CONSTRUCTING NUMBERS THROUGH MOMENTS IN TIME:

KANT'S PHILOSOPHY OF MATHEMATICS

A Thesis

by

PAUL ANTHONY WILSON

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF ARTS

August 2003

Major Subject: Philosophy

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Approved as to style and content by:

Michael Hand (Chair of Committee)	Christopher Menzel (Member)
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ABSTRACT

Constructing Numbers Through Moments in Time:

Kant's Philosophy of Mathematics. (August 2003)

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Among the various theses in the philosophy of mathematics, *intuitionism* is the thesis that numbers are constructs of the human mind. In this thesis, a historical account of intuitionism will be exposited--from its beginnings in Kant's classic work, *Critique of Pure Reason*, to contemporary treatments by Brouwer and other intuitionists who have developed his position further. In chapter II, I examine the ontology of Kant's philosophy of arithmetic. The issue at hand is to explore how Kant, using intuition and time, argues for numbers as mental constructs. In chapter III, I examine how mathematics for Kant yields synthetic a priori truth, which is to say an *informative* statement about the world whose truth can be known independently of observation. In chapter IV, I examine how intuitionism developed under the care of Brouwer and others (e.g. Dummett) and how Hilbert sought to address issues in Kantian philosophy of mathematics with his finitist approach. In conclusion, I examine briefly what intuitionism resolves and what it leaves to be desired.

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CHAPTER I

INTRODUCTION

1.1 The Significance of A Priori *Knowledge*

Though considered by many philosophers in the post-Quine era to be a dead issue, the epistemological problem of *a priori* knowledge still makes appearances in contemporary readings (see Boghossian and Peacocke, 2000). *A Priori* knowledge is that knowledge which is known independently of experience. The truths of the physical world are revealed to us through *a posteriori* knowledge, which is knowledge acquired through experience. Though Immanuel Kant (1724-1804) was the first to use the term, prominent discussion of a priori knowledge can be traced back at least as early as Gottfried Leibniz (1646-1716). Working in the rationalist tradition of his predecessors Rene Descartes (1596-1650) and Baruch Spinoza (1632-1677), Leibniz was concerned with knowledge of *necessary* truths. Leibniz defined a necessary truth as a proposition that was "true in all possible worlds." One such example is the proposition "2+2=4." We can imagine that *contingent* facts of the world might be different—it is conceivable that circumstances could have been such that the sky could have been green instead of blue, or that the earth could have been further from the sun than it in fact is. But these contingent truths that refer to facts of reality differ in kind from the kind of necessary truths that Leibniz relied upon for certainty.

David Hume (1711-1776), an empiricist, who divided knowledge into two categories. When Hume referred to "relations of ideas," he had in mind those kinds of

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truths that were a priori, necessary, and *analytic.* When Hume referred to "matters of fact," he had in mind those truths that were a posteriori, contingent, and *synthetic.* It was not Hume who used the terms "analytic" and "synthetic" specifically, but rather Kant who made the distinction between analytic truths and synthetic truths. A proposition is analytic if and only if the concept of the predicate is included in the concept of the subject. If the predicate is not contained in the concept of the subject, then the proposition is synthetic. "All bachelors are unmarried males" is an analytic statement by the nature of its subject-predicate relationship. We can know that all bachelors are unmarried males by knowing what a bachelor is. But "some bachelors are lawyers" is a synthetic statement because we do not learn that some bachelors are lawyers in virtue of the meanings of the words—we must confirm this statement through experience that we gain by observing the world.

If one is willing to accept the "Humean Fork," then analytic truths are significant insofar as they yield certainty; but many would not count these propositions as "knowledge" since the truths are merely true by definition, and sometimes dismissed as "trivially true." However, the great hope for rationalism was found in Kant, who argued for the possibility of the *synthetic a priori*, which is to say an *informative* statement about the world whose truth can be known independently of observation. Whereas the Humean position held that " 7+5=12" is an analytic statement, Kant argued that "7+5=12" is a synthetic statement, because the concept "12" is not contained in the subject "7+5." In other words, Kant was claiming that the concepts "7" and "5" are not included in the concept "12." Rather, he argued that "12" is a *new* item of knowledge that we obtain when we make a synthesis of "7" and "5." Regarding the ontology of numbers themselves, Kant did not believe that numbers were non-physical entities, but rather constructs of the mind. If Kant's position is correct, then there are a great many things we can know not just about the conceptual relationships between subjects and predicates, but about the world *with certainty*. But the question remains as to whether Kant's metaphysics and epistemology of numbers is defensible.

The analytic/synthetic distinction was downplayed in the early and middle 20th century by the logical positivists. Advocating a radical empiricism, the logical positivists argued that most knowledge was brought to us by the senses. Like Hume, logical positivists argued that those propositions that were a priori were also necessary and analytic, and that if a proposition had one of these three characteristics, it had all three as well. The common view of logical positivism was that mathematics was reducible to logic and that logic was analytic. Ayer (1952) explained to what extent analytic propositions were significant in telling us about the world:

> Like Hume, I divide all genuine propositions into two classes: those which, in his terminology, concern "relations of ideas," and those which concern "matters of fact." The former class comprises the *a priori* propositions of logic and pure mathematics, and these I allow to be necessary and certain only because they are analytic. That is, I maintain that the reason why these propositions cannot be confuted in experience is that they do not make any assertion about the empirical world, but simply record our determination to use *symbols* (italics mine) in a certain fashion. Propositions concerning empirical matters of fact, on the other hand, I hold to be hypotheses, which can be probable but never certain. And in giving an account of the method of their validation I claim also to have explained the nature of truth. (p. 31)

If the logical positivists downplayed the significance of a priority, analyticity and necessity, W.V. Quine, famous for his position of "epistemological holism," rejected the distinction altogether. Quine argued that we have a "web of beliefs," and that while certain truths of the web (such as those of logic and mathematics) are more integral to the structure of the web, they are nonetheless as subject to disconfirmation by experience as are the beliefs of science-or any other belief for that matter. For Quine all truths are ultimately empirical in nature, there are no *necessary* truths at all.

While this is an interesting answer to the problem of knowledge, it only brings with it more questions. What kind of experience could disconfirm the truth of "2+2=4?" Moreover, if mathematics is empirical in nature, how can we deduce the existence of infinite numbers with no experience from the world to give us this knowledge? The answers given to this problem of knowledge are unsatisfactory for many, because if anything retains its a priori status, only two things do: logic and mathematics. Laurence Bonjour (1998) rejects the thesis that all knowledge is a posteriori on the following grounds:

> Contrary to the tendency in recent times for even those who accept the existence of *a priori* justification to downgrade its epistemological importance, it is arguable that the epistemic justification of at least the vast preponderance of what we think of as empirical knowledge must involve an indispensable *a priori* component—so that the only alternative to the existence of *a priori* justification is skepticism of a most radical kind. (p.3)

If there were in fact pure a priori judgments, what kinds of statements would they be? More importantly, how is it possible that there should or could be such knowledge? Although Hume and Kant began the great categorization of knowledge, Bertrand Russell (1912) further elucidated the key difference between a priori knowledge and a posteriori knowledge. Regarding a principle of logic such as the law of non-contradiction, Russell explains, "Now what makes it natural to call this principle a law of *thought* is that it is by thought rather than outward observation that we persuade ourselves of its necessary truth (p.62)." Furthermore Russell explains, a priori knowledge deals "*exclusively with the* *relations of universals* (p. 75)." Since Plato's theory of forms, the term *universal* has referred to those abstract concepts like "justice" or "piety."

Radical empiricists will maintain that most knowledge is knowable a posteriori, and either downplay the significance of "truths of reason" or reject (as Quine did) the analytic/synthetic distinction altogether.   Roderick Chisholm (1966) considers the possibility that we actually know this alleged "truths of reason" through experience, rather than through our reason exclusively. Chisholm posed the question this way: "Why *not* say that such 'truths of reason' are thus known a posteriori? (p. 73)" He defended the status of a priori knowledge thusly:

> For one thing, some of these truths pertain to properties that have never been exemplified. If we take "square," "rectangular," and "circular" in the precise way in which these words are usually interpreted in geometry, we must say that nothing is square, rectangular, or circular; things in nature, as Plato said, "fall short" of having such properties. Hence, to justify "Necessarily, being square includes being rectangular and excludes being circular," we cannot even take the first of the three steps illustrated above; there being no squares, we cannot collect instances of squares that are rectangles and squares that are not circles. (p. 73)

Although we may have examples of *approximate* squares, rectangles, and circles that we know through experience, we never in fact experience a *true* square, rectangle, or circle. Chisholm's point is that there are many things that we can only have knowledge of by using reason. If we were to try and appeal to our observations of the physical world as proof of our knowledge of geometric shapes, we would have no instances of such shapes at all.

Another key difference between the a priori and the a posteriori is the way in which induction is used for empirical claims. By observing phenomena such as the rising of the sun, we can use induction to make knowledge claims. "The sun will rise tomorrow" is a strong inductive inference based on the observations of numerous past occasions when the sun has risen. But, as Chisholm points out, the process of induction does not justify a priori knowledge, but rather presupposes it:

> [A]pplication of induction would seem to presuppose knowledge of the "truths of reason." In setting out to confirm an inductive hypothesis, we must be able to recognize what its consequences would be. Ordinarily, to recognize these we must apply deduction; we take the hypothesis along with other things that we know and we see what is then implied. All of this, it would seem, involves apprehension of truths of reason—such truths as may be suggested by "For every state of affairs, *p* and *q,* the conjunctive state of affairs, composed of *p* and of either not-*p* or *q,* includes *q*," and "All *A*'s being *B* excludes some *A'*s not being *B*." Hence, even if we are able to justify some of the "truths of reason" by inductive procedures, any such justification will presuppose others, and we will be left with some "truths of reason" which we have not justified by means of induction. (pp.73-4)

If the distinction between analytic truths and synthetic truths is legitimate, then we are limited to either knowledge that is certain but uninformative, or propositions that may be informative but may be subject to disconfirmation in light of our experiences. However, there exists an exciting possibility about knowledge: if mathematics is a priori, it may be synthetic as well—which offers the promise of knowledge about the world, not just certainty about relations of ideas. While these issues are broadly metaphysical and epistemological in nature, they lay the foundation for another form of discourse—the philosophy of mathematics.

1.2 Various Theses in the Philosophy of Mathematics

Although its beginnings can be traced back to Pythagoras and Plato, the philosophy of mathematics is relatively new as a legitimate branch of philosophical discourse. Historically intractable problems of metaphysics and epistemology make themselves apparent when set within the context of the philosophy of mathematics. The ontological question, "what are numbers?" is fundamental to the philosopher of mathematics.

Moreover, there is the issue of how we come to know the truths of mathematical propositions, which relates to epistemology. According to Stewart Shapiro:

> The job of the philosopher is to give an account of mathematics and its place in our intellectual lives. What is the subject-matter of mathematics (ontology)? What is the relationship between the subject-matter of mathematics and the subject-matter of science which allows such extensive application and cross-fertilization? How do we manage to do and know mathematics (epistemology)? How can mathematics be taught? How is mathematical language to be understood (semantics)? (Shapiro 2000, 16)

Perhaps the two most crucial questions in the philosophy of mathematics are: 1) what is the ontological status of numbers and 2) what is the epistemological status of mathematical statements? As far as the metaphysics goes, one can hold the realist position or the anti-realist position. Realists in ontology claim that numbers have an actual mode of being and exist independently of the mind, whereas anti-realists in ontology claim that numbers are merely constructs of the human mind. Similarly in epistemology, one can be a realist or an anti-realist. Realists in epistemology claim that mathematical statements can be said to be either objectively true or false. Anti-realists in epistemology will claim either that mathematical statements are void of content or meaningless, or that mathematical truth is mind-dependent. It is worth noting that a particular ontology does not necessarily entail a particular epistemology, but nonetheless the challenge for the philosopher of mathematics has been to provide a coherent account of the ontology of numbers as well as the epistemology of mathematical propositions.

There have historically been three major schools of thought in the philosophy of mathematics. *Logicism* is the thesis that mathematics is in some way reducible to logic. Among the chief defenders of logicism are Russell and Frege, and the logical positivists. It is worth noting here that logicists often adopt a *platonist* or *realist* ontology of numbers. Even though the logicist thesis claims that mathematics is reducible to logic, these propositions are *about* abstract objects that are inert and have no causal relationship with the physical world. Set-theoretic realism, the thesis that numbers are in fact *sets* that exist independently of the mind, is a popular contemporary articulation of this position.

Formalism is the school of thought that claims that the "*essence* of mathematics is the manipulation of characters (Shapiro 2000, 140)." The formalist argues that mathematics is merely a game, i.e. the manipulation of symbols and characters. Again, hearkening back to the medieval debate over universals and particulars, the formalist thesis bears a striking resemblance to *nominalism*.

Finally, the school of *intuitionism* (also referred to as *idealism* or *constructivism*) holds that numbers are mental constructs and therefore are mind dependent. For the intuitionist, numbers do not exist in some strange platonic way; rather, numbers are ideas in the mind and are quite "real," but only insofar as an individual produces or creates these mental artifacts. The first major thinker to make implicit this thesis was Kant, although there was considerable development on this thesis by Brouwer.

There are attractive features about all three of the traditional positions in the philosophy of mathematics. If one appeals to the thesis of logicism, one may also adopt a platonist or realist ontology. Then there is the assurance that numbers have a kind of platonic existence, and therefore are permanent, mind-independent entities that are as "real" as concrete objects—they just have a different modality. However, when trying to defend this thesis, one is left with the burden of postulating an intangible object—or as Shapiro (2000) asks: "How can we know anything about a realm of causally inert, abstract objects?" (p. 133) Though many thinkers have appealed to the existence of sets because of the indispensable role they play in the sciences and even in our ordinary use of basic mathematics (e.g. Quine), we are no more certain of their existence than we are of God or of souls. However, the ontological mysteries that God and souls present us are just as puzzling as sets, so we are still ultimately appealing to mystery when arguing for their existence.

If there is a diametric opposite to logicism, then it is formalism. The formalist does not have to appeal to a platonic realm to justify its ontology, for there is no ontology to justify. If number theoretic and other mathematical statements are just pieces and rules of a game not too much unlike chess, then there is nothing further to discuss. But the formalist does have an intractable problem—just how is mathematics so useful in the world? If mathematics is just a game, then how does it map on to the universe, which is surely not a game-board created by the human mind?

In one particular way the most plausible mathematical thesis is that of intuitionism, which holds that numbers are mind-dependent entities. The intuitionist does not have to appeal to a platonic realm for its assurance of the existence of numbers, and the truth of mathematical statements is derived from the function of the mind and the way it perceives the world. The prototypical intuitionist was Kant, though his philosophy of mathematics was not without its problems. Broadly speaking, Kant based arithmetic on time and geometry on space, and made the controversial claim that mathematical judgments were items of *synthetic a priori* knowledge. Fault was found with Kant's view of geometry—after Kant's time, it was generally accepted that certain fundamental principles of Euclidean geometry were not indicative of physical space. Kant, working within the framework of his time, made the error of assuming that we have *a priori* knowledge of physical space (based on the now antiquated postulates of Euclid). However, Kant's insights into the nature of arithmetic and numbers and their relation to time are worth giving serious consideration. And even though Kantian intuitionism may be more plausible insofar as abstract entities are not relied upon, it is still a view that is difficult to defend. The task at hand, then, is to examine Kant's philosophy of mathematics in order to discover its strengths and weaknesses.

In the following essay, a historical account of intuitionism will be given- from its beginnings in Kant's classic work, *Critique of Pure Reason*, to contemporary treatments by Brouwer and other intuitionists who have developed his position further. In chapter II, I will examine the ontology of Kant's philosophy of arithmetic. The issue at hand will be to explore how Kant, using intuition and time, argues for numbers as mental constructs. In chapter III, I will examine how mathematics for Kant yields synthetic a priori truths. In chapter IV, I will examine how intuitionism developed under the care of Brouwer and others (e.g. Dummett) and how Hilbert sought to address issues in Kantian philosophy of mathematics with his finitist approach. In conclusion, I will examine briefly what intuitionism resolves and what it leaves to be desired.

CHAPTER II

THE ONTOLOGY OF NUMBERS

2.1 Time as A Priori

So what then, are numbers? The Kantian answer to this question involves two things: 1) the faculty of intuition (or perception) and 2) time—which can be understood as one of the two foundations of the faculty of intuition (the other foundation being space) (Palmer 1988, 210). Rather than assume that space and time are external and mind independent, he assumed that they were features of the mind's structure. Numbers, then, are constructed from our experience of the passage of time.

Kant's philosophy of time, as with much of his philosophy, came in direct response to Hume's radical empiricism. If Hume was right that we do not perceive space, time, or causality, then Kant's task was to answer the question of how perception is possible at all. So for Kant, space and time became those apparatuses of the mind that made perception possible. But essential to his thesis of mathematical objects and knowledge was the proposition that our knowledge of time was *a priori*.

For Kant, the a priori is defined in a more or less conventional way—empirical knowledge can only be derived through observation, whereas a priori knowledge can be derived without appeal to the physical world. As Walker (1978, 28) explains: "This gives us a different, and much more satisfactory, way of distinguishing the a priori from the empirical: a priori knowledge *can* be established without appeal to experience, whereas empirical knowledge cannot (emphasis mine)." Kant makes the case that our knowledge of time is a priori by way of two arguments. I shall call the first argument the "Presupposition Argument" and the latter the "Representational Argument."

The Presupposition Argument is Kant' s first attempt to show the a priority of time (and space). As Kant argues in *Critique of Pure Reason*, time and space do not have an empirical nature. First he addresses space:

> For in order that certain sensations be referred to something outside me (that is, to something in another region of space from that in which I find myself), and similarly in order that I may be able to represent them as outside and alongside one another, and accordingly as not only different but as in different places, the representation of space must be presupposed. The representation of space, cannot, therefore, be empirically obtained from the relations of outer appearance. On the contrary, this outer experience is itself possible at all only through that representation. (A23/B38)

Kant makes an analogous case for the a priority of time:

> Time is not an empirical concept that has been derived from any experience. For neither coexistence nor succession would ever come within our perception, if the representation were not presupposed as underlying them *a priori*. Only on the presupposition of time can we represent to ourselves a number of things as existing at one and the same time (simultaneously) or at different times (successively).

Kant's ideas in *Critique* were a response to those claims about the empirical world that Hume had already made. Hume's attack on the idea of causation can be answered similarly by Kant. While we do not perceive any causation "out there," Kant argues that causation is a category of the mind that makes it possible for us to make judgments about the world involving cause and effect. Similarly, we do not perceive time and space in the physical world because they compose the faculty of intuition. Whenever we refer to something in the world, there is a both a spatial and a temporal character to the judgment due to the faculty of intuition—not because of a time or space external to the mind.

Though similar, the Representational Argument makes a slightly different move. While Kant primarily argues that we cannot experience the world without presupposing space and time, he argues further that we cannot represent to ourselves the absence of space and time. He argues thus:

> Space is a necessary *a priori* representation, which underlies all outer intuitions. We can never represent to ourselves the absence of space, though we can quite well think it as empty of objects. It must therefore be regarded as the condition of the possibility of appearances, and not as a determination dependent upon them. It is an *a priori* representation which necessarily underlies all outer appearances. (A24/B39)

The argument for time is similar:

> Time is a necessary representation that underlies all intuitions. We cannot, in respect of appearances in general, remove time itself, though we can quite well think time as devoid of appearances. Time is, therefore, given *a priori*. In it alone is actuality of appearances possible at all. Appearances may, once and all vanish; but time (as the universal condition of their possibility) cannot itself be removed. (A31)

If the concepts of time and space are indispensable as conditions of experiencing the world, then Kant has succeeded in making his case. A world devoid of space and time would indeed be difficult to conceive of, but does it follow that such a world is *inconceivable?* And even if our judgments about the world presuppose time and space, does it follow that they are not components of the empirical world? Walker (1978) has responses for the Presupposition Argument and the Representational Argument. He claims that both arguments suffer from Kant's "tendency to interpret 'a priori' genetically." First, in response to the argument that time and space are presupposed when making judgments about the empirical world, Walker argues:

> Obviously we cannot think of objects as spatio-temporally located without having the ideas of space and time; but we may still have acquired these ideas by observing objects which now, after having performed the abstraction, we can think of as located spatio-temporally. In just the same way one cannot think of an object as red without

> having the idea of redness; but redness is Kant's paradigm of an empirical object, acquired by abstraction from the observation of things which once we have the concept we can describe as instances of redness. So the argument fails to show that space and time are a priori in the genetic sense (1978, 29)

Walker's criticisms of the Presupposition Argument are problematic. There is a faulty analogy between "redness" as an empirical idea and "time" (or "space") as an empirical idea. Even the staunch empiricist Hume would admit that we have empirical evidence of "redness" and do not have to appeal to anything other than our senses for an account of it. However, it seems as though the same does not apply for time and space. We do not have any physical instantiations of time or space in the same way that "redness" or "justice" is exemplified. Unless a case can be made that the ideas of time and space can be founded on some empirical phenomena, Kant's Presupposition Argument remains unscathed.

Next, Walker tries to reject the Representational Argument. On the Kantian claim that we can never represent to ourselves the absence of time and space, Walker replies:

> It is difficult to see this as more than a psychological remark, though Kant draws from it the conclusion that space and time are necessary a priori representations underlying all our experience. In fact it is both false and irrelevant. If it were true it could only be a contingent truth about us, and would no more prove that space and time were a priori than my inability to imagine a chiliagon shows the impossibility of any such figure. But it is false, because we can quite well imagine worlds which are not spatial; and I shall argue shortly that we can even imagine atemporal experience, though I admit this is more difficult (1978, 29).

Space and time may only be contingent foundations of human (or similar) minds. Perhaps this is nothing more than a "psychological fact" about us, but these foundations enable us to make such judgments. The issue is not whether these foundations of the human mind are contingent facts about us. What is at stake is whether time and space can be abstracted from our experiences of the physical world. Kant argues that they cannot. And even though Walker (1978, 35) makes the case that we can imagine a timeless world, the kind of world one envisions by virtue of this description seems to be essentially *non-physical.* Even Walker himself admits (although qualifiedly) "Experience in an entirely changeless world is not indeed easy to imagine, and one may feel inclined to dismiss the idea out of hand. But the fact that something is hard to imagine does not make it conceptually incoherent (1978, 37)." Let us suppose that we can imagine a world not in time or space. To do so without invoking something like a platonic realm of forms would be, I believe, quite difficult. Kant makes the claim that time and space are necessary foundations for perceiving the (physical) world. If we can imagine a world devoid of time and space and it ends up looking like Plato's heaven or God's heaven, it is no accident. But Kant was ultimately making a claim about the phenomenal world.

Even if they are known by us a priori, the burden rests upon Kant to give some explanation of how we in fact we have a priori knowledge of time and space.

An important historical note to make again here is that Hume's suggestion that we have no empirical knowledge of time and space is profoundly baffling. If we have no empirical knowledge of these fundamental ideas, where do they come from? If there is no evidence that time and space exist "out there" independently of the mind, then how is it that time and space can play such a crucial role in comprehension of the world? Surely their manifestations are not accidents (at least Kant did not believe so). Kant concluded that time and space must be features of our own consciousness. He referred to them as the two forms of the faculty of intuition.

2.2 The Faculty of Intuition

As in Walker's case, there have been attempts to prove that Kant was not successful in showing that space and time are a priori. Whether it can be conclusively demonstrated that Kant was successful remains to be seen. On his account, time and space are the two foundations for the faculty of intuition. Even if Kant is successful in making the case that time and space are the a priori foundations necessary for perceiving the world (and more importantly for our purposes here, for making synthetic a priori judgments), he is still left with the task of explaining the nature and function of the faculty of intuition. Moreover, Kant's use of the term "concept" is sometimes similar to his use of the word "intuition":

> Now among the manifold concepts that make up the highly complicated web of human knowledge, there are some which are marked out for pure *a priori* employment, in complete independence of all experience; and their right to be so employed always demands a deduction. For since empirical proofs do not suffice to justify this kind of employment, we are faced by the problem how these concepts can relate to objects which they yet do not obtain from any experience. (A85/B117)

Here Kant is setting up the outline for the "transcendental deduction," which is the "manner in which concepts can thus relate *a priori* to objects (B117)." In the following passage, he explicitly refers to space and time as "concepts":

> We are already in possession of concepts which are of two different kinds, and which yet agree in that they relate to objects in a completely *a priori* manner, namely, the concepts of space and time as forms of sensibility, and the categories as concepts of understanding. To seek an empirical deduction of either of these types of concept would be labour entirely lost. (B118)

A fair question to ask of Kant at this point is " are space and time concepts or intuitions?" On the one hand, space and time as *concepts* allow us to conceive of relations in general—i.e. we *conceive* of spatio-temporality, and conceive of how objects in general exist and stand in relation to one another spatially and temporally. Walker notes:

> (Kant) argues that we must use a priori concepts…For the data must be presented in space and time (or, presumably, in something analogous to them, to allow for the possibility of beings with a different mode of sensible intuition), and space and time themselves are a priori. So in becoming aware of the data, even at the most primitive level, we shall have to be aware of spatio-temporal relations, and for this we shall need a synthesis which is a priori, determined by a priori concepts. (1978, 79)

Intuitions, on the other hand, allow us to perceive *particular* objects in space and time. Körner (1955, 33) discusses how space and time are a priori *particulars* for Kant. What this means is that synthetic a priori judgments are made assuming that the judgments are referring to or describing particular objects or events not perceived via the senses. The belief that space and time are a priori particulars contrasts the view that space and time describe properties of an object or relations between objects. If space and time were by contrast properties of an object in the physical world, then they could be abstracted from our experiences, as could properties such as color or shape. Because they cannot, Kant argues that they are presupposed, and moreover that they are foundational to our experiences. In the following passage, Kant alludes to a distinction between particular perceptions that are allowed by invoking the intuitions of space and time as opposed to the general application of the concepts of space and time:

> Appearances, in their formal aspect, contain an intuition in space and time, which conditions them, one and all, *a priori.* They cannot be apprehended, that is, taken up into empirical consciousness, save through that synthesis of the manifold whereby the representations of a determinate space or time are generated, that is, through combination of the homogeneous manifold and consciousness of its synthetic unity. Consciousness of the synthetic unity of the manifold [and] homogeneous in intuition in general, in so far as the representation of an object first becomes possible by means of it, is, however, the concept of a magnitude (*quantum*). Thus even the perception of an object, as appearance, is only possible through the same synthetic unity of the manifold of the given sensible intuition as

> that whereby the unity of the combination of the manifold [and] homogeneous is thought in the concept of a *magnitude.* In other words, appearances are all without exception *magnitudes,* indeed *extensive magnitudes.* As intuitions in space or time, they must be represented through the same synthesis whereby space and time in general are determined. (B203)

Kant assumes that perception of the phenomenal world is indeed possible and so wants to answer the important question "how is perception possible?" Kant maintains that perception becomes possible via the synthesis of the "manifold" and the quanta or "magnitude." The manifold he is referring to is our consciousness which houses the foundational intuitions of space and time, and the quanta are the perceptions.

Young (1992) discusses how the Kantian intuition functions. Although the words "concept" and "intuition" are used in a similar manner, we need not infer that some kind of conflation of terms is occurring here. Young explains intuition—specifically how it relates to mathematical concepts:

> When he says that "mathematical definitions are constructions of concepts" that "contain an arbitrary synthesis" of things intuited (A729-30/B757-8), he is making the point in his own way. We cannot capture the content of a mathematical concept merely by listing *predicates* that the instances of that concept must satisfy. Instead, we must posit *objects* and represent them as standing in certain relations. Representing such objects involves intuition. In Kant's characteristic phrase, it involves representing a manifold, or multiplicity, in intuition. This manifold of things also has to be represented as related in certain ways, so as to constitute the thing we are conceiving. In Kant's phrase, the manifold also has to be "gone through in a certain way, taken up, and connected" (A77/B102) "Synthesis" is simply Kant's term for this form of representation, and it is in this sense that synthesis gives a mathematical concept its content. (p. 115)

Young's observation about how the construction of concepts "contain an arbitrary synthesis" is an interesting one. One might consider the central thesis of intuitionism (that numbers are constructed, not inhabitants of a platonic realm) to be problematic insofar as it raises the question, "how then does a number like 12 have an equal value for two individuals when the same number can be allegedly generated from two different minds?"

Young may have answered this question by way of his explanation of how numbers "stand in certain relations" to one another. For Kant, arithmetic involves the succession from one number to the next through moments in time. In a very genuine sense, this representation is completely dependent upon one's subjective consciousness and therefore arbitrary; it is the way the mind represents the phenomenon of generating numbers. The construction of the number 12 corresponds to how the numbers 7 and 5 stand in relation to one another. So although the synthesis is "arbitrary," the way humans minds represent such a relation is universal insofar as all humans perform the representation in the same way.

If mathematical judgments are products of the particular features of human minds, then it becomes problematic to square this with the idea that mathematical judgments are necessary and a priori. There is yet one other way that this apparently intractable problem may be dissolved. Kant would throughout the *Critique* contend that in addition to time and space, there are ten other categories of the mind that play a similar role in our understanding of our experiences. Guyer (1992) notes that space and time have a particular significance that the other categories do not:

> Kant claims that the problem of a transcendental deduction arises for the categories of the understanding in a way in which it does not for space and time as pure forms of intuition. He says this is so because, whereas *all* appearances or empirical intuitions are *given* to us already in spatial and temporal form, the applicability of any *concept, a fortiori* any *a priori* concept, to all empirical intuitions is not in the same way manifest in anything immediately given. (pp.124-5)

Guyer notes that the issue here is how space and time can yield different sorts of judgments than can other categories-namely, how can space and time enable seemingly *objective* judgments about the world when in fact they are subjective features of the mind:

> This difference may be marked by Kant's change from the claim that the *objective reality* of the categories must be deduced (A84/B116) to the claim that their *objective validity* must be demonstrated. Kant does not offer formal definitions of these terms, but usually employs them in contexts which suggest that a concept has *objective reality* if it has at least *some* instantiation in experience but *objective validity* only if it applies to *all* possible objects of experience. (p. 125)

Judgments that utilize the faculty of intuition, therefore, constitute a more inclusive sort of application. This offers some insight into why the construction of numbers is universal for all human minds, while judgments based on other categories of the mind may not have the same kind of universal significance.

Although the idea of time as a priori and the idea of time as one of two components of the faculty of intuition have been discussed, one more crucial issue remains to be examined concerning Kant's ontology of numbers. It has already been mentioned that numbers are not mind-independent entities, but entities nonetheless. For Kant, unlike the Platonist, numbers are those entities constructed from the mind that conceives them. How then are numbers constructed? Kant believes that we construct numbers from the passage of time itself, a number corresponding for each successive moment. In the following section, this final (and perhaps most significant) part of Kant's ontology of numbers will be discussed.

2.3 How Numbers Are Constructed

For Kant, as for later thinkers such as Edmund Husserl, there is a notable emphasis on the subjective character of time. In 1905, Husserl gave a series of lecture of the phenomenology of time consciousness. In these lectures, which were collected and published posthumously, Husserl draws distinctions between what is considered an objective or "world-time," and the internal time consciousness that is a feature of the human mind:

> It may further be an interesting study to establish how time which is posited in a time-consciousness as Objective is related to real Objective time, whether the evaluations of temporal intervals conform to Objective, real temporal intervals or how they deviate from them….When we speak of the analysis of time-consciousness, of the temporal character of objects of perception, memory, and expectation, it may seem, to be sure, as if we assume the Objective flow of time, and then really study only the subjective conditions of the possibility of an *intuition* (emphasis mine) of time and a true knowledge of time. What we accept, however, is not the existence of a world-time, the existence of a concrete duration, and the like, but time and duration appearing as such….To be sure, we also assume an existing time; this, however, is not the time of the world of experience but the *immanent time* of the flow of consciousness. (Husserl 1973, 23)

Husserl's position marks a striking similarity to that of Kant's insofar as he sees how space and time have a subjective character juxtaposed to an objective, absolute character. Like Kant, Husserl will make the claim that time (like space) is foundational to one's understanding of experiences, underlying all other judgments. First he considers the concept of space:

> What is meant by the exclusion of Objective time will perhaps become still clearer if we draw a parallel with space, since space and time exhibit so may noted and significant analogies. Consciousness of space belongs in the sphere of phenomenological givens, i.e., the consciousness of space is the lived experience in which "intuition of space" as perception and phantasy (sic) takes place…We discover relations….But these are not Objective-spatial relations. (pp.23-4)

He then makes an analogous case for time:

> We can now draw similar conclusions with regard to time. The phenomenological data are the apprehensions of time, the lived experiences in which the temporal in the Objective sense appears. Again, phenomenologically given are the moments of lived experience which specifically establish apprehensions of time as such, and, therefore, establish, if the occasion should

> arise, the specific temporal content (that which conventional nativism calls the primordially temporal). But nothing of this is Objective time. (p. 24)

There is a significant point to be made about the subjectivity of time-consciousness as it relates to the construction of numbers. If time is to be understood as Kant understood it to be a feature of consciousness, then it does not have an absolute foundation external to the mind. If numbers are derived from the passage of time, then all the more reason to concede that they are mind-dependent and do not belong in an external platonic realm. Furthermore, the intuitionist escapes one problem that the Platonist must face. The Platonist must answer the question of how an object such as a number that exists in a non-physical, static, inert realm of being has a place and function in the physical world. If we concede with the intuitionists that numbers are a feature of the consciousness (which is at least arguably part of the physical world), then this issue becomes much less problematic.

The Kantian construction of numbers is simple yet ingenious. If we let each identifiable moment constitute or represent a number, we can identify that number as 1. The number 1 stands in relation to all other numbers by means of succession. As we intuit one moment succeeding another, so too can we intuit the number 1 as it stands in relation to all other numbers in succession. For each passing moment that stands in relation to that initial number, we can name them as such: 2, 3, 4, 5, and so on. L E. J. Brouwer would go on to call this phenomenon the intuition of the “two-oneness.” It creates the numbers 1 and 2, and likewise all natural numbers. For each moment that passes, there exists the potential for another moment to pass. Likewise, for each ordinal number we can identify, there can be yet another added and identified in a given sequence. The fact that these numbers are *created* is significant insofar as it plays a role in Kant's epistemology and the possibility of the synthetic a priori, which will be discussed in the next chapter. But the fact that the natural numbers are created also has ontological significance—numbers are real, just not in the platonic sense. They are mind-dependent entities that come into being as one constructs them. To assert that numbers are mental constructs is not to commit to the position that a different "1" exists from person to person—rather, Kant seems to being making the case that respective "1s" in different minds are referring to a unit of passing time. This gives us our sense of order and succession, and thereby our numbers.

Kant derived his philosophy of geometry from his theory of space as a priori. With recent developments in non-Euclidean geometries, this view has been abandoned. However, Brouwer, Kant's successor in intuitionist thought, retained Kant's philosophy of arithmetic based on the theory of time as a priori as he believed this to be a viable mathematical principle. But as Parsons admits, "Kant does not discuss the philosophy of arithmetic at any great length, so that it is virtually impossible to understand him without making use of other material (1982,13)." Most of Kant's discussion of these issues deals almost exclusively with the a priori nature of time and space, and with the fact that time and space compose the faculty of intuition, which have been discussed here already at some length. Although Kant does not make reference explicitly to number in this passage, he does make a nod to the construction of mathematical concepts. Brouwer would call this the "two-oneness," or the construction of mathematical entities. The following is one such passage:

> The *a priori* method gives our rational and mathematical knowledge through the construction of a concept, the *a posteriori* method our merely empirical (mechanical) knowledge, which is incapable of yielding necessary and apodeictic propositions. Thus I might analyse my empirical concept of gold without gaining

> anything more than merely an enumeration of everything that I actually think in using the word, thus improving the logical character of my knowledge but not in any way adding to it. But I take the material body, familiarly known by this name, and obtain perceptions by means of it; and these perceptions yield various propositions which are synthetic but empirical. When the concept is mathematical, as in the concept of a triangle, I am in a position to construct the concept, that is, to give it *a priori* in intuition, and in this way to obtain knowledge which is at once synthetic and rational. (A722/B750)

By virtue of their character as concepts that are created, Kant maintained that mathematical knowledge was synthetic, another innovative component of his philosophy of mathematics and more generally of his epistemology. At this juncture, it is difficult to separate Kant's ontology of numbers from his epistemology of mathematical statements. I make this claim only because the construction of numbers involves the creation of a new entity (which deals with the ontology of numbers)—but we are simultaneously while considering the existence of a new entity considering a new item of knowledge. Kant infamously believed that mathematical statements such as 7+5=12 yielded knowledge that was synthetic a priori. In the next chapter, the epistemology, truth-value and meaning of such mathematical statements will be discussed.

CHAPTER III

THE TRUTH VALUE AND SEMANTICS OF MATHEMATICAL PROPOSITIONS

3.1 Mathematics as Synthetic A Priori

The metaphysics and epistemology of Kant are so closely tied together that it is difficult to talk about one at length without referring explicitly to the other. Kant noted that the world as we perceive it is distinct from world as it truly is. Kant makes the important claim that perception of the empirical world is made possible by the faculty of intuition. Without this faculty, we presumably could not perceive the world at all. The a priori "givens" of the mind (space and time) function as foundations that allow there to be perception of the world. These foundations allow for the phenomena, the "things as they appear," to be observed. The noumena is inhabited by "things as they are."

Kant and Plato are similar to the extent that both philosophers had a metaphysics that depicts reality as having two sides—the physical, perceptual world and the permanent, "real" world. But Kant's view of mathematical objects differs explicitly from that of Plato, insofar as Kant believed that numbers were mind-dependent entities and Plato believed that numbers were realities that existed in the realm of forms, i.e. the world of being. But Kant did not make the claim that numbers belonged to the realm of the noumena, but rather that numbers were mental constructs. But even if Kant owed some credit to Plato for his ontological commitments (insofar as both conceived of the world as having two parts), he developed a rather unique epistemology—one that allowed for knowledge that was *synthetic a priori.*

Although Hume is often credited with arousing Kant from his "dogmatic slumber" about the nature of reality and truth, Walker (1978) cites C.A. Crusius (1712-1755) as an influence on Kant perhaps as significant as Hume. Crusius was a German philosopher and contemporary of Kant who belonged to the Pietist movement, a sect of Lutheranism that Kant was also tied to since his childhood. While the two thinkers had biographical similarities, Crusius' influence on Kant's thinking was also remarkable. Crusius was an opponent of the kind of determinism that Leibniz and Wolff were unwillingly committed to by virtue of Leibniz's Principle of Sufficient Reason (the notion that for everything that exists, there is a reason why it does) (Walker, 1955, 3). This principle, shared by both Leibniz and Wolff, was for Leibniz a deduction from a supposedly more fundamental and self-evident logical truth—that something either is or is not (p.2). The rejection of the Principle of Sufficient Reason was significant for Kant insofar as both Crusius and Kant tried to find a place for human freedom in a world that seemed to indicate a necessary chain of cause of effect. But a more significant development would result eventually for Kant from Crusius' disagreements in *logic* with Leibniz and Wolff, as Walker explains:

> Crusius did not accept the Principle of Sufficient Reason entirely—he accepted a modified version of it which was supposed to leave a proper place for freedom. But he also, and for our purposes more importantly, pointed out the flaw in Wolff's supposed derivation of the Principle of Sufficient Reason, and showed in general that Wolff had tried to get far too much out of the Principle of Contradiction. Wolff had wanted it to be the sole ultimate and self-evident principle of reason; Crusius saw that it was by no means sufficient to yield the metaphysical claims Wolff needed to make, and had to be supplemented by other principles equally ultimate and equally self-evident. Crusius' achievement was, in effect (and still rather obscurely), to distinguish within the class of necessary truths those Kant would call analytic from those which while not analytic were still necessary. (p.3)

Even though Kant would go on to reject fundamental tenets of Crusius' metaphysics, it was Crusius' criticism of Wolff's logical inference that left a significant impression on Kant. As Walker notes, "For Crusius had asked a fundamental question, the question Kant was later to put as 'How is synthetic a priori knowledge possible?' (p.4)" It seemed as though Crusius' acceptance of a weaker logic (via rejection of what is referred to in the above passage as "the Principle of Contradiction," but can also be construed as the Law of the Excluded Middle) in some way opened for Kant a possibility of knowledge about the world that was not based on what Hume called "relations of ideas." It was Crusius who initially distinguished between those necessary truths that Kant would describe as analytic and those necessary truths that would not be described as analytic—thereby providing the possibility of synthetic a priori knowledge. Moreover, it is perhaps no coincidence that the very logic that Crusius was proposing would be adopted as part of the framework for intuitionist logic (we will return to this specific issue in chapter IV).

For synthetic a priori knowledge to be possible, it must be the case that mathematical statements are not merely true by definition or true by virtue of the subject-predicate relationship, i.e. not analytic truths. In other words, when one expresses the proposition "7+5=12," one must make the case that the concept of 12 is not included in the concepts 7, 5, and summation. The proponent of this view is making the case that a new item of knowledge is gained from the synthesis of the ideas of "7" and "5." Prima facie, this seems to be a difficult case to make. As Flew points out, the Kantian notion of "synthetic" has been criticized for a variety of reasons:

> Kant's distinction has been criticized: first, for being indeterminate—it is not clear what is or is not to be counted as thus contained; second for being

> inappropriately psychological, and hence for possibly yielding different determinations for different individuals; and third for assuming that all propositions must be of the subject-predicate form—the form to which the attention of the traditional formal logic was so largely confined. (1979,12)

A return will be made to these criticisms in some depth in section 3.2 and in chapter V. For our purposes now, the issues are 1) whether Kant can make the case for the possibility of synthetic a priori knowledge and 2) what kinds of propositions should count as synthetic a priori knowledge. Let us consider Kant's claim that mathematics should be included in this category.

The following passage from the *Critique* marks Kant's infamous claim that mathematical judgments (particularly sums) are synthetic:

> We might, indeed, at first suppose that the proposition 7+5=12 is a merely analytic proposition, and follows by the principle of contradiction from the concept of a sum of 7 and 5. But if we look more closely we find that the concept of the sum of 7 and 5 contains nothing save the union of the two numbers into one, and in this no thought is being taken as to what that single number may be which combines both. The concept of 12 is by no means already thought in merely thinking this union of 7 and 5; and I may analyse my concept of such a possible sum as long as I please, still I shall never find the 12 in it….For starting with the number 7, and for the concept of 5 calling in the aid of the fingers of my hand as intuition, I now add one by one to the number 7 the units which I previously took together to form the number 5, and…see the number 12 come into being. That 5 should be added to, I have indeed already thought in the concept of a sum=7+5, but not that this sum is equivalent to the number 12. Arithmetical propositions are therefore always synthetic. This is still more evident if we take larger numbers. For it is then obvious that, however we might turn and twist our concepts, we could never, by the mere analysis of them, and without the aid of intuition, discover what [the number is that] is the sum. (B15-16)

The issue for Kant seems to be not so much whether the antecedent integers that are added together are *identical* with the consequent sum. Rather, the issue is whether the concept "12" can be analyzed from the concepts of "7" and "5". For Kant, the number 12 "comes into being," i.e. is constructed by the mind. This may be the stalemate between those who accept the entirety of mathematics as analytic and those who concede with Kant that propositions of arithmetic are items of synthetic a priori knowledge: the issue seems to ride on whether there can be a sufficient distinction made between the identity of sums and their antecedent components and whether the sums can be analyzed from the concepts of the numbers themselves. It is fair to say that someone from the analytic camp could accuse Kant of making a distinction without a difference. But even if Kant is successful in proving that the number 12 "comes into being" when using our intuition, there is still another issue that remains about the epistemology of mathematical statements.

Can mathematical propositions, as understood by Kant, be true or false? Can a proposition that is created from one's subjective experience, such as "7+5=12," have objective truth? To make a brief return to an issue discussed in section 2.2, Kant does address the problem of these truths being "inappropriately psychological" by postulating a distinction between those truths that have *objective reality* and can be demonstrated to have *objective validity* (Guyer,1992,125). But is this a satisfactory account of knowledge? Can the mental constructs that come by way of intuition count as objective truth? Or is it the case that because these judgments are *subjective*, which is to say that they are similar in kind to judgments like "ice cream tastes good," that we cannot count them beliefs that can be true or false? Surely many individuals share judgments about the palatability of ice cream, but these shared beliefs are regarded as judgments that are not objectively verifiable. Even if *every* mind will make the judgment "7+5=12," the fact that it is shared belief alone cannot be substantiate objective truth (for consider the possibility that every mind could conceivably hold the belief "ice cream tastes good").

Conversely, if these judgments are products of one's subjective experience, how is it that everyone can make similar, if not identical judgments such as "7+5=12"?

We have already addressed the issue of whether numbers are *real* for Kant; the answer is yes, but they are mind-dependent entities. The next issue to address is whether Kant's epistemology of mathematics can give us knowledge of the world or just psychological facts about human consciousness. With a return to the issue referred to by Guyer, we will discuss this issue in the next section.

3.2 Truth or Artifacts of Human Intuition?

Kant's epistemology of mathematical statements may prove to be more problematic than his ontology. In *Language, Truth and Logic,* A.J. Ayer argues for a return to the original division of categories of knowledge as proposed by Hume. Ayer and his logical positivist contemporaries would argue that all "matters of fact" should be confined to empirical knowledge that is to be considered synthetic, contingent and a posteriori. Conversely, all "relations of ideas" will include those propositions that are considered analytic, necessary and a priori—and logic and mathematics should be considered eligible only for this category. His reasons for rejecting Kant's proposition that mathematics constitutes synthetic a priori knowledge are as follows (echoing the summary given by Flew earlier in 3.1):

> Kant does not give one straightforward criterion for distinguishing between analytic and synthetic propositions; he gives two distinct criteria, which are by no means equivalent. Thus his ground for holding that the proposition "7+5=12" is synthetic is, as we have seen, that the subjective intension of "7+5" does not comprise the subjective intension of "12"; whereas his ground for holding that "all bodies are extended" is an analytic proposition is that it rests on the principle of contradiction alone. That is, he employs a psychological criterion in the first of these examples, and a logical criterion in the second, and takes their equivalence for granted. But, in fact, a proposition which is synthetic according to the former criterion may very well be analytic according to the latter. For, as we have

> already pointed out, it is possible for symbols to be synonymous without having the same intensional meaning for anyone: and accordingly from the fact that one can think of the sum of seven and five without necessarily thinking of twelve, it by no means follows that the proposition "7+5=12" can be denied without self-contradiction. From the rest of his argument, it is clear that it is this logical proposition, and not any psychological proposition, that Kant is really anxious to establish. His use of the psychological criterion leads him to think he has established it, when he has not. (Ayer, 1946, 78)

This may be a death blow for Kant. One important criticism Ayer makes that Flew does not is that we can hardly deny the proposition "7+5=12" without self-contradiction. It seems that Ayer wishes to establish that this is one of the fundamental characteristics of analytic statements—they cannot be denied without being self-contradictory. However, Kant defined analyticity by explaining the subject-predicate relationship. This being the case, Ayer may be inadvertently referring to a characteristic of *necessity,* and not to analyticity as understood by Kant. Unless we wish to equate analyticity with necessity, or analyticity and necessity with a priority, we can try to keep these meanings distinct and preserve some of the integrity of Kant's proposal. But Ayer's criticism should not be dismissed so easily. The issue at hand is that there has historically not been a consensus on whether one can satisfactorily separate propositions in such a way, and Ayer has pointed out that Kant's distinctions leave much to be desired—and perhaps justifiably so.

If we concede like Hume and the logical positivists that logic and mathematics are properly confined to the analytic category, then the issue of truth is not problematic—such statements are necessarily true, but only because they are true by definition (and likewise they are *trivially* true). These kinds of statements, if belonging properly to this category, are not contingent facts about the world, but rather about concepts, their meanings, and their relation to other concepts. Kant's position was bold because he was arguing for a kind of truth that was necessary, but a truth about the world—not about relations of ideas or definitions. Even if we grant Ayer's criticism, there is something profoundly insightful about the Kantian attempt at postulating the synthetic a priori: it bridges the gap between the mathematical world and the physical world, so to speak. When we discussed the ontology of numbers in chapter II, we saw that the Platonist would struggle to explain how a number that existed in a non-physical realm would have significance and import in the physical world. One can demur from Platonism and embrace logicism, the view that mathematics is reducible to logic (which would entail the view that mathematics is not about any "matters of fact" but rather expressed analytic statements).

But just as the Platonists have trouble with their ontology, the logicists have trouble with their epistemology. After all, if mathematical statements are not about anything in the physical world, then how is it that we can use mathematics in conjunction with our experiences? Shapiro recognizes this problem with the traditional analytic/synthetic distinction as compared to Kant's proposal:

> There is an important tension in the traditional picture. On that view, mathematics is necessary and knowable a priori, but mathematics has *something* to do with the physical world. As noted, mathematics is essential to the scientific approach to the world, and science is empirical if anything is—rationalism notwithstanding. So how does a priori knowledge of necessary truths figure in ordinary, empirical knowledge-gathering?….According to Kant, mathematics relates to the forms of perception. It concerns the ways that we perceive the material world….arithmetic concerns the forms of spatial and temporal intuition. Mathematics is thus necessary because we cannot structure the world in any other way. We *must* perceive the world through these forms of intuition. No other forms are available to us. Mathematical knowledge is a priori since we do not need any particular experience with the world in order to grasp the forms of perceptual intuition. (Shapiro, 2000, 23)

If we cannot invoke the Kantian forms of intuition, then we are left with the problem of bridging the gap between a priori knowledge and how it applies to the empirical world. Benacerraf (1983) makes note of the significance of bridging this gap:

> I take it to be obvious that any philosophically satisfactory account of truth, reference, meaning, and knowledge must embrace them all and must be adequate for all the propositions to which these concepts apply. An account of knowledge that *seems* to work for certain empirical propositions about medium-sized physical objects but which fails to account for more theoretical knowledge is unsatisfactory—not only because it is incomplete, but because it may be incorrect as well, even as an account of the things it seems to cover quite adequately. To think otherwise would be, among other things, to ignore the interdependence of our knowledge in different areas. (p. 404)

So if even Kant has taken on more problems by invoking intuition as the foundation which makes mathematical judgments necessary and a priori, he has offered some hope of bridging the great chasm between, as Benacerraf puts it, knowledge of physical objects and "theoretical knowledge." If Kant were to have his way, then the forms of intuition give us a priori and necessary truths about the world, ergo synthetic truths. But can we take these "truths" as necessary, considering the possibility that they may be "inappropriately psychological?" Necessity as conceived by Leibniz is truth in all possible worlds—but should a judgment that may be classified as an artifact of human intuition count as a *necessary* truth? At this point we should return to Guyer's attempt at answering this question:

> This difference may be marked by Kant's change from the claim that the *objective reality* of the categories must be deduced (A84/B116) to the claim that their *objective validity* must be demonstrated. Kant does not offer formal definitions of these terms, but usually employs them in contexts which suggest that a concept has *objective reality* if it has at least *some* instantiation in experience but *objective validity* only if it applies to *all* possible objects of experience. (1992, 125)

Objective validity is still a troublesome notion. Kant wants to allow for a difference between a "thing-in itself" and a "thing-as-it-appears." Our a priori concepts of space and time allow us to perceive such things as physical objects, but can we properly understand these judgments as necessary and a priori if they are due to a *particular* kind of consciousness?

We can consider this problem from another angle. If Kant is correct that the forms of intuition make perception possible, then should we infer that alternate forms of intuition would make perception possible in some very different way? Let us suppose for the sake of argument that the forms of intuition that Kant allow us to conceive of the world in a way that is unique to humans. Therefore *all* humans will make judgments such as "7+5=12" and "this chair occupies space," but not in virtue of some absolute truth—rather only in virtue of the forms that allow such judgments. To this extent, we can only allow Kantian judgments their status of "necessary" and "a priori" with certain qualifications. Another apparatus could yield very different judgments—but nonetheless judgments that would be necessary and a priori for all entities with an alternate form of intuition. If we are to make sense of Kant's position in this way, then mathematical judgments are not just "inappropriately psychological;" it may be the case that their truth becomes contingent on the kind of intuition that allows for conditions of experiencing the world. It is not clear whether Kant can avoid the "inappropriately psychological" criticism as mentioned. If a different kind of consciousness can perceive the world in some different, but equally satisfactory way, then how can we give one kind of judgment a higher status than another?

So as we can see, the Kantian is not without some obstacles to overcome. Even though this is the case, Kant's thoughts in philosophy of mathematics had a great influence on a group who would become known as the intuitionists (a name not given to Kant during his time). L.E.J. Brouwer is considered to be the "father of intutionism," but is surely indebted to Kant's philosophy of arithmetic as the foundation of intuitionist thought. In recent years, intuitionism has found a spokesperson in Michael Dummett (though as we will see in the next chapter, Dummett's own arguments are radically different from the Kantian arguments). Moreover, Kant's philosophy of mathematics also had profound implications for the notion of infinity. Though not considered an intuitionist himself, David Hilbert's thesis of "finitism" fits comfortably with some aspects of Kantian intuitionism. Hilbert's program will be discussed along with Brouwer and Dummett in the next chapter.

CHAPTER IV

KANT'S INFLUENCE: INTUITIONISM AND ITS DEFENDERS

4.1 Brouwer

L.E.J. Brouwer was among the first group of mathematicians to follow in Kant's footsteps in believing that mathematics was an activity of the human intellect. Brouwer's philosophy of mathematics was an improvement upon Kant's in that he abandoned the notion that spatial perception gives us a priori knowledge of geometry. Instead, he kept Kant's philosophy of arithmetic and relied heavily on the notion that we acquire the natural numbers through temporal perception. In fact, Brouwer uses the techniques of Descartes to base geometry on the real numbers. The geometry of the "Cartesian Plane" works in the following manner: geometry is reduced to arithmetic by associating an ordered pair of real numbers with each point on the Euclidean plane. In doing so, Descartes reduces the study of relationships among these points to a study of relationships among pairs of real numbers. So interestingly enough, Brouwer's geometry (a la Descartes) turns out to be based on the Kantian intuition of time as well. He argues how his revisions of intuitionism in light of the abandonment of Kant's view of geometry make the theory more coherent in the following passage:

> But the most serious blow for the Kantian theory was the discovery of non-euclidean geometry, a consistent theory developed from a set of axioms differing from that of elementary geometry only in this respect that the parallel axiom was replaced by its negative….hence, it is not only impossible to show that the space of our experience has the properties of elementary geometry but it has no significance to ask for *the* geometry which would be true for the space of our experience…However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time. (Brouwer, 1983, 80)

Brouwer also introduces the term "two-oneness" to describe the phenomenon of constructing the real numbers:

> This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely…(1983, 80)

But Brouwer's development of Kantian mathematics did not stop there. Since for the intuitionist no mathematical entity exists until the mathematician constructs it, there are certain principles of intuitionism that will not square with classical mathematics and logic. The rejection of the law of excluded middle (either P is the case or P is not the case) stems from a difference in ontological and epistemological commitment between the intuitionists and the classicists. The principle of bivalence holds that *all* propositions are either true or false, and the law of excluded middle illustrates this. The classicist believes firmly that there are only two possible truth-values for any given proposition—but the intuitionist holds that not *every* proposition will have a *determinate* truth-value, and thus true or false cannot be appropriately assigned to every proposition (just which propositions the intuitionist has in mind we will examine in our discussion of Dummett in 4.3).

In short, the intuitionist logician demands a revision to bivalent logic. The law of the excluded middle (P V ~P) symbolizes that any proposition is either true or false; but this entails that some proposition will be true independently of it being considered by any mind. *Mathematical truth,* however, directly corresponds to the existence of mathematical objects. As we have already seen, the intuitionist does not presuppose that any mathematical object exists unless there is a proof given for it (i.e. unless it has been constructed). Likewise, a mathematical proposition cannot be deemed true or false by the intuitionist unless it refers to a mathematical object that has been constructed. Shapiro (2000) explains how the classicists and the intuitionists differ in ontology and epistemology:

> Intuitionists argue that excluded middle and the related inferences indicate a belief in the independent existence of mathematical objects and/or a belief that mathematical propositions are true or false independent of the mathematician… Some intuitionists reject this realism outright, while others just argue that mathematics should not presuppose any such metaphysical thesis. (p. 174)

There are some profound implications of the rejection of this law of logic. Symbolically, we can consider the following premise and conclusion, where P stands for any proposition and "~" stands for the negation of a proposition: ~ ~ P, therefore P. The classicist considers this inference, known as double negation elimination, to be valid. But Brouwer rejects this rule of classical logic. The negation of a proposition, such as P, can only be asserted when it is demonstrated that the assumption of P is contradictory. This being the case, a proposition such as ~ ~ P can be established by showing that ~P is contradictory. When one uses intuitionistic logic, ~ ~ P can be inferred from P; however P *cannot* be inferred from ~ ~ P (the only exception to this is when one *knows that* P is either true or false).

Intuitionism has implications for the concept of the infinite as well. If we consider one of Zeno's ancient paradoxes of motion, we are plagued by two conflicting ideas: 1) that there is an infinite number of points on a given line and 2) that we can move from point A to point B. But the truth of 1) seems to make 2) impossible: for we must move halfway between A and B before we get from A to B, and before that half that distance, and so on. Zeno's motion paradoxes illustrate a contradiction in believing in what the intuitionist would call "actual infinity" and accepting our common sense views about space and motion. Brouwer preferred to discuss the "potential infinite." Recall that for the intuitionist a number only exists once it has been constructed. The "actual infinity" referred to in Zeno's paradoxes is a somewhat philosophically dubious notion—just how many points are there in a given line? There are an infinite number of points, some might say. But the intuitionist would say that the numbers aren't "out there" in some realm—they are being created as we count, hence there is a *potential* infinity of numbers we can construct. The points on a line that come into being do so in the same way that numbers do for the intuitionist. Brown (1999) notes how Aristotle was perhaps the first to discover this fundamental principle of intuitionist thought, while at the same time offering a solution to the paradoxes of Zeno:

> Suppose I draw a line. How many points are there on it? You're likely to say, An infinite number of points. But Aristotle would say, None. However, Aristotle maintained, if I make a cut mark in this line, I create a point. Now it has one point. I can do this over and over again, creating many points. But, two things must be noted: first, at any stage in this process I have created only a finite number of points on the line, and second, no matter how many points I have created, I can always create more. (p. 123)

So for those who might find the concept of the actually infinite problematic, the intuitionist's proposal might seem satisfying. On the one hand, classical mathematics holds that for any line that extends from point A to B, there are an infinite number of points between A and B. As this theory applies to physical space, there are an infinite number of points in the hundred-yard length from goal line to goal line that a running back must pass before making a touchdown. Zeno uses his paradoxes to illustrate that there is an apparent problem in believing that 1) there are an infinite number of points along this line and 2) that motion along the line is possible. But on the other hand, if we adopt the view that points are constructed as we mark them along the line, we do not encounter the paradox that Zeno alludes to. There is no infinite number of points to traverse through, because the points are established (i.e. constructed) as we move to that area. There exists the hope that the paradox is dissolved, as we notice that our running back has not done the impossible by moving through an infinite number of points, but rather has established points along the way in his journey from end zone to zone.

Brouwer was not the only thinker to reflect on how Kantian intuitionism squared with the concept of the infinite. Although he was associated more closely with the formalist school of thought, David Hilbert's finitism was also greatly influenced by Kant's intuitionism. In the next section, we will discuss his thoughts on the finite and the infinite.

4.2 Hilbert and the Finitist Approach

Hilbert saw the principle of potential infinity to be correct. Yet at the same time, he wanted to preserve classical mathematics while incorporating this aspect of intuitionist thought. In particular, Hilbert wanted to preserve Cantor's transfinite set theory and famously proclaimed, "No one shall drive us out of the paradise which Cantor has created for us (1983,191)." Hilbert saw finitary mathematics as consisting of meaningful propositions. However, classical mathematics, which dealt with the concept of the actually infinite, was seen by Hilbert as problematic. For example, a statement such as $1<2$ is true; but a statement such as $x<2x$ can only be seen as void of meaning and thus possessing no truth value since we have not assigned an integer to the variable x. The variable in question represents an *infinite* quantity of numbers. Hilbert noted that it was problematic to assign truth-value to a statement with a variable that could stand for an infinite quantity of numbers because we do not know what the x is specifically referring to. But Hilbert wanted to incorporate finitary mathematics and Cantor's transfinite set theory primarily because with Cantor's theory comes enormous utility—as well as the great simplicity of classical logic. Brown explains: "In spite of their meaninglessness, statements involving the infinite can be added to meaningful, finite, true mathematics as supplements to make things run more smoothly or to derive new infinite results (1999, 66)."

The infinite was considered by Hilbert to be comprised of what he called *ideal elements*. Whereas finitary mathematics allows one to make meaningful statements using the natural numbers and the principles of addition, subtraction, multiplication and division, the ideal elements were any aspects of higher-level mathematics that made use of the actually infinite. *Ideal mathematics,* then, would include such things as set theory. The ideal elements were treated syntactically, which is to say *formally* (it is in this regard that Hilbert is often associated with formalism). The propositions containing ideal elements were not considered to be about anything, because the elements are only like chess pieces in a chess game. While these ideal elements were not *real* in the sense that finite constructs of numbers were, they serve the same purpose that theoretical entities in physics do; they may not be observable, but they have value for the physicist insofar as they help a theory work more smoothly (Brown, 1999, 66-67). Hilbert uses the method of *conservation*, whereby the introduction of a system of ideal elements is acceptable only by fulfillment of one condition. Hilbert maintains that conservation is achieved in a proof if the truths of finitary mathematics can be preserved when ideal elements are introduced in conjunction with the finitary. In other words, a conservative extension of a finite system is achieved if in a system of ideal elements you cannot derive some formula that leads to a contradiction in the system of finite numbers. If conservation can be achieved, then we can say of the system of ideal elements that it is *consistent.* Hilbert explains:

> There is just one condition, albeit an absolutely necessary one, connected with the method of ideal elements. That condition is a *proof of consistency*, for the extension of a domain by the addition of ideal elements is legitimate only if the extension does not cause contradictions to appear in the old, narrower domain, or, in other words, only if the relations that obtain among the old structures when the ideal structures are deleted are always valid in the old domain. (1983,199)

Consistency was of utmost importance to Hilbert, as *consistency implies existence*. But this position had some staunch opposition, and with good reason. We can examine the traditional concept of the God of Judaism and Christianity and characteristics such as omnipotence, omniscience and omnibenevolence come to mind. In reflecting on this depiction of God, we seem to have a consistent concept. But does this consistency imply that such a being *exists*? Prima facie, it seems dubious to count this concept in and of itself as a proof for God's existence, but if we are interpreting Hilbert correctly, this is what he has in mind. It is important to draw attention to the fact that Hilbert is not making the same move as defenders of the ontological proof for God's existence are; rather the analogy is that if a purported entity has qualities that are consistent with one another, this implies the existence of such an entity—and Anselm was instead arguing from the concept of perfection to the conclusion that in virtue of what a perfect being is that such a being exists necessarily. If we categorize Hilbert as a formalist, then this view is seen as less problematic. If consistency proofs are believed to demonstrate existence, then this may be a belief in the existence of ideal elements as *theoretical entities*—much in the same way an instrumentalist in the sciences may believe in theoretical entities to make his theory work. But if Hilbert's "consistency implies existence" policy is only theoretical and not literal, then he is still left with the intractable problem of how these theoretical entities can apply to something in the physical world.

Frege, for example, was quick to point out (by using the characteristics of God as an example) that a consistent concept does not imply that the concept in consideration has in bearing in reality (Brown 1999, 100). Gray (2000) explains the radical difference in approach between Frege and Hilbert:

> For Frege, existence was primarily a question of what objects there are in the world. Without objects, axiom systems were in his view void. Hilbert was radically of the other opinion: consistency implies existence. Whatever the murkiness of Hilbert's thinking about consistency, his philosophy of mathematics allows the existence of contradictory sets of objects, separately but not simultaneously. One may have Euclidean geometry and non-Euclidean geometry in mathematics, on Hilbert's view, because each has a consistent set of axioms. In fact, mathematicians had proved that if one accepts one of these geometries one must accept the other (they are, in the jargon, relatively consistent). But on Frege's view (most clearly in unpublished material) there is only one world and so only one geometry… (p.103)

We can examine this problematic feature of Hilbert's program from another angle. There is the matter of whether two separately consistent systems can be combined together and the conjunction makes for a consistent whole. Brown discusses this problem at some length:

> Suppose we have two theories, a theory of heat, H, and a theory of light, L. A scientific realist takes evidence for H and evidence for L to be evidence that they are each *true.* Given the truth of H and L separately, belief in the truth of the *conjunction* H & L follows on naturally. But the instrumentalist cannot be so sanguine, since evidence is understood merely as evidence that a theory is a good instrument. Thus, evidence that H is a good instrument and evidence that L is a good instrument need not be evidence that H & L is also a good instrument. For

> even though H and L are individually consistent (a precondition of being a good instrument), their conjunction need not be. (1999, 67)

Transfinite equations (e.g. $\exists x\ (2<X)$) deal with a finite number and an open variable. Prima facie, this proposition is easy enough to understand. We can read the statement as "there is some number that is greater than 2." We don't know what that number is, but there could be an infinite quantity of numbers that could replace the variable X to yield a true real statement. We can take this sentence to be *true* if we accept both the finitary and the non-finitary aspects of the equation (which amounts to a conjunction of two theories, F & I—jointly consistent). But how can we *prove* the equation until we know what X stands for? Hilbert tried to identify *truth* with provability; that is to say that a given theorem is true if one can give a proof for it. For less complex operations such as $2<3$, one can see the truth of this statement intuitively without the construction of a proof.

However, Gödel's first and second incompleteness theorems showed how any attempt to systematize arithmetic is incomplete. Gödel's first theorem can be understood in the following way: "if *D* is an effective deductive system that contains a certain amount of arithmetic, there are sentences in the language of *D* which are not decided by the rules of *D* (Shapiro, 2000, 131)." The second incompleteness theorem "asserts that no consistent theory (that contains a certain amount of arithmetic) can prove its own consistency (Shapiro, 2000, 167)." Finitary mathematics may be consistent—however Gödel's theorems show that one cannot use the formal system and axioms of finitary mathematics itself to prove consistency. This is what Hilbert's program tried to prove—and Gödel showed that consistency could not in and of itself be the sufficient condition for the complete formalization of mathematics that Hilbert wanted. Gödel, incidentally, would assert that our knowledge of mathematical truth could be founded upon an intuition of a platonic realm of numbers—hence, denying the quasi-formalist contention that "consistency implies existence". It is not sufficient that a system be consistent—there must be some justification of consistency that relies on more than the resources of the system itself. So in light of this, Gödel's incompleteness theorem was thought to undermine Hilbert's method of integrating finitary mathematics with transfinite set theory. There could be no successful merging of finite mathematics with ideal elements because consistency was the justification for the merge. Once the incompleteness theorem showed that mathematics could not be formalized in this way, Hilbert's program was on very shaky ground. Regarding Godel's first incompleteness theorem, Rucker (1982) explains "This theorem establishes not only that (Hilbert's Program) is incomplete, but that there is *no* finitely given formalized theory that can correctly answer all questions about the addition and multiplication of natural numbers (p. 279)." Rucker also considers Hilbert's loose association with formalism by noting:

> He thought that it was possible to find a complete formal system for mathematics, and that mathematics could then be viewed as a finitary symbol game based on this complete system. And in this he was wrong. No finitely given system can exhaust the riches of the actual infinite. The practicing mathematician's direct intuitions of infinite sets cannot be dispensed with. (1982, 280)

Brown also notices something problematic about the seemingly flawless finitary aspect of Hilbert's program:

> Hilbert associates trustworthy reasoning with the finite. But, clearly our grip on a finite lessens as the entity become (sic) larger and more complex. We can multiply two small numbers together and be confident of the answer. But what about the product of two finite numbers each over a billion digits long. It certainly does not correspond to any object of perception in the Kantian sense. The chances of making a computational error are considerable. I, for one, would be much happier betting on the truth of a transfinite proposition….than on an enormously complex finite example. Certainty cannot be simply identified with

> the finite; at best it can be linked to the very small. But if we confine ourselves to this, we won't have anything close to the classical mathematics we want and need (1999, 70)

Brown's point is well taken. If Gödel's attack on Hilbert is successful, then we are reduced only to finitary mathematics—and for that matter perhaps only finitary mathematics in small quantities. This will suffice for basic arithmetic (which is the most cogent component of Kant's philosophy of mathematics, as we have seen in light of the rejection of his views on geometry), but not for much else.

If Hilbert's program was dealt a death blow, then what recourse is there? Finitism in mathematics may have utility for elementary operations, but if Gödel's theorems hold then we are prohibited from 1) using a system based on consistency and 2) merging the finitary and ideal elements of mathematics. This is not a desirable position to be in. On the one hand, we can stay true to the finite aspect of mathematics and only accept the "potentially infinite" as Brouwer—but we may be helplessly reduced to a very weak mathematics if we do. On the other hand, we can try to include the infinite in our mathematics—but to do so the way that Hilbert tried would be in vain. In the next section we will look at the work of Michael Dummett as it relates to Kantian thought.

4.3 Contemporary Views: Michael Dummett

So far, the intuitionists that have been discussed have been concerned with mathematics as it pertains to one's own mental activity. Intuitionist thought has been criticized for being at the very least "inappropriately psychological" and at worst someone could claim it is analogous to the ontological thesis of idealism, the position that all reality is composed merely of minds and ideas. With Michael Dummett's intuitionism, mathematical truth and logical truth are concerned primarily with meaning:

"Any justification for adopting one logic rather than another as the logic for mathematics must turn on questions of *meaning* (Dummett, 1983, 97)." Insofar as this is significant for Dummett, we may call his position *semantic* anti-realism. Dummett considers two questions that are relevant to the foundations of intuitionist thought:

> Can the thesis that natural numbers are creations of the human thought be taken as a premiss for the adoption of an intuitionistic logic for number-theoretic statements? And another question was: What content can be given to the thesis that natural numbers are creations of human thought that does not prejudge the question what is the correct notion of truth for number-theoretic statements in general? (1983, 128-9)

Dummett argues that not every proposition has a determinate truth-value of true or false. He illustrates this view with examples pertaining to counterfactuals and subjunctive (future) conditionals. He asks us to consider the statement "if Fidel Castro were to meet President Carter, he would either insult him or speak politely to him (1983, 126)." Dummett is quick to point out that the truth value of this conditional is not determinately true or false, for it depends on other future conditions—where the meeting took place, what mood Castro will be in, etc. Therefore, we can examine a proposition that expresses this statement in symbolic form ($A \rightarrow B \cup C$) and not assert that it possesses a truth-value. Dummett holds that subjunctive conditionals are problematic for the following reason:

> If we yield to this [classical] line of thought, then we must hold that every statement formed by applying a decidable predicate to a specific natural number already has a definite truth-value, true or false, although we may not know it (1983, 127).

Subjunctive conditionals are like mathematical propositions containing a number that does not have a proof constructed for it. Not every proposition has a determinate truth-value—propositions containing subjunctive conditionals might not have a determinate truth-value, and similarly mathematical propositions containing reference to a number whose existence hasn't been proved might not have a determinate truth-value. But if we adopt the intuitionistic framework that Dummett is proposing we do not have the same problem:

> One who rejects the idea that there is already a determinate outcome for the application, to any specific case, of an effective procedure is, however, in a completely different position. If someone holds that the only acceptable sense in which a mathematical statement, even one that is effectively decidable, can be said to be true is that in which this means that we presently possess an actual proof or demonstration of it, then a classical interpretation of unbounded quantification over the natural numbers is simply unavailable to him. As is frequently remarked, the classical or platonistic coneption (sic) is that such quantification represents an infinite conjunction or disjunction: the truth value of the quantified statement is determined as the infinite sum or product of the truth-values of the denumerably many instances. Whether nor (sic) not this be regarded as an acceptable means of determining the meaning of these operators, the explanation presupposes that all instances of the quantified statement themselves already possess determinate truth-values: if they do not, it is impossible to take the infinite sum or product of these. (1983, 128)

Since Dummett is of course rejecting the idea that subjunctive conditionals have determinate truth-values, he will reject the idea that we can have an infinite sum of such statements. We can in this respect see how Dummett's focus on meaning closely parallels the intuitionist method of number construction. We cannot say of a statement dealing with future conditionals that it is either true or false. If we assume intuitionist logic and examine the proposition "If Fidel Castro were to meet President Carter, he would either insult him or speak politely to him," we cannot say of the statement that it is either true or false; it has an indeterminate truth value because no event in time can verify the claim. Analogously, we cannot say anything about a number unless a proof for the number has been constructed, as Dummett concludes:

> From what we have said about the intuitionistic notion of truth for mathematical statements, it has now become apparent that there is one way in which the thesis that natural numbers are creations of the human mind might be taken, namely as relating precisely to the appropriate notion of truth for decideable statements of

> arithmetic, which would provide a ground for rejecting a platonistic interpetation of number-theoretic statements generally, without appeal to any general thesis concerning the notion of meaning. This way of taking the thesis would amount to holding that there is no notion of truth applicable even to numerical equations save that in which a statement is true when we have actually performed a computation (or effected a proof) which justifies that statement. Such a claim must rest, as we have seen, on the most resolute skepticism concerning subjunctive conditionals: it must deny that there exists any proposition which is now true about what the result of a computation which has not yet been performed would be if it were to be performed. (1983,129)

All three of these thinkers are indebted to Kant for the legacy that he left behind. But as we have seen throughout the works discussed, intuitionism is not without its problems and criticisms. In the final chapter, I will briefly discuss two issues: what intuitionism resolves contrasted with what it leaves to be desired.

CHAPTER V

CONCLUSION

A Kantian philosophy of mathematics has its advantages and disadvantages. The thesis that numbers are constructed has many benefits—we do not have to appeal to a non-physical, ultimately inaccessible realm for our mathematical objects. Instead, we construct them;they are the products of our intellect. Moreover, construction of numbers offers the possibility of a bona fide solution to Zeno's paradoxes of motion. If we do not commit ourselves to the notion of the actually infinite, we can sensibly talk about fundamentals of space and motion—a result of Kantian intuitionism (although due in part to Aristotle's attempt to solve the paradoxes from a finitist perspective). Numbers are not mysterious entities that hold sway over the universe in a platonic realm; instead they are products of our perceptual apparatuses. Kantian intuitionism brings mathematics to a more "manageable" level—numbers are something immediately accessible to us, and therefore the mystery of how they apply to our experience is solved. It seems reasonable to suggest that, in this sense, constructed numbers have an advantage over numbers that are "not of this world," so to speak. Intuitionism also has an advantage over formalism insofar as the intuitionists have mathematical objects to speak of; the formalist's mathematical objects are symbols, and it becomes difficult to square this with the fact that these fictitious objects have application in the world.

But there are fundamental flaws to intuitionist thinking. Kantian intuitionism has been criticized as "psychologistic," i.e. "inappropriately psychological." If numbers are artifacts of the human mind, then one should categorize the intuitionist as an anti-realist in ontology. Even though a number may be "real" once it is constructed, it probably not appropriate to give it the same ontological status as a physical object (or even a set, or an object in a platonic realm). Moreover, we do not have actual infinity with this view. If intuitionist mathematics is our only tool, then we are limited to what we can do with mathematical propositions in comparison with any system that endorses classical logic.

Even if Kantian intuitionism suffers from serious criticisms, there is one final remark that should be made on its behalf: it questions the presuppositions of our most fundamental and pervasive of disciplines—mathematics and logic. Kant's influence enabled us to offer possible solutions to puzzles concerning the infinite, not to mention offering a insight into the nature of mathematical entities and mathematical truth and Even if disagreement remains, Kant's philosophy of mathematics offers another perspective in the search for truth in ontology and epistemology.

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