SYNTHESIS AND DESIGN OF PID CONTROLLERS

A Dissertation

by

HAO XU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2004

Major Subject: Electrical Engineering

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ABSTRACT

Synthesis and Design of PID Controllers. (December 2004)

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This dissertation presents research results on the synthesis and design of PID controllers for discrete-time systems and time-delayed systems. By using bilinear transformation and orthogonal transformation, earlier research results obtained in the continuous-time case are extended to discrete-time situation. The complete set of stabilizing PID controllers for the discrete-time systems is thus obtained. Moreover, this set remains to be a union of convex sets when one particular parameter is fixed. Thus a method to design robust and non-fragile digital PID controllers is proposed by following a similar design procedure for the continuous-time systems. In order to find the stabilizing controller set for systems with time-delays, the relationship between the Nyquist Criterion and Pontryagin's theory is investigated. The conditions under which one can correctly apply the Nyquist Criterion to time-delayed systems are derived. Then, the complete set of stabilizing PID controllers for arbitrary order LTI systems with time-delay up to a given value is obtained. Furthermore, the stability issue of a system with fixed-delay is also studied and a formula which provides complete knowledge of the distribution of the closed-loop poles is presented. Based on this formula, stabilizing P and PI controller sets for the system with fixed-delay can be computed.

To My Parents Ruixia Mao, Yongmao Xu and My Wife Xiruo Wang

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CHAPTER I

INTRODUCTION

This dissertation will develop methods to produce the set of all stabilizing digital PID controllers for a given Linear Time-Invariant (LTI) discrete-time plant and the set of all stabilizing PID controllers for an arbitrary order LTI plant with time-delay.

A. Background

Although many advanced control strategies have been developed over the last several decades, most real control systems in the world are operated by PID (Proportional-Integral-Derivative) controllers. In fact, more than 95% of the controllers used in process control applications are of the PID type [1]. Some of the reasons that PID controllers are so widely used in industry are its simple structure (fixed, low order), robustness to modeling errors, relatively good tracking and disturbance rejection. Despite the popularity of PID controllers, as a result of the gap rising between control theory and control engineering practice since the late 1950's [2], the theory related to PID designs did not receive much consideration until recently. Empirical techniques like Ziegler-Nichols tuning method are still used in most of the industrial PID designs while some of those techniques are known to give poor results in many cases [3, 4].

In an effort to bridge the gap between control theory and practice, in [2], the set of all stabilizing PID controllers for a given LTI plant described by a rational transfer function was computed. This was the first step to design an optimal PID controller. During this process, a generalized Hermite-Biehler Theorem was derived and used to compute the controller set. It turned out that the resulting set has

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some nice properties. For a given proportional gain k_p, the stabilizing set in the space of the two other parameters (k_i-k_d space) is a union of convex sets defined by groups of linear inequalities. Thus, the design of the optimal PID controller became a linear constrained optimization problem. In addition, the knowledge of this complete stabilizing set can be used to avoid the choice of controllers that are fragile. Several different designs, such as H_2, H_∞ optimum designs and robust non-fragile design, were carried out using that set.

B. Problems

Naturally, the above result should be extended to the following two cases:

1. discrete-time systems, and
2. time-delay systems.

For the first case, the reason we should consider discrete-time systems is the fact that the implementation of the PID is now based on a digital design [5]. However, even with a discrete-time version of the generalized Hermite-Biehler Theorem, in [2], only the constant gain case was solved. In this dissertation, instead of directly applying the generalized Hermite-Biehler Theorem, we convert the discrete-time problem to a continuous-time problem by using the bilinear transformation and then use the continuous-time generalized Hermite-Biehler Theorem to solve it.

For the second case, since almost all plants encountered in process control contain time-delays, finding the complete set of PID controllers that stabilize a given plant with time-delay is of considerable importance, both from the point of view of theory and practice. However, the synthesis results proposed in [2] cannot be applied directly to plants with time-delay because the generalized Hermite-Biehler Theorem presented there is for plants with rational transfer functions. Motivated by this, in [6], a version of Hermite-Biehler Theorem applicable to quasi polynomials [7, 8, 9] was used to compute the set of stabilizing PID parameters for a given first-order plant with time-delay. The resulting set is a trapezoid, a triangle or a quadrilateral in k_i-k_d space for different k_p's. Although this result was a breakthrough, the approach does not readily extend to the case of higher order plants with time-delay.

On the other hand, Nyquist Criterion ([10] in [11]) has often been used to *analyze* arbitrary order plants with time-delay. Its graphical simplicity provides a promising tool for attacking the *synthesis* problem of PID controllers. However, unlike Pontryagin's theory, the generalization of the Nyquist Criterion presented in the literature [10] lacks solid theoretical justification. This is because the proof of the generalization given in [10] may be inappropriate if the closed-loop system has an unbounded number of right half plane poles. In this dissertation, the conditions under which one can use the Nyquist Criterion are derived based on Pontryagin's theorems. Then a method to find the complete set of PID controllers to stabilize a given arbitrary order plant with time-delay is developed. As a starting point to design PID controllers for plants with interval delays or embedded delays, the complete set of stabilizing P, PI controllers for a plant with fixed-delay is also computed.

CHAPTER II

PREVIOUS RESULTS

In this chapter, we recall previous results on the computation of the complete set of stabilizing Proportional, Proportional-Integral (PI) and PID controllers for continuous-time systems without time-delay. These results can be found in [12, 13, 2, 14].

The system considered here is a simple feedback control system shown in Fig. 1. Here $C(s)$ is the controller while $G(s)$ is the plant with

$$G(s) = \frac{N(s)}{D(s)}.$$

$N(s)$ and $D(s)$ are coprime polynomials.

A. Stabilization Using P, PI Controllers

For Proportional controller

$$C(s) = k_p,$$

the closed-loop characteristic polynomial is

$$\delta(s, k_p) = D(s) + k_p N(s). \tag{2.1}$$

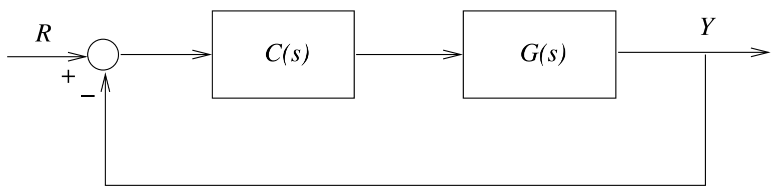

Fig. 1. Feedback control system.

Suppose $N(s)$ and $D(s)$ have degrees m and n respectively with $m \leq n$. Let $N(s)$ and $D(s)$ have the following even-odd decompositions:

$$\begin{aligned} N(s) &= N_e(s^2) + sN_o(s^2) \\ D(s) &= D_e(s^2) + sD_o(s^2). \end{aligned}$$

Define

$$N^*(s) = N(-s) = N_e(s^2) - sN_o(s^2).$$

Multiply both sides of (2.1) by $N^*(s)$ and substitute $s = j\omega$, we have

$$\delta^*(j\omega, k_p) = \delta(j\omega, k_p)N^*(s) = p(\omega, k_p) + jq(\omega),$$

where

$$\begin{aligned} p(\omega, k_p) &= p_1(\omega) + k_p p_2(\omega) \\ p_1(\omega) &= D_e(-\omega^2)N_e(-\omega^2) + \omega^2 D_o(-\omega^2)N_o(-\omega^2) \\ p_2(\omega) &= N_e(-\omega^2)N_e(-\omega^2) + \omega^2 N_o(-\omega^2)N_o(-\omega^2) \\ q(\omega) &= \omega[N_e(-\omega^2)D_o(-\omega^2) - D_e(-\omega^2)N_o(-\omega^2)]. \end{aligned}$$

Also define

$$\begin{aligned} p_f(\omega, k_p) &= \frac{p(\omega, k_p)}{(1+\omega^2)^{(m+n)/2}} \\ q_f(\omega) &= \frac{q(\omega)}{(1+\omega^2)^{(m+n)/2}}. \end{aligned}$$

The statement of the result requires the introduction of the following definitions.

Definition 1 *Let m, n and $q_f(\omega)$ be as already defined. Let*

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{l-1}$$

be the real, non-negative, distinct finite zeroes of $q_f(\omega)$ *with odd multiplicity. Define a sequence of numbers* $i_0,\ i_1,\ i_2,\ \cdots,\ i_l$ *as follows:*

1. *If* $N^*(j\omega_t) = 0$ *for some* $t = 1, 2, \cdots, l-1$*, then define*

$$i_t = 0;$$

2. *If* $N^*(s)$ *has a zero of multiplicity* k_n *at the origin, then define*

$$i_0 = \mathrm{sgn}[p_{1f}^{(k_n)}(0)]$$

where

$$p_{1f}(\omega) = \frac{p_1(\omega)}{(1+\omega^2)^{(m+n)/2}};$$

3. *For all other* $t = 0,\ 1,\ 2, \cdots, l$,

$$i_t \in \{-1,\ 1\}.$$

With above definition of i_t*, we define the set* A *as*

$$A := \begin{cases} \{\{i_0, i_1, \cdots, i_l\}\} & \textit{if } n+m \textit{ is even} \\ \{\{i_0, i_1, \cdots, i_{l-1}\}\} & \textit{if } n+m \textit{ is odd.} \end{cases}$$

Definition 2 *Let* m*,* n*,* $q(\omega)$ *and* $q_f(\omega)$ *be as already defined. Let*

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{l-1}$$

be the real, non-negative, distinct finite zeroes of $q_f(\omega)$ *with odd multiplicity. Also define* $\omega_l = \infty$*. For each string*

$$\mathcal{I} = \{i_0,\ i_1,\ \cdots\}$$

in A*, let* $\gamma(\mathcal{I})$ *denote the "imaginary signature" associated with the string* $\mathcal{I}$ *defined* *by*

$$\gamma(\mathcal{I}) := \begin{cases} [i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{l-1} 2i_{l-1} + (-1)^l i_l] \cdot (-1)^{l-1} \mathrm{sgn}[q(\infty)] \\ \quad \textit{for } m+n \textit{ even} \\ [i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{l-1} 2i_{l-1}] \cdot (-1)^{l-1} \mathrm{sgn}[q(\infty)] \\ \quad \textit{for } m+n \textit{ odd} \end{cases}$$

Definition 3 *The set of strings in A with a prescribed imaginary signature $\gamma = \psi$ is denoted by $A(\psi)$. The feasible strings for the Proportional controller stabilization problem is defined as*

$$F^* = A(n - (l(N(s)) - r(N(s)))),$$

where $l(N(s))$, $r(N(s))$ are the number of roots of $N(s)$ in the open left half and open right half planes, respectively.

Now we state the final result.

Theorem 1 *[2] The Proportional controller feedback stabilization problem is solvable for a given plant with transfer function $G(s)$ if and only if the following conditions hold:*

1. *F^* is not empty where F^* is as already defined, i.e., at least one feasible string exists and*

2. *There exists a string $\mathcal{I} = \{i_0, i_1, \cdots\} \in F^*$ such that*

$$\max_{i_t \in \mathcal{I}, i_t > 0} \left[-\frac{1}{G(j\omega_t)} \right] < \min_{i_t \in \mathcal{I}, i_t < 0} \left[-\frac{1}{G(j\omega_t)} \right]$$

where $\omega_0, \omega_1, \omega_2, \cdots$ are as already defined. Furthermore, if the above condition is satisfied by the feasible strings $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_s \in F^*$, then the set of all stabilizing

Proportional gains is given by $K = \cup_{r=1}^{s} K_r$*, where*

$$K_r = \left(\max_{i_t \in \mathcal{I}, i_t > 0} \left[-\frac{1}{G(j\omega_t)} \right], \min_{i_t \in \mathcal{I}, i_t < 0} \left[-\frac{1}{G(j\omega_t)} \right] \right), \ r = 1, 2, \cdots, s.$$

For PI controller

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}.$$

For each fixed k_p, it becomes a one parameter case. Then we can use the above method to obtain the stabilizing region of k_i for that k_p. By sweeping k_p, the complete stabilizing PI controller set can be obtained.

B. Stabilization Using PID Controllers

When the controller $C(s)$ is a PID controller, that is

$$C(s) = k_p + \frac{k_i}{s} + k_d s = \frac{k_d s^2 + k_p s + k_i}{s},$$

the closed-loop characteristic polynomial becomes

$$\delta(s, k_p, k_i, k_d) = sD(s) + (k_d s^2 + k_p s + k_i)N(s). \tag{2.2}$$

Suppose the degree of $\delta(s, k_p, k_i, k_d)$ is n and the degree of $N(s)$ is m. As before, with the same even-odd decompositions of $D(s)$ and $N(s)$, we multiply both sides of (2.2) by $N^*(s) = N(-s)$ and substitute $s = j\omega$ to obtain

$$\delta^*(j\omega, k_p, k_i, k_d) = \delta(s, k_p, k_i, k_d)N^*(j\omega) = p(\omega, k_i, k_d) + jq(\omega, k_p),$$

where

$$\begin{aligned} p(\omega, k_i, k_d) &= p_1(\omega) + (k_i - k_d\omega^2)p_2(\omega) \\ q(\omega, k_p) &= q_1(\omega) + k_p q_2(\omega) \end{aligned}$$

$$
\begin{aligned}
p_1(\omega) &= -\omega^2[N_e(-\omega^2)D_o(-\omega^2) - D_e(-\omega^2)N_o(-\omega^2)] \\
p_2(\omega) &= N_e(-\omega^2)N_e(-\omega^2) + \omega^2 N_o(-\omega^2)N_o(-\omega^2) \\
q_1(\omega) &= \omega[D_e(-\omega^2)N_e(-\omega^2) + \omega^2 D_o(-\omega^2)N_o(-\omega^2)] \\
q_2(\omega) &= \omega[N_e(-\omega^2)N_e(-\omega^2) + \omega^2 N_o(-\omega^2)N_o(-\omega^2)]
\end{aligned}
$$

Also define

$$
\begin{aligned}
p_f(\omega, k_i, k_d) &= \frac{p(\omega, k_i, k_d)}{(1+\omega^2)^{(m+n)/2}} \\
q_f(\omega, k_p) &= \frac{q(\omega, k_p)}{(1+\omega^2)^{(m+n)/2}}.
\end{aligned}
$$

Now, for each fixed k_p, we have following definitions.

Definition 4 *Let m, n and $q_f(\omega, k_p)$ be as already defined. For a given fixed k_p, let*

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{l-1}$$

be the real, non-negative, distinct finite zeroes of $q_f(\omega, k_p)$ with odd multiplicity. Define a sequence of numbers $i_0,\ i_1,\ i_2,\ \cdots,\ i_l$ as follows:

1. *If $N^*(j\omega_t) = 0$ for some $t = 1, 2, \cdots, l-1$, then define*

$$i_t = 0;$$

2. *If $N^*(s)$ has a zero of multiplicity k_n at the origin, then define*

$$i_0 = \mathrm{sgn}[p_{1f}^{(k_n)}(0)]$$

where

$$p_{1f}(\omega) = \frac{p_1(\omega)}{(1+\omega^2)^{(m+n)/2}};$$

3. For all other $t = 0,\ 1,\ 2, \cdots, l,$

$$i_t \in \{-1,\ 1\}.$$

With above definition of i_t*, we define the set* A_{k_p} *as*

$$A_{k_p} := \begin{cases} \{\{i_0, i_1, \cdots, i_l\}\} & \text{if } n+m \text{ is even} \\ \{\{i_0, i_1, \cdots, i_{l-1}\}\} & \text{if } n+m \text{ is odd.} \end{cases}$$

Definition 5 *Let* m*,* n*,* $q(\omega, k_p)$ *and* $q_f(\omega, k_p)$ *be as already defined. For a given fixed* k_p*, let*

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{l-1}$$

be the real, non-negative, distinct finite zeroes of $q_f(\omega, k_p)$ *with odd multiplicity. Also define* $\omega_l = \infty$*. For each string*

$$\mathcal{I} = \{i_0,\ i_1,\ \cdots\}$$

in A_{k_p}*, let* $\gamma(\mathcal{I})$ *denote the "imaginary signature" associated with the string* $\mathcal{I}$ *defined by*

$$\gamma(\mathcal{I}) := \begin{cases} [i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{l-1}2i_{l-1} + (-1)^l i_l] \cdot (-1)^{l-1}\operatorname{sgn}[q(\infty, k_p)] \\ \quad \text{for } m+n \text{ even} \\ [i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{l-1}2i_{l-1}] \cdot (-1)^{l-1}\operatorname{sgn}[q(\infty, k_p)] \\ \quad \text{for } m+n \text{ odd} \end{cases}$$

Definition 6 *The set of strings in* A_{k_p} *with a prescribed imaginary signature* $\gamma = \psi$ *is denoted by* $A_{k_p}(\psi)$*. For a given fixed* k_p*, the feasible strings for the PID controller stabilization problem is defined as*

$$F^*_{k_p} = A_{k_p}(n - (l(N(s)) - r(N(s)))).$$

Following is the main result.

Theorem 2 *[2] The PID controller feedback stabilization problem, with a given fixed k_p, is solvable for a given plant with transfer function $G(s)$ if and only if the following conditions hold:*

1. *$F^*_{k_p}$ is not empty where $F^*_{k_p}$ is as already defined, i.e., at least one feasible string exists and*
2. *There exists a string $\mathcal{I} = \{i_0, i_1, \cdots\} \in F^*_{k_p}$ and values of k_i and k_d such that $\forall t = 0, 1, 2, \cdots$ for which $N^*(j\omega_t) \neq 0$*

$$p(\omega_t, k_i, k_d)i_t > 0, \tag{2.3}$$

where $p(\omega, k_i, k_d)$ is as already defined. Furthermore, if there exist values of k_i and k_d such that the above condition is satisfied for the feasible strings $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_s \in F^*_{k_p}$, then the set of stabilizing (k_i, k_d) values corresponding to the fixed k_p is the union of the (k_i, k_d) values satisfying (2.3) for $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_s$.

Remark 1 *The admissible set for (2.3) is convex since the constraint set is linear. Thus, for each fixed k_p, the stabilizing controllers set in (k_i, k_d) space is a union of convex sets.*

C. Summary

In this chapter, previous research results on the characterization of stabilizing P, PI and PID controllers for a linear time-invariant continuous-time delay-free plant have been recalled. In the next several chapters of the dissertation, they are used as a starting point for computing the stabilizing controller set for discrete-time and time-delayed plant.

CHAPTER III

STABILIZING DIGITAL PID CONTROLLERS*

In [2], we obtained a complete characterization of the set of all stabilizing PID controller parameters for a continuous-time plant of arbitrary order by using the Generalized Hermite-Biehler Theorem. Extension of this result to the discrete-time case posed several problems [2]. First, a discrete-time version of the Generalized Hermite-Biehler Theorem applicable to rational functions had to be developed; even with this result in hand, only constant gain stabilization results could be obtained. In this chapter, we show that the discrete-time analogues of our earlier results on continuous-time PID stabilization can be obtained by applying our earlier results to a bilinearly transformed discrete-time system. It is remarkable to note that the linear programming nature of the continuous-time solution is preserved under the bilinear transformation and a suitable reparametrization.

The chapter is organized as follows. In Section A, we present some general results that can be used to ascertain the stability of a closed loop discrete-time system using a bilinear transformation. These results are specialized to the case of proportional (P), proportional-integral (PI) and PID controllers in Section B. Some illustrative examples are presented in Section C. Section D concludes this chapter.

A. Closed Loop Stability via the Bilinear Transformation

In the analysis of discrete-time systems, the problem of determining the Schur stability of a given polynomial can be converted to the problem of determining the Hurwitz stability of another polynomial using what is called a bilinear transformation [15]. There are several different bilinear transformations that can be used. Let us focus on a particular bilinear transformation $\mathcal{W}$ defined as follows. Given any polynomial $X(z)$,

$$\mathcal{W}(X(z)) = X(\frac{w+1}{w-1}) = Y(w)$$

where $Y(w)$ is a rational function of w.

As we will show in Lemma 1, the bilinear transformation $\mathcal{W}$ maps the roots of $X(z)$ located inside (on or outside) the unit circle to the zeros of $Y(w)$ in the open LHP (on the imaginary axis or in the open RHP). Additionally, a root or roots of $X(z)$ at $z = 1$ is mapped to a zero or zeros of $Y(w)$ at $w = \infty$. Thus the Schur stability of a polynomial $X(z)$ is equivalent to the Hurwitz stability of the numerator of $Y(w)$, provided the numerator and denominator of $Y(w)$ are of the same degree. Furthermore, provided $X(z)$ has no roots at $z = 1$, the root distribution of $X(z)$ with respect to the unit circle is identical to the root distribution of the numerator of $Y(w)$ with respect to the imaginary axis. These facts will play an important role in the sequel.

Let

$$\delta_z(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a given polynomial of degree n. Then $\mathcal{W}(\delta_z(z))$ is given by

$$\mathcal{W}\{\delta_z(z)\} = \frac{\delta(w)}{(w-1)^n} \tag{3.1}$$

where $\delta(w) = b_m w^m + b_{m-1} w^{m-1} + \cdots + b_1 w + b_0$ is a polynomial in w of degree $m \leq n$.

Lemma 1 *Let n_i, n_o, n_b be the numbers of roots of $\delta_z(z)$ located inside, outside and on the unit circle respectively. Furthermore, let m_l, m_r be the numbers of roots of $\delta(w)$ located in the open left half and open right half planes, and let m_b be the number* *of roots of $\delta(\omega)$ located on the imaginary axis. Then, we have*

1. $n - m =$ *the number of roots of* $\delta_z(z)$ *at* $z = 1$;

2. $n_i = m_l, n_o = m_r$;

3. $n_b = m_b + (n - m)$.

Proof: Let us rewrite the polynomial $\delta_z(z)$ in the factored form

$$\delta_z(z) = K \prod_{i=1}^{n} (z - z_i)$$

where z_i $i = 1, 2, \cdots, n$ are the roots of $\delta_z(z)$. Clearly,

$$\mathcal{W}(\delta_z(z)) = K \prod_{i=1}^{n} \mathcal{W}(z - z_i).$$

Let us now concentrate on the factor $\mathcal{W}(z - z_i)$. First, let us assume that $z_i \neq 1$. Then, from the definition of $\mathcal{W}$, we have

$$\begin{aligned} \mathcal{W}\{z - z_i\} &= \frac{w+1}{w-1} - z_i \\ &= \frac{(1 - z_i)w + 1 + z_i}{w - 1} \\ &= \frac{(1 - z_i)(w - \frac{z_i + 1}{z_i - 1})}{w - 1} \\ &= c_i \frac{w - w_i}{w - 1} \end{aligned} \tag{3.2}$$

where $c_i = 1 - z_i$, and $w_i = (z_i + 1)/(z_i - 1)$. If we assume $z_i = x_i + jy_i$, then

$$\begin{aligned} w_i &= \frac{z_i + 1}{z_i - 1} \\ &= \frac{x_i + 1 + jy_i}{x_i - 1 + jy_i} \\ &= \frac{x_i^2 + y_i^2 - 1}{(x_i - 1)^2 + y_i^2} - j \frac{2y_i}{(x_i - 1)^2 + y_i^2} \end{aligned} \tag{3.3}$$

Now, we consider the following three cases:

1. z_i is inside the unit circle.

   Then $x_i^2 + y_i^2 < 1$ so that from (3.3), it follows that $\mathrm{Re}[w_i] < 0$.

2. z_i is outside the unit circle.

   In this case, $x_i^2 + y_i^2 > 1$ so that from (3.3), it follows that $\mathrm{Re}[w_i] > 0$.

3. z_i is located on the unit circle.

   In this case, $x_i^2 + y_i^2 = 1$ so that from (3.3), it follows that $\mathrm{Re}[w_i] = 0$ and w_i lies on the imaginary axis.

Let us now consider the case $z_i = 1$. In this case, direct computation yields

$$\mathcal{W}\{z - z_i\} = \frac{2}{w - 1}.$$

Thus in this case, the numerator of $\mathcal{W}(z - z_i)$ has degree one less than its denominator. The proof of the lemma is obtained by applying the above observations to each of the factors $z - z_i$. ♣

We next examine how Lemma 1 can be used to study the closed loop stability of a discrete-time system. Suppose that the plant and the controller in a standard unity feedback discrete-time system are described by $P_z(z) = N_z(z)/D_z(z)$ and $C_z(z) = B_z(z)/A_z(z)$ respectively where $N_z(z)$, $D_z(z)$, $B_z(z)$, $A_z(z)$ are polynomials in z. Hence the characteristic equation of the closed loop system is given by

$$\delta_z(z) = A_z(z)D_z(z) + B_z(z)N_z(z). \tag{3.4}$$

Suppose that the polynomials $A_z(z)$, $B_z(z)$, $D_z(z)$, $N_z(z)$ have degrees n_c, m_c, n, m respectively. Furthermore, let us assume that $P_z(z)$ and $C_z(z)$ are proper so that

$$m_c \leq n_c, m \leq n. \tag{3.5}$$

Now applying the bilinear transformation to $P_z(z)$ and $C_z(z)$ we obtain

$$P(w) = \mathcal{W}\{P_z(z)\} = \frac{N(w)}{D(w)}$$
$$C(w) = \mathcal{W}\{C_z(z)\} = \frac{B(w)}{A(w)}$$

where, $A(w)$, $B(w)$, $D(w)$ and $N(w)$ are polynomials in w, and $P(w)$, $C(w)$ represent the new plant and controller in the w-domain.

Similarly, applying the bilinear transformation to $\delta_z(z)$ and taking into account the degree relationships in (3.5), we obtain

$$\begin{aligned} \mathcal{W}\{\delta_z(z)\} &= \mathcal{W}\{A_z(z)\}\mathcal{W}\{D_z(z)\} + \mathcal{W}\{B_z(z)\}\mathcal{W}\{N_z(z)\} \\ &= \frac{A_0(w)}{(w-1)^{n_c}} \cdot \frac{D_0(w)}{(w-1)^{n}} + \frac{B_0(w)}{(w-1)^{m_c}} \cdot \frac{N_0(w)}{(w-1)^{m}} \\ &= \frac{A_0(w) \cdot D_0(w) + B_0(w)(w-1)^{n_c-m_c} \cdot N_0(w)(w-1)^{n-m}}{(w-1)^{n+n_c}}. \end{aligned} \tag{3.6}$$

The following relationships are easily verified:

$$\begin{aligned} A(w) &= A_0(w), \\ B(w) &= B_0(w)(w-1)^{n_c-m_c}, \\ D(w) &= D_0(w), \\ N(w) &= N_0(w)(w-1)^{n-m}. \end{aligned}$$

Hence, the numerator of (3.6) can be expressed as

$$\delta(w) = A(w)D(w) + B(w)N(w). \tag{3.7}$$

This allows us to state the following result.

Lemma 2 *Suppose $\delta_z(z)$ in (3.4) has no roots at $z = 1$. Then the ($P_z(z)$,$C_z(z)$) closed loop system in the z-domain is Schur stable if and only if the $(P(w), C(w))$*

closed loop system in the w-domain is Hurwitz stable.

In the next section, we will make use of the above lemma to characterize the set of stabilizing P, PI and PID gains for a given discrete-time plant.

B. Computation of the Set of Stabilizing Gains: P, PI and PID

In this section, we derive discrete-time P, PI and PID stabilization results by applying our earlier approach developed for the continuous-time case [2] to appropriate bilinearly transformed systems in the w-domain.

To this end, we consider discrete-time P, PI and PID controllers defined by

$$\begin{aligned}
\mathrm{P}: C_z(z) &= k_p, \\
\mathrm{PI}: C_z(z) &= k_p + k_i \frac{1}{1-z^{-1}} = \frac{(k_p+k_i)z - k_p}{z-1}, \\
\mathrm{PID}: C_z(z) &= k_p + k_i \frac{1}{1-z^{-1}} + k_d \frac{1-2z^{-1}+z^{-2}}{1-z^{-1}} \\
&= \frac{(k_p+k_i+k_d)z^2 - (k_p+2k_d)z + k_d}{z^2 - z}.
\end{aligned}$$

Also consider their w-domain counterparts obtained by substituting $z = \frac{w+1}{w-1}$:

$$\begin{aligned}
\mathrm{P}: \frac{B(w)}{A(w)} &= \frac{k_p}{1}, \\
\mathrm{PI}: \frac{B(w)}{A(w)} &= \frac{k_i w + 2k_p + k_i}{2}, \\
\mathrm{PID}: \frac{B(w)}{A(w)} &= \frac{k_i w^2 + 2(k_p+k_i)w + 2k_p + k_i + 4k_d}{2w+2}.
\end{aligned}$$

According to (3.7), the corresponding w-domain closed loop characteristic polynomials are:

$$\begin{aligned}
\mathrm{P}: \delta(w) &= D(w) + k_p N(w), \\
\mathrm{PI}: \delta(w) &= 2D(w) + (k_i w + 2k_p + k_i)N(w),
\end{aligned}$$

$$\text{PID} : \delta(w) \;=\; (2w+2)D(w) + [k_i w^2 + 2(k_p + k_i)w + 2k_p + k_i + 4k_d]N(w).$$

In view of Lemma 2, it follows that as long as $\delta_z(z)$ has no roots at $z = 1$, the Hurwitz stability of each of the above w-domain polynomials will guarantee the Schur stability of the corresponding closed loop system. The pathological case of $\delta_z(z)$ having a root at $z = 1$ arises when a PI or a PID controller is being used and the plant has a zero at $z = 1$. However, in such a situation, there is an unstable pole-zero cancellation and so the discrete-time closed loop system is anyway internally unstable, regardless of the controller parameter values. Thus these cases can be handled by concluding instability directly without having to go through any bilinear transformation or subsequent procedures in the w-domain. For all other cases, we proceed as follows to find the controller parameter values that make $\delta(w)$ Hurwitz stable.

As in [2], in order to separate the parameters and prevent them from all showing up in both the real and imaginary parts of the w-domain characteristic polynomial, we multiply (3.7) by the factor $N(-w)$ to obtain

$$\delta^*(w) = N(-w)\delta(w).$$

We next provide the particular expressions for $\delta^*(w)$ corresponding to each of the three controllers being considered here. In the expressions to follow, the subscripts e and o indicate the polynomials corresponding to the even and odd parts of a polynomial, e.g. $N(w) = N_e(w^2) + wN_o(w^2)$, etc. For polynomials with two subscripts, the second subscript p, i or d indicates that the term represented by that polynomial depends on k_p, k_i or k_d; a second subscript of c indicates that the term represented by that polynomial is independent of k_p, k_i and k_d. Using this notation, we obtain the following expressions for $\delta^*(w)$:

(i) for a Proportional controller (P),

$$\begin{aligned}\delta^*(w) &= \delta_e^*(w^2, k_p) + w\delta_o^*(w^2) \\ &= [k_p\delta_{ep}(w^2) + \delta_{ec}(w^2)] + w\delta_{oc}(w^2),\end{aligned}$$

where

$$\begin{aligned}\delta_{ep}(w^2) &= N_e^2 - w^2 N_o^2 \\ \delta_{ec}(w^2) &= D_e N_e - w^2 D_o N_o \\ \delta_{oc}(w^2) &= D_o N_e - D_e N_o;\end{aligned}$$

(ii) for a Proportional-Integral controller (PI),

$$\begin{aligned}\delta^*(w) &= \delta_e^*(w^2, k_p, k_i) + w\delta_o^*(w^2, k_i) \\ &= [k_p\delta_{ep}(w^2) + k_i\delta_{ei}(w^2) + \delta_{ec}(w^2)] + w[k_i\delta_{oi}(w^2) + \delta_{oc}(w^2)],\end{aligned}$$

where

$$\begin{aligned}\delta_{ep}(w^2) &= 2(N_e^2 - w^2 N_o^2) \\ \delta_{ei}(w^2) &= N_e^2 - w^2 N_o^2 \\ \delta_{ec}(w^2) &= 2(D_e N_e - w^2 D_o N_o) \\ \delta_{oi}(w^2) &= N_e^2 - w^2 N_o^2 \\ \delta_{oc}(w^2) &= 2(D_o N_e - D_e N_o);\end{aligned}$$

(iii) for a Proportional-Integral-Derivative controller (PID),

$$\begin{aligned}\delta^*(w) &= \delta_e^*(w^2, k_p, k_i, k_d) + w\delta_o^*(w^2, k_p, k_i) \\ &= [k_p\delta_{ep}(w^2) + k_i\delta_{ei}(w^2) + k_d\delta_{ed}(w^2) + \delta_{ec}(w^2)] \\ &\quad + w[k_p\delta_{op}(w^2) + k_i\delta_{oi}(w^2) + \delta_{oc}(w^2)],\end{aligned}$$

where

$$\begin{aligned}
\delta_{ep}(w^2) &= 2(N_e^2 - w^2 N_o^2) \\
\delta_{ei}(w^2) &= (1+w^2)(N_e^2 - w^2 N_o^2) \\
\delta_{ed}(w^2) &= 4(N_e^2 - w^2 N_o^2) \\
\delta_{ec}(w^2) &= 2(N_e D_e + w^2 N_e D_o - w^2 N_o D_e - w^2 N_o D_o) \\
\delta_{op}(w^2) &= 2(N_e^2 - w^2 N_o^2) \\
\delta_{oi}(w^2) &= 2(N_e^2 - w^2 N_o^2) \\
\delta_{oc}(w^2) &= 2(N_e D_e + N_e D_o - N_o D_e - w^2 N_o D_o).
\end{aligned}$$

Note that for a proportional controller, k_p appears only in the even part of $\delta^*(w)$ and so we can use the approach of [2] to obtain the set of all k_p that make $\delta(w)$ Hurwitz stable. Similarly, we note that for a PI controller, the appearance of k_i in both the even and odd parts of $\delta^*(w)$ does not affect our computation using the method proposed in [2] for the continuous-time PI controller. All that one has to do is to sweep over k_i and find the stabilizing set of k_p's at each stage.

1. Reparametrization for the PID Case

In the case of the PID controller, the situation is a little more involved. Now there are two parameters, k_p and k_i which appear in both the even and odd parts. To simplify matters, we proceed as follows. Note that since $\delta_{op}(w^2) = \delta_{oi}(w^2)$, we can combine k_p and k_i together by using the substitution

$$k_i = k_s - k_p.$$

With this substitution, we have

$$\begin{aligned}\delta^*(w) &= \delta'_e(w^2, k_p, k_s, k_d) + w\delta'_o(w^2, k_s)\\ &= [k_p\delta'_{ep}(w^2) + k_s\delta'_{es}(w^2) + k_d\delta'_{ed}(w^2) + \delta'_{ec}(w^2)]\\ &\quad +w[k_s\delta'_{os}(w^2) + \delta'_{oc}(w^2)], \end{aligned} \tag{3.8}$$

where

$$\begin{aligned}\delta'_{ep}(w^2) &= (1-w^2)(N_e^2 - w^2N_o^2)\\ \delta'_{es}(w^2) &= (1+w^2)(N_e^2 - w^2N_o^2)\\ \delta'_{ed}(w^2) &= 4(N_e^2 - w^2N_o^2)\\ \delta'_{ec}(w^2) &= 2(N_eD_e + w^2N_eD_o - w^2N_oD_e - w^2N_oD_o)\\ \delta'_{os}(w^2) &= 2(N_e^2 - w^2N_o^2)\\ \delta'_{oc}(w^2) &= 2(N_eD_e + N_eD_o - N_oD_e - w^2N_oD_o).\end{aligned}$$

From (3.8), it is clear that we can now proceed as in [2], i.e. fix k_s, then use linear programming to solve for the stabilizing values of k_p and k_d. Now

$$\begin{bmatrix} k_p \\ k_d \\ k_s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix},$$

i.e. the triple (k_p, k_d, k_s) is a linear transformation on the triple (k_p, k_i, k_d). Furthermore, this transformation is invertible. Thus, once the stabilizing values of (k_p, k_d, k_s) have been obtained, the stabilizing values of (k_p, k_i, k_d) can be obtained using the following inverse transformation:

$$\begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_p \\ k_d \\ k_s \end{bmatrix}. \tag{3.9}$$

C. Examples

Example 1 *Consider a Proportional controller to stabilize the discrete-time system* $N_z(z)/D_z(z)$ *where*

$$\begin{aligned} N_z(z) &= 100z^3 + 2z^2 + 3z + 11 \\ D_z(z) &= 100z^5 + 2z^4 + 5z^3 - 41z^2 + 52z + 70. \end{aligned}$$

Solution: Using the bilinear transformation, we obtain

$$\begin{aligned} N(w) &= 116w^5 + 34w^4 - 88w^3 - 300w^2 + 148w + 90 \\ D(w) &= 188w^5 + 46w^4 + 1880w^3 + 308w^2 + 652w + 126. \end{aligned}$$

Applying the method of [2] to the above w-domain plant, we found that the set of stabilizing k_p's is given by

$$k_p \in (-0.4178, -0.1263).$$

This agrees with the result obtained in Example 9.5.3 of [2] where a discrete-time Generalized Hermite-Biehler Theorem was used. ♣

Example 2 *Consider a Proportional-Integral controller to stabilize the discrete-time system* $N_z(z)/D_z(z)$ *where*

$$\begin{aligned} N_z(z) &= z + 1 \\ D_z(z) &= z^2 - 1.5z + 0.5. \end{aligned}$$

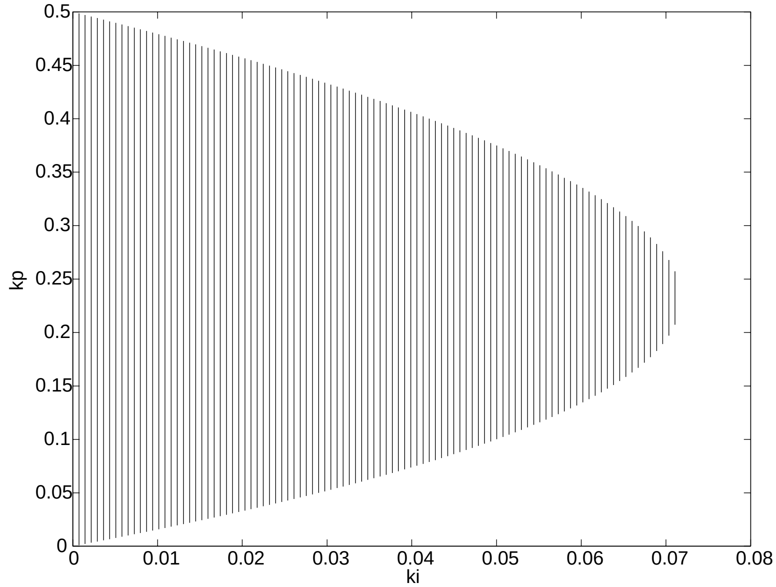

Fig. 2. The stabilizing region of (k_i, k_p).

Solution: Using the bilinear transformation, we obtain the w-domain plant $N(w)/D(w)$ where

$$\begin{aligned} N(w) &= 2w^2 - 2w \\ D(w) &= w + 3. \end{aligned}$$

As in the continuous-time case [2], we can determine the range of k_i values to be swept over by examining the odd part of $\delta^*(w)$. The range of k_i so determined is

$$0 < k_i < 0.0718.$$

For each value of k_i in this range, we obtained the set of stabilizing k_p values. The resulting stabilizing region is sketched in Fig. 2. ♣

Example 3 *Let us now use a PID controller to stabilize the same plant considered in Example 2.*

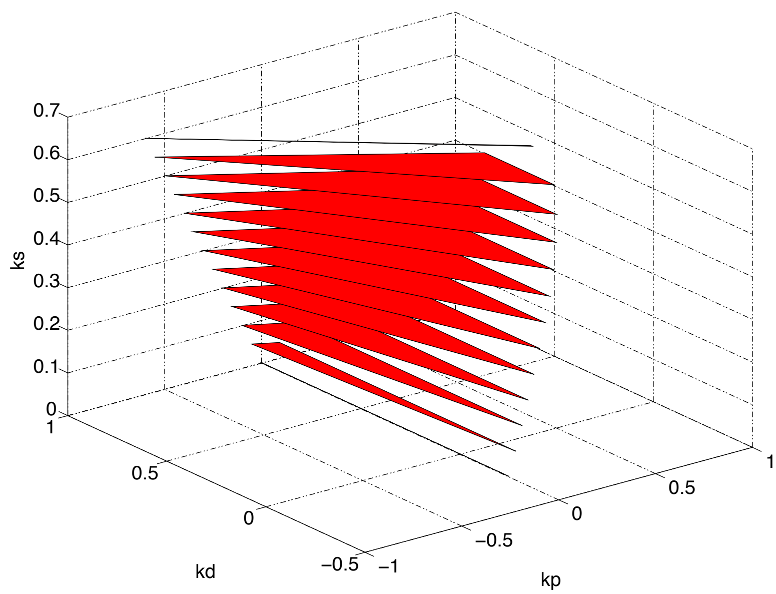

Fig. 3. The stabilizing region in the space of (k_p, k_d, k_s).

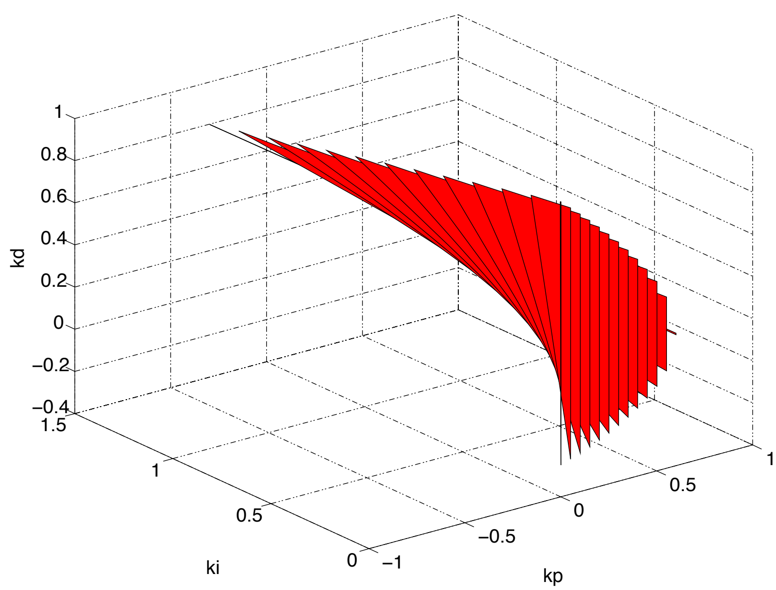

Fig. 4. The stabilizing region in the space of (k_p, k_i, k_d).

Solution: Fig. 3 shows the stabilizing regions in the space of (k_p, k_d, k_s), where $k_s = k_p + k_i$. After linear transformation as in (3.9), we obtained the stabilizing regions in the space of (k_p, k_i, k_d). These regions are sketched in Fig. 4. ♣

D. Summary

In this chapter, we have provided a complete solution to the problem of characterizing all PID gains that stabilize a given discrete-time plant of arbitrary order. The solution was obtained by applying our earlier continuous-time results to a bilinearly transformed system. This represents a significant advance over our earlier work [2] where, for the discrete-time case, only the constant gain problem could be tackled. For the PID case, after the reparametrization, the achieved set is also defined by a set of linear inequalities as in continuous-time case. In the next chapter, we will use this result to design the robust discrete-time PID controllers.

CHAPTER IV

DESIGN OF ROBUST NON-FRAGILE DIGITAL PID CONTROLLERS*

In previous chapter, we have obtained complete characterizations of the set of all stabilizing PID controllers for discrete-time plants [16] in addition to the continuous-time plants [2]. Also, in [14], it was shown that the results in [2] could be exploited to design robust and "non-fragile" PID controllers for continuous-time plants of the interval type. Such "non-fragile" designs can also be termed as controller-robust. In this chapter, we show how analogous results can be derived for the discrete-time case. The proposed approach makes use of a standard bilinear transformation followed by a special linear orthogonal one.

This chapter is organized as follows. In Section A, we develop a procedure for characterizing all stabilizing PID gains for discrete-time plants of both the fixed as well as the interval types. A method for designing non-fragile PID controllers for such plants is proposed in Section B. Section C contains an illustrative example and Section D summarizes this chapter.

A. Computation of Stabilizing PID Parameters for a Discrete-time Plant

For the discrete-time PID controller and Plant given in the previous chapter, we come to the point where the Hurwitz stability of the polynomial

$$\delta(w) = (2w+2)D(w) + [k_i w^2 + 2(k_p + k_i)w + 2k_p + k_i + 4k_d]N(w), \qquad (4.1)$$

decides the Schur stability of the discrete-time PID controlled system.

*©2004 IEEE. Reprinted, with permission, from "Plant-robust and controller-robust discrete-time PID design" by H. Xu, A. Datta and S. P. Bhattacharyya, *Proceedings of 2002 American Control Conference*, Vol. 5, pp. 3529-3533, May 2002.

As we did previously, consider the even-odd decomposition

$$N(w) = N_e(w^2) + wN_o(w^2).$$

Multiplying both sides of (4.1) by

$$N(-w) = N_e(w^2) - wN_o(w^2),$$

we obtain

$$\begin{aligned} \delta^*(w) &= N(-w)\delta(w) \\ &= \delta_e^*(w^2, k_p, k_i, k_d) + w\delta_o^*(w^2, k_p, k_i) \\ &= [k_p\delta_{ep}(w^2) + k_i\delta_{ei}(w^2) + k_d\delta_{ed}(w^2) + \delta_{ec}(w^2)] \\ &\quad +w[k_p\delta_{op}(w^2) + k_i\delta_{oi}(w^2) + \delta_{oc}(w^2)], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \delta_{ep}(w^2) &= 2(N_e^2 - w^2N_o^2) \\ \delta_{ei}(w^2) &= (1 + w^2)(N_e^2 - w^2N_o^2) \\ \delta_{ed}(w^2) &= 4(N_e^2 - w^2N_o^2) \\ \delta_{ec}(w^2) &= 2(N_eD_e + w^2N_eD_o - w^2N_oD_e - w^2N_oD_o) \\ \delta_{op}(w^2) &= 2(N_e^2 - w^2N_o^2) \\ \delta_{oi}(w^2) &= 2(N_e^2 - w^2N_o^2) \\ \delta_{oc}(w^2) &= 2(N_eD_e + N_eD_o - N_oD_e - w^2N_oD_o). \end{aligned}$$

1. Alternative Reparametrization of the PID Controllers

Observe from (4.2) that both the even and odd parts of $\delta^*(w)$ depend on at least two of the parameters k_p, k_i, k_d. Furthermore, from the fact that $\delta_{op}(w^2) = \delta_{oi}(w^2)$, we note that the coefficients of k_p and k_i appearing in the odd part of $\delta^*(w)$ are one and the same. Hence, we can reparametrize the PID parameters such that $k_p + k_i$ is defined to be a new parameter, say k_s. Thereafter, the approach developed in [2] for the continuous-time case can be used. This was the strategy followed in previous chapter. In this chapter, in addition, we would like to preserve the shape and size of the stabilizing regions. This can be achieved using the orthogonal transformation

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix}, \tag{4.3}$$

where $\theta \in [0, 2\pi)$. The corresponding inverse transformation is given by

$$\begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}. \tag{4.4}$$

To make $\cos\theta = \sin\theta$, we choose $\theta = \pi/4$ so that the above transformation and its inverse become:

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix} \tag{4.5}$$

and

$$\begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}. \tag{4.6}$$

Using (4.5), (4.2) can be rewritten as

$$\delta^*(w) \quad = \quad \delta'_e(w^2, k_1, k_2, k_3) + w\delta'_o(w^2, k_2)$$

$$
\begin{aligned}
&= [k_1\delta'_{e1}(w^2) + k_2\delta'_{e2}(w^2) + k_3\delta'_{e3}(w^2) + \delta'_{ec}(w^2)] \\
&\quad +w[k_2\delta'_{o2}(w^2) + \delta'_{oc}(w^2)],
\end{aligned}
$$

where

$$
\begin{aligned}
\delta'_{e1}(w^2) &= \frac{1}{\sqrt{2}}(1-w^2)(N_e^2 - w^2N_o^2) \\
\delta'_{e2}(w^2) &= \frac{1}{\sqrt{2}}(3+w^2)(N_e^2 - w^2N_o^2) \\
\delta'_{e3}(w^2) &= 4(N_e^2 - w^2N_o^2) \\
\delta'_{ec}(w^2) &= 2(N_eD_e + w^2N_eD_o - w^2N_oD_e - w^2N_oD_o) \\
\delta'_{o2}(w^2) &= 2\sqrt{2}(N_e^2 - w^2N_o^2) \\
\delta'_{oc}(w^2) &= 2(N_eD_e + N_eD_o - N_oD_e - w^2N_oD_o).
\end{aligned}
$$

Now for each fixed k_2, the stabilizing set of $(k_1,\ k_3)$ parameters can be obtained by solving a linear programming problem defined by a set of linear inequalities as in [2]. Then by sweeeping over k_2 and repeating the procedure at each stage, the entire set of stabilizing (k_1, k_2, k_3) values can be obtained.

2. Stabilizing PID Parameters for an Interval Plant Family

The approach developed in the last two subsections for a fixed plant can be easily extended to an interval plant family. Indeed, according to the Edge Theorem [17, 9], one particular set of controller parameters stabilizes the entire interval plant family if and only if it stabilizes the exposed edges of the polytope. So, when we are given an interval plant family and we compute the intersection of the stabilizing regions corresponding to each plant along the possible exposed edges, we will obtain the set of controller parameters that stabilize the entire plant family. Computationally, the only difference is that now for every fixed k_2, k_1 and k_3 will have to be determined by solving a linear programming problem with many more linear inequalities — one set coming from each of the "exposed edge plants."

B. PID Settings for a Controller-Robust Design

In this section, we consider the problem of designing PID controllers for which the closed loop systems are not destabilized by small perturbations in the PID settings. A controller for which the closed loop system is destabilized by small perturbations in the controller coefficients is said to be "fragile" [18]. Any controller that is to be practically implemented must necessarily be non-fragile (controller-robust) so that (1) round-off errors during implementation do not destabilize the closed loop; and (2) tuning of the parameters about the nominal design values is allowed. To carry out a controller-robust PID design, we will exploit the characterization of all stabilizing PID controllers for fixed and interval discrete-time plants developed in the last section.

Since we know the set of stabilizing PID controller parameters for a given plant or an interval plant family, we can choose the PID parameters to be at the center of the three dimensional ball of largest radius inscribed within that stabilizing region. The radius of this ball is the maximal l_2 parametric stability margin in the space of (k_1,k_2,k_3) and, indeed, in the space of (k_p,k_i,k_d), the latter being due to the orthogonal nature of the transformation (4.5). The method developed in [14] for finding the largest ball inside the PID stabilizing set for continuous-time plants can also be used here because, for a given plant (interval or otherwise) and a fixed value of k_2, the stabilizing regions of (k_1, k_3) are either convex polygons or intersections of half-planes [2, 16]. Even though the center of the largest ball inscribed inside the stabilizing region cannot be determined in closed form, it can be computed using the following algorithm.

Before presenting the algorithm, we first introduce some concepts. Consider a sphere $\mathcal{B}(\mathbf{x}, r)$ in the three dimensional (k_1,k_2,k_3) space with radius r and centered at $\mathbf{x} \triangleq (x_{k_1}, x_{k_2}, x_{k_3})$. Given any angle $\theta \in [-\pi/2, \pi/2]$, let $\mathcal{C}(\mathbf{x}, r, \theta)$ denote the circle with radius $r\cos\theta$, centered at $(x_{k_1}, x_{k_2}+r\sin\theta, x_{k_3})$ and parallel to the (k_1, k_3) plane. The sphere and circle are illustrated in Fig. 5. It is clear that

$$\mathcal{B}(\mathbf{x}, r) = \bigcup_{\theta\in[-\pi/2,\pi/2]} \mathcal{C}(\mathbf{x}, r, \theta). \tag{4.7}$$

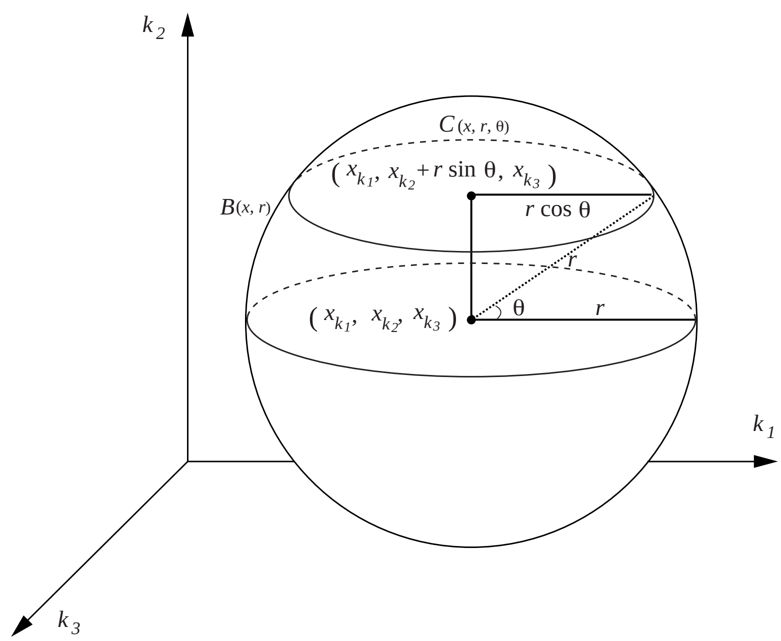

Fig. 5. A sphere $\mathcal{B}(\mathbf{x}, r)$ and the definition of the circle $\mathcal{C}(\mathbf{x}, r, \theta)$.

Now consider $\mathcal{C}(\mathbf{x}, r, \theta)$ with fixed x_{k_2}, r and θ so that $k_2 = x_{k_2} + r\sin\theta$ is fixed. Let the stabilizing (k_1, k_3) region associated with this fixed k_2 be given by the set of linear inequalities

$$\mathcal{P}_\theta = \{\mathbf{x}|\mathbf{a}_{\theta_i}^T\mathbf{x} \le b_{\theta_i}, i = 1, \ldots, m_\theta\} \tag{4.8}$$

where $\mathbf{a}_{\theta_i} \in \mathbf{R}^2$, $b_{\theta_i} \in \mathbf{R}$ and each inequality represents a half plane. Define $\mathbf{x}_c = [x_{k_1}, x_{k_3}]^T$. Then, from [14], $\mathcal{C}(\mathbf{x}, r, \theta)$ lies inside the stabilizing region $\mathcal{P}_\theta$ if and only if

$$\mathbf{a}_{\theta_i}^T \mathbf{x}_c + r\cos\theta \|\mathbf{a}_{\theta_i}\| \le b_{\theta_i}, (i = 1, \ldots, m_\theta) \tag{4.9}$$

holds. Let $\mathcal{S}_\theta$ denote the set of feasible solutions of (4.9). From the geometrical structure, we know that for all $\theta \in [-\pi/2, \pi/2]$, the centers of circles $\mathcal{C}(\mathbf{x}, r, \theta)$ have the same (k_1, k_3) coordinates. Since $\mathcal{S}_\theta$ is the set of feasible (k_1, k_3) coordinates of the centers associated with $\mathcal{C}(\mathbf{x}, r, \theta)$, it follows that $\mathcal{B}(\mathbf{x}, r)$ lies inside the stabilizing (k_1, k_2, k_3) region if and only if

$$\bigcap_{\theta \in [-\pi/2, \pi/2]} \mathcal{S}_\theta \neq \emptyset. \tag{4.10}$$

The above observations suggest a bisection algorithm for determining the maximum l_2 parametric stability margin while k_2 is fixed. Let r_{ub} be the upper bound for r. Since we have the complete characterization of all stabilizing $(k_1,\ k_2,\ k_3)$ values, we are able to determine the stabilizing range of k_2 explicitly. Let us assume that all stabilizing $k_2 \in [k_{2_{min}},\ k_{2_{max}}]$. Then for a fixed k_2, r_{ub} is given by the following formula:

$$r_{ub} = \min(k_2 - k_{2_{min}},\ k_{2_{max}} - k_2).$$

We propose the following bisection algorithm:

Step 1: Set $r_L = 0$ and $r_U = r_{ub}$;

Step 2: Set $r = \frac{r_L + r_U}{2}$;

Step 3: Sweep over all $\theta \in [-\frac{\pi}{2},\ \frac{\pi}{2}]$ and determine the set of all feasible solutions S_θ for (4.9) at each stage;

Step 4: If $\cap_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} S_\theta \neq \emptyset$, then set $r_L = r$; otherwise set $r_U = r$;

Step 5: If $|r_U - r_L| \le$ specified level then **STOP**; otherwise **GOTO Step 2**.

The above algorithm can be applied to determine the maximum l_2 parametric stability margin for any fixed k_2. Moreover, we can sweep over k_2 and choose that value of k_2 that gives the largest radius of the inscribed ball. Setting the $(k_1,\ k_2,\ k_3)$ values at the center of this ball will yield the maximum l_2 parametric stability margin in the space of $(k_1,\ k_2,\ k_3)$. The corresponding (k_p, k_i, k_d) values can be easily obtained from (4.6) to yield a maximally controller-robust PID controller having an l_2 parametric stability margin identical to that determined in the space of (k_1, k_2, k_3).

C. Example

Consider a PID controller to stabilize the discrete-time plant $N_z(z)/D_z(z)$ where

$$\begin{aligned} N_z(z) &= z + 1.5 \\ D_z(z) &= z^2 - 1.5z + 0.5. \end{aligned} \tag{4.11}$$

Using the bilinear transformation, the w-domain plant model becomes

$$\frac{N(w)}{D(w)} = \frac{2.5w^2 - 3w + 0.5}{w + 3}.$$

Applying the results of Section A, we obtain the set of stabilizing controller parameters in the space of (k_1, k_2, k_3). This set is shown in Fig. 6. The corresponding set in the space of the original PID parameters (k_p, k_i, k_d) is also shown in Fig. 7.

Next we design a controller-robust PID controller for this plant using the results of Section B. The maximally controller-robust PID parameters are $(k_p, k_i, k_d) = (0.2220, 0.0532, 0.3571)$ and the corresponding stability margin is 0.0516. If Δk_p, Δk_i, Δk_d denote the perturbations from these optimal values, then as long as

$$\Delta k_p^2 + \Delta k_i^2 + \Delta k_d^2 \leq 0.0516^2,$$

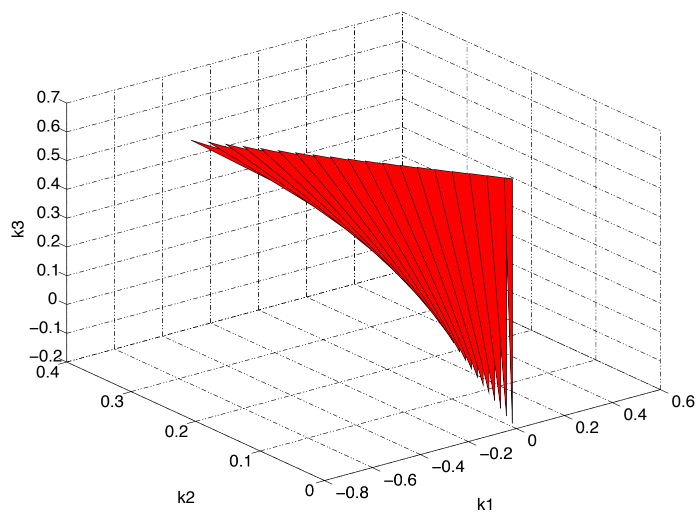

Fig. 6. The stabilizing region in the space of (k_1,k_2,k_3).

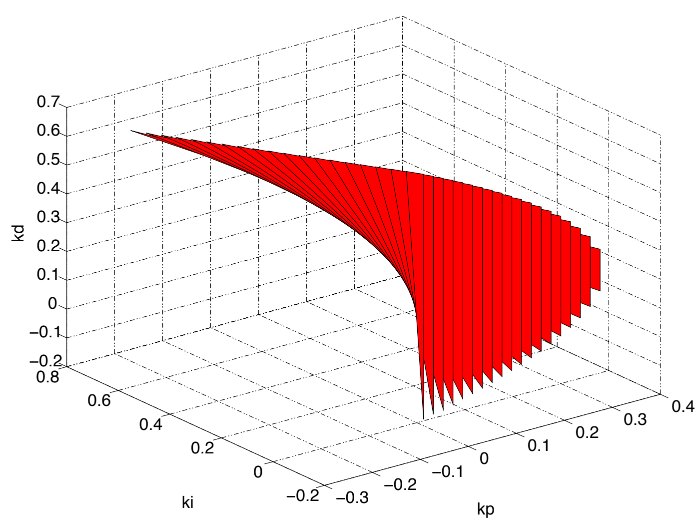

Fig. 7. The stabilizing region in the space of (k_p,k_i,k_d).

the closed loop system will remain stable.

Instead of the fixed plant (4.11), suppose we are now given the interval plant family

$$\frac{N_z(z)}{D_z(z)} = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0},$$

where $b_1 = 1$, $b_0 \in [1.4, 1.6]$, $a_1 \in [-1.6, -1.3]$, $a_0 \in [0.3, 0.6]$. Using the results of Section B, we obtain the robust, optimally non-fragile PID settings $(k_p, k_i, k_d) = (0.1906, 0.0405, 0.1840)$ with a corresponding stability margin of 0.0405.

D. Summary

In this chapter, we have presented a procedure for designing robust and non-fragile PID controllers for discrete-time interval plant families. These results significantly extend the earlier results [14] for the continuous-time case. It is our hope that these results will spur further research activity leading to effective approaches for addressing the issue of controller-robustness for other controller structures.

CHAPTER V

STABILIZING PID CONTROLLERS FOR SYSTEMS WITH INTERVAL TIME-DELAY*

In this chapter, the precise conditions under which one can use the generalized Nyquist Criterion are derived based on Pontryagin's theorems. Furthermore, a method to compute the complete set of PID controllers to stabilize a given arbitrary order plant with interval time-delay is developed.

A. Connection Between Pontryagin's Theory and the Nyquist Criterion

First, we will use an example to show that applying Tsypkin's results, which are standard in the control literature, to an arbitrary LTI plant with time-delay can lead to misleading conclusions, if not used carefully.

Example 4 *Given a system with nominal open-loop transfer function*

$$G(s) = \frac{2s+1}{s+2},$$

we can draw its Nyquist plot, as shown in Fig. 8. The closed-loop system is stable with unity negative feedback and the Nyquist plot intersects the unit circle at $\omega_0 = 1$*. Thus, from the graph, using Tsypkin's result, it would appear that the closed-loop system can tolerate a time-delay up to* $L_0 = \frac{\pi + \arg[G(j\omega_0)]}{\omega_0} = 3.7851$*. However, when we add a 1 second delay to the nominal transfer function, the closed-loop system becomes unstable, as shown in Fig. 9.*

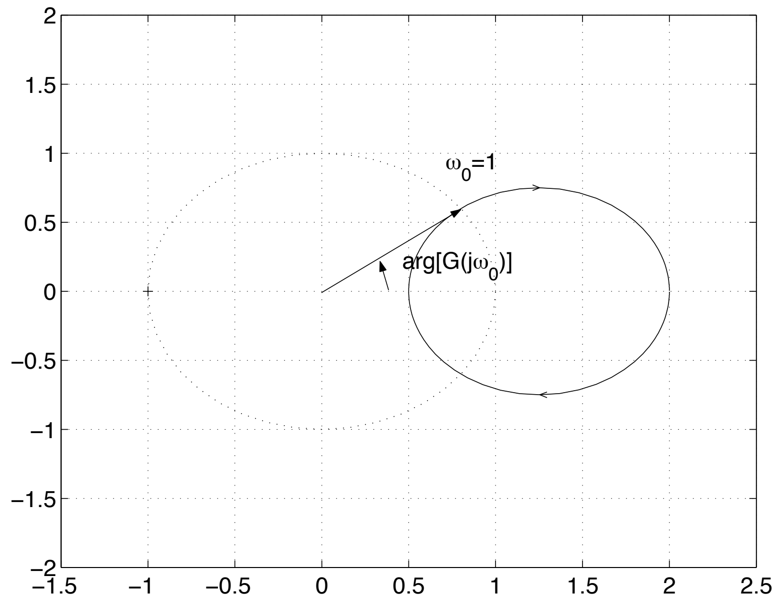

Fig. 8. Nyquist plot of a simple system.

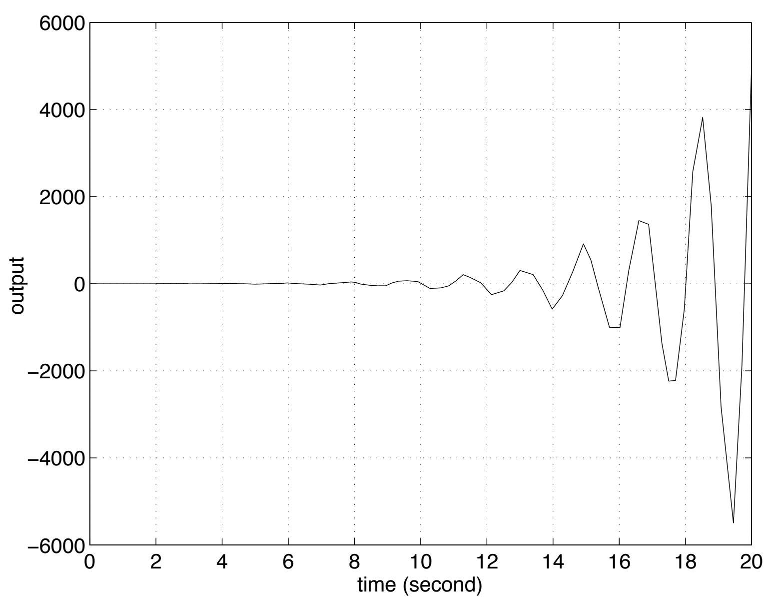

Fig. 9. Simulation of the system with 1 sec delay.

In this section, we use Pontryagin's Theorems to derive conditions under which a modified generalized Nyquist Criterion can be used to correctly analyze the stability of a system. This connection is important in its own right.

Let $h(z,t)$ be a polynomial in the two variables z and t with constant coefficients,

$$h(z,t) = \sum_{m=0}^{r} \sum_{n=0}^{s} a_{mn} z^m t^n. \tag{5.1}$$

The term $a_{rs} z^r t^s$ is called the principal term of the polynomial if $a_{rs} \neq 0$ and the exponents r and s each attain their maximum; that is for each other term $a_{mn} z^m t^n$ in (5.1), for $a_{mn} \neq 0$, either $r > m, s > n$, or $r = m, s > n$, or $r > m, s = n$. We can also write (5.1) as

$$h(z,t) = \chi_r^{(s)}(t) z^r + \chi_{r-1}^{(s)}(t) z^{r-1} + \cdots + \chi_1^{(s)}(t) z + \chi_0^{(s)}(t),$$

where $\chi_j^{(s)}(t), j = 0, 1, 2, \ldots, r$ are polynomials in t with degree at most equal to s.

We will use the following two theorems of Pontryagin [7] to clarify conditions under which the Nyquist Criterion can be used to study the stability of systems with time-delay.

Theorem 3 *[7] If the polynomial (5.1) has no principal term, then the function*

$$H(z) = h(z, e^z) \tag{5.2}$$

has an unbounded number of zeros with arbitrarily large positive real part.

Theorem 4 *[7] Let $H(z) = h(z, e^z)$, where $h(z,t)$ is a polynomial with principal term $a_{rs} z^r t^s$. If the function $\chi_r^{(s)}(e^z)$ has roots in the open right half plane, then the function $H(z)$ has an unbounded set of zeros in the open right half plane. If all the zeros of the function $\chi_r^{(s)}(e^z)$ lie in the open left half plane, then the function $H(z)$ has no more than a bounded set of zeros in the open right half plane.*

Remark 2 *We note that in Theorem 4, the situation when $\chi_r^{(s)}(e^z)$ has zero(s) on the imaginary axis is not mentioned. We will look into this more deeply. Let us look at the distribution of the zeros of $H(z)$ when $|z| \to \infty$. As $|z| \to \infty$, $H(z) = 0$ can be approximated as $\chi_r^{(s)}(e^z) = 0$. That means the roots of $\chi_r^{(s)}(e^z) = 0$ determine the zeros of $H(z)$ at infinity. According to [8, 19], those roots form certain chains and they go deep into the left half plane, the right half plane or go to infinity within strips with finite real parts. Thus, if $\chi_r^{(s)}(e^z)$ has zeros on the imaginary axis, $H(z)$ has root chains that approach the imaginary axis at infinity.*

The following theorem based on the above results gives us the conditions which should be satisfied when using the Nyquist Criterion with the conventional Nyquist contour (the contour consisting of the imaginary axis and a semicircle of arbitrarily large radius in the right half plane).

Theorem 5 *Given a unity feedback system with an open-loop transfer function*

$$G(s) = G_0(s)e^{-Ls} = \frac{N(s)}{D(s)}e^{-Ls}$$

where $N(s)$ and $D(s)$ are real polynomials of degree m and n respectively and L is a fixed delay, we have the following conclusions:

1. *If $n < m$, or, $n = m$ and $|\frac{b_n}{a_n}| \geq 1$, where a_n, b_n are the leading coefficients of $D(s)$ and $N(s)$ respectively, then the system is unstable according to Pontryagin's theorems.*

2. *If $n > m$, or, $n = m$ and $|\frac{b_n}{a_n}| < 1$, the conventional Nyquist Criterion is applicable and we can use it to check the stability of the closed-loop system.*

Proof: The characteristic equation of the closed-loop system is

$$\delta(s) = D(s) + N(s)e^{-Ls}. \tag{5.3}$$

Multiply (5.3) by e^{Ls} and let $z = Ls$ to obtain

$$\delta^*(z) = D_z(z)e^z + N_z(z), \tag{5.4}$$

here

$$D_z(z) = a_n L^{-n} z^n + a_{n-1} L^{-n+1} z^{n-1} + \cdots + a_1 L^{-1} z + a_0$$
$$N_z(z) = b_m L^{-m} z^m + b_{m-1} L^{-m+1} z^{m-1} + \cdots + b_1 L^{-1} z + b_0.$$

Note that both the above operations do not affect the number of RHP roots of the original equation with $L > 0$.

Now we will discuss the possible stability of (5.4) in the following three cases.

1. $deg[D_z(z)] < deg[N_z(z)]$, i.e., $n < m$.

   In this case, $\delta^*(z)$ does not have a principal term. According to Theorem 3, it has an unbounded number of RHP roots. The Nyquist Criterion is inapplicable but we already know that $\delta^*(z)$ is unstable.

2. $deg[D_z(z)] > deg[N_z(z)]$, i.e., $n > m$.

   $\delta^*(z)$ has the principal term $a_n L^{-n} z^n e^z$. The coefficient of z^n is

   $$\chi_n^{(1)}(e^z) = \frac{a_n}{L^n} e^z,$$

   which does not have roots in RHP and on the imaginary axis. Therefore, by Theorem 4, $\delta^*(z)$ can only have a bounded set of RHP zeros. This bounded set is also a finite set [8, 19], and the Nyquist Criterion can be used for stability analysis.

3. $deg[D_z(z)] = deg[N_z(z)]$, i.e., $n = m$.

   $\delta^*(z)$ has the principal term $a_n L^{-n} z^n e^z$ in this case too. However, the coefficient of z^n is

$$\chi_n^{(1)}(e^z) = \frac{a_n}{L^n} e^z + \frac{b_n}{L^n}.$$

To make $\chi_n^{(1)}(e^z) = 0$, we must have $e^z = -\frac{b_n}{a_n}$. Let $z = x + jy$ and $x, y \in \mathcal{R}$, then we have $e^x e^{jy} = -\frac{b_n}{a_n}$. The solutions are

- Case 1: $\frac{b_n}{a_n} > 0$. Then $e^x = |\frac{b_n}{a_n}|, e^{jy} = -1$ so that

$$x = \ln|\frac{b_n}{a_n}|, y = 2k\pi + \pi, k \in \mathbf{Z},$$

- Case 2: $\frac{b_n}{a_n} < 0$. Then $e^x = |\frac{b_n}{a_n}|, e^{jy} = 1$ so that

$$x = \ln|\frac{b_n}{a_n}|, y = 2k\pi, k \in \mathbf{Z}.$$

Depending on the value of $|\frac{b_n}{a_n}|$, we will arrive at different conclusions:

(a) If $|\frac{b_n}{a_n}| > 1$, then $\chi_n^{(1)}$ has RHP zeros. So, $\delta^*(z)$ has an unbounded set of RHP zeros. Again, the Nyquist Criterion is inapplicable but the closed-loop system is unstable.

(b) If $|\frac{b_n}{a_n}| < 1$, then $\chi_n^{(1)}$ only has LHP zeros. So, $\delta^*(z)$ has no more than a bounded and finite set of RHP zeros and the closed-loop stability is determinable from the Nyquist Criterion.

(c) If $|\frac{b_n}{a_n}| = 1$, then $\chi_n^{(1)}$ has zeros on the imaginary axis. So, $\delta^*(z)$ has root chains approaching the imaginary axis, so it is unstable [8, 19]. The Nyquist Criterion is inapplicable in this case.

Since $\delta^*(z)$ has the same number of RHP zeros as $\delta(s)$ for fixed $L > 0$, from the above analysis, we can see that in cases (1), (3a) and (3c), $\delta(s)$ is unstable, while in cases (2) and (3b), $\delta(s)$ has no more than a bounded set of zeros in the RHP, hence it is possibly stable.

So, only in cases (2) and (3b), the Nyquist Criterion can be used to ascertain possible stability. Thus Tsypkin's results and the proof of the Generalized Nyquist Criterion as given in [10] are valid only for these two cases. ♣

Remark 3 *In all fairness, it is appropriate to point out that most likely Tsypkin assumed the plant to be strictly proper, though he did not state it explicitly in the literature. Here, attaching a PID controller to a proper or strictly proper plant opens up the very real possibility of ending up with an improper or a proper open-loop transfer function. This is the reason that the above investigation had to be undertaken.*

Remark 4 *In case (1), (3a) and (3c), if we plot the Nyquist curve of the open-loop transfer function, the curve will encircle the unit circle, which includes the* $-1 + j0$ *point, an infinite number of times in clockwise direction. As a root counting procedure, the Nyquist Criterion is therefore unable to handle this situation. Some generalizations of the Nyquist Criterion can be found in [20, 21, 22], which addressed certain aspects of this issue. Here, we clarify the usage of the traditional Nyquist Criterion with the help of Pontryagin's Theorems.*

Remark 5 *In [20, 23, 24, 25], the discussion of "well-posedness" of the systems has reached a similar condition. Theorem 5 shows that this condition is valid not only for arbitrarily small delay but also for any value of delay. This condition also appears in [26].*

The above clarification sets the stage for determining all stabilizing P, PI and PID controllers for plants with time-delay using the Nyquist Criterion, which is the main purpose of this chapter.

B. Problem Formulation and Solution Approach

Problem Description: Consider a given LTI plant with time-delay L,

$$P(s) = P_0(s)e^{-Ls} = \frac{N(s)}{D(s)}e^{-Ls}$$

and a controller with a unity feedback fixed-structure, $C(s, \mathbf{k})$, where $\mathbf{k}$ is the vector of adjustable parameters of the controller. The problem of interest is to find the complete set of $\mathbf{k}$'s which can stabilize the system for any $L \in [0, L_0]$.

The approach developed in this chapter to solve this problem involves the following steps:

1. Find the complete set of $\mathbf{k}$'s which stabilize the delay-free plant $P_0(s)$ and denote this set as $\mathcal{S}_0$.

2. Define the set $\mathcal{S}_N$, which is the set of $\mathbf{k}$'s such that either $C(s, \mathbf{k})P_0(s)$ is an improper transfer function or $\lim_{s\to\infty} |[C(s, \mathbf{k})P_0(s)]| \geq 1$. Note that the elements in $\mathcal{S}_N$ make the closed-loop system unstable after the delay is introduced (Theorem 5). Exclude $\mathcal{S}_N$ from $\mathcal{S}_0$ and denote the new set by $\mathcal{S}_1$, i.e. $\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N$.

3. Compute the set $\mathcal{S}_L$:

$$\mathcal{S}_L = \{\mathbf{k}|\mathbf{k} \notin \mathcal{S}_N \text{ and } \exists L_1 \in [0, L_0], \omega_1 \in \mathbf{R}, \text{s.t.} C(j\omega_1)P_0(j\omega_1)e^{-jL_1\omega_1} = -1\}.$$

   From this definition, $\mathcal{S}_L$ is the set of $\mathbf{k}$'s which make $C(s, \mathbf{k})P(s)$ have a minimal critical delay that is less than or equal to L_0 [10].

4. The set $\mathcal{S}_R \triangleq \mathcal{S}_1 \backslash \mathcal{S}_L$ is the solution to our problem.

Theorem 6 *The set of controllers $C(s, \mathbf{k})$ denoted by $\mathcal{S}_R$ is the complete set of controllers in the unity feedback configuration that stabilize the plant $P(s)$ with delay L from 0 up to L_0.*

Proof: For any $\mathbf{k}_0 \in \mathcal{S}_R$, since $\mathcal{S}_R \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_0$, $\mathbf{k}_0 \in \mathcal{S}_0$, i.e. there is no closed-loop RHP pole when the controller $C(s, \mathbf{k}_0)$ is applied to the plant $P(s)$ with $L = 0$. Since $\mathbf{k}_0 \notin \mathcal{S}_N$, with the increase of L, there is no unbounded RHP closed-loop pole (Theorem 5) and the possible RHP closed-loop poles are the poles that come from the LHP by crossing the imaginary axis [10]. However, from $\mathbf{k}_0 \notin \mathcal{S}_L$, we know that there are no boundary crossing poles. So, the closed-loop system does not have RHP poles with L ranging from 0 to L_0 and it is, therefore, stable for those L's.

For any $\mathbf{k}_1 \notin \mathcal{S}_R$, it must fall into one or more of following categories.

1. $\mathbf{k}_1 \notin \mathcal{S}_0$, which means the controller cannot even stabilize the delay-free plant ($L = 0$).

2. $\mathbf{k}_1 \in \mathcal{S}_N$, the closed-loop system is unstable with any amount of delay (Theorem 5).

3. $\mathbf{k}_1 \in \mathcal{S}_L$, some closed-loop poles are on the imaginary axis for certain $L_1 \leq L_0$. These poles will either go into the RHP or return to the LHP. However, the stability at that L_1 has already been destroyed.

We can see from the above analysis that $\mathcal{S}_R$ is exactly the complete set of stabilizing controller parameters that we are looking for. ♣

Remark 6 *In the above procedures, if we have the knowledge of the complete stabilizing set for the system with a fixed delay $L_{\min}$, where $0 < L_{\min} < L_0$, and let $\mathcal{S}_1$ be this set. Also, let $\mathcal{S}_L$ be the set of $\mathbf{k}$'s which make $C(s, \mathbf{k})P(s)$ have a critical delay between $L_{\min}$ and L_0, i.e.*

$$\mathcal{S}_L = \{\mathbf{k} | \mathbf{k} \in \mathcal{S}_1 \text{ and } \exists L_1 \in [L_{\min}, L_0], \omega_1 \in \mathbf{R}, \text{s.t.} C(j\omega_1)P_0(j\omega_1)e^{-jL_1\omega_1} = -1\}.$$

Then the result $\mathcal{S}_R \triangleq \mathcal{S}_1 \backslash \mathcal{S}_L$ *is the complete stabilizing controllers set for the family of plants with interval delay* $[L_{\min}, L_0]$.

Remark 7 *In [27], the stability of a family of time-delay plants is analyzed by checking the boundary crossing of roots. Here, the same idea is used in the synthesis problem.*

In the following sections, we apply this general method to the special case of PID controllers to find all PID controllers which can stabilize a given plant with time-delay up to a certain value.

C. Proportional Controllers for Time-Delay Systems

Let us first consider using proportional controllers to stabilize an arbitrary plant with time-delay. We will then extend the result to PI and PID controllers. For a proportional controller, we have

$$C(s) = k_p,$$

and the plant is:

$$P(s) = P_0(s)e^{-Ls} = \frac{N(s)}{D(s)}e^{-Ls}.$$

Our objective is to find all the k_p's which stabilize $P(s)$ with time-delay $L \in [0, L_0]$.

To implement the method proposed in Section B, the key is to find $\mathcal{S}_L$. The Nyquist curve of the system crossing $(-1, 0)$ is equivalent to $C(j\omega)P_0(j\omega)e^{-jL\omega} = -1$ for certain L and ω. This, in turn, is equivalent to the following two conditions:

$$\arg[k_p P_0(j\omega)] - L\omega = 2h\pi - \pi, h \in \mathbf{Z} \tag{5.5}$$

$$|k_p P_0(j\omega)| = 1. \tag{5.6}$$

Here the argument function $\arg(\cdot) \in [-\pi, \pi)$ by convention. Also we only need to consider $\omega > 0$ since the Nyquist plot for $\omega < 0$ is symmetric. We are only interested in the minimal non-negative L which satisfies (5.5), so the phase condition(5.5) can be rewritten as

$$\arg[k_p P_0(j\omega)] - L\omega = -\pi.$$

Note that such a reasoning also applies to the PI and PID cases, to be considered later.

The two conditions above yield

$$L(\omega, k_p) = \frac{\arg[k_p P_0(j\omega)] + \pi}{\omega} \quad (5.7)$$

$$k_p(\omega) = \pm \frac{1}{|P_0(j\omega)|}. \quad (5.8)$$

For $k_p > 0$, we have

$$L(\omega, k_p) = L(\omega) = \frac{\arg[P_0(j\omega)] + \pi}{\omega}.$$

Solve $L(\omega) \leq L_0$ to get a set of ω, say Ω^+. From the magnitude condition (5.8), we can get a set of positive k_p's corresponding to Ω^+, and let us call this set $\mathcal{S}_L^+$. This set consists of all the positive k_p's that make the system have poles on the imaginary axis for some $L \leq L_0$.

Similarly, for $k_p < 0$, we will have a set Ω^- and a corresponding set $\mathcal{S}_L^-$.

Now, the combination of $\mathcal{S}_L^+$ and $\mathcal{S}_L^-$ is the complete set $\mathcal{S}_L$, i.e. $\mathcal{S}_L = \mathcal{S}_L^+ \cup \mathcal{S}_L^-$.

The above discussion leads to the following steps for computing $\mathcal{S}_R$.

1. Compute the delay-free stabilizing k_p set, $\mathcal{S}_0$, either by the Routh-Hurwitz Criterion or the method proposed in [2].

2. Find $\mathcal{S}_N$.

    - If $deg[N(s)] > deg[D(s)]$, $\mathcal{S}_N = \mathbf{R}$, which means $\mathcal{S}_R = \emptyset$.

- If $deg[N(s)] < deg[D(s)]$, $\mathcal{S}_N = \emptyset$.
- If $deg[N(s)] = deg[D(s)]$, $\mathcal{S}_N = \{k_p| \ |k_p| \geq |\frac{a_n}{b_n}|\}$, where a_n, b_n are the leading coeffients of $D(s)$ and $N(s)$ respectively.

3. Compute $\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N$.

4. Compute $\mathcal{S}_L$ according to the analysis in this section.

5. Compute $\mathcal{S}_R = \mathcal{S}_1 \backslash \mathcal{S}_L$.

D. PI Controllers for Time-Delay Systems

For a PI controller

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

and the open-loop transfer function becomes

$$G(s) = C(s)P(s) = C(s)P_0(s)e^{-Ls} = G_0(s)e^{-Ls}$$

where,

$$\begin{aligned} G_0(s) &= C(s)P_0(s) \\ &= \frac{k_p s + k_i}{s} \cdot \frac{N(s)}{D(s)} \\ &= (k_p s + k_i) \cdot \frac{N(s)}{sD(s)} \\ &= (k_p s + k_i) \cdot R_0(s), \end{aligned}$$

with $R_0(s) \triangleq \frac{N(s)}{sD(s)}$.

The magnitude and phase conditions

$$\begin{aligned} \arg[(k_i + jk_p\omega)R_0(j\omega)] - L\omega &= -\pi \\ |(k_i + jk_p\omega)R_0(j\omega)| &= 1 \end{aligned}$$

can be written as

$$L(\omega, k_p, k_i) = \frac{\arg[(k_i + jk_p\omega)R_0(j\omega)] + \pi}{\omega} \tag{5.9}$$

$$k_i = \pm\sqrt{\frac{1}{|R_0(j\omega)|^2} - k_p^2\omega^2}. \tag{5.10}$$

We can first fix k_p and define

$$M(\omega) = \frac{1}{|R_0(j\omega)|^2} - k_p^2\omega^2.$$

Thus

$$k_i = \pm\sqrt{M(\omega)}. \tag{5.11}$$

Note that since $k_i \in \mathbf{R}$, only those ω's with $M(\omega) \geq 0$ need consideration when we compute $\mathcal{S}_L$.

Substituting (5.11) into (5.9), we will have

$$L(\omega) = \frac{\arg\{[\pm\sqrt{M(\omega)} + jk_p\omega]R_0(j\omega)\} + \pi}{\omega}$$

Before proceeding further, we need to introduce some notation. For a given set in the controller parameter space, if one of the controller parameters appears as a subscript, then the new set represents the subset of the original one with that parameter fixed at some value. For example, $\mathcal{S}_{R,k_p}$ is a subset of $\mathcal{S}_R$ with k_p fixed at some value.

Based on the above discussion, the following steps can be used for computing $\mathcal{S}_R$:

1. Compute $\mathcal{S}_0$ using the results of [2].

2. Find $\mathcal{S}_N$.

   - If $deg[N(s)] > deg[D(s)]$, $\mathcal{S}_N = \mathbf{R}^2$, which means $\mathcal{S}_R = \emptyset$.

- If $deg[N(s)] < deg[D(s)]$, $\mathcal{S}_N = \emptyset$.
- If $deg[N(s)] = deg[D(s)]$, $\mathcal{S}_N = \{(k_p, k_i) | k_p, k_i \in \mathbf{R} \text{ and } |k_p| \geq |\frac{a_n}{b_n}|\}$, where a_n, b_n are the leading coeffients of $D(s)$ and $N(s)$ respectively.

3. Compute $\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N$.

4. For a fixed k_p, find $\mathcal{S}_{R,k_p}$.

   - First determine the sets Ω^+ and S^+_{L,k_p}:

$$\begin{aligned} \Omega^+ &= \{\omega | \omega > 0 \text{ and } M(\omega) \geq 0 \text{ and} \\ &\qquad L(\omega) = \frac{\arg\{[\sqrt{M(\omega)} + jk_p\omega]R_0(j\omega)\} + \pi}{\omega} \leq L_0\} \\ \mathcal{S}^+_{L,k_p} &= \{k_i | k_i \notin \mathcal{S}_{N,k_p} \text{ and } \exists\, \omega \in \Omega^+ \text{ s.t. } k_i = \sqrt{M(\omega)}\}. \end{aligned}$$

   - Next determine the sets Ω^- and S^-_{L,k_p}:

$$\begin{aligned} \Omega^- &= \{\omega | \omega > 0 \text{ and } M(\omega) \geq 0 \text{ and} \\ &\qquad L(\omega) = \frac{\arg\{[-\sqrt{M(\omega)} + jk_p\omega]R_0(j\omega)\} + \pi}{\omega} \leq L_0\} \\ \mathcal{S}^-_{L,k_p} &= \{k_i | k_i \notin \mathcal{S}_{N,k_p} \text{ and } \exists\, \omega \in \Omega^- \text{ s.t. } k_i = -\sqrt{M(\omega)}\}. \end{aligned}$$

   Compute $\mathcal{S}_{L,k_p} = \mathcal{S}^+_{L,k_p} \cup \mathcal{S}^-_{L,k_p}$ and $\mathcal{S}_{R,k_p} = \mathcal{S}_{1,k_p} \backslash \mathcal{S}_{L,k_p}$.

5. By sweeping over k_p, we will have the complete set of PI controllers that stabilize all plants with delay up to L_0:

$$\mathcal{S}_R = \bigcup_{k_p} \mathcal{S}_{R,k_p}.$$

E. PID Controllers for Time-Delay Systems

Here the PID controller takes the form

$$C(s) = k_p + \frac{k_i}{s} + k_d s = \frac{k_d s^2 + k_p s + k_i}{s},$$

and the open-loop transfer function becomes

$$G(s) = C(s)P(s) = C(s)P_0(s)e^{-Ls} = G_0(s)e^{-Ls}$$

where

$$\begin{aligned} G_0(s) &= C(s)P_0(s) \\ &= \frac{k_d s^2 + k_p s + k_i}{s} \cdot \frac{N(s)}{D(s)} \\ &= (k_d s^2 + k_p s + k_i) \cdot \frac{N(s)}{sD(s)} \\ &= (k_d s^2 + k_p s + k_i) \cdot R_0(s), \end{aligned}$$

with $R_0(s) \triangleq \frac{N(s)}{sD(s)}$.

The phase and magnitude conditions

$$\begin{aligned} \arg[(k_i - k_d\omega^2 + jk_p\omega)R_0(j\omega)] - L\omega &= -\pi \\ \text{and } |(k_i - k_d\omega^2 + jk_p\omega)R_0(j\omega)| &= 1 \end{aligned}$$

can be further reduced to:

$$L(\omega, k_p, k_i, k_d) = \frac{\pi + \arg\{[(k_i - k_d\omega^2) + jk_p\omega] \cdot R_0(j\omega)\}}{\omega} \tag{5.12}$$

$$k_i - k_d\omega^2 = \pm\sqrt{\frac{1}{|R_0(j\omega)|^2} - (k_p\omega)^2}. \tag{5.13}$$

Similar to the PI case, for fixed k_p, we define

$$M(\omega) = \frac{1}{|R_0(j\omega)|^2} - (k_p\omega)^2.$$

Then

$$k_i - k_d\omega^2 = \pm\sqrt{M(\omega)}. \tag{5.14}$$

As in the PI case, we only need to consider ω's with $M(\omega) \geq 0$ when we compute $\mathcal{S}_L$.

Substituting (5.14) into (5.12), we have

$$L(\omega, k_p, k_i, k_d) = L(\omega) = \frac{\pi + \arg\{[\pm\sqrt{M(\omega)} + jk_p\omega] \cdot R_0(j\omega)\}}{\omega}.$$

The following steps can then be used for computing $\mathcal{S}_R$:

1. Compute $\mathcal{S}_0$ using the results of [2].

2. Find $\mathcal{S}_N$.

   - If $deg[N(s)] > deg[D(s)] - 1$, $\mathcal{S}_N = \mathbf{R}^3$, which means $\mathcal{S}_R = \emptyset$.

   - If $deg[N(s)] < deg[D(s)] - 1$, $\mathcal{S}_N = \emptyset$.

   - If $deg[N(s)] = deg[D(s)] - 1$, $\mathcal{S}_N = \{(k_p, k_i, k_d)|k_p, k_i, k_d \in \mathbf{R}$ and $|k_d| \geq |\frac{a_n}{b_{n-1}}|\}$, where a_n, b_{n-1} are the leading coefficients of $D(s)$ and $N(s)$ respectively.

3. Compute $\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N$.

4. For a fixed k_p, determine the set $\mathcal{S}_{R,k_p}$ as follows:

   - First determine the sets Ω^+ and $\mathcal{S}^+_{L,k_p}$:

$$\begin{aligned}\Omega^+ \;=\; & \{\omega|\omega > 0 \text{ and } M(\omega) \geq 0 \text{ and} \\ & L(\omega) = \frac{\pi + \arg\{[\sqrt{M(\omega)} + jk_p\omega] \cdot R_0(j\omega)\}}{\omega} \leq L_0\}\end{aligned}$$

$$\begin{aligned}\mathcal{S}_{L,k_p}^{+} &= \{(k_i,k_d)|(k_i,k_d)\notin \mathcal{S}_{N,k_p} \\ &\qquad \text{and } \exists\, \omega \in \Omega^{+} \text{ s.t. } k_i - k_d\omega^2 = \sqrt{M(\omega)}\}.\end{aligned}$$

Note that $\mathcal{S}_{L,k_p}^{+}$ is a set of straight lines in the (k_i, k_d) space.

- Next determine the sets Ω^- and S_{L,k_p}^{-}:

$$\begin{aligned}\Omega^{-} &= \{\omega|\omega > 0 \text{ and } M(\omega) \geq 0 \text{ and} \\ &\qquad L(\omega) = \frac{\pi + \arg\{[-\sqrt{M(\omega)} + jk_p\omega]\cdot R_0(j\omega)\}}{\omega} \leq L_0\} \\ \mathcal{S}_{L,k_p}^{-} &= \{(k_i,k_d)|(k_i,k_d)\notin \mathcal{S}_{N,k_p} \\ &\qquad \text{and } \exists\, \omega \in \Omega^{-} \text{ s.t. } k_i - k_d\omega^2 = -\sqrt{M(\omega)}\}.\end{aligned}$$

Compute $\mathcal{S}_{L,k_p} = \mathcal{S}_{L,k_p}^{+} \cup \mathcal{S}_{L,k_p}^{-}$ and $\mathcal{S}_{R,k_p} = \mathcal{S}_{1,k_p} \backslash \mathcal{S}_{L,k_p}$.

5. By sweeping over k_p, we will have the complete set of PID controllers that stabilize all plants with delay up to L_0:

$$\mathcal{S}_R = \bigcup_{k_p} \mathcal{S}_{R,k_p}.$$

Remark 8 *The real PID controller has a small time constant stable pole which makes it proper. In addition, the real plant is usually strictly proper. Thus, step 2 and 3 can be omitted. The small time constant pole can be grouped with the transfer function of the plant and the above procedure can then be used to solve the PID problem.*

F. Examples

Here we present two numerical examples to illustrate the procedures in the previous sections. The first example computes proportional controllers to stabilize a third-order plant. The second one demonstrates the application of PID controllers to a fifth-order plant.

Example 5 *Find all proportional controllers that stabilize the plant*

$$P(s) = \frac{s^2 + 3s - 2}{s^3 + 2s^2 + 3s + 2} e^{-Ls}$$

with delay up to $L_0 = 1.8$.

Solution: For the delay-free plant, the stabilizing k_p range is

$$\mathcal{S}_0 = (-0.4093, 1).$$

Since $deg[N(s)] = 2 < 3 = deg[D(s)]$,

$$\mathcal{S}_N = \emptyset,$$

and

$$\mathcal{S}_1 = \mathcal{S}_0.$$

For $k_p > 0$,

$$\Omega^+ = [1.5129, +\infty),$$

(see Fig.10) and the corresponding

$$\mathcal{S}_L^+ = [0.4473, +\infty),$$

(see Fig.11).

For $k_p < 0$,

$$\Omega^- = [0.7359, 1.3312] \cup [2.6817, +\infty),$$

(see Fig.12) and the corresponding

$$\mathcal{S}_L^- = [-0.6025, -0.4082] \cup (-\infty, -1.3691],$$

(see Fig.13).

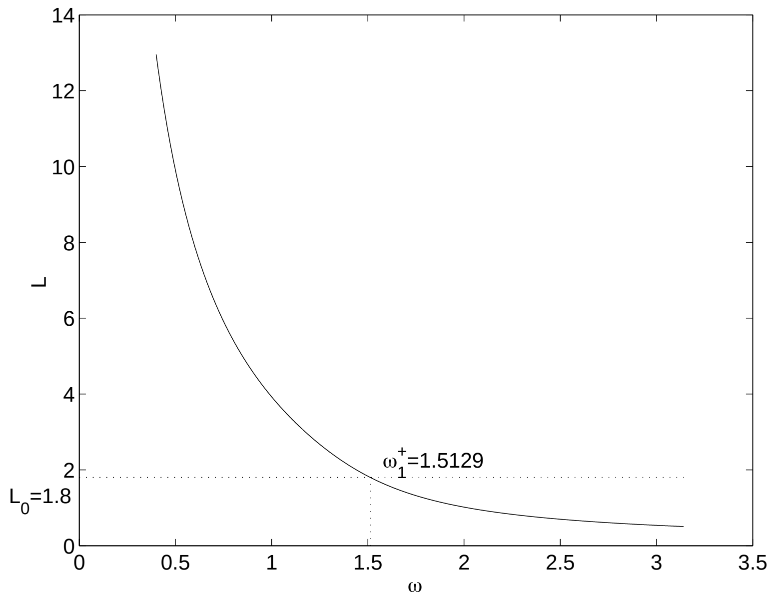

Fig. 10. $L(\omega)$ vs. ω for $k_p > 0$.

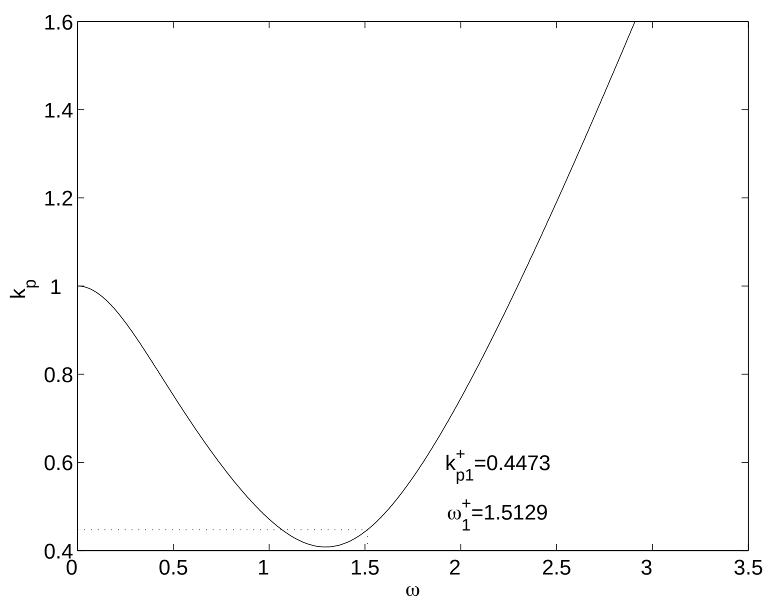

Fig. 11. $k_p(\omega)$ vs. ω.

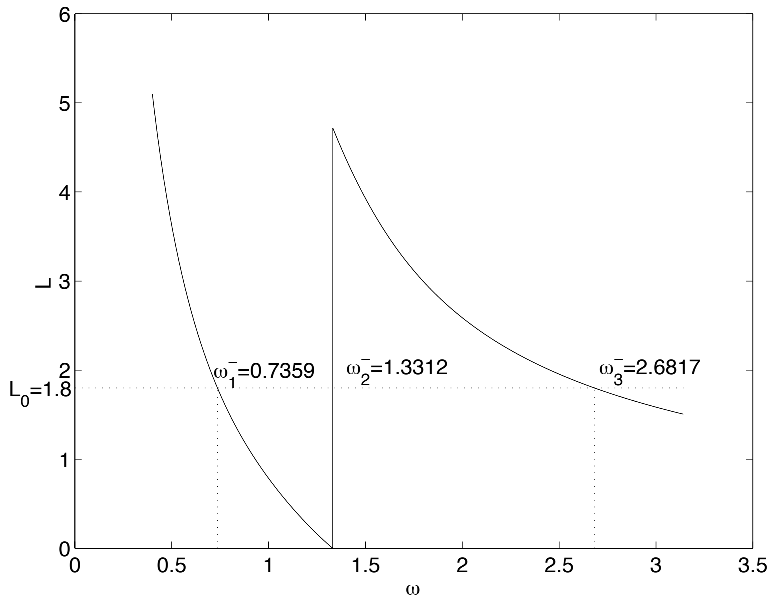

Fig. 12. $L(\omega)$ vs. ω for $k_p < 0$.

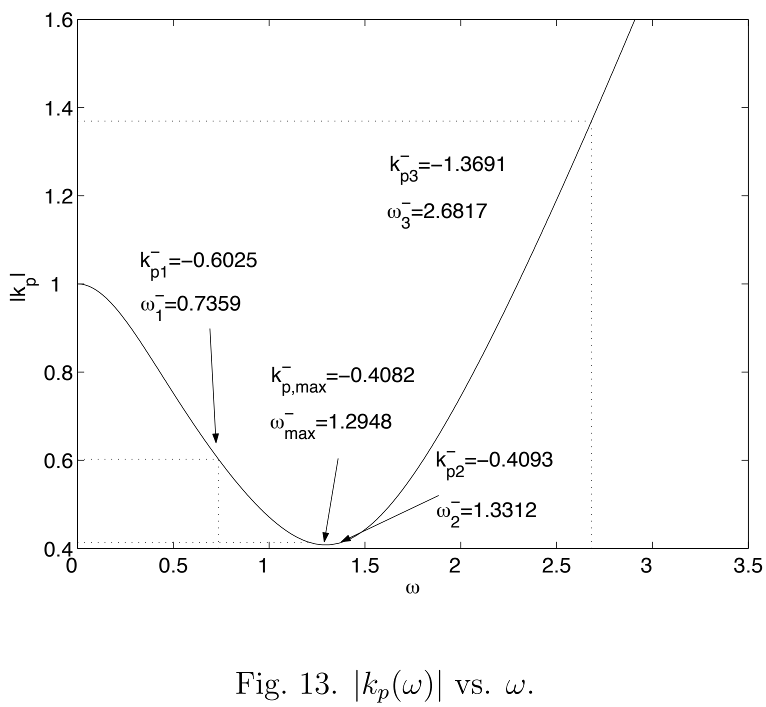

Fig. 13. $|k_p(\omega)|$ vs. ω.

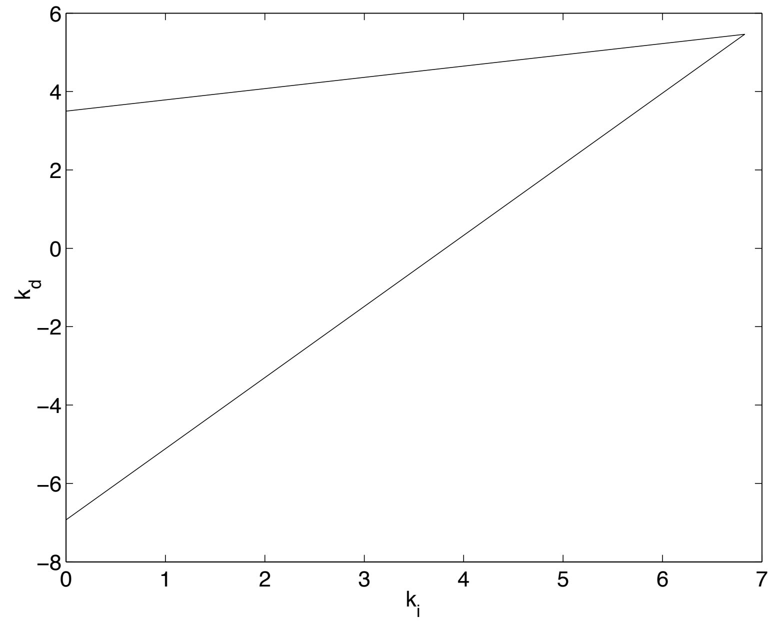

Fig. 14. Stabilizing region of(k_i, k_d) with $k_p = 1$ for delay free plant.

So, the stabilizing k_p for the plant with time-delay up to 1.8 is

$$\begin{aligned}\mathcal{S}_R &= \mathcal{S}_1 \backslash \mathcal{S}_L \\ &= (-0.4093, 1)\backslash([0.4473, +\infty) \cup [-0.6025, -0.4082] \cup (-\infty, -1.3691]) \\ &= (-0.4082, 0.4473). \quad \clubsuit\end{aligned}$$

Example 6 *Find all PID controllers that stabilize the plant*

$$P(s) = \frac{s^3 - 4s^2 + s + 2}{s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17} e^{-Ls}$$

with L *up to* $L_0 = 1$*, i.e., for all* $L \in [0, 1]$.

Solution: Fix $k_p = 1$. First, we can use the method proposed in [2] to get the stabilizing k_i, k_d values for the delay-free plant, $\mathcal{S}_{0,k_p}$, shown in Fig.14.

Since $deg[D(s)] - deg[N(s)] > 1$,

$$\mathcal{S}_N = \emptyset,$$

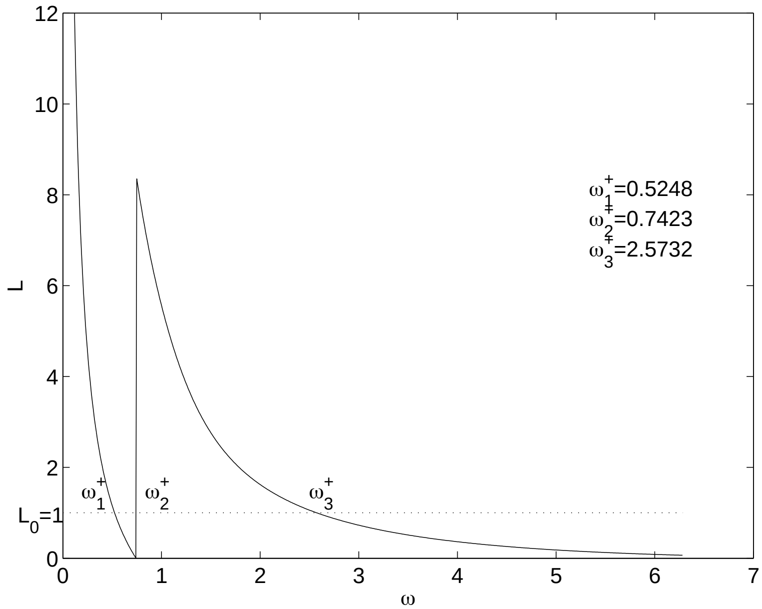

Fig. 15. $L(\omega)$ vs. ω with $k_i - k_d\omega^2 = \sqrt{M(\omega)}$.

and

$$\mathcal{S}_1 = \mathcal{S}_0.$$

For $k_i - k_d\omega^2 = \sqrt{M(\omega)} > 0$, the set of ω where $L(\omega) \le L_0$ is

$$\Omega^+ = [0.524825, 0.742302] \cup [2.57318, +\infty),$$

(see Fig.15). Also, we can find the corresponding values of $\sqrt{M(\omega)}$ (see Fig.16) and $\mathcal{S}^+_{L,k_p}$, i.e. the straight lines defined by

$$k_i - k_d\omega^2 = \sqrt{M(\omega)}$$

for $\omega \in \Omega^+$.

For $k_i - k_d\omega^2 = -\sqrt{M(\omega)} < 0$,

$$\Omega^- = [1.35894, 1.8659] \cup [4.37326, +\infty),$$

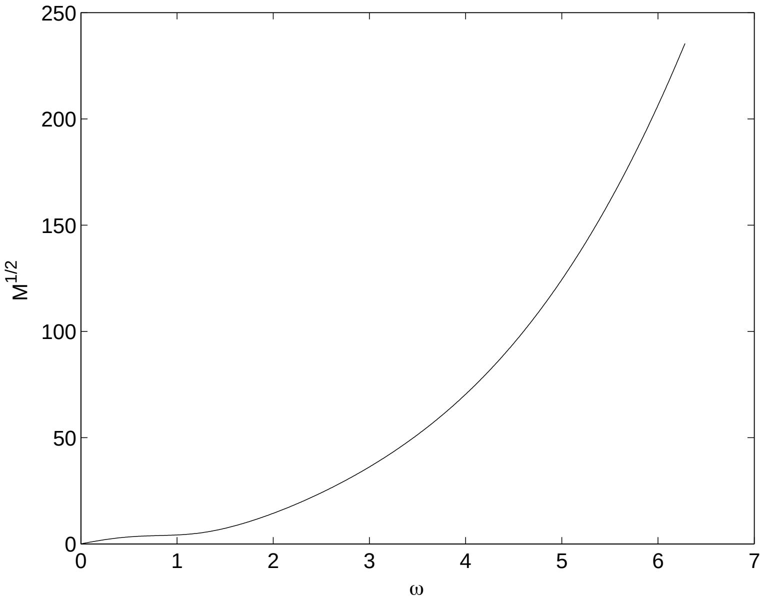

Fig. 16. $\sqrt{M(\omega)}$ vs. ω with $k_p = 1$.

(see Fig.17). Then we can get $\mathcal{S}^-_{L,k_p}$.

Finally, we can exclude $\mathcal{S}^+_{L,k_p}$ and $\mathcal{S}^-_{L,k_p}$ from $\mathcal{S}_{1,k_p}$ to get $\mathcal{S}_{R,k_p}$ (see Fig.18). ♣

G. A Special Case: First-Order Plant with Time-Delay

In this section, we show how the approach presented can be used to recover the results of [6].

Here, the problem is to determine all PID controllers that stabilize a first-order plant with time-delay up to L_0. To this end, consider the first-order plant with time-delay:

$$P(s) = \frac{k}{Ts+1}e^{-Ls}, \quad L \in [0, L_0].$$

The stabilizing PID parameters for the delay-free plant are:

$$\mathcal{S}_0 = \{(k_p, k_i, k_d) | k_p > -\frac{1}{k}, k_i > 0, k_d > -\frac{T}{k} \text{ or } k_p < -\frac{1}{k}, k_i < 0, k_d < -\frac{T}{k}\}$$

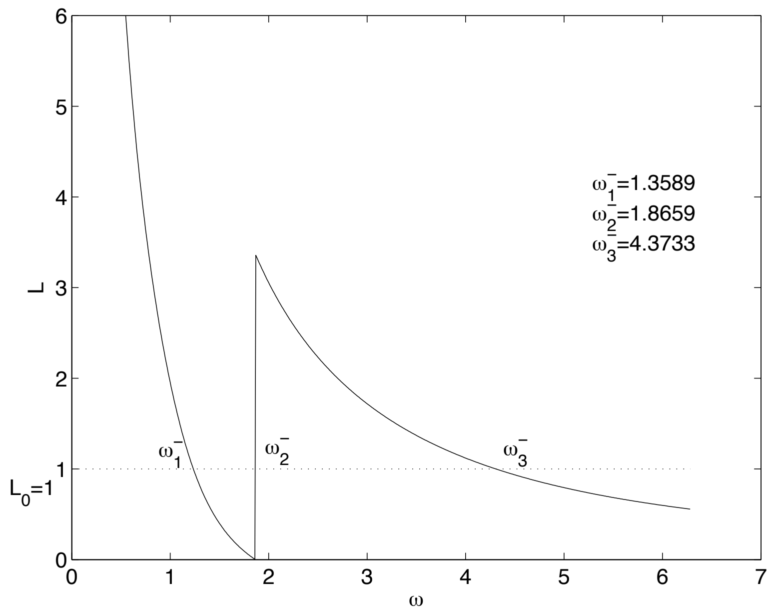

Fig. 17. $L(\omega)$ vs. ω with $k_i - k_d\omega^2 = -\sqrt{M(\omega)}$.

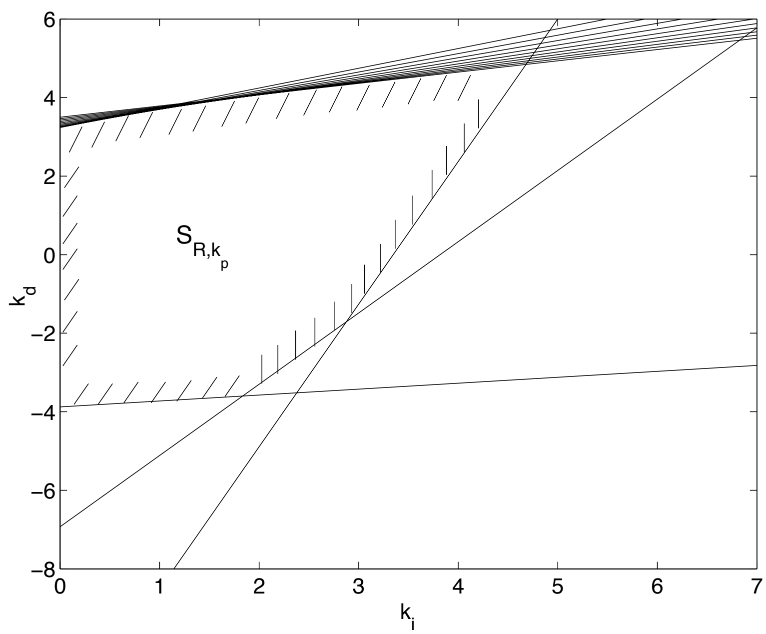

Fig. 18. Stabilizing region of(k_i, k_d) with $k_p = 1$ for plant with delay up to 1.

Since $deg[D(s)] - deg[N(s)] = 1$, $\mathcal{S}_N = \{(k_p, k_i, k_d) | k_p, k_i, k_d \in \mathbf{R} \text{ and } |k_d| \geq |\frac{T}{k}|\}$.

Without loss of generality, let us assume that $k > 0$. Then

$$\mathcal{S}_1 = \mathcal{S}_0 \backslash \mathcal{S}_N = \{(k_p, k_i, k_d) | k_p > -\frac{1}{k}, k_i > 0, \frac{T}{k} > k_d > -\frac{T}{k}\}$$

for $T > 0$, and

$$\mathcal{S}_1 = \{(k_p, k_i, k_d) | k_p < -\frac{1}{k}, k_i < 0, \frac{T}{k} < k_d < -\frac{T}{k}\}$$

for $T < 0$.

For the first-order plant

$$R_0(s) = \frac{N(s)}{sD(s)} = \frac{k}{Ts^2 + s}$$

and for a fixed k_p

$$M(\omega) = \frac{1}{|R_0(j\omega)|^2} - (k_p\omega)^2 = \frac{T^2\omega^4 + (1 - k^2k_p^2)\omega^2}{k^2}$$

For $M(\omega) \geq 0$, we must have $T^2\omega^2 + (1 - k^2k_p^2) \geq 0$

- When $1 - k^2k_p^2 \geq 0$, i.e., $|k_p| \leq 1/k$, all ω satisfy the requirement, that means we need to consider all $\omega > 0$.
- When $1 - k^2k_p^2 < 0$, i.e., $|k_p| > 1/k$. In this case, we only need to consider $\omega \geq \omega_s$, where $\omega_s = \sqrt{k^2k_p^2 - 1}/|T|$ and $M(\omega_s) = 0$.

Let us consider $T > 0$. Now we have two cases to consider.

1. Case 1: $k_i - k_d\omega^2 = \sqrt{M(\omega)}$. In this case,

$$L(\omega) = \frac{\pi + \arg[(\frac{\omega}{k}\sqrt{T^2\omega^2 + 1 - k^2k_p^2} + jk_p\omega) \cdot \frac{k}{-T\omega^2 + j\omega}]}{\omega} =: \frac{\alpha^+(\omega)}{\omega}$$

where $\alpha^+(\omega) \in [0, 2\pi)$.

First, let us check $L(\omega)$. Define

$$\begin{aligned}\alpha_1^+(\omega) &:= \arg(\frac{\omega}{k}\sqrt{T^2\omega^2+1-k^2k_p^2}+jk_p\omega) = \tan^{-1}\frac{kk_p}{\sqrt{T^2\omega^2+1-k^2k_p^2}}\\ \alpha_2^+(\omega) &:= \pi+\arg[\frac{k}{-T\omega^2+j\omega}] = \tan^{-1}\frac{1}{T\omega},\end{aligned}$$

where $\alpha_1^+(\omega)\in(-\pi/2,\pi/2)$ and $\alpha_2^+(\omega)\in(0,\pi/2)$.

- For $k_p \geq 0$, $\alpha_1^+(\omega)\in[0,\pi/2)$, thus

$$\alpha_1^+(\omega)+\alpha_2^+(\omega)\in(0,\pi)\subset[0,2\pi).$$

- For $-\frac{1}{k}<k_p<0$, $\alpha_1^+(\omega)\in(-\pi/2,0)$ and $|\alpha_1^+(\omega)|<|\alpha_2^+(\omega)|$, thus

$$\alpha_1^+(\omega)+\alpha_2^+(\omega)\in(0,\pi/2)\subset[0,2\pi).$$

Thus $L(\omega)$ can be decomposed as

$$L(\omega)=\frac{\alpha^+(\omega)}{\omega}=\frac{\alpha_1^+(\omega)+\alpha_2^+(\omega)}{\omega}. \tag{5.15}$$

Furthermore

- For $k_p \geq 0$, $\alpha_1^+(\omega)$ and $\alpha_2^+(\omega)$ are decreasing functions of ω. So $L(\omega)$ is also a decreasing function of ω.

- For $-\frac{1}{k}<k_p<0$, let us consider

$$\begin{aligned}\tan[\alpha_1^+(\omega)+\alpha_2^+(\omega)] &= \frac{\tan\alpha_1^+(\omega)+\tan\alpha_2^+(\omega)}{1-\tan\alpha_1^+(\omega)\tan\alpha_2^+(\omega)}\\ &= \frac{kk_pT\omega+\sqrt{T^2\omega^2+1-k^2k_p^2}}{T\omega\sqrt{T^2\omega^2+1-k^2k_p^2}-kk_p}.\end{aligned}$$

  Taking its derivative, we obtain

$$\frac{\mathrm{d}\tan[\alpha_1^+(\omega)+\alpha_2^+(\omega)]}{\mathrm{d}\omega}$$

$$
\begin{aligned}
&= \frac{T(1+T^2\omega^2)(-kk_pT\omega - \sqrt{T^2\omega^2+1-k^2k_p^2})}{(T\omega\sqrt{T^2\omega^2+1-k^2k_p^2} - kk_p)^2\sqrt{T^2\omega^2+1-k^2k_p^2}} \\
&< \frac{T(1+T^2\omega^2)(T\omega - \sqrt{T^2\omega^2+1-k^2k_p^2})}{(T\omega\sqrt{T^2\omega^2+1-k^2k_p^2} - kk_p)^2\sqrt{T^2\omega^2+1-k^2k_p^2}} \\
&< 0.
\end{aligned}
$$

Since $\alpha_1^+(\omega) + \alpha_2^+(\omega) \in (0, \pi/2)$, we have $\alpha_1^+(\omega) + \alpha_2^+(\omega)$ is a monotonically decreasing function of ω. So $L(\omega)$ is also a monotonically decreasing function of ω.

From the above analysis, we know, that for *any given* k_p in $\mathcal{S}_1$, $L(\omega)$ is a monotonically decreasing function of ω. This implies, there is only at most *one* ω which satisfies $L(\omega) = L_0$. We denote this ω when it exists by ω_1^+ (see Fig.19, Fig.20 and Fig.21). The quantity ω_1^+ along with the quantity ω_s, defined earlier, enables us to characterize Ω^+:

- For $-\frac{1}{k} < k_p \leq \frac{1}{k}$, $\Omega^+ = [\omega_1^+, +\infty)$.
- For $k_p > \frac{1}{k}$ and $L_0 \leq L(\omega_s)$, $\Omega^+ = [\omega_1^+, +\infty)$.
- For $k_p > \frac{1}{k}$ and $L_0 > L(\omega_s)$, $\Omega^+ = [\omega_s, +\infty)$.

Now, let us check the straight lines defined by $k_i - k_d\omega^2 = \sqrt{M(\omega)}$ in the (k_i, k_d) plane. The straight line

$$
k_i = \omega^2 k_d + \frac{\omega\sqrt{T^2\omega^2+1-k^2k_p^2}}{k}
$$

intersects the lines $k_d = \frac{T}{k}$ and $k_d = -\frac{T}{k}$ at $(k_{i,\omega}^+, \frac{T}{k})$ and $(k_{i,\omega}^-, -\frac{T}{k})$ respectively, where

$$
k_{i,\omega}^+ = \frac{\omega}{k}(\sqrt{T^2\omega^2+1-k^2k_p^2} + T\omega) \tag{5.16}
$$

$$
k_{i,\omega}^- = \frac{\omega}{k}(\sqrt{T^2\omega^2+1-k^2k_p^2} - T\omega). \tag{5.17}
$$

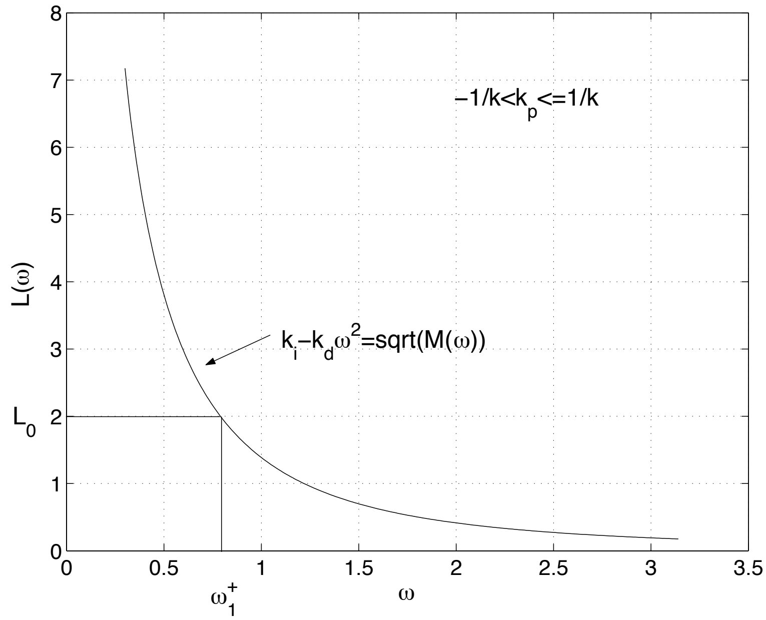

Fig. 19. First-order plant: $L(\omega)$ vs. ω with $k_i - k_d\omega^2 = \sqrt{M(\omega)}$.

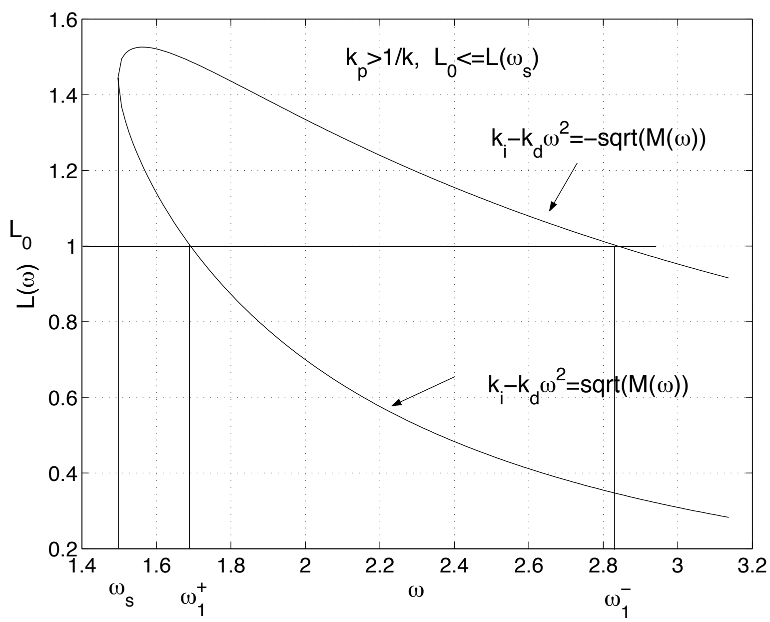

Fig. 20. First-order plant: $L(\omega)$ vs. ω for $k_p > \frac{1}{k}$ and $L_0 \leq L(\omega_s)$.

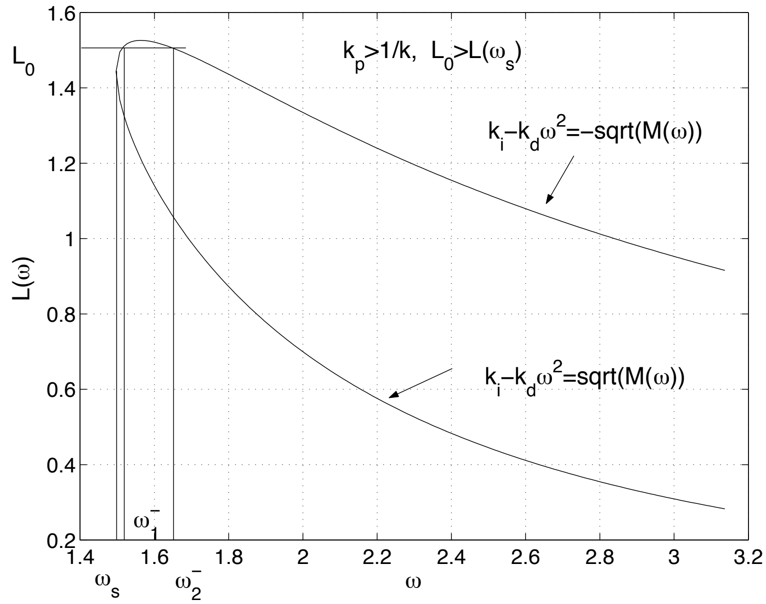

Fig. 21. First-order plant: $L(\omega)$ vs. ω for $k_p > \frac{1}{k}$ and $L_0 > L(\omega_s)$.

The derivative of $k_{i,\omega}^-$ is

$$\begin{aligned} \frac{\mathrm{d}k_{i,\omega}^-}{\mathrm{d}\omega} &= \frac{(\sqrt{T^2\omega^2 + 1 - k^2k_p^2} - T\omega)^2}{k\sqrt{T^2\omega^2 + 1 - k^2k_p^2}} \\ &\geq 0. \end{aligned}$$

From (5.16) and (5.17), it follows that $\frac{dk_{i,\omega}^+}{d\omega}$ is also non-negative. Thus $k_{i,\omega}^-$ and $k_{i,\omega}^+$ are both monotonically increasing functions of ω. From this, it follows that the set

$$\mathcal{S}_{L,k_p}^+ \triangleq \{(k_i, k_d) | k_i - k_d\omega^2 = \sqrt{M(\omega)}, \omega \in \Omega^+\} \cap \mathcal{S}_{1,k_p}$$

can be described as follows corresponding to the different values of k_p and L_0:

- For $-\frac{1}{k} < k_p \leq \frac{1}{k}$,

$$\mathcal{S}_{L,k_p}^+ = \{(k_i, k_d) | k_i \geq k_d(\omega_1^+)^2 + \sqrt{M(\omega_1^+)}\} \cap \mathcal{S}_{1,k_p}. \tag{5.18}$$

- For $k_p > \frac{1}{k}$ and $L_0 \leq L(\omega_s)$,

$$\mathcal{S}^+_{L,k_p} = \{(k_i, k_d) | k_i \geq k_d(\omega_1^+)^2 + \sqrt{M(\omega_1^+)}\} \cap \mathcal{S}_{1,k_p}.$$

- For $k_p > \frac{1}{k}$ and $L_0 > L(\omega_s)$,

$$\begin{aligned}
\mathcal{S}^+_{L,k_p} &= \{(k_i, k_d) | k_i \geq k_d(\omega_s)^2 + \sqrt{M(\omega_s)}\} \cap \mathcal{S}_{1,k_p} \\
&= \{(k_i, k_d) | k_i \geq k_d(\omega_s)^2\} \cap \mathcal{S}_{1,k_p} \text{ (since } M(\omega_s) = 0\text{, by definition)}
\end{aligned}$$

2. Case 2: $k_i - k_d\omega^2 = -\sqrt{M(\omega)}$. Here we first check the positions of these lines. They intersect $k_d = \frac{T}{k}$ and $k_d = -\frac{T}{k}$ at $(k^+_{i,\omega}, \frac{T}{k})$ and $(k^-_{i,\omega}, -\frac{T}{k})$ respectively, where

$$\begin{aligned}
k^+_{i,\omega} &= \frac{\omega}{k}(-\sqrt{T^2\omega^2 + 1 - k^2k_p^2} + T\omega) \\
k^-_{i,\omega} &= \frac{\omega}{k}(-\sqrt{T^2\omega^2 + 1 - k^2k_p^2} - T\omega).
\end{aligned}$$

Here, for $-\frac{1}{k} < k_p \leq \frac{1}{k}$, $k^+_{i,\omega} \leq 0$, which means that the lines $k_i - k_d\omega^2 = -\sqrt{M(\omega)}$ lie outside $\mathcal{S}_{1,k_p}$. So, $\mathcal{S}^-_{L,k_p} = \emptyset$ for these k_p's.

On the other hand, for $k_p > \frac{1}{k}$, $k^+_{i,\omega} > 0$, i.e. the lines $k_i - k_d\omega^2 = -\sqrt{M(\omega)}$ have a non-empty intersection with $\mathcal{S}_{1,k_p}$ and, therefore, affect the set of all stabilizing PID controllers for the system with time-delay.

We next proceed to determine this intersection. Now, the derivative of $k^+_{i,\omega}$ is

$$\begin{aligned}
\frac{\mathrm{d}k^+_{i,\omega}}{\mathrm{d}\omega} &= -\frac{(\sqrt{T^2\omega^2 + 1 - k^2k_p^2} - T\omega)^2}{k\sqrt{T^2\omega^2 + 1 - k^2k_p^2}} \\
&\leq 0.
\end{aligned}$$

So $k^+_{i,\omega}$ and $k^-_{i,\omega}$ are monotonically decreasing functions of ω and $k^+_{i,\omega}$ tends to zero as $\omega \to \infty$. This result will be used to determine $\mathcal{S}^-_{L,k_p}$. In order to do that, we also need to examine $L(\omega)$ when $k_p > \frac{1}{k}$. Now, in this case,

$$L(\omega) = \frac{\pi + \arg[(-\frac{\omega}{k}\sqrt{T^2\omega^2 + 1 - k^2k_p^2} + jk_p\omega) \cdot \frac{k}{-T\omega^2 + j\omega}]}{\omega} =: \frac{\alpha^-(\omega)}{\omega}$$

where $\alpha^-(\omega) \in [0, 2\pi)$. Define

$$\begin{aligned} \alpha_1^-(\omega) &= \arg(-\frac{\omega}{k}\sqrt{T^2\omega^2 + 1 - k^2k_p^2} + jk_p\omega) \\ &= \pi - \tan^{-1}\frac{kk_p}{\sqrt{T^2\omega^2 + 1 - k^2k_p^2}} \qquad (5.19) \\ &= \pi - \alpha_1^+(\omega) \\ \alpha_2^-(\omega) &= \pi + \arg[\frac{k}{-T\omega^2 + j\omega}] \\ &= \tan^{-1}\frac{1}{T\omega} \qquad (5.20) \\ &= \alpha_2^+(\omega) \end{aligned}$$

where $\alpha_1^-(\omega) \in (\pi/2, \pi)$ and $\alpha_2^-(\omega) \in (0, \pi/2)$ for $k_p > \frac{1}{k}$. Thus $\alpha_1^-(\omega) + \alpha_2^-(\omega) \in (\pi/2, 3\pi/2) \subset [0, 2\pi)$, so that $L(\omega)$ can be decomposed as

$$L(\omega) = \frac{\alpha^-(\omega)}{\omega} = \frac{\alpha_1^-(\omega) + \alpha_2^-(\omega)}{\omega}. \qquad (5.21)$$

We first evaluate $\tan[\alpha_1^-(\omega) + \alpha_2^-(\omega)]$:

$$\tan[\alpha_1^-(\omega) + \alpha_2^-(\omega)] = \frac{\sqrt{T^2\omega^2 + 1 - k^2k_p^2} - kk_pT\omega}{T\omega\sqrt{T^2\omega^2 + 1 - k^2k_p^2} + kk_p}$$

and its derivative:

$$\begin{aligned} &\frac{\mathrm{d}\tan[\alpha_1^-(\omega) + \alpha_2^-(\omega)]}{\mathrm{d}\omega} \\ = &\frac{T(1 + T^2\omega^2)(kk_pT\omega - \sqrt{T^2\omega^2 + 1 - k^2k_p^2})}{(T\omega\sqrt{T^2\omega^2 + 1 - k^2k_p^2} + kk_p)^2\sqrt{T^2\omega^2 + 1 - k^2k_p^2}} \\ > &\frac{T(1 + T^2\omega^2)(T\omega - \sqrt{T^2\omega^2 + 1 - k^2k_p^2})}{(T\omega\sqrt{T^2\omega^2 + 1 - k^2k_p^2} + kk_p)^2\sqrt{T^2\omega^2 + 1 - k^2k_p^2}} \text{ (since } kk_p > 1) \end{aligned}$$

$$> \quad 0$$

Since $\alpha_1^-(\omega)+\alpha_2^-(\omega) \in (\pi/2, 3\pi/2)$, $\alpha_1^-(\omega)+\alpha_2^-(\omega)$ is a monotonically increasing function of ω. Next, we evaluate the derivative of $L(\omega)$.

$$\begin{aligned}
\frac{\mathrm{d}L(\omega)}{\mathrm{d}\omega} &= \frac{\mathrm{d}}{\mathrm{d}\omega}[\frac{\alpha_1^-(\omega)+\alpha_2^-(\omega)}{\omega}] \\
&= \frac{1}{\omega^2}[\frac{kk_pT^2\omega^2}{(1+T^2\omega^2)\sqrt{T^2\omega^2+1-k^2k_p^2}} - \frac{T\omega}{1+T^2\omega^2} \\
&\quad -(\pi - \tan^{-1}\frac{kk_p}{\sqrt{T^2\omega^2+1-k^2k_p^2}} + \tan^{-1}\frac{1}{T\omega})] \\
&= \frac{1}{\omega^2}\{\frac{T\omega}{1+T^2\omega^2}(\frac{kk_pT\omega}{\sqrt{T^2\omega^2+1-k^2k_p^2}} - 1) - [\alpha_1^-(\omega)+\alpha_2^-(\omega)]\} \\
&= \frac{1}{\omega^2}\{\beta(\omega) - [\alpha_1^-(\omega)+\alpha_2^-(\omega)]\}
\end{aligned}$$

where

$$\beta(\omega) = \frac{T\omega}{1+T^2\omega^2}(\frac{kk_pT\omega}{\sqrt{T^2\omega^2+1-k^2k_p^2}} - 1)$$

For $\omega \leq 1/T$,

$$\begin{aligned}
&\frac{\mathrm{d}\beta(\omega)}{\mathrm{d}\omega} \\
= \; & \frac{T}{(1+T^2\omega^2)^2}[kk_p(1+T^2\omega^2)\frac{T\omega}{\sqrt{T^2\omega^2+1-k^2k_p^2}}(1 - \frac{T^2\omega^2}{T^2\omega^2+1-k^2k_p^2}) \\
& +(1-T^2\omega^2)(\frac{kk_pT\omega}{\sqrt{T^2\omega^2+1-k^2k_p^2}} - 1)] \\
< \; & \frac{T}{(1+T^2\omega^2)^2}[(1 - \frac{T^2\omega^2}{T^2\omega^2+1-k^2k_p^2}) + (\frac{kk_pT\omega}{\sqrt{T^2\omega^2+1-k^2k_p^2}} - 1)] \\
& \text{(using } \omega T \leq 1 \text{ and } kk_p > 1) \\
= \; & \frac{T^2\omega}{(1+T^2\omega^2)^2(T^2\omega^2+1-k^2k_p^2)}(kk_p\sqrt{T^2\omega^2+1-k^2k_p^2} - T\omega)
\end{aligned}$$

Since

$$\begin{aligned}(kk_p\sqrt{T^2\omega^2+1-k^2k_p^2})^2-(T\omega)^2 &= k^2k_p^2(1-k^2k_p^2)+k^2k_p^2T^2\omega^2-T^2\omega^2\\ &= (k^2k_p^2-T^2\omega^2)(1-k^2k_p^2)\\ &< 0,\end{aligned}$$

we have

$$\frac{\mathrm{d}\beta(\omega)}{\mathrm{d}\omega}<0.$$

For $\omega>1/T$, $\frac{T\omega}{1+T^2\omega^2}$ and $\frac{kk_pT\omega}{\sqrt{T^2\omega^2+1-k^2k_p^2}}-1$ are both positive while their derivatives are both negative so that when $\omega>1/T$, we have $\mathrm{d}\beta(\omega)/\mathrm{d}\omega<0$.

Thus, for all values of ω, $\beta(\omega)$ is a monotonically decreasing function of ω. At $\omega=\omega_s$,

$$\beta(\omega_s)-[\alpha_1^-(\omega_s)+\alpha_2^-(\omega_s)]=\infty-(\frac{\pi}{2}+\tan^{-1}\frac{1}{T\omega_s})=\infty>0,$$

and at $\omega=\infty$,

$$\beta(\infty)-[\alpha_1^-(\infty)-\alpha_2^-(\infty)]=0-(\pi-0)=-\pi<0.$$

Also as already shown, $\alpha_1^-(\omega)+\alpha_2^-(\omega)$ is a monotonically increasing function of ω. So, there is only one finite solution for the equation

$$\beta(\omega)-[\alpha_1^-(\omega)+\alpha_2^-(\omega)]=0$$

in the interval $(\omega_s,\ \infty)$. The above analysis suggests that $\mathrm{d}L(\omega)/\mathrm{d}\omega$ has only one finite zero, which indicates only *one* maximum point for $L(\omega)$ (see Fig.20 and Fig.21). Depending on the value of L_0, the sets $\mathcal{S}_{L,k_p}^-$ can be characterized as follows:

- For $L_0 \leq L(\omega_s)$, there is only one solution for $L(\omega) = L_0$, denoted by ω_1^- and $\Omega^- = [\omega_1^-, +\infty)$. With the knowledge about the positions of $k_i - k_d\omega^2 = -\sqrt{M(\omega)}$ that we acquired earlier (recall the monotonicity property of $k_{i,\omega}^+$ and $k_{i,\omega}^-$), we have

$$\begin{aligned} \mathcal{S}_{L,k_p}^- &= \{(k_i, k_d)|k_i - k_d\omega^2 = -\sqrt{M(\omega)}, \omega \in \Omega^-\} \cap \mathcal{S}_{1,k_p} \\ &= \{(k_i, k_d)|k_i \leq k_d(\omega_1^-)^2 - \sqrt{M(\omega_1^-)}\} \cap \mathcal{S}_{1,k_p}. \end{aligned} \tag{5.22}$$

- For $L(\omega_s) < L_0 < \max_{\omega \in (\omega_s, \infty)} L(\omega)$, there are two solutions for $L(\omega) = L_0$, denoted as ω_1^- and ω_2^- with $\omega_1^- < \omega_2^-$. So $\Omega^- = [\omega_s, \omega_1^-] \cup [\omega_2^-, +\infty)$, and

$$\begin{aligned} \mathcal{S}_{L,k_p}^- = &\{(k_i, k_d)|k_d(\omega_1^-)^2 - \sqrt{M(\omega_1^-)} \leq k_i \leq k_d(\omega_s)^2 \\ &\text{or } k_i \leq k_d(\omega_2^-)^2 - \sqrt{M(\omega_2^-)}\} \cap \mathcal{S}_{1,k_p}. \end{aligned} \tag{5.23}$$

- For $L_0 > \max_{\omega \in (\omega_s, \infty)} L(\omega)$, there is no solution for $L(\omega) = L_0$ and we have $\Omega^- = [\omega_s, +\infty)$ and

$$\mathcal{S}_{L,k_p}^- = \{(k_i, k_d)|k_i \leq k_d(\omega_s)^2\} \cap \mathcal{S}_{1,k_p}.$$

Now, we can compute $\mathcal{S}_{R,k_p} = \mathcal{S}_{1,k_p} \backslash (\mathcal{S}_{L,k_p}^+ \cup \mathcal{S}_{L,k_p}^-)$.

- For $-\frac{1}{k} < k_p \leq \frac{1}{k}$, $\mathcal{S}_{R,k_p}$ is defined by:

$$\begin{aligned} k_i &> 0 \\ -\frac{T}{k} &< k_d < \frac{T}{k} \\ k_i &< (\omega_1^+)^2 k_d + \sqrt{M(\omega_1^+)} \text{ (using (5.18))} \end{aligned}$$

  where ω_1^+ satisfies

$$L_0 = [\alpha_1^+(\omega_1^+) + \alpha_2^+(\omega_1^+)]/\omega_1^+ \text{ (see (5.15))}.$$

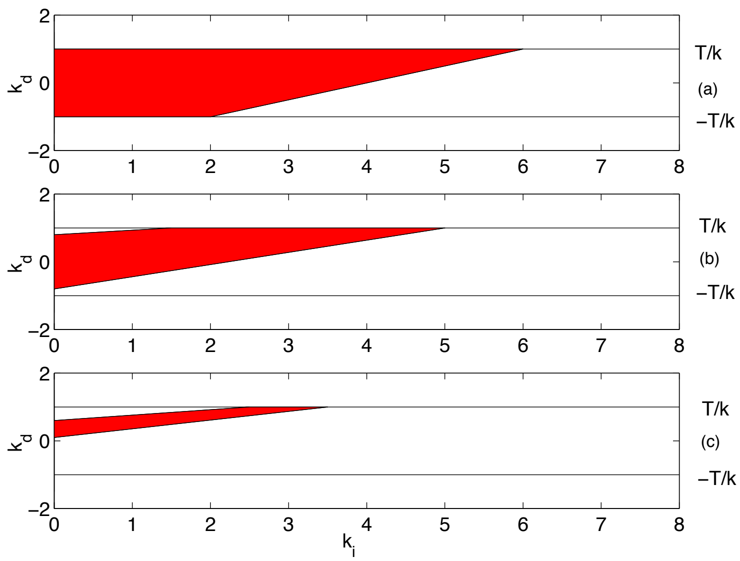

Fig. 22. First-order plant: stabilizing region of(k_i, k_d) with different k_p.

This region $\mathcal{S}_{R,k_p}$ is a trapezoid (see Fig.22(a)).

- For $k_p > \frac{1}{k}$ and $L_0 \leq L(\omega_s) = (\frac{\pi}{2} + \tan^{-1}\frac{1}{T\omega_s})/\omega_s$ (see (5.19), (5.20) and (5.21)) $\mathcal{S}_{R,k_p}$ is given by:

$$\begin{aligned} k_i &> 0 \\ k_d &< \frac{T}{k} \\ k_i &< (\omega_1^+)^2 k_d + \sqrt{M(\omega_1^+)} \text{ (using (5.18))} \\ k_i &> (\omega_1^-)^2 k_d - \sqrt{M(\omega_1^-)}, \text{ (using (5.22))} \end{aligned}$$

where ω_1^+ and ω_1^- satisfy

$$L_0 = [\alpha_1^+(\omega_1^+) + \alpha_2^+(\omega_1^+)]/\omega_1^+,$$

and

$$L_0 = [\alpha_1^-(\omega_1^-) + \alpha_2^-(\omega_1^-)]/\omega_1^-,$$

respectively. This set $\mathcal{S}_{R,k_p}$ is a quadrilateral (see Fig.22(b)).

- For $k_p > \frac{1}{k}$ and $L(\omega_s) < L_0 < \max_{\omega\in(\omega_s,\infty)}[\frac{\alpha_1^-(\omega)+\alpha_2^-(\omega)}{\omega}]$, $\mathcal{S}_{R,k_p}$ is given by:

$$\begin{aligned} k_i &> 0 \\ k_d &< \frac{T}{k} \\ k_i &< (\omega_1^-)^2 k_d - \sqrt{M(\omega_1^-)} (\text{using (5.23)}) \\ k_i &> (\omega_2^-)^2 k_d - \sqrt{M(\omega_2^-)}, (\text{using (5.23)}) \end{aligned}$$

where $\omega_1^- < \omega_2^-$ are solutions of the equation:

$$L_0 = [\alpha_1^-(\omega) + \alpha_2^-(\omega)]/\omega.$$

This set $\mathcal{S}_{R,k_p}$ is also a quadrilateral (see Fig.22(c)).

- For $k_p > \frac{1}{k}$ and $L_0 > \max \frac{\alpha_1^-(\omega)+\alpha_2^-(\omega)}{\omega}$, $\mathcal{S}_{R,k_p} = \emptyset$

The results show that with different k_p values, the stabilizing regions of (k_i, k_d) take on different but simple shapes. They agree with those in [6]

As for the case of an open-loop unstable plant, i.e., $T < 0$, the procedure to obtain the stabilizing regions is similar to the case when $T > 0$ and $k_p > \frac{1}{k}$.

H. Summary

In this chapter, we first clarified the conditions under which the Nyquist Criterion can be applied to time-delay systems. Based on this clarification, a method to compute the set of all P, PI and PID controllers to stabilize a given plant with time-delay was proposed. The procedure is simple and easy to understand. With this known PID stabilizing set in hand, further optimization (design) can be undertaken to satisfy various performance specifications, while meeting the stability constraint.

CHAPTER VI

STABILIZING PI CONTROLLERS FOR SYSTEMS WITH FIXED TIME-DELAYS

PID controllers are widely used in process control applications, and in many cases the plants have time-delays. For some systems with time-delay, Smith predictor combined with PID controllers can simplify the design procedure and achieve good results. The performance of such systems relies on the accurate modeling of the delay. In fact, they will not reject a d.c. load disturbance when there is a modeling error in the dead time [28] while pure PI or PID controllers will still keep this property. Thus, using direct PID controllers where applicable is still a good choice. In previous chapters, the complete set of PID controllers that stabilize a system with time-delay up to a given value L_0 was obtained. For that case, the delay is usually viewed as a modeling error. However, if the delay L_0 is very large, the obtained controllers set might be very small or even disappear since this set must stabilize all the systems with delay less than L_0, including the delay-free system. Designs based on such sets may not yield satisfactory result because of the extremely limited choice of available controller parameters. If we know there is also a lower bound of the delay $L_{\min}$ and the problem of finding the stabilizing set for $[L_{\min}, L_0]$ instead of $[0, L_0]$, a better controller might be found in this larger set. This is the case when there are embedded delays in the systems such as a flow-rate control system where the delay is caused by a long pipe. The thickness control in rolling mills (Example 8.3 in [29]) is also such a case. The general procedures proposed in the previous chapters can be used to achieve this provided we have the complete stabilizing controller set at a fixed delay $L = L_{\min}$ instead of the stabilizing controller set of the delay-free system.

The procedures to generate the stabilizing controller set for systems with fixed delays developed in this chapter are based on a direct analysis method in [26, 30]. This method is based on the fact that under certain conditions, we can count the RHP poles of the closed-loop system by tracking the number of roots crossing the imaginary axis at a finite number of frequencies. In [31], a formula has been given to compute the RHP poles. However, [31] did not include the situations where there are multiple closed-loop pure imaginary poles at the same frequency at a certain delay. Although [30] considered those situations, it did not clearly indicate the movement of the poles. By applying the Nyquist Criterion to the time-delay systems, which was validated previously in this dissertation, this chapter shows a complete picture of the crossing poles in different cases and gives the general formula to compute the number of RHP poles of the closed-loop system. Based on this formula complete stabilizing sets for P and PI controllers for embedded delay systems are found.

A. Stability Analysis of Time-Delay Systems

Consider a unity feedback system with open-loop transfer function:

$$P(s) = P_0(s)e^{-Ls} = \frac{N(s)}{D(s)}e^{-Ls}, \tag{6.1}$$

where

$$\begin{aligned} N(s) &= b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0, \\ D(s) &= a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0. \end{aligned}$$

The closed-loop characteristic quasi-polynomial is

$$\delta(s, L) = D(s) + N(s)e^{-Ls}. \tag{6.2}$$

From the discussion in previous chapters, we know that the necessary condition for this system with positive delay to be stable is

$$
\begin{aligned}
& m < n \\
\text{or} \quad & m = n \text{ and } |b_n| < |a_n|.
\end{aligned} \tag{6.3}
$$

Under this condition, we have

1. With the introduction of delay, an infinite number of new roots of (6.2) appear in the LHP.

2. For a given delay, the number of RHP roots is finite and those roots are in a finite or bounded region.

3. With the increase of delay, root crossings between LHP and RHP only happen at the imaginary axis.

4. The Nyquist Criterion with the conventional contour can be used for stability analysis of the time-delay system.

Thus, we can calculate the number of RHP poles of the system with delay L by using the following guideline:

$$N_L = N_0 + N^+ - N^-, \tag{6.4}$$

where N_L is the number of RHP poles of the system with delay L, N_0 is the number of RHP poles of the delay-free system, and N^+ (N^-) is the number of the poles crossing from LHP (RHP) to RHP (LHP) when the delay is increased from 0 to L.

When the Nyquist plot crosses the -1 point, i.e. there are pure imaginary closed-loop poles, from [26], we know that the solutions of the magnitude condition

$$W(\omega^2) \equiv |D(j\omega)|^2 - |N(j\omega)|^2 = D(j\omega)D(-j\omega) - N(j\omega)N(-j\omega) = 0 \tag{6.5}$$

are possible crossing frequencies at those delays L which satisfies

$$\cos \omega L = \mathrm{Re}\left\{-\frac{D(j\omega)}{N(j\omega)}\right\}, \quad \sin \omega L = \mathrm{Im}\left\{\frac{D(j\omega)}{N(j\omega)}\right\}. \tag{6.6}$$

Here, the corresponding values of L (critical delays) can also be written as

$$L = L_m(\omega) + 2l\pi/\omega, \quad l = 0, 1, 2, ...,$$

while $L_m(\omega) =$ is the smallest non-negative solution of (6.6), and for each ω satisfing (6.5) there are an infinite number of values of L. On the other hand, for a given $L_0 > 0$, we can define a σ for each crossing frequency ω:

$$\sigma(\omega) = \left\lceil \frac{L_0 - L_m(\omega)}{2\pi/\omega} \right\rceil, \tag{6.7}$$

where $\lceil \cdot \rceil$ is the ceiling function. Obviously, $\sigma(\omega)$ is the number of times when root crossing happens at $j\omega$ with the delay increasing from 0 to L_0 (including the root crossing at $L = 0$ if there are root crossing when the delay is introduced).

Remark 9 *A special case here is when $W(\omega^2) = 0$ has $\omega = 0$ as one of its roots. In this case, if $a_0 = -b_0$, this means the closed-loop system will always have a pole at the origin with or without delay. Thus the system is always unstable. On the other hand, if $a_0 = b_0$, it does not give us a L_m and will not affect our analysis.*

Equation (6.5) gives us a finite number of real roots. We only need to consider those positive real ω's since the Nyquist plot and the distribution of the closed-loop poles are symmetric for systems with real coefficients. These roots can be classified according to the behavior of the function $W(\omega^2)$ at those points. The following lemma describes the movement of the closed-loop poles.

Lemma 3 *Suppose at a certain $L > 0$, the closed-loop system has poles on the imaginary axis at $\pm j\omega_s$ (Nyquist plot crosses -1 point at $\pm\omega_s$). When the delay is changed* *from $L - dL$ to $L + dL$ (dL is a infinitesimal positive value), the root crossings occur as following:*

1. *If $W(\omega^2)$ crosses the ω-axis from above to below (Nyquist plot cuts the unit circle from inside), one pair of poles cross from RHP to LHP. Such an ω_s is a "stabilizing" frequency.*

2. *If $W(\omega^2)$ crosses the ω-axis from below to above (Nyquist plot cuts the unit circle from outside), one pair of poles cross from LHP to RHP. Such ω_s is a "destabilizing" frequency.*

3. *If $W(\omega^2)$ touches the ω-axis without crossing it, there is no pole crossing the imaginary axis. Such ω_s is a "touching" frequency.*

Proof: From the Nyquist plot for positive ω, the presence of positive delay will make the plot shift by an angle of $L\omega$ clockwisely while preserving the magnitude of the plot. The larger the delay and the frequency, the larger the phase-shift. Thus for case 1, the Nyquist plot shifts as in Fig. 23. We can see obviously that the movement of the plot gives us -1 change in the number of clockwise encirclements around -1 point. Thus the complete Nyquist plot has a -2 change in the number of clockwise encirclements, which means that two RHP poles cross the imaginary axis into the LHP for increasing L at that value. Similarly, for case 2, as in Fig. 24, the number of clockwise encirclements increases by 2, i.e. the number of RHP poles increases by 2. For case 3, the change of delay from $L - dL$ to $L + dl$ does not change the number of encirclements and the number of RHP poles remains the same (see Fig. 25 and Fig. 26). ♣

Remark 10 *From the proof of the lemma, we can see that the directions of the root crossings at the solutions of $W(\omega^2) = 0$ are fixed and they are independent of the delay.*

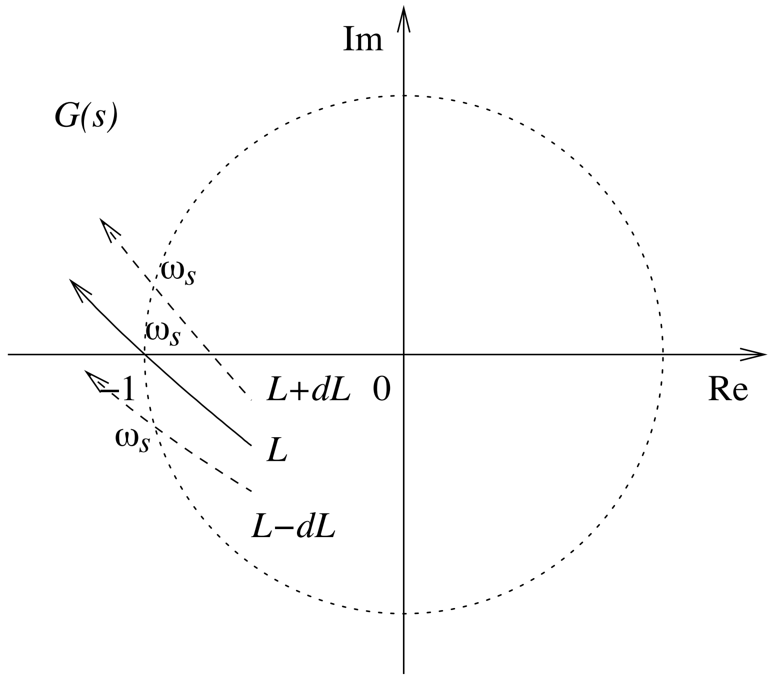

Fig. 23. Nyquist plot cuts the unit circle from inside.

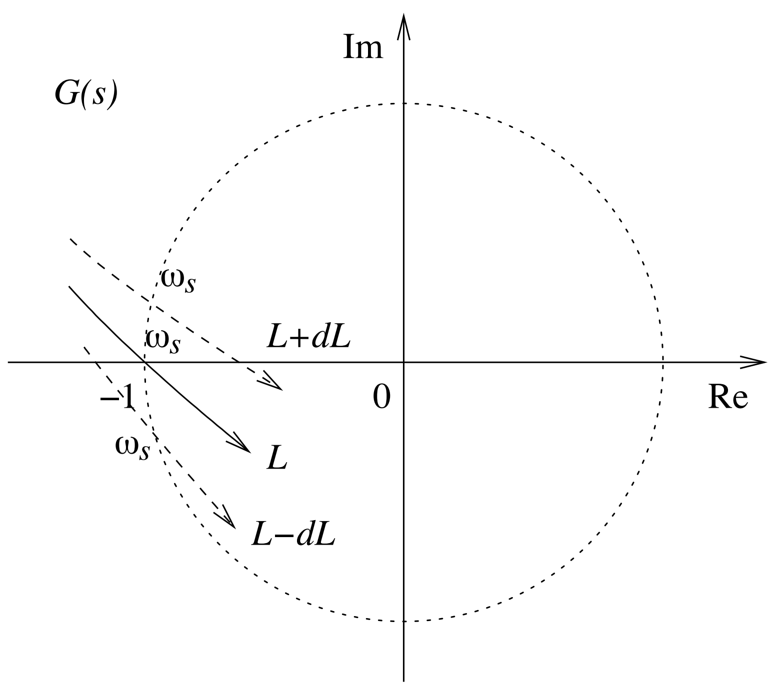

Fig. 24. Nyquist plot cuts the unit circle from outside.

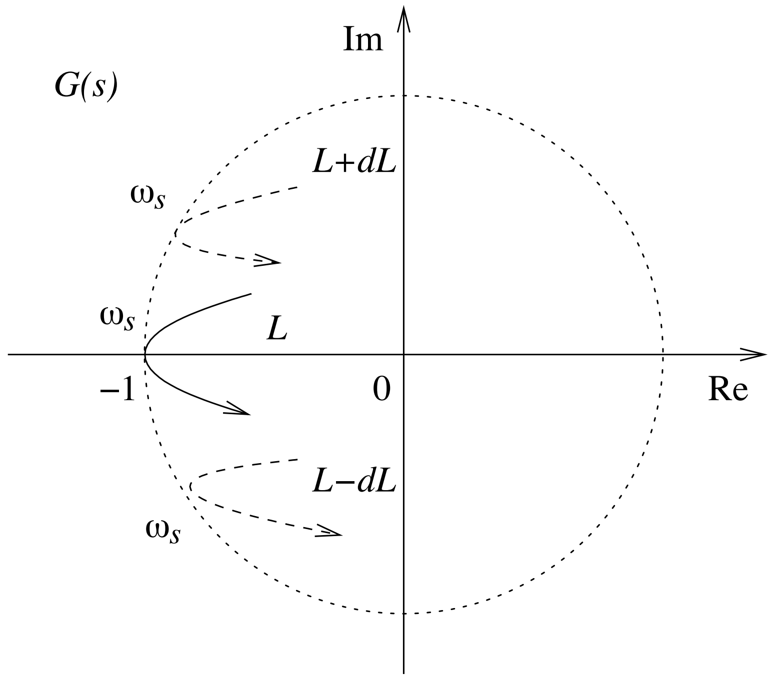

Fig. 25. Nyquist plot touches the unit circle from inside.

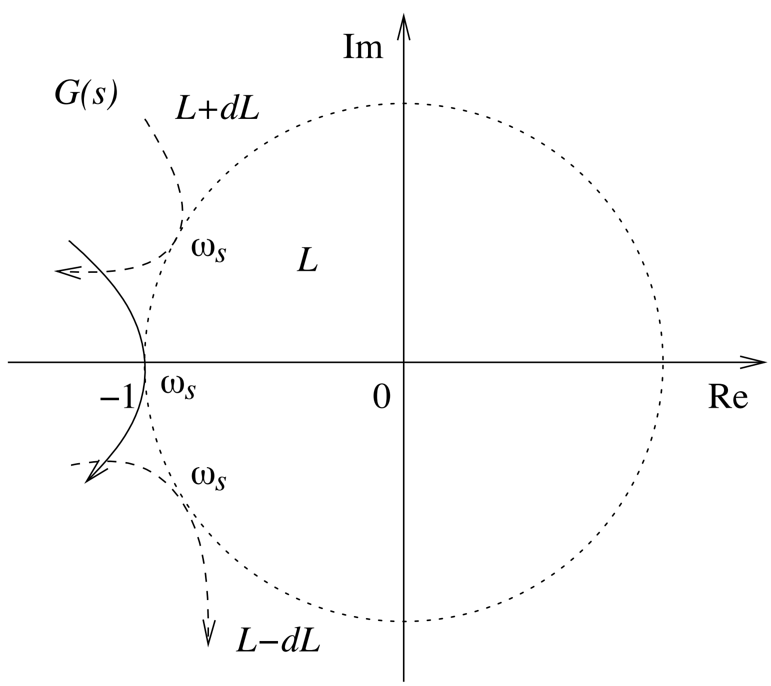

Fig. 26. Nyquist plot touches the unit circle from outside.

Closed-loop poles will always move from RHP into LHP at stabilizing frequencies and from LHP into RHP at destabilizing frequencies. The crossing frequencies (stabilizing or destabilizing frequencies) are the roots of $W(\omega^2) = 0$ *with odd multiplicity and the touching frequencies are roots with even multiplicity.*

Remark 11 *Under condition (6.3), the Nyquist plot of the system will end inside the unit circle when* $\omega = \infty$*. If there are crossing frequencies, at the largest crossing frequency, the Nyquist plot will always cut the unit circle from outside. Thus the largest crossing frequency is always a destabilizing frequency. Because* $W(\omega^2)$ *crosses the* ω*-axis at different directions for two adjacent crossing frequencies, those crossing frequencies are successively destabilizing, stabilizing, etc. in descending order.*

If the system has pure imaginary closed-poles, the following lemma can be used to determine their movement when an infinitesimal positive delay dL is introduced.

Lemma 4 *Suppose the delay-free system has a pair of pure imaginary closed-loop poles at* $\pm j\omega_s$*, each with multiplicity* m*. With the introduction of an infinitesimal positive delay* dL

1. *If* ω_s *is a stabilizing frequency, then there will be* $m - 1$ *new RHP closed-loop poles.*

2. *If* ω_s *is a destabilizing frequency, then there will be* $m + 1$ *new RHP closed-loop poles.*

3. *If* ω_s *is a touching frequency, then there will be* m *new RHP closed-loop poles.*

Proof: We will use a modified Nyquist contour as in Fig. 27. The delay-free Nyquist plot will have $m/2$ counter-clockwise encirclements around the -1 point. For case 1, m is odd. Suppose $m = 2n + 1$ as shown in Fig. 28. With the presence of the

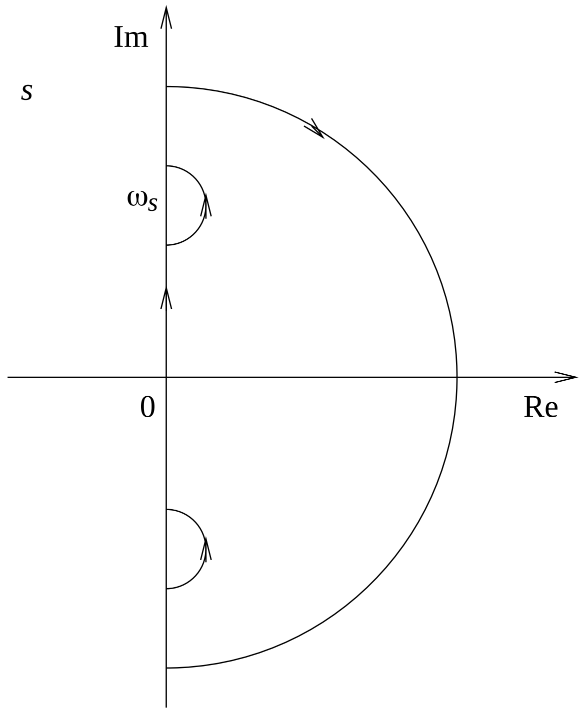

Fig. 27. A modified Nyquist contour.

delay dL, the number of counter-clockwise encirclements will decrease by n, i.e. the number of clockwise encirclements will increase by n. Thus, the complete Nyquist plot will have $2n = m - 1$ more clockwise encirclements. So, $m - 1$ closed-loop poles enter the area enclosed by the modified Nyquist contour or the open RHP. For case 2, m is also odd. Let $m = 2n + 1$ as in Fig. 29. Similarly, we can see that number of the clockwise encirclements will increase by $2(n + 1) = m + 1$, i.e. $m + 1$ new closed-loop poles appear in the open RHP. For case 3, m is even. Suppose $m = 2n$ as in Fig. 30 and Fig. 31, we can conclude that m closed-loop poles will enter the open RHP. ♣

Assume that the system has open loop transfer function (6.1). The corresponding $W(\omega^2) = 0$ has following positive roots which are crossing frequencies,

$$0 < \omega_1 < \omega_2 < \cdots < \omega_p < +\infty,$$

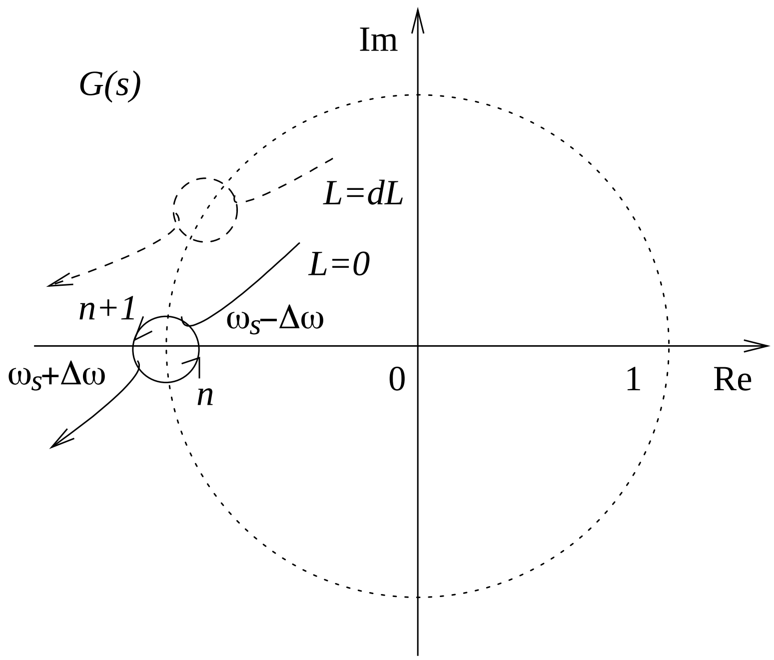

Fig. 28. Nyquist plot at stabilizing frequency.

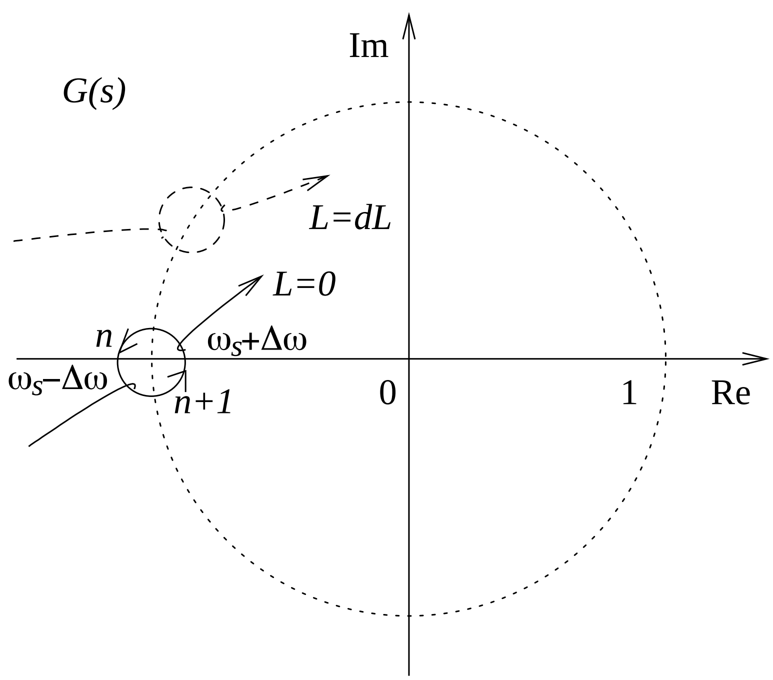

Fig. 29. Nyquist plot at destabilizing frequency.

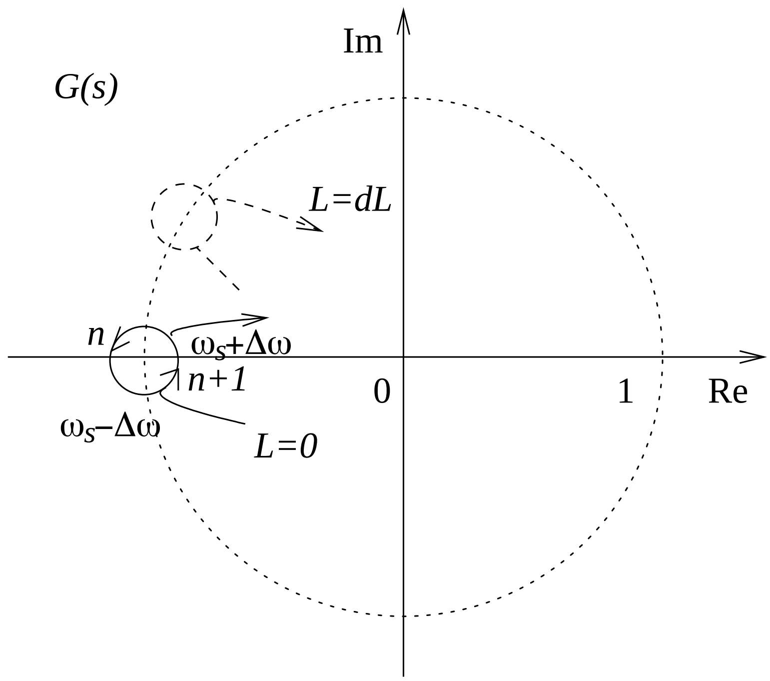

Fig. 30. Nyquist plot at touching frequency, case 1.

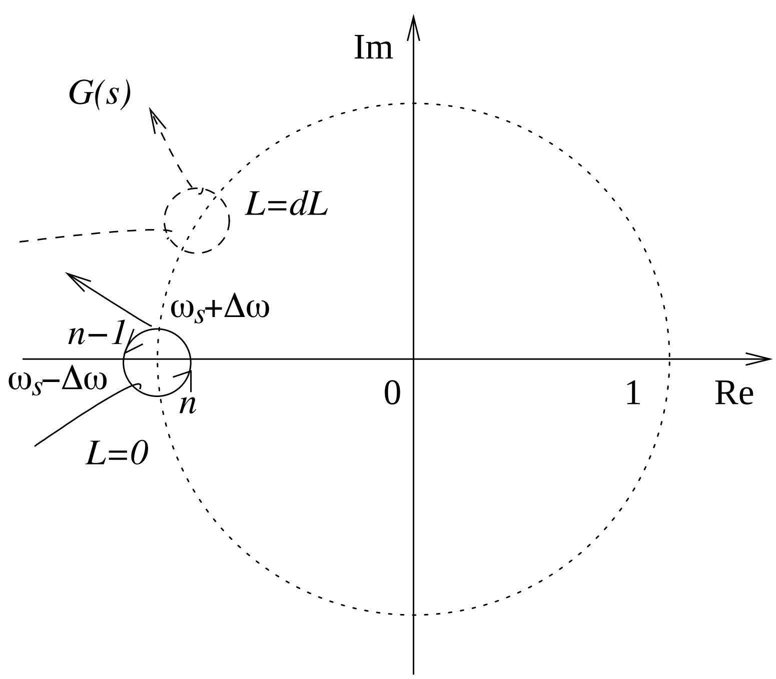

Fig. 31. Nyquist plot at touching frequency, case 2.

and following positive roots which are touching frequencies,

$$\nu_1,\ \nu_2,\ \cdots,\ \nu_q.$$

The delay-free system has pure imaginary closed-loop poles at $\pm j\mu_k$ with multiplicity n_k, $k = 1, 2, \cdots, t$. Note that

$$\{\mu_1,\ \mu_2, \cdots, \mu_t\} \subseteq \{\omega_1,\ \omega_2, \cdots, \omega_p, \nu_1,\ \nu_2, \cdots, \nu_q\}.$$

Let r be the number of the μ_k's which are destabilizing frequencies, and s be the number of the stabilizing frequencies among μ_k's.

Theorem 7 *Let Δ be the number of RHP closed-loop poles of the system (6.1) at delay $L = L_0 > 0$. Assume system satisfies condition (6.3) and L_0 is not a critical delay, then*

$$\Delta = \Delta_0 + \Delta_X, \tag{6.8}$$

where

$$\begin{aligned} \Delta_X &= \sum_{k=1}^{t} n_k + s - r + 2\sum_{k=1}^{p}(-1)^{p-k}\sigma_k \\ &= \sum_{k=1}^{t} n_k + s - r + 2\sum_{k=1}^{p}(-1)^{p-k}\left\lceil \frac{L_0 - L_m(\omega_k)}{2\pi/\omega_k} \right\rceil, \end{aligned}$$

Δ_0 is the number of RHP closed-loop poles of the delay-free system, $L_m(\omega_k)$ is the smallest non-negative solution of (6.6) with $\omega = \omega_k$, and $\sigma_k = \sigma(\omega_k)$ is as defined in (6.7).

Proof: With the introduction of time-delay, if μ_k is a stabilizing frequency, $n_k - 1$ poles will move into RHP at $\pm j\mu_k$ according to Lemma 4. Similarly, $n_k + 1$ or n_k poles will move into RHP if μ_k is a destabilizing frequency or touching frequency.

Thus, at that moment, the number of RHP poles will increase by

$$\sum_{k=1}^{t} n_k + r - s. \tag{6.9}$$

As analyzed before, the largest crossing frequency ω_p is destabilizing frequency when the system satisfies (6.3), and the crossing frequencies are successively destabilizing, stabilizing, etc. in descending order. From the definition, σ_k represents the number of times when root crossing happens at $j\omega_k$ with the delay increasing from 0 to L_0. Thus with the increase of the delay, there are $2\sum_{k=1}^{p}(-1)^{p-k}\sigma_k$ poles moving into RHP. However, if some $\omega_i = \mu_j$ is a destabilizing frequency, 2 RHP poles which were included in (6.9) have been counted again here and they should be subtracted. Similarly, for some $\omega_i = \mu_j$ which is a stabilizing frequency, 2 extra RHP poles have been subtracted here and they should be added back. Thus, when the delay is introduced and then increases to L_0, the change of the number of RHP poles is

$$\begin{aligned} & \sum_{k=1}^{t} n_k + r - s + 2\sum_{k=1}^{p}(-1)^{p-k}\sigma_k - 2r + 2s \\ = & \sum_{k=1}^{t} n_k + s - r + 2\sum_{k=1}^{p}(-1)^{p-k}\sigma_k \\ = & \Delta_X \end{aligned}$$

Following the root counting guideline (6.4), combine Δ_X with the number of original RHP poles of the delay-free plant, Δ_0, we will have the number of RHP poles when delay is L_0. So $\Delta = \Delta_0 + \Delta_X$. ♣

Remark 12 *Since the poles move into RHP in pairs, it is obvious that if the delay-free closed-loop system has an odd number of RHP poles, it will not be stable at any amount of positive delay L.*

B. Stabilization of Time-Delay Systems with Proportional Controllers

Now we will use a proportional controller

$$C(s) = k_p$$

to stabilize a given plant

$$P(s) = P_0(s)e^{-Ls} = \frac{N(s)}{D(s)}e^{-Ls},$$

at $L = L_0$.

Let us first consider the delay-free system. k_p space can be divided into a finite number of open sets

$$K_{0,1}, \;\; K_{0,2}, \;\cdots\;, \;\; K_{0,r},$$

s.t. for any $k_p \in K_{0,i}, \; i = 1, 2, \cdots, r$, the number of open RHP poles of the delay-free system is constant; this number is denoted as $\Delta_{0,i}$.

$W(\omega^2) = 0$ is equivalent to

$$k_p = \pm\frac{1}{|P_0(j\omega)|},$$

or

$$U(\omega^2) + k_p^2 V(\omega^2) = 0 \tag{6.10}$$

where

$$\begin{aligned} U(\omega^2) &= D(j\omega)D(-j\omega), \\ V(\omega^2) &= -N(j\omega)N(-j\omega). \end{aligned}$$

The approach used in [2] can be used here to determine the number of positive real roots of (6.10) and the corresponding range of k_p^2. Note that except on those finite

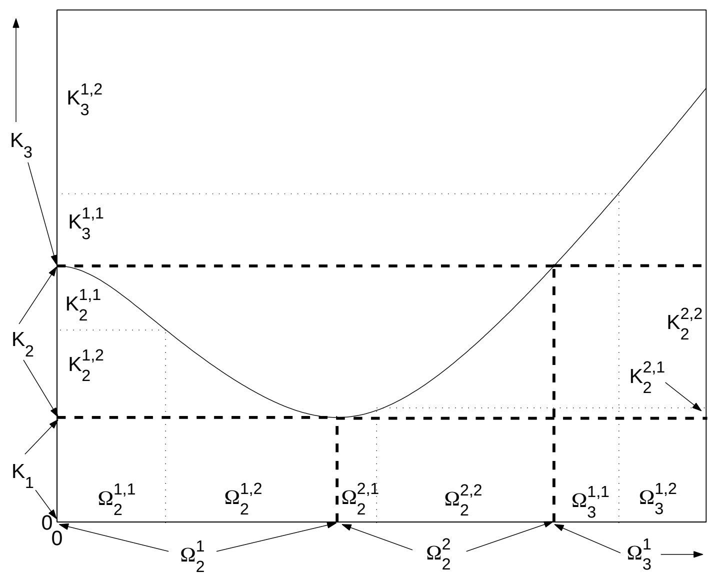

Fig. 32. Illustrative partition of k_p.

number of breakaway points, for all other k_p values, the positive real roots of (6.5) are simple roots, i.e. they are all crossing frequencies.

Thus, we can divide k_p space into disjoint open intervals

$$K_1, \; K_2, \; \cdots \;, \; K_t,$$

(see Fig. 32) s.t. for any $k_p \in K_i, \; i = 1, 2, \cdots, t$, the number of crossing frequencies is constant; this number is denoted as n_i.

For such an interval K_i, if $n_i > 0$, the corresponding n_i frequencies distribute in n_i disjoint open intervals in ω space

$$\Omega_i^1, \; \Omega_i^2, \; \cdots \;, \; \Omega_i^{n_i},$$

(see Fig. 32 and Fig. 33) s.t. Ω_i^j is on the left side of Ω_i^{j+1} and for a fixed $k_p \in K_i$, there is one and only one crossing frequency in each of such interval. These intervals

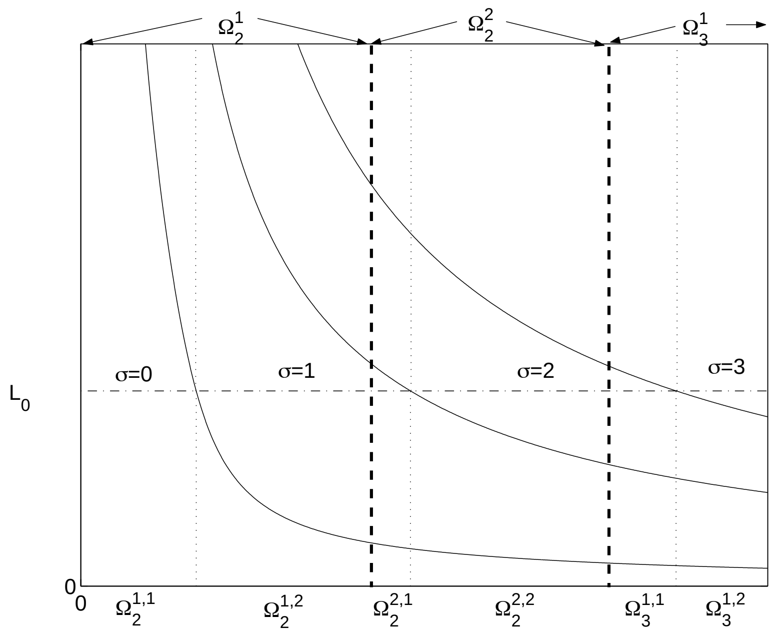

Fig. 33. Illustrative partition of ω.

are called frequency segments.

Each frequency segment Ω_i^j can be further divided into disjoint open intervals, or frequency subsegments

$$\Omega_i^{j,1},\ \ \Omega_i^{j,2},\ \cdots,$$

(see Fig. 33) s.t. $\Omega_i^{j,l}$ is on the left side of $\Omega_i^{j,l+1}$ and over each subsegment σ is constant. The number of subsegments of Ω_i^j is denoted as n_i^j. The number σ corresponding to the subsegment $\Omega_i^{j,l},\ l = 1, 2, \cdots, n_i^j$, is denoted as $\sigma_i^{j,l}$. Also denote the k_p corresponding to $\Omega_i^{j,l}$ as $K_i^{j,l}$, which is also an open interval and $\cup_{l=1}^{n_i^j} K_i^{j,l} = K_i$ (see Fig. 32).

Pick one frequency subsegment from each of these n_i frequency segments

$$\Omega_i^{1,h_1},\ \ \Omega_i^{2,h_2},\ \cdots\ ,\ \ \Omega_i^{n_i,h_{n_i}},$$

where $h_j \in [1, 2, \cdots, n_i^j],\ \ j = 1, 2, \cdots, n_i$. A string $I = \{h_0, h_1, h_2, \cdots, h_{n_i}\}$ is a selection of n_i subsegments. If

$$K_i(I) = \bigcap_{j=1}^{n_i} K_i^{j,h_j} \neq \emptyset,$$

then

$$\Delta_X(I) = 2\sum_{j=1}^{n_i}(-1)^{n_i-j}\sigma_i^{j,h_j}$$

is the increase in the number of RHP poles of the closed loop system when the delay is increased from 0 to L_0 for a fixed $k_p \in K_i(I)$.

If $n_i = 0$, there is no crossing frequency for any $k_p \in K_i$, then $I = \emptyset$, $K_i(I) = K_i$ and $\Delta_X(I) = 0$.

If there is a $K_{0,s},\ s = 1, 2, \cdots, r$, s.t. the corresponding $\Delta_{0,s} = -\Delta_X(I)$ and

$$K_i^*(I) = K_{0,s} \bigcap K_i(I) \neq \emptyset,$$

then for any $k_p \in K_i^*(I)$, the number of RHP poles with delay L_0 is 0. Such a string I is called an admissible string and the set of all the admissible strings with respect to K_i is denoted as F_i.

The final set is

$$K = \bigcup_{i=1}^{t} \left[\bigcup_{I \in F_i} K_i^*(I) \right] = \bigcup_{i=1}^{t} K_i^*. \tag{6.11}$$

Theorem 8 *The set K defined in (6.11) is the complete stabilizing controller set except on the boundary points of*

$$K_{0,1},\ K_{0,2},\ \cdots,\ K_{0,r},$$

and

$$K_i^{j,l},\ i = 1, 2, \cdots, t;\ j = 1, 2, \cdots, n_i;\ l = 1, 2, \cdots, n_i^j.$$

Proof: For a $k_{p0} \in K_i$, suppose $k_{p0} \in K_{0,s}$. $W(\omega^2)$ of this system then has n_i simple positive real roots, according to the definition of K_i. Denote them as

$$0 < \omega_1 < \omega_2 < \cdots < \omega_{n_i},$$

and $\omega_j \in \Omega_i^{j,h_j}$.

If k_{p0} can stabilize the system, according to Theorem 7, provided the fact that in this case there is no touching frequency and no imaginary closed-loop poles for the delay-free system. For this string

$$I = \{h_1,\ h_2,\ \cdots,\ h_{n_i}\},$$

the number of RHP poles is $\Delta_{0,s} + \Delta_X(I) = 0$. Thus $\Delta_{0,s} = -\Delta_X(I)$. Also, since

$$K_i^*(I) = K_{0,s} \bigcap K_i(I) = K_{0,s} \bigcap \left[\bigcap_{j=1}^{n_i} K_i^{j,h_j} \right] \ni k_{p0},$$

$K_i^*(I) \neq \emptyset$. I is an admissible string of K_i.

On the other hand, if $k_p^* \in K_i^*(I^*)$ for some admissible string

$$I^* = \{h_1^*,\ h_2^*,\ \cdots,\ h_{n_i}^*\},$$

then $k_p^* \in K_i^{j,h_j^*}$, $j = 1,\ 2, ...,\ n_i$ and there exists some s^*, $s^* \in \{1, 2, \cdots, r\}$, s.t. $k_p^* \in K_{0,s^*}$ and

$$\Delta_X(I^*) = 2 \sum_{j=1}^{n_i} (-1)^{n_i - j} \sigma_i^{j,h_j^*} = -\Delta_{0,s^*}$$

From the definition of frequency subsegments, there is one and only one ω_j^* in each Ω_i^j s.t. $W(\omega_j^{*2}) = 0$. So ω_j^*'s are crossing frequencies and $\sigma(\omega_j^*) = \sigma_i^{j,h_j^*}$. For this k_p^*,

$$\Delta = \Delta_{0,s^*} + 2 \sum_{j=1}^{n_i} (-1)^{n_i - j} \sigma_i^{j,h_j^*} = \Delta_{0,s^*} + \Delta_X(I^*) = 0,$$

so the system is stable. ♣

Now for those k_p's which are the terminals of $K_{0,i}$ and $K_i^{j,l}$, we can check their stability individually or simply using the following analysis.

1. If k_p's are terminals of $K_{0,i}$, they can be classified as

    (a) The one when the delay-free closed-loop system has poles at the origin. There will be poles at the origin with or without delay as analysed before. So it is unstable.

    (b) The one lowers the order of the closed-loop system. Such situation is usually classified as unstable.

    (c) The one when there are pure imaginary poles for the delay-free closed-loop system. It is actually the situation 2b. We will check it there.

2. For terminals of $K_i^{j,l}$.

    (a) If it is introduced because one of its corresponding ω is at the intersection of $L(\omega)$ and $L = L_0$, the system is unstable because there are poles on the imaginary axis.

    (b) If it is introduced because one of its corresponding ω is at the intersection of $L(\omega)$ and $L = 0$, then there is no imaginary axis poles and we can decide the stability by the stability of the adjacent region.

    (c) If it is one of terminals of K_i's, we will check it in the following case.

3. For the terminals of K_i, if the corresponding ω's are not the intersection points of $L(\omega)$ and $L = L_0$ as in 2a, the stability or instability of the system is the same as that of the adjacent region.

C. Stabilization of Time-Delay Systems with PI Controllers

Here we will use a PI controller

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

to control the time-delay plant.

For a fixed k_p, we will have

$$k_i = \pm\sqrt{\frac{\omega^2}{|P_0(j\omega)|^2} - k_p^2\omega^2} = \pm\sqrt{M(\omega^2)}, \tag{6.12}$$

$$L_m(\omega) = \frac{\arg\{[\pm\sqrt{M(\omega^2)} + jk_p\omega]P_0(j\omega)/j\omega\} + \pi}{\omega}, \tag{6.13}$$

where

$$M(\omega^2) = \frac{\omega^2}{|P_0(j\omega)|^2} - k_p^2\omega^2.$$

Furthermore, (6.12) is equivalent to

$$U(\omega^2) + k_i^2 V(\omega^2) = 0, \tag{6.14}$$

where

$$\begin{aligned} U(\omega^2) &= \omega^2[D(j\omega)D(-j\omega) - k_p^2 N(j\omega)N(-j\omega)], \\ V(\omega^2) &= -N(j\omega)N(-j\omega). \end{aligned}$$

Now as a one-parameter problem, we can use the exaxct approach we used in proportional controller case to get the stabilizing set of k_i for this fixed k_p. By sweeping over k_p, we then have the complete stabilizing set of PI controllers for the plant.

D. Example

In the following example, we will use Proportional controllers to stabilizing a system with fixed time-delay. It is used here to demonstrate the procedures proposed in Section C.

Example 7 *Find all the Proportional controllers which stabilize the plant*

$$P(s) = \frac{s^2 + 3s - 2}{s^3 + 2s^2 + 3s + 2} e^{-Ls},$$

with $L = L_0 = 1.8$*.*

Solution: First, k_p can be divided into $K_{0,i}$ according to the RHP poles of the delay-free closed-loop system.

i	$K_{0,i}$	$\Delta_{0,i}$
1	$(-\infty, -0.4093)$	2
2	$(-0.4093, 1)$	0
3	$(1, +\infty)$	1

For $k_p > 0$, it can also be divided into K_i according to the number of real positive roots of $W(\omega)$, Fig. 34.

i	K_i	n_i
1	$(0, 0.4082)$	0
2	$(0.4082, 1)$	2
3	$(1, \infty)$	1

1. For K_1, since $n_1 = 0$, the set of all possible strings is

$$\{\emptyset\}.$$

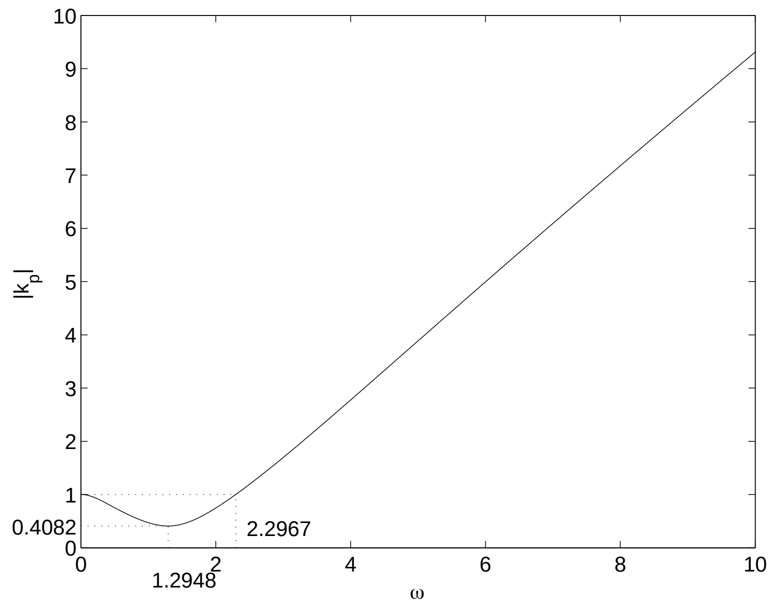

Fig. 34. Partition k_p.

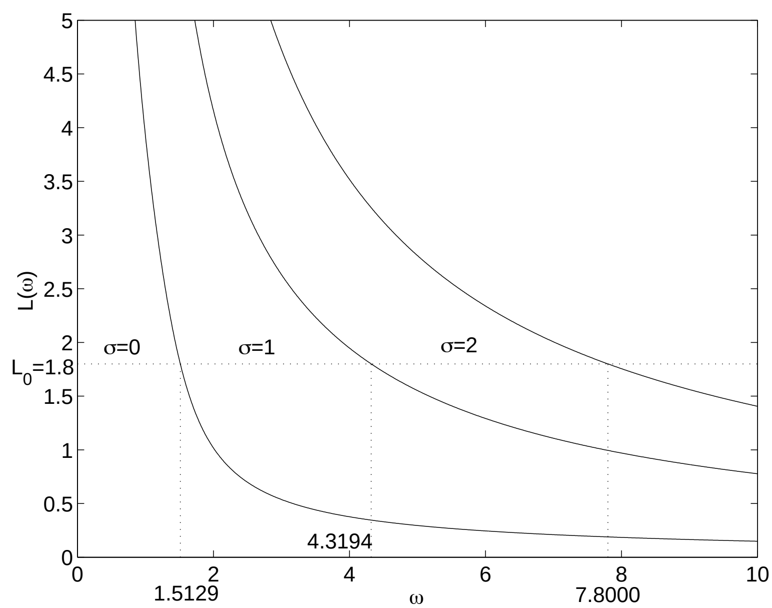

Fig. 35. Different σ's over ω for $k_p > 0$.

For $I = \emptyset$, $K_1(I) = K_1$ and $\Delta_X(I) = 0$. There exists $K_{0,2}$ with $\Delta_{0,2} = 0$ and

$$K_1^*(I) = K_{0,2} \cap K_1(I) = (0, 0.4082) \neq \emptyset.$$

Thus the set of admissible strings is

$$F_1 = \{\emptyset\},$$

and

$$K_1^* = (0, 0.4082).$$

2. For K_2, from Fig. 35, we have following subsegments

$$\begin{array}{llll} n_2^1 = 1 & \Omega_2^{1,1} = (0, 1.2948) & K_2^{1,1} = (0.4082, 1) & \sigma_2^{1,1} = 0 \\ n_2^2 = 2 & \Omega_2^{2,1} = (1.2949, 1.5129) & K_2^{2,1} = (0.4082, 0.4473) & \sigma_2^{2,1} = 0 \\ & \Omega_2^{2,2} = (1.5129, 2.2967) & K_2^{2,2} = (0.4473, 1) & \sigma_2^{2,2} = 1 \end{array}$$

So, the set of all possible strings is

$$\{\{1,1\}, \{1,2\}\}.$$

For $I = \{1, 1\}$,

$$K_2(I) = K_2^{1,1} \cap K_2^{2,1} = (0.4082, 1) \cap (0.4082, 0.4473) = (0.4082, 0.4473),$$

and $\Delta_X(I) = 0$. There exists $K_{0,2}$ with $\Delta_{0,2} = 0$ and

$$K_2^*(I) = K_{0,2} \cap K_2(I) = (0.4082, 0.4473) \neq \emptyset.$$

Thus, this string is an admissible string.

For $I = \{1, 2\}$,

$$K_2(I) = K_2^{1,1} \cap K_2^{2,2} = (0.4082, 1) \cap (0.4473, 1) = (0.4473, 1),$$

and $\Delta_X(I) = 2$. Since $\Delta_X(I) > 0$, there does not exist a $\Delta_{0,s} = -\Delta_X(I) < 0$. Thus, this string is not an admissible string.

So set of the admissible strings is

$$F_2 = \{\{1,1\}\},$$

and

$$K_2^* = \bigcup_{I \in F_2} K_2^*(I) = (0.4082, 0.4473).$$

3. For K_3, the subsegments are

$$\begin{array}{llll} n_3^1 = \infty & \Omega_3^{1,1} = (2.2967, 4.3194) & K_3^{1,1} = (1, 3.1288) & \sigma_3^{1,1} = 1 \\ & \Omega_3^{1,2} = (4.3194, 7.8000) & K_3^{1,2} = (3.1288, 6.9608) & \sigma_3^{1,2} = 2 \\ & \ldots & \ldots & \ldots \end{array}$$

The set of all possible strings is

$$\{\{1\}, \{2\}, \{3\}, \cdots\}.$$

Since for any possible string I, $\Delta_X(I) > 0$, there does not exist an admissible string in the above set. Thus, the set of the admissible strings is

$$F_3 = \emptyset,$$

and

$$K_3^* = \emptyset.$$

Thus, for $k_p > 0$ we have

$$K = \bigcup_{i=1}^{3} K_i^* = (0, 0.4082) \cup (0.4082, 0.4473).$$

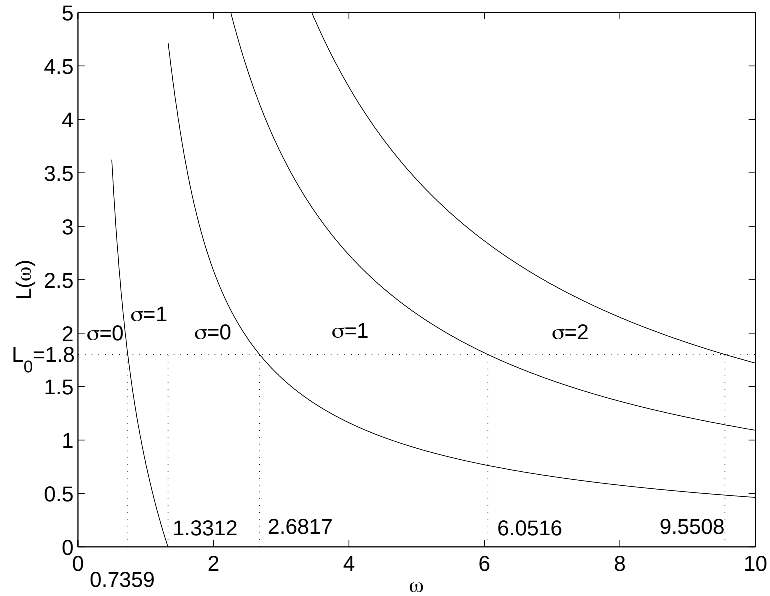

Fig. 36. Different σ's over ω for $k_p < 0$.

For $k_p < 0$, similarly we have,

i	K_i	n_i
1	$(-0.4082, 0)$	0
2	$(-1, -0.4082)$	2
3	$(-\infty, -1)$	1

1. For K_1, the set of all possible strings is

$$\{\emptyset\}.$$

For $I = \emptyset$, $K_1(I) = K_1$ and $\Delta_X(I) = 0$. There exists $K_{0,2}$ with $\Delta_{0,2} = 0$ and

$$K_1^*(I) = K_{0,2} \cap K_1(I) = (-0.4082, 0) \neq \emptyset.$$

Thus the set of admissible strings is

$$F_1 = \{\emptyset\},$$

and

$$K_1^* = (-0.4082, 0).$$

2. For K_2, from Fig. 36, there are following subsegments

$$\begin{array}{lllll} n_2^1 = 2 & \Omega_2^{1,1} = (0, 0.7359) & K_2^{1,1} = (-1, -0.6025) & \sigma_2^{1,1} = 0 \\ & \Omega_2^{1,2} = (0.7359, 1.2948) & K_2^{1,2} = (-0.6025, -0.4082) & \sigma_2^{1,2} = 1 \\ n_2^2 = 2 & \Omega_2^{2,1} = (1.2948, 1.3312) & K_2^{2,1} = (-0.4093, -0.4082) & \sigma_2^{2,1} = 1 \\ & \Omega_2^{2,2} = (1.3312, 2.2967) & K_2^{2,2} = (-1, -0.4093) & \sigma_2^{2,2} = 0 \end{array}$$

The set of all possible strings is

$$\left\{ \begin{array}{cc} \{1,1\} & \{1,2\} \\ \{2,1\} & \{2,2\} \end{array} \right\}.$$

For $I = \{1, 1\}$, $\Delta_X(I) = 2 > 0$. Thus it is not an admissible string.

For $I = \{1, 2\}$, $K_2(I) = (-1, -0.6025)$ and $\Delta_X(I) = 0$. There exists $K_{0,2}$ with $\Delta_{0,2} = 0$ but

$$K_2^*(I) = K_{0,2} \cap K_2(I) = \emptyset.$$

Thus, it is not an admissible string.

For $I = \{2, 1\}$, $K_2(I) = (-0.4093, -0.4082)$ and $\Delta_X(I) = 0$. There exists $K_{0,2}$ with $\Delta_{0,2} = 0$ and

$$K_2^*(I) = K_{0,2} \cap K_2(I) = (-0.4093, -0.4082) \neq \emptyset.$$

Thus, it is an admissible string.

For $I = \{2, 2\}$, $K_2(I) = (-0.6025, -0.4093)$ and $\Delta_X(I) = -2$. There exists $K_{0,1}$ with $\Delta_{0,1} = 2$ and

$$K_2^*(I) = K_{0,1} \cap K_2(I) = (-0.6025, -0.4093) \neq \emptyset.$$

Thus, it is an admissible string.

So the set of the admissible strings is

$$F_2 = \{\{2, 1\}, \{2, 2\}\},$$

and

$$K_2^* = \bigcup_{I \in F_2} K_2^*(I) = (-0.4093, -0.4082) \cup (-0.6025, -0.4093).$$

3. For K_3, the subsegments are

$$\begin{array}{llll} n_3^1 = \infty & \Omega_3^{1,1} = (2.2967, 2.6817) & K_3^{1,1} = (-1.3691, -1) & \sigma_3^{1,1} = 0 \\ & \Omega_3^{1,2} = (2.6817, 6.0516) & K_3^{1,2} = (-5.0516, -1.3691) & \sigma_3^{1,2} = 1 \\ & \Omega_3^{1,3} = (6.0516, 9.5508) & K_3^{1,3} = (-8.8355, -5.0516) & \sigma_3^{1,3} = 2 \\ & \ldots & \ldots & \ldots \end{array}$$

The set of all possible strings is

$$\{\{1\}, \{2\}, \{3\}, \cdots\}.$$

For $I = \{1\}$, $K_3(I) = (-1.3691, -1)$ and $\Delta_X(I) = 0$. There exists $K_{0,2}$ with $\Delta_{0,2} = 0$ but

$$K_3^*(I) = K_{0,2} \cap K_3(I) = \emptyset.$$

Thus, it is not an admissible string.

For any other string I, $\Delta_X(I) > 0$. Thus they are not admissible strings.

So the set of the admissible strings is

$$F_3 = \emptyset,$$

and

$$K_3^* = \emptyset.$$

Thus, for $k_p < 0$ we have

$$K = \bigcup_{i=1}^{3} K_i^* = (-0.4082, 0) \cup (-0.4093, -0.4082) \cup (-0.6025, -0.4093).$$

The final set is

$$\begin{aligned} K &= (-0.6025, -0.4093) \cup (-0.4093, -0.4082) \cup (-0.4082, 0) \\ &\quad \cup (0, 0.4082) \cup (0.4082, 0.4473) \end{aligned}$$

Among those terminals, 0.4082 and −0.4082 are of case 3 and −0.4093 is of case 1c. The regions surrounding them are stablizing regions, thus they are stable parameters. If we include the trivial point $k_p = 0$, we will have the complete stabilizing k_p set,

$$K = (-0.6025, 0.4473)$$

If we compare this result with the result of the same example in previous chapter where the stabilizing k_p range for delay from 0 to 1.8 is $(-0.4082, 0.4473)$, the set obtained here is larger. It includes some region where delay-free system or systems with less delay are unstable. The simulation results verified this. With $k_p = -0.5$, the system is stable at $L = 1.8$, Fig. 37, but unstable at $L = 0.5$, Fig. 38. ♣

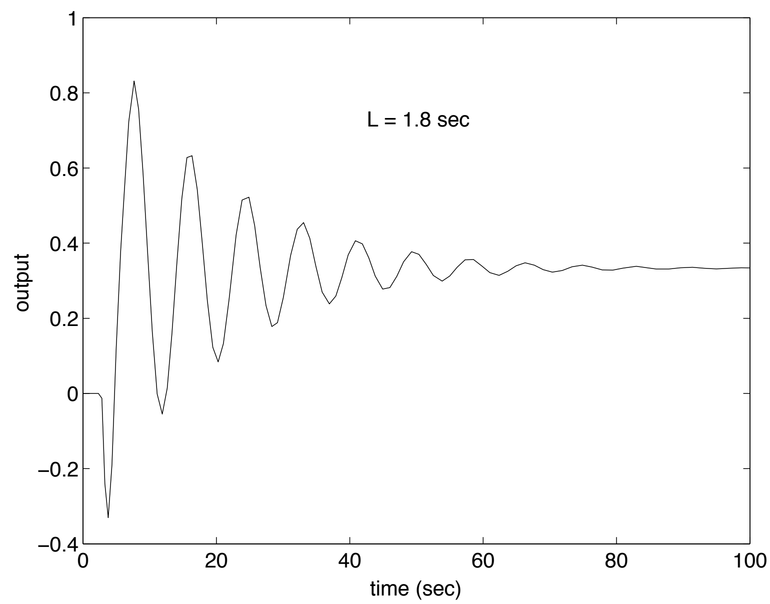

Fig. 37. Simulation of the system with $k_p = -0.5$ and $L = 1.8$.

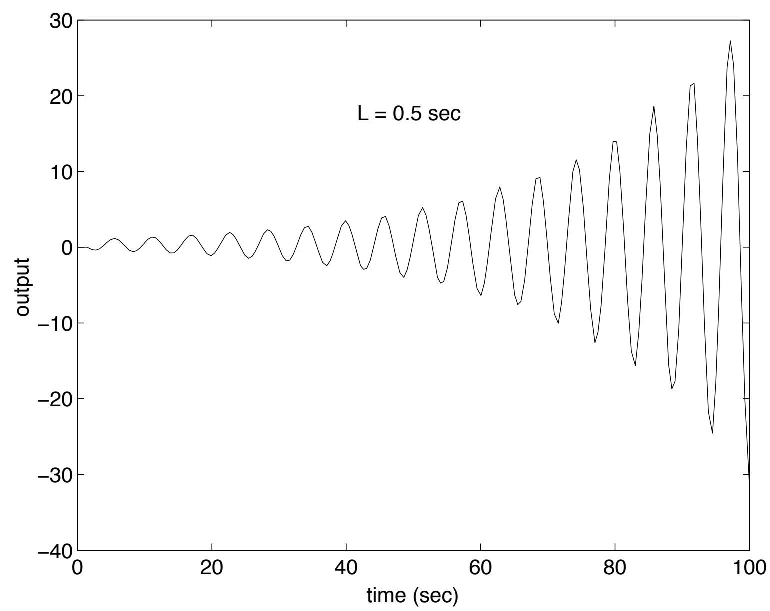

Fig. 38. Simulation of the system with $k_p = -0.5$ and $L = 0.5$.

E. Summary

In this chapter, a direct method to analyze the stability of LTI systems with fixed time-delay has been investigated. This approach has been used to develop the procedures to find the stabilizing P, PI controllers set for systems with fixed time-delay. Further extension to PID controllers will be sought after. Combined with the results of the previous chapter, the stabilizing PID controllers set for systems with interval time-delay can also be computed.

CHAPTER VII

CONCLUSION

In [2], the characterization of the entire set of stabilizing PID controllers gave us not only the starting point of the designs of PID controllers, but also the insight into the structure of such sets. Thus, there are two kinds of approaches to design a PID controller based on the stabilizing controllers set obtained.

1. Choose PID controllers which optimize given performance indices from the entire stabilizing PID controllers set, for example, the design of H_2 and H_∞ optimal PID controllers [2]. These methods involve the solutions of nonlinear programming problems (usually linear constrained nonlinear programming problems).

2. Integrate the design of PID controllers with certain stability requirements (robustness, non-fragility, stability margins) into the procedures of seeking stabilizing controllers set. These methods utilize the structure properties of the stabilizing PID controllers set (a union of convex sets for a fixed parameter) and usually involve the solutions of series of linear programming problems. For example, the design of a PID controller which has an optimal position in the stabilizing controllers set. [2, 14].

Following the strategy in [2], this dissertation aimed to address the synthesis and design issues of discrete-time PID control systems and PID control systems with time-delays by investigating the entire stabilizing PID controllers set for these two types of systems.

For discrete-time systems, we used bilinear transformation to convert the discrete-time controllers and plants from z-domain to w-domain. Then we studied the Hurwitz stability of the characteristic polynomials in that domain instead of the Schur stability of the original discrete-time characteristic polynomials. After the reparametrization of the PID controllers with the orthogonal transformation, we applied the same procedure used in continuous-time case and obtained the stabilizing set with the same shape and size of the set in original parameters space. In addition, this set has the same property of the continuous-time case, that is, it is also a union of convex sets for a fixed parameter. Naturally, the design approaches mentioned before can both be used here. Particularly, we presented the design of a robust and non-fragile discrete-time PID controller. Given some performance indices, we can also find the optimal PID controllers within this stabilizing controllers set.

For time-delay case, we used a generalized Nyquist Criterion to find the stabilizing PID controllers set for a given plant with interval delay up to certain value. The procedure is to exclude all the controllers which make the system marginally stable at a lesser delay than L_0 from the stabilizing controller set for the delay-free system. Thus, given some suitable performance indices, the optimal PID controllers can be found on this set. Although the obtained stabilizing PID controllers set is not a set defined by groups of linear inequalities with a fixed parameter, such set can be approximated by groups of linear inequalities. Such approximation will of course simplify the search of a optimal controller. Furthermore, systems with embedded delays are also considered in this dissertation. We introduce a formula to analyze the stability of a LTI system with a fixed-delay. This formula gives the complete information of the distribution of the closed-loop poles. Based on that, the solutions for Proportional and Proportional-Integral controllers have been given.

Future research work can be conducted in the following aspects:

1. Synthesis of PID controllers for systems with multiple delays, especially com-

mensurate delays.

The distributed systems with delays are common control subjects. The computer network control problems fall in this category. Thus, the synthesis problems of systems with commensurate delays are of practical meaning.

2. Further designs of PID controllers for time-delayed systems.

   The ultimate objective to seek the complete stabilizing PID controllers set is to design optimal PID controllers. In [26], various integral of a squared error (ISE) type performance indices for time-delayed systems were investigated. Optimal controllers can be designed by minimizing these indices over the available stabilizing region.

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