Quantum state measurement using phase-sensitive amplification in a driven three-level atomic system

Mashhood Ahmad,[1] Shahid Qamar,[2] and M. Suhail Zubairy[1,3]

[1]*Department of Electronics, Quaid-i-Azam University, Islamabad, Pakistan*

[2]*Department of Physics and Applied Mathematics, Pakistan Institute of Engineering and Applied Sciences, Nilore, Islamabad, Pakistan*

[3]*Department of Physics, Texas A&M University, College Station, Texas 77843-4242*

Phase-sensitive amplification in a three-level atomic system exhibits interesting features. For example, in the zero-detuning limit and for sufficiently strong driving field, this system becomes an ideal parametric amplifier, whereas, for a weak driving field it is a phase-insensitive amplifier [Ansari *et al.*, Phys. Rev. A **41**, 5179 (1990)]. In this paper, we show that this system could be used to measure the quantum state of the radiation field inside a cavity. To reconstruct the quantum state, we amplify it through a three-level atomic system and in the parametric limit, when noise in both the quadratures approaches to zero measure the amplified field quadrature. The complete quadrature distribution is obtained by measuring the quadratures for different values of the driving field phases. The inverse Radon transformation is then employed to reconstruct the original quantum state. Our scheme is insensitive to the problems associated with nonunit detector efficiency in homodyne detection measurement.

I. INTRODUCTION

During recent years, a number of models have been proposed for the measurement of nonclassical states of the electromagnetic field. These models incorporate techniques based upon absorption and emission spectroscopy [2], dispersive atom-field coupling [3], conditional measurement of an atom in a micromaser [4] and others [5]. A measurement scheme based upon quantum tomography was proposed by Vogal and Risken [6]. In this scheme, the field quadrature distribution is measured via optical homodyne detection, and the Wigner function of the given quantum state is then reconstructed from these measurements by using inverse Radon transformation. A knowledge of the Wigner function reveals the complete quantum state of the system [7,8]. This scheme was applied successfully to experimentally measure the vacuum and the squeezed states of the radiation field [9,10]. However, measurements of the quantum states are highly sensitive to the noise associated with the detector inefficiencies [11,12]. An important question in this regard is how to overcome the nonunit efficiency of the detectors.

In this paper, we present a model for quantum state measurement using a two-photon phase-sensitive amplification by three-level atoms in the cascade configuration, where coherence is induced between the top and the bottom levels by driving the atoms continuously with a strong external field. Under the limits of the strong driving field and zero detuning, this system amplifies one quadrature of the field at the expense of deamplification in the conjugate quadrature. Furthermore, the noise in both the quadratures approaches zero, and hence the amplifier becomes identical to an ideal degenerate parametric amplifier [1]. Under the two-photon resonance condition, the amplification for a particular quadrature phase can be obtained by controlling the phase of the external driving field. In this study it is shown that this system could be used for the reconstruction of the quantum state of the field inside the cavity.

The Wigner function of the quantum state can be reconstructed by calculating the complete distribution function $\omega(x, \theta)$ for the quadrature values $x(\theta)$ with θ varying from 0 to π. The maximum amplification with reduced noise for different quadrature phases is obtained by driving the system for different values of the phase φ of the classical field, correspondingly. We have calculated the quadrature distribution for an arbitrary quantum state after its amplification by a phase-sensitive linear amplifier. The distribution function of the noise-free quadrature is then used to reconstruct the quantum state of the field using the inverse Radon transformation, well known in quantum tomography. We apply this model to a Schrödinger-cat state [13–15], and discuss its reconstruction after its amplification through a two-photon phase-sensitive linear amplifier in the zero-detuning limit. This model enables us to overcome the problems arising due to the nonunit efficiency of detectors in the homodyne measurement scheme. It is worthwhile to mention here that, recently, Leonhardt and Paul [16] have also proposed an interesting scheme for quantum state measurement. Their scheme was based on antisqueezing the propagating field with respect to the quadrature of interest, using a degenerate optical parametric amplifier that also allows one to overcome the problems associated with the nonunit efficiency of the detectors. Here we would like to point out that our scheme allows us to measure the quantum state of the field inside the cavity. In our earlier papers, we proposed a model for the observation of quantum interferences associated with the Schrödinger-cat state [17] and the measurement of quantum state [18] using two-photon correlated emission laser (CEL) [19–22].

II. MODEL AND EQUATION OF MOTION FOR FIELD-DENSITY MATRIX

Our amplifier consists of three-level atoms in cascade configuration as shown in Fig. 1. The atoms in state $|a\rangle$ are
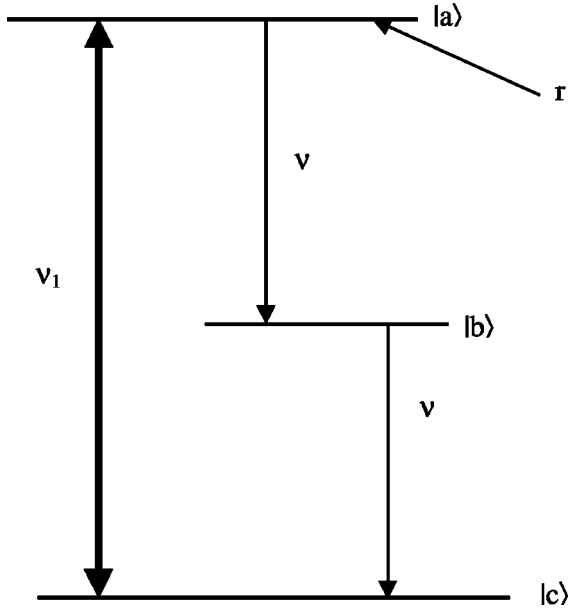
FIG. 1. Energy diagram of a three-level atomic system in a cascade configuration.

injected into the cavity in such a manner that only one atom at a time is present inside the cavity. The transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ are dipole allowed, whereas the transition $|a\rangle \rightarrow |c\rangle$ is dipole forbidden. We assume that the transition $|a\rangle \rightarrow |c\rangle$ may be induced by employing a sufficiently strong resonant external driving field. We are considering a linear amplifier; therefore, we treat $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ transitions quantum mechanically up to second order in the coupling constant, and the $|a\rangle \rightarrow |c\rangle$ transition semiclassically to all orders.

The Hamiltonian for the atom-field system is given by

$$H = H_0 + V, \tag{1}$$

where

$$H_0 = \sum_{i=a,b,c} \hbar \omega_i |i\rangle\langle i| + \hbar \nu a^\dagger a, \tag{2}$$

and the interaction Hamiltonian is given by

$$V = \hbar g \left[(|a\rangle\langle b| + |b\rangle\langle c|)a + a^\dagger(|b\rangle\langle a| + |c\rangle\langle b|) \right.$$
$$\left. - \frac{\hbar \Omega}{2} (e^{-i\varphi - i\nu_1 t}|a\rangle\langle c| + e^{i\varphi + i\nu_1 t}|c\rangle\langle a|) \right]. \tag{3}$$

Here a ($a^\dagger$) is the destruction (creation) operator for the field mode of frequency ν, g is the atom-field coupling constant, which is assumed to be equal for both transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$; and Ω is the Rabi frequency of the driving classical field with ν_1 and φ as its frequency and phase, respectively. We assume that the atoms are initially pumped incoherently to the upper level $|a\rangle$ at rate r. For simplicity, the decay rate γ is considered to be the same for all three levels. In the zero-detuning and two-photon resonance (2ν

$=\nu_1$) conditions, the evolution of the reduced density matrix for the field is given by the master equation [1]

$$\dot{\rho}_F = -[\alpha_{11}^* aa^\dagger \rho_F - (\alpha_{11} + \alpha^*_{11})a^\dagger \rho_F a + \alpha_{11}\rho_F aa^\dagger]$$
$$-[\alpha_{22}^* a^\dagger a \rho_F - (\alpha_{22} + \alpha^*_{22})a\rho_F a^\dagger + \alpha_{22}\rho_F a^\dagger a]$$
$$-[\alpha_{12}^* aa \rho_F - (\alpha_{12}^* + \alpha_{21})a\rho_F a + \alpha_{21}\rho_F aa]e^{-i\Phi}$$
$$-[\alpha_{21}^* a^\dagger a^\dagger \rho_F - (\alpha_{12} + \alpha^*_{21})a^\dagger \rho_F a^\dagger$$
$$+ \alpha_{12}\rho_F a^\dagger a^\dagger]e^{-i\Phi}, \tag{4}$$

where

$$\alpha_{11} = \frac{g^2 r}{(\gamma^2 + \Omega^2)}, \tag{5}$$

$$\alpha_{12} = i\frac{g^2 r\Omega(\Omega^2 - 2\gamma^2)}{\gamma(\gamma^2 + \Omega^2)(4\gamma^2 + \Omega^2)}, \tag{6}$$

$$\alpha_{21} = i\frac{g^2 r\Omega}{\gamma(\gamma^2 + \Omega^2)}, \tag{7}$$

and

$$\alpha_{22} = \frac{3g^2 r\Omega^2}{(\gamma^2 + \Omega^2)(4\gamma^2 + \Omega^2)}. \tag{8}$$

The first term in Eq. (4) corresponds to the gain, and the second to the absorption in the system. The third and fourth terms are due to the coherent excitation of the atomic states. This coherence is produced by the classical driving field, and is responsible for the phase sensitivity in the system. The phase Φ in Eq. (4) is given as $\Phi = \varphi + (\nu_1 - 2\nu)t$. In the following, we assume a two-photon resonance, i.e., $2\nu = \nu_1$. Under this condition, we have $\Phi = \varphi$.

In order to reconstruct the quantum state of the field, we need a noise-free amplification of the cavity field. In homodyne detection, we measure only one quadrature component of the field, therefore, we require noise-free amplification only with respect to that particular quadrature. Here we calculate the amplifier gain for a field quadrature $x(\theta)$ for our three-level atomic system, where the coherence is established by driving it through a strong external field of phase φ. Using Eq. (4), we obtain

$$\frac{d\langle a\rangle}{dt} = \rho_1 \langle a\rangle + \rho_2 \langle a^\dagger\rangle, \tag{9}$$

where

$$\rho_1 = (\alpha_{11} - \alpha_{22}^*), \tag{10}$$

$$\rho_2 = (\alpha_{12} - \alpha_{21}^*)\exp(i\varphi). \tag{11}$$

The parameters α_{11} and α_{12} are the same as defined via Eqs. (5) and (6), and α_{21}^* and α_{22}^* can be obtained by taking the

complex conjugate of Eqs. (7) and (8). The generalized quadrature of the field is defined as

$$x(\theta) = \frac{1}{2}[a \exp(-i\theta) + a^\dagger \exp(i\theta)], \qquad (12)$$

Using Eq. (4), we obtain

$$\frac{d\langle x(\theta)\rangle}{dt} = \sqrt{\rho_1^2 + |\rho_2|^2 + 2\rho_1|\rho_2|\cos\left(\varphi - 2\theta + \frac{\pi}{2}\right)} \\ \times \langle x(\theta + \psi)\rangle, \qquad (13)$$

where

$$\psi = \tan^{-1}\left(\frac{|\rho_2|\sin\left(\varphi - 2\theta + \frac{\pi}{2}\right)}{\rho_1 + |\rho_2|\cos\left(\varphi - 2\theta + \frac{\pi}{2}\right)}\right). \qquad (14)$$

It is clear from Eq. (13) that an exact solution can be obtained for $\psi = 0$. Under this condition, the solution of the Eq. (13) reads as

$$\langle x(\theta)\rangle_t = \sqrt{G}\langle x(\theta)\rangle_0. \qquad (15)$$

where

$$G = \exp\left[2t\sqrt{\rho_1^2 + |\rho_2|^2 + 2\rho_1|\rho_2|\cos\left(\varphi - 2\theta + \frac{\pi}{2}\right)}\right]. \qquad (16)$$

For $\psi = 0$, we have two possible choices of $[\varphi - 2\theta + (\pi/2)]$ i.e., 0 or π. Under these conditions, the parameter G reduces to the same expressions for the gain parameters as mentioned in Ref. [1]. It follows form Eq. (16) that an optimum gain can be obtained for $[\varphi - 2\theta + (\pi/2)] = 0$. This condition indicates that an amplified quadrature with phase θ can be obtained by adjusting the phase φ of the classical driving field accordingly. In order to reconstruct the quantum state, we require a set of amplified noise-free quadrature val-

ues $x(\theta)$, with θ varying from 0 to π. In the homodyne detection measurements, the phase θ of the quadrature field is given by the phase of the local oscillator.

III. RECONSTRUCTION OF THE WIGNER FUNCTION

In this section, we discuss the reconstruction of the Wigner function of the initial quantum state after its amplification through a three-level atomic system in the zero-detuning limit. The zero-detuning condition requires that level $|b\rangle$ lies exactly in between the upper level $|a\rangle$ and the ground-state level $|c\rangle$. To reconstruct the Wigner function, we need the quadrature distribution $\omega(x,\theta)$ for values of $x(\theta)$ which can be measured in the homodyne detection scheme. The quadrature distribution $\omega(x,\theta)$ for the amplified field can be obtained from the Wigner function $W(\alpha,t)$ of the cavity field. The time evolution of the Wigner function of the field can be evaluated by writing the master equation (4) in terms of its Fokker-Planck equation for the Wigner distribution, and by finding its time-dependant solution. The Wigner function $W(\alpha,t)$ in terms of the density operator ρ_F is given by the equation [23]

$$W(\alpha,t) = \pi^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^2\beta Tr[\exp\{-\beta(\alpha^* - a^\dagger) \\ + \beta^*(\alpha - a)\}\rho_F]. \qquad (17)$$

Using Eq. (17), we can rewrite the master equation (4) as a Fokker-Planck equation for the Wigner function,

$$\frac{\partial}{\partial t}W = -\frac{A}{2}\left[\frac{\partial}{\partial\alpha_x}A_{11}\alpha_x + \frac{\partial}{\partial\alpha_x}A_{12}\alpha_y + \frac{\partial}{\partial\alpha_y}A_{21}\alpha_x \\ + \frac{\partial}{\partial\alpha_y}A_{22}\alpha_y + \frac{\partial^2}{\partial\alpha_x^2}B_{11} + \frac{\partial^2}{\partial\alpha_y^2}B_{22} + \frac{\partial^2}{\partial\alpha_x\partial\alpha_y}B_{12} \\ + \frac{\partial^2}{\partial\alpha_y\partial\alpha_x}B_{21}\right]W, \qquad (18)$$

where

$$A = \frac{2a\gamma\left\{\gamma^2(2\gamma^2 - \Omega^2)^2 + \Omega^2(\gamma^2 + \Omega^2)^2 + 2\gamma\Omega(2\gamma^2 - \Omega^2)(\gamma^2 + \Omega^2)\cos\left(\varphi - 2\theta + \frac{\pi}{2}\right)\right\}^{1/2}}{(\gamma^2 + \Omega^2)(4\gamma^2 + \Omega^2)}, \qquad (19)$$

$$A_{11} = \frac{2a\gamma[2\gamma^3 - \gamma\Omega^2 - (\gamma^2\Omega + \Omega^3)\sin\varphi]}{A(\gamma^2 + \Omega^2)(4\gamma^2 + \Omega^2)}, \qquad (20)$$

$$A_{12} = A_{21} = \frac{2a\gamma\Omega\cos\varphi}{A(4\gamma^2 + \Omega^2)}, \qquad (21)$$

$$A_{22} = \frac{2a\gamma[2\gamma^3 - \gamma\Omega^2 + (\gamma^2\Omega + \Omega^3)\sin\varphi]}{A(\gamma^2 + \Omega^2)(4\gamma^2 + \Omega^2)}, \qquad (22)$$

$$B_{11} = \frac{-a\gamma^2}{2A(4\gamma^2+\Omega^2)}\left(\frac{2(\gamma^2+\Omega^2)-3\gamma\Omega\sin\varphi}{(\gamma^2+\Omega^2)}\right), \tag{23}$$

$$B_{12} = B_{21} = \frac{-a\gamma^2}{2A(4\gamma^2+\Omega^2)}\frac{3\gamma\Omega\cos\varphi}{(\gamma^2+\Omega^2)}, \tag{24}$$

$$B_{22} = \frac{-a\gamma^2}{2A(4\gamma^2+\Omega^2)}\left(\frac{2(\gamma^2+\Omega^2)+3\gamma\Omega\sin\varphi}{(\gamma^2+\Omega^2)}\right), \tag{25}$$

and $a=2g^2r/\gamma^2$ is the linear gain coefficient in the absence of the driving field. The formal solution of Eq. (18) for any arbitrary initial quantum state is given by [24]

$$W(\alpha,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^2\beta W(\beta,0) W_c(\alpha,t;\beta,0), \tag{26}$$

where the conditional probability $W_c(\alpha,t;\beta,0)$ is given by

$$W_c(\alpha,t;\beta,0) = \frac{\sqrt{4K_1K_2-K_{12}^2}}{2\pi(G-1)}\exp\left[\frac{1}{G-1}K_1(\alpha_x-\sqrt{G}\beta_x)^2\right.$$
$$+K_2(\alpha_y-\sqrt{G}\beta_y)^2$$
$$\left.+K_{12}(\alpha_x-\sqrt{G}\beta_x)(\alpha_y-\sqrt{G}\beta_y)\right]. \tag{27}$$

Parameters K_1, K_2, and K_{12} in Eq. (27) are given by

$$K_1 = \frac{A_{21}B_{12}-A_{11}B_{22}}{2(B_{12}B_{21}-B_{11}B_{22})}, \tag{28}$$

$$K_2 = \frac{A_{12}B_{21}-A_{22}B_{11}}{2(B_{12}B_{21}-B_{11}B_{22})}, \tag{29}$$

and

$$K_{12} = \frac{A_{11}B_{21}+A_{22}B_{12}-A_{12}B_{22}-A_{21}B_{11}}{2(B_{12}B_{21}-B_{11}B_{22})}. \tag{30}$$

Parameter G in Eq. (27) is the same as defined by Eq. (16). On substituting the values of ρ_1 and ρ_2 from Eqs. (10) and (11) in Eq. (16), and using the values of $\alpha_{11}, \alpha_{12}, \alpha_{21}^*$, and α_{22}^* from Eqs. (5)–(8), we obtain the following expression for the gain parameter G:

$$G = \exp\left(\frac{2a\gamma t\left\{\gamma^2(2\gamma^2-\Omega^2)^2+\Omega^2(\gamma^2+\Omega^2)^2+2\gamma\Omega(2\gamma^2-\Omega^2)(\gamma^2+\Omega^2)\cos\left(\varphi-2\theta+\frac{\pi}{2}\right)\right\}^{1/2}}{(\gamma^2+\Omega^2)(4\gamma^2+\Omega^2)}\right). \tag{31}$$

Here we are interested in the measurement of the quadrature distribution $\omega(x,\theta)$ when the initial quantum state is amplified through a phase-sensitive three-level atomic system. A homodyne detector measures the quadrature component given by Eq. (12). In a balanced homodyne detection measurement scheme, the quadrature phase θ is characterized by the phase of the local oscillator. A complete distribution $\omega(x,\theta)$ for the quadrature component $x(\theta)$ is determined by scanning the field quadrature over a range of phase θ varying from 0 to π. Such distributions have recently been measured in quantum optical tomography [9]. It was shown by Vogel and Risken that the Wigner function $W(\alpha,t)$ and the generalized quadrature distribution $w(x,\theta)$ for the field hold a one-to-one correspondence with each other, which is given by the following [6]:

$$\omega(x,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\alpha\, d\eta\, W(\alpha,t)\exp[-i\eta(x-\alpha_x\cos\theta-\alpha_y\sin\theta)]. \tag{32}$$

On substituting the expression for the Wigner function $W(\alpha,t)$ from Eq. (26) into Eq. (32), we obtain

$$\omega(x,\theta) = \sqrt{\frac{2}{\pi(G-1)\xi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\beta W(\beta,0)\exp\left(-\frac{2}{(G-1)\xi}[x-\sqrt{G}(\beta_x\cos\theta+\beta_y\sin\theta)]^2\right), \tag{33}$$

where the parameter G is the gain factor as defined earlier via Eq. (31) and the parameter ξ is given by the following:

$$\xi = \frac{(4\gamma^6-\gamma^4\Omega^2+4\gamma^2\Omega^4)\{(4\gamma^6-\gamma^4\Omega^2-5\gamma^2\Omega^4)-(4\gamma^5\Omega-7\gamma^3\Omega^3-2\gamma\Omega^5)\sin(\varphi-2\theta)\}}{(4\gamma^6-\gamma^4\Omega^2-5\gamma^2\Omega^4)^2-(4\gamma^5\Omega-7\gamma^3\Omega^3-2\gamma\Omega^5)^2}. \tag{34}$$

It is clear from the expression for the gain parameter G [given by Eq. (31)] that an optimum gain can be obtained if we choose $\varphi - 2\theta = -\pi/2$. Under this condition, the quadrature distribution becomes

$$
\omega(x,\theta) = \sqrt{\frac{2}{\pi(G'-1)\xi'}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2\beta W(\beta,0)
$$
$$
\times \exp\left(\frac{-2(x-\sqrt{G'}(\beta_x \cos\theta + \beta_y \sin\theta))^2}{(G'-1)\xi'}\right),
$$
(35)

where

$$
G' = \exp\left(\frac{2a\gamma t\{\gamma(2\gamma^2 - \Omega^2) + \Omega(\gamma^2 + \Omega^2)\}}{(\gamma^2 + \Omega^2)(4\gamma^2 + \Omega^2)}\right) \quad (36)
$$

and

$$
\xi' = \frac{[4\gamma^6 - \gamma^4\Omega^2 + 4\gamma^2\Omega^4]}{[4\gamma^6 - \gamma^4\Omega^2 - 5\gamma^2\Omega^4] - [4\gamma^5\Omega - 7\gamma^3\Omega^3 - 2\gamma\Omega^5]}.
$$
(37)

When $\Omega \gg \gamma$ i.e., when the Rabi frequency of the classical driving field is much larger than the atomic level width γ, the expressions for the parameters G' and ξ' reduce as $G' \rightarrow \exp(2at\gamma/\Omega)$ and $\xi' \rightarrow 0$. However, the gain in the conjugate quadrature G'' (which can be obtained by choosing $\varphi - 2\theta = \pi/2$) would reduce as $\exp(-2at\gamma/\Omega)$. Under the conditions $\Omega/\gamma \rightarrow \infty$ and $at \rightarrow \infty$, $at\gamma/\Omega$ becomes finite, the noise in both the quadratures approaches to zero and $G' = 1/G''$. Thus the amplifier becomes identical to a degenerate parametric amplifier [1]. The noise free amplification is obtained by driving the system with a classical field of phase φ for $\Omega \gg \gamma$. The quadrature with maximum gain can be obtained by choosing the phase of the local oscillator θ such that $\theta = \varphi/2 - \pi/4$. The complete distribution for $\omega(x,\theta)$ can then be obtained by driving the amplifier with a classical field of phase φ ranging form $\pi/2$ to $5\pi/2$. Once the noise-free quadrature distribution of the amplified quantum state is measured in balanced homodyne detection scheme, then the corresponding Wigner function can be reconstructed by carrying out the inverse Radon transformation familiar in optical tomographic imaging [6]:

$$
W(\alpha_x, \alpha_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\pi} \omega(x,\theta)|\eta|
$$
$$
\times \exp[i\eta(x - \alpha_x\cos\theta - \alpha_y\sin\theta)]dx\,d\eta\,d\theta.
$$
(38)

The Wigner function of the amplified quantum state can be obtained if we substitute Eq. (35) into Eq. (38), and this results

$$
W(\alpha_x, \alpha_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\pi} d^2\beta W(\beta,0)|\eta|d\eta d\theta
$$
$$
\times \exp\left[-\frac{(G'-1)\xi'}{8}\eta^2 - i\eta[(\alpha_x - \sqrt{G'}\beta_x)\right.
$$
$$
\left. \times \cos\theta + (\alpha_y - \sqrt{G'}\beta_y)\sin\theta]\right].
$$
(39)

For $G' = 1$, which corresponds to $at = 0$ [see Eq. (36)], we obtain the Wigner function for the original quantum state. Equation (39) can be rewritten in terms of the rescaled variables $\alpha'_x = \alpha_x/\sqrt{G'}$, $\alpha'_y = \alpha_y/\sqrt{G'}$, and $\eta' = \eta/\sqrt{G'}$ as follows

$$
W(\alpha'_x, \alpha'_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\pi} d^2\beta d\eta' d\theta W(\beta,0)|\eta'|
$$
$$
\times \exp\left(-\frac{(1-1/G')\xi'}{8}\eta'^2 - i\eta'\right.
$$
$$
\left. \times [(\alpha'_x - \beta_x)\cos\theta + (\alpha'_y - \beta_y)\sin\theta]\right).
$$
(40)

In the parametric limit, when G' approaches a finite value and $\xi' \rightarrow 0$ [see Eqs. (36) and (37)], it is clear that we recover the original quantum state. This shows that the quantum state can be fully recovered after its amplification through a phase-sensitive three-level atomic system in the parametric limit. Only an appropriate rescaling of the measured distribution is required.

Here, we consider the Schrödinger-cat state, which is the superposition of two coherent states $|\xi_0\rangle$ and $|-\xi_0\rangle$, which are 180° out of phase with respect to each other,

$$
\Psi_0 = \sqrt{N}[|\xi_0\rangle + |-\xi_0\rangle], \quad (41)
$$

where $N^{-1} = 2[1 + \exp(-2\xi_0^2)]$ is the constant of normalization and ξ_0 is taken as real for the sake of simplicity. The Wigner function $W(\beta,0)$ of this state is defined as [25]

$$
W(\beta,0) = \frac{1}{\pi[1 + \exp(-2\xi_0^2)]}\{\exp[-2(\beta_x - \xi_0)^2 - 2\beta_y^2]
$$
$$
+ \exp[-2(\beta_x + \xi_0)^2 - 2\beta_y^2]
$$
$$
+ 2\exp(-2\beta_x^2 - 2\beta_y^2)\cos(4\xi_0\beta_y)\}.
$$
(42)

The Wigner function of the amplified Schrödinger-cat state can be obtained by using the expression for $W(\beta,0)$ in Eq. (40). In terms of the rescaled variables $\alpha'_x = \alpha_x/\sqrt{G'}$, $\alpha'_y = \alpha_y/\sqrt{G'}$, and $\eta' = \eta/\sqrt{G'}$, it is given by
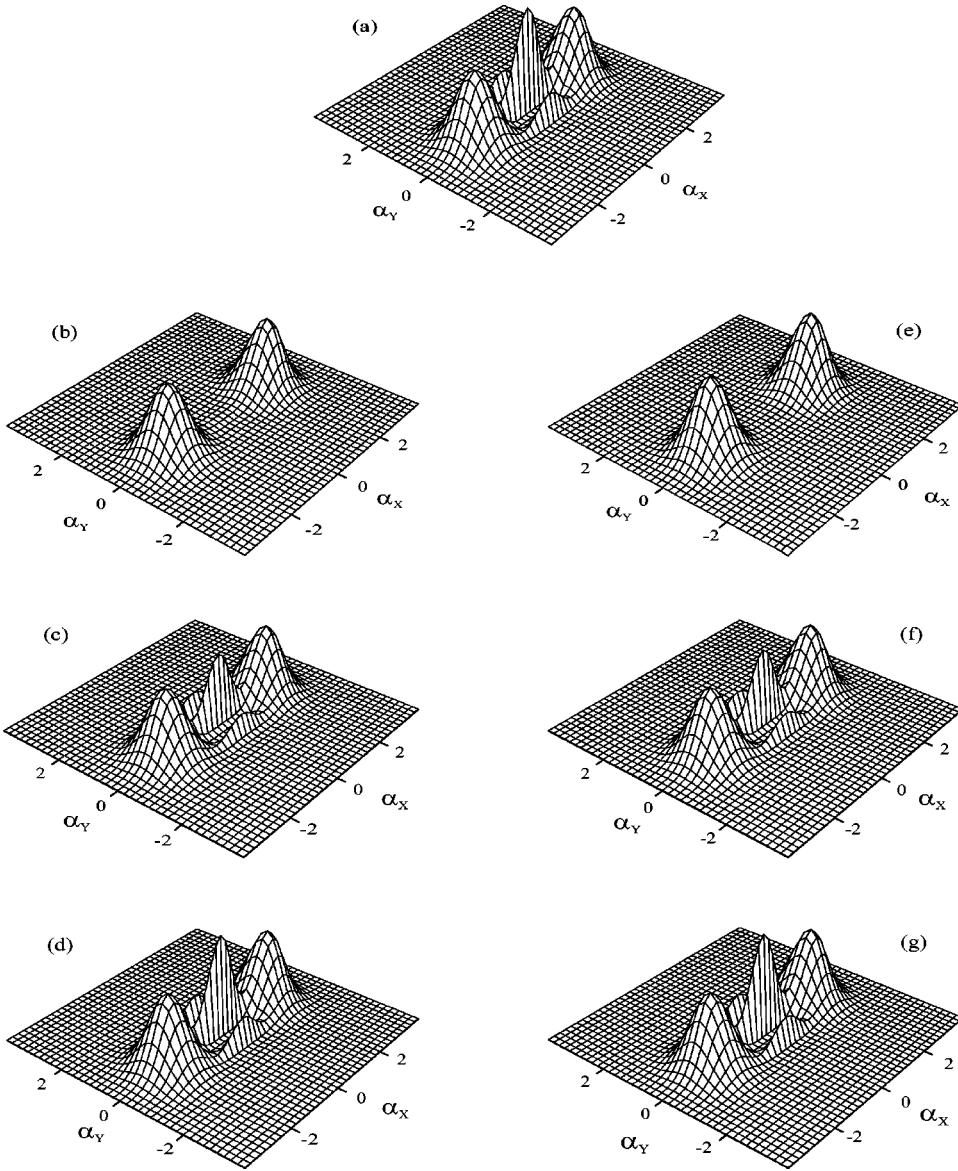
FIG. 2. Plots of the Wigner distribution function $W(\alpha_x, \alpha_y)$ for a Schrödinger-cat state in the zero-detuning limit, (a) for $\xi_0 = 2$ and $at = 0$. Here, we obtain the well-known structure associated with the Wigner function of the Schrödinger-cat state. In (b)–(d), we show the plots for $\xi_0 = 2$, $at = 1$, and $\Omega/\gamma = 1, 15$ and 30, respectively. Due to the increase in Ω/γ, the system grows from a phase-insensitive amplifier to a parametric amplifier and the original Schrödinger-cat state is almost fully recovered for $\Omega/\gamma = 30$. In (e)–(g), we show the plots of the Wigner function for the same set of parameters as mentioned in (a) except for $at = 10$ and $\Omega/\gamma = 1, 30$, and 90, respectively. The figure clearly shows that the original Schrödinger-cat state is almost fully recovered for $\Omega/\gamma = 90$.

$$W(\alpha'_x, \alpha'_y) = \frac{1}{8\pi^2(1 + \exp(-2\xi_0^2))} \int_{-\infty}^{+\infty} \int_0^{\pi} d\eta' d\theta |\eta'| \left[\exp\left(\frac{-(1 + (1 - 1/G')\xi')}{8} \eta'^2 - i\eta'[(\alpha'_x + \xi_0)\cos\theta + \alpha'_y \sin\theta] \right) \right.$$

$$+ \exp\left(\frac{-(1 + (1 - 1/G')\xi')}{8} \eta'^2 - i\eta'[(\alpha'_x - \xi_0)\cos\theta + \alpha'_y \sin\theta] \right)$$

$$+ \exp(-2\xi_0^2) \times \left\{ \exp\left(\frac{-(1 + (1 - 1/G')\xi')}{8} \eta'^2 - i\eta'[\alpha'_x \cos\theta + (\alpha'_y + i\xi_0)\sin\theta] \right) \right.$$

$$\left. \left. + \exp\left(\frac{-(1 + (1 - 1/G')\xi')}{8} \eta'^2 - i\eta'[\alpha'_x \cos\theta + (\alpha'_y - i\xi_0)\sin\theta] \right) \right\} \right]. \tag{43}$$

It is clear that in the parametric limit, when G' approaches a finite value and $\xi' \to 0$, we recover the Wigner function for the original Schrödinger-cat state. In the forthcoming section, we present the results of our numerical simulations.

IV. RESULTS AND DISCUSSION

In Fig. 2(a), we show the plot of the Wigner function for $\xi_0 = 2$ and $at = 0$ in the zero-detuning limit. The figure clearly shows the Wigner function of the initial Schrödinger-

cat state. Here two Gaussian hills at $\alpha_x = \pm 2$ correspond to the location of the two coherent states, and the oscillations perpendicular to the line joining the two hills arise due to the superposition of these states. These oscillations are an unambiguous signature of the quantum interference in a Schrödinger-cat state. The finite efficiency of the detectors tends to wipe out these nonclassical features during the measurement process. Figures 2(b)–2(d) show the plots of Wigner function for $\xi_0 = 2$, $at = 1$, and $\Omega/\gamma = 1, 15$, and 30, respectively, in the limit of zero detuning. The results show that for $\Omega/\gamma = 1$ (when the system exhibits the behavior of phase-insensitive amplifier) the well known oscillatory behavior of the Wigner function completely vanishes. However, with the increase in Ω/γ the system approaches a parametric limit, and the oscillations start to appear. For $\Omega/\gamma = 30$, the original Wigner function is almost fully recovered. In Figs. 2(e)–2(g), we present plots of the Wigner function for $\xi_0 = 2$, $at = 10$, and $\Omega/\gamma = 1$, 30, and 90, respectively. A comparison of Figs. 2(e)–2(g) with Figs. 2(b)–2(d) shows that for $at = 10$, the complete Wigner function is obtained for $\Omega/\gamma = 90$. This shows that an increase in at requires a larger value of Ω/γ for a complete reconstruction of the original quantum state.

The Wigner function is reconstructed by measuring the quadrature distribution of the amplified quantum state, and then taking the inverse Radon transformation. The measurement of the quadrature distribution can be realized in a balanced homodyne detection scheme [26]. During the measurement, the field leaks through the end mirror of the cavity. To ensure that the field does not leak through the cavity during amplification, the time scales in the experiment have to be properly adjusted. We have $a = 2g^2 r/\gamma^2$ and $G' \rightarrow \exp(2at\gamma/\Omega)$ (in the parametric limit); combining these, we obtain

$$ t \rightarrow \frac{\gamma\Omega\ln G'}{4g^2 r}, \tag{44} $$

which corresponds to the total time for amplification. This time should be small compared to the life time (t_c) of the cavity, i.e., $t \ll t_c$.

In conclusion, we propose a scheme for the measurement of the Wigner function of the quantum state of radiation field inside a cavity. Our scheme is based on amplification of the signal through a three-level atomic system (in the zero-detuning limit), where the coherence is established by driving the atoms continuously through a strong external classical field. It is shown that in the parametric limit this system will allow us to fully recover the Wigner function of the initial quantum state. Only an appropriate rescaling of the measured distribution is required. As an example, we apply this scheme to a Schrödinger-cat state and successfully reconstruct its Wigner function. This scheme overcomes the problems of non-ideal detector efficiency.

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