\langle b_1,b_2\rangle,\ C=\langle c_1,c_2\rangle$.

We pick a general element $p\in\sigma_2(\hat{X})$ and write $p=a_1\otimes b_1\otimes c_1+a_2\otimes b_2\otimes c_2$.

Then using (2.2.1), we obtain

$$\begin{aligned}\hat{T}_{[p]}\sigma_2(X) =&\hat{T}_{[a_1\otimes b_1\otimes c_1]}X + \hat{T}_{[a_2\otimes b_2\otimes c_2]}X\\ =&(A\otimes b_1\otimes c_1 + a_1\otimes B\otimes c_1 + a_1\otimes b_1\otimes C)\\ &+ (A\otimes b_2\otimes c_2 + a_2\otimes B\otimes c_2 + a_2\otimes b_2\otimes C)\\ =&A\otimes B\otimes C.\end{aligned}$$

and these affine tangent spaces intersect only at the origin. Thus dim $\sigma_2(X) = 2\cdot 4 - 1 = 7$ and $\sigma_2(X) \cong \mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$.

□

Definition 2.2.15. (see Definition 2.6 in [5]) Let $P_i \in \hat{X}_i$ be a general point of $\hat{X}_i$, $1 \le i \le k$. When for any $j \in \{1, \dots, k\}$ and $Q_j \in \hat{X}_j$, $\hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k$ contains $\hat{T}_{[Q_j]}X_j$ only if $[Q_j] \in \{[P_1], \dots, [P_k]\}$, we say $\mathbf{J}(X_1, \dots, X_k)$ is *not tangentially weakly defective*. Otherwise, we say that $\mathbf{J}(X_1, \dots, X_k)$ is *tangentially weakly defective*.

Remark. By semicontinuity (see Theorem $III.12.8$ of [16]), if for one particular set of general points $\{P_1, \dots, P_k\}$, $\hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k$ contains $\hat{T}_{[Q_j]}X_j$ only if $[Q_j] \in \{[P_1], \dots, [P_k]\}$, then $\mathbf{J}(X_1, \dots, X_k)$ is tangentially weakly defective.

Remark. Notice that if $\mathbf{J}(X_1, \dots, X_k)$ is defective, then it is tangentially weakly defective, but the converse is not true (see Example 4.0.5).

Remark. The equality in Theorem 2.2.11 is equivalent to

$$\hat{T}^{\perp}_{[P]}\mathbf{J}(X_1, \dots, X_k) = \bigcap_{1\le i\le k} \hat{T}^{\perp}_{[P_i]}X_i.$$

Moreover $\hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k \supset \hat{T}_{[Q_j]}X_j$ is equivalent to

$$\bigcap_{1 \le i \le k} \hat{T}^{\perp}_{[P_i]}X_i \subset \hat{T}^{\perp}_{[Q_j]}X_j;$$

so if $\mathbf{J}(X_1, \ldots, X_k)$ is tangentially weakly defective, then every hyperplane in $\bigcap_{1 \le i \le k} \hat{T}^{\perp}_{[P_i]}X_i$ is also tangent at $[Q_j] \in X_j$. We will need the following generalization of of Proposition 2.4 in [4].

Corollary 2.2.16. *If $\boldsymbol{J}(X_1, \ldots, X_k)$ is not tangentially weakly defective, then for general $[\varphi] \in \boldsymbol{J}(X_1, \ldots, X_k)$, the decomposition*

$$[\varphi] = [P_1 + \cdots + P_k],$$

with $[P_i] \in X_i$, $1 \le i \le k$, is essentially unique.

Proof. The proof proceeds like the proof of Proposition 2.4 in [4]. Assume the contrary, let us take a general point $[\varphi] \in \mathbf{J}(X_1, \ldots, X_k)$ and

$$[\varphi] = [P_1 + \cdots + P_k] = [Q_1 + \cdots + Q_k],$$

with $[Q_i] \in X_i$, $1 \le i \le k$, and at least one of them, say $[Q_j] \in X_j$, not belong to $\{[P_1], \ldots, [P_k]\}$. Then by Lemma 2.2.1, $\hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k$ also contains the tangent space of X_j at $[Q_j]$. Hence we get a contradiction, and Corollary 2.2.16 follows. □

Definition 2.2.17. (see Page 6 in [5]) The secant variety $\sigma_k(X)$ is *weakly defective* if the general hyperplane which is tangent to X at some k general points $[P_1], \ldots, [P_k]$, is also tangent at some other point $[Q] \neq [P_1], \ldots, [P_k]$. Here general means in an open subset of the set of hyperplanes which are tangent to X at k general family of points $[P_1], \cdots, [P_k]$.

Remark. By Theorem 2.2.11, if $\mathbf{J}(X_1, \ldots, X_k)$ is defective then $\mathbf{J}(X_1, \ldots, X_k)$ is weakly defective, but the converse is not necessarily true (see Example 4.0.6). To sum up, we have the following relationship

$$\{\text{defectivity}\} \subset \{\text{tangentially weak-defectivity}\} \subset \{\text{weak-defectivity}\}$$

2.3 Infinitesimal study of subspace varieties

For the benefit of the reader, we recall the following standard notations (see any textbook of algebraic geometry, for example [32]).

Let $V = \mathbb{C}^n$, $G(m, V)$ denote the Grassmannian of m-planes through the origin in V. It is a smooth compact algebraic variety of dimension $m(n-m)$.

The *trivial bundle* $G(m, V) \times V$, $V \cong \mathbb{C}^n$ over $G(m, V)$ contains the *universal subbundle* $\mathcal{S}$ of rank m that consists of the pairs (E, v) with $v \in E$. The *quotient bundle* $\mathcal{Q}$ over $G(m, V)$ of rank $n-m$ whose fiber over E is canonically isomorphic to V/E. These fit into the exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow G(m, V) \times V \longrightarrow \mathcal{Q} \longrightarrow 0.$$

The following Lemma is well known:

Lemma 2.3.1. *There is a canonical bundle isomorphism*

$$TG(m, V) = \mathcal{Q} \otimes \mathcal{S}^*$$

corresponding to the canonical isomorphism

$$T_E G(m,V) \cong V/E \otimes E^*.$$

Proof. In this case, we will use the definition of tangent spaces at a point x as the tangent vectors of curves starting at x. Recall that the *Plücker* embedding is a map $G(m,V) \mapsto \mathbb{P}(\wedge^m V)$ sends a subspace E of V to its m-th exterior power $\wedge^m E$. The image of $G(m,V)$ under the *Plücker* embedding are precisely all lines in $\mathbb{P}(\wedge^m V)$ which have a generator of the form $e_1 \wedge \cdots \wedge e_m$ for $e_i \in V$. So pick a point $E' = \langle e_1 \wedge \cdots \wedge e_m \rangle$ of $G(m,V)$. Complete $e_1, \ldots, e_m$ to a basis $e_1, \ldots, e_n$ of V. Given a map $\varphi : E' \mapsto V$, we can define a curve by $\varphi(t) = \langle (e_1 + t\varphi(e_1)) \wedge \cdots \wedge (e_m + t\varphi(e_m)) \rangle$. Since $\varphi(0) = E'$, this determines a tangent vector $\varphi'(0) = \sum_{i=1}^m e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots e_m$. Two curves given by $\varphi(t), \varphi'(t)$ determine the same tangent vector if and only if the image of their difference $\varphi(t) - \varphi'(t)$ lies in $\langle e_1, \ldots, e_m \rangle$, so the tangent space naturally contains $Hom(E, V/E)$ as a subspace. But this subspace has dimension $m(n-m)$, which is the dimension of $G(m,V)$, so in fact they are equal. Hence we conclude that the tangent bundle of $G(m,V)$ is $\mathcal{Q} \otimes \mathcal{S}^*$.

□

Lemma 2.3.2. *Let $A_1, \ldots, A_n$ and $A'_1, \ldots, A'_n$ be as in Definition 2.2.6. For general $\varphi \in A'_1 \otimes \ldots \otimes A'_n$, we have*

$$\hat{T}_\varphi(\widehat{Sub}_{k_1,\ldots,k_n}(A_1 \otimes \cdots \otimes A_n)) = (A'_1 \otimes \cdots \otimes A'_n) + \sum_{1 \le i \le n} (A_i \otimes \varphi(A'^*_i)), \qquad (2.3.1)$$

where $\varphi(A'^*_i), 1 \le i \le n$ is defined as in Definition 2.1.3.

Proof. First, we recall the following Kempf-Weyman desingularization for $\widehat{Sub}_{k_1,\ldots,k_n}(A_1 \otimes$

$\ldots \otimes A_n)$ as in section 7.4.2 of [21].

Consider the product of Grassmannians $B = G(k_1, A_1) \times \cdots \times G(k_n, A_n)$ and the bundle $\mathcal{S} := \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n \to_p B$, which is the tensor product of the tautological subspace bundles pulled back to B. A point of $\mathcal{S}$ is of the form $(E_1, \ldots, E_n; T)$ where $E_j \subset A_j$ is a k_j -plane, and $T \in E_1 \otimes \cdots \otimes E_n$. Consider the projection $q : \mathcal{S} \to A_1 \otimes \cdots \otimes A_n, (E_1, \ldots, E_n; T) \to T$. The image of q is $\widehat{Sub}_{k_1,\ldots,k_n} A_1 \otimes \cdots \otimes A_n$. If T is a smooth point in $\widehat{Sub}_{k_1,\ldots,k_n} A_1 \otimes \cdots \otimes A_n$, then dim $T(A_i^*) = k_i$ for all $1 \leq i \leq n$ and $E_i = T^t(A_i^*)$ is the unique preimage of T under q. Thus the map $q : \mathcal{S} \to \widehat{Sub}_{k_1,\ldots,k_n} A_1 \otimes \cdots \otimes A_n$ is a Kempf-Weyman desingularization of $\widehat{Sub}_{k_1,\ldots,k_n} A_1 \otimes \cdots \otimes A_n$.

We have the following diagram:

$$\begin{array}{ccc} & \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n & \\ \swarrow p & & \searrow q \\ G(k_1, A_1) \times \cdots \times G(k_n, A_n) & & \widehat{Sub}_{k_1,\ldots,k_n} A_1 \otimes \cdots \otimes A_n \end{array} \tag{2.3.2}$$

From the Kempf-Weyman desingularization described in Chapter 7.2 of [32], and using Lemma 2.3.1 for B, for general $\varphi \in A'_1 \otimes \ldots \otimes A'_n$, we deduce Lemma 2.3.2. □

Proposition 2.3.3. *Let A_i be complex vector spaces, for all $1 \leq i \leq n$; The join $\boldsymbol{J}(Sub_{k_1^1,\ldots,k_n^1}(A_1 \otimes \cdots \otimes A_n), \ldots, Sub_{k_1^m,\ldots,k_n^m}(A_1 \otimes \cdots \otimes A_n))$*

(ι) is non-defective if dim $A_i \geq \sum_{1 \leq j \leq m} k_i^j$, $1 \leq j \leq m$, for each $1 \leq i \leq n$;

($\iota\iota$) is defective if $\prod_{1 \leq i \leq n}$ dim $A_i < \sum_{1 \leq j \leq m} \prod_{1 \leq i \leq n} k_i^j$;

Also

($\iota\iota\iota$) the defect of the join is at least $\sum_{1 \leq j \leq m} \prod_{1 \leq i \leq n} k_i^j - \prod_{1 \leq i \leq n}$ dim A_i.

Proof. (ι) If dim $A_i \geq \sum_{1 \leq j \leq m} k_i^j$, $1 \leq j \leq m$, without loss of generality, we assume equality holds. Splitting $A_i = A_i^1 \oplus \cdots \oplus A_i^m$, for each $1 \leq i \leq n$, such that dim $A_i^j =$

k_i^j and dim $A_i = \sum_{1\leq j\leq m} k_i^j$, and further taking $\varphi_t \in A_1^t \otimes \cdots \otimes A_n^t, 1 \leq t \leq m$, it follows from Lemma 2.3.2 that

$$
\begin{aligned}
&\hat{T}_{\varphi_t}(\widehat{Sub}_{k_1^t,\ldots,k_n^t}(A_1 \otimes \ldots \otimes A_n)) \\
&= (A_1^t \otimes \cdots \otimes A_n^t) \bigoplus_{1\leq s\leq n} ((A_s^1 \oplus \cdots \oplus \widehat{A_s^t} \oplus \cdots \oplus A_s^m) \otimes \varphi_t(A_s^{t*})) \\
&\subset (A_1^t \otimes \cdots \otimes A_n^t) \bigoplus_{1\leq s\leq n} ((A_s^1 \oplus \cdots \oplus \widehat{A_s^t} \oplus \ldots \oplus A_s^m) \otimes A_1^t \otimes \cdots \otimes \widehat{A_s^t} \otimes \cdots \otimes A_n^t).
\end{aligned}
$$

Since

$$
\begin{aligned}
&(A_1^t \otimes \cdots \otimes A_n^t) \bigoplus_{1\leq s\leq n} ((A_s^1 \oplus \cdots \oplus \widehat{A_s^t} \oplus \cdots \oplus A_s^m) \otimes A_1^t \otimes \cdots \otimes \widehat{A_s^t} \otimes \ldots \otimes A_n^t) \\
&\bigcap \sum_{t'\neq t} ((A_1^{t'} \otimes \cdots \otimes A_n^{t'}) \bigoplus_{1\leq s\leq n} ((A_s^1 \oplus \cdots \oplus \widehat{A_s^{t'}} \oplus \cdots \oplus A_s^m) \otimes A_1^{t'} \otimes \cdots \otimes \widehat{A_s^{t'}} \otimes \cdots \otimes A_n^{t'})) \\
&= \{0\},
\end{aligned}
$$

we deduce

$$
\hat{T}_{\varphi_t}(\widehat{Sub}_{k_1^t,\ldots,k_n^t}(A_1 \otimes \cdots \otimes A_n)) \bigcap \sum_{1\leq t'\neq t\leq m} \hat{T}_{\varphi_{t'}}(\widehat{Sub}_{k_1^{t'},\ldots,k_n^{t'}}(A_1 \otimes \cdots \otimes A_n)) = \{0\}.
$$

Using Theorem 2.2.11, we have

$$
\begin{aligned}
&\hat{T}_{\sum_{1\leq t\leq m}\varphi_t}(\mathbf{J}(\widehat{Sub}_{k_1^1,\ldots,k_n^1}(A_1 \otimes \cdots \otimes A_n), \ldots, \widehat{Sub}_{k_1^m,\ldots,k_n^m}(A_1 \otimes \cdots \otimes A_n))) \\
&= \bigoplus_{1\leq t\leq m} \hat{T}_{\varphi_t}(\widehat{Sub}_{k_1^t,\ldots,k_n^t}(A_1 \otimes \cdots \otimes A_n)).
\end{aligned}
$$

Thus $\mathbf{J}(Sub_{k_1^1,\ldots,k_n^1}(A_1 \otimes \cdots \otimes A_n), \ldots, Sub_{k_1^m,\ldots,k_n^m}(A_1 \otimes \cdots \otimes A_n))$ is non-defective.

(Proof of $\iota\iota - \iota\iota\iota$) When $\prod_{1\leq i\leq n} \dim A_i < \sum_{1\leq j\leq m}\prod_{1\leq i\leq n} k_i^j$, we have

$$\hat{T}_{\varphi_t}(Sub_{k_1^t,\ldots,k_n^t}(A_1\otimes\cdots\otimes A_n)) = (A_1^t\otimes\cdots\otimes A_n^t) + \sum_{1\leq s\leq n}(A^s\otimes\varphi_t(A^{s*})),$$

where $\varphi_t \in A_1^t\otimes\cdots\otimes A_n^t, 1\leq t\leq m$.

And there exists $t_1, t_2 \in \{1,\ldots,m\}$, such that

$$(A_1^{t_1}\otimes\cdots\otimes A_n^{t_1}) \cap (A_1^{t_2}\otimes\cdots\otimes A_n^{t_2}) \neq \{0\},$$

which implies

$$\hat{T}_{\varphi_{t_1}}(Sub_{k_1^{t_1},\ldots,k_n^{t_1}}(A_1\otimes\cdots\otimes A_n)) \cap \hat{T}_{\varphi_{t_2}}(Sub_{k_1^{t_2},\ldots,k_n^{t_2}}(A_1\otimes\cdots\otimes A_n)) \neq \{0\}.$$

Therefore, $\mathbf{J}(Sub_{k_1^1,\ldots,k_n^1}(A_1\otimes\cdots\otimes A_n),\ldots,Sub_{k_1^m,\ldots,k_n^m}(A_1\otimes\cdots\otimes A_n))$ is defective with defect at least $\sum_{1\leq j\leq m}\prod_{1\leq i\leq n} k_i^j - \prod_{1\leq i\leq n}\dim A_i$. $\square$

Proposition 2.3.4. *Let A, B and C be complex vector spaces of dimensions a, b, c respectively. And let a', a'' be nonnegative integers, such that $a', a'' < a$, as well as b', b'', c', c''. If*

$$a' \leq (b-b')(c-c'),\ b' \leq (a-a')(c-c'),\ c' \leq (a-a')(b-b'),$$
$$a'' \leq (b-b'')(c-c''),\ b'' \leq (a-a'')(c-c''),\ c'' \leq (a-a'')(b-b''),$$

then $\mathbf{J}(Sub_{a',b',c'}(A\otimes B\otimes C), Sub_{a'',b'',c''}(A\otimes B\otimes C))$ has defect $(a'+a''-a)^+(b'+b''-b)^+(c'+c''-c)^+$, where $x^+ = x$ if $x\geq 0$ and 0 if $x<0$.

For the proof of Proposition 2.3.4, we need some preliminary considerations.

Let A, B and C be three complex vector spaces, of dimensions a, b, c respectively, and further let A be sum of two spaces E_A and F_A, of dimension a', a'' respectively and $E_A \cap F_A = A_0$. Let A_1 and A_2 respectively denote choices of complementary spaces in E_A and F_A respectively. The vector spaces $E_B,\ F_B,\ B_0,\ B_1,\ B_2$, and $E_C,\ F_C,\ C_0,\ C_1,\ C_2$ are defined in a similar manner. That is:

$$\begin{aligned} &A = A_1 \oplus A_0 \oplus A_2,\ B = B_1 \oplus B_0 \oplus B_2,\ C = C_1 \oplus C_0 \oplus C_2. \\ &E_A = A_1 \oplus A_0,\ F_A = A_2 \oplus A_0,\ \dim\, E_A = a',\ \dim\, F_A = a'', \\ &E_B = B_1 \oplus B_0,\ F_B = B_2 \oplus B_0,\ \dim\, E_B = b',\ \dim\, F_B = b'', \\ &E_C = C_1 \oplus C_0,\ F_C = C_2 \oplus C_0,\ \dim\, E_C = c',\ \dim\, F_C = c''. \end{aligned} \tag{2.3.3}$$

Note that $a' + a'' = a$, if and only if A_0 is $\{0\}$, and similarly for B_0, C_0.

Remark. As a special case of Proposition 2.3.3, if $a' + a'' \le a$, $b' + b'' \le b$, and $c' + c'' \le c$, $\mathbf{J}(Sub_{a',b',c'}(A \otimes B \otimes C), Sub_{a'',b'',c''}(A \otimes B \otimes C))$ is non-defective.

Lemma 2.3.5. *There exist a rational map*

$$f : E \otimes V \to G(e, V),$$

where $\dim E = e$*, such that for* $\varphi \in E \otimes V$*, where* $\varphi : E^* \to V$ *is injective, we have*

$$f(\varphi) = \varphi(E^*) \subset V,$$

and the open subset

$$U := \{\varphi | \varphi : E^* \to V \text{ is injective}\}$$

is the locus where f *is regular.*

Proof. Let dim $V = v$. In suitable bases, the image $f(U)$ is the $GL(e)$-orbit space of $\mathrm{Mat}(e, v)$, where $\mathrm{Mat}(e, v)$ denotes matrices of size $e \times v$ and of rank e. For example, when $I = \{1, \dots, e\}, X \in \mathrm{Mat}(e, v)$, each orbit in the affine open set is uniquely represented by a matrix

$$(X_I)^{-1}X = \begin{bmatrix} 1 & 0 & \dots & 0 & * & \dots & * \\ 0 & 1 & \dots & 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & * & \dots & * \end{bmatrix}$$

in which the $e(v - e)$ entries $*$ serve as coordinates on $\mathbb{C}^{e(v-e)}$. Note that each $*$ is a $GL(a')$- invariant rational form on $\mathrm{Mat}(e, v)$. This orbit coincides with a e-dimensional subspaces of a fixed v-dimensional vector space, which is $G(e, v)$. Therefore $f(U) \subseteq G(e, v)$ is open. □

Recall some well known results of *Shubert varieties* in Grassmannians.

Definition 2.3.6. For any $k \leq n - m$ and any subspace $W \subseteq \mathbb{C}^n$ of codimension $m + k - 1$, the subvariety

$$\tau(W) := \{E \in G(m, n) \mid E \cap W \neq \{0\}\} \tag{2.3.4}$$

has codimension k in $G(m, n)$. The subvariety $\tau(W)$ is one of an important family of subvarieties of the Grassmannian known as *Shubert varieties*. We have a cell decomposition of the Grassmannian: if we set $V_i = \{e_1, \dots, e_i\} \subseteq \mathbb{C}^n$, then the set of $\Lambda \in G(m, n)$ whose intersection with each V_i is of a specified dimension. For generic $\Lambda \in G(m, n), \Lambda \bigcap V_i$ will be zero for $i \leq n-m$ and $(i+m-n)$-dimensional thereafter.

Proof of Proposition 2.3.4

By (2.3.3) and using Lemma 2.3.2, for general $\varphi_E \in E_A \otimes E_B \otimes E_C$, $\varphi_F \in F_A \otimes F_B \otimes F_C$, we have

$$\hat{T}_{\varphi_E}(\widehat{Sub}_{a',b',c'}(A \otimes B \otimes C)) = E_A \otimes E_B \otimes E_C \tag{2.3.5}$$
$$\oplus A_2 \otimes \varphi_E(E_A^*)(\subset A_2 \otimes E_B \otimes E_C)$$
$$\oplus B_2 \otimes \varphi_1(E_B^*)(\subset B_2 \otimes E_A \otimes E_C)$$
$$\oplus C_2 \otimes \varphi_1(E_C^*)(\subset C_2 \otimes E_A \otimes E_B),$$

and similarly

$$\hat{T}_{\varphi_F}(\widehat{Sub}_{a'',b'',c''}(A \otimes B \otimes C)) = F_A \otimes F_B \otimes F_C \tag{2.3.6}$$
$$\oplus A_1 \otimes \varphi_F(F_A^*)(\subset A_1 \otimes F_B \otimes F_C)$$
$$\oplus B_1 \otimes \varphi_F(F_B^*)(\subset B_1 \otimes F_A \otimes F_C)$$
$$\oplus C_1 \otimes \varphi_F(F_C^*)(\subset C_1 \otimes F_A \otimes F_B).$$

Therefore

$$A_0 \otimes B_0 \otimes C_0 \subset$$
$$\hat{T}_{\varphi_E}(\widehat{Sub}_{a',b',c'}(A \otimes B \otimes C)) \cap \hat{T}_{\varphi_F}(\widehat{Sub}_{a'',b'',c''}(A \otimes B \otimes C)) \tag{2.3.7}$$
$$\subset (A_0 \otimes B_0 \otimes C_0) \oplus (A_1 \otimes B_0 \otimes C_0) \oplus (A_2 \otimes B_0 \otimes C_0) \oplus (A_0 \otimes B_2 \otimes C_0)$$
$$\oplus (A_0 \otimes B_0 \otimes C_1) \oplus (A_0 \otimes B_0 \otimes C_2) \oplus (A_2 \otimes B_0 \otimes C_1) \oplus (A_1 \otimes B_2 \otimes C_0)$$
$$\oplus (A_2 \otimes B_1 \otimes C_0) \oplus (A_1 \otimes B_0 \otimes C_2) \oplus (A_0 \otimes B_1 \otimes C_2) \oplus (A_0 \otimes B_2 \otimes C_2)$$
$$\oplus (A_0 \otimes B_1 \otimes C_0).$$

We first need to prove that the first inclusion in (2.3.7) is actually an equality. For this purpose, we want to choose sufficiently general $\varphi_E,\ \varphi_F$ to avoid a possible larger intersection of (2.3.5) and (2.3.6).

Let $p,\ p'$ be general elements in $\hat{T}_{\varphi_E}(\widehat{Sub}_{a',b',c'}(A\otimes B\otimes C))\cap\hat{T}_{\varphi_F}(\widehat{Sub}_{a'',b'',c''}(A\otimes B\otimes C))$, and use (2.3.5) and (2.3.6) to represent p, p' respectively as

$$
\begin{aligned}
&p = v_0 + v_1 + v_2 + v_3,\ \text{with}\\
&v_0 \in E_A\otimes E_B\otimes E_C,\ v_1 \in A_2\otimes\varphi_E(E_A^*),\\
&v_2 \in B_2\otimes\varphi_E(E_B^*),\ v_3 \in C_2\otimes\varphi_E(E_C^*);
\end{aligned}
$$

and

$$
\begin{aligned}
&p' = v_0' + v_1' + v_2' + v_3',\ \text{with}\\
&v_0' \in F_A\otimes F_B\otimes F_C,\ v_1' \in A_1\otimes\varphi_F(F_A^*),\\
&v_2' \in B_1\otimes\varphi_F(F_B^*),\ v_3' \in C_1\otimes\varphi_F(F_C^*).
\end{aligned}
$$

From (2.3.7), we have

$$v_1 \in A_2\otimes(B_0\otimes C_0\oplus B_1\otimes C_0\oplus B_0\otimes C_1),$$

and hence

$$v_1 \in (A_2\otimes\varphi_E(A^*))\bigcap(A_2\otimes(B_0\otimes C_0\oplus B_1\otimes C_0\oplus B_0\otimes C_1)).$$

.

Now consider

$$\varphi_E(A^*) \subseteq E_B \otimes E_C \text{ and } U = (B_0 \otimes C_0) \oplus (B_1 \otimes C_0) \oplus (B_0 \otimes C_1),$$

and note that the codimension of U in $E_B \otimes E_C$ is $(b - b')(c - c') \geq a'$. We also consider the Schubert subvariety

$$\tau(U) := \{E \in G(a', b'c') \mid E \cap U \neq \{0\}\},$$

which has codimension $(b - b')(c - c') - a' + 1$ in $G(a', b'c')$.

By virtue of Lemma 2.3.5, and note that in this case, $E = E_A$, $V = E_B \otimes E_C$, letting $\varphi = \varphi_E^{A^*}$, for general $\varphi_E^{A^*}$, we have

$$f(\varphi_E^{A^*}) \cap U = \{0\}.$$

But the image $f_E(\varphi_E^{A^*})$ is in the complement of variety $\tau(U)$. Therefore, we obtained in this way a Zariski-open dense set of general $\varphi_E^{A^*}$. In the same way, we can obtain a Zariski-open dense sets of general $\varphi_E^{B^*}$, $\varphi_E^{C^*}$. It follows that

$$(A_2 \otimes \varphi_E(A^*)) \bigcap (A_2 \otimes (B_0 \otimes C_0 \oplus B_1 \otimes C_0 \oplus B_0 \otimes C_1)) = \{0\},$$

and in consequence $v_1 = \{0\}$. Similarly, we have $v_2 = \{0\}$, $v_3 = \{0\}$; for the same reason, $v_1' = \{0\}, v_2' = \{0\}$, $v_3' = \{0\}$. Taking the intersection of those $\varphi_E^{A^*}$, $\varphi_E^{B^*}$, $\varphi_E^{C^*}$, we obtain φ_E. In the same way, we get φ_F, that give rise to

$$p = v_0 \in E_A \otimes E_B \otimes E_C,$$

and respectively

$$p' = v_0' \in F_A \otimes F_B \otimes F_C.$$

Therefore, using (2.3.3), we obtain

$$\hat{T}_{\varphi_E}(\widehat{Sub}_{a',b',c'}(A \otimes B \otimes C)) \cap \hat{T}_{\varphi_F}(\widehat{Sub}_{a',b',c'}(A \otimes B \otimes C))$$
$$= A_0 \otimes B_0 \otimes C_0.$$

By virtue of (2.3.7), $A_0 \otimes B_0 \otimes C_0$ is the intersection of $\hat{T}_{\varphi_E}(\widehat{Sub}_{a',b',c'}(A \otimes B \otimes C))$ and $\hat{T}_{\varphi_F}(\widehat{Sub}_{a',b',c'}(A \otimes B \otimes C))$ for general φ_E and φ_F, and this completes the proof of Theorem 2.3.4.

3. BLIND SIGNAL SEPARATION IN SIGNAL PROCESSING

3.1 Blind source separation

Tensors are well-known to arise in signal processing as higher order *cumulants* in independent component analysis [7], and have been used successfully in blind source separation [12]. The signal processing application considered here is of a different nature but also has a natural tensor decomposition model.

Example 3.1.1. *How does our central nervous system detect where a muscle is and how it is moving?(see Chapter* 13 *Section* 1 *of [21])*

The muscles send of electrical signals through two types of transmitters in the nerves, called primary and secondary, as the first type sends stronger signals. There are two things to be recovered, the function $p(t)$ of angular position and $\dot{p}(t)$ of angular speed. (These are to be measured at any given instance so your central nervous system cannot simply take a derivative.) One might think one type of transmitter sends information about $\dot{p}(t)$ and the other about $p(t)$, but the opposite was observed, there is some kind of mixing: say the signals sent are respectively given by functions $y_1(t), y_2(t)$. Let $p(t) = x_1(t), \dot{p}(t) = x_2(t)$, then it was observed there is a matrix A, such that

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and the central nervous system somehow decodes $x_1(t), x_2(t)$ from $y_1(t), y_2(t)$.

In general, let $\mathbb{R}^m$ have coordinates $y_1, \ldots, y_m$ (which may be thought of as m quantities that can be measured at any time t, such are called stochastic processes, see Chapter 1 Section 2 of [27]), and a probability measure $d\mu(y)$. Say we expect that there are exactly $r < m$ independent quantities from which the y_j are constructed and we would like to find them. That is, we have an equation of the form (see Chapter 2 Section 1 of [25]):

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} = x_1(t) \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_r(t) \begin{bmatrix} a_{1r} \\ \vdots \\ a_{mr} \end{bmatrix} \tag{3.1.1}$$

One would like to recover $[x_1(t), \ldots, x_r(t)]^T$, plus the matrix A, from knowledge of the function $y_1(t), \ldots, y_m(t)$ alone. Note that the $x_i(t)$ are like eigenvectors, in the sense that they are only well defined up to scale and permutation, so recover means modulo this ambiguity.

Definition 3.1.2. (see Section 2.1 of [23]) Let $X_1, \ldots, X_n$ be random variables with respect to the same probability distribution μ. The *moments* and *cumulants* of the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ are symmetric tensors of order k defined by

$$\begin{aligned} m_k(\mathbf{X}) &= \left[E(x_{i_1} x_{i_2} \cdots x_{i_k})\right]_{i_1,\ldots,i_k=1}^n \\ &= \left[\int \cdots \int x_{i_1} x_{i_2} \cdots x_{i_k} \; d\mu(x_{i_1}) \cdots d\mu(x_{i_k})\right]_{i_1,\ldots,i_k=1}^n \end{aligned}$$

and

$$\kappa_k(\mathbf{X}) = \left[\sum\nolimits_{\substack{A_1 \sqcup \cdots \sqcup A_p \\ = \{i_1,\ldots,i_k\}}} (-1)^{p-1}(p-1)! E\left(\prod\nolimits_{i \in A_1} x_i\right) \cdots E\left(\prod\nolimits_{i \in A_p} x_i\right)\right]_{i_1,\ldots,i_k=1}^n$$

respectively. The sum above is taken over all possible partitions $\{i_1, \ldots, i_k\} = A_1 \sqcup \cdots \sqcup A_p$. It is not hard to show that both $m_k(\mathbf{X})$ and $\kappa_k(\mathbf{X}) \in S^k(\mathbb{R}^n)$. For $n = 1$, the quantities $\kappa_k(\mathbf{X})$ for $k = 1, 2, 3, 4$ have well-known names: they are the expectation, variance, skewness, and kurtosis of the random variable X, respectively.

For moments and cumulants in probability, we refer to Chapter XII Section 8 of [31] and Chapter 4 of [8]. For the use of cumulants in signal analysis, we refer the reader to Appendix $11-A$ of [26]. Cumulants have several important properties that make them useful and justify their slight additional complexity relative to moments. The first (also true of moments) is multilinearity. If x is a $\mathbb{C}^n$-valued random variable and $A \in \mathbb{C}^{m\times n}$, then we have

$$k_d(Ax) = A \cdot k_d(x), \tag{3.1.2}$$

where A is the multilinear action.

The second is independence. If $x_1, \ldots, x_p$ are mutually independent of variables $y_1, \ldots, y_p$, we have

$$k_d(x_1 + y_1, \ldots, x_p + y_p) = k_d(x_1, \ldots, x_p) + k_d(y_1, \ldots, y_p). \tag{3.1.3}$$

Now the moments of x are

$$m(x)_{i_1,\ldots,i_p} = \int_0^T \cdots \int_0^T x_{i_1} \cdots x_{i_p} d\mu^p. \tag{3.1.4}$$

Note that $[m(x)_{i,j}]$ satisfy linearity (3.1.2), such that

$$[m(Ax)_{i,j}] = A[m(x)_{i,j}]A^T.$$

The 2nd and 3rd order cumulants are

$$
\begin{aligned}
k(x)_{ij} &:= m(x)_{ij} - m(x)_i m(x)_j \\
&= \int_0^T \int_0^T x_i x_j d\mu^2 - (\int_0^T x_i d\mu)(\int_0^T x_j d\mu).
\end{aligned}
$$

If $x_1, \ldots, x_r$ are statistically independent, that is

$$k(x)_{ij} = 0, \ \forall i \neq j,$$

then by definition 3.1.2, we have

$$
\begin{aligned}
k(x)_{ijk} =& m(x)_{ijk} - (m(x)_i m(x)_{jk} + m(x)_j m(x)_{ik} + m(x)_k m(x)_{ij}) \\
& + 2m(x)_i m(x)_j m(x)_k.
\end{aligned}
$$

From (3.1.2), we can obtain a system of $\binom{m+2}{3}$ linear equations for $mr + r$ unknowns

$$
\begin{aligned}
k(y)_{ijk} &= k(Ax)_{ijk} \\
&= a_{i\alpha} a_{j\beta} a_{k\gamma} k(x)_{\alpha\beta\gamma}.
\end{aligned}
$$

But since $x_1, \ldots, x_r$ are statistically independent, we have

$$k(y)_{ijk} = k(Ax)_{ijk} = a_{i\alpha} a_{j\alpha} a_{k\alpha} k(x)_{\alpha\alpha\alpha}.$$

Recall if $x_1, \ldots, x_r$ are statistically independent quantities, we have (see Chapter 12 of [21])

$$\mathrm{R}_S(k_3(x)) = r.$$

Then we can decompose the order three symmetric tensor (cubic polynomial) into a sum of r cubes

$$k_3(y) = (\sum a_{i1}a_{j1}a_{k1})k(x)_{111} + \cdots + (\sum a_{ir}a_{jr}a_{kr})k(x)_{rrr};$$

with probability one, there will be a unique decomposition.

Blind deconvolution [17] is related to the above blind source separation modeling in two respects. First, a convolution (see Chapter 6 of [28]) with a finite impulse response (see Chapter 1 Section 2 of [26]) can always be written as the product with a Toeplitz matrix, which means that the modeling (3.1.1) still holds valid, provided matrix A is subject to the Toeplitz structure (see Chapter 3 Section 3 of [6]). Second, if the source process is linear, then extracting the sources is equivalent to computing the linear prediction residue (see Chapter 7 Section 1 of [6]). Then the problem reduces to an unstructured static separation as in (3.1.1).

3.2 Block term analysis

In spread-spectrum systems (see Chapter 13 of [15]) that employ an antenna array at the receiver, the received data are naturally represented by the third-order tensor that shows the signal along the temporal, spectral and spatial axis (see Chapter 3 Section 5 of [15]). It was shown for Direct Sequence Code Division Multiple Access (DS-CDMA) systems that (see Chapter 13 Section 4.5 of [15]), in simple propagation scenarios that do not cause Inter-Symbol-Interference (ISI) (see Chapter 6 Section 5 of [15]), every user contributes a rank-1 term to the received data. Consequently, in a non-cooperative setting multiple access can be realized through the computation of a CP decomposition. In propagation scenarios that do involve ISI, rank-1 terms are a too restrictive model. When reflections only take place in the far field of the receive array, multiple access can be realized through the computation of a decomposition in multilinear rank $(1,\ L_1,\ L_1), \ldots, (1,\ L_R,\ L_R)$ terms [12]. In a more general type of block term tensor decomposition was used to deal with cases where reflections do not only take place in the far field. The same ideas can be applied to other systems with at least triple diversity (see Chapter 7 of [15]).

Let us consider R users transmitting at the same time within the same bandwidth (see Chapter 4 Section 2.2 of [27]), frames of K symbols spread by DS-CDMA codes of length J, towards an array of I antennas (see Chapter 15 of [27] and Chapter 13 Section $2-3$ of [15]). In a direct path -only propagation scenario, the assumption that the channel is noiseless and memoryless leads to the following instantaneous data model without Inter-Chip-Interference (see Chapter $7, 10$ of [27]):

$$T_{ijk} = \sum_{r=1}^{R} a_k^r c_i^r s_j^r,$$

where T_{ijk} is the sample of the signal received by the k-th antenna at the i-th chip-sampling instant within the j-th symbol period. The scalar a_k^r is the fading factor between user r and antenna element k, s_j^r is the j-th symbol transmitted by the r-th user and c_i^r is the i-th chip of the CDMA code assigned to user r (see Chapter 7 Section 1 of [27]). We now consider a multipath propagation scenario with large delay spread (see Chapter 9 of [27]). We assume that for a given user, the multipath channel is the same for all antennas, up to a multiplicative fading factor a_k^r, which is valid when the multipath reflectors are in the far field of the antennas (see Chapter 10 of [15]). If we denote by x_{ijk}^r the i-th chip of the signal received by the k-th antenna during the j-th symbol period for the r-th user, we get:

$$x_{ijk}^r = a_k^r \sum_{l=1}^{L} h_{i+(l-1)J}^r s_{j-l+1}^r,$$

where h^r contains the coefficients obtained by convolution (see Chapter 6 of [28]) between the impulse response of the r-th channel and the r-th CDMA code. L is the number of interfering symbols. So $h^r_{i+(l-1)J}$ is the coefficient of the overall impulse response at the chip rate corresponding to the i-th chip and the l-th interfering symbol (see Chapter 1 Section 2.3 of [27]). We finally get the expression for one sample of the overall received signal by summing the contributions of R users:

$$T_{ijk} = \sum_{r=1}^{R} a^r_k \sum_{l=1}^{L} h^r_{i+(l-1)J} s^r_{j-l+1}.$$

In general, the tensor $T \in \mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K$ will admit a block term decomposition into a sum of R elements of $\sigma_R(Sub_{1,L,L}\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$.

Setting $K = \tilde{K} + L - 1$, $L = \lceil \frac{L'}{J} \rceil$ and $L = \tilde{J} + L' - 1$ and letting matrices H_r, S_r to be

$$H_r \doteq \begin{bmatrix} h_{1r} & h_{J+1,r} & h_{2J+1,r} \cdots & h_{J+L'-1,r} & \cdots & 0 \\ h_{2r} & h_{J+2,r} & h_{2J+2,r} \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ h_{Jr} & h_{2J,r} & h_{3J,r} \cdots & \cdots & 0 & 0 \end{bmatrix}$$

$$S_r^T \doteq \begin{bmatrix} s_{1r} & s_{2r} & \cdots & s_{\tilde{K}r} & 0 & \cdots 0 & 0 & \cdots 0 \\ 0 & s_{1r} & s_{2r} & \cdots & s_{\tilde{K}r} & 0 & \cdots 0 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ 0 & \cdots 0 & s_{1r} & s_{2r} & \cdots & s_{\tilde{K}r} & 0 & \cdots 0 \end{bmatrix},$$

then we know the $J \times K$ matrix $X_r = [x^r_{jk}] = H_r \cdot S_r^T$ has rank at most L because the matrix S_r does (see Chapter 13 Section 2 of [21]). When block term decomposition

(1.1.1) is unique, say $(h_{ir}) \cdot (s_{jr}) = (x^r_\alpha)$ and assuming (h_{ir}) is invertible, we can recover s_{jr} [12], that is $S^T_r = P_r \cdot X_r$, where P_r is the (unknown) inverse of H_r.

Generally, from (1.1.1), once we know the product $X_r = H_r \cdot S^T_r$, the entries of S_r can be found by exploiting the Toeplitz structure of S_r (see Chapter 3 Section 3 of [6]). Define as a new unknown the Moore-Penrose pseudo-inverse (see Chapter 3 Section 4 of [29]) $P_r \doteq pinv(H_r)$ of H_r. Then we have: $S^T_r = P_r \cdot X_r$. We can express the constraints that the entries of S_r are constant along the diagonals as linear equations in X and solve the resulting set of equations in least-squares (see Chapter 6 of [6]).

Example 3.2.1. *Let* $I = J = 2,\ K = 4,\ L = 2,\ R = 2$*,*$(\sigma_2(Sub_{1,2,2}\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4))$

$$T = \sum_{r=1}^{2} a_r \otimes (\begin{bmatrix} h_{1r} & h_{3r} \\ h_{2r} & 0 \end{bmatrix} \cdot \begin{bmatrix} s_{1r} & s_{2r} & s_{3r} & 0 \\ 0 & s_{1r} & s_{2r} & s_{3r} \end{bmatrix}).$$

Given a 2×4 *matrix*

$$X_r = \begin{bmatrix} x_{1r} & x_{2r} & x_{3r} & x_{4r} \\ x_{5r} & x_{6r} & x_{7r} & x_{8r} \end{bmatrix}$$

then

$$\begin{aligned} \begin{bmatrix} s_{1r} & s_{2r} & s_{3r} & 0 \\ 0 & s_{1r} & s_{2r} & s_{3r} \end{bmatrix} &= \begin{bmatrix} h_{1r} & h_{3r} \\ h_{2r} & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} x_{1r} & x_{2r} & x_{3r} & x_{4r} \\ x_{5r} & x_{6r} & x_{7r} & x_{8r} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{h_{2r}} \\ \frac{1}{h_{3r}} & -\frac{h_{1r}}{h_{2r}h_{3r}} \end{bmatrix} \cdot \begin{bmatrix} x_{1r} & x_{2r} & x_{3r} & x_{4r} \\ x_{5r} & x_{6r} & x_{7r} & x_{8r} \end{bmatrix}, \end{aligned}$$

one can recover the entries of S_r from the three linear equations

$$
\begin{aligned}
s_{1r} &= \frac{x_{5r}}{h_{2r}} = \frac{x_{2r}}{h_{3r}} - \frac{x_{6r}h_{1r}}{h_{2r}h_{3r}}, \\
s_{2r} &= \frac{x_{6r}}{h_{2r}} = \frac{x_{3r}}{h_{3r}} - \frac{x_{7r}h_{1r}}{h_{2r}h_{3r}}, \\
s_{3r} &= \frac{x_{7r}}{h_{2r}} = \frac{x_{4r}}{h_{3r}} - \frac{x_{8r}h_{1r}}{h_{2r}h_{3r}}, \\
0 &= \frac{x_{8r}}{h_{2r}} = \frac{x_{1r}}{h_{3r}} - \frac{x_{5r}h_{1r}}{h_{2r}h_{3r}}.
\end{aligned}
$$

The result guarantees that the CDMA system can handle more simultaneous users (in the case of perfect data without noise, see Chapter 7 of [27]), and that there is also a sharp bound on the number of users that the system can theoretically handle. If there are a finite number of presentations (but more than just one) then it remains to find out which presentation is the true one. We can assume that this can in principle be detected by checking for each presentation whether it yields meaningful results (e.g. one could check whether the estimates of the transmitted symbols belong to the constellation that is being used, see Chapter 5 Section 2.9 of [27]).

4. SOME ILLUSTRATIVE EXAMPLES

In this section, we exhibit several examples to clarify both the basic concepts introduced in the previous sections as well as the relations between them.

First, we exhibit two examples of a defective secant variety for which the defect can be computed directly according to the formula provided by Proposition 2.3.4, although the conditions in that proposition are not justified.

Example 4.0.2. *The secant variety* $\sigma_2(Sub_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11}))$ *has defect* 1.

Proof. To facilitate our exposition, let A, B and C be complex vector spaces of dimensions $3, 5, 11$ respectively.

First, we note that for $\varphi_1 \in A' \otimes B' \otimes C'$ and $\varphi_2 \in A'' \otimes B'' \otimes C''$, where A', A'' are 2 dimensional subspaces of A; B', B'' are 3 dimensional subspaces of B and C', C'' are 6 dimensional subspaces of C.

Since $\dim\, A' \cap A'' \geq 1,\ \dim\, B' \cap B'' \geq 1,\ \dim\, C' \cap C'' \geq 1$, from Lemma 2.3.2, we have

$$\begin{aligned}&\dim\, \hat{T}_{\varphi_1}(\widehat{Sub}_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11})) \cap \hat{T}_{\varphi_2}(\widehat{Sub}_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11}))\\ &\geq \dim\, (A' \otimes B' \otimes C') \cap (A'' \otimes B'' \otimes C'') \geq 1.\end{aligned}$$

So if there exists a pair of points on $\sigma_2(Sub_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11}))$, whose tangent spaces have a one dimensional intersection, we can claim the defect is exactly 1.

Choose a pair of tensors $\{\varphi_1, \varphi_2\}$, such that

$$\varphi_1 = a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) + a_2 \otimes (b_2 \otimes c_2 - b_3 \otimes c_3) \in E_A \otimes E_B \otimes E_C,$$

$$\varphi_2 = a_3 \otimes (b_2 \otimes c_2 + b_4 \otimes c_4 + b_5 \otimes c_5) + a_2 \otimes (b_2 \otimes c_2 - b_5 \otimes c_5) \in F_A \otimes F_B \otimes F_C,$$

where $\{a_1, a_2, a_3\}$, $\{b_1, \ldots, b_5\}$ and $\{c_1, \ldots, c_{11}\}$ are fixed bases for A, B and C, and $E_A = \langle a_1, a_2 \rangle$, $F_A = \langle a_2, a_3 \rangle$, $E_B = \langle b_1, b_2, b_3 \rangle$, $F_B = \langle b_2, b_4, b_5 \rangle$, $E_C = \langle c_1, \ldots, c_6 \rangle$, $F_C = \langle c_2, c_7, \ldots, c_{11} \rangle$.

It is clear that

$$\begin{aligned}
&\hat{T}_{\varphi_1}(\widehat{Sub}_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11})) \\
&= (E_A \otimes E_B \otimes E_C) \\
&+ (A \otimes \langle b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3 \rangle) \\
&+ (B \otimes \langle a_1 \otimes c_1, a_1 \otimes c_2 + a_2 \otimes c_2, a_1 \otimes c_3 - a_2 \otimes c_3 \rangle) \\
&+ (C \otimes \langle a_1 \otimes b_1, a_1 \otimes b_2 + a_2 \otimes b_2, a_1 \otimes b_3 - a_2 \otimes b_3 \rangle),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\hat{T}_{\varphi_2}(\widehat{Sub}_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11})) \\
&= (F_A \otimes F_B \otimes F_C) \\
&+ (A \otimes \langle b_2 \otimes c_2 + b_4 \otimes c_4 + b_5 \otimes c_5 \rangle) \\
&+ (B \otimes \langle a_3 \otimes c_4, a_3 \otimes c_2 + a_2 \otimes c_2, a_3 \otimes c_5 - a_2 \otimes c_5 \rangle) \\
&+ (C \otimes \langle a_3 \otimes b_4, a_3 \otimes b_2 + a_2 \otimes b_2, a_3 \otimes b_5 - a_2 \otimes b_5 \rangle).
\end{aligned}$$

Hence

$$\hat{T}_{\varphi_1}(\widehat{Sub}_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11})) \cap \hat{T}_{\varphi_2}(\widehat{Sub}_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11})) = \langle a_2 \otimes b_2 \otimes c_2 \rangle.$$

Therefore, φ_1 and φ_2 as chosen above are sufficiently general, and the defect is 1. $\square$

Example 4.0.3. *The secant variety* $\sigma_2(Sub_{2,2,4}\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^7)$ *has defect* 1.

Proof. To facilitate our exposition, let A, B and C be complex vector spaces of dimensions $3, 3, 7$ respectively.

First, we note that for $\varphi_1 \in A' \otimes B' \otimes C'$ and $\varphi_2 \in A'' \otimes B'' \otimes C''$, where A', A'' are 2 dimensional subspaces of A; B', B'' are 3 dimensional subspaces of B and C', C'' are 6 dimensional subspaces of C.

Since $\dim\, A' \cap A'' \geq 1,\ \dim\, B' \cap B'' \geq 1,\ \dim\, C' \cap C'' \geq 1$, from Lemma 2.3.2, we have

$$\begin{aligned}
&\dim\, \hat{T}_{\varphi_1}(Sub_{2,2,4}\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^7) \cap \hat{T}_{\varphi_2}(Sub_{2,2,4}\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^7)\\
&\geq \dim\, (A' \otimes B' \otimes C') \cap (A'' \otimes B'' \otimes C'') \geq 1.
\end{aligned}$$

So if there exists a pair of points on $\sigma_2(Sub_{2,2,4}\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^7)$, whose tangent spaces has a one dimensional intersection, we can claim the defect is exactly 1.

Choose a pair of tensors $\{\varphi_1, \varphi_2\}$, such that

$$\begin{aligned}
\varphi_1 =& a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4) \in A' \otimes B' \otimes C',\\
\varphi_2 =& a_1' \otimes (b_1' \otimes c_1' + b_2' \otimes c_3') + a_2 \otimes (b_1 \otimes c_2' + b_2' \otimes c_4) \in A'' \otimes B'' \otimes C'',
\end{aligned}$$

where

$$\begin{aligned}
A' =& \langle a_1, a_2 \rangle,\ A'' = \langle a_1', a_2 \rangle,\\
B' =& \langle b_1, b_2 \rangle,\ B'' = \langle b_1, b_2' \rangle,\\
C' =& \langle c_1, c_2, c_3, c_4 \rangle,\ C'' = \langle c_1', c_2', c_3', c_4 \rangle,
\end{aligned}$$

By a similar computation in the example of $\sigma_2(Sub_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11}))$, it is clear that

$$\hat{T}_{\varphi_1}(\widehat{Sub}_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11})) \cap \hat{T}_{\varphi_2}(\widehat{Sub}_{2,3,6}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{11})) = \langle a_2 \otimes b_1 \otimes c_4 \rangle.$$

Therefore, φ_1 and φ_2 as chosen above are sufficiently general, and the defect is 1. □

The following is a baby example for the study of block term tensor decomposition of multilinear rank $(1, L, L)$.

Example 4.0.4. *The secant variety $\sigma_2(Sub_{1,L,L}\mathbb{C}^2 \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$, $J, K > L$ is not defective.*

Proof. To facilitate our exposition, let A, B and C be complex vector spaces of dimensions $2, J, K$ respectively.

First, we note that for $\varphi_1 \in A' \otimes B' \otimes C'$ and $\varphi_2 \in A'' \otimes B'' \otimes C''$, where A', A'' are 1 dimensional subspaces of A; B', B'' are L dimensional subspaces of B and C', C'' are L dimensional subspaces of C. Write

$$\begin{aligned}\varphi_1 =& a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + \cdots + b_L \otimes c_L) \in A' \otimes B' \otimes C';\\ \varphi_2 =& a'_1 \otimes (b'_1 \otimes c'_1 + b'_2 \otimes c'_2 + \cdots + b'_L \otimes c'_L) \in A'' \otimes B'' \otimes C'',\end{aligned}$$

where $\{a_1, a_2\}$, $\{b_1, \ldots, b_J\}$ and $\{c_1, \ldots, c_K\}$ are fixed bases for A, B and C, and $E_A = \langle a_1 \rangle$, $F_A = \langle a_2 \rangle$, $E_B = \langle b_1, \ldots, b_L \rangle$, $F_B = \langle b'_1, \ldots, b'_L \rangle$, $E_C = \langle c_1, \ldots, c_L \rangle$, $F_C = \langle c'_1, \ldots, c'_L \rangle$, where $\{b'_1, \ldots, b'_L\} \subset \{b_1, \ldots, b_J\}$ is not the same as $\{b_1, \ldots, b_L\}$, and $\{c'_1, \ldots, c'_L\} \subset \{c_1, \ldots, c_J\}$ is not the same as $\{c_1, \ldots, c_L\}$. Then we have

$$\begin{aligned}\hat{T}_{\varphi_1}(Sub_{1,L,L}\mathbb{C}^2 \otimes \mathbb{C}^J \otimes \mathbb{C}^K) = & \sum_{1 \le i \le 2, 1 \le j \le L, 1 \le k \le L} \langle a_1 \otimes b_j \otimes c_k \rangle + \langle a_2 \otimes \sum_{1 \le i \le L} b_i \otimes c_i \rangle \\ & + \sum_{1 \le j \le L, 1 \le k \le L} \langle b'_j \otimes a_1 \otimes c_k \rangle + \sum_{1 \le j \le L, 1 \le k \le L} \langle c'_j \otimes a_1 \otimes b_k \rangle,\end{aligned}$$

$$\hat{T}_{\varphi_1}(Sub_{1,L,L}\mathbb{C}^2 \otimes \mathbb{C}^J \otimes \mathbb{C}^K) = \sum_{1\le j\le L, 1\le k\le L} \langle a_2 \otimes b'_j \otimes c'_k\rangle + \langle a_1 \otimes \sum_{1\le i\le L} b'_i \otimes c'_i\rangle$$
$$+ \sum_{1\le j\le L, 1\le k\le L} \langle b_j \otimes a_2 \otimes c'_k\rangle + \sum_{1\le j\le L, 1\le k\le L} \langle c_j \otimes a_2 \otimes b'_k\rangle.$$

It is clear that

$$\hat{T}_{\varphi_1}(Sub_{1,L,L}\mathbb{C}^2 \otimes \mathbb{C}^J \otimes \mathbb{C}^K) \cap \hat{T}_{\varphi_2}(Sub_{1,L,L}\mathbb{C}^2 \otimes \mathbb{C}^J \otimes \mathbb{C}^K) = \{0\},$$

thus by semi-continuity, $\sigma_2(Sub_{1,L,L}\mathbb{C}^2 \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$ is not defective. □

Example 4.0.5. *The secant variety $\sigma_2(Sub_{2,2,2}(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$ is tangentially weakly defective, although it is non-defective.*

Proof. The fact that $\sigma_2(Sub_{2,2,2}(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$ is not defective follows from Remark of Proposition 2.3.4. So now we pass to the proof that $\sigma_2(Sub_{2,2,2}(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$ is tangentially weakly defective.

Let A, B and C be complex vector spaces of dimensions $4, 4, 4$ respectively. Choose the splitting $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, $C = C_1 \oplus C_2$, where each one of $A_1, A_2, B_1, B_2, C_1, C_2$ has dimension 2.

Since $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ (see Theorem 5.5.1.1 in [21]), there exists a general pair $\{\varphi_1, \varphi_2\} \in \widehat{Sub}_{2,2,2}(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$, such that

$$\varphi_1 = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2,$$

and

$$\varphi_2 = a_3 \otimes b_3 \otimes c_3 + a_4 \otimes b_4 \otimes c_4,$$

where $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ are bases for A_1, B_1, C_1 respectively and similarly

$\{a_3, a_4\}$, $\{b_3, b_4\}$, $\{c_3, c_4\}$ are bases for A_2, B_2, C_2. And note that $\varphi_1+\varphi_2$ is a general point in $\sigma_2(Sub_{2,2,2}(\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4))$. From Theorem 2.2.11, we have

$$\begin{aligned}
&\hat{T}_{\varphi_1+\varphi_2}(\sigma_2(\widehat{Sub}_{2,2,2}(\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4)))\\
=&\bigoplus_{1\le i,j,k\le 2}\langle a_i\otimes b_j\otimes c_k\rangle\oplus\bigoplus_{3\le i\le 4,1\le j\le 2}\langle a_i\otimes b_j\otimes c_j\rangle\\
&\oplus\bigoplus_{1\le i\le 2,3\le j\le 4}\langle a_i\otimes b_i\otimes c_j\rangle\oplus\bigoplus_{3\le j\le 4,1\le k\le 2}\langle a_k\otimes b_j\otimes c_k\rangle\\
&\oplus\bigoplus_{3\le i,j,k\le 4}\langle a_i\otimes b_j\otimes c_k\rangle\oplus\bigoplus_{1\le i\le 2,3\le j\le 4}\langle a_i\otimes b_j\otimes c_j\rangle\\
&\oplus\bigoplus_{3\le i\le 4,1\le j\le 2}\langle a_i\otimes b_i\otimes c_j\rangle\oplus\bigoplus_{1\le j\le 2,3\le k\le 4}\langle a_k\otimes b_j\otimes c_k\rangle.
\end{aligned}$$

Define $\psi=a_1\otimes b_1\otimes c_1+a_3\otimes b_3\otimes c_3$, which is a third general point in $\widehat{Sub}_{2,2,2}\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4$, and note that

$$\begin{aligned}
&\hat{T}_{\psi}(\widehat{Sub}_{2,2,2}(\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4))\\
=&\bigoplus_{i,j,k\in\{1,3\}}\langle a_i\otimes b_j\otimes c_k\rangle\oplus\bigoplus_{i=2,4,j=1,3}\langle a_i\otimes b_j\otimes c_j\rangle\\
&\oplus\bigoplus_{j=2,4,i=1,3}\langle b_j\otimes a_i\otimes c_i\rangle\oplus\bigoplus_{k=2,4,\ i=1,3}\langle c_k\otimes a_i\otimes b_i\rangle.
\end{aligned}\tag{4.0.1}$$

It is straightforward to compute that

$$\hat{T}_{\varphi_1}(\widehat{Sub}_{2,2,2}(\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4))+\hat{T}_{\varphi_2}(\widehat{Sub}_{2,2,2}(\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4))\supset\hat{T}_{\psi}(\widehat{Sub}_{2,2,2}(\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4)).$$

This implies $\sigma_2(Sub_{2,2,2}(\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4))$ is tangentially weakly-defective. □

Remark. Although tangentially weakly defective does not imply non-uniqueness, the decomposition is not unique here. The reason is trivial:

$$\begin{aligned}&(a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2) + (a_3 \otimes b_3 \otimes c_3 + a_4 \otimes b_4 \otimes c_4)\\ =&(a_1 \otimes b_1 \otimes c_1 + a_3 \otimes b_3 \otimes c_3) + (a_2 \otimes b_2 \otimes c_2 + a_4 \otimes b_4 \otimes c_4).\end{aligned}$$

Example 4.0.6. *The secant variety $\sigma_2(Sub_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$ is weakly defective, but not tangentially weakly defective.*

Proof. Part 1: Let $A,\ B$ and C denote complex vector spaces of dimensions $2, 4, 4$ respectively.

We need to prove that for any general hyperplane H tangent to $\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$ at a general pair of points $\{\varphi_1, \varphi_2\}$ is also tangent to $\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$ at some general point ψ, satisfying $[\psi] \neq [\varphi_1], [\varphi_2]$.

Choose general points $\varphi_1, \varphi_2 \in \widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$, and $\varphi_1 + \varphi_2 \in \hat{\sigma}_2(Sub_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$ is also general. Without loss of generality, we assume

$$\begin{aligned}\varphi_1 &= a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2),\\ \varphi_2 &= a_2 \otimes (b_3 \otimes c_3 + b_4 \otimes c_4),\end{aligned}$$

where $A = \langle a_1, a_2 \rangle,\ B = \langle b_1, \ldots, b_4 \rangle$ and $C = \langle c_1, \ldots, c_4 \rangle$. Note that $A = A_1 \oplus A_2,\ B = B_1 \oplus B_2,\ C = C_1 \oplus C_2$, where $\{a_1\}$, $\{b_1,\ b_2\}$, $\{c_1, c_2\}$ are bases for A_1, B_1, C_1, respectively, and similarly $\{a_2\}$, $\{b_3,\ b_4\}$, $\{c_3, c_4\}$ are bases for A_2, B_2, C_2.

For $\varphi_p \in A_p \otimes B_p \otimes C_p, p = 1, 2$, we have

$$\hat{T}_{\varphi_p}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4) \tag{4.0.2}$$
$$= (A_p \otimes B \otimes C_p) + (A_p \otimes B_p \otimes C) + (A \otimes \varphi_p(A_p^*)),$$

and

$$\hat{T}^{\perp}_{\varphi_p}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4) = (A_p^{\perp} \otimes B_p^{\perp} \otimes C^*) \oplus (A^* \otimes B_p^{\perp} \otimes C_p^{\perp})$$
$$\oplus (A_p^{\perp} \otimes B^* \otimes C_p^{\perp}) \oplus (A_p^{\perp} \otimes (\varphi_p(A_p^*)^{\perp} \cap (B_p^* \otimes C_p^*))).$$

Then using Theorem 2.2.11, we have

$$\hat{T}^{\perp}_{\varphi_1+\varphi_2}(\sigma_2(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$$
$$=\hat{T}^{\perp}_{\varphi_1}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4) \cap \hat{T}^{\perp}_{\varphi_2}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$$
$$=(A_2^* \otimes (\varphi_1(A_1^*)^{\perp} \cap (B_1^* \otimes C_1^*))) \oplus (A_1^* \otimes (\varphi_2(A_2^*)^{\perp} \cap (B_2^* \otimes C_2^*))),$$

which implies

$$\hat{T}^{\perp}_{\varphi_1+\varphi_2}(\sigma_2(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)))$$
$$=\langle a_2^* \otimes b_1^* \otimes c_2^*,\ a_2^* \otimes b_2^* \otimes c_1^*,\ a_2^* \otimes b_1^* \otimes c_1^* - a_2^* \otimes b_2^* \otimes c_2^*, \tag{4.0.3}$$
$$a_1^* \otimes b_4^* \otimes c_3^*,\ a_1^* \otimes b_3^* \otimes c_4^*,\ a_1^* \otimes b_4^* \otimes c_4^* - a_1^* \otimes b_3^* \otimes c_3^* \rangle.$$

Due to (4.0.3), every hyperplane tangent to $\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$ at φ_1 and φ_2 is of the form

$$H = a_2^* \otimes (\lambda_1 b_1^* \otimes c_2^* + \lambda_2 b_2^* \otimes c_1^* + \lambda_3(b_1^* \otimes c_1^* - b_2^* \otimes c_2^*))$$
$$+ a_1^* \otimes (\mu_1 b_4^* \otimes c_3^* + \mu_2 b_3^* \otimes c_4^* + \mu_3(b_4^* \otimes c_4^* - b_3^* \otimes c_3^*)),$$

where all of $\lambda_i, \mu_j, 1 \leq i, j \leq 3$ are not zero.

It is straightforward to calculate that H is tangent to $\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$ at ψ, where

$$\psi = a_1 \otimes (-\lambda_2(b_1 \otimes c_2) + \lambda_1(b_2 \otimes c_1) + \lambda_3(b_1 \otimes c_1 + b_2 \otimes c_2)),$$

clearly $[\psi] \neq [\varphi_1], [\varphi_2]$.

This concludes the proof that $\sigma_2(Sub_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$ is weakly defective.

We pass now to the proof that $\sigma_2(Sub_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$ is not tangentially weakly defective.

Part 2: Let $\psi = a' \otimes (b' \otimes c' + b'' \otimes c'') \in A' \otimes B' \otimes C'$ be a general point in $Sub_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$, where $A' = \langle a' \rangle$, $B' = \langle b', b'' \rangle$ and $C' = \langle c', c'' \rangle$; obviously we have

$$\hat{T}_\psi(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)) \tag{4.0.4}$$
$$= (A' \otimes B \otimes C') + (A' \otimes B' \otimes C) + (A \otimes \psi(A'^*)).$$

First, let $\{\varphi_1, \varphi_2\} \in \widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$. Without loss of generality, we can consider the general pair $\{\varphi_1, \varphi_2\}$ as in Part 1.

Also according to Remark 2.2,

$$\hat{T}_{\varphi_1}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)) + \hat{T}_{\varphi_2}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)) \supset \hat{T}_{\psi}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)), \tag{4.0.5}$$

is equivalent to that

$$\hat{T}^{\perp}_{\varphi_1}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)) \cap \hat{T}^{\perp}_{\varphi_2}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)) \subset \hat{T}^{\perp}_{\psi}(\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)). \tag{4.0.6}$$

Hence we need to prove that these inclusions imply that $[\psi]$ is either $[\varphi_1]$ or $[\varphi_2]$.

Express a', c' as

$$a' = x_1 a_1 + x_2 a_2,\ c' = z_1 c_1 + \cdots + z_4 c_4.$$

and we first treat the case when $x_1,\ x_2$ are both nonzero. A hyperplane

$$H_1 = a_2^* \otimes b_1^* \otimes c_2^*$$

and respectively

$$H_2 = a_2^* \otimes b_2^* \otimes c_1^*$$

is tangent to $\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$ at ψ, only if, using the symbol $\vdash$ introduced in (2.1.1), we have

$$a_2^* \otimes b_1^* \otimes c_2^* \vdash (x_1 a_1 + x_2 a_2) \otimes b_1 \otimes (z_1 c_1 + \cdots + z_4 c_4) = x_2 z_2 = 0,$$

and respectively,

$$a_2^* \otimes b_2^* \otimes c_1^* \vdash (x_1a_1 + x_2a_2) \otimes b_2 \otimes (z_1c_1 + \cdots + z_4c_4) = x_2z_1 = 0,$$

where

$$(x_1a_1 + x_2a_2) \otimes b_j \otimes (z_1c_1 + \cdots + z_4c_4) \in A' \otimes B \otimes C',\ j = 1, 2.$$

This implies $z_2,\ z_1 = 0$ and by symmetry, $z_k = 0$, for $1 \leq k \leq 4$. Then $c' = 0$, so $b' = 0$ by symmetry; for the same reason $c'' = b'' = 0$, hence $\psi = 0$.

Next without loss of generality, we treat the case when $a' = a_1$; taking $H = a_1^* \otimes b_4^* \otimes c_3^*,\ a_1^* \otimes b_3^* \otimes c_4^*$, we obtain $z_3,\ z_4 = 0$, so $c' \in C_1$. By symmetry, $c'' \in C_1$; and for the same reason, $b', b'' \in B_1$. Thus we have $\psi \in A_1 \otimes B_1 \otimes C_1$. Similarly when $a' = a_2$, we have $\psi \in A_2 \otimes B_2 \otimes C_2$.

From the above analysis, H is tangent to $\widehat{Sub}_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$ at ψ only if $\psi \in \bigcup_{p=1,2} A_p \otimes B_p \otimes C_p$. But without loss of generality, if $\psi \in A_1 \otimes B_1 \otimes C_1$, in the previous case, (4.0.5) becomes

$$\begin{aligned} &\bigoplus_{p=1,2} \{(A_p \otimes B \otimes C_p) + (A_p \otimes B_p \otimes C) + (A \otimes \varphi_p(A_p^*))\} \\ &\supset (A_1 \otimes B \otimes C_1) + (A_1 \otimes B_1 \otimes C) + (A \otimes \psi(A_1^*)), \end{aligned}$$

and hence $[\psi(A_1^*)] = [\varphi_1(A_1^*)]$, which implies $[\psi] = [\varphi_1]$. By semicontinuity, this concludes the proof that $\sigma_2(Sub_{1,2,2}(\mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4))$ is not tangentially weakly-defective. □

5. PROOF OF MAIN RESULTS

This section contains a proof of Theorem 1.2.1. However, before passing to the proof, it worth noting that the geometric meaning of the basic hypothesis of Theorem 1.2.1, namely

$$\sum_{1\le r\le R} (J\cdot L_r + L_r\cdot (K-L_r) + (I-1)) < IJK,$$

is equivalent to the join $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K),\ldots,Sub_{1,L_R,L_R}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K))$ not filling its ambient space.

It is relatively easy to determine cases where tensors have at most a finite number of decompositions.

5.1 Proof that Condition A implies partial uniqueness

Let A, B and C be complex vector spaces of dimensions I, J, K respectively. Choose general $\varphi_p \in \widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K), 1\le p\le R$. Without loss of generality, we assume

$$\varphi_p = (a_1+\lambda^p a_2)\otimes(b_{p,1}\otimes c_{p,1} + b_{p,2}\otimes c_{p,2} + \cdots + b_{p,L_p}\otimes c_{p,L_p}) \in A_p\otimes B_p\otimes C_p,$$

where $\{a_1+\lambda^p a_2\}$ are bases for A_p, $\{b_{p,1},\ldots,b_{p,L_p}\}\subset\{b_1,\ldots,b_J\}$, $\{c_{p,1},\ldots,c_{p,L_p}\}\subset\{c_1,\ldots,c_K\}$ are bases for B_p, C_p, where $1\le p\le R$.

Using Lemma 2.3.2, it is straightforward to compute

$$\begin{aligned}&\hat{T}_{\varphi_p}(\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K))\\ =&(A\otimes\langle \sum_{1\le j\le L_p} b_{p,j}\otimes c_{p,j}\rangle) + (\langle a_1+\lambda^p a_2\rangle\otimes B\otimes C_p) + (\langle a_1+\lambda^p a_2\rangle\otimes B_p\otimes C)\end{aligned}$$

and by Theorem 2.2.11, we have

$$
\begin{aligned}
&\hat{T}_{\sum_{p=1}^{R}\varphi_p}(\hat{\mathbf{J}}(Sub_{1,L_1,L_1}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K),\ldots,Sub_{1,L_R,L_R}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K)))\\
=&\sum_{p=1}^{R}\{(A\otimes\langle\sum_{1\leq j\leq L_p} b_{p,j}\otimes c_{p,j}\rangle) \qquad (5.1.1)\\
&+(\langle a_1+\lambda^p a_2\rangle\otimes B\otimes C_p)+(\langle a_1+\lambda^p a_2\rangle\otimes B_p\otimes C)\}.
\end{aligned}
$$

We need to prove that (5.1.1) is a direct sum at least for this set of points $\{\varphi_1,\ldots,\varphi_R\}$.

Since $\binom{J}{L_R},\binom{K}{L_R}\geq R$, we can choose bases such that for any pair $p\neq q$, there exists $b_s\in\{b_{p,1},\ldots,b_{p,L_p}\}$ not belong to $\{b_{q,1},\ldots,b_{q,L_q}\}$; and similarly there exists $c_s\in\{c_{p,1},\ldots,c_{p,L_p}\}$ not belong to $\{c_{q,1},\ldots,c_{q,L_q}\}$ for any pair $p\neq q$. Then we have

$$(\sum_{1\leq j\leq L_p} b_{p,j}\otimes c_{p,j})\cap(B\otimes C_q+B_q\otimes C)=\{0\}.$$

Therefore,

$$
\begin{aligned}
&(A\otimes\langle\sum_{1\leq j\leq L_p} b_{p,j}\otimes c_{p,j}\rangle)\bigcap\\
&(\sum_{\forall q\neq p}\{(A\otimes\langle\sum_{1\leq j\leq L_p} b_{q,j}\otimes c_{q,j}\rangle)+(\langle a_1+\lambda^q a_2\rangle\otimes B\otimes C_q)\\
&+(\langle a_1+\lambda^q a_2\rangle\otimes B_q\otimes C)\})\\
&=\sum_{\forall q\neq p}\langle a_1+\lambda^q a_2\rangle\otimes(\langle\sum_{1\leq j\leq L_p} b_{p,j}\otimes c_{p,j}\rangle\cap(B\otimes C_q+B_q\otimes C))\\
&=\{0\},
\end{aligned}
$$

as well as

$$\{(\langle a_1+\lambda^p a_2\rangle\otimes B\otimes C_p)+(\langle a_1+\lambda^p a_2\rangle\otimes B_p\otimes C)\}$$
$$\bigcap\sum_{p\neq q}\{(\langle a_1+\lambda^q a_2\rangle\otimes B\otimes C_q)+(\langle a_1+\lambda^q a_2\rangle\otimes B_q\otimes C)\}=\{0\}.$$

Thus (5.1.1) is a direct sum, and by semicontinuity, this concludes the proof of non-defectivity. Now Condition A follows from Case 2.2.13.

5.2 Proof that Condition B implies non-uniqueness

Using Proposition 2.3.3, and from Corollary 2.2.13, we deduce Case B.

5.3 Proof that Condition C implies generic uniqueness

It is sufficient to prove the case $I=2, J=K=\sum_{r=1}^{R}L_r$. Otherwise we replace the equality in (5.3.2) with the inclusion $\supset$.

Let $A,\ B$ and C be complex vector spaces of dimensions I,J,K respectively. Split $B=\bigoplus_{1\leq q\leq R}B_q$ and $C=\bigoplus_{1\leq r\leq R}C_r$, where for $1\leq q,r\leq R$, B_q and C_r are of dimensions L_q,L_r, respectively.

Choose a general set $\{\varphi_p\in\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K):1\leq p\leq R\}$. Without loss of generality, we can assume

$$\varphi_p=(a_1+\lambda^p a_2)\otimes(b_{p,1}\otimes c_{p,1}+b_{p,2}\otimes c_{p,2}+\cdots+b_{p,L_p}\otimes c_{p,L_p})\in A_p\otimes B_p\otimes C_p,$$

for any $1\leq p\leq R$, where $\{a_1+\lambda^p a_2\}$, $\{b_{p,1},\ldots,b_{p,L_p}\}$ and $\{c_{p,1},\ldots,c_{p,L_p}\}$ are bases for A_p, B_p, C_p. Note that for a general set $\{\varphi_p\in A_p\otimes B_p\otimes C_p:1\leq p\leq R\}$, we have

$$\hat{T}_{\varphi_p}(\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K)) \tag{5.3.1}$$
$$=(A\otimes\varphi_p(A_p^*))+(A_p\otimes B\otimes C_p)+(A_p\otimes B_p\otimes C),$$

and

$$\hat{T}^{\perp}_{\varphi_p}(\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K))=(A_p^{\perp}\otimes B_p^{\perp}\otimes C^*)\oplus(A_p^*\otimes B_p^{\perp}\otimes C_p^{\perp})$$
$$\oplus(A_p^{\perp}\otimes B_p^*\otimes C_p^{\perp})\oplus(A_p^{\perp}\otimes(\varphi_p(A_p^*)^{\perp}\cap(B_p^*\otimes C_p^*))).$$

Then due to Theorem 2.2.11, we deduce

$$\hat{T}^{\perp}_{\sum\varphi_p}(\mathbf{J}(\widehat{Sub}_{1,L_1,L_1}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K),\ldots,\widehat{Sub}_{1,L_R,L_R}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K)))$$
$$=\bigcap_{1\le p\le R}\widehat{T}^{\perp}_{\varphi_p}(\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K))$$
$$=\bigoplus_{1\le p\le R}A_p^{\perp}\otimes(\varphi_p(A_p^*)^{\perp}\cap(B_p^*\otimes C_p^*)) \tag{5.3.2}$$
$$=\bigoplus_{1\le p\le R,\ j\ne k}\langle\lambda^p a_1^*-a_2^*\rangle\otimes\langle b_{p,j}^*\otimes c_{p,k}^*,b_{p,j}^*\otimes c_{p,j}^*-b_{p,k}^*\otimes c_{p,k}^*\rangle.$$

For any $1\le s\le R$, let

$$\psi_s=a'\otimes(b_1'\otimes c_1'+\cdots+b_{L_s}'\otimes c_{L_s}')\in A'\otimes B'\otimes C', \tag{5.3.3}$$

be a general point of $\widehat{Sub}_{1,L_s,L_s}\mathbb{C}^I\otimes\mathbb{C}^J\otimes\mathbb{C}^K$, where $A'=\langle a'\rangle$, $B'=\langle b_1',\ldots,b_{L_s}'\rangle$ and

$C' = \langle c'_1, \ldots, c'_{L_s} \rangle$; note that

$$\hat{T}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K) \tag{5.3.4}$$
$$= (A' \otimes B \otimes C') + (A' \otimes B' \otimes C) + (A \otimes \psi(A'^*)).$$

Also according to Remark 2.2, the relation

$$\hat{T}_{\varphi_1}(\widehat{Sub}_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) + \cdots + \hat{T}_{\varphi_R}(\widehat{Sub}_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \tag{5.3.5}$$
$$\supset \hat{T}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)),$$

is equivalent to

$$\bigcap_{1 \leq p \leq R} \widehat{T}^{\perp}_{\varphi_p}(\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \subset \hat{T}^{\perp}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)),$$

Hence we need to prove that these inclusions imply that $\psi_s \in \{\varphi_1, \ldots, \varphi_R\}$.

Denote c' any one of $c'_1, \ldots, c'_{L_s}$ in (5.3.3) and write a', c' as

$$a' = x_1 a_1 + x_2 a_2,$$
$$c' = \sum_{1 \leq h \leq L_1} z_{1,h} c_{1,h} + \cdots + \sum_{1 \leq h \leq L_R} z_{R,h} c_{R,h}.$$

We treat first the case when $(x_1,\ x_2) \neq (1,\ \lambda^p)$ for any $1 \leq p \leq R$.

A general hyperplane in (5.3.2) is a linear combination of $(\lambda^p a_1^* - a_2^*) \otimes b_{p,j}^* \otimes c_{p,k}^*$ and $(\lambda^p a_1^* - a_2^*) \otimes (b_{p,j}^* \otimes c_{p,j}^* - b_{p,k}^* \otimes c_{p,k}^*)$. In particular,

$$(\lambda^p a_1^* - a_2^*) \otimes b_{p,j}^* \otimes c_{p,k}^* \in \hat{T}^{\perp}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$$

(i.e. is tangent to $\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$ at ψ_s) only if

$$\begin{aligned}
&(\lambda^p a_1^* - a_2^*) \otimes b_{p,j}^* \otimes c_{p,k}^* \vdash \\
&(x_1 a_1 + x_2 a_2) \otimes b_{p,j} \otimes (\sum_{1 \le h \le L_1} z_{1,h} c_{1,h} + \cdots + \sum_{1 \le h \le L_R} z_{R,h} c_{R,h}) \\
&= (\lambda^p x_1 - x_2) z_{t,p_k} = 0,
\end{aligned}$$

where

$$(x_1 a_1 + x_2 a_2) \otimes b_{p,j} \otimes (\sum_{1 \le k \le L} z_{t,r_k} c_{t,r_k}) \in A' \otimes B \otimes C'.$$

Therefore $z_{t,h} = 0, 1 \le t \le R$ for any h, so $c' = 0$, that is $c'_1 = \cdots = c'_{L_s} = 0$; for the same reason $b'_1 = \cdots = b'_{L_s} = 0$, hence $\psi_s = 0$ in this case.

Next, we consider the case when $a' = a_1 + \lambda a_2$; taking a hyperplane in (5.3.2)

$$H_p = (\lambda^p a_1^* - a_2^*) \otimes b_{p,j}^* \otimes c_{p,k}^*, \ p \neq 1,$$

we obtain $z_{t,h} = 0, 2 \le t \le R$ for any h. So we deduce

$$c' = \sum_{1 \le h \le L_1} z_{1,h} c_{1,h} \in C_1,$$

for every $c' \in \{c'_1, \ldots, c'_{L_s}\}$. By symmetry, we have $b'_1, \ldots, b'_{L_s} \in B_1$. Thus $\psi_s \in A_1 \otimes B_1 \otimes C_1$. In this case, (5.3.5) becomes

$$\begin{aligned}
&\bigoplus_{1 \le p \le R} \{(A_p \otimes B \otimes C_p) + (A_p \otimes B_p \otimes C) + (A \otimes \varphi_p(A_p^*))\} \\
&\supset (A_1 \otimes B \otimes C_1) + (A_1 \otimes B_1 \otimes C) + (A \otimes \psi_s(A_1^*)),
\end{aligned}$$

this is valid if and only if

$$[\psi_s(A_1^*)] = [\varphi_1(A_1^*)]$$

and then $[\psi_s] = [\varphi_1]$.

Finally, it remains to consider the case when $a' = a_1 + \lambda^q a_2,\ 1 < q \leq R$. Then clearly, we have $\psi_s \in A_q \otimes B_q \otimes C_q$, so $[\psi_s] = [\varphi_q]$. Now by semicontinuity, we can conclude the proof that $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), \ldots, Sub_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is not tangentially weakly-defective for $I = 2,\ J = K = \sum_{s=1}^R L_s$. From Corollary 2.2.16, we obtain Case C.

Remark. An alternative least square algorithm for the computation of decompositions under the above Case C is given in [12].

5.4 Proof that Condition D implies generic uniqueness

The proof will be nearly the same as that for condition C. (The difference is that we cannot apply Corollary 2.2.16 directly in the proof of Condition D.) As there, it is sufficient to prove the case $I = 2$. Otherwise we change the equality in (5.4.2) into the inclusion $\supset$.

Let A, B and C denote vector spaces of dimensions I, J, K respectively. Split $A = A_1 \oplus A_2$, $B = B_1 \oplus B_0 \oplus B_2$ and $C_1 \oplus C_0 \oplus C_2$, where A_1, A_2 are of dimension one, B_1, B_0, B_2, and C_1, C_0, C_2 are of dimension $L_1 - l_b$, l_b, $L_2 - l_b$, $L_1 - l_c$, l_c, $L_2 - l_c$, respectively, for some $0 \le l_b, l_c < \min\{L_1, L_2\}$.

For general $\varphi_p \in A_p \otimes (B_p \oplus B_0) \otimes (C_p \oplus C_0), p = 1, 2$, we have

$$\begin{aligned} &\hat{T}_{\varphi_p}(\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \\ =&(A \otimes \varphi_p(A_p^*)) + (A_p \otimes B \otimes (C_p \oplus C_0)) + (A_p \otimes (B_p \oplus B_0) \otimes C); \end{aligned} \tag{5.4.1}$$

and also

$$\begin{aligned} &\hat{T}^{\perp}_{\varphi_1}(\widehat{Sub}_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \\ =&(A_2^* \otimes B_2^* \otimes C^*) \oplus (A_1^* \otimes B_2^* \otimes C_2^*) \oplus (A_2^* \otimes (B_1 \oplus B_0)^* \otimes C_2^*) \\ &\oplus (A_2^* \otimes (\varphi_1(A_1^*)^{\perp} \cap ((B_1^* \oplus B_0^*) \otimes (C_1^* \oplus C_0^*)))), \end{aligned}$$

respectively

$$\begin{aligned} &\hat{T}^{\perp}_{\varphi_2}(\widehat{Sub}_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \\ =&(A_1^* \otimes B_1^* \otimes C^*) \oplus (A_2^* \otimes B_1^* \otimes C_1^*) \oplus (A_1^* \otimes (B_2 \oplus B_0)^* \otimes C_1^*) \\ &\oplus (A_1^* \otimes (\varphi_2(A_2^*)^{\perp} \cap ((B_2^* \oplus B_0^*) \otimes (C_2^* \oplus C_0^*)))). \end{aligned}$$

Choose general points $\varphi_p \in \widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$, for $1 \le p \le 2$. Without loss of generality, we assume $l_b \ge l_c$, and then

$$\begin{aligned}\varphi_1 &= a_1 \otimes (b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1-l_b} \otimes c_{1,L_1-l_b} + b_{0,1} \otimes c_{1,L_1-l_b+1} + \cdots + b_{0,l_b} \otimes c_{0,l_c}) \\ &\in A_1 \otimes (B_1 \oplus B_0) \otimes (C_1 \oplus C_0) \cong \mathbb{C} \otimes \mathbb{C}^{L_1} \otimes \mathbb{C}^{L_1},\end{aligned}$$

and respectively

$$\begin{aligned}\varphi_2 &= a_2 \otimes (b_{2,1} \otimes c_{2,1} + \cdots + b_{2,L_1-l_b} \otimes c_{2,L_1-l_b} + b_{0,1} \otimes c_{2,L_1-l_b+1} + \cdots + b_{0,l_b} \otimes c_{0,l_c}) \\ &\in A_2 \otimes (B_2 \oplus B_0) \otimes (C_2 \oplus C_0) \cong \mathbb{C} \otimes \mathbb{C}^{L_2} \otimes \mathbb{C}^{L_2},\end{aligned}$$

where $A_i = \langle a_i \rangle (i = 1, 2)$; $\{b_{0,1}, \ldots, b_{0,l_b}\}$, $\{b_{1,1}, \ldots, b_{1,L_1-l_b}\}$, $\{b_{2,1}, \ldots, b_{2,L_2-l_b}\}$, $\{c_{0,1}, \ldots, c_{0,l_c}\}$, $\{c_{1,1}, \ldots, c_{1,L_1-l_c}\}$ and $\{c_{2,1}, \ldots, c_{2,L_2-l_c}\}$ are bases for B_0, B_1, B_2, C_0, C_1 and C_2 respectively, where $J + l_b = L_1 + L_2$ and $K + l_c = L_1 + L_2$.

Using these bases, we obtain

$$\begin{aligned}&\hat{T}^{\perp}_{\varphi_1}(\widehat{Sub}_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \cap \hat{T}^{\perp}_{\varphi_2}(\widehat{Sub}_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \\ =&(A_2^* \otimes (\varphi_1(A_1^*)^{\perp} \cap (B_1^* \otimes C_1^*))) \oplus (A_1^* \otimes (\varphi_2(A_2^*)^{\perp} \cap (B_2^* \otimes C_2^*))) \\ =&\bigoplus_{j \ne k} \langle a_1^* \otimes b_{2,j}^* \otimes c_{2,k}^*, a_2^* \otimes b_{1,j}^* \otimes c_{1,k}^*, \\ &a_1^* \otimes (b_{2,j}^* \otimes c_{2,j}^* - b_{2,k}^* \otimes c_{2,k}^*), a_2^* \otimes (b_{1,j}^* \otimes c_{1,j}^* - b_{1,k}^* \otimes c_{1,k}^*)\rangle.\end{aligned} \tag{5.4.2}$$

Let

$$\psi_s = a' \otimes (b'_1 \otimes c'_1 + \cdots + b'_{L_s} \otimes c'_{L_s}) \in A' \otimes B' \otimes C', \tag{5.4.3}$$

be a general point in one of $\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$, $s = 1, 2$, where $A' = \langle a' \rangle$,

$B' = \langle b'_1, \ldots, b'_{L_s} \rangle$ and $C' = \langle c'_1, \ldots, c'_{L_s} \rangle$. Using Lemma 2.3.2, we have

$$\begin{aligned} &\hat{T}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \qquad (5.4.4) \\ &= (A' \otimes B \otimes C') + (A' \otimes B' \otimes C) + (A \otimes \psi(A'^*)). \end{aligned}$$

If

$$[\varphi_1 + \varphi_2] = [\psi_1 + \psi_2], \qquad (5.4.5)$$

we have

$$\begin{aligned} &\hat{T}_{\varphi_1}(\widehat{Sub}_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) + \hat{T}_{\varphi_2}(\widehat{Sub}_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \qquad (5.4.6) \\ &\supset \hat{T}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)), \end{aligned}$$

and according to Remark 2.2, it is equivalent to that

$$\begin{aligned} &\hat{T}^{\perp}_{\varphi_1}(\widehat{Sub}_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \cap \hat{T}^{\perp}_{\varphi_2}(\widehat{Sub}_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \qquad (5.4.7) \\ &\subset \hat{T}^{\perp}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)). \qquad (5.4.8) \end{aligned}$$

Hence we need to prove that these inclusions imply that $\psi_s \in \{\varphi_1, \varphi_2\}$.

Express a $c' \in \{c'_1, \ldots, c'_{L_r}\}$ in (5.4.3) as

$$c' = \sum_{1 \le h \le l_c} z_{0,h} c_{0,h} + \sum_{1 \le h \le L_1 - l_c} z_{1,h} c_{1,h} + \sum_{1 \le h \le L_2 - l_c} z_{2,h} c_{2,h},$$

and write $a' = x_1 a_1 + x_2 a_2$. We treat first the case that both x_1, x_2 are nonzero. From (5.4.4), a general hyperplane in (5.4.2) is a linear combination of $a_1^* \otimes b_{2,j}^* \otimes c_{2,k}^*, a_2^* \otimes b_{1,j}^* \otimes c_{1,k}^*, a_1^* \otimes (b_{2,j}^* \otimes c_{2,j}^* - b_{2,k}^* \otimes c_{2,k}^*), a_2^* \otimes (b_{1,j}^* \otimes c_{1,j}^* - b_{1,k}^* \otimes c_{1,k}^*)$.

Note that for

$$(x_1a_1 + x_2a_2) \otimes b_{1,j} \otimes (\sum_{1\leq h\leq l_c} z_{0,h}c_{0,h} + \sum_{1\leq h\leq L_1-l_c} z_{1,h}c_{1,h} + \sum_{1\leq h\leq L_2-l_c} z_{2,h}c_{2,h})$$

in $A' \otimes B \otimes C'$, we have

$$H_1 = a_2^* \otimes b_{1,j}^* \otimes c_{1,k}^* \in \hat{T}_{\psi_s}^{\perp}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$$

(i.e. H_1 is tangent to $\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$ at ψ_s) only if

$$\begin{aligned}
&a_2^* \otimes b_{1,j}^* \otimes c_{1,k}^* \vdash \\
&(x_1a_1 + x_2a_2) \otimes b_{1,j} \otimes (\sum_{1\leq h\leq l_c} z_{0,h}c_{0,h} + \sum_{1\leq h\leq L_1-l_c} z_{1,h}c_{1,h} + \sum_{1\leq h\leq L_2-l_c} z_{2,h}c_{2,h}) \\
&= x_2z_{1,k} = 0,
\end{aligned}$$

and respectively

$$H_2 = a_1^* \otimes b_{2,j}^* \otimes c_{2,k}^* \in \hat{T}_{\psi_s}^{\perp}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$$

only if

$$\begin{aligned}
&a_1^* \otimes b_{2,j}^* \otimes c_{2,k}^* \vdash \\
&(x_1a_1 + x_2a_2) \otimes b_{2,j} \otimes (\sum_{1\leq h\leq l_c} z_{0,h}c_{0,h} + \sum_{1\leq h\leq L_1-l_c} z_{1,h}c_{1,h} + \sum_{1\leq h\leq L_2-l_c} z_{2,h}c_{2,h}) \\
&= x_1z_{2,k} = 0.
\end{aligned}$$

Therefore $z_{1,h} = 0$, for $1 \leq h \leq L_1 - l_c$, and $z_{2,h} = 0$, for $1 \leq h \leq L_2 - l_c$. So $c' = \sum_{1\leq h\leq l_c} z_{0,h}c_{0,h} \in C_0$. For the same reason $b' \in B_0$ for every $b' \in \{b'_1, \ldots, b'_{L_s}\}$.

Thus $\psi_s \in A_p \otimes B_0 \otimes C_0,\ p = 1, 2$. But since dim B_0, dim C_0 are both less than min $\{L_1,\ L_2\}$, we get a contradiction to the fact that ψ_s has multilinear rank $(1, L_1, L_1)$ or $(1, L_2, L_2)$. Thus $x_1,\ x_2$ cannot be both nonzero.

Next, without loss of generality, we consider the case when $a' = a_1$. Taking the hyperplane

$$H_2 = a_1^* \otimes b_{2,j}^* \otimes c_{2,k}^*,$$

in (5.4.2), we obtain by a similar computation that $z_{2,h} = 0$, for each $1 \leq h \leq L_2 - l_c$, so

$$c' = \sum_{1 \leq h \leq l_c} z_{0,h} c_{0,h} + \sum_{1 \leq h \leq L_1 - l_c} z_{1,h} c_{1,h} \in C_1 \oplus C_0;$$

for the same reason, $b_h' \in B_1 \oplus B_0$, and

$$\psi_s \in A_1 \otimes (B_1 \oplus B_0) \otimes (C_1 \oplus C_0).$$

Now (5.4.5) is valid if and only if

$$[\psi_s(A_1^*)] = [\varphi_1(A_1^*)],$$

and then $[\psi_s] = [\varphi_1]$. Thus we obtain Case D.

5.5 Proof that Condition E implies generic uniqueness

The proof will be nearly the same as that for Condition D. As there, it is sufficient to prove the case $I = R$, $K = \sum_{r=1}^{R} L_r$. Let A, B and C denote vector spaces of dimensions I, J, K respectively. Choose splitting $A = \bigoplus_{1\le p\le R} A_p$ and $C = \bigoplus_{1\le r\le R} C_r$. Further fix a basis $\{b_1, \ldots, b_J\}$ for B.

Choose general points $\varphi_p \in \widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$ for $1 \le p \le R$. Without loss of generality, for $1 \le p \le R$, we can assume

$$\varphi_p = a_p \otimes (b_{p,1} \otimes c_{p,1} + b_{p,2} \otimes c_{p,2} + \cdots + b_{p,L_p} \otimes c_{p,L_p}) \in A_p \otimes B_p \otimes C_p,$$

where $\{a_p\}$, $\{b_{p,1}, \ldots, b_{p,L_p}\} \subset \{b_1, \ldots, b_J\}$, $\{c_{p,1}, \ldots, c_{p,L_p}\}$ are bases for A_p, B_p, C_p, respectively.

Then for $\varphi_p \in A_p \otimes B_p \otimes C_p, 1 \le p \le R$, it is clear that

$$\begin{aligned}&\hat{T}^{\perp}_{\varphi_p}(\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) = (A_p^{\perp} \otimes B_p^{\perp} \otimes C^*) \oplus (A_p^* \otimes B_p^{\perp} \otimes C_p^{\perp})\\ &\oplus (A_p^{\perp} \otimes B_p^* \otimes C_p^{\perp}) \oplus (A_p^{\perp} \otimes (\varphi_p(A_p^*)^{\perp} \cap (B_p^* \otimes C_p^*))).\end{aligned}$$

Due to Theorem 2.2.11 and since $J > 2L_R \ge L_i + L_r$, there exists $b^*_{q,j} \notin B_i^* \oplus B_r^*$ for any i, r, where $i \ne r$, such that

$$\begin{aligned}&\hat{T}^{\perp}_{\sum \varphi_p}(\mathbf{J}(\widehat{Sub}_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), \ldots, \widehat{Sub}_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)))\\ &\supset \bigoplus_{i\ne r} \langle a_i^* \otimes b^*_{q,j} \otimes c^*_{r,k}\rangle, 1 \le j \le L_q, 1 \le k \le L_r. \qquad (5.5.1)\end{aligned}$$

Let

$$\psi_s = a' \otimes (b'_1 \otimes c'_1 + \cdots + b'_{L_s} \otimes c'_{L_s}) \in A' \otimes B' \otimes C', \qquad (5.5.2)$$

be a general point in one of $\widehat{Sub}_{1,L_s,L_s}\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K,\ 1 \leq s \leq R$, where $A' = \langle a' \rangle$, $B' = \langle b'_1, \ldots, b'_{L_s} \rangle$ and $C' = \langle c'_1, \ldots, c'_{L_s} \rangle$, and note that

$$\begin{aligned} &\hat{T}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K) \\ &= (A' \otimes B \otimes C') + (A' \otimes B' \otimes C) + (A \otimes \psi(A'^*)). \end{aligned} \tag{5.5.3}$$

If

$$[\varphi_1 + \cdots + \varphi_R] = [\psi_1 + \cdots + \psi_R], \tag{5.5.4}$$

we have

$$\begin{aligned} &\hat{T}_{\varphi_1}(\widehat{Sub}_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) + \cdots + \hat{T}_{\varphi_R}(\widehat{Sub}_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \\ &\supset \hat{T}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)), \end{aligned} \tag{5.5.5}$$

and according to Remark 2.2, it is equivalent to that

$$\bigcap_{1 \leq p \leq R} \widehat{T}^{\perp}_{\varphi_p}(\widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \subset \hat{T}^{\perp}_{\psi_s}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)).$$

Hence we need to prove that these inclusions imply that $\psi_s \in \{\varphi_1, \cdots, \varphi_R\}$.

Express a' and c' in one of the $\{c'_i : 1 \leq i \leq L_s\}$ occurring in (5.5.2) as

$$\begin{aligned} a' &= x_1 a_1 + \cdots + x_R a_R, \\ c' &= \sum_{1 \leq h \leq L_1} z_{1,h} c_{1,h} + \cdots + \sum_{1 \leq h \leq L_R} z_{R,h} c_{R,h}. \end{aligned}$$

Since $J > 2L_R,\ \binom{J}{L_R} \geq R$, without loss of generality, we first treat the case when both x_1, x_2 are nonzero.

A general hyperplane in (5.5.1) is a linear combination of $a_i^* \otimes b_{q,j}^* \otimes c_{r,k}^*$, and for $b_{q,j}^* \notin B_i^* \oplus B_r^*$, we have

$$H = a_i^* \otimes b_{q,j}^* \otimes c_{r,k}^* \in \hat{T}_{\psi_s}^{\perp}(\widehat{Sub}_{1,L_s,L_s}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)),$$

only if

$$\begin{aligned}
& a_i^* \otimes b_{p,j}^* \otimes c_{r,k}^* \vdash \\
& (x_1 a_1 + \cdots + x_R a_R) \otimes b_{p,j} \otimes (\sum_{1 \le h \le L_1} z_{1,h} c_{1,h} + \cdots + \sum_{1 \le h \le L_R} z_{R,h} c_{R,h}) \\
& = x_i z_{t,h} = 0,\ t \neq i,\ \text{for } i = 1, 2,
\end{aligned}$$

where

$$\begin{aligned}
& (x_1 a_1 + \cdots + x_R a_R) \otimes b_{p,j} \otimes (\sum_{1 \le h \le L_1} z_{1,h} c_{1,h} + \cdots + \sum_{1 \le h \le L_R} z_{R,h} c_{R,h}) \\
& \in A' \otimes B \otimes C'.
\end{aligned}$$

Therefore $z_{t,h} = 0$, when $t \neq 1, 2$. Then $c_1' = \cdots = c_{L_s}' = 0$, $b_1' = \cdots = b_{L_s}' = 0$, hence we have $\psi_s = 0$ in this case.

Next without loss of generality, we treat the case $a' = a_1$. Taking the hyperplane $H = a_1^* \otimes b_{q,j}^* \otimes c_{r,k}^*$, we obtain by similar computation that $z_{t,h} = 0$, when $t \neq 1$. So $c' = \sum_{1 \le h \le L_1} z_{1,h} c_{1,h} \in C_1$. For the same reason, we have $b' \in B_1$, for every $b' \in \{b_1', \ldots, b_{L_s}'\}$ and consequently $\psi_s \in A_1 \otimes B_1 \otimes C_1$.

Now (5.5.4) is valid only if

$$[\psi_s(A_1^*)] = [\varphi_1(A_1^*)]$$

and then $[\psi_s] = [\varphi_1]$. Generally when $a' = a_q$, we have $[\psi_s(A_q^*)] = [\varphi_q(A_q^*)]$, which implies $[\psi_s] = [\varphi_q]$. Thus we obtain Case E.

6. A CRITERION OF UNIQUENESS

We will give a new proof of a criterion of uniqueness for block term tensor decomposition, due to De Lathauwer [14]. Note that the uniqueness condition in our Corollary 2.2.16 concerns a larger class of decompositions.

Theorem 6.0.1. *([14], Theorem 2.3) Assume $I \geq R$, and let X_{j_s} denote matrices of rank L_{j_s}, then* (1.1.1) *is essentially unique if and only if for any $X_{j_1}, \cdots, X_{j_s}$, we have*

$$\langle X_{j_1}, \dots, X_{j_s} \rangle \cap \sigma_{L_{j_t}}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1}) \subset \{X_{j_1}, \dots, X_{j_s}\}, \ 1 \leq t \leq s. \tag{6.0.1}$$

Proof. Assume the contrary that $Y = \sum_{r=1}^{R} a'_r \otimes \tilde{X}_r$ is different from (1.1.1). Since $a_1, \dots, a_R$ are independent, we have $a'_r = \sum_{j=1}^{R} \alpha_j^r a_j$, where α_j^r are not all zero. Hence

$$Y = \sum_{r=1}^{R} a_r \otimes X_r = \sum_{r=1}^{R} a_r \otimes (\sum_{j=1}^{R} \alpha_j^r \tilde{X}_j).$$

Thus $X_r = \sum_{j=1}^{R} \alpha_j^r \tilde{X}_j$. Taking the inverse of $[\alpha_j^r]$, we have $\tilde{X}_r = \sum_{j=1}^{R} \tilde{\alpha}_j^r X_j$. Consequently, there exists $r, j_1, j_2 \in \{1, \dots, R\}$ such that $j_1 \neq j_1$ and $\tilde{\alpha}_{j_1}^r \cdot \tilde{\alpha}_{j_2}^r \neq 0$. Then we obtain

$$\tilde{X}_r \in \langle X_{j_1}, \dots, X_{j_s} \rangle \cap \sigma_{L_r}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1}).$$

But $\tilde{X}_r$ is not belong to $\{X_{j_1}, \dots, X_{j_s}\}$, which contradicts to (6.0.1). $\square$

7. SUMMARY OF THIS DISSERTATION

Blind signal separation [1, 7, 8, 17] is a key problem in signal processing. In order to make the factorization of the data matrix essentially unique, one needs to impose constraints on the mixing matrix and the sources (see Chapter 4 of [24]).

CP decomposition (see Section 3 of [18]) has been used in signal processing, but there are severe limitations to its use. A problem for CP decomposition is when the different signals transmitted from a user interfere (see Chapter 6.2 of [27]) with each other, so the result needs to be deconvolved (see Chapter 2 of [25]).

In this dissertation, we further study a tensor decomposition that De Lathauwer [9, 10, 13, 14] have recently introduced, namely the decomposition of a three way tensor in multilinear rank $(1,\ L_1,\ L_1), \ldots, (1,\ L_R,\ L_R)$ terms, and a new technique for blind signal separation based on this decomposition. The technique involves computing the decomposition of the data tensor in multilinear rank $(1,\ L_1,\ L_1), \ldots, (1,\ L_R,\ L_R)$ terms. Based on algebraic geometric method [4], we deduce several conditions for generic uniqueness of tensor decompositions of multilinear rank $(1,\ L_1,\ L_1), \ldots, (1,\ L_R,\ L_R)$ terms. So more flexibility in applicable situations occurs using this decomposition (referring to [12, 9, 10, 13, 14]).

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