WACHSPRESS VARIETIES

A Dissertation

by

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ABSTRACT

Barycentric coordinates are functions on a polygon, one for each vertex, whose values are coefficients that provide an expression of a point of the polygon as a convex combination of the vertices. Wachspress barycentric coordinates are barycentric coordinates that are defined by rational functions of minimal degree. We study the rational map on $\mathbb{P}^2$ defined by Wachspress barycentric coordinates, the *Wachspress map*, and we describe polynomials that set-theoretically cut out the closure of the image, the *Wachspress variety.* The map has base points at the intersection points of non-adjacent edges.

The Wachspress map embeds the polygon into projective space of dimension one less than the number of vertices. Adjacent edges are mapped to lines meeting at the image of the vertex common to both edges, and base points are blown-up into lines. The deformed image of the polygon is such that its non-adjacent edges no longer intersect but both meet the exceptional line over the blown-up corresponding base point.

We find an ideal that cuts out the Wachspress variety set-theoretically. The ideal is generated by quadratics and cubics with simple expressions along with other polynomials of higher degree. The quadratic generators are scalar products of vectors of linear forms and the cubics are determinants of 3×3 matrices of linear forms. Finally, we conjecture that the higher degree generators are not needed, thus the ideal is generated in degrees two and three.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

1.1 Introduction

Introduced by Möbius in 1827 [13], barycentric coordinates for triangles are classical, but their generalization to arbitrary polygons is an area of current research. Barycentric coordinates are functions on a polygon, one for each vertex, that express a point in the polygon as a convex combination of the vertices. In other words, if $v_1, \ldots, v_N$ are the vertices of an N-gon, a polygon with N sides, then barycentric coordinates are functions $\beta_1, \ldots, \beta_N$ on the N-gon such that

$$p = \sum_{i=1}^{N} \beta_i(p) v_i \quad \text{and} \quad \sum_{i+1}^{N} \beta_i(p) = 1,$$

with the coefficients $\beta_i(p)$ non-negative for each point p in the N-gon.

There is only one way to define barycentric coordinates for triangles; in this case each coordinate function β_i is a linear polynomial. However, they are not unique for larger polygons. For a square, there is one type of barycentric coordinates where each β_i is a piecewise linear function and another type where each β_i is a quadratic polynomial, see Example (1.8). Rarely can barycentric coordinates for non-triangles be expressed as functions as simple as polynomials. The next best that one could hope for are rational expressions. Rational barycentric coordinates exist for any convex polygon and they are our focus in this work.

While convenient to work with mathematically, these rational coordinates are often not the best choice for applications. Barycentric coordinates are used in geometric modeling to deform shapes. Rational coordinates sometimes produce deformations that are too crude; others with more complicated expressions often produce better quality deformations. For many applications there is current research investigating which barycentric coordinates have the best properties for those applications [4, 5]. Deforming planar shapes can be accomplished by placing the shape inside of a polygon, then the vertices are moved and barycentric coordinates are used to extend the motion to the entire shape [17]. This technique is employed frequently by computer animation studios [10]. Barycentric coordinates are fundamental in the construction of *multisided surface patches*, which are used in computer aided geometric design [12]. A patch is a deformed image of a planar polygon that has been transformed in a controlled way to have desired smoothness properties and to satisfy certain boundary conditions. Much of this theory was developed for and pioneered by the automobile industry [12].

In 1975 Eugene Wachspress introduced rational barycentric coordinates for polygons in his work on finite elements [14]. In 1996 Joe Warren showed that Wachspress's coordinates are in some sense the simplest possible. He showed they are the unique rational barycentric coordinates of minimal degree [15, 16].

Wachspress coordinates define a rational map β, the *Wachspress map*, on the projective plane $\mathbb{P}^2$ whose value at a point p of the polygon is the N-tuple of barycentric coordinates of p considered as a point in $\mathbb{P}^{N-1}$. This map has $N(N-3)/2$ base points occurring at the polygon's *external vertices*, points where non-adjacent edges meet. It is a birational isomorphism onto its image whose inverse τ is projection from a codimension three plane $\mathcal{C}$, called the *center of projection*. The Zariski closure of β, the *Wachspress variety* $\mathcal{W}$, is our main object of study. It is a surface in $\mathbb{P}^{N-1}$ and our goal is to describe an ideal that cuts out $\mathcal{W}$ set-theoretically.

In Chapter 2, we use linear algebra to compute the ideal of $\mathcal{W}$ in degree two. These quadratics cut out the union $\mathcal{W} \cup \mathcal{C}$. In Chapter 3, cubics in the ideal are described that cut out the Wachspress variety in $\mathcal{C}$. Finally, we show that the quadratics from Chapter 2, the cubics from Chapter 3, and some additional higher dimensional polynomials cut out the Wachspress variety set-theoretically, and later conjecture that the higher degree polynomials are not needed so the Wachspress variety is cut out in degrees two and three. The Wachspress quadrics and cubics admit an elegant description. Each quadratic is expressed as a scalar product of two vectors of linear forms while each cubic is the determinant of a 3×3 matrix of linear forms. Later in Chapter 3 we describe syzygies among the Wachspress quadratics and cubics and work out Betti diagrams for the Wachspress variety in some special cases. Lastly, we look at examples of Wachspress varieties and the ideal generated by the Wachspress quadratics and cubics for some small values of N.

For a quadrilateral, $\mathcal{W}$ is a quadric hypersurface in $\mathbb{P}^3$ and the center of projection is a point on $\mathcal{W}$. For the pentagon, $\mathcal{W}$ is the intersection of two quadric hypersurfaces and $\mathcal{C}$ is a line. The Wachspress variety was examined in [7] for a particular hexagon. There it was observed that in this case $\mathcal{W}$ is cut out by three quadratics and one cubic in $\mathbb{P}^5$. The variety defined by the quadratics is $\mathcal{W} \cup \mathcal{C}$, and, in $\mathcal{C}$, the cubic cuts out the intersection curve $\mathcal{W} \cap \mathcal{C}$. Also, it was noted that the intersection curve $\mathcal{W} \cap \mathcal{C}$ is a reducible cubic in $\mathcal{C}$. We will see that the this curve is reducible because of the symmetries of this hexagon.

The Wachspress variety has interesting geometry. The image of vertex v_i under β is $\hat{v}_i := [0 : \cdots : 1 : \cdots : 0]$ where the 1 is in the ith position, and the image of the edge through v_i and v_{i+1} is the line through $\hat{v}_i$ and $\hat{v}_{i+1}$ in $\mathcal{W} \subseteq \mathbb{P}^{N-1}$. The Wachpress map deforms the N-gon in such a way that non-adjacent edges no longer meet, although they do meet a common line which is the exceptional divisor over the corresponding external vertex.

In $\mathbb{P}^2$ there is a unique degree $N-3$ curve, the *adjoint curve* $\mathcal{A}$, passing through the $N(N-3)/2$ external vertices. The Wachspress map takes the adjoint curve to the curve obtained by intersecting $\mathcal{W}$ with $\mathcal{C}$. The proper transform of the adjoint curve under β is $\mathcal{W} \cap \mathcal{C}$. The τ-fiber over a generic point of $\mathcal{A}$ is a point on $\mathcal{C}$ while over an external vertex it is a line passing through $\mathcal{C}$.

1.2 Background

We quickly review some fundamental ideas from algebraic geometry. The presentation here is based on Harris [8]. Complex affine n-space, $\mathbb{A}^n$, is the set of n-tuples of complex numbers. An affine variety in $\mathbb{A}^n$ is the zero set of a collection of polynomials in n variables. Complex projective n-space $\mathbb{P}^n$ is the space of lines in $\mathbb{A}^{n+1}$ through the origin. More precisely, it is $\mathbb{A}^{n+1} \setminus \{0\}$ with points x and y identified if there is a nonzero complex number λ with $x = \lambda y$. A point in $\mathbb{P}^n$ will be denoted by $[x_0 : \cdots : x_n]$. A polynomial is *homogeneous* if all its terms have the same degree. A projective variety in $\mathbb{P}^n$ is the zero set of a collection of homogeneous polynomials in $n + 1$ variables. A variety is *irreducible* if it can not be written as the union of two of its proper subvarieties. Every variety has a unique decomposition into irreducible subvarieties. The subvarieties in this decomposition are the *components* of the variety.

Let $R = \mathbb{C}[x_0, \ldots, x_n]$ be the ring of polynomials in $n+1$ variables with coefficients in $\mathbb{C}$. This ring is graded; it is the direct sum

$$R = \bigoplus_{d=0}^{\infty} R_d,$$

where R_d is the vector space of homogeneous polynomials in R of degree d along with the zero polynomial. An ideal in R is *homogeneous* if it can be generated by homogeneous polynomials. For an ideal $I \subseteq R$ we set $I_k := I \cap R_k$, the degree k piece of I. We will assume that all ideals are homogeneous.

A homogeneous ideal I defines a projective variety $\mathbb{V}(I)$, the common zero locus of all polynomials in I. Given a variety $X \subseteq \mathbb{P}^n$, the ideal of *all* polynomials vanishing on X is denoted $I(X)$. If $I \subseteq R$ is an ideal and $X = \mathbb{V}(I)$ we say that I *cuts out* X. An ideal I may cut out a variety X but not contain *all* polynomials vanishing on X. Primarily because of this, we consider three different refinements of the notion of an ideal cutting out a variety. If $I = I(X)$ we say I cuts out X *ideal-theoretically.* The *saturation* of I is the ideal $\mathrm{sat}(I) = \{f \in R \mid (x_0, \dots, x_n)^k I \subseteq I \text{ for some } k\}$. It arises because the ideal $(x_0, \dots, x_n)$ defines the empty set in $\mathbb{P}^n$. Observe that we have $I \subseteq \mathrm{sat}(I) \subseteq I(\mathbb{V}(I))$. If $\mathrm{sat}(I) = I(X)$, then we say I cuts out X *scheme-theoretically.* Finally, if $\mathbb{V}(I) = X$ we say I cuts out X *set-theoretically.* Among these, we have the relations:

$$\text{ideal-theoretic} \Rightarrow \text{scheme-theoretic} \Rightarrow \text{set-theoretic.}$$

We illustrate this distinction:

Example 1.1. Set $X = \{[0 : 0 : 1]\}$. This is a variety in $\mathbb{P}^2$, for example, it is cut out ideal-, scheme-, and set-theoretically by its ideal $I(X) = (x_0, x_1)$. The ideal $(x_0, x_1, x_2) \cdot I(X) = (x_0^2, x_1^2, x_0x_1, x_0x_2, x_1x_2)$ cuts out X set- and scheme-theoretically but not ideal-theoretically. The ideal (x_0^2, x_1^2) only cuts out X set-theoretically.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties. A collection of n-variate polynomials $\phi_0, \dots, \phi_m$ of the same degree such that $\mathbb{V}(\phi_0, \dots, \phi_m) \cap X = \emptyset$ defines a morphism $\phi : X \to Y$ given by $x \longmapsto [\phi_0(x) : \cdots : \phi_n(x)]$. Two such morphisms are equivalent if their values agree on an open subset of X. If $X \nsubseteq \mathbb{V}(\phi_0, \dots, \phi_m)$ the map ϕ defines a *rational map* $\phi : X \dashrightarrow Y$. Morphisms are special cases of rational maps. A rational map does not necessarily have a value at all points of X. Two rational maps are equivalent if they agree on an open subset of X.

For a representation of the rational map ϕ the image $\phi(X)$ may not be a projective variety; however, it is contained in its *Zariski closure*, the smallest projective variety containing $\phi(X)$ or, equivalently, the intersection of all projective varieties containing $\phi(X)$.

A birational isomorphism is a rational map $\phi : X \dashrightarrow Y$ such that there exists another rational map $\psi : Y \dashrightarrow X$ with both compositions $\phi \circ \psi$ and $\psi \circ \phi$ equivalent to the identity map on X and Y, respectively.

The product $X \times Y$ is a projective variety whose subvarieties are zero sets of collections of bihomogeneous polynomials. The graph of a rational map $\phi : X \dashrightarrow Y$ denoted Γ_ϕ is the closure in $X \times Y$ of the subset

$$\{(x, y) \mid y = \phi(x) \text{ on some open set where the map is defined}\}.$$

If ϕ is defined by polynomials $\phi_0, \ldots, \phi_m$ and the saturation of the ideal generated by these polynomials is the ideal of the variety $Z := \mathbb{V}(\phi_0, \ldots, \phi_m)$, then Γ_ϕ is called the *blowup of* X *along* Z. Let $\pi : \Gamma_\phi \to X$ be projection onto the first coordinate. The fiber $\pi^{-1}(Z)$ is the *exceptional divisor* of the blowup.

It follows from Hironaka's Theorem [9] that any rational map $\phi : X \dashrightarrow \mathbb{P}^m$ may be *resolved* by a sequence of blowups. More precisely:

Theorem 1.2. *Given a rational map $\phi : X \dashrightarrow \mathbb{P}^m$ there is a sequence of varieties $X = X_1, \ldots, X_k$, subvarieties $Z_i \subseteq X_i$ with X_{i+1} a blowup of X_i along Z_i, and projection maps $\pi_i : X_{i+1} \to X_i$ such that the composition $\pi_k \circ \pi_{k-1} \circ \cdots \pi_1 \circ \phi : X_k \to \mathbb{P}^m$ is a morphism.*

Example 1.3. Consider the rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by $[x_0 : x_1 : x_2] \longmapsto [x_0, x_1]$. Since the ideal (x_0, x_1) is the ideal of the single point $\{[0 : 0 : 1]\} \in \mathbb{P}^2$, the graph of ϕ is the blowup of $\mathbb{P}^2$ along the variety $\mathbb{V}(x_0, x_1) = \{[0 : 0 : 1]\}$. The map ϕ is undefined at $[0 : 0 : 1]$, but according to Theorem 1.2 we can resolve ϕ. Let

$\Gamma_\phi = \mathbb{V}(x_0y_1 - x_1y_0) \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ be the graph of ϕ and $\pi_2 : \mathbb{P}^2 \to \mathbb{P}^1$ the projection onto the second coordinate. We claim π_2 resolves ϕ. Let $\pi_1 : \Gamma_\phi \longmapsto \mathbb{P}^2$ be the projection to the first coordinate, the *blowup map*. Let $P := ([x_0 : x_1 : x_2], [y_0 : y_1]) \in \Gamma_\phi$. Then $\phi \circ \pi_1(P) = \phi([x_0 : x_1 : x_2]) = [x_0 : x_1]$ and $\pi_2(P) = [y_0 : y_1]$. Note that these two values to agree by the defining equation of the graph.

$$\begin{tikzcd}
\Gamma_\phi \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathbb{P}^2 \arrow[d, dashed, "\phi"] \\
& \mathbb{P}^1
\end{tikzcd}$$

Theorem 1.2 simplifies greatly if X is a surface, which is the case for the applications of this theorem used in this work, in fact X will be $\mathbb{P}^2$, the subvarieties Z_i being blown up are just points, and the exceptional divisors are copies of $\mathbb{P}^1$.

1.3 Barycentric Coordinates

Let Δ be an N-gon with vertices $v_1, \ldots, v_N$ and indices taken modulo N so that, for example, $v_{N+1} = v_1$.

Definition 1.4. Functions $\{\beta_i : \Delta \to \mathbb{R} \mid 1 \leq i \leq N\}$ are *barycentric coordinates* if

1. $\beta_i(p) \geq 0$ 2. $p = \sum_{i=1}^{N} \beta_i(p) v_i$ 3. $\sum_{i=1}^{N} \beta_i(p) = 1$

for all $p \in \Delta$.

Example 1.5. (Line Segment)

Let Δ be the line segment between two vertices v_1 and v_2 in $\mathbb{R}^2$. The functions

$$\beta_1(p) = \tfrac{d(v_2,p)}{d(v_1,v_2)} \qquad \beta_2(p) = \tfrac{d(p,v_1)}{d(v_1,v_2)},$$

where d is the Euclidean distance function, are barycentric coordinates.

Example 1.6. (Triangle)

Barycentric coordinates of triangles can be described in terms of areas of subtriangles shown in Figure 1.1. The barycentric coordinate for the i-th vertex is $\beta_i = A_i/(A_0 + A_1 + A_2)$ where A_i is the area indicated in the figure.

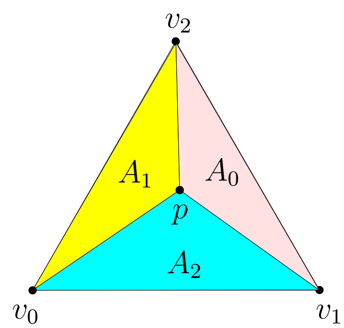

Figure 1.1: Barycentric coordinates for a triangle

Example 1.7 (Simplex)**.** There is a similar description of the barycentric coordinates of simplices by splitting up into subsimplices.

Barycentric coordinates for simplices are unique; i.e., there is only one collection of functions $\beta_1, \ldots, \beta_N$ satisfying Definition 1.4. To see this, fix a point p of an $(N-1)$-simplex in $\mathbb{R}^{N-1}$. The unknown coefficients $\beta_1(p), \ldots, \beta_N(p)$ have N independent conditions imposed on them by (2) and (3), hence are uniquely determined. However, barycentric coordinates for general polygons are not unique, many different types have been studied [4, 11, 15]. To illustrate non-uniqueness, we next describe two different ways to define barycentric coordinates for a square.

Example 1.8. (Square)

We describe two sets of barycentric coordinates for the square in Figure 1.2. Here is a piecewise linear set:

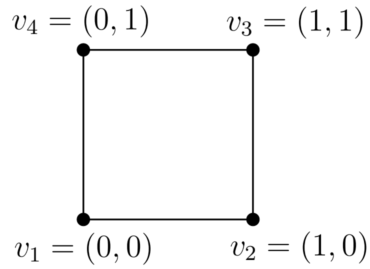

Figure 1.2: A square

$$\beta_1 = \begin{cases} 1-x, & \text{if } x \leq y, \\ 1-y, & \text{if } x > y. \end{cases} \qquad \beta_2 = \begin{cases} x-y, & \text{if } x \leq y, \\ 0, & \text{if } x > y. \end{cases}$$

$$\beta_3 = \begin{cases} y, & \text{if } x \leq y, \\ x, & \text{if } x > y. \end{cases} \qquad \beta_4 = \begin{cases} 0, & \text{if } x \leq y, \\ y-x, & \text{if } x > y. \end{cases}$$

and here is a set given by quadratic polynomials:

$$\beta_1 = (1-x)(1-y) \qquad \beta_2 = y(1-x)$$

$$\beta_3 = xy \qquad \beta_4 = x(1-y).$$

$\square$

The second set of barycentric coordinates in Example 1.8 are the *Wachspress barycentric coordinates* for the square. Eugene Wachspress developed Wachspress coordinates for general polygons in 1975 [14] for an application to approximation theory. In the case of the square they are a bidegree (1,1) tensor product Bézier patch [3]. Warren generalized them to higher dimensional polytopes in 1996 [15]. These coordinates are algebraic; each coordinate β_i is a rational function. They are the unique *rational* functions satisfying (1)-(3) in Definition 1.4 of minimal degree [16].

Wachspress coordinates admit a geometric description similar to that of barycentric coordinates for the triangle in terms of areas of subtriangles. Let $A(a, b, c)$ denote the area of the triangle with vertices a, b, and c. For $1 \leq j \leq N$ set $\alpha_j := A(v_{j-1}, v_j, v_{j+1})$ and $A_j := A(p, v_j, v_{j+1})$.

Definition 1.9. For $1 \leq i \leq N$, the functions

$$\beta_i = \frac{b_i}{\sum_{j=1}^{N} b_j},$$

where $b_i = \alpha_i \prod_{j \neq i-1, i} A_j$ are *Wachspress barycentric coordinates* for the N-gon Δ, see Figure 1.3.

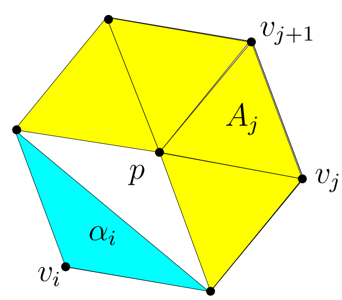

Figure 1.3: Wachspress coordinates for a polygon

Remark 1.10. To simplify our expressions and take advantage of multilinear algebra we identify each vertex v_i with its lift $(1, v_i)$ at height 1 in three dimensional space. The vertices now span a cone through the origin with edge $[v_i, v_{i+1}]$ spanning a facet with normal vector $n_i := v_i \times v_{i+1}$. We redefine α_j and A_j to be the determinants $\det(v_{j-1}, v_j, v_{j+1})$ and $\det(v_j, v_{j+1}, p)$, respectively, which agrees with the previous definitions and allows us to define Wachspress coordinates for non-convex polygons. The non-negativity property of barycentric coordinates fails when Δ is non-convex, but this is not a problem for us. Each A_j is a linear polynomial in p; in particular, set $p := (1, x, y)$, then $\ell_j := A_j = n_j \cdot p = \det(v_j, v_{j+1}, p)$. The linear polynomial ℓ_j cuts out the line supporting the edge $[v_j, v_{j+1}]$. We see that the numerator b_i of each Wachspress coordinate is the product of $N-2$ linear forms. With this algebraic definition we can allow complex vertices. It is important to note that all algebraic results we describe in this work hold in this generality, but most applications will take the convex $\mathbb{R}^2$ case. Our results however do require the condition that no three edges are concurrent, which is equivalent to the condition that $|n_i\ n_j\ n_k| := \det(n_i, n_j, n_k) \neq 0$ for all distinct indices $1 \leq i, j, k \leq N$.

Definition 1.11. The *dual cone* to Δ is the cone spanned by the normals $n_1, \ldots, n_N$ and is denoted Δ^*.

1.4 Wachspress Varieties

We homogenize the numerators of Wachspress coordinates with a new variable z and let $\mathbb{P}^\Delta$ be the projective space with coordinates indexed by the vertices of the polygon Δ.

Definition 1.12. The *Wachspress map* is the rational map $\beta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^\Delta$ sending $[z, x, y]$ to $[b_1(z, x, y), \ldots, b_N(z, x, y)]$.

This map assigns to each point of Δ its Wachspress coordinates. The *Zariski closure* of a subset S of $\mathbb{P}^\Delta$ is the smallest projective variety containing S.

Definition 1.13. The *Wachspress variety* $\mathcal{W}$ is the Zariski closure of the image of the map β.

Lemma 1.14. *The map β has $N(N-3)/2$ base points occurring at the external vertices of Δ, see Figure 1.4.*

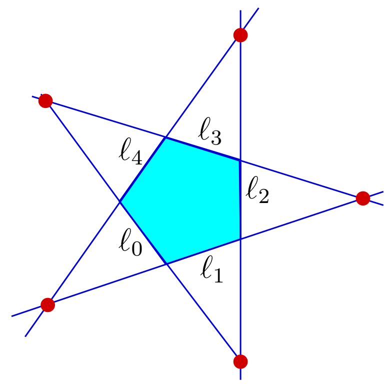

Figure 1.4: The base points of β

Proof. By definition of of Wachspress coordinates, the external vertices are base-points as every Wachspress coordinate vanishes at them. Now assume that p is an arbitrary basepoint of β. Since $b_1(p) = 0$ there exists an $i_1 \neq 1, N$ such that $\ell_{i_1} = 0$. We also have $b_{i_1}(p) = 0$, which means that there is some $i_2 \neq i_1 - 1, i_1$ such that $\ell_{i_2}(p) = 0$. Since $i_2 \neq i_1$ we can conclude that for any base point p at least two of the lines $\ell_1, \ldots, \ell_N$ vanish at p. For an arbitrary basepoint p, we know that two lines ℓ_i and ℓ_j vanish at p. Suppose these lines are adjacent; say without loss of generality that $j = i - 1$. Then p lies on the adjacent lines ℓ_{i-1} and ℓ_i. But this means that p is the vertex v_i, and this is not a basepoint since $b_i(v_i) \neq 0$. The non-adjacent edges of Δ are in one-to-one correspondence with the diagonals of the dual cone Δ^*, thus there are $N(N-3)/2$ base points. $\square$

Theorem 1.15. *The degree of the Wachspress variety of an N-gon is*

$$\frac{1}{2}(N^2 - 5N + 8).$$

Proof. We will use the following result known as the *Degree Formula*: If the dimension of the image of a rational map between projective spaces is two, the map is degree 1 and defined by degree d polynomials, then the degree of the image is

$d^2 - \{$the number of base points counted with multiplicity$\}$ [6]. The degree of the Wachspress map is 1, is defined by $(N-2)$ polynomials, and the number of base points is $N(N-3)/2$.

We show that the base points have multiplicity one. Let $p := \ell_i \cap \ell_j$ be a base point. We can choose affine coordinates for $\mathbb{P}^2$ so that neither ℓ_i nor ℓ_j is the line at infinity. Then these lines form a normal crossing and we can choose our coordinates (x, y) such that $\ell_i = x$, $\ell_j = y$, and $p = (0,0)$. In these coordinates we have

$$b_i = \alpha_i \prod_{m \neq i-1, i} \ell_m, \quad \text{and}$$

$$\frac{\partial b_i}{\partial y}(p) = \ell_{i+1}(p) \cdots \ell_{j-1}(p)\ell_{j+1}(p) \cdots \ell_{i-1}(p).$$

If this partial derivative were zero, then p would lie on at least three edges which is only possible if three edges are concurrent and this is not the case by our assumptions on Δ . Finally, we observe that $(N-2)^2 - N(N-3)/2 = \frac{1}{2}(N^2 - 5N + 8)$. $\square$

1.5 The Wachspress Variety as a Blow-up of $\mathbb{P}^2$

By Theorem 1.2 in the case of surfaces there is a blow-up $\mathbb{P}^2$ at a finite number of points that resolves β. A set of points that will accomplish this for us is the set Y of the $N(N-3)/2$ external vertices of Δ, which are the base points of β.

$$\begin{tikzcd}
\mathrm{Bl}_Y(\mathbb{P}^2) \arrow[d, "\pi"] \arrow[dr, "\tilde{\beta}"] & & \\
\mathbb{P}^2 \arrow[r, dashed, "\beta"] \arrow[rr, dashed, bend right, "id_{\mathbb{P}^2}"'] & \mathcal{W} \arrow[r, dashed, "\tau"] & \mathbb{P}^2
\end{tikzcd}$$

Since β is a birational isomorphism onto its image, resolving β by blowing up $\mathbb{P}^2$ along Y yields an isomorphism $\tilde{\beta} : \mathrm{Bl}_Y(\mathbb{P}^2) \to \mathcal{W}$ taking an exceptional line $\pi^{-1}(p)$ over a base point p to the line $\tau^{-1}(p)$ in $\mathcal{W}$.

1.6 Adjoint Polynomials

A polygon Δ defines a polyhedral cone in three space by putting the polygon in a plane at height one and taking all rays that pass through the origin and a point on an edge of Δ. A triangulation of the polygon Δ will give a triangulation of the corresponding cone into simplices. Each triangle in a triangulation of polygon corresponds to a three-simplex S spanned by vertices v_i, v_j, and v_k, whose volume is $a_S := |v_i\ v_j\ v_k| := \det(v_i, v_j, v_k)$.

Definition 1.16. Let C be a cone defined by a polygon, $v(C)$ its vertex set, and $T(C)$ a triangulation of C. The *adjoint* of C is defined by

$$\mathcal{A}(z) := \sum_{S \in T(C)} a_S \prod_{v \in v(C) \setminus v(S)} (v \cdot z).$$

The adjoint is a tri-variate homogeneous polynomial of degree $N - 3$.

Example 1.17. We calculate the adjoint polynomial of a quadrilateral using the triangulation in Figure 1.5. The adjoint in this case is

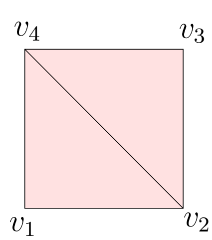

Figure 1.5: A triangulation of a quadrilateral

$$\mathcal{A}(\mathbf{z}) = |v_1\ v_2\ v_4|\ v_3 \cdot \mathbf{z} + |v_2\ v_3\ v_4|\ v_1 \cdot \mathbf{z},$$

where we take $\mathbf{z} = (z_1, z_2, z_3)$ to be coordinates on $\mathbb{P}^2$.

Theorem 1.18. *(Warren [15])*

Adjoints are independent of the triangulation used.

The following will be used in several proofs throughout this work.

Lemma 1.19. *Any vectors $v_1, v_2, v_3,$ and v_4 in three space satisfy*

$$|\, v_2\, v_3\, v_4 \,|\, v_1 - |\, v_1\, v_3\, v_4 \,|\, v_2 + |\, v_1\, v_2\, v_4 \,|\, v_3 - |\, v_1\, v_2\, v_3 \,|\, v_4 = 0.$$

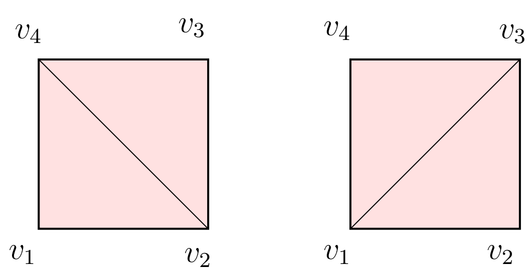

Figure 1.6: Triangulations of quadrilateral

Proof. Theorem 1.18 applied to the adjoints computed using the two triangulations of the quadrilateral in Figure 1.6 yields

$$|\, v_1\, v_2\, v_4 \,|\, v_3 \cdot z + |\, v_2\, v_3\, v_4 \,|\, v_1 \cdot z = |\, v_1\, v_3\, v_4 \,|\, v_2 \cdot z + |\, v_1\, v_2\, v_3 \,|\, v_4 \cdot z,$$

for and $z \in \mathbb{P}^2$. Subtracting and taking out the z factor produces

$$\big(\, |\, v_2\, v_3\, v_4 \,|\, v_1 - |\, v_1\, v_3\, v_4 \,|\, v_2 + |\, v_1\, v_2\, v_4 \,|\, v_3 - |\, v_1\, v_2\, v_3 \,|\, v_4\big) \cdot z = 0.$$

This is zero for all z only if the factor in parentheses is zero, proving the result. It is also not hard to show directly that the result holds if the vectors are not affinely independent. ☐

Theorem 1.20. *(Wachspress [14], Warren [15])*

Wachspress coordinates reduce to linear interpolation on the edges of Δ.

This means that any point, p, on the edge $[v_i, v_{i+1}]$ of Δ is written $p = \beta_i(p)v_i + \beta_{i+1}v_{i+1}$ where $\beta_{i+1}(p) = \frac{|p-v_i|}{|v_{i+1}-v_i|}$ and $\beta_i = \frac{|v_{i+1}-p|}{|v_{i+1}-v_i|}$. Here we take note of an important fact: the Wachspress coordinates of the vertex v_i are $(0, \dots, 0, \beta_i(v_i) = 1, 0, \dots, 0)$.

Lemma 1.21. *There are no linear algebraic relations among Wachspress coordinates.*

Proof. Suppose $\sum_{i=1}^{N} c_i\beta_i = 0$ for some $c_i \in \mathbb{C}$. For each i all Wachpress coordinates vanish at v_i except β_i. The dependence relation reduces to $c_i\beta(v_i) = 0$ at v_i. Since $\beta_i(v_i) \neq 0$, we have $c_i = 0$. Therefore there are no linear relations among Wachspress coordinates. ☐

1.7 Image of Adjoint Curve Contained in Center

We conclude this Chapter by looking at what happens to the adjoint curve in $\mathbb{P}^2$ under the Wachspress map β. In Warren's work it is noted that the denominator of each Wachspress coordinate is the adjoint of the dual polygon $\mathcal{A}(\Delta^*)$. This follows since both $\sum_{i=1}^{N} b_i$ and $\mathcal{A}(\Delta^*)$ are degree $N-3$ polynomials interpolating the $N(N-3)/2$ base points of β. From this observation we can show the following.

Lemma 1.22. *The image of adjoint curve under the Wachspress mapping β is contained in center of projection.*

Proof. The adjoint curve A is the curve in $\mathbb{P}^2$ cut out by $\sum_{S\in T(\Delta^*)} a_S \prod_{n_j\notin S} \ell_j$ for any triangulation $T(\Delta^*)$ of Δ^*. For any point $z := [z_0 : z_1 : z_2] \in \mathbb{P}^2$ we have

$$\sum_{i=1}^{N} \frac{b_i(z)v_i}{\mathcal{A}(\Delta^*)(z)} = z$$

where defined by definition of barycentric coordinates. Thus the equation

$$\sum_{i=1}^{N} b_i(z)v_i = \begin{pmatrix} z\ \mathcal{A}(\Delta^*)(z) \\ x\ \mathcal{A}(\Delta^*)(z) \\ y\ \mathcal{A}(\Delta^*)(z) \end{pmatrix}$$

must hold for all z. The right hand side vanishes because z is on the adjoint curve. Thus $0 = \sum_{i=1}^{N} b_i(z)v_i = \pi(\beta(z))$ and hence $\beta(x) \in \mathcal{C}$ as desired. $\square$

Since the center of projection $\mathcal{C}$ has codimension three and the Wachspress variety $\mathcal{W}$ is two-dimensional, we expect that the intersection $\mathcal{C}\cap\mathcal{W}$ is a curve on $\mathcal{W}$. Since the image $\beta(\mathcal{A}(\Delta^*))$ of the adjoint curve is clearly contained $\mathcal{W}$ and by Lemma 1.22 contained in $\mathcal{C}$ we conclude that $\mathcal{C}\cap\mathcal{W} = \beta(\mathcal{A}(\Delta^*))$

2. WACHSPRESS QUADRATICS

2.1 Introduction

We describe the quadratic polynomials that vanish on the Wachspress variety $\mathcal{W}$, the *Wachspress quadratics*, and study the geometry of the variety they cut out. We characterize these quadratics by showing they must vanish on a certain linear space and finding a set of monomials that support them. To understand the geometry of the variety they define, we will describe the variety's irreducible decomposition.

The set of polynomials vanishing on the variety $\mathcal{W}$ is the *Wachspress ideal* I. We construct a generating set for I_2 consisting of polynomials that are each expressed as a scalar product with a fixed vector τ. This vector τ defines a rational map $\mathbb{P}^\Delta \dashrightarrow \mathbb{P}^2$, also denoted by τ, given by

$$\mathbf{x} \longmapsto \sum_{i=1}^{N} x_i v_i,$$

where $\mathbf{x} = [x_1 : \cdots : x_N] \in \mathbb{P}^\Delta$, called the *linear projection*. By Property 2 of the definition of barycentric coordinates, the composition $\tau \circ \beta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is equivalent to the identity map on $\mathbb{P}^2$ hence $\dim(\mathcal{W}) = 2$. Since $v_i \in \mathbb{C}^3$, the vector τ is a triple of linear forms in $\mathbb{C}[\mathbb{P}^\Delta]$. The linear subspace $\mathcal{C}$ of $\mathbb{P}^\Delta$ where the linear projection is undefined is called the *center of projection*. The ideal $I(\mathcal{C})$ of $\mathcal{C}$ is generated by the three linear forms defining τ.

Lastly, we show that $\mathbb{V}(\langle I_2 \rangle) = \mathcal{W} \cup \mathcal{C}$ is an irreducible decomposition. This will be useful in Chapter 3, where polynomials are described that cut out $\mathcal{W}$. These polynomials consist of a generating set for I_2 and cubics that cut out $\mathcal{W}$ in $\mathcal{C}$.

2.2 Diagonal Monomials

Polynomials in I_2 are supported on a special set of quadratic monomials. A

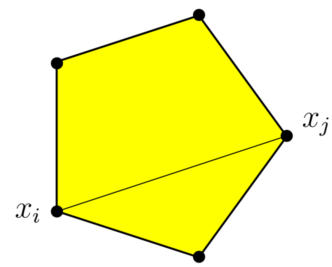

Figure 2.1: A diagonal monomial

diagonal monomial is a monomial $x_i x_j$ in $\mathbb{C}[\mathbb{P}^\Delta]$ such that $j \notin \{i-1, i, i+1\}$. Identifying variable x_i with vertex v_i, a diagonal monomial is a diagonal in Δ, see Figure 2.1.

Lemma 2.1. *Polynomials in I_2 are linear combinations of diagonal monomials.*

Proof. Let Q be a polynomial in I_2. Then $Q(\beta) = Q(b_1, \ldots, b_N) = 0$. On the edge $[v_k, v_{k+1}]$ all the b_i vanish except b_k and b_{k+1}. Thus on this edge, the expression $Q(\beta) = 0$ is

$$c_1 b_k^2 + c_2 b_k b_{k+1} + c_3 b_{k+1}^2 = 0 \tag{2.1}$$

for some constants c_1, c_2, and c_3 in $\mathbb{C}$. Recall that $b_i(v_j) = 0$ if $i \neq j$ and $b_i(v_i) \neq 0$ for each i. Evaluating Equation 2.1 at v_k and v_{k+1}, we conclude $c_1 = c_3 = 0$. At an interior point of edge $[v_k, v_{k+1}]$ neither b_k nor b_{k+1} vanishes. This implies that $c_2 = 0$. A similar calculation on each edge shows that all coefficients of non-diagonal terms in Q are zero. $\square$

2.3 The Map to $I(\mathcal{C})_2$

We define a surjective map onto $I(\mathcal{C})_2$. Computing the dimensions of the image and kernel is central to characterizing Wachspress quadratics. We first use the map to calculate the dimension of the vector space of polynomials in $I(\mathcal{C})_2$ that are supported on diagonal monomials. Later we argue that I_2 has the same dimension, implying that that these vector spaces are equal, which yields the desired characterization of Wachspress quadratics.

Let $\mathbb{C}[\mathbb{P}^\Delta]_1^3$ denote the space of triples of linear forms on $\mathbb{P}^\Delta$. Define the map $\Psi : \mathbb{C}[\mathbb{P}^\Delta]_1^3 \to I(\mathcal{C})_2$ by $F \mapsto F \cdot \tau$, where $\cdot$ is the scalar product.

Lemma 2.2. *The kernel of Ψ is three-dimensional.*

Proof. Let σ_1, σ_2, and σ_3 be independent linear forms generating $I(\mathcal{C})$. Let $\mathbb{C}[\mathcal{C}]_2$ be the degree two piece of the coordinate ring of $\mathcal{C}$. Then we have $\dim(\mathbb{C}[\mathcal{C}]_2) = \binom{N-4+2}{2} = \binom{N-2}{2}$. To see this, note that $\mathcal{C}$ is projectively equivalent to coordinate plane cut out by the ideal (x_{N-2}, x_{N-1}, x_N), so $\mathbb{C}[\mathcal{C}] \cong \mathbb{C}[x_1, \ldots, x_N]/(x_{N-2}, x_{N-1}, x_N) \cong \mathbb{C}[x_1, \ldots, x_{N-3}]$, and $\dim(\mathbb{C}[x_1, \ldots, x_{N-3}]_2) = \binom{N-2}{2}$. Now observe that $\dim(I(\mathcal{C})_2) = \dim(\mathbb{C}[\mathbb{P}^\Delta]_2) - \dim(\mathbb{C}[\mathcal{C}]_2) = \binom{N+1}{2} - \binom{N-2}{2} = 3N - 3$. Since any element of $I(\mathcal{C})_2$ is a combination of the three linear forms defining $I(\mathcal{C})$ with linear form coefficients, Ψ is surjective thus we have $\ker(\Psi) = \dim(\mathbb{C}[\mathbb{P}^\Delta]_1^3) - I(\mathcal{C})_2 = 3N - (3N - 3) = 3$. $\square$

Next, we examine the image of a vector under the map Ψ and describe conditions so that this image is supported on diagonal monomials. Let $\mathcal{D}$ be the vector subspace of $\mathbb{C}[\mathbb{P}^\Delta]_2$ spanned by all diagonal monomials. If $w_i \in \mathbb{C}^3$ for $i = 1, \ldots, N$,

$$F = \sum_{i=1}^{N} x_i w_i$$

is an element of $\mathbb{C}[\mathbb{P}^{\Delta}]_1^3$. The projection τ is the triple:

$$\sum_{i=1}^{N} x_i v_i.$$

Thus,

$$\Psi(F) = F \cdot \tau = (\sum_{i=1}^{N} x_i w_i) \cdot (\sum_{i=1}^{N} x_i v_i) = \sum_{i,j=1}^{N} (w_i \cdot v_j + w_j \cdot v_i) x_i x_j. \tag{2.2}$$

For this image to be in $\mathcal{D}$ the coefficients of non-diagonal monomials must vanish;

$$w_i \cdot v_i = 0 \quad \text{and} \quad w_i \cdot v_{i+1} + w_{i+1} \cdot v_i = 0 \quad \text{for all } i. \tag{2.3}$$

Lemma 2.3. *The dimension of the vector space $\mathcal{D} \cap I(\mathcal{C})_2$ is $N - 3$.*

Proof. We show the conditions in Equation (2.3) give $2N$ independent conditions on the $3N$-dimensional vector space $\mathbb{C}[\mathbb{P}^{\Delta}]_1^3$, and the solution space is $\Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)$, thus $\dim(\Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)) = N$. The conditions are represented by the matrix equation:

$$\begin{pmatrix} v_1 \cdot w_1 \\ \vdots \\ v_N \cdot w_N \\ v_1 \cdot w_2 + v_2 \cdot w_1 \\ \vdots \\ v_N \cdot w_1 + v_1 \cdot w_N \end{pmatrix} = \overbrace{\begin{pmatrix} v_1^T & 0 & \cdots & 0 \\ 0 & v_2^T & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & v_N^T \\ v_2^T & v_1^T & & 0 \\ 0 & & \ddots & \\ v_N^T & & & v_1^T \end{pmatrix}}^{M} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix},$$

where the v_i and w_i are column vectors the superscript T indicates transpose. The matrix M in the middle is a $2N \times 3N$ matrix, and the proof will be complete if the rows are shown to be independent. Denote the rows of M by $R_1, \ldots, R_N, R_{N+1} \ldots, R_{2N}$ and let $\gamma_1 R_1 + \cdots + \gamma_N R_N + \gamma_{N+1} R_{N+1} + \cdots + \gamma_{2N} R_{2N}$ be a dependence relation among the rows. The first three columns give the dependence relation $\gamma_1 v_1 + \gamma_{N+1} v_2 + \gamma_{2N} v_N = 0$. Since v_N, v_1, and v_2 are adjacent vertices of a polyhedral cone, they must be independent, so γ_1, γ_{N+1}, and γ_{2N} must be zero. Looking at the other columns we can similarly show the rest of the γ_i's are zero.

Since the restriction $\Psi : \Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2) \to \mathcal{D} \cap I(\mathcal{C})_2$ remains surjective we deduce $\dim(\mathcal{D} \cap I(\mathcal{C})_2) = \dim(\Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)) - \dim(\ker(\Psi)) = N - 3$. $\square$

2.4 Wachspress Quadratics

We first compute the dimension of the space of Wachspress quadratics I_2. For this we study a surjective map with kernel I_2. Then we give an explicit set of quadratics that span I_2. Lastly, we prove that I_2 consists of the quadratic polynomials that are supported on diagonal monomials and vanish on the center of projection.

Set $\gamma(i) := \{1, \ldots, N\} \setminus \{i-1, i\}$ and $\gamma(i,j) := \gamma(i) \cap \gamma(j)$. The image of a diagonal monomial $x_i x_j$ under the pullback map $\beta^* : \mathbb{C}[\mathbb{P}^\Delta] \to \mathbb{C}[\mathbb{P}^2]$ is

$$b_i b_j = \alpha_i \alpha_j \prod_{k \in \gamma(i)} \ell_k \prod_{m \in \gamma(j)} \ell_m = \alpha_i \alpha_j \prod_{k=1}^{N} \ell_k \prod_{m \in \gamma(i,j)} \ell_m,$$

and each has the common factor $P := \prod_{k=1}^{N} \ell_k$. To find the quadratic relations among Wachspress coordinates it suffices to find linear relations among products $\prod_{m \in \gamma(i,j)} \ell_m \in \mathbb{C}[\mathbb{P}^2]_{N-4}$ for diagonal pairs i, j.

Define the map $\phi : \mathcal{D} \to \mathbb{C}[\mathbb{P}^2]_{N-4}$ by

$$x_i x_j \longmapsto \frac{b_i b_j}{P},$$

and extending by linearity. This is simply β^* restricted to $\mathcal{D}$ and divided by P. By Lemma 2.1 it follows that $I_2 = \ker(\phi) \subseteq \mathcal{D}$.

Example 2.4. (Surjectivity of ϕ for the Hexagon)

For a hexagon we have $\phi : \mathcal{D} \cong \mathbb{C}^9 \to \mathbb{C}^6 \cong \mathbb{C}[\mathbb{P}^2]_2$. We show the image is six-dimensional by exhibiting six diagonal monomials with independent images. We consider the images of diagonal monomials not including x_1. Let $p_{ij} := \ell_i \cap \ell_j$ be the external vertex at the intersection of non-adjacent edges ℓ_i and ℓ_j. The $(i,j)^{th}$ entry in Table 2.1 is the value of the image of the diagonal monomial in column j at the external vertex in row i, a star $*$ represents a nonzero number, and a blank space is zero. The external vertices p_{ij} used are those with $j \neq 6$, and they arranged with indices in lexicographic order down the rows. The lower triangular nature of Table

Table 2.1: Values of images of diagonal monomials at intersection points

	x_2x_4	x_2x_5	x_2x_6	x_3x_5	x_3x_6	x_4x_6
p_{13}	$*$					
p_{14}	$*$	$*$				
p_{15}		$*$	$*$			
p_{24}	$*$	$*$		$*$		
p_{25}		$*$	$*$	$*$	$*$	
p_{35}				$*$	$*$	$*$

2.1 demonstrates the linear independence of the images of the diagonal monomials that appear across the top row. Successively evaluating any dependence relation at the intersection points shows that all coefficients are zero.

The same holds for any polygon.

Lemma 2.5. *The map* $\phi : \mathcal{D} \to \mathbb{C}[\mathbb{P}^2]_{N-4}$ *is surjective with* $\dim(\ker \phi) = N - 3$. *It follows that* $\dim(I_2) = N - 3$.

Proof. There are $N - 3$ diagonal monomials that have x_1 as a factor. We show that the images of the remaining

$$N(N-3)/2 - (N-3) = (N-3)(N-2)/2 = \dim(\mathbb{C}[\mathbb{P}^2]_{N-4})$$

diagonal monomials are independent. Set $p_{ij} := \ell_i \cap \ell_j$. In Table 2.2, a star, $*$, represents a nonzero number, a blank space is zero. The $(i, j)^{th}$ entry in Table 2.2 represents the value of the image of the diagonal monomial in column j at the external vertex in row i. The external vertices not lying on ℓ_N are arranged down the rows with their indices in lexicographic order. The lower triangular nature of Table 2.2

Table 2.2: Values of images of diagonal monomials at external vertices

	x_2x_4	$\cdots$	x_2x_N	x_3x_5	$\cdots$	x_3x_N	$\cdots$	$x_{N-3}x_{N-1}$	$x_{N-3}x_N$	$x_{N-2}x_N$
$p_{1,3}$	$*$									
$\vdots$		$\ddots$								
$p_{1,N-1}$			$*$							
$p_{2,4}$	$*$			$*$						
$\vdots$					$\ddots$					
$p_{2,N-1}$			$*$			$*$				
$\vdots$							$\ddots$			
$p_{(N-4)(N-2)}$								$*$		
$p_{(N-4)(N-1)}$								$*$	$*$	
$p_{(N-3)(N-1)}$								$*$	$*$	$*$

shows the independence of the images. We have found $\dim(\mathbb{C}[\mathbb{P}^2]_{N-4})$ independent images and hence ϕ is surjective. This is a map from a vector space of dimension

$N(N-3)/2$ to one of dimension $(N-2)(N-3)/2$. Since this map is surjective the kernel has dimension $N(N-3)/2-(N-2)(N-3)/2 = N-3$. $\square$

There is a generating set for I_2 where each generator is a scalar product with the vector τ. The other vectors in these scalar products are

$$\Lambda_k = \frac{x_{k+1}}{\alpha_{k+1}} n_{k+1} - \frac{x_k}{\alpha_k} n_{k-1} \in \mathbb{C}[\mathbb{P}^\Delta]^3_1.$$

Lemma 2.6. *The vectors* $\{\Lambda_1 \ldots, \Lambda_N\}$ *form a basis for the space* $\Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)$.

Proof. Suppose that $\sum_{k=1}^N c_k \Lambda_k = 0$ is a linear dependence relation among the Λ_k. The coefficient of a variable x_k is

$$\frac{1}{\alpha_k}(c_{k-1}n_k - c_k n_{k-1}).$$

By the dependence relation this must be zero, which implies that n_{k-1} and n_k are scalar multiples. This is impossible since they are normal vectors of adjacent facets of a polyhedral cone. Hence, $c_{k-1} = c_k = 0$ for all k which shows that the Λ_k are independent.

In the proof of Lemma 2.3 we showed that $\dim(\Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)) = N$ and we have just shown $\dim(\langle \Lambda_k \mid k = 1, \ldots, N\rangle) = N$. We now show $\langle \Lambda_k \mid k = 1, \ldots, N\rangle \subseteq \Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)$, which proves the result since two vector spaces of the same dimension with one contained in the other are equal. The conditions stated in Equation (2.3) are what is required for $\Lambda_k \in \mathbb{C}[\mathbb{P}^\Delta]^3_1$ to lie in $\Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2$. We show these conditions are satisfied for each Λ_k.

Set $w_i := 0$ if $i \neq k, k+1$, $w_i := -n_{k-1}/\alpha_k$ if $i = k$, and $w_i = n_{k+1}\alpha_{k+1}$ for each fixed k. Then

$$\Lambda_k = \frac{x_{k+1}}{\alpha_{k+1}} n_{k+1} - \frac{x_k}{\alpha_k} n_{k-1} = \sum_{i=1}^{N} w_i x_i.$$

If $i \neq k, k+1$, then clearly $w_i \cdot v_i = 0$. Since $n_{k-1} \cdot v_k =$ and $n_{k+1} \cdot v_{k+1} = 0$, we obtain $w_i \cdot v_i = 0$ for each i. We now show that $w_i \cdot v_{i+1} + w_{i+1} \cdot v_i = 0$ holds for $i = k$. We have

$$\begin{aligned} -\frac{n_{k-1}}{\alpha_k} \cdot v_{i+1} + \frac{n_{k+1}}{\alpha_{k+1}} \cdot v_i &= -\frac{v_{k-1} \times v_k \cdot v_{k+1}}{\alpha_k} + \frac{v_{k+1} \times v_{k+2} \cdot v_k}{\alpha_{k+1}} \\ &= -\frac{\det(v_{k-1}, v_k, v_{k+1})}{\alpha_k} + \frac{\det(v_{k+1}, v_{k+2}, v_k)}{\alpha_{k+1}} = 0, \end{aligned}$$

as $\alpha_j = \det(v_{j-1}, v_j, v_{j+1})$. Thus the w_i satisfy the conditions in Equations (2.3) and we conclude $\Lambda_k \in \Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)$. $\square$

Theorem 2.7. ***(Characterization of Wachspress Quadratics)***
The Wachspress quadratics are characterized as the quadratic polynomials in $\mathbb{C}[\mathbb{P}^\Delta]$ that are diagonally supported and vanish on the center of projection. Further, the quadratics $Q_k = \Lambda_k \cdot \tau$ for $k = 1, \ldots, N$ span I_2.

Proof. Let $\mathbf{z}$ be the vector (z, x, y). By definition of Wachspress coordinates,

$$\tau(\beta(\mathbf{z})) = \sum_{i=1}^{N} b_i(\mathbf{z}) \cdot v_i = \mathbf{z} \cdot \sum_{i=1}^{N} b_i(\mathbf{z}).$$

We have

$$\begin{aligned} \Lambda_k(\beta(\mathbf{z})) &= \alpha_{k+1} n_{k-1} b_k - \alpha_k n_{k+1} b_{k+1} \\ &= \alpha_k \alpha_{k+1} (n_{k-1} \prod_{j \neq k-1,k} \ell_j - n_{k+1} \prod_{j \neq k,k+1} \ell_j) \\ &= \alpha_k \alpha_{k+1} \prod_{j \neq k-1,k,k+1} \ell_j \ \ (n_{k-1}\ \ell_{k+1} - n_{k+1}\ \ell_{k-1}) \end{aligned}$$

$$= P\ [n_{k-1}\ (n_{k+1} \cdot \mathbf{z}) - n_{k+1}\ (n_{k-1} \cdot \mathbf{z})],$$

where $P = \alpha_k \alpha_{k+1} \prod_{j \neq k-1,k,k+1} \ell_j$. Set $\overline{P} := P\ \sum_{i=1}^{N} b_i(\mathbf{z})$. Then we have

$$\begin{aligned} Q_k(\beta(\mathbf{z})) &= \tau(\beta(\mathbf{z})) \cdot \Lambda_k(\beta(\mathbf{z})) \\ &= \overline{P}\ \mathbf{z} \cdot [n_{k-1}\ (n_{k+1} \cdot \mathbf{z}) - n_{k+1}\ (n_{k-1} \cdot \mathbf{z})] \\ &= \overline{P}\ [(n_{k-1} \cdot \mathbf{z})(n_{k+1} \cdot \mathbf{z}) - (n_{k+1} \cdot \mathbf{z})(n_{k-1} \cdot \mathbf{z})] = 0. \end{aligned}$$

We have just shown that $Q_k \in I_2$. By Lemma 2.6 we know $\langle \Lambda_k \rangle = \Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)$. Observe that $\langle Q_1, \ldots, Q_N \rangle = \Psi(\langle \Lambda_k \rangle) = \mathcal{D} \cap I(\mathcal{C})_2$. Thus we have $\dim(\langle Q_1 \ldots, Q_N \rangle) = N - 3$ and by Lemma 2.5 $\dim(I_2) = N - 3$. Therefore, since $\langle Q_1 \ldots, Q_N \rangle \subseteq I_2$, we can conclude that $\langle Q_1, \ldots, Q_N \rangle = I_2 = \mathcal{D} \cap I(\mathcal{C})_2$. $\square$

2.5 Irreducible Decomposition of $\mathbb{V}(\langle I_2 \rangle)$

We describe the decomposition of $\mathbb{V}(\langle I_2 \rangle)$ into its irreducible components. First observe that the variety $\mathcal{W}$ is irreducible because it is the closure of the image of an irreducible variety under a rational map. We show that if a point of $\mathbb{V}(\langle I_2 \rangle)$ does not lie in the linear space $\mathcal{C}$, then it lies in $\mathcal{W}$. We begin with some useful lemmas.

Lemma 2.8. *For any i, j, and k we have*

$$|n_i\ n_j\ n_k| = |v_j\ v_k\ v_{k+1}| \cdot |v_i\ v_{i+1}\ v_{j+1}| - |v_{j+1}\ v_k\ v_{k+1}| \cdot |v_i\ v_{i+1}\ v_j|$$

Proof. Apply the vector and scalar triple product formulas $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) -$

$\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ and $|\mathbf{a}\ \mathbf{b}\ \mathbf{c}| = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$,

$$\begin{aligned}
|n_i\ n_j\ n_k| &= n_i \times n_j \cdot n_k = (n_i \times (v_j \times v_{j+1})) \cdot n_k \\
&= [v_j(n_i \cdot v_{j+1}) - v_{j+1}(n_i \cdot v_j)] \cdot n_k \\
&= (v_j \cdot n_k)(n_i \cdot v_{j+1}) - (v_{j+1} \cdot n_k)(n_i \cdot v_j) \\
&= |v_j\ v_k\ v_{k+1}| \cdot |v_i\ v_{i+1}\ v_{j+1}| - |v_{j+1}\ v_k\ v_{k+1}| \cdot |v_i\ v_{i+1}\ v_j|.
\end{aligned}$$

□

Corollary 2.9.

$$|n_i\ n_j\ n_{j+1}| = \alpha_{j+1}|v_i\ v_{i+1}\ v_{j+1}|$$

Proof. This follows from Lemma 2.8 and the definition of α_{j+1}. □

Corollary 2.10.

$$|n_{i-1}\ n_i\ n_{i+1}| = \alpha_i\alpha_{i+1}$$

Proof. This follows from Lemma 2.8 and the definition of α_i and α_{i+1}. □

Lemma 2.11. *Let $x = [x_1 : \cdots : x_N] \in \mathbb{V}(\langle I_2 \rangle) \setminus \mathcal{C}$. If $\tau(x)$ is a base point $p_{ij} = n_i \times n_j$, then x lies on the exceptional line $\hat{p}_{ij}$ over p_{ij}.*

Proof. Since indices are cyclic we assume that $i = 1$. Thus $\tau(x) = p_{1,j} = n_1 \times n_j$ for some $j \notin \{N, 1, 2\}$. The relation $q_1(x) = \Lambda_1 \cdot \tau(x) = \Lambda_1 \cdot (n_1 \times n_j) = 0$ yields

$$L_1 := x_2\ n_2 \cdot p_{1,j} - x_1\ n_N \cdot p_{1,j} = 0. \tag{2.4}$$

The relation $q_j(x) = 0$ implies,

$$L_j := x_{j+1} \mid n_{j+1}\ n_1\ n_j| - x_j \mid n_2\ n_1\ n_j \mid = 0. \tag{2.5}$$

Also,

$$q_2(x) = (x_3 n_3 - x_2 n_1) \cdot n_1 \times n_j = x_3 \mid n_3 \; n_1 \; n_j \mid = 0,$$

implying $x_3 = 0$ since $\mid n_3 \; n_1 \; n_j \mid \neq 0$ if $j \neq 3$. Assume $x_k = 0$ for $3 \leq k < j-1$. Note that

$$q_k(x) = (x_{k+1} n_{k+1} - x_k n_{k-1}) \cdot n_1 \times n_j = x_{k+1} \mid n_{k+1} \; n_1 \; n_j \mid = 0,$$

hence $x_{k+1} = 0$ since $\mid n_{k+1} \; n_1 \; n_j \mid \neq 0$ and by induction $x_k = 0$ for $3 \leq k \leq j-1$. An analogous argument shows that $x_k = 0$ for $j+2 \leq k \leq N$. Hence x lies on the line $\mathbb{V}(L_1, L_j, x_k \mid k \notin \{1, 2, j, j+1\})$, which is the exceptional line $\hat{p}_{1,j}$. □

Theorem 2.12. *The subset $\mathbb{V}(\langle I_2 \rangle) \backslash \mathcal{C}$ is contained in $\mathcal{W}$. It follows that the variety $\mathbb{V}(\langle I_2 \rangle)$ has irreducible decomposition $\mathcal{W} \cup \mathcal{C}$.*

Proof. Let $x = [x_1 : \cdots : x_N] \in \mathbb{V}(\langle I_2 \rangle) \setminus \mathcal{C}$. The Wachspress quadratics give the relations

$$x_{r+1} \; n_{r+1} \cdot \tau = x_r \; n_{i-1} \cdot \tau \tag{2.6}$$

for each $r = 1, \ldots N$. Claim: For each $k \in \{1, \ldots, N\}$ that $b_k(\tau(x)) = \mathcal{A}(\tau(x)) \, x_k$

where the triangulation in Figure 2.2 is used to express the adjoint $\mathcal{A}$. By definition of the Wachspress map β we obtain

$$\beta(\tau(x)) = \mathcal{A}(\tau(x)) \, x. \tag{2.7}$$

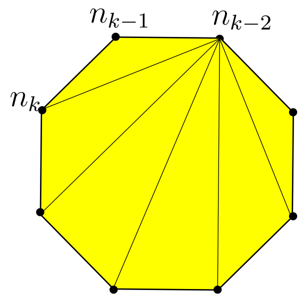

Figure 2.2: Triangulation used for adjoint

Provided $\mathcal{A}(\tau(x)) \neq 0$, the result follows since $\beta(\tau(x)) \in \mathbb{P}^\Delta$ is a nonzero scalar multiple of x, hence x is in the image of the Wachspress map and thus lies on $\mathcal{W}$. If $x \in \mathbb{V}(\langle I_2 \rangle) \setminus \mathcal{C}$ and $\mathcal{A}(\tau(x)) = 0$, then by Equation (2.7) $\beta(\tau(x)) = 0$ and hence $\tau(x)$ is a basepoint of β. Thus $\tau(x) = n_i \times n_j$ for some diagonal pair (i, j). By Lemma 2.11 x lies on an exceptional line and hence lies on $\mathcal{W}$.

We prove the claim. Since all indices are cyclic it suffices to assume $k = 3$. We introduce the notation: $|n_{ijk}| := |n_i \; n_j \; n_k| = \det(n_i, n_j, n_k)$ and

$$n_{i_1,\ldots,i_m} \cdot \tau := \prod_{j=1}^{m} (n_{i_j} \cdot \tau).$$

This is the product of m linear forms in the coordinates of $\mathbb{P}^\Delta$, and with this notation $b_3(\tau) = n_{1,4,5,\ldots,N} \cdot \tau$. For each $r \in \{3, \ldots, N\}$ define

$$\begin{aligned} S_r : \;&=\; (n_{4,\ldots,r} \cdot \tau)\; n_1 \cdot \Bigg[\sum_{i=3}^{r} v_i (n_{r+1,\ldots,N} \cdot \tau)\; x_i \; + \\ &\qquad \sum_{i=r+1}^{N} v_i (n_{r-1,\ldots,i-2} \cdot \tau)\; (n_{i+1,\ldots,N} \cdot \tau)\; x_r \Bigg], \end{aligned}$$

where we set $n_{i,\ldots,j} \cdot \tau = 1$ if $j < i$. We show $x\,\mathcal{A}(\tau(x)) = S_3 = S_N = b_3(\tau(x))$.

We first show that $S_3 = x_3\, \mathcal{A}(\tau)$. Observe that $x_3\, \mathcal{A}(\tau)$ is

$$|n_{123}|\ (n_{4 \cdots N} \cdot \tau)\ x_3 + \sum_{i=4}^{N} |n_{1,i-1,i}|\ (n_{2,\ldots,i-2} \cdot \tau)\ (n_{i+1,\ldots,N} \cdot \tau)\ x_3, \tag{2.8}$$

where we have expressed the adjoint $\mathcal{A}$ using the triangulation in Figure 2.2. Applying the scalar triple product to the determinants $|n_{123}|$ and $|n_{1,i-1,i}|$ in the expression (2.8),

$$n_1 \cdot (n_2 \times n_3)\, (n_{4 \cdots N} \cdot \tau)\, x_3 + \sum_{i=4}^{N} n_1 \cdot (n_{i-1} \times n_i)(n_{2,\ldots,i-2} \cdot \tau)\, (n_{i+1,\ldots,N} \cdot \tau) x_3. \tag{2.9}$$

Factoring an n_1 and noting that $n_i \times n_{i+1} = v_{i+1}$, (2.9) becomes

$$n_1 \cdot \left[v_3 (n_{4 \cdots N} \cdot \tau)\ x_3 + \sum_{i=4}^{N} v_i (n_{2,\ldots,i-2} \cdot \tau)(n_{i+1,\ldots,N} \cdot \tau)\, x_3 \right] = S_3.$$

Now we show $S_N = b_3(\tau)$. Since $n_{N+1,\ldots,N} \cdot \tau = 1$

$$S_N = (n_{4,\ldots,N} \cdot \tau) n_1 \cdot \left(\sum_{i+3}^{N} v_i (n_{N+1,\ldots,N} \cdot \tau) x_i \right) = (n_{4,\ldots,N} \cdot \tau) n_1 \cdot \left(\sum_{i+3}^{N} v_i x_i \right). \tag{2.10}$$

Observing that $n_1 \cdot \sum_{i=1}^{2} x_i v_i = 0$ we see that (2.10) is

$$(n_{4,\ldots,N} \cdot \tau)\ (n_1 \cdot \tau) = n_{1,4,\ldots,N} \cdot \tau = b_3(\tau).$$

We now claim that for $r \in \{3, \ldots, N-1\}$ we have $S_r = S_{r+1}$. Indeed,

$$\begin{aligned} S_r = {} & (n_{4,\ldots,r} \cdot \tau)\, n_1 \cdot \Bigg[\sum_{i=3}^{r} v_i(n_{r+1,\ldots,N} \cdot \tau)\ x_i + \\ & \sum_{i=r+1}^{N} v_i(n_{r,\ldots,i-2} \cdot \tau)\, (n_{i+1,\ldots,N} \cdot \tau)(n_{r-1} \cdot \tau)\ x_r \Bigg] \\ & = (n_{4,\ldots,r} \cdot \tau)\, n_1 \cdot \Bigg[\sum_{i=3}^{r} v_i(n_{r+1,\ldots,N} \cdot \tau)\, x_i + \\ & \sum_{i=r+1}^{N} v_i(n_{r,\ldots,i-2} \cdot \tau)\, (n_{i+1,\ldots,N} \cdot \tau)\, (n_{r+1} \cdot \tau)\ x_{r+1} \Bigg], \end{aligned} \tag{2.11}$$

where we have applied relation (2.6) to the last term. Next we factor out $n_{r+1} \cdot \tau$ to obtain

$$(n_{4,\ldots,r+1} \cdot \tau)\, n_1 \cdot \Bigg[\sum_{i=3}^{r} v_i(n_{r+2,\ldots,N} \cdot \tau)\, x_i + \sum_{i=r+1}^{N} v_i(n_{r,\ldots,i-2} \cdot \tau)\, (n_{i+1,\ldots,N} \cdot \tau)\, x_{r+1} \Bigg]$$

Lastly, since the expressions in both summations agree at the index $i = r+1$ we can shift the indices of summation,

$$(n_{4,\ldots,r+1} \cdot \tau)\, n_1 \cdot \Bigg[\sum_{i=3}^{r+1} v_i(n_{r+2,\ldots,N} \cdot \tau)\ x_i + \sum_{i=r+2}^{N} v_i(n_{r,\ldots,i-2} \cdot \tau)\, (n_{i+1,\ldots,N} \cdot \tau)\, x_{r+1} \Bigg],$$

which is precisely S_{r+1}, proving the claim. The claim shows that $S_3 = S_N$, hence (2.7) holds and we conclude that x lies in $\mathcal{W}$ if $\mathcal{A}(\tau(x)) \neq 0$. $\square$

3. THE WACHSPRESS CUBICS

3.1 Introduction

Theorem 2.12 shows that the Wachspress quadratics do not suffice to cut out the Wachspress variety $\mathcal{W}$. We now construct cubics, the *Wachspress cubics*, that lie in the Wachspress ideal and are not contained in the ideal generated by the Wachspress quadratics I_2. These cubics have elegant expressions as determinants of 3×3 matrices of linear forms. The proof that they lie in the ideal of $\mathcal{W}$ uses the adjoint polynomial $\mathcal{A}$ of the dual polygon Δ^* and that adjoints are independent of the triangulation used to express them, see Theorem 1.18. In the next chapter we show that Wachspress quadratics, cubics, and certain higher degree forms cut out $\mathcal{W}$ set-theoretically (the higher degree forms are conjectured to be not needed). The proof centers on the construction of several rational maps that are equivalent to the linear projection τ on $\mathcal{W}$. In the second half of this chapter, we construct these rational maps.

3.2 Construction of Wachspress Cubics

A cubic monomial $x_ix_jx_k$ in $\mathbb{C}[\mathbb{P}^\Delta]$ is a Δ-*monomial* if every pair of the variables x_i, x_j, x_k forms a diagonal monomial. The triple of indices of the variables in a Δ-monomial is a Δ-*triple*. Identifying variable x_i with vertex v_i, a Δ-monomial is a triangle inscribed in Δ formed by diagonals.

Notation 3.1. The set $\gamma(i)$ is $\{1,\ldots,N\}\setminus\{i-1,i\}$ and we set $\gamma(i,j,k) := \gamma(i)\cap\gamma(j)\cap\gamma(k)$.

Lemma 3.2. *Evaluating a Δ-monomial at Wachspress coordinates yields*

$$x_ix_jx_k(\beta) = b_ib_jb_k = P^2\prod_{m\in\gamma(i,j,k)}\ell_m, \tag{3.1}$$

where P is the product of all the linear forms ℓ_i defining the edges of Δ.

Recall

$$\Lambda_r = \frac{x_{r+1}}{\alpha_{r+1}} n_{r+1} - \frac{x_r}{\alpha_r} n_{r-1}$$

as in Chapter 2, see Lemma 2.6.

Theorem 3.3. *For a Δ-triple i, j, k the polynomial*

$$w_{i,j,k} := \det(\Lambda_i, \Lambda_j, \Lambda_k), \tag{3.2}$$

lies in the Wachspress ideal.

Example 3.4. For $N = 6$ there are two Δ-triples $(1, 3, 5)$ and $(2, 4, 6)$ and hence we obtain the two cubics $w_{1,3,5}$ and $w_{2,4,6}$.

The cubics $w_{i,j,k}$ will be referred to as *Wachspress cubics*. Before taking on the proof of Theorem 3.3 let us first perform a preliminary calculation and make some observations. There are no triangular triples if $N < 6$; hence, there are no Wachspress cubics for such N. Thus when discussing Δ-triples we are implicitly assuming $N \geq 6$.

Remark 3.5. By making a change of variable, replacing x_i with x_i/α_i, we ignore the constants α_i in what follows.

Preliminary Calculation 3.6. Using the definition of the Λ's and the multilinearity of determinant,

$$\begin{aligned} \det(\Lambda_i, \Lambda_j, \Lambda_k) \quad = \quad & |n_{i+1}\ n_{j+1}\ n_{k+1}| x_{i+1}x_{j+1}x_{k+1} - |n_{i+1}\ n_{j+1}\ n_{k-1}| x_{i+1}x_{j+1}x_k - \\ & |n_{i+1}\ n_{j-1}\ n_{k+1}| x_{i+1}x_j x_{k+1} + |n_{i+1}\ n_{j-1}\ n_{k-1}| x_{i+1}x_j x_k - \\ & |n_{i-1}\ n_{j+1}\ n_{k+1}| x_i x_{j+1}x_{k+1} + |n_{i-1}\ n_{j+1}\ n_{k-1}| x_i x_{j+1}x_k + \\ & |n_{i-1}\ n_{j-1}\ n_{k+1}| x_i x_j x_{k+1} - |n_{i-1}\ n_{j-1}\ n_{k-1}| x_i x_j x_k. \end{aligned}$$

We now prove Theorem 3.3.

Proof. The three equations $i+1=j-1$, $j+1=k-1$, and $k+1=i-1$ involving the indices of our Δ-triple yield four cases:

1. All three hold 2. Two hold 3. One holds 4. None hold.

We prove Theorem 3.3 in each of these four cases separately.

Case 1: The Δ-triple (i,j,k) satisfies Case 1 if and only if $N=6$. For $N=6$ there are only two Δ-triples; $(1,3,5)$ and $(2,4,6)$, hence $w_{1,3,5}$ and $w_{2,4,6}$ are the only Wachspress cubics. We show that $w_{1,3,5}$ vanishes on Wachspress coordinates. The case of $w_{2,4,6}$ is similar. All but two of the determinants in Preliminary Calculation 3.6 vanish, leaving

$$w_{1,3,5} = \det(\Lambda_1, \Lambda_3, \Lambda_5) = |n_2\ n_4\ n_6| x_2x_4x_6 - |n_6\ n_2\ n_4| x_1x_3x_5. \tag{3.3}$$

Notice that the coefficients are equal, thus we finish the proof by showing that $x_1x_3x_5 - x_2x_4x_6$ vanishes on Wachspress coordinates. The monomials $x_1x_3x_5$ and $x_2x_4x_6$ evaluated at Wachspress coordinates are $b_1b_3b_5$ and $b_2b_4b_6$, respectively. Using Lemma 3.2 we observe that

$$b_1b_3b_5 = P^2 \prod_{m\in\gamma(1,3,5)} \ell_m = P^2 = P^2 \prod_{m\in\gamma(2,4,6)} \ell_m = b_2b_4b_6,$$

hence $x_1x_3x_5 - x_2x_4x_6$ vanishes on Wachspress coordinates.

Case 2: We can assume without loss of generality $i+1\neq j-1$, $j+1=k-1$, and

$k+1=i-1$. Four coefficients vanish in the Preliminary calculation, yielding

$$\begin{aligned} w_{ijk} &= |n_{i+1}\ n_{j+1}\ n_{i-1}|x_{i+1}x_{j+1}x_{i-1} - |n_{i+1}\ n_{j-1}\ n_{i-1}|x_{i+1}x_jx_{i-1} \\ &+ |n_{i+1}\ n_{j-1}\ n_{j+1}|x_{i+1}x_jx_{i-2} - |n_{i-1}\ n_{j-1}\ n_{j+1}|x_ix_jx_{i-2}. \end{aligned}$$

Evaluating this at Wachspress coordinates yields,

$$\begin{aligned} w_{ijk}\circ\beta &= |n_{i+1}n_{j+1}n_{i-1}| \prod_{m\in\gamma(i+1,j+1,i-1)} \ell_m + |n_{i+1}n_{j-1}n_{i-1}| \prod_{m\in\gamma(i+1,j,i-1)} \ell_m - \\ &\quad |n_{i+1}n_{j-1}n_{j+1}| \prod_{m\in\gamma(i+1,j,i-1)} \ell_m - |n_{i-1}n_{j-1}n_{j+1}| \prod_{m\in\gamma(i,j,i-1)} \ell_m \\ &= P^2\left(\prod_{m\in\gamma(i-1,i+1,j+1,j)} \ell_m\right)(|n_{i+1}n_{j+1}n_{i-1}|\ell_{j-1} - |n_{i+1}\ n_{j-1}\ n_{i-1}|\ell_{j+1} + \\ &\quad |n_{i+1}\ n_{j-1}\ n_{j+1}|\ell_{i-1} - |n_{i-1}n_{j-1}n_{j+1}|\ell_{i+1}) \\ &= P^2\left(\prod_{m\in\gamma(i-1,i+1,j+1,j)} \ell_m\right)\Big[\big(|n_{i+1}n_{j+1}n_{i-1}|\ell_{j-1} + |n_{i-1}n_{j+1}n_{j-1}|\ell_{i+1}\big) - \\ &\quad \big(|n_{i+1}\ n_{j-1}\ n_{i-1}|\ell_{j+1} + |n_{i+1}\ n_{j+1}\ n_{j-1}|\ell_{i-1}\big)\Big] \end{aligned}$$

where $P=\prod_{i=1}^N \ell_i$. The last factor is the difference of the two adjoints respect to

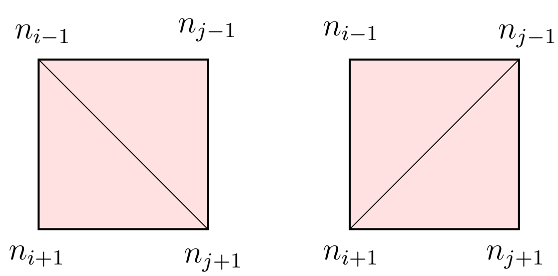

Figure 3.1: Case 2 triangulation

the triangulations of the quadrilateral in Figure 3.1.

Case 3: Assume without loss of generality $i+1\neq j-1$, $j+1\neq k-1$, and

$k+1=i-1$. In this case two coefficients vanish in the Preliminary calculation and after evaluating at Wachspress coordinates we obtain,

$$\begin{aligned} w_{ijk}\circ\beta \;=\;& |n_{i+1}n_{j+1}n_{i-1}| \prod_{m\in\gamma(i+1,j+1,k+1)} \ell_m - |n_{i+1}n_{j+1}n_{k-1}| \prod_{m\in\gamma(i+1,j+1,k)} \ell_m - \\ & |n_{i+1}n_{j-1}n_{i-1}| \prod_{m\in\gamma(i+1,j,k+1)} \ell_m + |n_{i+1}n_{j-1}n_{k-1}| \prod_{m\in\gamma(i+1,j,k)} \ell_m + \\ & |n_{i-1}n_{j+1}n_{k-1}| \prod_{m\in\gamma(i,j+1,k)} \ell_m - |n_{i-1}n_{j-1}n_{k-1}| \prod_{m\in\gamma(i,j,k)} \ell_m \\ =\;& P^2\Bigg(\prod_{\substack{m\in\gamma(i,j,k\\ i+1,j+1,k+1)}} \ell_m\Bigg)\big(\, |n_{i+1}n_{j+1}n_{i-1}|\ell_{j-1}\ell_{k-1} - \\ & |n_{i+1}n_{j+1}n_{k-1}|\ell_{i-1}\ell_{j-1} - |n_{i+1}n_{j-1}n_{i-1}|\ell_{j+1}\ell_{k-1} + \\ & |n_{i+1}n_{j-1}n_{k-1}|\ell_{j+1}\ell_{i-1} + |n_{i-1}n_{j+1}n_{k-1}|\ell_{i+1}\ell_{j-1} - \\ & |n_{i-1}n_{j-1}n_{k-1}|\ell_{i+1}\ell_{j+1}\big) \end{aligned}$$

The last factor is the difference of adjoints with respect to the triangulations of the

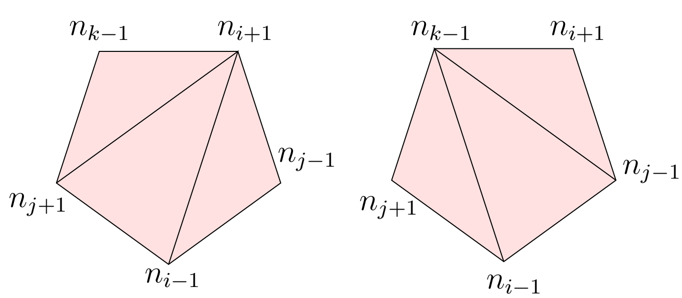

Figure 3.2: Case 3 triangulation

pentagon in Figure 3.2.

Case 4: In this case evaluation at Wachspress coordinates yields,

$$w_{ijk}\circ\beta = |n_{i+1}n_{j+1}n_{k+1}| \prod_{m\in\gamma(i+1,j+1,k+1)} \ell_m - |n_{i+1}n_{j+1}n_{k-1}| \prod_{m\in\gamma(i+1,j+1,k)} \ell_m -$$

$$
\begin{aligned}
&|n_{i+1}n_{j-1}n_{k+1}| \prod_{m\in\gamma(i+1,j,k+1)} \ell_m + |n_{i+1}n_{j-1}n_{k-1}| \prod_{m\in\gamma(i+1,j,k)} \ell_m - \\
&|n_{i-1}n_{j+1}n_{k+1}| \prod_{m\in\gamma(i,j+1,k+1)} \ell_m + |n_{i-1}n_{j+1}n_{k-1}| \prod_{m\in\gamma(i,j+1,k)} \ell_m + \\
&|n_{i-1}n_{j-1}n_{k+1}| \prod_{m\in\gamma(i,j,k+1)} \ell_m - |n_{i-1}n_{j-1}n_{k-1}| \prod_{m\in\gamma(i,j,k)} \ell_m \\
&= P^2 \left(\prod_{\substack{m\in\gamma(i,j,k \\ i+1,j+1,k+1)}} \ell_m \right) \big(|n_{i+1}n_{j+1}n_{k+1}|\ell_{i-1}\ell_{j-1}\ell_{k-1} - \\
&|n_{i+1}n_{j+1}n_{k-1}|\ell_{i-1}\ell_{j-1}\ell_{k+1} - |n_{i+1}n_{j-1}n_{k+1}|\ell_{i-1}\ell_{j+1}\ell_{k-1} + \\
&|n_{i+1}n_{j-1}n_{k-1}|\ell_{i-1}\ell_{j+1}\ell_{k+1} - |n_{i-1}n_{j+1}n_{k+1}|\ell_{i+1}\ell_{j-1}\ell_{k-1} + \\
&|n_{i-1}n_{j+1}n_{k-1}|\ell_{j+1}\ell_{i-1}\ell_{k+1} + |n_{i-1}n_{j-1}n_{k+1}|\ell_{i+1}\ell_{j+1}\ell_{k-1} - \\
&|n_{i-1}n_{j-1}n_{k-1}|\ell_{i+1}\ell_{j+1}\ell_{k+1} \big)
\end{aligned}
$$

The last factor is the difference of adjoints expressed using the triangulations of

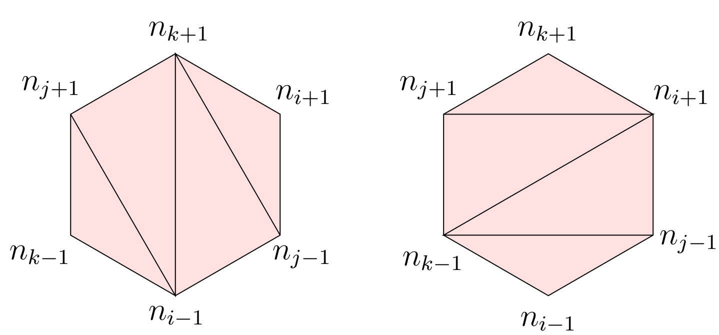

Figure 3.3: Case 4 triangulation

the hexagon in Figure 3.3. □

3.3 The Approach for Obtaining a Set-Theoretic Result

Let $\hat{I}$ be the ideal generated by the Wachspress quadratics and cubics and $\mathbb{T}^\Delta$ the algebraic torus in $\mathbb{P}^\Delta$. By construction we know that $\mathcal{W} \subseteq \mathbb{V}(\hat{I})$. To obtain our set-theoretic result, Theorem 4.4, we must show that $\mathbb{V}(\hat{I}) \subseteq \mathcal{W}$. We have already shown in Chapter 2 that $V(\hat{I}) \backslash \mathcal{C} \subseteq \mathcal{W}$. Showing that $\mathbb{V}(\hat{I}) \cap \mathbb{T}^\Delta \subseteq \mathcal{W}$ will be treated in Section 4.1 of Chapter 4. In the next section we learn how to deal with points that lie in the center $\mathcal{C}$ and we will be able conclude that $(\mathbb{V}(\hat{I}) \cap \mathcal{C}) \smallsetminus \mathbb{T}^\Delta \subseteq \mathcal{W}$.

3.4 Another Expression for the Projection τ When N is Odd

Our goal is to construct rational maps that are equivalent to the linear projection τ on the Wachspress variety $\mathcal{W}$. The construction of the maps differs slightly depending on the parity of the number of edges N of the polygon Δ. We first direct our attention to the odd case. Let Δ be an N-sided polygon with $N = 2k + 1$. We now define monomials that we be used to construct the projections.

Definition 3.7. Define the monomial

$$M_i = \prod_{j=1}^{k} x_{i+2j}.$$

For example, with $k = 4$, the monomial M_1 is $x_3x_5x_7x_9$ and M_2 is $x_1x_4x_6x_8$. The essential property of these monomials is revealed when we evaluate them at Wachspress coordinates.

Lemma 3.8. *The monomial M_i evaluated at Wachspress coordinates $(b_1, \ldots, b_N)$ is $P^{k-1}\, \ell_i$ where $P = \prod_{j=1}^{N} \ell_j$.*

Proof. Follows directly by evaluating $M_i(b_1, \ldots, b_N)$. ☐

We are ready to define a collection of maps that are equivalent to the linear projection on $\mathcal{W}$. For $i = 1, \ldots, N$ define the rational map $\sigma_i : \mathbb{P}^\Delta \dashrightarrow \mathbb{P}^2$ by

$$\sigma_i := \frac{(n_i \times n_{i+1})}{A_i} M_{i-1} + \frac{(n_{i+1} \times n_{i-1})}{A_i} M_i + \frac{(n_{i-1} \times n_i)}{A_i} M_{i+1},$$

where $A_i = |n_{i-1}\ n_i\ n_{i+1}|$. The indeterminacy locus of σ_i is $\mathbb{V}(M_{i-1}, M_i, M_{i+1}) \subseteq \mathbb{P}^{\Delta} \setminus \mathbb{T}^{\Delta}$.

Theorem 3.9. *The map σ_i is equivalent to τ on $\mathcal{W}$.*

Proof. A generic point on $\mathcal{W}$ has the form $\beta(\mathbf{z}) = [b_1(\mathbf{z}), \ldots, b_N(\mathbf{z})]$ for some point $\mathbf{z} = [z_0, z_1, z_2] \in \mathbb{P}^2_{\mathbb{C}}$. The linear projection τ maps $\beta(\mathbf{z})$ to $\mathbf{z}$ on an open set. We show that the same holds for σ_i. By Lemma 3.8,

$$\begin{aligned}\sigma_i(\beta(\mathbf{z})) &= \frac{n_i \times n_{i+1}}{A_i} M_{i-1}(\beta(\mathbf{z})) + \frac{n_{i+1} \times n_{i-1}}{A_i} M_i(\beta(\mathbf{z})) + \frac{n_{i-1} \times n_i}{A_i} M_{i+1}(\beta(\mathbf{z})) \\ &= \frac{P^{k-1}}{A_i} \Big((n_i \times n_{i+1})\ell_{i-1}(\mathbf{z}) + (n_{i+1} \times n_{i-1})\ell_i(\mathbf{z}) + (n_{i-1} \times n_i)\ell_{i+1}(\mathbf{z}) \Big).\end{aligned}$$

We claim that $(n_i \times n_{i+1})\ell_{i-1}(\mathbf{z}) + (n_{i+1} \times n_{i-1})\ell_i(\mathbf{z}) + (n_{i-1} \times n_i)\ell_{i+1}(\mathbf{z}) = A_i\, \mathbf{z}$. We prove the claim by showing that

$$\Big(n_i \times n_{i+1})\ell_{i-1}(\mathbf{z}) + (n_{i+1} \times n_{i-1})\ell_i(\mathbf{z}) + (n_{i-1} \times n_i)\ell_{i+1}(\mathbf{z}) \Big) \cdot e_j = A_i z_j \tag{3.4}$$

for the standard basis vectors $e_1 = [1, 0, 0]$, $e_2 = [0, 1, 1]$, and $e_3 = [0, 0, 1]$. Observe that the left hand side of Equation 3.4 is

$$\begin{aligned}&\Big(n_i \times n_{i+1})\ell_{i-1}(\mathbf{z}) + (n_{i+1} \times n_{i-1})\ell_i(\mathbf{z}) + (n_{i-1} \times n_i)\ell_{i+1}(\mathbf{z}) \Big) \cdot e_j \\ &= |n_i\ n_{i+1}\ e_j|\ n_{i-1} \cdot \mathbf{z} + |n_{i+1}\ n_{i-1}\ e_j|\ n_i \cdot \mathbf{z} + |n_{i-1}\ n_i\ e_j|\ n_{i+1} \cdot \mathbf{z} \\ &= \Big(|n_i\ n_{i+1}\ e_j|\ n_{i-1} + |n_{i+1}\ n_{i-1}\ e_j|\ n_i + |n_{i-1}\ n_i\ e_j|\ n_{i+1} \Big) \cdot \mathbf{z}.\end{aligned} \tag{3.5}$$

By applying Lemma 1.18 we see that Equation 3.5 is $|\ n_{i-1}\, n_i\, n_{i+1}\ |\ e_j \cdot \mathbf{z} = A_i z_j$, proving the claim. We have shown that the values of τ and σ_i agree on the open set $\mathcal{W} \setminus (\mathbb{V}(P) \cup \mathcal{C})$, thus they are equivalent on $\mathcal{W}$. ∎

Lemma 3.10. *The polynomials*

$$d_i := |n_i\, n_{i+1}\; n_{i+2}|\; M_{i-1} - |n_{i-1}\, n_{i+1}\; n_{i+2}|\; M_i + |n_{i-1}\; n_i\; n_{i+2}|\; M_{i+1} - |n_{i-1}\; n_i\; n_{i+1}|\; M_{i+2}$$

for $i = 1, \ldots, N$ vanish on $\mathcal{W}$.

Proof. We evaluate d_i at Wachspress coordinates:

$$\begin{aligned}
d_i(\beta) =& |n_i\; n_{i+1}\; n_{i+2}|\; M_{i-1}(\beta) - |n_{i-1}\; n_{i+1}\; n_{i+2}|\; M_i(\beta) + \\
& |n_{i-1}\; n_i\; n_{i+2}|\; M_{i+1}(\beta) - |n_{i-1}\; n_i\; n_{i+1}|\; M_{i+2}(\beta) \\
=& P^{k-1}\big(|n_i\, n_{i+1}\, n_{i+2}|\ell_{i-1} - |n_{i-1}\, n_{i+1}\; n_{i+2}|\ell_i + \\
& |n_{i-1}\, n_i\, n_{i+2}|\ell_{i+1} - |n_{i-1}\; n_i\; n_{i+1}|\ell_{i+2}\big) \\
=& P^{k-1}\big(|n_i\; n_{i+1}\; n_{i+2}|\ell_{i-1} + |n_{i-1}\; n_i\; n_{i+2}|\ell_{i+1} - \\
& (|n_{i-1}\; n_{i+1}\; n_{i+2}|\ell_i + |n_{i-1}\; n_i\; n_{i+1}|\ell_{i+2})\big) \\
=& P^{k-1}\big(|n_i\, n_{i+1}\, n_{i+2}|n_{i-1} + |n_{i-1}\, n_i\, n_{i+2}|n_{i+1} - \\
& |n_{i-1}\, n_{i+1}\, n_{i+2}|n_i - |n_{i-1}\, n_i\, n_{i+1}|n_{i+2}\big) \cdot \mathbf{z}.
\end{aligned}$$

By Lemma 1.19 the factor in parentheses in the last line is zero. □

Let J be the ideal generated by the Wachspress quadratics, Wachspress cubics, and the d_i.

Conjecture 3.11. *The polynomials d_i lie in the ideal $\hat{I}$; hence, $J = \hat{I}$.*

Lemma 3.12. *The rational maps $\sigma_1, \ldots, \sigma_N$ are equivalent on $\mathbb{V}(J)$.*

Proof. It suffices to show that $\sigma_i \cong \sigma_{i+1}$ modulo J for any $i = 1, \ldots, N$. We show that the difference $\sigma_i - \sigma_{i+1}$ is zero modulo J.

$$\begin{aligned}\sigma_i - \sigma_{i+1} =& |n_i\ n_{i+1}\ n_{i+2}|\ M_{i-1} - |n_i\ n_{i+1}\ n_{i+2}|\ M_{i+2} \\ & \big(|n_i\ n_{i+1}\ n_{i+2}|\ n_{i+1} \times n_{i-1} - |n_{i-1}\ n_i\ n_{i+1}|\ n_{i+1} \times n_{i+1}\big)\ M_i + \\ & \big(|n_i\ n_{i+1}\ n_{i+2}|\ n_{i-1} \times n_i - |n_{i-1}\ n_i\ n_{i+1}|\ n_{i+2} \times n_i\big)\ M_{i+1} \\ =& |n_i\ n_{i+1}\ n_{i+2}|\ M_{i-1} - |n_i\ n_{i+1}\ n_{i+2}|\ M_{i+2} \\ & n_{i+1} \times \big(|n_i\ n_{i+1}\ n_{i+2}|\ n_{i-1} - |n_{i-1}\ n_i\ n_{i+1}|\ n_{i+2}\big)\ M_i \\ & \big(\ |n_i\ n_{i+1}\ n_{i+2}|\ n_{i-1} - |n_{i-1}\ n_i\ n_{i+1}|\ n_{i+2}\big) \times n_i\ M_{i+1}.\end{aligned}$$

Notice that the two factors enclosed in parenthesis above are the same and by Lemma 1.19 are equal to $|n_{i-1}\ n_{i+1}\ n_{i+2}|\ n_i - |n_{i-1}\ n_i\ n_{i+2}|\ n_{i+1}$. Thus we have

$$\begin{aligned}\sigma_i - \sigma_{i+1} = (n_i \times n_{i+1})\ \big(&|n_i\ n_{i+1}\ n_{i+2}|\ M_{i-1} - |n_{i-1}\ n_{i+1}\ n_{i+2}|\ M_i \\ &+ |n_{i-1}\ n_i\ n_{i+2}|\ M_{i+1} - |n_{i-1}\ n_i\ n_{i+1}|\ M_{i+2}\big) = (n_i \times n_{i+1}) d_i.\end{aligned}$$

Thus the maps σ_i and σ_{i+1} agree on J since $d_i \in J$. Note that we have actually shown that $A_i\sigma_i = A_{i+1}\sigma_{i+1}$ and more generally it follows that $A_i\sigma_i = A_j\sigma_j$. □

Remark 3.13. We obtained in the proof of the preceding lemma the equation $A_i\sigma_i = A_j\sigma_j$; however, we will assume we have $\sigma_i = \sigma_j$, ignoring the constant A_i to simplify future arguments. The constants can be carried along without effecting our arguments but make for tedious bookkeeping.

Lemma 3.14. *For any indices $i, j \in \{1, \ldots, N\}$ we have $\ell_j(\sigma_i) = M_j$.*

Proof. It is immediate from the definition of σ_i that $\ell_i(\sigma_i) = M_i$. By Lemma 3.12, on $\mathbb{V}(J)$, we have

$$\ell_j(\sigma_i) = \ell_j(\sigma_j) = M_j.$$

$\square$

Lemma 3.15. *For $x = [x_1 : \cdots : x_N] \in \mathbb{V}(J)$ and for any $i \in \{1, \ldots, N\}$,*

$$\beta \circ \sigma_i(x) = \big(\prod_{s=1}^{N} x_s\big)^{k-1}\ x.$$

Proof. It suffices to show that for any $j \in \{1, \ldots, N\}$,

$$(b_j \circ \sigma_i)(x) = \big(\prod_{s=1}^{N} x_s\big)^{k-1}\ x_j.$$

Observe,

$$(b_j \circ \sigma_i)(x) \quad = \quad \prod_{s \neq j-1, j} \ell_s(\sigma_i(x)) = \prod_{s \neq j-1, j} M_s = \big(\prod_{s=1}^{N} x_s\big)^{k-1}\ x_j.$$

To see the last equality it suffices to set $j = 1$. Observe $M_{i-1}M_i = \prod_{j \neq i} x_j$ for any j; hence,

$$\begin{aligned}(b_1 \circ \sigma_i)(x) = \prod_{s \neq 1, N} M_s &= (M_2 M_3)(M_4 M_5) \cdots (M_{N-3} M_{N-2}) M_{N-1} \\ &= (\prod_{j \neq 3} x_j)(\prod_{j \neq 5} x_j) \cdots (\prod_{j \neq N-2} x_j) x_1 x_3 \cdots x_{N-2} = P^{k-1} x_1.\end{aligned}$$

$\square$

3.5 Another Expression for the Projection τ When N is Even

We now find maps analogous to the σ_i's in the case where the polygon Δ has an even number of edges. Let $N = 2k$ be even. Define the monomials $M_{i,j}$ for

$1 \leq i, j \leq N$ with i and j of opposite parity as

$$M_{i,j} = x_{i-1} \prod_{m=1}^{\frac{j-i-1}{2}} x_{i+2m} \prod_{m=1}^{\frac{N-j+i-3}{2}} x_{j+2m} \tag{3.6}$$

if $i < j$ and $j - i > 1$, and

$$M_{i,i+1} = x_{i-1} \prod_{m=1}^{k-2} x_{i+2m+1}. \tag{3.7}$$

Example 3.16. Let $k = 4$ and hence $N = 8$. Then,

$$\begin{array}{llll} M_{1,2} = x_4x_6x_8 & M_{1,6} = x_3x_5x_8 & M_{2,3} = x_1x_5x_7 & M_{2,7} = x_1x_4x_6 \\ M_{1,4} = x_3x_6x_8 & M_{1,8} = x_3x_5x_7 & M_{2,5} = x_1x_4x_7, & \end{array}$$

and with $k = 5$,

$$\begin{array}{llll} M_{1,2} = x_4x_6x_8x_{10} & M_{1,6} = x_3x_5x_8x_{10} & M_{2,3} = x_1x_5x_7x_9 & M_{2,7} = x_1x_4x_6x_9 \\ M_{1,4} = x_3x_6x_8x_{10} & M_{1,8} = x_3x_5x_7x_{10} & M_{2,5} = x_1x_4x_7x_9 & M_{2,9} = x_1x_4x_6x_8 \end{array}$$

Notice that for i even (or odd) $M_{i-1,i}$ is the product of all even (or odd) indexed variables except x_i. The important fact about these monomials is that under evaluation at Wachspress coordinates $M_{i,j}$ is $P^{k-2}\ell_i\ell_j$. We can now define maps analogous to the σ_i's. For $1 \leq i \leq N$ define the rational maps

$$\zeta_i(x) := \frac{(n_{i+3} \times n_{i+5})}{\hat{A}_i} M_{i,i+1} + \frac{(n_{i+5} \times n_{i+1})}{\hat{A}_i} M_{i,i+3} + \frac{(n_{i+1} \times n_{i+3})}{\hat{A}_i} M_{i,i+5},$$

where $\hat{A}_i =\mid n_{i+1}\, n_{i+3}\, n_{i+5} \mid$.

Remark 3.17. Observe that $\ell_i(\zeta_{i-1}(x)) = M_{i-1,i}$.

The polynomials in Lemma 3.18 are completely analogous to the polynomials d_i in Lemma 3.10 in the odd case, so we also denote them by d_i.

Lemma 3.18. *The following polynomials d_i for $i = 1, \dots, N$ vanish on $\mathcal{W}$.*

$$\begin{aligned} d_i :=& |n_{i+3} n_{i+5} n_{i+7}| x_{i+1} M_{i,i+1} - |n_{i+1} n_{i+3} n_{i+5}| x_{i+2} M_{i+2,i+7} - \\ & |n_{i+1} n_{i+5} n_{i+7}| x_{i+1} M_{i,i+3} + |n_{i+1} n_{i+3} n_{i+5}| x_{i+1} M_{i+1,i+5}. \end{aligned}$$

Proof. The proof proceeds in the same manner as the proof of Lemma 3.10 and is omitted. ☐

Conjecture 3.19. *The polynomials d_i for $i = 1, \dots, N$ lie in $\hat{I}$.*

Let J be the ideal generated by Wachspress quadratics, Wachspress cubics, and the d_i.

Lemma 3.20. *For $1 \le i \le N$ the rational maps ζ_i and ζ_{i+2} are equivalent on the variety $\mathbb{V}(J)$. Further, we have $\hat{A}_i \, x_{i+1} \zeta_i = \hat{A}_{i+2} \, x_{i+2} \zeta_{i+2}$ modulo J.*

Proof. We show that the difference $\hat{A}_{i+2} \, x_{i+1} \zeta_i - \hat{A}_i \, x_{i+2} \zeta_{i+2}$ is zero modulo J.

$$\begin{aligned} & \hat{A}_{i+2} \, x_{i+1} \zeta_i - \hat{A}_i \, x_{i+2} \zeta_{i+2} \\ =& |n_{i+3} n_{i+5} n_{i+7}| (n_{i+3} \times n_{i+5}) x_{i+1} M_{i,i+1} + |n_{i+3} n_{i+5} n_{i+7}| (n_{i+5} \times n_{i+1}) M_{i,i+3} + \\ & |n_{i+3} n_{i+5} n_{i+7}| (n_{i+1} \times n_{i+3}) x_{i+1} M_{i+1,i+5} - |n_{i+1} n_{i+3} n_{i+5}| (n_{i+5} \times n_{i+7}) x_{i+2} M_{i+2,i+3} - \\ & |n_{i+1} n_{i+3} n_{i+5}| (n_{i+7} \times n_{i+3}) x_{i+2} M_{i+2,i+5} - \\ & |n_{i+1} n_{i+3} n_{i+5}| (n_{i+3} \times n_{i+5}) x_{i+2} M_{i+2,i+7} \end{aligned} \tag{3.8}$$

It is not difficult to check directly from the definitions that $x_{i+1} M_{i,i+3} = x_{i+2} M_{i+2,i+3}$

and $x_{i+1}M_{i+1,i+5} = x_{i+2}M_{i+2,i+5}$. Using this we can combine term in (3.8) to obtain

$$\begin{aligned}
&|n_{i+3}n_{i+5}n_{i+7}|(n_{i+3}\times n_{i+5})x_{i+1}M_{i,i+1} - |n_{i+1}n_{i+3}n_{i+5}|(n_{i+3}\times n_{i+5})x_{i+2}M_{i+2,i+7}+\\
&\big(|n_{i+3}n_{i+5}n_{i+7}|(n_{i+5}\times n_{i+1}) - |n_{i+1}n_{i+3}n_{i+5}|(n_{i+5}\times n_{i+7})\big)x_{i+1}M_{i,i+3}+\\
&\big(|n_{i+3}n_{i+5}n_{i+7}|(n_{i+1}\times n_{i+3}) - |n_{i+1}n_{i+3}n_{i+5}|(n_{i+7}\times n_{i+3})\big)x_{i+1}M_{i+1,i+5}\\
=&|n_{i+3}n_{i+5}n_{i+7}|(n_{i+3}\times n_{i+5})x_{i+1}M_{i,i+1} - |n_{i+1}n_{i+3}n_{i+5}|(n_{i+3}\times n_{i+5})x_{i+2}M_{i+2,i+7}+\\
&n_{i+5}\big(|n_{i+3}n_{i+5}n_{i+7}|n_{i+1} - |n_{i+1}n_{i+3}n_{i+5}|n_{i+7}\big)x_{i+1}M_{i,i+3}+\\
&\big(|n_{i+3}n_{i+5}n_{i+7}|n_{i+1} - |n_{i+1}n_{i+3}n_{i+5}|n_{i+7})\big)n_{i+3}x_{i+1}M_{i+1,i+5}.
\end{aligned}$$

The two factors in parentheses are the same and by Lemma 1.19 are both equal to $|n_{i+1}n_{i+3}n_{i+5}|n_{i+5} - |n_{i+1}n_{i+5}n_{i+7}|n_{i+3}$. The last line in the Equation above becomes,

$$\begin{aligned}
&|n_{i+3}n_{i+5}n_{i+7}|(n_{i+3}\times n_{i+5})x_{i+1}M_{i,i+1} - |n_{i+1}n_{i+3}n_{i+5}|(n_{i+3}\times n_{i+5})x_{i+2}M_{i+2,i+7}+\\
&n_{i+5}\times\big(|n_{i+1}n_{i+3}n_{i+5}|n_{i+5} - |n_{i+1}n_{i+5}n_{i+7}|n_{i+3}\big)x_{i+1}M_{i,i+3}+\\
&\big(|n_{i+1}n_{i+3}n_{i+5}|n_{i+5} - |n_{i+1}n_{i+5}n_{i+7}|n_{i+3}\big)\times n_{i+3}x_{i+1}M_{i+1,i+5}\\
=&(n_{i+3}\times n_{i+5})\big(|n_{i+3}n_{i+5}n_{i+7}|x_{i+1}M_{i,i+1} - |n_{i+1}n_{i+3}n_{i+5}|x_{i+2}M_{i+2,i+7}-\\
&|n_{i+1}n_{i+5}n_{i+7}|x_{i+1}M_{i,i+3} + |n_{i+1}n_{i+3}n_{i+5}|x_{i+1}M_{i+1,i+5}\big).
\end{aligned}$$

□

It remains to show that the maps ζ_i and ζ_j are equivalent for indices i and j of opposite parity. To show this it suffices to let $i = 1$ and $j = 2$. The rational maps ζ_1 and ζ_2 are represented by triples of forms of degree $k-1$ in $\mathbb{C}[\mathbb{P}^\Delta]$. Let

$\zeta_1 := [f_1 : f_2 : f_3]$, $\zeta_2 = [g_1 : g_2 : g_3]$, and let M_{fg} be the matrix

$$\begin{bmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{bmatrix}.$$

Lemma 3.21. *The minors of the matrix M_{fg} vanish on $\mathcal{W}$.*

Proof. The matrix M_{fg} is simply the 2×3 matrix with rows ζ_1 and ζ_2. By definition, evaluation at Wachspress coordinates yields a matrix with rows $\zeta_1(\beta(z)) = P\,z$ and $\zeta_2(\beta(z))$ for any $z \in \mathbb{P}^2$ where P is the product of all the linear forms defining the edges of Δ. This matrix clearly has rank one for all z, hence each minor of M_{fg} vanishes on Wachspress coordinates so vanishes on $\mathcal{W}$. $\square$

Lemma 3.22. *The rational maps ζ_1 and ζ_2 are equivalent modulo the ideal generated by the minors of the matrix M_{fg}. Thus modulo these minors the rational maps differ by a rational function $c(x)$; i.e., $\zeta_2(x) = c(x)\zeta_1$. The rational function $c(x)$ can be expressed as $\frac{f_i}{g_i}$ for $i = 1, 2, 3$ and, further, we can assume that $c(x)$ is defined and nonzero for $x \notin \mathbb{T}^\Delta$.*

Proof. If $x \in \mathbb{V}(J) \setminus \mathbb{T}^\Delta$ then we can assume without loss of generality that $f_1(x) \neq 0$ since the indeterminacy locus of ζ_1 is contained in $\mathbb{T}^\Delta$. Suppose that $g_1(x) = 0$. Then since $g_1 f_2(x) = g_2 f_1(x)$, then $g_2(x) = 0$. Now since $g_1 f_3(x) = g_3 f_1(x)$, then $g_3(x) = 0$. This means that x is in the indeterminacy locus of ζ_2 and hence does not lie in the torus $\mathbb{T}^\Delta$. This is a contradiction, so if $f_1(x) \neq 0$, then $g_1(x) \neq 0$. Therefore, since we assume in this section that $x \notin \mathbb{T}^\Delta$ throughout, we can assume without loss of generality that $c(x) = \frac{f_1}{g_1}$ and this quantity is defined as well as non-zero. $\square$

Let

$$P_{ev} = \prod_{i=1}^{k} x_{2i} \qquad \text{and} \qquad P_{od} = \prod_{i=1}^{k} x_{2i-1}$$

be the product of all even and odd-indexed variables, respectively.

Lemma 3.23. *The polynomial $P_{od} - P_{ev}$ vanishes on $\mathcal{W}$.*

Proof. This follows immediately by evaluating at Wachspress coordinates. $\square$

Conjecture 3.24. *The polynomial $P_{od} - P_{ev}$ lies in the ideal $\hat{I}$.*

We redefine J to be the ideal generated by Wachspress quadratics, Wachspress cubics, the d_i, the three minors of M_{fg}, and $P_{od} - P_{ev}$. We now aim to show that for $x \in \mathbb{V}(J) \setminus \mathbb{T}^{\Delta}$ that $\beta(\zeta_m(x)) = x$ for $m = 1, \ldots, N$. First note it suffices to set $m = 1$ and show that $b_j(\zeta_1(x)) = x_j$ for any $j = 1, \ldots, N$. We wish to evaluate the expression

$$b_j(\zeta_1(x)) = \prod_{i \neq j-1,j} \ell_i(\zeta_1).$$

The next lemma shows us how to evaluate each factor $\ell_i(\zeta_1)$ in this product.

Lemma 3.25. *If $i \geq 2$ is even, then*

$$\ell_i(\zeta_1(x)) = \frac{x_3 x_5 \cdots x_{i-1}}{x_2 x_4 \cdots x_{i-2}} M_{i-1,i},$$

and if $i \geq 2$ is odd, then

$$\ell_i(\zeta_1(x)) = \frac{1}{c(x)} \frac{x_4 x_6 \cdots x_{i-1}}{x_3 x_5 \cdots x_{i-2}} M_{i-1,i}.$$

Finally, for $i = 1$,

$$n_1 \cdot \zeta_1(x) = \frac{1}{c(x)} \frac{x_1 x_3 \cdots x_{N-1}}{x_2} = \frac{1}{c(x)} \frac{x_2 x_4 \cdots x_N}{x_2} = \frac{1}{c(x)} x_4 x_6 \cdots x_N$$

Proof. This follows by applying Lemma 3.20 and Remark 3.17. $\square$

Example 3.26. Let $k = 3$. We work with the map ζ_1, expressing each ζ_i in terms of ζ_1.

$$\zeta_1 = \tfrac{1}{c(x)}\zeta_2 \qquad \zeta_1 = \tfrac{x_3}{x_2}\zeta_3 \qquad \zeta_1 = \tfrac{1}{c(x)}\zeta_2 = \tfrac{x_4}{c(x)x_3}\zeta_4$$
$$\zeta_1 = \tfrac{x_3x_5}{x_2x_4}\zeta_5 \quad \zeta_1 = \tfrac{x_4x_6}{c(x)x_3x_5}\zeta_6$$

Using the above computations and Remark 3.17 we can calculate

$$\ell_2(\zeta_1) = M_{1,2} \qquad \ell_3(\zeta_1) = \tfrac{1}{c(x)}M_{2,3}$$
$$\ell_4(\zeta_1) = \tfrac{x_3}{x_2}M_{3,4} \qquad \ell_5(\zeta_1) = \tfrac{x_4}{c(x)x_3}M_{4,5}$$
$$\ell_6(\zeta_1) = \tfrac{x_3x_5}{x_2x_4}M_{5,6} \quad \ell_1(\zeta_1) = \tfrac{x_4x_6}{c(x)x_3x_5}M_{6,1}.$$

Further, we can use this to compute the following:

$$\begin{aligned}
b_1(\zeta_1) =& \ell_2(\zeta_1)\ell_3(\zeta_1)\ell_4(\zeta_1)\ell_5(\zeta_1)\\
=&(M_{1,2})(\frac{1}{c(x)}M_{2,3})(\frac{x_3}{x_2}M_{3,4})(\frac{x_4}{c(x)x_3}M_{4,5})\\
=&\frac{1}{c(x)^2}\frac{x_4}{x_2}M_{1,2}M_{2,3}M_{3,4}M_{4,5}\\
=&\frac{1}{c(x)^2}\frac{x_4}{x_2}\frac{P_{ev}}{x_2}\frac{P_{od}}{x_3}\frac{P_{ev}}{x_4}\frac{P_{od}}{x_5}\\
=&\frac{1}{c(x)^2}\frac{P_{ev}^2P_{od}^2}{x_2^2x_3x_5} = \frac{1}{c(x)^2}\frac{P^2}{x_2^2x_3x_5}\\
=&\frac{1}{c(x)^2}\frac{P^2}{x_2^2M_{6,1}}, \text{ and}
\end{aligned}$$

$$
\begin{aligned}
b_2(\zeta_1) =& \ell_3(\zeta_1)\ell_4(\zeta_1)\ell_5(\zeta_1)\ell_6(\zeta_1) \\
=& (\frac{1}{c(x)}M_{2,3})(\frac{x_3}{x_2}M_{3,4})(\frac{x_4}{c(x)x_3}M_{4,5})(\frac{x_3x_5}{x_2x_4}M_{5,6}) \\
=& \frac{1}{c(x)^2}\frac{x_3x_5}{x_2^2}M_{2,3}M_{3,4}M_{4,5}M_{5,6} \\
=& \frac{1}{c(x)^2}\frac{x_3x_5}{x_2^2}\frac{P_{od}}{x_3}\frac{P_{ev}}{x_4}\frac{P_{od}}{x_5}\frac{P_{ev}}{x_6} \\
=& \frac{1}{c(x)^2}\frac{P_{ev}^2P_{od}^2}{x_2^2x_4x_6} = \frac{1}{c(x)^2}\frac{P^2}{x_2^2x_4x_6} \\
=& \frac{1}{c(x)^2}\frac{P^2}{x_2^2M_{1,2}}.
\end{aligned}
$$

Now we claim that

$$\frac{1}{x_1}b_1(\zeta_1) = \frac{1}{x_2}b_2(\zeta_1). \tag{3.9}$$

From our calculation above and after some canceling, Equation 3.9 reduces to

$$\frac{1}{x_1M_{6,1}} = \frac{1}{x_2M_{1,2}},$$

but this simply says that

$$\frac{1}{P_{od}} = \frac{1}{P_{ev}} \Rightarrow P_{od} = P_{ev},$$

which we know is true.

Example 3.27. Let $N = 8$.

$$
\begin{aligned}
n_1(\zeta_1) &= \tfrac{1}{c(x)}\tfrac{x_4x_6x_8}{x_3x_5x_7}M_{8,1} & n_2(\zeta_1) &= M_{1,2} \\
n_3(\zeta_1) &= \tfrac{1}{c(x)}M_{2,3} & n_4(\zeta_1) &= \tfrac{x_3}{x_2}M_{3,4} \\
n_5(\zeta_1) &= \tfrac{1}{c(x)}\tfrac{x_4}{x_3}M_{4,5} & n_6(\zeta_1) &= \tfrac{x_3x_5}{x_2x_4}M_{5,6} \\
n_7(\zeta_1) &= \tfrac{1}{c(x)}\tfrac{x_4x_6}{x_3x_5}M_{6,7} & n_8(\zeta_1) &= \tfrac{x_3x_5x_7}{x_2x_4x_6}M_{7,8}
\end{aligned}
$$

$$
\begin{aligned}
b_1(\zeta_1) =& \ell_2(\zeta_1)\ell_3(\zeta_1)\ell_4(\zeta_1)\ell_5(\zeta_1)\ell_6(\zeta_1)\ell_7(\zeta_1) \\
=& M_{1,2}\frac{1}{c(x)}M_{2,3}\frac{x_3}{x_2}M_{3,4}\frac{1}{c(x)}\frac{x_4}{x_3}M_{4,5}\frac{x_3x_5}{x_2x_4}M_{5,6}\frac{1}{c(x)}\frac{x_4x_6}{x_3x_5}M_{6,7} \\
=& \frac{1}{c(x)^3}\frac{x_4x_6}{x_2^3}\, M_{1,2}M_{2,3}M_{3,4}M_{4,5}M_{5,6}M_{6,7} \\
=& \frac{x_4x_6}{c(x)^3}\frac{1}{x_2^3}\frac{P_{ev}}{}\frac{P_{od}}{x_3}\frac{P_{ev}}{x_4}\frac{P_{od}}{x_5}\frac{P_{ev}}{x_6}\frac{P_{od}}{x_7} \\
=& \frac{1}{c(x)^3}\frac{1}{x_2^3}\frac{P_{ev}}{}\frac{P_{od}}{x_3}\frac{P_{ev}}{}\frac{P_{od}}{x_5}\frac{P_{ev}}{}\frac{P_{od}}{x_7} = \frac{1}{c(x)^3}\frac{1}{x_2^3}\frac{P^3}{M_{8,1}}
\end{aligned}
$$

Similarly, we will have

$$
b_2(\zeta_1) = \frac{1}{c(x)^3}\frac{1}{x_2^3}\frac{P_{ev}}{}\frac{P_{od}}{x_3}\frac{P_{ev}}{}\frac{P_{od}}{x_5}\frac{P_{ev}}{}\frac{P_{od}}{x_7} = \frac{1}{c(x)^3}\frac{1}{x_2^3}\frac{P^3}{M_{1,2}}.
$$

Observe that we have,

$$
\frac{1}{x_1}b_1(\zeta_1) = \frac{1}{x_2}b_2(\zeta_2). \tag{3.10}
$$

After canceling, Equation 3.10 reduces to

$$
\frac{1}{x_1}\frac{1}{M_{8,1}} = \frac{1}{x_2}\frac{1}{M_{1,2}},
$$

and this reduces to

$$
P_{od} = P_{ev},
$$

which does hold.

The two examples computed above can easily be generalized to conclude the following.

Lemma 3.28.

$$b_i(\zeta_1) = \frac{1}{c(x)^{k-1}} \frac{P^{k-1}}{x_2^{k-1} M_{i-1,i}}$$

We have seen that $b_i(\zeta_1) = x_i f(x)/c(x)^{k-1}$ for some monomial $f(x)$. From this it follows that $\beta(\zeta_1(x)) = f(x)/c(x)^{k-1}\, x$ and this is equal to x in $\mathbb{P}^\Delta$ since $f(x)/c(x)^{k-1}$ is defined and does not vanish on the complement of the torus $\mathbb{T}^\Delta$.

4. Conclusion

4.1 Intersection with a Coordinate Hyperplane

We have shown in the previous chapter that any point $x \in \mathbb{V}(J) \setminus \mathbb{T}^\Delta$ lies on $\mathcal{W}$. To obtain our set-theoretic result, Theorem 4.4, we must prove that the intersection of any coordinate hyperplane $V(x_i)$ with $V(J)$ is contained in $\mathcal{W}$. This will allow us to conclude that $V(J) = \mathcal{W}$. The ideal $\hat{I}$ is generated by the Wachspress quadratics and cubics and the ideal K is generated by the Wachspress cubics. In this chapter we first investigate how $V(\hat{I})$ intersects a coordinate hyperplane. This will allow us to conclude or main result Theorem 4.4.

Later in the chapter we use the expression of the Wachspress quadratics and cubics as scalar products and determinants to describe a set of syzygies among them. We conclude by looking at some examples of Wachspress varieties and the ideal we have found that cuts them out for small N.

Lemma 4.1. *For any $i = 1, \ldots, N$ the ideal $\langle x_i \rangle + K$ is the monomial ideal generated by x_i and all Δ-monomials not involving x_i.*

Proof. It suffices to show this for $i = 1$. If $x_1 = 0$, then we have $\Lambda_1 = x_2 n_2$ and $\Lambda_N = x_N n_1$. From this it immediately follows that $w_{135} = |n_{246}| x_2 x_4 x_6$ and $w_{N-4,N-2,N} = |n_{N-5,N-3,N-1}| x_{N-4} x_{N-2} x_N$. Using that $x_1 = 0$ and $x_2 x_4 x_6 = x_{N-4} x_{N-2} x_N = 0$, we can recursively show that each of the remaining Wachspress cubics w_{ijk} either reduce to zero or to a Δ-monomial. In fact every Δ-monomial not involving x_1 will occur in this way. $\square$

Lemma 4.2. *The intersection of $\mathbb{V}(K)$ with the coordinate hyperplane $V(x_i)$ is a union of three dimensional coordinate planes.*

Proof. It suffices to let $i = 1$. The monomial ideal generated by x_1 and all $\Delta-$monomials not involving x_1 has primary decomposition consisting of all ideals of the form $(x_1, x_{i_1}, \ldots, x_{i_{N-5}})$ where each $\Delta-$monomial not involving x_1 involves a variable in the set $\{x_{i_1}, \ldots, x_{i_{N-5}}\}$. $\square$

Lemma 4.3. *$V(\hat{I}) \cap \mathbb{V}(x_i)$ is the union of $\deg \mathcal{W} = (N^2 - 5N + 8)/2$ lines. It contains all edge images except $\hat{\ell}_{i-1}$ and $\hat{\ell}_i$. It also contains the $(N-3)(N-4)/2$ blown up base points $\hat{p}_{j,k}$ where neither j nor k is in $\{i-1, i\}$.*

Proof. We calculate the lines that are expected when $x_1 = 0$. We expect to get all edge images $\hat{\ell}_j$ except $\hat{\ell}_1$ and $\hat{\ell}_N$, so $N - 2$. We also expect to get all blow ups of the base points p_{ij} that do not meet either ℓ_1 or ℓ_N. There are $N(N-3)/2$ base points and $N - 3$ of them meet ℓ_1 and another $N - 3$ meet ℓ_N. So we expect to get

$$N(N-3)/2 - 2(N-3) = (N-3)(N-4)/2 \tag{4.1}$$

blown up base points. In total then we expect to get

$$N - 2 + (N-3)(N-4)/2 = (N^2 - 5N + 8)/2 \tag{4.2}$$

lines. The ideal generated by the Wachspress cubics and x_i is the monomial ideal generated by x_i and all Δ-monomials not involving x_i. By Lemma 4.2 this ideal's components are all ideals of the form $(x_{i_1}, \ldots, x_{i_{N-4}})$ where $x_1 = 1$ and at least one of the variables of $\mathfrak{m}$ is contained in $\{i_2, \ldots, i_{N-4}\}$ for each Δ-monomial $\mathfrak{m}$ not involving x_1. Suppose we are on one such component $(x_{i_1}, \ldots, x_{i_{N-4}})$. The component cuts out a three dimensional coordinate plane. We impose Wachspress quadratics on this component. Only four variables are nonzero on each component so most quadratics reduce to zero since many of the vectors Λ_i will vanish. The remaining ones will cut down the three plane to a blown-up line or edge image. $\square$

We now state our main result whose proof follows from the lemmas and theorems in this and previous chapters.

Theorem 4.4. *The Wachspress variety $\mathcal{W}$ is cut out set-theoretically by the ideal J.*

Once Conjectures 3.11, 3.19, and 3.24 are proven this result will be strengthened to the following.

Conjecture 4.5. *The Wachspress variety $\mathcal{W}$ is cut out set-theoretically by the ideal generated by Wachspress quadratics and cubics.*

4.2 Syzygies and Betti Diagrams

Given the expressions of the Wachspress quadratics and cubics as determinants and scalar products, we can use basic vector calculus identities to easily construct syzygies among them. Let a, b, and c be vectors in three space. From vector calculus we have the identities:

$$\mid a\, b\, c \mid = a \cdot (b \times c) = (a \times b) \cdot c$$

$$b(a \cdot c) = a \times (b \times c) + c(a \cdot b).$$

Theorem 4.6. *The relation $\pi\, w_{i,j,k} = (\Lambda_i \times \Lambda_j)q_k - (\Lambda_i \times \Lambda_k)q_j + (\Lambda_j \times \Lambda_k)q_i$ of vectors of three linear forms holds for any Δ-triple (i, j, k). In each coordinate, this vector relation yields a syzygy among the Wachspress quadratics and cubics.*

Proof. Using the vector identities above we can write

$$\begin{aligned}
\pi\, w_{i,j,k} =& \pi(x)(\Lambda_i \cdot \Lambda_j \times \Lambda_k) \\
=& \Lambda_i \times (\pi(x) \times (\Lambda_j \times \Lambda_k)) + (\Lambda_j \times \Lambda_k)\Lambda_i \cdot \pi(x) \\
=& \Lambda_i \times [\Lambda_j(\pi(x) \cdot \Lambda_k) - \Lambda_k(\pi(x) \cdot \Lambda_j)] + (\Lambda_j \times \Lambda_k)q_i \\
=& \Lambda_i \times [\Lambda_j q_k - \Lambda_k q_j] + (\Lambda_j \times \Lambda_k)q_i \\
=& (\Lambda_i \times \Lambda_j)q_k - (\Lambda_i \times \Lambda_k)q_j + (\Lambda_j \times \Lambda_k)q_i.
\end{aligned}$$

Think of both sides of the equation in the statement of the theorem as column vectors of three forms. The top row writes $w_{i,j,k}$ times the first generator of the center $\mathcal{C}$ as a combination of the Wachspress quadrics q_i, q_j, and q_k. The second row expresses w_{ijk} times the second generator similarly and so on. □

Theorem 4.6 accounts for many of the syzygies in the first syzygy module of $\hat{I}$. In fact there is computational evidence supporting the next conjecture.

Conjecture 4.7. *The Koszul syzygies and those from Theorem 4.6 generate the first syzygy module of $\hat{I}$.*

It remains to investigate the higher syzygies. The crucial information about syzygies is displayed in Betti diagrams. Using the computer algebra system Singular [2], Betti diagrams for Wachspress varieties for n-gons with $n \leq 12$. Some examples of these diagrams are displayed in Table 4.1. It follows from the Auslander-Buchsbaum For-

Table 4.1: Betti diagrams for $n = 5, 6$, and 7, respectively

	0	1	2
0 :	1	-	-
1 :	-	2	-
2 :	-	-	1

	0	1	2	3
0:	1	-	-	-
1:	-	3	-	-
2:	-	1	6	3

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	4	-	-	-
2:	-	4	21	20	6

mula [1] that a variety is Cohen-Macaulay if the length of its minimal free resolution is equal to its codimension. Thus, the Betti diagrams in Table 4.1 are evidence for the following conjecture.

Conjecture 4.8. *Wachspress varieties are Cohen-Macaulay.*

4.3 Examples for Small N

We end this Chapter with a close look at Wachspress varieties for N-gons with $N = 3, 4, 5, 6$, and 7.

Example 4.9. Let $N = 3$. The Wachspress coordinates for a triangle are $b_1 = \ell_2$, $b_2 = \ell_3$, and $b_3 = \ell_1$. Since the linear forms ℓ_i cut out the edges of a triangle, the Wachspress map $\beta = [\ell_2, \ell_3, \ell_1]$ is an automorphism of $\mathbb{P}^2$. Hence, $\mathcal{W} = \mathbb{P}^2$ and the center of projection $\mathcal{C}$ is empty.

Example 4.10. Let $N = 4$. The Wachspress coordinates for a quadrilateral are $b_1 = \ell_2\ell_3$, $b_2 = \ell_3\ell_4$, $b_3 = \ell_1\ell_4$, and $b_4 = \ell_1\ell_2$. In this case $\mathcal{W}$ is cut out by one Wachspress quadratic and, thus, is a quadric surface in $\mathbb{P}^3$. The center $\mathcal{C}$ is a point lying on $\mathcal{W}$. The adjoint curve $\mathcal{A}$ is a line through the two base points and is contracted to the center point by the Wachspress map β.

Example 4.11. Let $N = 5$. the Wachspress coordinates for a pentagon are $b_1 = \ell_2\ell_3\ell_4$, $b_2 = \ell_3\ell_4\ell_5$, $b_3 = \ell_1\ell_4\ell_5$, $b_4 = \ell_1\ell_2\ell_5$, and $b_5 = \ell_1\ell_2\ell_3$. There are five Wachspress quadratics but only two are linearly independent, and there are no Wachspress cubics. In this case $\mathcal{W}$ is cut out by two quadratics in $\mathbb{P}^4$. The center of projection $\mathcal{C}$ is a line contained in $\mathcal{W}$. It is the image of the adjoint curve under β in this case. The adjoint curve for the pentagon is the unique conic through the five base points and is mapped to the center line by β.

Example 4.12. Let $N = 6$. The Wachspress coordinates for a hexagon are each a product of four linear forms. This is the first case where $\mathcal{W}$ is not cut out in degree two. The variety $\mathcal{W}$ is cut out by three Wachspress quadratics and one Wachspress cubic which in this case happens to be a binomial. The Wachpress cubic has the simple form $\alpha_2\alpha_4\alpha_6\, x_1x_3x_5 - \alpha_1\alpha_3\alpha_5\, x_2x_4x_6$. This is the first case where $\mathcal{C}$ is not contained in $\mathcal{W}$. The center is a two-plane that meets $\mathcal{W}$ in a degree three curve that is the image of cubic adjoint curve through the nine base points. The reducibility of the intersection $\mathcal{C} \cap \mathcal{W}$ in [7] stems from the hexagon in that example having three parallel sets of edges. In this case three of the base points are collinear, each lying on the line at infinity, leading to a reducible adjoint curve through the nine base points.

Example 4.13. Let $N = 7$. The Wachspress coordinates for a heptagon are each a product of five linear forms. The Wachspress variety is cut out by four quadratics and four cubics in $\mathbb{P}^6$. The center $\mathcal{C}$ is a three plane meeting $\mathcal{W}$ in the image $\beta(\mathcal{A})$ of the adjoint curve interpolating the fourteen base points.

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