THE STUDY OF MIDDLE SCHOOL TEACHERS' UNDERSTANDING AND USE OF MATHEMATICAL REPRESENTATION IN RELATION TO TEACHERS' ZONE OF PROXIMAL DEVELOPMENT IN TEACHING FRACTIONS AND ALGEBRAIC FUNCTIONS

A Dissertation

by

ZHONGHE WU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2004

Major Subject: Curriculum and Instruction

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Approved as to style and content by:

Gerald O. Kulm (Chair of Committee)	Robert M. Capraro (Member)
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ABSTRACT

The Study of Middle School Teachers' Understanding and Use of Mathematical Representation in Relation to Teachers' Zone of Proximal Development in Teaching Fractions and Algebraic Functions. (August 2004)

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Chair of Advisory Committee: Dr. Gerald Kulm

This study examined teachers' learning and understanding of mathematical representation through the Middle School Mathematics Project (MSMP) professional development, investigated teachers' use of mathematics representations in teaching fractions and algebraic functions, and addressed patterns of teachers' changes in learning and using representation corresponding to *Teachers' Zone of Proximal Development* (TZPD).

Using a qualitative research design, data were collected over a 2-year period, from eleven participating 6th and 7th grade mathematics teachers from four school districts in Texas in a research-designed professional development workshop that focused on helping teachers understand and use of mathematical representations. Teachers were given two questionnaires and had lessons videotaped before and after the workshop, a survey before the workshop, and learning and discussion videotapes during the workshop. In addition, ten teachers were interviewed to find out the patterns of their changes in learning and using mathematics representations.

The results show that all teachers have levels of TZPD which can move to a higher level with the help of capable others. Teachers' knowledge growth is measurable and follows a sequential order of TZPD. Teachers will make transitions once they grasp the specific content and strategies in mathematics representation. The patterns of teacher change depend on their learning and use of mathematics representations and their beliefs about them.

This study advocates teachers using mathematics representations as a tool in making connections between concrete and abstract understanding. Teachers should understand and be able to develop multiple representations to facilitate students' conceptual understanding without relying on any one particular representation. They must focus on the conceptual developmental transformation from one representation to another. They should also understand their students' appropriate development levels in mathematical representations.

The findings suggest that TZPD can be used as an approach in professional development to design programs for effecting teacher changes. Professional developers should provide teachers with opportunities to interact with peers and reflect on their teaching. More importantly, teachers' differences in beliefs and backgrounds must be considered when designing professional development. In addition, professional development should focus on roles and strategies of representations, with ongoing and sustained support for teachers as they integrate representation strategies into their daily teaching.

DEDICATION

In memory of my parents Peilan Wu and Cuizhen Xu

For their loving support and sacrifice they made

To my wife Shuhua An and my son Andrew Wu

For their patience and understanding

To my brothers and sisters

For their loyalty and encouragement

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CHAPTER I

INTRODUCTION

Background

Mathematics is a study of science, and a relationship between numbers and the spatial, focusing on reasoning and logical thinking. Whitehead (1929) defines mathematics as a "science of order." This definition identifies the features of mathematics as generalization, simplification, representation, communication, and application, and determines its wide applications in technology, economics, social studies and other fields. For many centuries, mathematics has been used everywhere in the world as a universal language.

The natural features of mathematics challenge teachers to view mathematics differently and require teaching and learning mathematics with imagination and creativity. As Green (1995) describes, "To learn and to teach, one must have an awareness of leaving something behind while reaching toward something new and this kind of awareness must be linked to imagination" (p. 14). Green (1995) believes that the imagination is the gateway through which understanding and meanings derived from past experiences connect present learning. In modern society, as an important subject for K-12 schools worldwide, mathematics teaching and learning need to develop a gateway that opens the door for understanding new learning. To achieve success in school mathematics education and reach the goal of mathematics teaching and learning,

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This dissertation follows the style and format of the *Journal for Research in Mathematics Education.*

teachers need to play a key role in implementing mathematics curriculum in their classroom teaching.

Throughout the history of mathematics education, teachers and teaching have been found to be the major factors relating to students' mathematical achievement (Mullis et al., 2000; Stigler, & Hiebert, 1999). To address the important role of teaching, Stigler and Hiebert (1999) state, “Teaching is the next frontier in the continuing struggle to improve schools” (p. 2). Since mathematics teachers’ knowledge is a base for effective teaching, teachers should acquire sufficient knowledge to enhance their abilities and improve their quality of teaching. According to the National Council of Teachers of Mathematics (NCTM, 2000), “Effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies” (p. 17). To develop the knowledge of effective teaching, teachers need to not only have opportunities to participate in professional development, but more importantly attend high quality professional development, with focuses on the growth and improvement of teachers’ knowledge.

In current mathematics teacher professional development programs, a question has arisen about how effectively professional development meets mathematics teachers’ and students’ needs. Many studies find that traditional professional development has little influence on improving classroom teaching and learning (Carpenter, & Fennema, 1991; National Educational Research Policies and Priorities Board, 1999). The problems framed and methods preferred in many professional development programs have produced knowledge represented in forms that make it difficult for teachers to use

(Hiebert, Gallimore, & Stigler, 2002) and have resulted in "the tension between formal research and practical inquiry" (Stocks, & Schofield, 1997, p. 285). These problems call for substantive changes in the delivery of professional development. Even with changes and extensive training provided for teachers, currently professional development is still not productive in terms of effective teaching (Stocks, & Schofield, 1997) and students' mathematical proficiency development. This critical issue on professional development challenges mathematics educators to seek a new and effective way to provide teachers with specific, structural, and practical knowledge that could enhance their knowledge in a wide scope and multiple dimensions, enable them to make a transformation in content, pedagogical, and pedagogical content knowledge, and help them arrive at higher levels of *Teachers' Zone of Proximal Development* (TZPD) in teaching, based on and extended from the theory of *Zone of Proximal Development* (ZPD) in student learning (Vygotsky, 1978). Teachers' knowledge growth under effective professional development, in turn, helps them implement a standards-based curriculum that is based on the National Council of Teachers of Mathematics *Principles and Standards for School Mathematics* (NCTM, 2000).

Brief History of Standards-Based Mathematics Curriculum Development

The history of mathematics education reflects the teaching and learning of mathematics in the social and economic movements. In the last 50 years, mathematics education reform in the United States has experienced several trends. The new mathematics reform movement in the 1950s, aimed at cultivating highly technological human resources, had two main views in the understanding of mathematics: "precision of language, and discovery of generalization" (Osborne, & Crosswhite, 1970, p. 254). In this new mathematics movement, logical structures and abstract symbols were almost exclusively used in the classrooms to convey new concepts at all levels. The curricula developers did not consider the needs and characters of children's learning. Consequently, the new mathematics movement failed, for it focused on highly structured and abstract principles and rules, which ignored learners' needs. Most importantly, the new mathematics movement failed because of its lack of adequate training for teachers. The results of the new mathematics movement resulted in a movement to "go back to the basics" (Troutman, & Lichtenberg, 2003).

In 1957, the Soviet Union's launching of the "Sputnik" satellite, which challenged the U.S. to improve the quality of science education, including mathematics education, provoked an increase in public attention on education problems. "The recognition of grave deficiencies in the school mathematics program highlighted dramatically by the spectacular performance of Sputnik, sparked a new era of thought and action concerning the instructional programs in school mathematics" (Gibb, Karnes, & Wren, 1970, p. 327). Although several curricula were established as a result of this pressure, the most important movement was the establishment of the School Mathematics Study Group (SMSG) in 1958. Because the SMSG set a good example, showing that mathematicians and educators can work together holding the same view of mathematics curriculum development, it successfully developed the secondary mathematics curriculum. With teachers' involvement, the SMSG provided detailed materials for students and teachers, and teachers' training became an essential consideration in curriculum development. Considering teachers' needs and providing guidelines for classroom practices were the key factors for the SMSG's success.

During the 1970s, as technology developed and classrooms acquired a more diverse student population, mathematics education faced new challenges for reform. This reform was different from previous ones: it applied cognitive learning theory to restructure instructional practice, which called for more attention on individual students' needs and on individualized instruction (Troutman, & Lichtenberg, 2003). After the 1970s, a series of documents released from professional organizations initiated a prelude to standards-based curriculum reform. The "*Agenda for Action,*" published by the National Council of Teachers of Mathematics (NCTM, 1980) called for new mathematics reform in eight areas, including problem solving and technology usage. "*Everybody Counts"* (MSEB, 1989) provided new ideas on mathematics education. These documents enlightened many researchers and educators' views about teaching and learning mathematics: quality mathematics education was beneficial to the nation's future. Three documents on improving quality mathematics education, based on constructivism, in the 1980s and 1990s, entitled *"Curriculum and Evaluation Standards for School Mathematics"* (NCTM, 1989), "*Professional Standards for Teaching Mathematics*" (NCTM, 1991), and "*Assessment Standards for School Mathematics*" (NCTM, 1995), launched by NCTM, further addressed reform in mathematics education in detail and provided a fresh view for mathematics education reform in the United States. One of a number of remarkable documents, *Benchmarks for Science Literacy* (AAAS Project 2061, 1993), launched by Project 2061 of the American Association for the Advancement of Science, provided benchmarks for what all students should know or be able to do in science, mathematics, and technology by the end of grades 2, 5, 8, and 12. This document built a strong base for students to "become literate in science, mathematics, and technology by graduation from high school" (AAAS, 1993, p. vii). In 2000, the NCTM released the *Principles and Standards for School Mathematics* (PSSM), which acted as a milestone in the history of mathematics education. The features of PSSM (NCTM, 2000) are: (1) it is based on the previous three NCTM Standards; (2) it is research based, and constructivism and social constructivism are the backbone of PSSM; (3) it has six principles as its base, aimed at high quality in mathematics education; (4) it separates the different grade levels into K-2, 3-5, 6-8, and 9-12, stating the important mathematics ideas for the different grade levels; (5) not only does it have five content standards, but it also has five process standards to guide teachers using different approaches in classroom teaching; and (6) it adds an electronic version, in which teachers are able to view mathematics with technology and to visualize mathematics concepts to produce the best ideas for effective teaching. For the middle school level, it not only provides detailed mathematics content areas, but it also addresses a variety of ideas on how to make a transition between arithmetical and algebraic thinking for middle school students in order to achieve success in learning mathematics.

The Content of PSSM and Mathematical Representation

The content of PSSM is divided into two parts: principles and standards. PSSM uses six fundamental principles as its guidelines for school mathematics: equity, curriculum, teaching, learning, assessment, and technology. The equity principle states that the mathematics curriculum should be available to every student and should include high expectations for all students. The curriculum principle advises that “mathematics should be coherent, focus on important mathematics, and should be well articulated across the grades” (NCTM, 2000, p. 15). The learning principle dictates that the mathematics curriculum should have let students understand and be able to apply the mathematics they have learned. The assessment principle addresses that assessment needs to support students’ learning and teachers’ teaching. The technology principle advocates the use of technology for learning and teaching mathematics. Among these principles, the teaching principle shows that mathematics teaching relies on the quality of teachers to understand students and mathematics, including four essential components: (1) mathematics teachers should carefully analyze and consider students’ learning characteristics; (2) mathematics teachers should prepare mathematics classes coherently in lessons, units and subjects, and have content and pedagogical, as well as knowledge of students’ thinking; (3) mathematics teachers should create a classroom environment in which students have the confidence necessary for learning mathematics; and (4) mathematics teachers should use “discourse” to let students participate in mathematics learning (NCTM, 2000). In addition, PSSM illustrates the essential concepts and procedures of learning mathematics and clearly states that learning mathematics is dependent on personality and society characteristics. Students’ differences in learning mathematics require mathematics teachers to understand their different learning backgrounds and learning styles and to design various approaches to meet students' needs and direct them in learning mathematics. Students' understanding the knowledge of mathematics and realizing its value is the key to learning mathematics successfully. It is essential that a teacher helps students build valuable mathematics attitudes and beliefs to help them not only in learning mathematics but also in their future lives.

PSSM also illustrates five content standards and five process standards. The content standards are: number and operations, algebra, geometry, measurement, and data analysis and probability; the process standards are: problem solving, reasoning and proof, communication, connection, and representation. These standards are not separated, but are connected, interwoven, and communicate with each other since mathematics is developed from and connected by problem solving, reasoning, representation, communication, and connection. Most importantly, PSSM advocates mathematics representation, as an important mathematics process standard, connects this standard with others in a dynamic system of processes for K-12 mathematics education.

Researchers have found that mathematical representation is a tool that can help students overcome difficulties in developing and understanding mathematical concepts (Goldin, 2003; Monk, 2003; Smith, 2003). Ability in students' use of representations in mathematics learning reveals the level of internalized understanding and helps teachers in finding their students' learning *Zone of Proximal Development* (ZPD) because "how learners represent and connect pieces of knowledge is a key factor in whether they will understand it deeply and can use it in problem solving" (National Research Council, 2001, p. 117).

The Theoretical Model for the Study

For decades, various studies have focused on many aspects of mathematics classroom teaching, but generally few cases have been concerned with how specific content and pedagogical knowledge have merged in mathematics representations (Demana, & Leitzel, 1988; Goldin, 2003, Greeno, & Hall, 1997; Smith, 2003). What is needed is a research study that blends the concern for the realities of classroom instruction and teachers' knowledge such as mathematics representations of teaching with the concern for individual students' representations, using the rich analysis of the structure of mathematics representations. These concerns called for a research-based investigation that examines teachers' active and thoughtful engagement in learning mathematics representations through professional development, and how professional development helps teachers gain knowledge from both mathematics representations and instructional strategies, which could facilitate the transformation of teachers' knowledge into powerful and meaningful forms of representations that make sense to their students and enhance students' learning.

Researchers have developed different teaching models based on effective teaching (Carpenter, & Fennema, 1991; Simon, 1997). According to Carpenter and Fennema (1991), students' learning depends on the teacher's decision making, and the teacher's decision relies on the teacher's knowledge and beliefs, which, in turn, is based on students' cognitions. Therefore, they believe that effective teaching is based on students' behaviors. However, the scope of teachers' knowledge and beliefs are too broad and very complex, and much detail is needed to address effective teaching. In the development of the Mathematics Teaching Cycle (MTC), Simon (1997) emphasizes the interaction between teachers and students in classroom teaching and learning. According to Simon (1997), teachers not only need to learn the mathematical aspects of teaching, but also how to set goals to direct students' mathematical thinking and discussion. In order to achieve these goals, teachers need to organize their classes using good communication with students. The teacher, as a constructivist, should be able to communicate with students using mathematical teaching strategies in which teachers' knowledge of mathematics activities contributes to the learning trajectory, and the interaction between teacher and students develops students' mathematics learning and promotes the learning of new concepts (Simon, 1997). Therefore, he believes that identifying and developing the content of activities is an important task for the effectiveness of professional development.

In the sense of mathematics representation not only as it directly relates to teachers' content and pedagogical knowledge, but also as it builds a close connection between effective teaching and student learning for particular content areas, this study develops an effective teaching model, based on the importance of mathematics representations in teaching and learning mathematics. Since the core objective of teaching is to enhance student learning with understanding, this new model places mathematics representation at the center of a network of effective teaching (see Figure 1). Therefore, this study focuses on the components of professional development, knowledge of representation, and effective teaching via mathematics representations in particular content in relation to the *Teachers' Zone of Proximal Development* (TZPD) in the model.

Figure 1 shows that teachers' knowledge (content, pedagogical, and pedagogical content knowledge) and teachers' beliefs are the main knowledge bases for effective teaching. However, as teachers learn new knowledge and gain more teaching experience, the interaction between these two will change the teachers' beliefs. Yet all of their knowledge and beliefs may be unorganized, and their content and pedagogical knowledge could be unconnected (An, Kulm, & Wu, 2004). Therefore, professional development plays a fundamental role in connecting teachers' knowledge to their beliefs, and importantly, in building and structuring their knowledge using specific mathematics content and strategies -- Knowledge of representation is defined as strategies of using representations to illustrate and convey mathematics ideas.

The components of mathematical representations in Figure 1 generally involve communication, mathematics reasoning, connection, and problem solving. They are interwoven with representation and modified representation in a transferable, interpretable, visual, sense making, communicative, and deliverable form to students. Teachers' mathematics representation knowledge also comes from teachers' and students' interaction which Simon (1997) referred to as "inquiry into students' mathematics, facilitation of discourses, problem posing, and interactive constitution of classroom practices" (p. 79). Students' learning occurs as a result of interaction between teachers and students. The model, therefore, matches the theory of *Zone of Proximal Development* (ZPD).

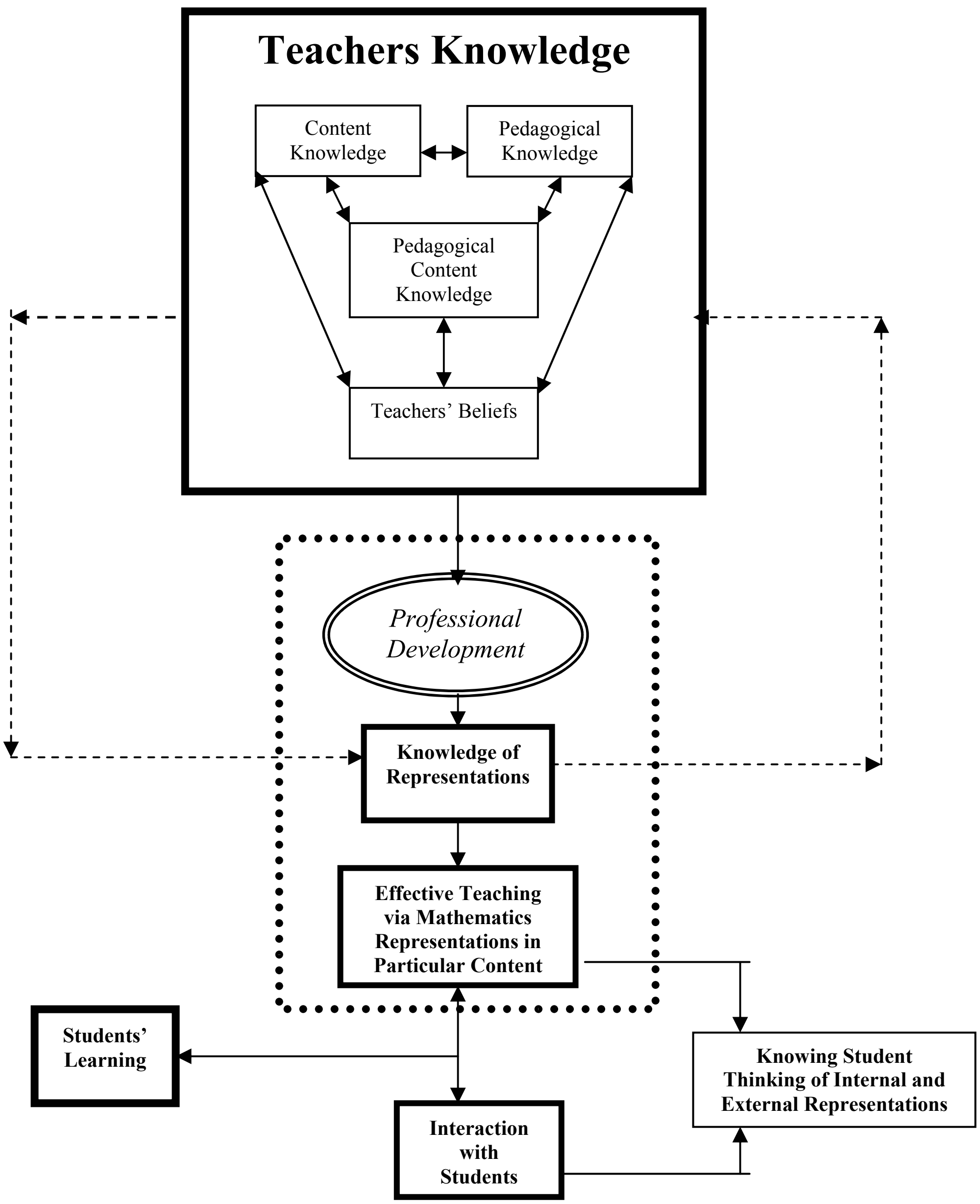


Figure 1. Effective teaching via the mathematics representation model.

Representation in the Zone of Proximal Development (ZPD) for Learning

In mathematics education, two questions are often asked: In what ways do students understand mathematics best? In what ways do teachers teach effectively? To address these questions, curriculum and instruction in mathematics education have applied the *Zone of Proximal Development* (ZPD) theory (Vygotsky, 1978) in many areas (Albert, 2000; Steele, 1999). Vygotsky (1978) described ZPD as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined by problem solving under adult guidance or in collaboration with more capable peers" (p. 86). According to his view, the ZPD designates a range of tasks that learners cannot yet perform independently, but can perform with the help of others. By applying the theory of ZPD, professional development programs as capable others could foster teachers' skills and capacities, which would gradually become internalized as learning proceeds from originally independent to collaborative learning situations.

Using representations in mathematics teaching is one of the examples of ZPD applications. First, knowledge of representation is very important for effective teaching and learning. According to Shulman (1986), "What is also needed is knowledge of the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations -- in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). In mathematics, representation is not a single dimensional trait, rather it is related to everything in teaching and learning mathematics, i.e., a good representation makes meaningful sense for mathematics understanding (Goldin, 2003). Therefore, mathematics representations, such as concrete models, diagrams, graphs, charts, and symbolic expressions, contribute to effective mathematics teaching and learning.

Second, there is a close relationship between representations and the development of ZPD. According to Cuoco (2001), there are two kinds of representations in the learning process: external and internal. External representation refers to the representations that people can easily communicate to others using physical marks; internal representation refers to the images that people create in their minds for mathematics objects and processes (Cuoco, 2001). Learners' internal representations could be developed through the use of external representations; therefore, internal representation is a higher level than external representation. "When a child [learner] is ready to build an internal representation of a concept or relationship, the child [learner] is said to be in the ZPD" (Troutman, & Lichtenberg, 2003, p.16). Like internal and external representation, people's intrapsychological functioning of ZPD is a higher mental function developed through interpsychological functioning of ZPD because "interpsychological functioning was social at some point before becoming an intrapsychological, truly mental function" (Vygotsky, 1981, p. 162). The levels of mastering representations determine the development of ZPD. In mathematics learning, "when students gain access to mathematical representations and ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically" (NCTM, 2000, p. 67) and to internalize their understanding. The power of mathematics representations will not only place children in their ZPDs but also help them develop to their fullest potential and, in turn, increase their level of ZPDs. Therefore, the term "representation" refers "to the act of capturing a mathematical concept or relationship in some forms" (NCTM, 2000, p. 67), such as a diagram, graphical display, or symbolic expression (Goldin, 2003).

According to Vygotsky (1962), a mediator (e.g., teacher) is an important ingredient in the learning process. In classroom teaching practice, mathematical representation strategies help teachers change their role according to students' ZPDs reflected in mathematical representations. By working with students, the teacher helps them use various representations to internalize mathematics concepts and relationships, which not only helps students overcome their difficulties in learning but also enables students to reach a potential beyond what they could achieve independently. The important part of this process is that teachers should encourage students to use different representations according to their levels of ZPD to solve problems or to portray, clarify, or extend mathematical ideas. In order to achieve this goal, teachers must have profound and structured content, pedagogical, and pedagogical content knowledge (PCK), i.e., they must have their own higher levels of TZPD in the knowledge of representation.

Model of the Zone of Proximal Development (ZPD) for Teaching

Although the theory of the *Zone of Proximal Development* (ZPD) was originally developed with children's learning, Vygotsky (1978) acknowledged that a learning discrepancy exists between solitary and social problem solving as he developed his notion of ZPD (Forman, & Cazden, 1985). Vygotsky (1978) defined ZPD as the distance between the actual developmental level of independent learning and the potential development level of learning through collaboration with more capable peers. According to Vygotsky (1978), learning consists of the internalization of the social interaction process, and learning development proceeds when interpsychological regulation is transformed into intrapsychological regulation (Forman, & Cazden, 1985).

Applying Vygotsky's ZPD principle for students, this study designed *Teachers' Zone of Proximal Development* (TZPD). There are three stages of TZPD corresponding to three different levels: when teachers' knowledge develops under individual learning without interpsychological function, their knowledge development is limited. In this stage, teachers' knowledge is limited in mathematical content, and sometimes even with inadequate content knowledge. When teachers' knowledge is developed together with and guided by capable others, their knowledge is in the interpsychological process, in which their content knowledge and pedagogical knowledge are developed. When teachers continue to develop their knowledge together with and guided by capable others and reflect their new learning in teaching practices, they gradually reach their potential in intrapsychological function, and their pedagogical content knowledge is developing (see Table 1).

Table 1
The Relationship between TZPD and Development of Teachers' Knowledge

Level	TZPD	Teachers' knowledge
1	Zone 1: Learning without interpsychological function.	a) Teachers' knowledge development is limited in content areas. b) Their content knowledge may be inadequate.
2	Zone 2: Learning with interpsychological function.	a) Teachers' knowledge is developed when acquired with and guided by capable others. b) Both teachers' content and pedagogical knowledge are developed.
3	Zone 3: Learning in intrapsychological function.	a) Teachers' knowledge is continually developed when acquired with and guided by capable others. b) Teachers reflect and apply new learning in teaching and gradually reach their potential. c) Their pedagogical content knowledge is developing.

To examine the effectiveness of this learning process of TZPD, this study applied the theory of ZPD to teachers' learning processes in mathematics representations. When teachers learn representations through individual learning, they engage in their personal learning in a limited scope of zone, in which their knowledge of representation is limited, unstructured, and unconnected, that is not sufficient to teach mathematics effectively. Through structured professional development, teachers interact with peers and are guided by capable others, and their learning is transformed to interpsychological regulation, in which teachers can provide clear representations, though not comprehensible with limited variety and connection; by continuing to collaborate with more capable others, teachers have opportunities to reflect and internalize their knowledge of representation in the intrapsychological function in which teachers are able to provide various representations accurately and comprehensibly to students. The process of transition from interpsychological to intrapsychological function leads teachers to make progress to a higher and broader zone of ZPD, which is called TZPD in this study, defined as teacher learning occurs in collaboration with more capable others (see Figure 2).

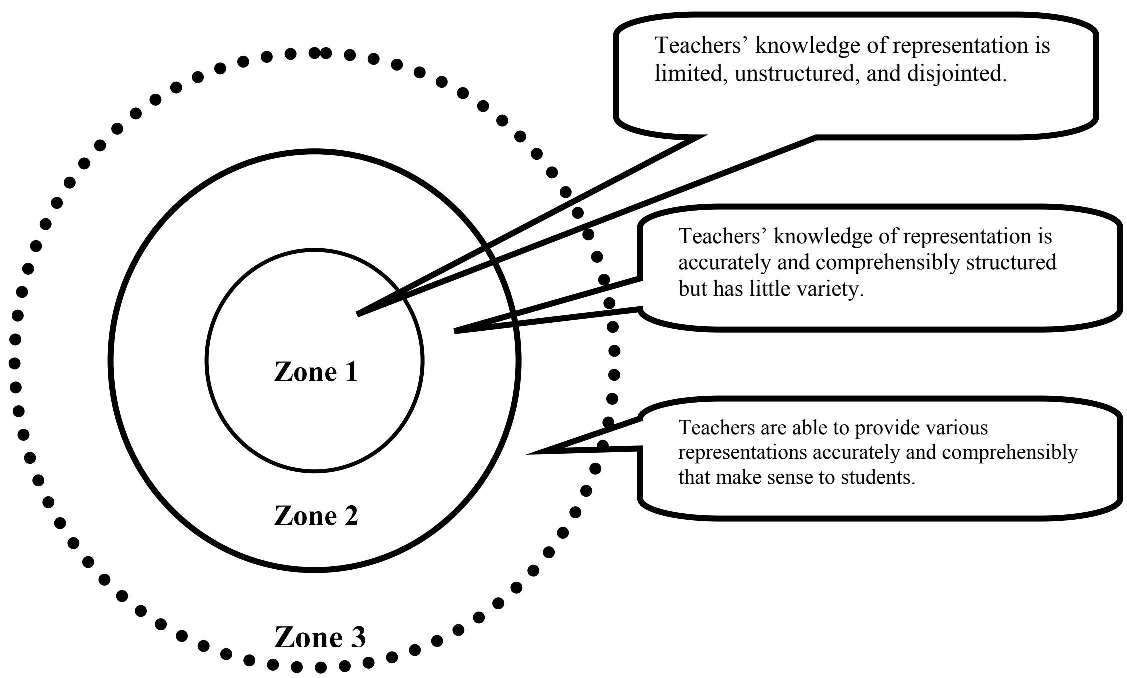


Figure 2. Model of TZPD.

In this study, teachers' knowledge of representations is categorized into three levels corresponding to the three zones of TZPD. Zone 1 is considered lower level; Zone 2 is at the middle level; and Zone 3 is at a high level. At the lower level of TZPD, teachers' knowledge of representation is unstructured and disjointed with limited accuracy; at the middle level of TZPD, teachers' knowledge is accurately and comprehensibly structured but has little variety; at the high level of TZPD, teachers are able to provide various representations accurately and comprehensibly according to students' learning needs that make sense to students. The different zones from level one to three are not separate zones, and are movable in a spiral movement from lower to higher levels. The higher zones are always built upon lower zones and are broader than the previous one. To make progress from one level to another in TZPD, an effective professional development model focusing on specific mathematical content in connection with representation plays an important role. During this effective professional development, teachers have opportunities to learn various strategies for developing representations from collaboration with more capable others and to restructure their knowledge and engage in interpsychological regulation. To fully master the skills of use of representation in classroom teaching, teachers will apply the knowledge of representations in their teaching. By interacting with students, teachers verify and solidify their knowledge of representation and reflect on their learning in their teaching practice, thus helping them build intrapsychological regulations, in which they reorganize and refine their knowledge into the connected network of representation, and they internalize their knowledge of representation and transform it into a higher level of TZPD. The levels of teachers' knowledge of mathematics representation not only reflect their mathematics understanding and ability in mathematics teaching but also indicate their zones of TZPD. Teachers' high level of TZPD in mathematics representations will in turn help them make instructional decisions for effective teaching.

Purpose of the Study

In mathematics education, representation can be defined as a configuration of signs, characters, icons, or objects that represent mathematics ideas (Cuoco, 2001; Goldin, 2003). In teaching mathematics, the aim of teachers' roles is helping students build mathematical conceptual understanding from idiosyncratic presentation, to meaningful mathematical representation, which can be seen in the role of the inductive representation bridge (Capraro, 2004; Smith, 2003).

Given the potential importance of mathematics representations, it is fundamental that professional development, aiming at specific, structured, and connected content and pedagogical knowledge, help teachers expand their knowledge and improve their teaching. Although some studies have called for pedagogical content knowledge, which is the connection between content and pedagogical knowledge, this connection mainly focuses on the knowledge of students' cognition (An, 2000; Fennema, & Franke, 1992; Shulman, 1986). With knowledge of students' mathematical thinking, how teachers convey their knowledge into powerful forms that could be acceptable and understood by students is still unclear. The process of conveying teachers' knowledge is the process of transformation that could be achieved by knowledge of representation. Given the potential importance of mathematics representations, it is also fundamental to understand how teachers understand mathematics representations and how they implement them in effective teaching; especially, it is vital to examine how teachers develop their knowledge of representations in *Teachers' Zone of Proximal Development* (TZPD) in teaching from professional development. The example of teacher professional development model designed by Middle School Mathematics Project (MSMP) focuses on teachers' content and pedagogical knowledge change and on taking action for classroom teaching. The features of the MSMP professional development are (1) analyzing current teaching content and difficulties (e.g., fractions and algebraic functions); (2) stating changes in teaching for this contents; (3) focusing on mathematics representation in teaching; (4) making plans for implementing standards-based teaching; and (5) assessing the impact of the new way of teaching.

Since there is no study focused on examining TZPD for K-12 teachers from professional development, it is imperative to investigate how to develop TZPD in mathematics representations through professional development. Without professional development aiming explicitly at helping teachers build higher levels of TZPD, understanding the transformation of specific content and processes such as mathematics representations in fractions and algebraic functions, knowing how to convey mathematics concepts in the form of representations, using them well and properly, and being able to assess them using various approaches, it is unlikely that teachers will be able to teach mathematics effectively and fully implement a standards-based curriculum, the curriculum advocated by NCTM (2000). Furthermore, the absence of current research on TZPD, especially on the transition in the use of representations, results in the lack evidence for how teachers develop TZPD and how they use mathematics representations in classroom teaching, and in what respect the representations contribute to the improvement of students' learning. Therefore, the ultimate goal of this study is to discover how and to what extent teachers' knowledge of representation fit into their TZPD sufficiently in order to help them improve their teaching. Specifically, the purpose of this study is a) to examine teachers' change in understanding of mathematics representations through the MSMP professional development workshop, b) to investigate how teachers change their use different types of mathematics representations to teach fractions and algebraic functions through the MSMP professional development workshop, and c) to assess patterns of teachers' changes after the MSMP professional development workshop in using mathematics representations in classroom teaching.

Delimitation of the Study

There are two delimitations in this study: (1) although videotape has been viewed as reliable sources for collecting data, it may be difficult to view the whole class context; therefore, the researcher's judgment may have some degree of bias; (2) participants were asked to voluntarily participate in Middle School Mathematics Project (MSMP); therefore, the data in this study represents only the results obtained from those who participated this study and results.

Definition of Terms

Terms used in this study are defined as follows:

Content knowledge is the knowledge of specific content areas in mathematics.

Pedagogical knowledge is the knowledge of teaching techniques, instructional materials, and classroom management and organization management (NCTM, 2000).

Pedagogical content knowledge is the knowledge of effective teaching, including knowing student thinking, preparing instruction, and mastering instruction delivery (An, 2004).

Teachers' beliefs are teachers' view about mathematics teaching and learning.

Mathematics representation is a configuration of signs, characters, icons, or objects that represent mathematics ideas (Cuoco, 2001; Goldin, 2003).

Knowledge of representation is the knowledge about how mathematical ideas can be represented to teach students effectively (NCTM, 2000).

Interpsychological function is a social interaction involved in learning development.

Intrapsychological function is introspection and reflection in the personal learning process.

ZPD (Zone of Proximal Development) is the distance between the actual development level as determined by independent problem solving and the level of potential development as determined by problem solving under adult guidance or in collaboration with more capable others (Vygotsky, 1978).

TZPD (Teachers' Zone of Proximal Development is teacher learning occurs by collaborating with peers and through guidance by others with more experience.

Research Questions

This study poses the following questions: (1) What do teachers understand about mathematical representations through the MSMP professional development workshop? How does this understanding of representation fit into TZPDs? (2) How do teachers use their new knowledge of mathematics representations gained from the MSMP professional development workshop in classroom teaching? How does this practice relate to their levels of TZPDs? (3) In what ways do teachers make changes in their actual classroom teaching in terms of TZPD levels in using mathematics representation after the MSMP professional development workshop? Are there any patterns of changes in classroom teaching?

CHAPTER II

REVIEW OF LITERATURE

Introduction

Current mathematics reform calls for changes in mathematics teaching and learning (NCTM, 2000). These changes are transitions involving fundamental shifts in reconceptualizing both mathematical teaching and learning (Cooney, & Shealy, 1997). The notion of teacher change has many dimensions. Although there have been numerous studies on mathematics teacher change (e.g., Nelson, 1997), there has been little research on the changes in teachers' knowledge of mathematical representation from professional development.

In order to successfully implement standards-based curricula and teach mathematics effectively, teachers need to develop their knowledge of representation, which helps teachers "thoroughly overhaul their thinking about what it means to know and understand mathematics, the kinds of tasks in which their students should be engaged, and, finally, their own role in the classroom" (Smith, 2001, p. 4). In addition the knowledge of representation "helps teachers make curricular judgments, respond to students' questions, and look ahead to where concepts are leading and plan accordingly" (NCTM, 2000, p.17). However, current teacher professional development rarely focuses on teachers' knowledge growth and changes in representations in the specific mathematical content areas that they are teaching. There are many ways of reforming professional development in the area of teachers' knowledge, teachers' collaboration etc.; however, it is impossible for successful professional development to focus effectively on an excessive number of goals. The Middle School Mathematics Project (MSMP) focuses on the goal that merges teachers' mathematics content and pedagogical knowledge with mathematical pedagogical content knowledge to help teachers gain mathematical knowledge of representation in classroom teaching (MSMP, 2001). This focus on teacher changes, in particular mathematical content for the professional development, is essential and especially important for middle school teachers because "teachers, particularly at the elementary and middle school levels, often have limited knowledge of the mathematical ideas that are central to the curriculum they are teaching" (Smith, 2001, p. 42). Therefore, this study examined the changes in teachers' knowledge of representations through the MSMP professional development workshop in their teaching practice.

Teachers' Professional Development

Professional development in mathematics education has long been a debated topic. The main argument is how and to what extent professional development helps teachers teach mathematics effectively. Researchers have developed many professional development frameworks (Jones et al., 1994), principles (Clark, 1994), and models (Wallace, Cederberg, & Allen, 1994); however, many of these frameworks, principles, and models are difficult to match to the needs of teaching practices. Weissglass's (1994) citation of Maria Montessori's words makes clear this difficulty, "nothing is more difficult for a teacher than to give up her [or his] old habits and prejudices" (p. 67). Weissglass (1994) further indicates that culture has profound implications for school education and teacher change, and obtaining emotional support is an immeasurable factor that often varies and is therefore undependable. Most professional development programs "collectively do not form a cohesive and cumulative program" and "much of the time and money invested in such programs, however, is not used effectively" (National Research Council, 2001, p. 431). In order to effect teacher change, professional development should find a way to reduce the resistance to change and motivate teachers to learn and use multiple representations to reach all students.

Teachers' change and knowledge growth are based on the proper design and the understanding of function of professional development, because teachers' knowledge gain from professional development comes from content knowledge change, instructional strategic change, and change in viewpoint on support materials of mathematical teaching. Teachers' change is based on, according to MSMP (2001), the growth of teachers' mathematics knowledge and skills, student learning, and curriculum materials. *Principles and Standards for School Mathematics* (NCTM, 2000) and *Benchmarks for Science Literacy* (AAAS, 1993) both indicate that "mathematics education should focus on a carefully specified and coherent set of important concepts and skills that all students should learn" (MSMP, 2001, p. 2). Research has found that teachers need to have specific mathematical knowledge to teach mathematics with understanding (Carpenter et al., 1999; Kaput, 1999), and teachers' knowledge has been considered as the most important factor for effective teaching. However, a report from National Research Council (2001) revealed that very few teachers currently have the specialized knowledge to teach mathematics effectively. Stein, Smith and Silver's (1999) critical view about current professional development concludes that "neither form (mandated district-sponsored staff development and elective participation provided by the university) was designed to transmit a specific set of ideas, techniques, or materials to the teacher" (p. 23). Therefore, the quality of professional development must "focus on research-based knowledge of children's thinking within specific mathematics domains" (Chambers, & Hankes, 1994, p. 294). Furthermore, professional development should and must (1) focus on building the capacity to understand subject matter and guide students' development of concepts; (2) use practical and related support (grade level, subject matter); and (3) use a knowledge base for immediate use and future use (Stein, Smith, & Silver, 1999). Once teachers understand the concept of the particular mathematics content and domain they are teaching, the change will be possible. Professional development that focuses on specialized knowledge and teachers' change is the key for teachers' practice in effective teaching.

How do we help teachers acquire specialized knowledge and make changes and transitions inside and outside of the classroom? According to the theory of ZPD (Vygotsky, 1978), learning new knowledge is based on prior knowledge and learning interaction. Professional development plays an important role in connecting teachers' prior knowledge (content knowledge) to their new learning (pedagogical knowledge) and in promoting learning interaction among teachers. For many years, besides teacher education programs in college, professional development has been the main approach used in teachers' knowledge growth. However, its ineffectiveness has been shown by a number of studies (An, 2004; National Research Council, 2001; Stigler, & Hiebert, 1997). There are many reasons for the absence of a systematically effective approach to teacher professional development, the main reason often being the failure is not to tie teacher professional development closely to specific content and teaching strategies. In addition, only when teachers are allowed to see themselves collectively and directly improving their teaching practice by improving content and pedagogical knowledge and by improving students' opportunities to learn, will teacher professional development be effective (Stigler, & Hiebert, 1997). The basis for this change is to change the focus of professional development to specific content and teaching strategies, such as understanding and use of mathematics representation. Enhancing teachers' ability in understanding and using mathematics representation is essential for professional development because much of the history of mathematics is concerned with creating and refining representational systems (Lesh, Landau, & Hamilton, 1983). Therefore, learning mathematics with understanding simply means getting a solid grounding in mathematics representations which provides a gateway to successful learning. With profound knowledge of mathematics representation, teachers also acquire and reshape their knowledge in both content area and pedagogy strategies, which leads to the transition in teaching practices.

Teachers' Role in Implementing a Standards-Based Mathematical Curriculum

Project 2061's study (AAAS, 2000) shows the information that curriculum materials also play a big role in effective teaching. Effective teaching requires teachers to align their beliefs with new standards and implement new standards in classroom teaching. "Teachers are asked to focus on mathematical concepts, multiple representations of those concepts" (Sherin, 2002, p. 122) in order to improve teaching;

however, “teachers who use reform-based curricula do not always appear to be implementing reform in the way intended” (Sherin, 2002, p.122). As a result, “the mathematics performance in the U.S. population has never been seen as satisfactory, and today dissatisfaction with that performance has become intense, and it is growing” (RAND, 2003, p. 2). Today, with diverse backgrounds and educational objectives, a teaching paradigm is needed to be proposed as a framework to solve these problems. Reform mathematics teaching, therefore, requires that teachers change what they teach and how they teach by using valuable strategies such as mathematics representation to meet the needs of different levels of learning. The obstacle to the change is that “either teachers do not have enough content knowledge, or what they do know is not the ‘right’ content knowledge” (Sherin, 2002, p.123). Many aspects of teaching and learning mathematics “limit the power and utility of representations as tools for learning and doing mathematics” (NCTM, 2000, p. 14). Therefore, transforming and adapting new knowledge of representation that makes connections with others is essential for teaching and learning mathematics. Teachers’ change and transition are necessary for improving the quality of teaching and learning mathematics. However, the approach for improving teaching is not based on assuming that teaching will change when surrounding elements change. Rather, it is based on the direct study and experience of teaching, with the goal of steady improvement in the students’ mathematics learning (Stigler, & Hiebert, 1997).

However, teachers’ change is not as easy as learning new methods of teaching and instituting a new curriculum to follow. Teachers who are not optimistic about their students will have no reason or will be resistant to change “because they seldom engage in reflective practice, they will have little evidence of any need of change, and because they have low expectations of their students, they will not be surprised when their students fail to learn" (Hillocks, 1999, p. 134). One possibility that will help teachers to develop professional skills is to provide the opportunities for teachers to reflect on their own and others' work and to be aware of the difference between these reviews. Reflective teachers' thinking about their own teaching may improve their classroom teaching effectively. Teachers' reflecting on their own teaching is an effective way to help teachers' learning in professional development (Stein, Smith, & Silver, 1999). The Middle School Mathematics Project (MSMP) provides a good example of reflective teachers' thinking about their own teaching.

Standards-based curriculum challenges teachers not only to change their roles from traditional teaching to a new standard for teaching, but also to feel comfortable with the new standards. However, in implementing a standards-based curriculum, teachers might encounter many difficulties including building a teaching environment and fostering positive beliefs and attitudes to meet the needs of the new standards. This change is especially difficult in the very beginning. Experience from Show-me project indicates that as teachers continue to engage in professional development that support standards-based curricula and intertwines mathematics, pedagogy, and assessment, the focus of teachers' concerns shift to student learning and the interrelationship between teaching and learning (Show-Me, 2003). Although there are three components – content, teaching and learning, and assessment – normally involved in teacher professional development, the core components involve helping teachers deepen their knowledge and skills for effective teaching and learning mathematics.

In order to teach effectively, in addition, the textbook also plays an important role in teaching practice because teachers' mathematics knowledge greatly influences how they evaluate and implement the textbook. This knowledge manifests itself in how the teachers plan their instruction, interact with students, and use the textbook in the classroom (Manouchehri, & Goodman, 2000, p. 1).

Cognitive View of Building Student Understanding via Mathematical Representation

Mathematics representation is based on cognitive learning, as Monk (2003) explained, "complexities of representation as a cognitive and social process and of how it is inextricably linked with the knowledge people have of the situation being presented" (p. 250). Vygotsky's theory of ZPD in turn states that cognitive development and the ability to use thought to control learning actions require cultural communication systems and learning to use these systems to regulate thought processes. The ZPD theory also reveals that learning takes place when students are working in their Z*one of Proximate Development* (Slavin, 2003), in which "skilled teachers pay attention to their student's words, written work, use of manipulative materials, or use of calculator and computers as they try to understand individual conceptions and misconceptions" (Goldin, & Shteingold, 2001, p. 6). Teaching mathematics effectively means that teachers understand the effects on their students' learning of external representations and structured mathematics activities they teach as well as student's internal mathematics representations that reflect student's internal development and thinking insight.

According to Schallert and Martin (2002), learning is intrinsically tied to motivation. However, motivation is formed by different environmental factors and social influences. From a cognitive perspective, motivation can increase learners' energy level, direct their goals, promote initiative in activities, and affect the strategies employed by different learners. Expectancy and task value are the most important two factors in achievement behavior. Tasks are related to competence since learners choose a certain task and expectancies are related to the learning goals and achievement. In constructivism, "learners do a great deal with the information they acquire, actively trying to organize and make sense of it" (Ormrod, 1999, p. 171). "The essence of the constructivist perspective lies in its portrayal of learners as organic rather than mechanistic sense-making" (Schallert, & Martin, 2002, p. 12). Prior knowledge, as a component of constructivism, plays a big role in the learning process. Learners can connect new information to prior knowledge only when they actually have knowledge related to the things that they are learning. Fosnot (1996) suggested that learning is the process of development, in which reflection is a key to internal representation. Other researchers have also discussed teachers' roles in preparing a learning environment that creates motivating conditions for students, produces problem a solving situation, promotes acquisition and retrieval of prior knowledge, and fosters a positive learning attitude in a social environment (Phye, 1997), allowing multiple perspectives in classroom, connecting learning to students real life experience, and encouraging self-awareness and ownership (Honebein, 1996).

Learning mathematics should be a process of development (Honebein, 1996), which should be interesting to students, should challenge their imagination, and should produce creative solutions in their specific interest. "Learning is seen less as the result of information provided to the students by the teachers, and more as the result of students' active efforts to make things comprehensible for themselves" (Goldsmith, & Shifter, 1997, p. 28). Concepts of epistemology provide teachers with a better understanding of how "to listen and observe students' working on mathematical problems, and how to reflect on their own processes of understanding" (Goldsmith, & Shifter, 1997, p. 28). Understanding is developed to the extent that students see the meaning of mathematics in what they are doing. To understand students' learning, teachers should better understand the psychological aspects of learning mathematics, and should use new teaching techniques and strategies for the reconstruction of the new notions of teaching and learning of mathematics.

In the constructive learning process, mathematics representation functions as a pathway to understanding. Teachers also need to understand that essential mathematics representations are (1) the form of representation people use to represent what they think, influencing both the processes and products of thinking; (2) different forms of representation develop into different cognitive skills; (3) the selection of a form of representation influences not only what they are able to represent but also what they are able to see; (4) forms of representation can be combined to enrich the array of resources students can respond to; and (5) each form of representation can be used in a different way, and each ways calls on the use of different skills and forms of thinking (Eisner, 1997).

The Role of Mathematics Representation in Systematic Mathematics Learning

What is the role of mathematics representation? In traditional mathematics classrooms, the emphasis has been on the use of abstract and procedure representations. For instance, the long-division algorithm represents the division procedure. Under a standards-based curriculum, students are given more freedom in creating and using representations. For instance, tables, graphs, and verbal explanations are commonly used in standards-based curricula. Monk (2003) observes, "Graphs, along with diagrams, charts, number sentences, and formulas are increasingly seen as useful tools for building understanding and for communicating both information and understanding" (p. 250). Although representations are used in both traditional and reformed classrooms to represent the ideas of the problem in order to find a solution (Smith, 2003), the paths are different. Only when teachers personally experience the standards-based curriculum and are able to see the difference, can their change possibly occur. That is what the MSMP professional development is designed for: teachers' reflection on their own learning and teaching bringing changes to their own teaching.

Representation is not only a network system in which the representational structures are related to each other, but also is a pathway to understanding of mathematics, which determines its features of permeation, connection, diversity, and evolution in mathematics learning. The feature of permeation shows that representations can be seen in every content area; connection allows representations to link one content area to another, such as using area-geometry representations to address the concept of multiplication; diversity encourages multiple perspectives and manifold representations created in learning mathematics; and evolution address the continuous growth of the presentation as learning progresses. The NCTM (2000) standards show these features of representation, in which the process standards (problem solving, reasoning, communication, connection, and representation) are linked to each other with mathematical content standards (numbers, algebra, geometry, measurement, and data analysis). Both teachers and students are facing the challenges to fully understand and grasp representations in mathematics learning, as NCTM (2000) states, "instructional programs from pre-kindergarten through grade 12 should enable all students to create and use representations to recognize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; use representations to model and interpret physical, social, and mathematical phenomena" (NCTM, 2000, p. 67).

In mathematics learning, an important goal for people in using representation is to be able to communicate with each other using multiple forms of representation. For instance, a graph can be used as a tool for communicating information and understanding and making meaning in mathematics. Both communicating and making meaning play important roles in all kinds of representations. Monk (2003) gives the example of graphs to illustrate the importance of using representations with the explanation of "(1) using graphs, students can explore aspects of a context that are not otherwise apparent; (2) the process of representing a context can lead to questions about the context itself; (3) using graphs to analyze a well-understood context can deepen a student's understanding of a graph and graphing; (4) students can construct new entities and concepts in a context beginning with important features of a graph; and (5) students can elaborate on their understanding of both a graph and its context through an iterative and interactive process of exploring both; and a group can build shared understanding through joint reference to the graph of phenomena in a context" (pp. 252-256).

There are two forms of representations: internal and external representations according to Troutman and Lichtenberg (2003). Internal representation is the result of cognitive development, which "are composed of networks of concepts and relationships" and "external representation is used to describe things that can be represented outside of the human mind" (p.10). However, in reality, internal representations sometimes play an important role in learning and teaching mathematics. Troutman and Lichtenberg (2003) indicate, "A goal of education is to help individuals create internal representations that accurately mirror external representations" (p.10). According to Greeno and Hall (1997), "forms of representation can be considered as useful tools for constructing understanding and for communicating information and understanding" (p. 362). Very often, in the reality of classrooms, students sometimes act on reflect their internal representation and interpret their own representations in multiple ways at various levels, serving construction and communication purposes.

Teachers' Transition in Using Mathematical Representation

The transition in teachers' mathematical representation closely relates to their belief and value system. Changing their current system relies heavily on professional development. Teachers need to understand students' representations in order to teach effectively because effective systems of representation include personal change in beliefs and values in mathematics and about themselves in relation to mathematics (Goldin, & Shteingold, 2001).

Students in middle school have difficulty doing mathematics because they have difficulty in selecting the proper mathematical representations. Sometimes they select the representation that they are familiar with; obviously, they do not necessarily select the most appropriate representation for their mathematical learning. For instance, students' written work is their representation because it represents students' thinking. Very often, students see the different representations as different mathematical problems (Janvier, 1987). The typical problem that students have is that they cannot see the relationships between a particular representation and corresponding or equivalent representations. Teachers need to learn strategies that articulated well with the representation in order to help students learn effectively and to effectively lead students to draw particular connections between mathematical questions in order to understand the mathematics concept.

Teachers should be aware that there are five distinct types of representation systems interrelated with each other that occur in mathematics learning and teaching: (1) experience based "scripts" – interpreting and solving problems based on real world events; (2) the manipulative model – elements have little meaning, but building relationships and operations makes sense in mathematics; (3) pictures or diagrams – can be internalized as images; (4) spoken languages – including sublanguages related to domains like logic, etc.; (5) written symbols – can involve specialized sentences and phrases (e.g., X + 3 = 7) (Lesh, Post, & Behr, 1987). Translating one representation to another requires the establishment of a relationship between different representations, i.e., equivalence representations.

The relationship between representation and understanding is another area to look at in using representation to teach and learn mathematics, as Janvier (1987) indicated "representation can be considered as a combination of three components: symbols (written), real objects, and mental images" (p. 68). The teachers' role includes helping students see the similarity and difference between multiple problem contexts. Thus teachers need to understand all kinds of mathematics representations and their relationships. The interaction of teachers and students shows that "to help students' progress valuing the representation, the teacher needs to understand how children view and relate to different mathematics representations" (Smith, 2003, p. 264), i.e., from mathematics to the real world.

Although some studies have discussed mathematical representations, teachers' representations topics are less focused. When claiming that teachers are the most important factor for effective teaching, we necessarily note that there is a need for developing teachers' mathematical representation concepts with an effective teaching model. In the book, *The Middle Path in Math Instruction*, An (2004) describes teaching mathematics by using the concrete model, and indicates that "through building and solving mathematical models, students internalize their learning and abstract their thinking" (p. 219).

Constructing Understanding through Mathematics Representation

Pedagogically, teaching mathematics using representation requires an understanding of internal and external representations. This teaching focus is the motive for using external representations in mathematics teaching (Dufour-Janvier, Bednarz, & Belanger, 1987). It is important for teachers to understand that "(1) Representations are an inherent part of mathematics, such as functions and Cartesian graphics, which represent the concepts by using these tools; (2) Representations are the multiple concretizations of a concept. For instance, multiple representations will represent only one concept" (Dufour-Janvier, Bednarz, & Belanger, 1987, p. 111). The ultimate goal of mathematics education is to help students build internal representations that accurately reflect external representations and include concepts and relationships (Troutman, & Lichtenberg, 2003). Proficiency in mathematics representation involves an understanding of mathematics concepts and operations (NRC, 2001).

To illustrate mathematics representations in specific contents, the following examples address the difficulties students often have in learning mathematics and how representations help students understand mathematics concepts:

Classroom researchers have brought to our attention learning and teaching fractions effectively. Different interpretations and representations of fractions often confuse students. Lamon (2001) summarized five different constructions for fractions:

part/whole comparison, measures, operators, quotients, and ratios and rates. With the part/whole concept, Kerslake (1986) observes that students often conceive of a fraction as a number, and they might think of it either as two numbers or not a number. An (2004) suggests using the unit fraction concept to help students represent fractions.

The concept of function is still considered to be a complicated area for algebra beginners, especially for symbolic representation. The representation of algebra often involves the translation from verbal information into symbolic expression and equation (NRC, 2001). How to use representation to express the function by connecting symbolic and numerical representation is an important question in teaching middle school mathematics. Yerushalmy and Shternberg (2001) state that modeling situations before using symbolic representations for the functions will help students better understand algebra functions. Meyer (2001) further describes the four principles for a particular situation. First, contexts simulate representations, which allow students to engage in meaningful activities; second, students reflect on the abstraction of algebra by the pictures, charts, tables, and equations. In this stage, teachers should allow students to use informal strategies in order to understand algebra concepts; third, during interaction between teacher and students, students further understand many representations for the situation; and fourth, a connection between abstract and concrete algebra functions should take place. Markovits, Eylon and Bruckheimer (1988) discuss two aspects of difficulty in learning functions. First, the relationship between the domain and range determines that every element of the domain has exactly one element in the range; second, functions have multiple representations -- graphs, equations, tables, and the arrow diagram. Some students confuse function and linear function and always think that function is linear function. Other researchers argue that many difficulties of learning algebra come from the students' view of the difference between algebra and arithmetic. However, algebra is not separate from arithmetic and indeed algebra generalizes arithmetic operations in many ways. Booth (1988) summarizes the difficulties in students' learning algebra: 1) Arithmetic way in viewing algebra; 2) Notation and conventions; and 3) Letters and variables. School algebra has to deal with students' difficulties in understanding variables and their operations, and teachers need to create various representations to help students understand algebra.

The ability to read and interpret graphs is a basic skill and apparently is not being effectively taught. In investigating the ways that students make sense of information represented through graphical representations and make connections between related pairs of graphs, Friel and Bright (1995) find that students need to talk more about graphs. It is important to understand how students think through the structure of their representations. Teachers need to create an environment to let students become involved in a variety of representations in data analysis and to understand the structure and meaning of these representations.

In a reformed curriculum, teachers must cover materials which have not been traditionally taught in the middle grades, and they must teach it in new and creative ways. Therefore, teachers need to be trained to use new and creative ways. In professional development, teachers need to learn important mathematics representations and the strategies in specific content areas which will reach various levels of students and help students achieve understanding and proficiency in mathematics.

Summary

It is impossible to effectively focus on too many educational reform goals in professional development. Effective professional development should have a single focus. Professional development that focuses on teachers' change is the key for teachers' practice in effective teaching. However, without focusing on the specific content knowledge, it is unfeasible for professional development to reach the goal of teachers' change. Specific content knowledge such as mathematics representations is an effective approach that helps teachers deepen their knowledge and strengthen the abilities of teaching. Through well-designed professional development that focuses on specific content and strategies, teachers engage in an inquiry process of learning and understanding of various representations. This teacher learning process broadens teachers' views, promotes reflection on their own teaching, and enables teachers to arrive at a higher TZPD. Mathematics representation is also a pathway to understanding mathematics for all levels of students. It helps students overcome obstacles in learning, makes abstract mathematics ideas concrete and meaningful, and assists students in reaching a high level of understanding and proficiency in mathematics.

CHAPTER III

METHODOLOGY

Background of the Study

This study is a part of 5-year longitudinal study investigating middle school teachers' instruction using mathematics representations as they learned teaching strategies and skills through the MSMP professional development. This study explores the impact of the teachers' changes in implementing a standards-based curriculum by examining the important roles of mathematics representations, and investigated how teachers used the mathematics representations learned from the MSMP professional development workshop to help students improve their learning of mathematics.

The MSMP professional development program that served as the basis for this study is the Middle School Mathematics Project (MSMP), which focuses on helping teachers understand mathematics representation and constructing models of mathematical representation in well-defined content domains. Teachers' textbook usage was the basis for studying instructional practice in MSMP. Three different textbooks were used in four different school districts. The *Connected Mathematics Project* [CMP] (Lappan, Fey, Fitzgerald, Friel, & Phillip, 2002) is funded by the National Science Foundation to develop a middle school mathematics curriculum for 6th to 8th grades. CMP is "a curriculum built around mathematical problems that help students develop understanding of important concepts and skills" (Grant et al., 2003, p. 4) in NCTM's five content areas. *Math Thematics* [MT] (Billstein, & Williamson, 1999), also funded by the National Science Foundation, is designed to be "mathematically accurate, utilize technology, and provide students with bridges to science and other mathematical fields" (Show-Me, 2001, Philosophy – MATH Thematics section). *Mathematics: Applications and connections* [MAC] (Glencoe/McGraw-Hill, 1998) is designed to show students how mathematics relates to the real world and how mathematics connects to other subject areas. These textbooks served as basic instructional materials for teachers' use in their teaching, and teachers used strategies that the textbooks provided for their classroom teaching. During the summer 2003 workshop (part of the MSMP professional development program), teachers not only learned particular areas of content of mathematics representations in fractions and algebraic functions, but also reflected on their teaching by watching and analyzing their own videos of teaching that were recorded before the workshop. While textbooks were important in helping teachers connect concepts in the MSMP professional development to their instructional practice, in this study textbook use was not important to the research questions.

In this study, data were collected over a 2-year period, and 11 middle school teachers and their classes served as sources and objects for data collection and analysis. Videotapes, audiotapes, surveys, and questionnaires served as the main source for the study. Interviews were conducted to further investigate how teachers made transitions in understanding and in using mathematics representations and to examine the relationship between teachers' understanding and the use of representations in classroom teaching.

Participants and Setting

A total of eleven 6th and 7th grade middle school teachers in Texas participated in this study. The criteria of inclusion for subjects were: 1) volunteered to participate and were willing to contribute to this study; 2) taught at the same grade level for two years; and 3) attended a full week of the MSMP professional development workshop held in the summer of 2003 or a four-hour make up session of the workshop held in the beginning of fall semester 2003 (three of them attended it). The workshop aimed at showing specific content knowledge of representations in fractions and algebraic functions. For the analysis of teachers' learning and understanding and the use of representations as well as their knowledge of changes, data from all 11 teachers were examined. For the detailed analysis of the patterns of change, 10 out 11 teachers were interviewed to further understand their views about learning and using mathematical representations.

The 11 teachers came from four school districts. Each district has similar characteristics of textbooks, grades, teaching content, and source of students at the same grade level. Table 2 shows the distribution of participants in two grade levels at four school districts. Table 3 indicates the use of three types of textbooks: CMP, MT, and MAC.

Table 2
Distribution of Participants and Content at Two Grade Levels

	District A	District B	District C	District D	Total
6^{th} (N)	3	3	1	-	7
7^{th} (A)	1	1	1	1	4
Total	4	4	2	1	11

Note: 1) N: Numbers refers to fractions in this study; A: Algebra refers to patterns of changes in this study. 2) Numerals in rows are the numbers of participants from each school district. 3) "-" indicates there is no participant.

Table 3
The Use of Three Types of Textbooks

	District A	District B	District C	District D	Total
CMP	3	-	-	-	3
MT	-	4	-	-	4
MAC	1	-	2	1	4
Total	4	4	2	1	11

Note: CMP: Connected Mathematics Project (Prentice Hall, 2002); MT: Math Thematics (McDougal, 1999); MAC: Mathematics: applications and connections (Glencoe, 1998); "-" indicates there is no participant.

Procedure

The MSMP Professional Development Workshop

This study was part of a Middle School Mathematics Project (MSMP) 5-year longitudinal study. It investigated middle school teachers' instruction as they learned teaching strategies and skills through the MSMP professional development workshop and explored the impact of the teachers' changes on implementing a standards-based curriculum, advocated by the National Council of Teachers of Mathematics (NCTM, 2000). The MSMP professional development program that served as the basis for this study focused on helping teachers understand mathematics representation, design probing and guiding questions, and construct model lessons that apply these strategies toward well-defined content learning goals.

Two workshops were held for teachers. The content of the 2002 workshop included reviewing the distinction between a topic and a specific learning goal, and studying Texas Essential Knowledge and Skills (TEKS, 1997) for middle school mathematics standards and corresponding American Association for the Advancement of Science benchmarks (AAAS, 1993) and National Council of Teachers of Mathematics standards (NCTM, 2000). The workshop also provided opportunities for teachers to review related research to deepen their understanding of key mathematical ideas, prerequisites, and students' common misconceptions in the selected learning content (e.g., fractions and algebraic functions). After participation in the workshops, teachers implemented learned knowledge in their teaching, and their lessons were videotaped in the 2002-2003 school year.

The MSMP professional development workshop held during the summer of 2003 by professional development staff of the MSMP focused on representations and questioning. During the MSMP professional development, teachers viewed and discussed video examples of appropriate representations for equivalent fractions and algebraic patterns of change. They then viewed and analyzed their own videotaped lessons that had been taken during school year 2002-2003 and identified ways to improve their use of strategies of representations. Finally, they reflected on their teaching and designed new lessons that incorporated the ideas they had studied.

The purpose of the MSMP professional development workshop in summer 2003 was to help teachers: 1) learn mathematical representation teaching strategies to help their students relate representations to the ideas in the learning goals; and 2) use the knowledge of mathematical representation to help students improve their mathematical thinking about the ideas in the learning goals in the fraction and algebra areas.

During the MSMP professional development for summer 2003, participants learned learning goals on equivalent fractions: “use, interpret, and compare numbers in several equivalent forms such as integers, fractions, decimals, and percents” (AAAS, 1993, p. 291). By the end of the professional development, participants had drafted plans for improving the use of representations. The purposes of teachers’ using mathematical representations are: to probe teachers’ understanding of ideas underlying models of equivalent fractions and to demonstrate that an idea can be represented in a variety of ways to stimulate teachers’ thinking about different ways to represent equivalent fractions. For example, participants were asked to represent 0.75 = 6/8 and 4/7 = 8/14 in different ways. The reason for posing these questions was also related to students’ learning: students often have difficulties dividing a circle into sevenths, so they switch to using a rectangle, but often end up with the rectangle showing sevenths that are smaller than the rectangle showing fourteenths. Discussing these types of questions between teachers during the workshop helped teachers clarify differences among a measurement model (use of number line), an area model, a set model, an angle model, and perhaps even a ratio model of equivalence.

In algebraic patterns of change, the teachers also learned that "symbolic equations can be used to summarize how the quantity of something changes over time or in response to other changes" (AAAS, 1993, p. 274). Similar to the equivalent fractions, the teachers were asked to use all kinds of mathematical representations to illustrate their mathematical ideas. The purposes for doing that were: to probe teachers' ideas for representing patterns of change and to understand how symbolic equations can be used to represent patterns of changes and to demonstrate the patterns.

As teachers discussed their representations in two content areas, they were encouraged to consider such questions as: 1) how each representation fits in with the learning goal; 2) what ideas are and are not represented; 3) strengths of the models in representing the ideas; and 4) limitations of the models (e.g., something that might be confusing to students or might promote misconceptions). For example, considering comprehensibility, some representations used to illustrate equivalence may not be transparent to all students. Teachers' learning and discussion during the workshop were videotaped for data collection purposes.

Video of Classroom Teaching after Workshop

During the 2002-2003 school year, each teacher in the 6^{th} and 7^{th} grades had three to five classes videotaped on teaching equivalent fractions (6^{th}) or algebraic patterns of change (7^{th}). The 2003 MSMP professional development workshop also utilized videotapes for analyzing teachers' learning and understanding processes. After the summer of 2003 workshop, participating teachers' lessons were videotaped to examine their improvement by using strategies of representations for the school year 2003-2004. All teachers from the 6^{th} and 7^{th} grades were videotaped during three lessons on the same teaching content. In their lessons, teachers were expected to integrate learned mathematical standards from the workshop in 2003 as learning goals, particularly those standards relating to the Texas Essential Knowledge and Skills (TEKS).

Interviews with Teachers

Ten teachers (one teacher was not able to be interviewed at that particular time for a particular reason) were interviewed in order to find out in what ways teachers make transitions in understanding and using mathematics representations, patterns of teachers' change, and relationships between teachers' understanding and use of representations and their students' thinking and learning. Teachers were asked a set of questions on their view of the MSMP professional development and the mathematical representation during an hour interview. Each interview was audio-taped to make sure of the accuracy of data collection.

Instrumentation

To examine teachers' growth and changes in understanding and use of mathematics representations, this study designed questionnaires after the workshop, a set of questions for interviews with teachers, and criteria for measuring teachers' knowledge both in learning and use of mathematical representation in the *Teachers' Zone of Proximal Development* (TZPD). This study also used MSMP existing data of questionnaires before the workshop and a survey of teacher preparation, attitudes, and support structures before the workshop, videotapes during the workshop and videotaped lessons before and after the workshop.

Questionnaires

Two questionnaires were designed to assess teachers' understanding and use of mathematics representations: a questionnaire before MSMP professional development and a questionnaire after MSMP professional development. Both questionnaires consisted of two problems on equivalent fractions and algebra functions, one for each area. The purpose of the first questionnaire was to examine teachers' understanding and use of mathematics representation; the second questionnaire focused on assessing teachers' knowledge of students' thinking on mathematics representations and their misconceptions and strategies in making corrections (see Appendix A: Questionnaires before and after Professional Development).

Survey

Before the MSMP professional development workshop, a survey of teacher preparation, attitudes, and support structures was provided to teachers. There were two parts to the survey**:** content, pedagogy, and experience, which consisted of four questions, and support structures, which consisted of two questions. The goals of the survey were to examine how various factors influence the way teachers approach the teaching of mathematics. These factors include that textbook developers create materials that teachers can use more effectively, educational researchers design better pre-service and in-service teacher education programs, and school administrators provide the support teachers need to help them improve their teaching (see Appendix B: Survey of Teacher Preparation, Attitudes, and Support Structures).

Interview

Besides videotaped lessons, teachers were interviewed to further explore their growth and changes in knowledge of mathematics representations. Interview questions consisted of six questions. The objectives of the interview questions were to examine in what ways teachers make transitions in understanding and in using mathematics representations, patterns of teachers' change, and relationships between teachers' understanding and use of representations and their students' mathematical thinking and learning (see Appendix C: List of Interview Questions).

Videotapes

Teachers' participation in workshops was videotaped to furnish more information on how teachers grow and change their knowledge of mathematics representations. Indicators of learning and understanding of mathematics representation consisted of three categories: 1) knowing accurate representations help students identify specific learning goals and addresses important mathematics ideas; 2) knowing comprehensible representations promote students with diverse backgrounds, abilities, and interests in learning mathematics; and 3) knowing variety of accurate and comprehensible representations help students build abstract understanding based on concrete model.

Teachers' classroom teaching was videotaped before and after the workshop. The observation criteria for videotapes were constructed according to the levels of instructional use of mathematical representations. Indicators of use of mathematics representation in effective classroom teaching consisted of three categories: 1) identifies specific mathematical learning goals and addresses important mathematics ideas; 2) using comprehensible representations promotes students with diverse backgrounds, abilities, and interests with their prior knowledge in learning mathematics , encourage them to prove and justify their mathematical reasoning in real life situations, and provides them with opportunities to share their mathematical ideas; and 3) using variety of accurate and comprehensible representations addresses students' mathematical misconceptions and builds students' conceptual understanding based on abstract model.

Data Collection

Data were collected over a 2-year period from videotapes, teacher surveys, questionnaires, and interviews with teachers. A total of 11 middle school teachers and their classes served as sources and objects for data collection and analysis. During the 2002-2003 school year, each teacher in the 6^{th} and 7^{th} grades had three to five class videotapes on teaching fractions (6^{th}) or algebraic patterns of change (7^{th}); therefore, a total of 42 videotapes were used and transcribed in this study. Eleven videotapes from the 2003 MSMP professional development workshop were collected. These video and audio tapes also transcribed for analyzing teachers' learning and understanding processes. During 2003-2004 school year, most teachers from the 6^{th} and 7^{th} grades were videotaped for three lessons on the same teaching content; therefore, a total of 27 videotapes were used and transcribed in this study. Teachers' interviews were audio-taped, and a total of 10 audio-tapes from interviews were transcribed for the purpose of data analysis. Therefore, a total of 90 videotapes and audiotapes were transcribed and coded in this study.

Data Analysis

This study employed a qualitative method of data analysis. The data analysis consisted of three intended uses: teachers' learning and understanding of mathematical representations; teachers' use of mathematical representations; and teachers' change in mathematical representation knowledge. Each use corresponded to *Teachers' Zone of Proximal Development* (TZPD) and focused on two content areas, equivalent fractions and algebraic patterns of changes. Multiple data resources were used and analyzed for each area. Data from videotapes, surveys, and questionnaires served as the main source for the study. Teachers' responses from interviews were used to further illustrate the ways in which teachers made transitions in understanding and in using mathematics representations, and to understand the patterns of change.

To provide an organization for analyzing and interpreting the data, this study proposed a common basis for summarizing and analyzing each teacher's knowledge of understanding and using mathematical representations as well as their changes according to the framework of three levels of TZPDs. In order to limit the data coding bias, two researchers coded data separately and discussed and categorized teachers' knowledge of representation under three aspects. Therefore, reliability and validity are ensured by using triangulation of data, member checks, peer examination, and purposive samplings (Anfara, 2002).

The criteria used in analyzing and coding data for examining teachers' knowledge in understanding and using mathematics representations in two content areas were based on and adapted from the Project 2061 criteria (AAAS, 2000, Appendix C Methodology), mainly aimed at helping students relate representations to mathematical ideas. Teachers' levels of mathematics representations were based on accuracy and comprehensibility. Accuracy refers to the things being represented truthfully, and comprehensibility means the ideas being represented in an easier and meaningful way to students. The standards of two content areas were adapted from *Benchmarks for science literacy* (AAAS Project 2061, 1993): numbers and operations and algebra symbolic equations.

Measures of Teacher Learning and Understanding Representations

To answer research question one of, "What do teachers understand about mathematical representations through the MSMP professional development workshop? How does this understanding of representation fit into TZPDs?", the study used questionnaires before and after professional development as well as the videotapes of the duration of the workshop, teachers' lesson videotapes before and after the workshop, and a survey of teacher preparation, attitudes, and support structures to examine teachers' learning and understanding of mathematical representations for equivalent fractions and algebraic patterns of change. This study used questionnaires as the main evidence tool, and surveys and videotapes as supplementary materials to determine teachers' levels of understanding of mathematical representations corresponding to their levels of TZPDs relating to teachers' knowledge (content, pedagogical, and pedagogical content knowledge), which vary along the following two dimensions: accuracy and comprehensibility.

In order to have accurate data, this study designed the criteria for measuring teachers' knowledge in learning and understanding representation in TZPD (see Table 4).

Table 4
The Criteria for Measuring Teachers' Knowledge in Learning and Understanding Representations in TZPD

Levels	**TZPD and Knowledge**	**Description of knowledge in mathematical representations**
1	Zone 1: - Learning *without interpsychological function.* - Knowledge is limited in *content area*.	Limited knowledge of mathematical representations in the above categories. *The representation is unclear and disjointed.*
2	Zone 2: - Learning *with interpsychological function.* - Knowledge is developed in both *content and pedagogical areas.*	Knowing mathematical representations accurately in the above categories. *The representation is clear but is not comprehensible with little variety and connection.*
3	Zone: - Learning *in intrapsychological function.* - Knowledge is developed for *pedagogical content knowledge.*	Knowing various mathematical representations accurately and comprehensibly in the above categories. *The representation is accurate and comprehensible with variety and connection.*

Using these criteria, teachers' responses from questionnaires and survey were analyzed and coded into three categories that were grouped in three levels of TZPD. Similarly, videotapes of the summer 2003 workshop were transcribed and analyzed, and teachers' reflection journals during the workshop were also examined to verify their responses in questionnaires and survey. This procedure took about two and half months.

Measures of Teacher Use of Representations in Classroom Teaching

To address research question two of, "How do teachers use their new knowledge of mathematics representations gained from the MSMP professional development workshop in classroom teaching? How does this practice relate to their levels of TZPDs?", this study used videotapes of classroom teaching before and after the summer 2003 workshop as the main evidence to determine teachers' levels of use of mathematical representations in classroom teaching corresponding to their levels of TZPDs, which vary in the following two dimensions: accuracy and comprehensibility. From videotapes, this study not only examined the levels of teachers' use of representation as a useful tool to direct their students' conceptual understanding, but also explored and discovered the relationships between their use of mathematics representations and their levels of knowledge of TZPD as related to teachers' knowledge (content, pedagogical, and pedagogical content knowledge). The categories vary alone the two dimensions of accuracy and comprehensibility in different teaching tasks. Table 5 was constructed to analyze the levels of teachers' using mathematical representations in classroom teaching corresponding to their levels of TZPDs.

Table 5
The Criteria of Measuring Teachers' Using Representation in Teaching

Levels	**TZPD and Knowledge**	**Description of using mathematical representations**
1	Zone 1: - Learning *without interpsychological function.* - Knowledge is limited in *content knowledge*.	Limited use of mathematical representations in classroom teaching in the above categories. *The use of representation is unclear and disjointed.*
2	Zone 2: - Learning in transition *from interpsychological to intrapsychological function.* - Knowledge is developed in *content and pedagogical knowledge.*	Using some mathematical representations accurately, but not comprehensibly in the above categories. *The representation is clear but is not comprehensible with little variety and connection.*
3	Zone: - Learning *with intrapsychological function.* - Knowledge is developed in *pedagogical content knowledge.*	Uses various mathematical representations accurately and comprehensibly in classroom teaching in the above categories. *The representation is accurate and comprehensible with variety and connection.*

Case Examples

To answer research question three of, "In what ways do teachers make changes in their actual classroom teaching in terms of TZPD levels in using mathematics representation after the MSMP professional development workshop? Are there any patterns of changes in classroom teaching?", this study used interviews with teachers as case examples along with videotapes in classroom teaching before and after professional development and a survey before professional development. Case examples compared teachers' different levels of performance in mathematical representations and provided an illustration of the characteristics of patterns of teachers' changes and a picture of teachers' ongoing learning.

From videotapes, this study intended to find the patterns of teachers' change in understanding and use of mathematical representation in teaching, involvement in the innovation, and development within the particular context with representation. The discussion and interaction with students in classroom teaching allowed teachers to know students' mathematical thinking and view learning and teaching mathematics from different angles. The instrument for interviews is a set of questions on the view of the MSMP professional development and learning and use of mathematics representations. All interviews were audio-taped, and transcriptions were made of the interviews. The responses to interview questions were also analyzed through the use of concept mapping to clarify teachers' understanding and use of representations.

CHAPTER IV

RESULTS

In order to discover how and to what extent teachers' knowledge of representation fits into their TZPD sufficiently to help them improve their teaching, this study examined teachers' understanding of mathematics representations through the MSMP professional development, investigated how teachers use different types of mathematics representations to teach fractions and algebraic functions, and assessed how teachers can make changes in using mathematics representations after the MSMP professional development.

In this study, data were collected over a 2-year period, and 11 middle school teachers and their classes served as sources and objects for data collection and analysis. To examine teachers' growth and changes in the understanding and use of mathematics representations, teachers were assessed by two questionnaires before and after the workshop, a survey of teacher preparation, attitudes, and support structures before the workshop, videotape lessons before and after the workshop, and learning and discussion video and audio tapes during the workshop. During the summer of 2003 MSMP professional development workshop, teachers not only learned particular mathematics content representations, in equivalent fractions and patterns of change in algebraic functions, but also reflected on their teaching by watching and analyzing their own video lessons teaching that were recorded before the workshop. In addition, ten teachers were interviewed in order to find out in what ways teachers can make transitions in understanding and using mathematics representation, and the patterns of their changes.

This chapter seeks to provide information on teachers' changes in the understanding and use of mathematics representations, to address the teachers' change in classroom instruction, and to present the relationship between teachers' change in learning and using mathematical representation before and after a professional development workshop. All data were analyzed and coded according to three levels of TZPD using a particular framework (see Table 4). In this chapter, the results of data analysis from questionnaires, videotapes, surveys, and interviews are reported in three sections a) teachers' change in learning and understanding representations, b) teachers' change in classroom instruction, and c) the patterns of teachers' change in learning and using representations according to research questions in this study: (1) What do teachers understand about mathematical representations through the MSMP professional development workshop? How does this understanding of representation fit into TZPDs? (2) How do teachers use their new knowledge of mathematics representations gained from the MSMP professional development workshop in classroom teaching? How does this practice relate to their levels of TZPDs? (3) In what ways do teachers make changes in their actual classroom teaching in terms of TZPD levels in using mathematics representation after the MSMP professional development workshop? Are there any patterns of changes in classroom teaching? Section one and two presented results for three levels: the changes in teachers' learning or use of representations in Zone 1, the changes in teachers' learning or use of representations in Zone 2, and the changes in teachers' learning or use of representations in Zone 3. Section three addressed the patterns of teachers' change in learning and using representations.

Teachers' Changes in Learning and Understanding Representations

To find out how teachers make changes in learning and understanding representations, questionnaires were used as the main evidence. Surveys and videotapes were used as supplementary materials to determine teachers' levels of understanding of mathematical representations corresponding to three levels of TZPD relating to teachers' knowledge.

Table 6 compares the different levels of TZPD relating to teachers' knowledge in learning representations before and after the workshop, and indicates changes in the zones.

Table 6
Changes of Teachers' Learning Representations

Teachers	A	B	C	D	E	F	G	H	I	J	K
Zone level before workshop	2	1	3	2	2	2	1	1	2	1	3
Zone level after workshop	3	2	3	3	3	3	2	2	3	2	3
Change in learning (Zone)	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N

Note: Letters A to K represent eleven teachers. Numerals in rows indicate three levels of *Teachers' Zone of Proximal Development* (TZPD): Zone 1, Zone 2, and Zone 3. Y: Learning zone changed; N: Learning zone did not change (in this study, the highest zone is Zone 3).

Table 7 shows the number of movements of changes in teachers' knowledge in learning representations. Nine out of eleven teachers made a move from a lower zone level to a higher zone level; four teachers moved from Zone 1 to Zone 2; five teachers progressed from Zone 2 to Zone 3, and two teachers maintained their level in Zone 3, which is the highest level in this study (see Table 7).

Table 7
Movement of Change in Teachers' Learning Representations

Teachers	B	G	H	J	A	D	E	F	I	C	K
Movement of change in learning Zone	1-2	1-2	1-2	1-2	2-3	2-3	2-3	2-3	2-3	3-3	3-3
Number of Teachers in Change	4				5					2	

Note: Letters A to K represent eleven teachers. Numerals in row 2 indicate the movement of changes in three levels of *Teachers' Zone of Proximal Development* (TZPD). Numerals in row 3 addressed the numbers of teachers changing their learning zones.

The Changes of Teacher Learning Representation in Zone 1

Table 6 shows three out 11 teachers' knowledge levels made a move from Zone 1 to Zone 2. The teachers who were placed in Zone 1 indicated their level of mathematical representations as basically accurate but abstract. They mainly used symbolic representations to represent mathematical ideas. Teachers' knowledge in Zone 1 also indicated that they had mathematical content knowledge, but they lacked pedagogical knowledge, which made it difficult for them to convey their knowledge to the students.

Teacher B is a 6^{th} grade mathematics teacher with 12 years of teaching experience, and her field of concentration is in elementary education. Her learning and understanding indicated that she has gained the knowledge of mathematical representations to go from Zone 1 to Zone 2 through the MSMP professional development workshop. For instance, before the workshop, she used different ways (table, number line, words, and symbolic equations) to address mathematical ideas using different representations; however, only the symbolic equation was accurate; tabular, number line, or word representations were either not stating the problem situation clearly, or not involving important mathematical ideas. The following representations were from teacher B's answers in one of the questions in the questionnaire before the MSMP professional development (see Figure 3):

1. Symbolic form: Bob = 45 + 1X = 100 X = 55 Sec.

   Andy = 2.5X = 100 X = 40 Sec.

2. Tabular form:

Bob	Andy
1 m per sec.	2.5 m per sec.

3. Number line form

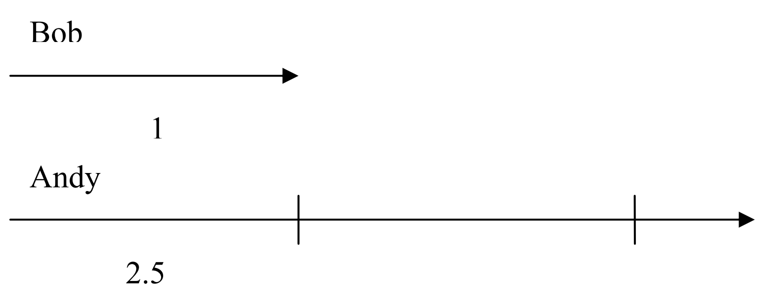


Figure 3. Teacher B's representation before professional development.

Three forms of representations from teacher B represented the same mathematical idea; however, only the symbolic form was accurate. The tabular and number line forms were not accurate. Although teacher B answered the question correctly, that Andy won, her answer to the second question "what distance would make the race fair for both boys?" was not even relevant to the question: 30 seconds.

After the workshop, teacher B was able to clarify student misconceptions about equivalent fractions by saying "the equivalency of fractions using equal sizes of a whole." She also recognized that students need to have a visual aid such as a pizza or pictures to help them build concrete understanding. She believed that during the learning process, the teacher should "continue to work more abstractly, moving towards the concrete. [The lesson should] allow students to become more involved in the discussion of the learning process and express their own outcomes." She also realized that "the representation will become a great meaningful tool for this particular lesson." Through learning mathematical representations, she understood much better that mathematical representations "make learning easier and more understandable [for students]."

Teachers G and H are in the same school, and both teach 6th grade mathematics. Their teaching styles followed almost the same pattern as their learning and use of mathematical representations. They understood the equivalent fractions; however, their representations are limited to symbolic forms using mathematical procedures to address the given questions. For instance, to solve equivalent fraction 0.75 = 6/8, teacher G accurately used long-division algorithm to change 6/8 to 0.75. For the equivalent fraction 4/7 = 8/14, she used the same approach to get each fraction to 0.571…. Evidently, teachers G's representation was not a comprehensible and visual form for students' understanding. The videotapes of her teaching after the workshop showed that she used more comprehensible mathematical representations in her classroom teaching. For instance, she used symbolic forms to represent equivalent fractions, used manipulatives to help students build concrete understanding of equivalent fractions, and was able to make the connection between symbolic form and manipulative concrete models. Her reflection on workshops also indicated that she has learned that comprehensible representations help students better understand the math concept. "To make sure the students can prove the equivalency in more than one way. Let one way be a pictorial model and have them justify their answers," she stated in her reflection during the workshop.

Teacher J is a 7th grade mathematics teacher with three years of teaching experience, and a field concentration of elementary education in reading. In the question of equivalent fractions in the pre-questionnaire, she used the following forms to address the procedure of 0.75 = 6/8 and 4/7 = 8/14:

$$0.75 = \frac{75}{100} = \frac{3}{4}, \text{ and then } \frac{75}{100} \div \frac{25}{25}$$

$$0.75 = \frac{75}{100} \div \frac{25}{25} = \frac{3}{4} \text{ x } \frac{2}{2} = \frac{6}{8}$$

She stated, "Top and bottom were multiplied by the same # [number]."

$$\frac{4}{7} \text{ x } \frac{2}{2} = \frac{8}{14}$$

She also stated that: "again both top and bottom multiplied by the same # [number]."

Her knowledge of mathematical representation indicated that she understood mathematical procedures and rules, but lacked comprehensible representations. In learning reflection during the workshop, she said, "I liked the activity the workshop provided. We dealt with every aspect (tables, graphs, equations, ordered pairs). [In using these representations], the students have many different ways to find the answers." The changes in teacher J's knowledge also showed in her answers on the post questionnaire that involved students' mistakes about algebraic functions. Her responses were different in terms of mathematical representation compared to the pre-questionnaire. She realized,

> Andy was thinking that because the one tank is emptying faster and they would never be the same, he did not realize that the tanks started at different levels. John realized the tanks started at different levels. In his representation, he showed that tanks will meet at a certain point, do not change at a constant rate. Edward was thinking that 50 will only be taken out once, and 25 will be only added once.

She also understood that using multiple mathematical representations is an effective way to help students correct their mistakes:

> I think that drawing a graph together would help the students understand that the tanks are changing at a constant rate. It will also help them see that just because the tanks are different, doesn't mean they will never be the same. Andy needs to watch the two tanks make the change, even if it is a picture. John needs to see that the two change at the same pace. Edward would need to see that this is happening over several hours, and the tanks are changing each hour.

Her responses indicated that she has gained profound knowledge about mathematical representation not only in accurate but also in comprehensible ways.

In summary, these four teachers' knowledge of mathematical representation progressed from Zone 1 to Zone 2 because their knowledge shifted from symbolic representation toward knowing mathematical representation accurately and comprehensibly, though with little variety and connection.

Changes of Teachers' Learning Representation in Zone 2

Teachers in Zone 2 knew mathematical representations accurately and comprehensibly; however, their representations had little variety and connection, which made it difficult for them to convey their knowledge to reach students with diverse backgrounds. Table 6 shows that five teachers made a move from Zone 2 to Zone 3.

Teacher A is a 6^{th} grade mathematics teacher with 15 years of teaching experience and a field of concentration of elementary education in science. In his classroom teaching before the workshop, he clearly indicated that his favorite mathematical representations were equations and tables, which evidently served to promote in part students learning of mathematics with understanding. During the workshop he realized that multiple representations are effective ways to promote students learning with diverse backgrounds in mathematics. The interview data confirmed his thought:

> Representation is the way you teach the concept, and you can't do it mathematically, but help them [students] understand the concept, instead of just learning mathematical process because they don't understand it. Given that the situation is not traditional, using a certain process may not be able to solve the problem. If they understood the concept and know how it works, in a nontraditional problem, they are able to solve the problem. They are more likely to succeed in the learning."

Teacher E is a 7th grade mathematics teacher with eight years of teaching experience and a field concentration in secondary mathematics. Her responses from the questionnaire and classroom teaching showed that her mathematical representation knowledge is accurate and comprehensible. She liked to use charts and equations to represent mathematical ideas. For instance, in the given question of Andy and Bob race in the pre-questionnaire, she used two kinds of mathematical representation: charts and number lines (see Figure 4).

1. Chart form

Time	**Andy**	**Bob**
0	0	45
10	25	55
20	50	65
30	**75**	**75**
40	100	85

2. Number line form

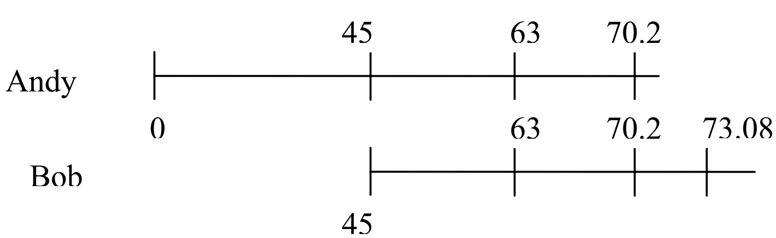


Figure 4. Teacher E's representation before professional development.

Clearly, for teacher E, her chart representation were accurate and comprehensible for the given situation; however, when she tried to answer the same question using the number line, it was not clear and comprehensible. In her number line representation, she included mathematical procedures such as long-division algorithm to indicate how she got the numbers. This is not visual enough for students to understand the mathematical idea. During the workshop, she reflected clearly that multiple representations help students build mathematical understanding. “[We need to] be sure to generate linear equations both from the graph and from the table,” she said. In the given question after the workshop, she was able to not only point out each student’s misconceptions but also find a way to help them by using three types of mathematical representations: table, graph, and numerical forms.

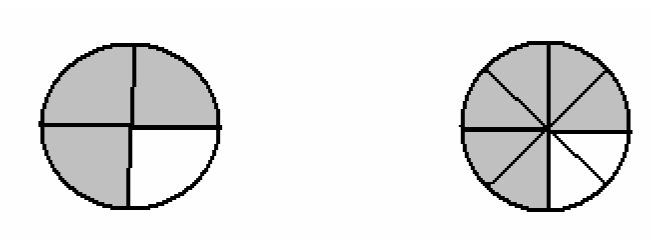
Teacher F is a 6th grade mathematics teacher with 25 years of teaching experience and a field concentration in elementary education of reading. Before the workshop, her representations on given questions were accurate and comprehensible by using abstract form and area models. The following were her answers to the given questions (see Figure 5).

She accurately represented the two equivalent fractions; however, she could use multiple forms to represent them in different ways. During the workshop, she realized that charts also helped students understand mathematics better. “We do a lot of graphs and charts in 6th grade,” she said on reflecting on the workshop. She was also able to recognize students’ mistakes in comparing equivalent fractions by saying “the picture of fractions should be the same” when comparing two fractions. She completely understood that multiple representations are effective ways to help students who have different abilities understand mathematics better.

1) 0.75 = 6/8:

Abstract way: 6.00 ÷ 8 = 0.75 therefore, 0.75 = 6/8

Area model:



0.75 or $\frac{75}{100} = \frac{3}{4}\ \frac{Qearters}{Quarters}$ in a dollar 6/8

2) 4/7 = 8/14

Abstract way: $\frac{4}{7}$ x 2 = $\frac{8}{14}$ (2 should be $\frac{2}{2}$) or $\frac{4}{7} = \frac{8}{14}$ (using cross multiplication: 56 = 56)

Area model:

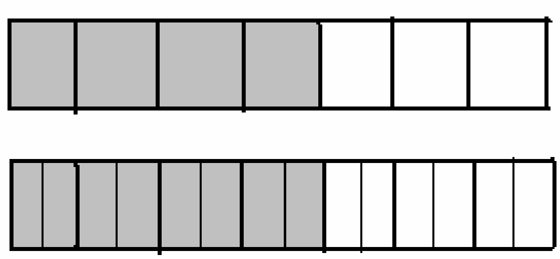


Figure 5. Teacher F's representation before professional development.

Teacher I is a 6th grade mathematics teacher with 27 years teaching experience and a field concentration of elementary education. Her knowledge of mathematical representations before the workshop was accurate and comprehensible, but with a lack of variety. Although she tried to use a variety of mathematical representations, some forms of representations were not accurate. The following are her representations on pre-questionnaire for the given problem: 0.75 = 6/8 and 4/7 = 8/14 (see Figure 6).

1	25	25/100	¼
2	50	50/100	2/4 = ½
3	75	75/100	3/4

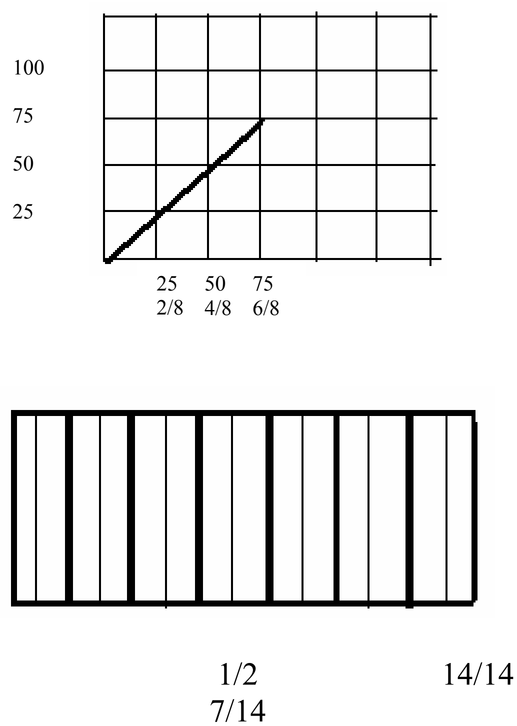


Figure 6. Teacher I's representation before professional development.

In answering equivalent fraction of 0.75 = 6/8, she used a grid to represent the 100, and then shaded 75 to represent 75/100; however, she did not state that 75/100 = 0.75. She used the proportion to represent the equivalent fractions: $\frac{75}{100} = \frac{3}{4}$ and $\frac{6 \div 2}{8 \div 2} = \frac{3}{4}$. She also used the table to represent the given question clearly. However, when she tried to use graphs, the answer was not clear.

In answering equivalent fraction of 4/7 = 8/14, she did accurately follow the procedure by using cross multiplication and proportion methods: $\frac{4}{7} = \frac{8}{14} \rightarrow$ cross multiplication 4 x 14 = 7 x 8 → 56 = 56. In the other form, she represented it in another proportion: $\frac{14}{14} \text{ x } \frac{4}{7} = \frac{56}{98}$ and $\frac{7}{7} \text{ x } \frac{8}{14} = \frac{56}{98}$. However, when she tried to a use rectangle area representation, it was not clear. Although she marked a few fractions on the rectangle, there was no connection between these fractions to address given equivalent fractions.

In Zone 2, teachers' answers indicated that they use area models of rectangles and circles, tables, cross multiplication, ratios, and grid papers accurately and comprehensively. However, their representations were not multiple, and can only reach part of their students' understanding of mathematical ideas. From the workshop, they all learned to use the strategies of multiple mathematical representations to improve their way of teaching as one teacher, who reflected during the interview said, "I learned problem solving by using a different way[s] to solve it, like graphs, flash cards, pictures, and tables. This gives me a different way[s] to teach the class." These teachers made progress toward a higher level zone.

Changes of Teachers' Learning Representation in Zone 3

Teachers in Zone 3 demonstrated that their knowledge of mathematical representation is accurate and comprehensible, with variety and connection. Their knowledge of mathematical representations showed that using representations not only directs students' conceptual and procedural understanding but also reaches students with different learning abilities.

Teacher K is a 7^{th} grade mathematics teacher with five years of teaching experience and a field concentration in secondary mathematics education. In responding to the given question in the pre-questionnaire, she used every possible representation she could to demonstrate the important mathematical ideas. She used symbolic equations, tables, number lines, and graphs to describe the problem situations and her mathematical representations are accurate and comprehensible with variety, which could promote learning mathematics for students with diverse backgrounds and interests. In her reflection on the workshop, she pointed out particularly how NCTM standards that the workshop covered helped her, "I really enjoyed seeing the NCTM web site. I did not realize it had so many good things and lesson type things on it." During the interview, she especially addressed the fact that mathematical representation helped her to reach economically disadvantaged kids in her teaching area:

> Kids in here are different; it [mathematical representation] gives me an idea how to approach [mathematical ideas for] different kids. We have very high level poverty here; we need to be aware how these kids understand math. I have

> probably 96% socially economically disadvantaged students. MSMP provides information in each different environment to teach effectively.

Teacher C is a 6th grade mathematics teacher with four years of teaching experience and a field concentration in elementary education. Before the workshop, she not only understood the importance of multiple mathematical representations but also used different representations according to students' needs. She stated the following during the interview:

> I remember we discussed students' misconceptions during professional development, I think this was the best part for me in the professional development. If we can be aware of this problem in the classroom, it would be very beneficial to the teaching. Professional development staffs have talked about students' misconceptions, but did not go further (because of limited time in the make-up session). I think it would be helpful if they can go further, especially for the new teachers as well as in undergraduate and graduate courses. Misconceptions help you to understand where you will go in terms of instruction.

Teacher C's comments indicated that teachers in Zone 3 needed more challenges in professional development in order to fully discover their potential and go beyond their current knowledge.

In summary, although every teacher in this study had different levels of TZPD before the workshop, they made a move to reach their fully potential after engaging in a workshop program, in which teachers experienced interpsychological function by interacting with and learning from other teachers and capable others. From watching their own video lessons and reflecting on their teaching, teachers experienced intrapsychological function that helped them to arrive at new zones.

Teachers' Change in Classroom Instruction

The results for teachers' use of mathematical representations in classroom teaching showed that eight out of 11 teachers moved from lower levels to higher levels or remained in the higher levels. Only three out of 11 teachers did not move in the similar direction as their learning path.

Table 8 compares the different levels of TZPD related to teachers' knowledge in using representations in classroom teaching before and after the workshop, and indicates its changes in zones.

Table 8
Changes of Teachers' Using Representations in Classroom Teaching

Teachers	A	B	C	D	E	F	G	H	I	J	K
Zone level before workshop	2	1	3	2	2	2	1	1	2	1	2
Zone level after workshop	3	1	3+	3	3	2	2	2	3	2	2
Change in using (Zone)	Y	N	Y	Y	Y	N	Y	Y	Y	Y	N

Note: Letters A to K represent eleven teachers. Numerals in rows indicate three levels of *Teachers' Zone of Proximal Development* (TZPD): Zone 1, Zone 2, and Zone 3. Y: Learning zone changed; N: Learning zone did not change.

Table 9 shows the number of movement changes in teachers' knowledge in using representations in classroom teaching. Eight out of 11 teachers made a move from a lower to a higher or stayed at high level: three teachers moved from Zone 1 to Zone 2, five teachers progressed from Zone 2 to Zone 3, and one of them maintained high level 3, which is the highest level in this study.

Table 9
Movement of Change for Teachers' Using Representations

Teachers	G	H	J	A	D	E	I	C	B	F	K
Movement of change in using Zone	1-2	1-2	1-2	2-3	2-3	2-3	2-3	3-3+	1-1	2-2	2-2
Number of Teachers in Change	3			5					3		

Note: Letters A to K represent eleven teachers. Numerals in row 2 indicate the movement of changes in three levels of *Teachers' Zone of Proximal Development* (TZPD). Numerals in row 3 address the numbers of teachers changing their learning zones

The Changes of Teachers' Using Representation in Zone 1

Teachers' instruction in Zone 1 basically involved accurate but not comprehensible use of mathematical representation. These kinds of representations result in difficulties for students in learning mathematics with understanding most of the time, especially for students with lower learning abilities.

Teacher B used the same CMP textbook to teach fractions, percents, and decimals before and after the workshop. Her teaching style indicated that she followed a procedure and used memorization to teach the lesson. Although the textbook required students to prove their reasons, she usually gave the students answers without their exploration. The following was an example of her classroom teaching on comparing fractions using fraction strips:

> Problem: At the end of the fourth day of their fund-raising campaign, the teachers at Thurgood Marshall School had raised $270 of the $360 they needed to reach their goal. Three of the teachers got into a debate about how they would report their progress.
>
> - Ms. Mendoza wanted to announce that the teachers had made it three fourths of the way to their goal.
> - Mr. Park said that six eighths was a better description.

- Ms. Christos suggested that two thirds was really the simplest way to describe the teachers' progress.

A. Which of the three teachers do you agree with? Why?
B. How could the teacher you agreed with in part A prove his or her case?
(CMP, 2002, Pits and Pieces I, Investigation 2.1)

During students discovering the questions, teacher B gave them the answers without students' exploration.

Teacher: Ms. Mendoza is correct. Not only is Ms. Mendoza correct, Mr. Park is correct [also]. ¾ and 6/8 are equivalent [at this time, the term of equivalent fraction had not be introduced yet]. Have you looked at your ruler? You can see equivalence. If you fold your 4^{th} fraction strip, you will see the 8^{th}.

At this time, one student did not get it. Teacher B worked with him and explained to him what should be correct.

Teacher: Ms. Mendoza and Mr. Park are correct. Ms. Christos is not correct because 270 and 360 do not match 2/3.

In this class students were not motivated to learn mathematics because they did not follow her direction for understanding.

Teacher: You don't have energy. Do we need to stretch?

The following conversation between teacher B and one student indicated that her instruction in word mathematical representation without any visual part did not make sense for this student.

Teacher: Now, let's look at Ms. Christos, we look at $360 is our total, and we want to divide $360 into what [without students' answer]? Three equal parts.
[Student was silent].
Teacher: Okay Now, what about $270? Ms. Christos said what? She said 2/3 is really the simplest way to describe the progress they were already made. 2/3?
[Student was silent and the teacher B asked the student].
Teacher: What do you think Ms. Christos is coming out?
[Student was silent].
Teacher: $90. Now see how many parts.
Student 1: One [mistakenly].

Teacher: There are four parts. OK. Now how many parts are for Christos?
Student 1: …
Teacher: See right on the board, and I have it: two thirds. OK. You've got one third, what would be another?
Student 1: …
Teacher: $90. Another one third, so it would be two thirds. This would be what? How much 90 and 90?
Student 1: 90 and 90?
Teacher: No. 90 plus 90.
Student 1: 190.
Teacher: No. one hundred what? 180, OK. Three thirds would be what?
Student 1: Three thirds would be what? Are you going to put another 90?
Teacher: How much it would be?
Student 1: ... 270.
Teacher: Do we make 270 a weight by two thirds?
Student 1: Yes.
Teacher: No, No, No. We got three thirds when we get 270. So can we say that we can use two thirds to describe how much we sold so far?
Student 1: No.
Teacher: No. We can't because basically according to Ms. Christos, we only sold how much?
Student 1: 270.
Teacher: No. How much? Two thirds would be what?
Student 1: 180
Teacher: $180, which is not we sold already. And we sold how much?
Student 1: 270.
Teacher: 270. So instead of use 2/3, Ms. Christos should use what?
Student 1: …
Teacher: What bring us to 270?
Student 1: 3? 4?
Teacher: No. We are looking at thirds right now. Three fourths is correct.

Although teacher B had learned new strategies from the workshop, her teaching did not change. For instance, in teaching equivalent fractions, there was no clear learning goals in teacher B's class. The representation that she mainly used in the classroom was mathematical procedure, and she also asked students to memorize the procedures (e.g., moving left two places to make changes from a percent to a decimal) to solve the problem. Although she did give students opportunities to explore the questions that the textbook provided, many times she simply gave the answers to the students. Students' reactions from her class showed that many of them had a hard time understanding the mathematical concepts. When students had difficulties, teacher B simply helped them to recall something they have been asked previously. In teaching equivalent fractions, the important mathematical concept is to understand the same size and same shape when comparing the fractions. In order to understand this concept, students are required to construct a new fraction strip by using original fraction strips that have different sizes. Her teaching showed that she gave the students answers without students' exploration leading to their own findings.

Although the above conversation was only with one student, the whole class was lost at this point. Without appropriate mathematical representation, it is hard for students to understand the mathematical concept beyond the computation. For teacher B's abstract representation, some students may understand, though evidently many students in this classroom were lost. When teaching converting percents and decimals, she asked students to memorize moving decimal places instead of helping them to gain conceptual understanding.

Teacher: Percent is how many places behind a decimal. In order to get rid of a decimal, what I am going to do is to move two places, and then add a percent sign. Alright, let's look at that chart here. For $2000 on up, 18% of the people are willing to pay, which means 18 out of 100 and .18 in decimal form. To change from percent into a decimal you drop the percent sign and move the decimal point two positions to the left.

[…]

Teacher: When we are given a fraction and asked to turn that into a decimal or a percent, what must we do? You divide!

Although she clearly stated the procedure for converting percents and decimals, students were confused in many places. For instance, teacher B asked one student:

Teacher: Are you sure this is 30 out of hundred? This is three hundredth, not 30 hundredth. So you move the decimal two places to the right. Then you get 3%, not 30%.

Although in the rest of the lesson, the students followed the procedure to fill in the blank that converts between fractions, decimals, and percents, the students' understanding of these concepts was questionable.

In the second year, teacher B taught the same lesson after she had gained the knowledge of mathematical representations; however, her teaching style has not changed. In teaching comparing fractions, although she gave students more opportunities to explore the mathematics ideas in the group discussion, she easily gave students answers without their proof. The following was the same lesson content as she taught in the previous year:

Teacher: Let's look at the question again. Mr. Mendoza and Mr. Park are correct because ¾ of \$360 and 6/8 of \$360 are both equal to \$270. Okay. Ms. Christos is incorrect because 2/3 of \$360 is only \$240 on meeting their goal. … Now how can you prove your answers? Alright, if you divide \$360 into four equal parts, that is 360 ÷ 4, each part equals to what?
Students: 90.
Teacher: \$90 for each part. If we take three parts, we get \$270. For Mr. Park, if we divide 360 into eight parts, we get 45 for each part. We take six parts, we also get \$270.

Although she did address the calculation correctly, she missed the important mathematics concept of using the same size and same shape area to compare the fractions. In the abstract representation forms, the observations on both video lessons before and after the workshop showed teacher B hasn't made progress in using representations in her teaching. Therefore her TZPD stayed at the same level.

Teachers G and H are teaching at the same grade level in the same school and have the same path to teach. Their teaching approaches are similar and their teaching changes also followed a similar pattern. Before the workshop, their teaching approaches followed the patterns of warm-up, teaching new lesson, and student practice. The warm up normally included a few computational problems for students to practice; the teaching of a new lesson normally followed the textbook (MT) to do almost exactly the same thing; and the students' practice usually took about 20 minutes. Since their teaching followed in content and steps almost exactly as the textbook, the representations were from the book, which focused on much more manipulatives by using pattern blocks to investigate the equivalent fractions and other mathematical concepts. Although students had the chance to use manipulatives, their role was to follow the teachers' direction, in which their understanding or misunderstanding was covert. As a result, many students' responses to questions reflected that they do have many problems during their class-work practices. The following lesson from teacher G was a very typical example for both teachers.

1. Warm-up: change improper fractions to mixed numbers: 6/5, 17/4, 10/5, 25/6, 18/6, 13/9, and 23/7. The approach she taught students was to use the long division of numerators and denominators, and then to follow the procedure to remember to put the remainder as a new numerator. For instance, to change 23/7 to a mixed number, teacher G divided 23 by 7 and got remainder 2, and then she wrote the result as $3\frac{2}{7}$.

2. Teaching a new lesson: making equivalent fractions. To follow the instruction from the book, teacher G passed pattern blocks to the students and directed students to make changes between hexagons, rhombuses, and triangles in order for students to understand the equivalent fractions. For instance, a certain hexagon was partly covered by trapezoids, and the fraction was 6/8, what fraction would it be if the same picture was covered by triangles or rhombuses? Although students had the chance to use manipulatives, they did not get a chance to discover the concept behind the manipulatives. The teacher led students in discussions, but most of the time she did the work. The students copied the teacher's work in order to follow her instructions. The main representation that she used was what she called the butterfly approach (cross multiplication) to check the equivalent fractions.

3. Student practice in the classroom: after teaching a new lesson, teacher G gave students the class-work (extended as homework). She did a few of them with students. The lack of comprehensibility and variety of mathematical representations caused difficulties for some students in understanding the concept of equivalent fractions in this classroom. In the class discussion, only a few students were involved.

Teacher H had almost the same approach as teacher G except she tried to use more questions to challenge students' proofs. However, without multiple mathematical representation skills, students' proofs were limited to cross multiplication (butterfly approach). Although they used manipulatives, the connection between concrete and abstract was missing in the teaching.

After the workshop, their teachers' approaches changed in terms of using mathematical representations when they taught the same lesson and used the same textbook. Although they did not improve the warm-up part, they did use some concrete visual forms to represent fractions in the new lesson. For instance, teacher H asked many more students to come to the front to show their understanding of equivalent fractions by using manipulatives. The important part was that she asked students to represent equivalent fractions and prove their answers beyond using manipulatives. The following conversation between teacher H and her students indicated the proof having different representations:

Teacher: The first time we did when I used the blue color [pattern blocks], and it comes to 6/12; the second time I did use green color, and it comes out 12/24; what can you tell me about these two fractions? Are they equivalent? How do I know that?
Student 1: Because 6/12 equals one half and 12/24 also equals to one half.
Teacher: She's telling me that 6/12 if I reduced it equal one half. If I reduce 12/24, it also equals one half. How do I prove it using my pattern blocks that it is the same? What do I do to prove it?
Student 2: When we use the six blue and 12 green, they are covering the same area.
Teacher: Yes. We are just using the different colors. She is right they are all half, and 6/12 are equivalent to 12/24. Give me another fraction that would be equivalent to these fractions.
Student 3: 25/50
Teacher: Another one.
Student 4: 50/100
Teacher: Give me another one.
Student 5: 2/4
Teacher: Another one.
Student 6: 3/6.
Teacher: We can go on and on, and all these fractions, we call equivalent fractions because they are the same. If we reduce them, they all come to one half.

In teacher G's class, she also used picture representations to bridge the conceptual understanding for equivalent fractions. Teacher G used trapezoids, parallelograms, and triangles to represent the same picture in the given example, and discussed the difference representations with students one by one.

Teacher J, a 7th grade mathematics teacher with three years of teaching experience and a field concentration in elementary education in reading, did not know much about teaching strategies using mathematical representations before the workshop. Her typical teaching style was to almost completely follow the textbook (MAC). For instance, in order to address the topic of solving equations such as X + 3 = 5, she used the "zero pairs" approach that was introduced in the textbook and some manipulatives to ask students to follow the procedure. Her lesson indicated that her approach did not reach students with diverse backgrounds. The abstract representations in the text did not help her to explain clearly why one side needs to be taken off and moved to another in order to figure out what X is. After her lecture and during students' working on problems, several students, especially minority students, had difficulties doing work on their own. Evidently, her approach did not reach the group of students who might have lower learning ability. Similar to other teachers who went from Zone 1 to Zone 2, Teacher J was able to follow ideas in the textbook to use some representations, but they were abstract with little variety, and not comprehensible to the students. After the workshop, teacher J used many more students' representations to direct her teaching by calling students to come forth with their mathematical ideas, though her teaching is still driven by the textbook.

Changes of Teachers' Using Representation in Zone 2

Teachers in Zone 2 showed that they used mathematical representations in their instruction accurately and comprehensively, but with a lack of variety. Teachers' instruction in this zone can only reach part of students' learning with understanding. However, with the workshop that focused on specific strategies of representations, four teachers made moves in their TZPD toward to the next level, and two teachers did not make any moves in Zone 2.

Teacher A, a 6th grade mathematics teacher with 15 years of teaching experience and a field concentration in elementary education in science, provided students with opportunities to prove their mathematical ideas using fraction strips. From student representations, he knew much more about students' mathematical thinking. The following conversation was an example from the lesson of using fraction strips to compare fractions (CMP) before the workshop.

Teacher: We not only need the answers, but also need you to prove it.
[Students discussed as groups]
Teacher: Which teacher do you agree with? Let's take a vote.
[Students voted]
Teacher: Adam.
Adam: Because ¾ and 6/8 are the same thing, all you have to do is to multiply it by two.
Teacher: Let's not use multiplication, instead use your strips. How [can] you show a four-year-old kid that you are correct?
Adam: In first one, 3 of 4 are shaded.
Teacher: You are saying using 4th strip folded, will it show you?
[The teacher demonstrated this on the transparency]
Teacher: You prove this. How about the next one?
Adam: If you take 8th strip, you divide into eight parts; it shows you 6/8.
Teacher: You see both of these did not work for 2/3. Name another strip that works.
Billy: 9/12.
Teacher: You took the 12th strip, and you will find it also works.

Teacher A found one student's mistakes and tried to correct it for the whole class.

Teacher: What is wrong with this fraction? Alright, Billy looked at the 12^{th} strip, but he had 13^{th} strips. What he did is using 13^{th} strip and fold one hided and made it the 12^{th}. What is the problem?
Student 1: Because one of them is too short.
Teacher: The total thing has to be the same [the teacher compared and concluded they are not the same]. 1/3 and 4/12. Are they equal?
Students: Yes.
Teacher: If you took the 3^{rd} strip and the 12^{th} strip, what do you find out?
Students: They are the same

Although teacher A did not use comprehensible representation very often in his teaching, he did use student representations to address the ideas of comparing fractions. However, after the workshop, he taught the same lesson in a different class and made it much more of a challenge for students to represent their mathematical ideas. The following conversation was from his lesson of comparing fractions after the workshop.

Teacher: Last lesson we talked about fund raising and the goal is $360 and they got $270. Ms. Mendoza said ¾, Mr. Park said 6/8, and Ms. Christos said 2/3. Look at the questions. What I would like you to do is to discuss this, and then each group reports your findings. I need you to answer why?
Teacher: I heard several comments: fraction strips don't fit.
Student 1: They are too long.
Teacher: What is goal? What is the whole?
Students: 360.
Teacher: What could you do?
Students: ...
Teacher: Could you make your own strip?
Students: …

Teacher A challenged students to prove their answers.

Teacher: How many of you said that Ms. Mendoza was correct. Charley, explain your group's reason.
Charley: Ms. Mendoza and Mr. Park are all correct and Ms. Christos is wrong.
Teacher: If you announce the result, what do you agree with?
Charley: Ms. Mendoza.

Teacher: Why do you agree with her?
Charley: ¾ makes more sense.
[The teacher measures it on the board]
Teacher: If you use the piece of paper, the fraction strips go this far [more than 360]. Will this give you a different answer? It could be.
Charley: We divide 360 by 4, we get 90, then we use 90 times 3, we get 270, therefore we say ¾ is correct.
Teacher: They try to do it mathematically. What about you?

Teacher A called on another student (Daniel) and addressed his mis-understanding.

Daniel: Ms. Christos is correct [wrong].
Teacher: Why do you select Ms. Christos?
Daniel: We put every answer ¾, 6/8, and 2/3. We know ¾ = 6/8, so we select 2/3.
Teacher: So you take 2/3 by default. Your group?
[Teacher A called on another group]
Group: We use strips.

Teacher A used fraction strips to address the mathematical concept beyond fraction strips.

Teacher: Right now, I am not saying which group is right and wrong. Let's look at question b and then we can come back to this. How can you prove your answer? A lot of times in your lives you are expected to prove your reasoning.
Adam: 2/3 is not working because the empty one is less than 1/3.
[Adam pointed the fraction strips].
Ashley: We agree with Ms. Mendoza because we can make the strips and to measure it.
Edward: We switch to Ms. Mendoza.
Teacher: How important is that fraction strip fit on there?
Students: Important.
Teacher: Why?
Students: …
Teacher: You talked about whole. What is whole in this case?
Students: 360
Student 1: You can also use your fingers to measure it.
Teacher: What fraction we can use to represent their progress? What is difference between 4^{th}, 8^{th}, and 12^{th}?
Teacher: The whole is the same but the piece is different.

In this lesson, students were very excited about fractions and raised some questions about fractions. Teacher A pointed out students' misconceptions by looking at patterns and made a simple way to compare the fractions.

Teacher D, a 7th grade mathematics teacher with three years of teaching experience and a field concentration in elementary education in mathematics, taught writing expressions and equations in algebra. She used the ten blocks as a concrete model to represent the algebraic equation, and then she used the letters to represent the same equation in a formal way.

Teacher: What is the equation?
Student 1: It is math sentence, number of sentence.
Teacher: How do you know?
Teacher: What is our goal for solving equation?
Student 2: Solving variables.
Teacher: We will do drawing and mathematics part today.

Teacher D used the model to show the equation, and she used equations with a T table representation to show the two sides in the equation. She asked students to do the practice and asked them to remember the "zero pairs", which is a typical procedure that MAC textbook introduced in solving algebraic equations. She then called two pairs of students to come to the front and show the work of solving $2 + p = 5$. Students copy the procedure from her and explained the zero pairs. Since teacher D followed the textbook for her instruction and the representation from textbook is too abstract, some students did have problems to understanding "zero pairs." After the workshop, her lesson in representations had more variety by using graphs and other forms which made students understand better. The following example is from her lesson after the workshop to show the concept of proportional linear equations:

Teacher: 10g = h, what does the graph looks like?

She called one student to come up to the front to show the graph by drawing a linear line in the square box.

Teacher: How about 2n + 10 = h?

She called students to the front to draw the graph, but the students made mistakes in drawing the curve. She called several students to come to the front to make their representations, and she explained students' different representations, including mis-represented mathematical ideas. She used students' representations to make clear linear equations until most of the students understood the mathematical ideas.

Teacher E, a 7th grade mathematics teacher with eight years of teaching experience and a field concentration in secondary mathematics, taught algebra. She provided students with opportunities to represent algebraic ideas; however, the representations she used were still too abstract for most of her students because most of her students are Hispanic.

Teacher: How do we write equation from the table? Let's see the table

X	-2	-1	0	1	2
Y	1	0	-1	-2	-3

After teacher E corrected one student's representation x = -1 and y = 0 to x = 0 and y = -1 (she had introduced this approach in the last lesson).

Teacher: What does y equal?
[At this point, this student made -1 = 0 with teacher's help].
Teacher: Is this a true statement? We want to make a true statement and we need to put in some thing. What we need to put in? -1 on right side, so -1 = 0 -1
Teacher: We want to say y = x -1
[At this time, the student did not understand this representation].

Teacher: We have equation y = x -1. How do we get it? That is our best guess. Then we pick up another point.
Student 1: x = -1 and y = 0
Teacher: We put the value in. 0 = -1 -1. Is this a true statement?
[At this time, teacher E let students play with these numbers trying to make a true statement].

Teacher E's mathematical representations on equations were so abstract that they were not comprehensible to students. Furthermore, the guess and check approach made it even harder for students to understand the mathematical concept.

Teacher: Did any one get the correct answer?
Student 2: This is hard.

At this time, students did not know about slop, so it was hard for them to build the equation. Although they sometimes made a true mathematical statement, it did not make sense to them mathematically. Although the teacher made an effort to explain, students were totally confused at this time.

Unclear mathematical representations in teacher E's lesson made students confused in understanding how to generalize the question from the given data. Students did not have any ideas about the relationship among variables. Teacher E did not make the connection between the representations.

After the workshop, teacher E used multiple representations to help students understand the mathematical concept, and students made progress in understanding mathematical concepts. The following was an example of the lesson similar to previous one on equations.

[Two students demonstrated their work on the board using tables to find patterns of linear relationships and then filled in the missing numbers].
Teacher: What is the equation for the following table?

X = 1	2	3	5	7	9 …
Y = ½	1	1 ½	2 ½	3 ½	4 ½ …

Student 1: Y = ½ x or .5 x or x/2

Students knew that the easy way to write the equation is to choose Y = KX when X = 1.

Teacher: What if x = 22?
Student 1: Y = 11
Teacher: How about this table? What is the pattern? What is the equation?

X = 1	2	5	7…
Y = 5	10	25	35…

Student 2: Y = 5X

This class had almost no lecture because students are so diverse with lower ability; the teacher had to use different representations to work with students to address the patterns of changes and the variables with changes. Because students had difficulties in understanding variables, the teacher used proportional relationships to help students see the patterns of changes, which is in the form of Y = KX. It was clear that students made progress.

Teacher F's teaching followed almost the same pattern before and after the workshop: students had a warm-up with computational practice, and the teacher addressed the fraction ideas using money and pizza as examples. Although the teacher used manipulatives (e.g., pattern blocks), there was little connection between the manipulatives and conceptual understanding. Therefore, students were limited in concrete understanding.

Teacher: Let's think about six on your own to see what you know about six, and then share your ideas.
Student 1: Hexagon.
Student 2: Six months is half a year.
Teacher: I'd like you to make the left figure using pattern blocks [in MT book].

On red shape, the teacher drew the trapezoid, which equals three triangles.

Teacher: So 1/6 → 3/18 because total of triangles are 18 (6, 6, 6), and you can use 1/6 times (3/3) to get 3/18 also, and you can use cross multiplication to check your answers.

The teacher demonstrated the cross multiplying procedure. She also demonstrated the procedure on the board for three other similar questions. On ¾ =9/12, the teacher made the circle to show students 9 out of 12 on the circle. The teacher drew improper fractions on the board: 8/6 = 6/6 +2/6 and 16/12 = 12/12 + 4/12.

Teacher: Two ways you can use to check your answers: using the calculator and using cross multiplication.

The teacher showed the cross multiplying approach to prove the equivalent fractions.

Teacher: Last thing we are going to do is practice without using calculators, so close your book, do the work as homework. We need to find equivalent fractions: 12/18 = 4/6. How can we make the 12/18 to the 4/6?

After students made concrete models using pattern blocks, students worked on the questions to transfer them to the abstract form. However, the questions included improper fractions. She called students to come to the front to make the equivalent fractions. During the practice, the teacher showed the concrete model again to make sure students had this connection. Teacher F made sure that students used equal sizes to make equivalent fractions. For improper fractions, she drew the rectangle with six parts and then made another one to shade with two parts that she did the same way for 16/12. However, students did not get a chance to do that, so the teacher did not know whether the students understood or not. Another feature of her teaching is that she used the teacher-centered teaching approach, in which students did their work, but they did not have the opportunities to prove their work, even with other students, although students got enough time to practice.

Although teacher I's teaching appeared to be monotonous before the workshop, her teaching after the workshop was effectively changed by using representation strategies. The following is an example of her lesson before the workshop.

Teacher: Today we are going to talk about mixed numbers and improper fractions. Some day you might be a manager of a piece of property. When you manage property in cold place where it is snowing, and you use the salt to melt the snow. You have four ounces of salt. On the package, it says it will melt nine square feet ice. But you only have one ounce salt in the bag. How many square feet we are going to melt?

Students: Divide

Teacher: That sounds good to me. We divide 9/4? What do you know about this fraction? They are big on top and small on bottom. This is improper fraction: when the top number is greater than bottom number. We are going to use manipulatives to change improper fraction to mixed number and from mixed numbers to improper fractions.

The teacher put the hexagon and triangle transparency on the overhead projector. Each hexagon equals six triangles.

Teacher: How many triangles are equivalent to a hexagon?

The teacher called one student to the overhead to make the equivalent shapes using triangles and a hexagon.

Teacher: How many does it take him to make this?

Student 1: Six.
Teacher: We have problem in here because we have one more. How many sixth we have in here?
Student 1: Seven.
Teacher: 7/6. What fraction do we come up?
Student 1: Improper fraction.
Teacher: How can I change improper fraction to one whole and one left over?
Student 1: Divide.
Teacher: Seven divided by 6 equals to $1\frac{1}{6}$. Does this make sense to you? Let's do another shape using pattern blocks.

The teacher used hexagon and trapezoids (each hexagon equals to two trapezoids), and called one girl to come up to do it.

Teacher: How many does it take to make a hexagon?
Student 2: Two.
Teacher: We have three extra. Who can tell me how much we have? [One student comes to make two and half].
Student 3: We need to change this to numbers. It is 5/2. We divide 5 by 2 and get $2\frac{1}{2}$.
Teacher: Now you need to practice on this and you can work with your partners.

The teacher used a work-sheet that was filling blanks and including the picture and fractions. She explained the procedure again before students started to work. She was walking in the classroom to help students.

Teacher: Let's check your answers. What is your first step?
Teacher: Let's do the mixed number to improper fraction. Let's do $3\frac{2}{3}$. How can we go backward?

She put pattern blocks again on the overhead projector as students could visualize this.

Students: 3 x 3 + 2 = 11, so 11/3.

The teacher did not comment on student's methods, instead she said, *Teacher:* Let's try to use manipulatives

The teacher used pattern blocks and a calculator to show the mixed numbers and improper fractions.

Teacher: I will let some of you come up like teachers to see if you did it right. The first question is 1 2/5 change to improper fractions.

One student comes to the front and does 5 x 1 + 2 = 7 → 7/5 using manipulatives.

Teacher: How about 12/7?

Another student comes to the front and does 12 / 7 = $1\frac{5}{7}$ using manipulatives.

Teacher: Let's see if we can do this mentally. Change $3\frac{1}{2}$ to an improper fraction by multiplying and adding.

Student: 3 x 1 + 2 = 7, so 7/2.

The teacher made the connection between concrete and abstract understanding.

After the workshop, her lessons appeared to have much more mathematical representations. The following showed the example of her post lessons:

Teacher: Today we are going to learn about improper fractions and mixed numbers. First, I want to you to draw me the picture of $3\frac{1}{4}$. Draw me a model and what you think it should be.

This great problem challenged the students, and she checked students' understanding and called one student to come up to show the result on the board. One student drew three whole squares and ¼ square. It was really a good picture; actually he drew 13 pieces out of 4.

Teacher: What is the first thing he did here?

Students: Divide the number 4.
Teacher: How many 4^{th} do you see in his picture?
Students: 4, 4, 4, and 1. 13.
Teacher: We also can do another thing. Can you tell me what do you see here?
[Teacher has written 4/4 + 4/4 + 4/4 + ¼ = $3\frac{1}{4}$].
Student 1: 1 + 1 + 1 + ¼ = $3\frac{1}{4}$
Teacher: Is there anything I can do to come to $3\frac{1}{4}$?
Student 1: 4 x 3 + 1
Teacher: Good job.
Teacher: Now we pick up whole number and get a fraction, so we will have the mixed number. Now I want you to do the improper fraction and draw the model and show me the change to the mixed number using your model.

The teacher mentioned that each part in the whole has to be congruent. This really helped students understand the mathematical concept. By this time the teacher has the opportunity to walk around in the classroom to check every one's work. From the conversation, the teacher also helped students review the geometry by drawing figures, for instance, a pentagon. The teacher also corrected one student's figure because it was not congruent. She called one student to come to the front to demonstrate his work to change improper fraction 17/5 to mixed numbers. This student chose the rectangle and divided it into five equal parts, with one rectangle he shaded and wrote 5/5 = 1. He did same thing in the second, and third. He then counted pieces that were shaded until 17. For the final one, he did 2/5, and then he put a little "+" sign and wrote the conclusion of mixed number $3\frac{2}{5}$. This representation showed his conceptual understanding. At this time, the teacher made sure that students understood models with equal size, and knew that 17/5 and $3\frac{2}{5}$ are equivalent fractions.

She called one student to do another problem to change improper fraction 28/5 to mixed numbers. This student used the circle as a model, and tried to draw a congruent part. She got six circles and divided each circle into five parts. She shaded the first five circles, and shaded last one in three parts. She proved her answer by writing each circle as 5/5 = 1 and the last one as 3/5 and then wrote $5\frac{3}{5}$. At this time some students said she made mistakes. However, the teacher let her prove her answers. The student proved the solution by using one whole and models.

Teacher: Very good. I will use yours as an example. How can we make $5\frac{3}{5}$ to 28/5?

Students: 5 x 5 + 3.

After the students understood, the teacher got more questions just for computational practice. This was a very effective way to focus on both conceptual and procedure understanding. When the teacher led students in computing $\frac{28}{5}$ to get $5\frac{3}{5}$, she did challenge them by asking where five is coming from (denominator), which promoted students' rethinking again.

Teacher: What determines the denominator in the fraction?

When students got the answer, she confirmed that 5 is a whole. In this lesson, she did connect the math concepts to picture representations, symbolic representation and verbal representation. In the last few minutes, she called students to come to the board to present the result. One African American student drew a very interesting picture - Pentagon with squares and triangles. He did the same way as previous students to make it equivalent from 13/5 to $2\frac{3}{5}$ and then made them in symbolic form.

Teacher: How can we change the mixed number to a whole number?
Students: Divide
Teacher: 12/4
Students: 3.
Teacher: We also learned to change improper fractions to a mixed number $3\frac{1}{4}$.
Students: 13 divided by 4 is $3\frac{1}{4}$.

The above examples showed teacher I changes and growth in using mathematics representation in teaching improper fractions and mixed numbers.

Although teacher K's understanding of mathematical representations has remained at a high level, her using mathematical representations in instructions, however, did not change and remained at the middle level. Teacher K's instruction was driven by the textbook (MAC) almost completely, which is naturally abstract. The following example was from one of the post lessons.

Teacher: Okay. Today we are going to solve addition and subtraction equations. We have done part of them before. Let's see this X + 4 = 6. What we are going to do to solve this equation?

The teacher solves the equation using the procedure of "zero pairs" and gets the answer: X + 4 = 6 → X + 0 = 6 – 4 = 2 → X = 2.

Teacher: How do I check this? What we do is 2 + 4 = 6 → 6 = 6. Therefore, the other way we can do is using the model.

The teacher was modeling X + 4 = 6 by using one cup and four small circles inside, with a positive sign. She circled them together, and then made an equal sign to six small circles with positive sign inside on the right-hand side. She used the model of zero pairs to make one cup with four pairs of positive and negative circled with signs on the left-hand side, the same thing on right hand side, and then cancels the pairs. However, one student was confused with the procedure.

Student 1: I don't know this, and it confuses me.

The teacher went over the steps again and tried to convince students that this zero pair works.

Teacher: I try to get rid of this 4 in left hand side of X + 4 = 6, in order to zero this out, I use negative numbers and let X be alone.
Student 1: Where do you get the negative from?
Teacher: In order to get rid of the four positive.
Student 1: Oh, OK.
Teacher: Did you get that? Whatever I do on the left side I have to do it on the right side.
Teacher: Now we have X – 2 = 3. What we are going to do?
Student 1: You add.
Teacher: We add +2 in both sides, and X – 0 = 3 + 2 = 5, then we check it 5 – 2 = 3 → 3 = 3.

The teacher got a few more examples to have students practice using the same method.

Teacher: 12 ¾ + y = 321/8. Two ways you can do it: common denominators and decimals. What is common denominator?
Students: 8 → 4 x 2
Teacher: How do you know that it is 8?
Students: Because … factor,
Teacher: Thank you. Because factors 4 → 2, 2 and 8 → 2, 4, therefore, the common denominator is 8.

The teacher and students were solving this problem and got the answer $y = 19\frac{3}{8}$.

Teacher: Can we simplify it? 3 → 3 and 8 → 2, 2, 2. There is no common. So it is in its simplest form.

The conversation between the teacher and some students showed that teacher K used abstract procedures to teach solving equations, which cannot reach most of student's understanding. Teacher K needs to provide a variety of representations and achieve a connection between representations and procedures in order to make growth and move to the next level of TZPD.

Changes of Teachers' Using Representation in Zone 3

Teachers in Zone 3 usually use mathematical representation in classroom teaching accurately and comprehensibly with a variety of forms to represent the mathematical ideas according to students' needs. Teachers are able to apply the new knowledge learned from the workshop in their classroom teaching practice.

There was only one teacher, teacher C, in Zone 3. Although she has a higher level of ZPTD, she still made progress in teaching after the workshop. Teacher C was able to connect the prior knowledge and new lesson in her teaching, and she used the textbook only as a reference. In her classroom teaching, the learning goals were clearly stated, and the interaction between her and her students indicated active learning and teaching in her classroom. From students' informal representations, she understood what their weaknesses and strengths were for their understanding mathematical concepts. In the equivalent fraction lesson, she provided students with opportunities to discover the mathematical ideas, and then let them present in the classroom. Although students' representations of mathematical ideas appear to have mistakes, her correction of these mistakes benefited the whole class.   For low ability students, she not only encouraged them to use different representations to understand the mathematics concept but to also connect these informal representations to their conceptual understanding. In her class, students used different examples that related to their life experiences and made mathematical representations meaningful. Letting students prove the mathematical ideas by using mathematical representations was another one of her teaching strategies. The following conversation between her and her students indicated that her lesson promoted student learning with understanding.

Teacher: I need for you to get out the work package. You do not need fraction strips today.
Teacher: Open the book to page 19 [CMP]. The title is comparing fractions. Can someone read the paragraph for me?
[One boy reads, and another student reads]
Teacher: What does this tell us? What are problems they are having? You guys' job is to tell me which teacher you think is correct. You need to be able to prove why their way is the best way. I will give you 5 minutes on your own to answer questions A and B on page 19, then you will have another 5 minutes to talk to your partners about it: What is similar and what is different? Then we will talk about this in class and you can present your ideas.

Students worked either on their own or together. The teacher walked around to help students, and tried to find students' difficulties and asked students questions.

Teacher: Right now we are going to get your ideas. Raise your hand.
Student 1: I vote on Ms. Mendoza because she is right. Mr. Park is also right.
Teacher: Hold down. How many have you agreed with Ms. Mendoza?

She noticed only one student disagreeing, and she called on her.

Teacher: Kathy, who did you agree with?
Kathy: Ms. Christos.
Teacher: I want to know your reason for that. [To all class] How about Mr. Park.? Nobody agrees with him?
[Some students raise hands]
Teacher: How many of you agree with both of them [Ms. Mendoza and Mr. Park]?
[She counts 25 out of 29]
Teacher: I'd like to hear from you why you agree with them. Who would like to share your idea to just pick Ms. Mendoza?
Student 1: I choose Ms. Mendoza because …

[The student has difficulties to prove it].
Teacher: How do you figure out what fractions they have gotten?
Student 1: I used fraction strips.
Teacher: Why don't you show us what you have down?

The teacher wanted to see the student's proof because she liked to see students comparing two fractions of different sizes. The student showed the fraction strips on the overhead projector. He used 12^{ths}, but the new fraction strip is smaller and covered 8 pieces of 12^{ths}. Clearly the gray color covered six pieces of new fraction strip. So he said that it is 6/8.

Teacher: Now I need some one to come here explain both Mr. Park and Ms. Mendoza, Jenny?
[Jenny wrote on the board: 270 and 360 divided by 10 and got 27 and 36. She put 27/36, then factored 27: 1, 3, 9, and 27 and factored 36: 1, 2, 3, 4, 6, 9, 12, 18, and 36. → the common factor is 9 → 27/9 = 3 and 36/9 = 4 → ¾].
Teacher: I'd like one more person to show Mr. Park and Ms. Mendoza being equally correct. Sam?
[Sam divided 360 by 4 = 90, and then he used 90 x 3 = 270].
Teacher: What happens to Mr. Park's?
[At this time, the student were not clear].
Teacher: What happens to Ms. Christos?
[Sam divided 360 by 3 and got 120, and then he multiplied120 x 2 = 240].
Sam: That's wrong because we try to get 270.
Teacher: Nice job!

The teacher went back to the one student who disagreed at the very beginning.

Teacher: Tanya, I know you have changed your mind, but if you want to show us your thought. Is it OK?
Tanya: I can say it.

The teacher listened very carefully. The difficulty that this student had was exact fraction size concept because she looked at the whole fraction strip in the book, which has different sizes compared to her fraction strips, and then decided 270 is 2/3 as Ms.

Christos did. Teacher C led students prove their reasoning and understand the mathematical concept behind the proof.

Patterns of Teacher Change in Learning and Using Representations

The results of this study show that teacher change in use of mathematical representations in instruction does not completely follow the same pattern as their learning of mathematical representations. The interviews with teachers furnished more information on the patterns of changes, and teachers' responses are summarized in Table 10.

Summary of Teachers' Change in Learning and Using Mathematical Representations

Table 10 summarizes the changes in teachers' learning and using mathematical representations before and after the MSMP professional development. In addition, Table 10 shows teachers' educational and teaching experience background and indicates key factors that impact the relationship of change. Although all eleven teachers made a change in their learning of mathematical representations through the MSMP professional development workshop, eight of them made a change in their use of mathematical representations and only three teachers did not make a change in using representation; therefore, their change patterns between learning and using mathematics representations are different. Teachers' instructional changes do not follow a similar path as their learning changes.

Table 10 also shows the key factors that contributed to teachers' changes in this study. For teachers who made changes in learning and using representations, the following aspects were the key factors: to use new strategies learned from the workshop to reach students, to further understand students' thinking, and to interact with students. In addition, using a textbook with new beliefs and new strategies learned from the workshop about teaching and using representation is also an important factor. For teachers who did not make a move in their TZPD in using representations in their teaching, there were mainly two reasons: teachers had internal resistance to change because of previous teaching experience; teachers changes were affected by external factors because of the isolated school location led teaching driven by a low quality textbook. The examination of the relationship between teacher learning changes and instructional changes indicates the following patterns.

Table 10

The Relationship between Teachers' Change in Learning and Using Mathematical Representations before and after the MSMP Professional Development Workshop

Teacher	**Change in learning**	**Change in use**	**Change in Both**	**Education Background**	**Teaching experience**	**Relationship between learning and using representation**	**Key factor impacting the relationship**
A	2-3	2-3	+ +	Elementary in science	15yrs	Learning and use both improved to a high level	Using new strategies to reach students
B	1-2	1-1-	+ -	Elementary Ed in English and mathematics	15 yrs	Learning improved to a middle level, but use remained low	Previous experience in teaching
C	3-3	3-3+	+ +	Elementary Ed in general	4 yrs	Learning remained at high levels and using moved even higher	Using new strategies to further understand students' thinking
D	2-3	2-3	+ +	Elementary Ed in general	3 yrs	Learning and use both improved to a high level	Using text book with new strategies
E	2-3	2-3	+ +	Secondary math	8 yrs	Learning and use both improved to a high level	Using new strategies to interact with students
F	2-3	2-2	+ -	Elementary Ed in reading and Special Ed	25 yrs	Learning improved to a high level, but use did not change	Previous experience in teaching
G	1-2	1-2	+ +			Learning and use improved to a middle level	Using text book with new strategies
H	1-2	1-2	+ +	Elementary Ed in math	7 yrs	Learning and use improved to a middle level	Using textbook with new strategies
I	2-3	2-3	+ +	Elementary Ed in general	27 yrs	Learning and use both improved to a high level	Beliefs about teaching and using new strategies to reach students
J	1-2	1-2	+ +	Elementary Ed in reading	3 yrs	Learning and use improved to a middle level	Teaching driven by low quality textbook
K	3-3	2-2	+ -	Secondary math	5 yrs	Learning remained at a high level, but use did not change	Teaching driven by low quality textbook

Notes: In the Change in Learning column, the two numbers indicate the movement of teachers' knowledge in learning mathematical representation before and after the workshop. In the Change in Use column, the two numbers indicate the movement of teachers' use of mathematical representations in classroom teaching before and after the workshop.

Influence of Teachers' Diverse Backgrounds

The examination of the patterns of change in this study revealed the relationship between teachers' background and classroom practice. Teachers in this study have diverse backgrounds that contribute to their understanding and using mathematical representations. Two out of three teachers who did not move to a higher level in using representations in their instruction have relatively long teaching experience. For example, teacher B has had 15 years of teaching experience and teacher F has had 25 years of teaching experience. Even though they have acquired new knowledge about mathematical representation, and their knowledge of mathematical representation in learning has moved to higher levels, and have good beliefs about mathematical representations, their long-time teaching experience still drives them to keep the old teaching approach, and their instructional approaches do not change at all. However, this pattern does not apply to other teachers who did move to a higher level in using representation. This study finds that the location of the school and the opportunity to engage in social interaction are also important factors in teachers' change. For example, teacher K, who has had five years of teaching experience and a major in secondary mathematics education has relatively strong mathematical content knowledge, which helps her gain higher level of knowledge in mathematical representations. However, the school district she worked for was relatively small and isolated compared to the other school districts, which might have limited her opportunities for interacting with others and learning new teaching strategies. She did not know about NCTM and other standards until attending the workshop. Her comments from her reflection during the workshop explained why she barely tried the new approach to teach her classes, "I really enjoyed seeing the NCTM web site. I did not realize that it had so many good things and lesson type things on it."

The pattern of the longer time teaching and being more influenced by previous teaching experience does not apply to teacher I. Teacher I, who has had with 27 years of teaching experience and a major in elementary education, was always eager to try something new in her teaching, which contributed to her very positive disposition in using mathematics representation. In addition, she had teaching experience in 7th grade mathematics, which helped her become aware of the needs of 6th grade students in mathematics learning. She explained it during the interview, "I taught 7th grade so I know what they should know as 6th graders. They need to be able to explain something they have learned. For instance, what is 1/5 mean? It means 1 out of 5 students in our classroom in science class."

It is interesting to note that teachers with less experience tend to be driven by textbooks. For example, four (K, J, H, D) out of six teachers who had teaching experience of less than ten years heavily rely on textbooks in teaching. Among teachers leaning on textbooks, only 50% of them (H, D) used new strategies from the workshop, which provides suggestion to textbook and curriculum developers of paying more attention to integrating teaching strategies in textbooks, since these textbooks seriously influence teachers who are new and do not have much teaching experience.

Although teachers with less experience incline to be driven by textbooks, the results of this study show that 50% of teachers (five out of 10: C, D, E, H, and J) with less than 10 years of teaching experiences were able to use new strategies from the workshop in teaching practice, whereas 20% of teachers (two out of 10: A and I) with teaching experience of more than 10 years also applied new strategies in their teaching. However, among teachers (A, C, D, E, I) who have the highest level (Zone 3) of TZPD in using representations, three of them (C, D, E) have less than 10 years of teaching experiences, which indicates that teachers with less teaching experience more easily make a change in using representation and achieve higher levels of TZPD than teachers with longer teaching experience. In addition, those who have relatively high mathematics or science backgrounds appeared to easily move to higher levels in learning and instructions (A and E).

Impact of Teachers' Definitions about Mathematical Representation

The interviews with teachers furnished more information on the patterns of changes. During the interview, teachers were asked to define their own understanding about mathematical representations. Their responses indicated different levels of clarity of mathematical representations and their knowledge about mathematical representation.

Table 11
Teachers' Definitions of Mathematical Representation

Teachers	Definition of mathematical representation
A	Using any sensory means to represent mathematical concepts.
B	Mathematical representation is anything that you can use and apply to make sure that students have a good understanding of the concepts.
C	Using pictures, diagrams or expressions to help someone to understand the mathematical ideas.
D	Sometimes it is in words, sometimes it is in numbers, pictures, equations, so it depends on each student's use.
E	It should not be a matter of different forms: picture, symbolic form. Students should be able to use them in different situations and transfer them into other forms.
F	A picture that shows what we are talking about in the classroom.
H	Showing the why for the answer. Not just formula, show me mathematically with the picture, and something how you explain your answers. I guess show the WHY.
I	Representation can be a thought, it can be an expression in sentences, it can be put on paper, and it can be designed.
J	Any visual picture that you use will be a representation whether is a graph, or manipulatives, even a textbook.
K	It's varied. It can be a simple equation, it can be graph, chart. I try to use different forms so my kids can understand better.

Note: Teacher G's interview data is not available.

Table 11 shows that seven out of 10 teachers whose representation knowledge in learning and using representations moved to higher levels had clearer definitions about mathematical representations. Teacher A believed that representation is using any sensory means to represent the mathematical concept; teacher C regards mathematics representation as using pictures, diagrams or expressions to help students to understand the mathematics; teacher D knows the form of representations depended on each student, it can be in picture form, oral communication, or equation form; teacher E argues that "It should not be a matter of different forms: picture, symbolic form. Students should be able to use them in different situations and transfer them into other forms;" teacher H states that mathematics representation is to "Showing the why for the answer. Not just formula, show me mathematically with the picture, and something how you explain your answers. I guess show the WHY;" and teacher I believed a representation has multiple forms: "it can be a thought, an expression in sentences, and something that can be put on paper, it can be designed."

All seven teachers understood the specific goals of representation. For instance, teachers A, C, D, E, and I understood the role of mathematical representation. "It [mathematical representation] helps visualize and understand," the teacher A explained. Teacher C described the benefits and limitations of using mathematical representation:

> Representations explain many whys behind the mathematical ideas. So representations helps answer the most WHY questions. Limitations would be that some people rely on it too much, and they don't transfer into the calculations because they are not comfortable with the numbers. Representation is a tool, not a purpose, however many students use as purpose, which does not help them.

Teacher D supplemented her words about benefits of using mathematical representations, "[mathematical representations are able] to make connections for what students interested in learning. If they are not interested, you lost them. You just waste your energy."

Teachers' definitions of mathematical representation not only indicated their beliefs and understanding of mathematical representation but also indicated their ability in using mathematical representations in their instructions. Their disposition levels of mathematical representations showed the levels of their knowledge in mathematical representation. For example, although teacher J has changed representation level from 1 to 2, her view on mathematical representations was still limited to within textbook. She believed that the textbook is one of good representations; therefore, her instructions were driven by the textbook. “I think it is good for the memory tool because kids might remember the model or picture, I hope that even if they do not remember any thing, the picture will still help them.” She also realized that “Finding representation for each kid might be a limitation; one kid likes this way, and another kid likes that way. Representation is not everything.” In her instruction, she tried to use different representations, and finally follow the textbook representations to address the mathematics concept. Teacher J’s definition of mathematical representations indicated her beliefs about mathematical representations.

Teacher B’s definition of mathematical representations was not clear, which was reflected on her instruction also. Table 10 showed that her level of learning and using representation did not make any move after the workshop; it stayed at level 1. Look at the response from teacher B:

> Every thing I do for my students should represent my teaching. When I say math, it means what ever we are working on. Now we are doing geometry, I have to represent geometrical figures and formulas we are working with too, make sure my children have a good concept that are being taught. Mathematics representation, we use manipulatives, overhead, all these things represent what I am trying to teach. My representation is the lesson from the book. I think mathematical representation is anything that you can use and apply to make sure that students have a good understanding of the concept I am trying to teach.

Teacher F believed that “A picture that shows what we are talking about in classroom.” This view showed her representation from a narrow viewpoint; therefore, her level of using representations in the classroom did not increase after the workshop, which is still at level 2.

Teachers' Views on Benefits and Limitations of Using Representations

In answering interview question 3 about what the benefits and limitations of using representation in teaching mathematics are, all teachers recognized the benefits of using representation; even the teachers whose TZPD level did not make a move still made a positive comment. However, the following showed that teacher levels of TZPD correspond to the degrees of the benefits of using representation. Teachers A, C. D, E, and I had TZPDs in Zone 3 for using representations, they showed clear benefits of using representation learned from the workshop. For example, teacher D believed that using representations increases students' interest:

> To be able to make connection what students' interested in learning. If they are not interested in, you lost them [students]. You just waste your energy. It is hard to always to capture their interests. It is important to me because I don't think they are paying more attention if they are not interested.

Teacher D found that "Once they [students] are comfortable with one representation, it is hard to change them by using other representations. Sometimes it could be very hard to change to the abstract." To help students understand mathematics abstractly, "I always try to put the abstract next to the concrete in showing them how to connect concrete and abstract look likes," she said.

Teacher A believed that "representation helps visualize and understand," and teacher E agreed that it helps students with various learning abilities, "The benefits are different kids understand different representations." Teacher I looked further: "I always look at new ways to improve life. We need to help kids to make bigger picture about representations, this picture should help them with their lives."

Teacher C whose both TZPD levels were at Zone 3 had a different view: "Representations explain many whys behind mathematical ideas. So representations help answer the most WHY questions."

For teachers whose TZPD levels were in Zone 2, they were able to see some benefits, but their comments were not specific and did not see representations help students at the conceptual understanding level. For example, teacher H said, "They can apply to new situations." Teacher J believed that "It is good for memory tool because kids might remember model or picture. I hope even though they do not remember anything, the picture will still help them." Teacher F who's TZPD at level 2 in using representation did not make a move commented, "Learn more and understand more, and it is fun for having different representations."

With a TZPD level in using representation in Zone 1, Teacher B recognized that representations make learning easier, but commented, "I don't believe representation is completely universal. For instance, area for triangle, in whole world taught this as ½ base time height, but should we teach this in same way? My advanced students said that they will say 0.5 base times height. This is the different mathematical representation based on prior knowledge." Obviously, influenced by previous experience in teaching, her view of representation was still limited in abstract form.

All teachers in this study recognized that there are limitations when using representation in teaching, except teacher I and teacher B. First, the textbook or other teaching materials have some limitations on representations. "You limited by what materials you have," teacher A said. Second, numerical representation is sometimes ignored. Teacher E pointed out the importance of numerical representation:

> The limitations are less focusing on the computational skills. Sometimes we get the representation, we lose the numerical concentration. But numerical is a representation. A lot of times we think representations are the pictures, graphs, we ignore the numerical representations. Many people do not think computational skills are representation. But numerical representations are out of anything. Including NCTM standards, there are increasing use alternative representation, by extension, people just assumed that we need less numerical representations. The concepts are picture, graph, and numerical.

Third, representation is not purpose for leaning mathematics. Teacher C addressed the goal of representation as a tool, not the purpose: "Limitations would be that some people rely on representation too much, and they don't transfer their meaning into the calculations because they are not comfortable with the numbers. Representation is a tool, not a purpose, however many students use it as a purpose, which does not help them."

The above three limitations all came from teachers who had TZPDs in Zone 3. In addition, teachers in Zone 2 also had their view about the limitation of using representations: representations are not everything for every learner. Teacher H said, "It may not follow learning styles, and not every student is able to do it. Some students are not thinking that way." Teacher J agreed with teacher H, "Finding for representation for each kid might be a limitation, one kid likes this way, and another kid likes that way. Representation is not everything."

It is interesting to note that teacher I and teacher B believed that there is no any limitation in using representations. Teacher I's using mathematical representations is at Zone 3, and Teacher B's using mathematical representations is at Zone 1.

In summary, teachers' views about representations were consistent with their level of TZPD. Teachers in a higher zone in TZPD in using representation gained specific and clear benefits in teaching; they also could observe limitations in using representation; teachers in a lower zone in TZPD or who did not change zone in using representation only see general benefits of using representation in teaching, and were not able to tell the limitations of using representations. This also showed the positive relationship between learning and using representation.

Teachers' View on Making Accurate and Comprehensible Representations for Students

During the interviews, teachers were asked to answer question 4: What do you do to make sure representations are accurate in showing the ideas in learning goals (TEKS)? What do you do to assure representations are comprehensible to students?

Teachers provided a variety of views on making accurate and comprehensible representations for students. However, teachers' responses related to their levels of TZPDs in learning and using representations. Teachers A, C, D, E, and I had both levels moved to Zone 3, provided their unique views about accurate and comprehensible representations. For example, teacher A believed that a teacher should understand the representation first before teaching it:

> Just understand yourself before you teach it. When I prepare my lesson, I make sure I understand. As a teacher, if you don't understand the problem you are

> going to teach, then you cannot make representation accurately. Teachers not only need to understand the content – be accurate, but also need to make it easier for the students – comprehensible. This just depends on teachers' experience; if students do not understand it, I will try different methods. I will know my students' understanding by class work or home work.

Teacher E realized the key in using representation is that "they [students] were able to explain it." She explained further: "students need to see the sequence of mathematics using informal methods to prove their answers."

Agreeing with teacher E, teacher D firmly believed the role of concrete models in using accurate and comprehensible representations, "If they are able to explain to me in the picture what is going on, I will let them to do the abstract part." Furthermore she considered that understanding in concrete form determines students' understanding in abstract form:

> In my experience, if they can understand concretely a certain representation for that form, eventually they can understand abstractly. If they don't understand it in the concrete form, then they are not going to understand abstractly. If they are stuck with concrete, they are going to be stuck with abstract. So it is important that they get the concrete part. A lot of time, students resist the concrete part, as a teacher, you need to do it.

From analyzing students' work, teacher D further commented on students' levels of understanding and preferences for various representations:

> Some students feel more comfortable with graphs, some students feel more comfortable with tables and charts, and others felt comfortable with more abstract forms. So if they are allowed to choose whichever they feel comfortable with and they will be able to explain it more clearly or express the views better.

She also pointed out, "Sometimes, they are misunderstanding; as a teacher, you need to correct them right away, but you always try to build on what they are comfortable with. We need other representations too, but always start with the forms they like."

However, teacher C argued that sometime students understand numerical representation instead of picture representation, and she used the example in the post-questionnaire to address her idea:

> The picture does not prove that $\frac{1}{4} = 4/16$. So I think that students do understand the number in terms of mathematics, but do not understand the representation [in picture]. So this student does not use representation accurately. For comprehensible, I think that something are understood by students.

Teacher I believed multiple ways of representations help students learn mathematics with understanding. For example, to find GCF (Greatest Common Factor), she was not limited the approach in the textbook, which only provided one way to use representation. "You can use the table that lists factors, and do a T chart. The thing is you try to reach a different perception. What is the key? The key is to help child's understand there is no particular one. You have to use multiple representations," she explained.

Other teachers such as teacher F, H, and J, who had a middle level of TZPD in using representations, viewed accurate and comprehensible of representations differently compared to teachers at a high level of TZPD. Most of them followed the curriculum closely and did not really understand it and barely had their own ways. For example, teacher F said, "Look at your learning goal first, and then try to make the lesson plan as close as possible. You can tell if they understand or not just by the questions that they asked. I give them 1 or 2 questions ask them do on their own. Students seem to be learning better this year." Teacher H also followed TEKS, "I make sure what I am teaching from TEKS, and I need to understand the TEKS. We just make sure the material and questions we ask are appropriate from TEKS." The results of following

TEKS without understanding are “I try to show everything to them in many ways. Maybe one way they don’t understand, then do manipulatives, pictures, make a concrete model, then go to abstract.” Teacher J had the same view as teacher F, which is to ask students question to understand if students are using comprehensible representations, “I think we need to monitor students and get them to understand it. I think you just need to get the response from kids. If I put something on the board, none of them even have a clue, and then we have to change different ways. Ask questions to see the best way for kids’ understanding.”

Teacher B, who had a lower level of TZPD in using representation, recognized that she learned a lot from students and believed that assessment will make her to know students’ understanding.

> We do instruction and make sure everything is in TEKS, and that is requirement here. I try to make sure we have a good understanding of it. To make comprehensible, I am through assessment. I am very good at looking at students’ face to assess their understanding. I use different kind of assessment. I may send them to the board to show me how they work this problem; I also do a lot of one to one during advisory class. Through one to one with students, I can see whether they understand or not. What the concept lesson is. We do a lot of drawing and research, in which students do a lot of presentations; it tells me whether they have a good understanding. I get a lot of information from students’ sides. For instance, fraction with name of dog and nanny with inside house and outside house. Just for easier memory, for the numerator and denominator. This way kids remember them.

Impact of the MSMP Workshop

During the interviews, teachers were asked to report how much MSMP professional development helps in their learning using mathematics representation. All teachers addressed their views differently:

Teacher A found that “something we did in classroom and have confirmed by the workshop.” Teacher B used some strategies such as fraction strips that were learned from professional development last summer in the lesson on comparison of fractions. Teachers E and F learned problem solving by using different ways to solve it, like graphs, flesh cards, pictures, and tables. “That gives me the different ways to teach class. I still remember the example of fraction strips that were covered in the workshop,” teacher F commented. Teacher E believed that “using strategies to get conceptual understanding from MSMP is really helpful,” she benefited not only the problem solving relating to representation MSMP provided but also the lesson plan writing, which allowed her to look at the overall picture of the lesson. Furthermore, teacher E said, “That has been very much brought out form MSMP. Our school district did a lot of this in terms of problem solving. MSMP also presented some curricula we are using and some activities we are doing.”

The MSMP workshop also helped teachers I, H, F, and D in teaching. For example, teacher I created the activity relating to professions and was able to ask kids to come up with different representations in teaching mathematics. Teacher D also tried to use more representations in her lesson and she explained, “I try to not standing there tell them this how you do it. Try to let them think and try to let them come to ideas by themselves. Some kids like that, and some kids don’t. You have to set atmosphere where you want them to go. Ask more questions without give them the answers.”

Teacher C still remembered that they discussed students' misconceptions during the MSMP professional development, and she thought this the best part for her in the workshop:

> I think if we can be aware of this problem in classroom, it would be very beneficial to the teaching. I think it would be very good if they can go further, especially for the new teachers as well as undergraduate and graduate courses. Misconceptions help you to understand where you will go how you go in terms of instruction.

From interacting with colleagues and watching their own video lessons and reflecting on their teaching in the workshop, all teachers in this study recognized that they learned various strategies and ideas that helped them in teaching. Teacher A believed the opportunity to watch their own videotapes is "a good way to see the reaction in the classroom," and teacher I commented, "I like to have other teachers have comments on my teaching. It is a good feedback, it might not be always positive, but it keeps you humble." Teacher B agreed, "Feedback is a growth." Teacher E recognized the difficulty at the beginning, "I really think it is a very good strategy. At very beginning, it was incredibly difficult, but once you saw it, after you get over at this point, then you realize that there are a lot of things that is worth to see."

Through watching their own and other teachers' video lessons, teachers would get positive feedback and see the benefits of changes, and question their own practices. Teacher E summarized the benefits of it:

> MSMP provides opportunities to see other teachers have done, how other teachers presented. It just helps you broaden your view: oh maybe I should try that. I think this is the key to teach diverse group. Not one thing that you can reach to the kids. You have to have the different approaches. I have never been videotaped and I had never seen myself videotapes. Watching someone else video tapes teaching is completely new thing to me.

Teacher I's comments reflected all teachers' views about the MSMP workshop: "MSMP is one of the most exciting projects in my teaching career. I use graph more and let students gather information to make graphs."

The professional workshop in this study engaged teachers in interpsychological and intrapsychological processes. Through watching videos and self-reflection, analysis of theirs own and others' teaching and peer discussion, teachers transferred from an interpsychological function to an intrapsychological function, in which teachers developed their TZPD.

In summary, the MSMP professional development workshop on specific content and strategies in using representations not only enhanced teaching knowledge of mathematics representation, but also enhanced their abilities in effective teaching, which was reflected in the movement of teachers' ZPTD levels. Teachers who had a high level in their TZPD had a clear or specific understanding of mathematics representations and were able to use various representations in their teaching practice; teachers who had a lower level of TZPD or did not change their levels, had unclear or general understanding of mathematics representation and were not able to use mathematics representations effectively in their teaching, mainly because they only either followed the textbooks or based their teaching on previous teaching experience, which caused their resistance to change. The results showed a positive consistent relationship between the level of understanding and the level of using mathematics representations, which means the higher level of understanding in representation, and the higher level of use in instruction.

CHAPTER V

DISCUSSION AND CONCLUSION

Introduction

This study investigated three aspects of teachers' knowledge of mathematics representation, relating to their levels of *Teachers' Zone of Proximal Development* (TZPD). First, this study examined how mathematics teachers, as learners, learn and understand mathematics representations through the MSMP professional development workshop that was designed and focused on the specific contents of fractions and algebraic functions, and that concentrated on the knowledge of mathematics presentation, a process standard advocated by NCTM (2000). Second, this study observed how teachers use mathematical representations learned from the MSMP workshop in their classroom teaching. Lastly, the study addressed the relationship and patterns of teachers' change in understanding and using representation corresponding to their TZPDs. The subjects were 11 mathematics teachers in 6th and 7th grades from four school districts in Texas. Multiple data sources were collected and used from questionnaires before and after the MSMP professional development workshop, as well as the videotapes of the duration of the workshop, teachers' lesson videotapes before and after workshop, a survey of teacher preparation before the workshop, and interviews with teachers.

The goal of this study was to investigate how teachers learn and use the mathematics representations to help students improve their learning of mathematics, to investigate how they made transitions in using mathematics representations by examining their classroom teaching and their beliefs on the role of mathematics representation and the professional workshop. This chapter discusses the major findings from the results according to three research questions in this study.

Changes of Learning and Understanding Representations in TZPD

The results from this study indicated that, in general, teachers' learning is like that of students', depending on the transformation of interpsychological regulation to intrapsychological regulation, which constructs *Teachers' Zone of Proximal Developmen*t (TZPD) that is confirmed and supported by Vygotsky's theory of *Zone of Proximal Development* (Vygotsky,1978).

Teachers' Zone of Proximal Development (TZPD) in Learning

The results of this study showed that all learners (teachers) have their own *Zone of Proximal Development.* Teachers are life-long learners, and every teacher in this study also has a *Zone of Proximal Development* (ZPD) at a certain level that shows their unique features in the knowledge of teaching. The *Teachers' Zone of Proximal Development* (TZPD) showed the learners' development distance between actual development and potential development (see Figure 2). In order to teach effectively, the important task for teachers is to make progress to move from a lower level zone to a higher level zone, which means to move from their actual development zone to a potential development zone. With the help of capable others, the learners' potential development is always greater than their independent development. In this study, teachers' knowledge growth proved that they can reach their potentials at different levels with help from capable others. Table 7 in Chapter IV indicated 82 % of the teachers in this study as having moved from their actual development zone to their potential development zone in learning, and 18% remaining at a high level zone. This result indicated that every teacher in this study can learn and can be a successful learner; it also showed that learning environment plays a vital role in learners' development, and this environment should be created by professional development.

To design a productive professional development program, clearly, in the teacher education program (both in-service and pre-service), teacher educators not only need to know teachers' or prospective teachers' actual development levels but also need to realize their potential development levels. In addition, a helpful professional development seeks in relation to create a social and cultural environment that respects every teacher as a capable other to participant learners, which means learners can learn from each other, peer interaction and discussion, and reflection. The study showed that peer discussion and self-reflection of teachers' watching their own videotapes during the MSMP workshop helped teachers to see their own and others' teaching strategies, which plays an important role in teachers' knowledge development. This model of teachers' knowledge development is a process and transformation from interpsychological to intrapsychological function (Vygotsky, 1978).

Teachers' Changes in Learning Need to Have Interpsychological and Intrapsychological Processes

According to Vygotsky (1978), learning consists of the internalization of the social interaction process. The workshop on teachers' knowledge of understanding and using mathematics representation in this study provided an important opportunity for teachers to engage in the social interaction process. Teachers not only acquired knowledge of mathematics representations, but also enhanced their knowledge of teaching from discussion and interaction with others that included all teachers at different levels of TZPD in this study. Many studies have addressed the importance and benefits of this social nature of learning that occurs during social interaction with those who may have more expertise than the learner (Stocks, & Schofield, 1997; Wood, Cobb, &Yackel, 1991). This has confirmed the belief that mathematics teacher learning is "a process of enculturating the learner into the practices of an intellectual community" (Stocks, & Schofield, 1997, p. 284). In this study, although all eleven teachers had different levels of TZPD at the beginning in both understanding and use of mathematics representation, the MSMP professional development workshop helped them form an intellectual community and make progress toward a higher zone level on mathematics representation.

However, engaging in social interaction process in the intellectual community by itself is not enough for learners to make a change and reach their potential. One of the important factors that facilitates this change is intrapsychological regulation (Vygotsky, 1978) that requires learners to be introspection and reflect on their learning. In this study, teachers were encouraged to watch and reflect on their own teaching video lessons and to be introspection on their teaching according to students' learning goals and standards, and teachers were also provided with opportunities to reflect on their lesson with others and watch other teachers' video lessons. Teachers' comments in chapter IV represented their positive views about the process of intrapsychological regulation. Through watching their own and other teachers' video lessons, teachers would get positive feedback and see the benefits of changes and question their own teaching practices, which motivates teachers' desire to change (Cobb et al., 1990; Stocks, & Schofield, 1997).

The MSMP workshop in this study provided an interpsychological process for teachers. Through watching videos and self-reflection, analysis of theirs and others' teaching, and peer discussion, teachers see their weaknesses and find strategies to improve their teaching, and thus their learning is transformed into an intrapsychological function from an interpsychological function, in which learning development proceeds (Forman, & Cazden, 1985), and teachers' TZPD reach their potential level.

Changes in Learning Need to Meet Teachers' Needs: Analysis of Students' Work

The results showed that there are multiple approaches to teachers engaging in learning in professional development. In this study, however, teachers also benefited more from analyzing students' misconceptions during the workshop. Some teachers felt that it was the best part of the learning experience and was very beneficial to the teaching because teachers could be aware of these problems in the classroom teaching. During the interview with teachers, they expressed their interests and hoped the workshop could go further on error analysis, especially for the new teachers as well as teacher education program.

Through analyzing students' errors, teachers understand their students' mathematical thinking process and have better ideas on their weaknesses and strengths in learning mathematics. Furthermore, error analysis helped teachers understand where and how they will go to prepare instruction and to teach with a focus.

NCTM (2000) calls that "To improve their mathematics instruction, teachers must be able to analyze what they and their students are doing and consider how those actions are affecting students' learning" (p.19). With a solid of knowledge of students' thinking, especially on addressing misconceptions, teachers' pedagogical content knowledge will be enhanced, deeply impacting effective teaching (An, 2004).

Although the error analysis strategy provided in the workshop in this study was only part of the professional development, teachers' responses from pre and post questionnaires and their classroom teaching indicated that they have gained substantial knowledge on their students' mathematical thinking. With the knowledge of students' thinking, teachers in this study knew their students' levels of understanding and preferences on various representations, and were able to use of variety of representations to fit different levels of students' thinking and help them understand mathematics in different ways.

According to Ashlock (2002), "Errors are a positive thing in the process of learning" (p. 9). It is powerful to collaborate with colleagues to observe, analyze, and discuss teaching and students' mathematical thinking; however, it is often neglected in professional development in the U.S. (Stigler, & Hiebert, 1999). The workshop in this study set a model for implementing the error analysis in professional development in the U.S. The results of this study showed the evidence of teacher learning from analyzing students' work and errors on mathematics representation and suggested that professional development in the U.S. should design programs encouraging teachers to focus on analysis of students' misconceptions in specific content area and use the results of error analysis and knowledge of students' mathematical thinking to prepare effective teaching.

Use of Mathematical Representations in Instruction in TZPD

The results of teachers' use of mathematical representations in classroom teaching showed that the professional development that focused on mathematics representation not only enhanced teachers' learning, but also changed their teaching. Table 9 in Chapter IV showed that 72% of teachers in this study moved their TZPD in using mathematics representation from lower levels to higher levels or remained at the higher levels. Only three out of 11 teachers did not move in a similar direction as their learning path. What factors are closely related to the levels of TZPD in using mathematics representation in the classroom? The following discussion addresses three factors that influence teachers' practice in using mathematics representations.

Changes in Instruction Needs to Change Teachers' Beliefs

Although teachers' changes should include many other factors such as school, personality, and students, the study addressed two factors of the complexity of teacher change: external and internal factors. External factors involve interpsychological functions that are related to professional development. Internal factors are determined by intrapsychological functions that are reflected in teachers' beliefs. The results of this study showed that a teacher belief is the key for instructional change (teachers' definitions about mathematical representation, teachers' views on benefits and limitations of using mathematical representations, and teachers' views on making accurate and comprehensible representation for students). To draw a whole picture of teachers' beliefs takes much effort in understanding of their relationships between content, pedagogy, and pedagogical content knowledge. To change teachers' beliefs, the entire teaching culture must be changed. Researchers recognize that today's teachers were educated in an era in which traditional and abstract approaches to teaching were the norm (Stocks, & Schofield, 1997). This taken-for-granted assumption impedes teacher change. Can teachers' beliefs on mathematical education be changed? How can teachers' beliefs about mathematical education be changed? These questions are very complex and challenging, but certainly can be explored by focusing on specific content and strategies. This study showed the evidence of teacher change from a carefully well-designed workshop that focused on mathematics representation in learning and using in fractions and algebraic functions. This study confirmed that applying *Teachers' Zone of Proximal Development* (TZPD) should be an effective approach for professional developers to design effective professional development programs that bring about changes in teachers' beliefs simply because their learning in the *Zone of Proximal Development* (ZPD) is more effective than learning under self development or traditional approaches that rely on a transmission-reception model of learning in professional development (Stocks, & Schofield, 1997). Tharp and Gallimore (1988) confirmed the effectiveness of this kind of learning as a process of moving from assisted performance to unassisted performance through a ZPD. Under research-designed guide and workshops, teachers are able to learn specific teaching strategies, in particular, mathematics content and to apply the new learning in their individual teaching. In the process of changing TZPD, teachers' intrapsychological function is based on their interpsychological function. During the workshop, peer discussion and self-reflection enabled teachers to engage in both interpsychological and intrapsychological processes helping them to see their weaknesses and understand that teaching is for students' learning with understanding. With a sound and productive intrapsychological function, teachers value new ideas, have confidence, and are motivated to change. Their beliefs about teaching will be changed as they make progress in their learning. However, like students, teachers are different in their backgrounds and abilities. To reach teachers' diverse backgrounds and their potentials in knowledge and development, a carefully designed professional development program is needed for fostering teachers' belief changes preceding changes in practice (Richardson, Anders, Tidwell, & Lloyd, 1991) and for supporting teachers' learning and applying new strategies in their instruction.

Teachers' Level of TZPD in Using Representations Related to PCK

Teachers' use of mathematical representations for teaching depends on their mathematics pedagogical content knowledge and their beliefs in representation. This study showed that teachers' content knowledge is a base, but is not sufficient for effective teaching. Only when teachers believe that certain strategies (e.g., representation) work in their teaching practice, may their changes in teaching then become possible. For example, teacher B, F, and K's knowledge of representation in learning all moved from lower levels to a higher level, which indicated they have gained content knowledge of representation of fractions and algebraic patterns of changes; however, their use of representation in their teaching did not make the same move as their knowledge of learning did; therefore, their knowledge of using representations, which is pedagogical content knowledge did not move to a higher level. In this study, mathematics teachers' pedagogical content knowledge is not only referred to as the connection between content and pedagogy (An, Kulm, & Wu, 2004), but also attached the feature of productive disposition in applying knowledge, which is a habit of thought in teaching in this study. Some studies have related disposition to attitudes and beliefs about mathematics education (NRC, 2001; Resnick, 1987). This study found that there is a positive relationship between productive dispositions.

The teachers whose TZPDs were in lower level zones or did not make instructional changes have background characters related to disposition in using mathematics representation. For teachers B and F, their teaching are driven by previous teaching experience because both of them have more than 15 years of teaching experience (teacher B has 15 years, and teacher F has 25 years); for teachers J and K, although their teaching experience is less than six years, their teaching is more likely driven by textbooks, according to their interviews. Therefore, these teachers do not have a productive disposition in using mathematics representation because of two factors: teachers with a long period of teaching experience tend to block their desire to change, and teachers with a short amount of teaching experience also find it easy to limit their knowledge to what is in the textbook. As a result, their habits of thinking in teaching are bounded in scope and the use less new ideas and new strategies in teaching, and thus their disposition in using representations is not productive.

To have profound pedagogical content knowledge in using representations, the study addressed disposition as an additional aspect of PCK, playing an important role in using representation in teaching. It is not enough for teachers to make transitions in using mathematics representation once they have grasped specific mathematics content. Teachers also need to have a good habit of thinking about their teaching and root this habit into their daily work. Once using representations becomes a productive disposition, teachers' change in using representations will take place automatically, and they will be able to reach higher zones. The interviews with teachers who had higher zones in using representations in this study confirmed the importance of disposition and implied that good professional development should not only provide opportunities for teachers to learn and understand strategies in using mathematics representation, but also foster a productive disposition for teachers.

Using Representation Appropriately in Teaching

The results of this study brought out some of the teachers' views about mathematics representations (see Table 11). Teachers' definitions of mathematical representation revealed their beliefs and understanding of mathematical representation, which exhibited their disposition in mathematical representations and the ways of using mathematical representations in their teaching.

Mathematics representation is a configuration of signs, characters, icons, or objects that represent mathematics ideas (Cuoco, 2001; Goldin, 2003). The manifold aspects of mathematics representations determine the features of a good representation as being comprehensible, accurate, and transferable, and showing mathematics ideas with pattern, design, arrangement, or relationship in multiple forms. Here multiple forms can be categorized into two basic models: 1) visual models such as tables, charts, graphs, pictures, and concrete materials; 2) abstract model or numerical form, such as mathematics equations and formulas. The knowledge of representation for teachers in this study consists of strategies of using representations to illustrate mathematical ideas. Effective strategies include various familiar representations, understanding mathematics ideas that underlie representations, selecting the proper representation that fits into students' development level, and being able to transfer representations from one to another under a common goal.

One important goal of using representation is to promote conceptual understanding. Helping students move from a concrete understanding to an abstract understanding is a key task in middle school mathematics. Although teachers in this study were able to learn and use multiple representations to help students construct concrete understanding, teachers' use of concrete representations (e.g., pictures, graphs, charts, and tabular forms) often stopped at the concrete level, only reaching students' concrete understanding.

The limitation in using representations is closely related to teachers' knowledge of mathematical representations. Mathematics is a structural and abstract language that requires a solid understanding of mathematical concepts. Teachers should expand their views to understand that any concrete and visual representation is a tool; the final destination is conceptual understanding. Therefore, the definition of mathematical representation is the any form that addresses mathematical ideas, but this form must connect to the mathematical concept. Using only concrete or visual representations also sometimes result in students' relying on or limiting themselves to any particular representation (e.g., pizza or money for fractions). Although multiple mathematical representations help students who have diverse backgrounds, abilities, and interests in learning mathematics, teachers should select the most appropriate representation that addresses the need of each individual student.

With respect to mathematics representations as a language, Vygotsky's ZPD encourages us to look at the learning environment in which language (e.g., linguistic words) acts as a very special function in children's intellectual development. The findings in this study indicated that this theory could also be applied to the adult learning process. In teachers' knowledge growth, mathematical language – mathematical representations (e.g., symbolic systems, pictures, graphs, charts, and tables) play an important role in teachers' development for effective teaching. According to Vygotsky (1978), language, as a tool, plays a key role in humans' intellectual development. From this perspective, mathematical representation is not only defined as a tool, but is also a language that connects students' conceptual understanding. The use of language requires teachers to fully master the main feature of mathematics representation that is to be comprehensible, accurate, and transferable. In addition, teachers should understand that many forms of mathematical representations have the characteristic of visualization, which directly develops learners' mathematical perception; however, the connection with the abstract representation will produce students' conceptual understanding.

Patterns of Change in Classroom Teaching and TZPD Levels

This study explored the ways teachers make changes in their actual classroom teaching in terms of TZPD levels in using mathematics representations and found some patterns of changes in classroom teaching.

Sequential Order of Teacher Change in Learning and Using Representation in TZPD

The results of this study showed that teachers' changes in understanding and using mathematical representations in TZPD follow the sequential order in zones, which indicates that there is no jumping between zones. Although few teachers' changes in instructions did not move toward a higher zone as their changes in learning representations increased, there is no single case showing that a teacher's instruction change moved backward. From learning representations from the workshop, teachers attained rich knowledge in representations, which provided a strong base for their changes in teaching using representations. This indicated that only by teachers' knowledge growth, may their instructional changes become possible. Teachers' knowledge change is a gradual process that follows the order of zones and evolves one by one over time. To measure teachers' change, this study designed three levels of TZPD in both the learning and use of mathematics representations (see Table 4 & 5). Using these levels of TZPD allowed this study to understand changes both within and among teachers and to learn the patterns of changes between learning and using representations.

This study showed that each of the TZPD levels builds on the previous one. Each level portrays the degree of teachers' understanding and using representations.

Analysis of teachers' TZPD provides the relationships between changes in learning and using representations. This study indicated that all teachers in this study were able to acquire the new knowledge of mathematical representations, and their knowledge growth in learning and understanding mathematical representations moved to higher levels (zones) compared to their previous levels. On the basis of their learning, eight teachers were able to move in their TZPD in using representation in teaching, and only three teachers did not make a move to a higher level; their teaching was driven by their previous teaching experiences of more than 15 years and textbooks.

Many studies have confirmed the benefits of using the levels to measure teacher change. Franke, Fennema and Carpenter (1997) used four levels of Cognitively Guided Instruction (CGI) to describe the details of teacher change in two areas: beliefs and classroom practice. Based on the work of Vygotsky, Tharp and Gallimore (1988) proposed a sequence of teacher change stages in four levels to measure changes in practice or thinking at each stage. However, Franke, Fennema and Carpenter (1997) pointed out the drawback in Tharp and Gallimore's four stages: "Thoughts and action are not explicitly distinguished at each stage" (p. 257). Unlike other studies, rather than focusing on teacher change in general, this study focused on teacher changes in specific content areas (fractions and algebraic functions) and on specific strategies in mathematics representations. It focuses on three dimensions: accuracy, comprehensibility, and variety in representations at three levels of TZPD. This study constructed a concrete measurement of teachers' knowledge in learning and using representations and provided a powerful framework in understanding how and to what degree teachers make changes, and their relationship between the changes. The findings in this study in TZPD provided a systematic and measurable way to understand teacher change in a complex and challenging context.

Impact of Teachers' Disposition on Teacher Changes

Although teachers' changes in the learning and use of mathematical representations follow different patterns, in general, their disposition about mathematical representations contributed to their knowledge growth and their use of the knowledge. In this study, teachers' disposition refers to teachers' habit for thinking, their attitude, confidence, and their beliefs.

The results of this study showed that teachers' use of mathematical representations depends on their beliefs about mathematical representations. Those who have changed both in their learning and instruction tended to have more positive views on mathematics representations. Eight teachers who made a positive change believed that mathematical representation helped them reach different students, especially lower ability students. They also believed that using multiple mathematical representations enabled students to share their mathematical thinking. Three of them were very surprised to find out that their lower ability students have more interests in experiencing mathematics after learning mathematics representation.

The findings show that mathematical content knowledge is a factor in teachers having positive views and confidence to use their knowledge learned from the workshop in mathematical teaching. The findings also show that teachers with different mathematical backgrounds and teaching experiences had different views in using a new teaching strategy and used representations in different ways. Teachers with more teaching experience tended to have strong mathematical teaching habits focusing on abstract form and had a hard time integrating new strategies in teaching. Two out of three teachers whose changes followed a negative correlation had over 15 years of teaching experience, and one of them is not majoring in mathematics education. Although their responses from the interview showed their willingness to use new strategies to teach, their actual classroom teaching still followed the old teaching approach.

Textbook driven teaching is another obstacle affecting teachers' change in using mathematical representation. The degree of reliance on textbooks for their instruction reflects teachers' disposition. The extent of using textbooks indicates the teachers' confidence about using new knowledge – mathematical representations. In general, teachers with more content knowledge as well as their knowledge of mathematical representation use the textbook as a reference, and they integrate new knowledge learned from the workshop for active teaching. However, all teachers from this study learned new strategies from the workshop, but some of them still relied on the textbooks, reflecting a lack of mathematical confidence. Evidently, teachers who lack mathematical content knowledge easily incline to follow text books more than those who have relatively strong mathematical content knowledge. Workshop aligned with teachers' textbooks seems to be a good way to foster a productive disposition and influence teachers' applying strategies in their teaching.

Using Representation as a Tool

Although mathematical representations are effective strategies in teaching mathematics, moving from concrete understanding to abstract understanding is a key process for middle school mathematics learning. This study showed that representations (as well as problem solving, connecting, communication, and reasoning) are tools and bridges in students' learning of mathematics, and the final destination of learning should be conceptual understanding. Teachers with clear learning goals easily addressed important mathematical ideas with the help of proper representations. It is also clear that multiple mathematical representations help students with diverse backgrounds, abilities, and interests, especially ESL (English as Second Language) students, to learn mathematics because mathematical representations allowed students to use different formal or informal forms to represent their mathematical ideas. Teachers need to make sure to use mathematical representations as tools to reach students' conceptual understanding and need to make sure students do not rely on any one particular representation (e.g., fractions are the pizza or money).

Teachers who made a change in using representations in this study clearly realized the function of mathematics representation as a tool in teaching. Teacher A explained, "Representation is a tool, not a purpose; however, many students use it as purpose, which does not help them." With this perspective on mathematics representation, the professional workshop should design clear goals and provide teachers with opportunities to explore various representations and use them to solve problems. Once teachers have experience using various representations to convey the same mathematics idea and solve the same problem, they will see the role of representations as a tool and will use it to reach conceptual understanding.

In addition, the use of representations as a tool also illustrates the features of representations: numerical and visual functions. Teachers in this study showed concern in using too much picture representation and ignoring numerical representation. This concern indicated that they understood the goal of teaching mathematics -- to foster logical and reasoned thinking. The professional workshop should focus on students' thinking logically and numerically in mathematics learning, centered on using numerical representation as a base for students' conceptual understanding.

Interaction between the teacher and students plays an important role in effective teaching. The study showed that higher levels of knowledge of mathematical representation may not be enough for effective teaching. Only when teachers understand their students' appropriate development level in the knowledge of mathematical representations and use them as tools, will effective teaching becomes possible.

Implications for Further Research

The current educational reform movement calls for radical changes in professional development (NRC, 2001). How can we structure professional development so that it is effective in bringing about a major change in teaching? What factors facilitate teacher change? This study explored how mathematics teachers, as learners, can learn and understand and use mathematics strategies in teaching and learning such as mathematics representation learned from the workshop that focused on specific content and teaching strategies to promote and support students' mathematical thinking, learning, and understanding. The results from this study furnishes curriculum developers with insightful ideas that professional development programs may develop effective teaching models through specific mathematical processes such as mathematics representation to help teachers build effective teaching strategies. The study presents powerful evidence and contributes to both theoretical and practical issues. In particular, it contributes to current research in two areas: provides insight into the application of the Theory of the *Zone of Proximal Development* (ZPD) and focuses on using mathematics representation in teachers' instructional practice.

Application of the Theory of Zone of Proximal Development (ZPD)

This study examines teachers' knowledge growth in mathematics representations in terms of their ZPDs and how teachers' ZPDs in mathematics representation help students' learning. The study of teacher change in understanding and using representations has laid the groundwork for attempting to understand what professional development can provide for teachers to make changes in teaching, and how teacher changes can be measured. The most important contribution of this study to teachers' knowledge is that it has created a framework with three levels of TZPD that are more than lists of criteria to describe teachers' learning and using representations; the framework is sequenced in order corresponding to evolving structures (Franke, Fennema, & Carpenter, 1997). Three levels of TZPD for representation can be used for examining teachers' knowledge in other mathematics processes, such as problem solving, proof and reasoning, communication, and connection. The framework of this study provides a basis for interpretation, transformation, and reframing of teachers' knowledge (Franke, Fennema, & Carpenter, 1997). In addition, it provides a basis and model of how to measure teachers' knowledge.

Focusing on Using Mathematics Representation in Teachers' Instructional Practice

Mathematics representation is one of five process standards in NCTM (2000) standards, which addresses the important role of mathematics representation in instruction. The results of this study indicate that effective teaching should use mathematics representation to provide students with meaningful and visual mathematics ideas that might help students learn mathematics with understanding. However, although many teachers learned various mathematics representations from the workshop and have seen the important role of mathematics representations in teaching, translating various representations and merging understanding of learning into effective numerical representations is still problematic. The challenging task for future research is to explore ways to help teachers make a transformation between visual representation and numerical representation and build a connection between concrete understanding and conceptual understanding.

The findings in this study also suggest that developing teachers' (pre- and in-service) understanding of mathematical representations can be an effective strategy for helping teachers make a fundamental change. However, teachers' changes depend on their background of mathematics content knowledge. Professional developers need to consider teachers' differences as well as their background when designing professional development programs. Regarding teachers' changes, TZPD theory emphasizes the factors outside teachers' thinking i.e., help from capable others; teachers' thinking from inside certainly is an important factor that needs to be seriously considered.

This study demonstrates a model for building on teachers' knowledge growth and explains how teachers' mathematics representation knowledge develops through the MSMP professional development, and elaborates the characteristics of the use of knowledge of mathematics representation and how mathematics representations may encourage students in learning mathematics.

Limitations

Understanding and using mathematics representations in this study provides an avenue for understanding teacher change. However, modifications that should be considered for future studies in this area include the following:

Changes in Students' Achievement

This study examined teachers' changes in understanding and using mathematical representations before and after the MSMP professional development; however, it would be interesting to examine any change in student achievement corresponding to their teachers' changes (data for students' achievement is currently not available this study).

Quality of Ongoing Professional Development Focusing on Analysis of Students' Errors

The study introduces a new model of professional development that focuses on specific content knowledge and strategies of mathematics representations. The workshop provided insights for teachers for effective teaching. However, the results of this study show that teachers were accustomed to analyzing students' misconceptions;

this indicates that teachers need more practical knowledge that relates to their teaching. Analyzing students' errors not only helps teachers know the weaknesses of their students, but also provides information for the effectiveness of instruction, as well as enhancing their knowledge of students' thinking, which builds their pedagogical content knowledge. In addition, professional development should provide a great deal of ongoing support to teachers as they attempt to implement new approaches in teaching mathematics on a regular basis. This ongoing workshop builds on teachers' knowledge and allows them to communicate their ideas on the use of mathematics representation and make sense of mathematics teaching.

Conclusion

As many innovative professional development programs attempt to develop new strategies for enhancing teachers' knowledge, it is important to incorporate specific strategies that focus on specific content areas. This study explored such a model that focused on strategies in mathematics representation in the specific content areas of fractions and algebraic functions. Specifically, this study examined the issues of teacher learning and understanding mathematical representations through the MSMP professional development workshop and how this understanding of representation fit into *Teachers' Zone of Proximal Development* (TZPD). In addition, this study investigated how teachers use their new knowledge of mathematics representations in classroom teaching and how this practice relates to their levels of TZPDs. Finally, this study discovered the patterns of relationship in teacher changes in classroom teaching.

This study found that all teachers in this study have their levels of TZPD, and their knowledge growth can be moved to a higher level with help by capable others who may be teachers in a social learning community. Teachers' knowledge growth, which is measurable, follows a sequential order of the three levels of TZPD.

The examination of teachers' use of mathematical representations in teaching indicates it depends on their mathematical content backgrounds and their beliefs on representations. Teachers are able to make transitions in using mathematics representation once they have grasped specific mathematics content and strategies, and teacher patterns' changes depend on their learning and understanding of mathematics representation during the professional development and their beliefs about mathematics representations.

Grounded in the beliefs that teachers must understand and use mathematics representations in their teaching, this study calls for substantial revamping of professional development, focusing on the roles and strategies of representations with ongoing and sustained support for teachers as they integrate representation strategies in their daily teaching.

This study notices that moving from a concrete understanding to an abstract understanding is a key challenge for teachers, and that mathematical representations are effective strategies in helping students with diverse backgrounds. The study advocates that teachers regard mathematics representation as a useful tool or bridge that connects students' concrete understanding to an abstract understanding and focuses on conceptual understanding as the final destination. Teachers need to understand and be able to develop multiple representations to facilitate students' conceptual understanding and need to make sure students do not rely on any one particular representation. Teachers also need to focus on the development of the transformation from one representation to another, which leads conceptual understanding. In addition, teachers with higher levels of knowledge of mathematical representation may find this is not be enough for effective teaching; only when teachers understand their students' appropriate development level in mathematical representation, may effective teaching then become possible.

Undergoing teacher changes is a complex process, in which teachers engage in interpsychological functions and transfer teaching skills to intrapsychological functions. *Teachers' Zone of Proximal Development* (TZPD) should be used as an effective approach for professional developers to design effective professional development programs to effect teacher changes. To draw the whole picture of teacher change takes much effort, and must include understanding the relationship between content, pedagogy, pedagogical content knowledge, and teachers' beliefs. Professional developers need to provide teachers with opportunities to interact with peers and to reflect on their own teaching, and more importantly, to consider teachers' differences in beliefs as well as in background when designing professional development programs.

Above all, it is important for professional development to consider how to structure effective programs so that they are consistent with the needs of teachers with diverse backgrounds and are effective in bringing about major changes in their classroom teaching.

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APPENDIX A

Questionnaires before and after Professional Development

1. Questionnaires before Professional Development workshop

Name: __________________ School: ____________________

Show different ways in which one might represent each of the following equivalences:

A) 0.75 = 6/8 B) 4/7 = 8/14

Show one or two different ways in which one might represent the following situation:

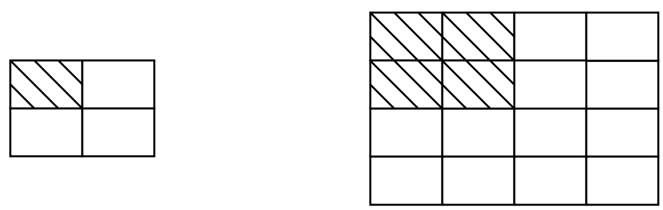
> Bob challenges his older brother, Andy, to a 100-meter race. Bob's average running speed is 1 meter per second. Andy's average running speed is 2.5 meters per second. Andy decides to give Bob a 45-meter head start.

Who will win the race? What distance would make the race fair (competitive) for both boys?

2. Questionnaires after Professional Development

Name: ____________________ School: ______________________

(For 6th grade teachers) John is a 12-year-old student in 6th grade and has average ability. John was asked to use a diagram to show that ¼ is equivalent to 4/16. Look at John's written work for this problem:



$\frac{1}{4}$ $\frac{4}{16}$

Therefore, $\frac{4}{16} = \frac{1}{4}$

When the teacher asked him to explain, John said, "Both pictures show one-fourth of the whole rectangle, so four-sixteenths is equivalent to one-fourth."

1. What prerequisite knowledge might John not understand?

2. What questions or tasks would you ask John in order to determine what he understands about the meaning of fraction equivalence?

3. What real-world example of equivalent fractions is John likely to be familiar with that you could use to help him?

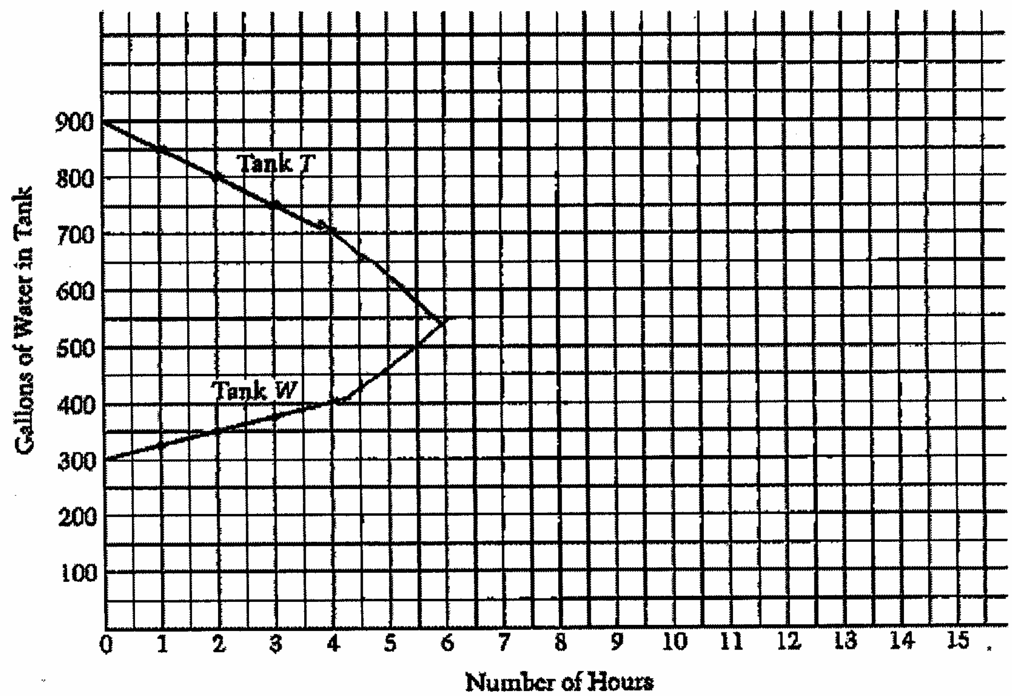
(For 7th grade teachers) Andy, John, and Edward are middle school students and have average ability. Look at their answers for this problem:

> Two large storage tanks, T and W contain 900 and 300 gallons water, respectively. T starts losing water at the rate of 50 Gallons per hour, at the same time additional water starts flowing into W at the rate of 25 gallons per hour. Assume that the rates of water loss and water gain continue. At what number of hours will the amount of water in T be equal to the amount of water in W?

Andy's answer: they will be never equal because T is always faster than W

T tank	W tank
50	25
100	50
...	…
…	…

John's answer: they will meet at 6 hours because



(Source: NAEP, 1999)

Edward's answer: they will not be equal because T Tank is 900 – 50 = 850 and W Tank is 300 + 25 = 325.

1. What might each of the students be thinking?

2. What representations would you provide to each student for them to understand patters of changes in algebra learning? How would you correct each student's misconception about patterns of changes?

APPENDIX B

Survey of Teacher Preparation, Attitudes, and Support Structures

The research project that you are involved with is a study of mathematics teaching and learning. As part of this study we are interested in knowing how various factors influence the way teachers approach the teaching of mathematics. What we learn from this study will help textbook developers create materials that teachers can use more effectively, education researchers design better pre-service and in-service teacher education programs, and school administrators provide the support teachers need to help them improve their teaching. Please answer the following questions regarding your preparation for teaching, the support that you receive, and your beliefs and attitudes about mathematics teaching.

I. Content, Pedagogy, and Experience

1. Please check the courses that you took in high school. Check all that apply.
   __ Algebra
   __ Geometry
   __ Trigonometry
   __ Pre Calculus
   __ Advanced Placement Calculus
   __ Other (list) ______________________________

2. Please check the courses that you took in college or graduate school. Check all that apply.
   (*Courses taken that focused on the teaching of mathematics will be addressed in later questions*.)
   __ College Algebra
   __ Trigonometry/Elementary Functions
   __ Calculus
   __ Advanced Calculus
   __ Real Analysis
   __ Differential Equations
   __ Geometry
   __ Probability and Statistics
   __ Abstract Algebra
   __ Number Theory
   __ Linear Algebra
   __ Applications of Mathematics/Problem Solving
   __ History of Mathematics
   __ Discrete Mathematics
   __ Other (list) ____________________________________

3. Please indicate the level of preparation that you have received for teaching mathematics:

   a) Approximately how many hours of in-service professional development have you received that is directly related to the textbook that you are using in your class? ______________________

   b) Approximately how many hours of in-service professional development have you received that is related to mathematics teaching but not directly related to the textbook you are using in your class? ______________________

   c) Approximately how many hours of in-service professional development have you received that is related to teaching in general but not directly related to the teaching of mathematics? ______________________

   d) How many years have you been teaching mathematics? ______________________

   e) How many years have you been teaching mathematics at the grade level for this study? ______________________

   f) Please list the courses you have taken at the undergraduate or graduate levels that were devoted primarily to the teaching of mathematics. ______________________

   ____

   ____

g) Please list the courses you have taken at the undergraduate or graduate levels that were devoted either to teaching other than mathematics or to more general educational issues. ______________________

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4. Please indicate how prepared you feel to do each of the listed classroom practices in your mathematics teaching. Also indicate the level of support provided by your textbook/teacher's guide, professional development and formal courses for following these practices.

Classroom Practice	**Preparation**				**Support I Have Received**								
	Unprepared←→highly prepared				From Textbook and Teacher's Guide			From Professional Development			From Formal Coursework		
	1	2	3	4	poor	adequate	strong	poor	adequate	strong	poor	adequate	strong
Explain to students the ways in which what they are learning in mathematics is important to them.													
Take students' prior knowledge into account when planning instruction.													
Develop students' conceptual													

understanding of mathematics.													
Develop students' computational skills.													
Encourage students to express their ideas and share their understanding of mathematics with others.													
Provide students with opportunities to apply what they have learned to new situations.													
Teach students of diverse backgrounds, abilities, and interests.													
Gather accurate information about how well students are accomplishing targeted learning goals.													

II. **Support Structures**

1. Which of the following textbooks are you currently using in your classroom? Please check if it is a primary text or supplemental material.

	Primary Text	**Supplemental Material**
Connected Mathematics		
Glencoe – Mathematics Applications and Connections		
Middle Grades MathThematics		
Mathematics in Context		
Other ________________________		

	______________	☐	☐
Other	______________	☐	☐
Other	______________		

2. Please explain how you use both your primary and supplementary texts (i.e., for additional basic skill exercises, to provide additional real-life problems, etc.)

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APPENDIX C

List of Interview Questions

Name:	School:
Date:	Years Teaching Years in Middle Level Education:
Grade/s and Subjects teaching:	Degree:
Concentration:	Certification:

1. Could you please tell me how much MSMP professional development helps in terms of *mathematics representation* in the following aspects
    a. Being able to explain to students the ways in which what they are learning in mathematics is important for them?
    b. Being able to take students' prior knowledge into account when planning instruction?
    c. Being able to develop students' conceptual understanding of mathematics?
    d. Being able to develop students' computational skills?
    e. Being able to encourage students to express their ideas and share their understanding of mathematics with others?
    f. Being able to provide students with opportunities to apply what they have learned to new situations?
    g. Being able to teach students of diverse backgrounds, abilities, and interests?
    h. Being able to gather accurate information about how well students are accomplishing targeted learning goals?
2. What is your definition of mathematical representation? (follow up: concrete, picture, and symbol)
3. In teaching mathematics, what are the benefits of using representation? Are there any limitations using representations?
4. What do you do to make sure representations are accurate in showing the ideas in learning goals (TEKS)? Follow-up: Can you give an example? What do you do to assure representations are comprehensible to students? Follow-up: Can you give an example?
5. Have you used any of the representation strategies or ideas learned from professional development last summer? Give the examples. What do you think watching yourself others teaching videotapes? How about others watch your teaching videotapes?
6. There are five process standards in the NCTM standard: problem solving, reasoning and proof, communication, connection, and representation. In the TEKS, representation is categorized as one of the underlying processes and mathematical tools. In what ways is representation similar to the communication, problem solving, reasoning and proof, and connection as a process standard?

VITA

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Education
Ph.D. (Curriculum & Instruction) Texas A&M University

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B.S. (Mechanical Engineering) Wuhan University Marine Mechanical Engineering Institute

Professional Experience

2004 – Present	Instructor, *Texas A&M University,* College Station, TX. Taught mathematics problem solving course for undergraduate students.
2003 - 2004	*Graduate Research Assistant* Assistant for the Middle School Mathematics Project (MSMP), an IERI-funded five -year project (2001-2006) that is a national effort to help teachers grow content and pedagogical knowledge through professional development.
2001- 2003	*Graduate Research Assistant* Assistant for the Show-Me Project, a NSF-funded eight-year project (1997-2005) that is a national effort to support dissemination and implementation of NSF-sponsored middle grade mathematics curricula.
9/2000 - 11/2000	Instructor, *Southern California Institute of Technology,* Anaheim, CA. Taught college algebra.
1999 – 2000	Teacher, *Brazos School for Inquiry and Creativity*, Bryan, Texas. Taught middle and secondary grades mathematics.
1998 – 1999	Teacher, *Jane Long Middle School,* Bryan, Texas. Taught 7th grade mathematics.