SOME RESULTS ON THE 1D LINEAR WAVE EQUATION WITH VAN DER POL TYPE NONLINEAR BOUNDARY CONDITIONS AND THE KORTEWEG-DE VRIES-BURGERS EQUATION

A Dissertation

by

ZHAOSHENG FENG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2004

Major Subject: Mathematics

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ABSTRACT

Some Results on the 1D Linear Wave Equation with van der Pol Type Nonlinear Boundary Conditions and the Korteweg-de Vries-Burgers Equation. (August 2004)

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Many physical phenomena can be described by nonlinear models. The last few decades have seen an enormous growth of the applicability of nonlinear models and of the development of related nonlinear concepts. This has been driven by modern computer power as well as by the discovery of new mathematical techniques, which include two contrasting themes: (i) the theory of dynamical systems, most popularly associated with the study of chaos, and (ii) the theory of integrable systems associated, among other things, with the study of solitons.

In this dissertation, we study two nonlinear models. One is the 1-dimensional vibrating string satisfying $w_{tt} - w_{xx} = 0$ with van der Pol boundary conditions. We formulate the problem into an equivalent first order hyperbolic system, and use the method of characteristics to derive a nonlinear reflection relation caused by the nonlinear boundary conditions. Thus, the problem is reduced to the discrete iteration problem of the type $u_{n+1} = F(u_n)$. Periodic solutions are investigated, an invariant interval for the Abel equation is studied, and numerical simulations and visualizations with different coefficients are illustrated.

The other model is the Korteweg-de Vries-Burgers (KdVB) equation. In this dissertation, we proposed two new approaches: One is what we currently call First

Integral Method, which is based on the ring theory of commutative algebra. Applying the Hilbert-Nullstellensatz, we reduce the KdVB equation to a first-order integrable ordinary differential equation. The other approach is called the Coordinate Transformation Method, which involves a series of variable transformations. Some new results on the traveling wave solution are established by using these two methods, which not only are more general than the existing ones in the previous literature, but also indicate that some corresponding solutions presented in the literature contain errors. We clarify the errors and instead give a refined result.

To my lovely son, Sebert Xi Feng, and my dear wife, Xiaohui Wang

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When I finished writing the last word of this dissertation, I can not help having so warm feelings in my mind. In the past few years, there are several things kindhearted and beautiful which can never be moved out of my mind. Among them, first and foremost, I would like to express my sincere gratitude to my advisor, Dr. Goong Chen, for his kind support, precious and valuable advice, and patient guidance during my graduate study and in the preparation of this dissertation. It is he who not only led me into an interesting research field—chaotic dynamical system, chose a challenging topic for me and never failed to share his enthusiasm and knowledge in a most generous way, but also led me to the understanding of "how to do a human being", which can not be learned from any textbook. These will undoubtedly constitute an invaluable asset for a young researcher in his future research and life!

I am very grateful to the members of my advisory committee, Drs. Joseph Pasciak, Ciprian Foias, and Donald Friesen for their service and constructive comments on this dissertation. From each of their classes or seminars, I learned a lot, some of which I have used in some of my works. I am also grateful for useful discussions with Dr. Sze-Bi Hsu. A special of thanks goes to Dr. Jianxin Zhou for his kind help and suggestions.

I would like to take this opportunity to thank the Department of Mathematics and College of Science at Texas A&M University for providing me with a generous Teaching Assistantship, which allowed me to concentrate on my research. Our department head, Dr. Al Boggess; Director of Graduate Studies, Dr. Thomas Schlumprecht; and Assistant for Graduate Program, Ms. Monique Stewart, especially deserve special thanks for their always-available administrative assistance.

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While I was writing this dissertation, my beautiful wife, Xiaohui Wang, was pregnant and studying for her Ph.D in statistics. I thank her for her love, endurance, and many sacrifices on a daily basis. Without her love and loving support, this dissertation would not be completed.

I deeply know that somehow, saying thank you or expressing gratitude just does not seem adequate.

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CHAPTER I

INTRODUCTION

The study of nonlinear models in electronic systems and mechanics fluids has always been an important area of research by scientists and engineers [1]. Applications of nonlinear models range from atmospheric science to condensed matter physics and to biology, from the smallest scales of theoretical particle physics up to the largest scales of cosmic structure [2-5]. In recent years, the primary emphasis of such research appears to be focused on the chaotic phenomena. Through numerical simulations, chaos has been shown to exist in many second order ordinary differential equations arising from nonlinear vibrating springs and electronic circuits, see [6, 7], for example. The mathematical justifications required in rigorously establishing the occurrence of chaos are technically very challenging. Some successful examples can be found in [8, 9]. While important progress in the development of mathematical chaos theory for nonlinear ordinary differential equations is being made, relatively little has been done in the mathematical study of chaotic vibration in mechanical systems governed by partial differential equations (PDEs) containing nonlinearity. On the other hand, for some realistic nonlinear models, such as the KdV-type equations, one of the basic physical problems is how to obtain traveling wave solutions to those nonlinear systems. For instance, under what conditions, the traveling solitary wave solutions to nonlinear equations can be expressed explicitly. We know that the kink soliton can be used to calculate energy and momentum flow and topological charge in the quantum field [10]. Although in the last few decades, some perfect methods for finding traveling wave solutions of nonlinear equations have been proposed, such as

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Hirota's dependent variable transformation [3], the B$\breve{a}$cklund transformation [4], the inverse scattering transform [4], Painlevé analysis [11, 12], the real exponential method developed by Hereman and Takaoka [13], the homogeneous balance method [14-16], the tanh-function method [17-19], and several ansatz methods [20-23]. Nevertheless, not all nonlinear equations can be handled by using these approaches. Therefore, seeking new and more efficient methods for dealing with various nonlinear equations has been another interesting and important subject for a rather diverse group of scientists.

In this dissertation, our attention first is focused on the chaotic behavior of a system, which is a partial differential equation with a van der Pol nonlinear boundary condition. Chaos in partial differential equations is very challenging to investigate, and few results are available in this particular case. We mainly follow Chen's ideas as described in [24-27] together with other innovative mathematical techniques to investigate periodic solutions for a first order hyperbolic system, the invariant interval for the Abel equation in Banach Space, and numerical analysis. Then we extend our study to KdVB equation, which arises from many different physical contexts as a model equation incorporating the effects of dispersion, dissipation and nonlinearity. Typical examples are provided by a wide class of nonlinear Galilean-invariant systems under the weak-nonlinearity and long-wavelength approximations [28], the propagation of waves on an elastic tube filled with a viscous fluid [29], the flow of liquids containing gas bubbles [30] and turbulence [31-32 et al.]. We proposed two new approaches as well as applying Painlevé analysis to deal with KdVB equation. Some new results are presented and some errors regarding traveling wave solutions to KdVB equation in the previous literature are corrected and a refined result is presented.

A. Objectives

1. Model Problem 1: The 1D Wave Equation with a van der Pol Boundary Condition

Since the work of Chen et al. [24], quite a number of papers concerning the 1D wave equation

$$w_{xx}(x,t) - w_{tt}(x,t) = 0, \qquad 0 < x < 1, \quad t > 0, \tag{1.1}$$

with the boundary and initial conditions

$$\begin{cases} w_x(0,t) = -\eta w_t(0,t), & \eta > 0, \quad \eta \neq 1, \quad t > 0, \\ w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), & 0 < \alpha < 1, \quad \beta > 0, \\ w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), & 0 < x < 1 \end{cases} \tag{1.2}$$

have received considerable attention. Note that a cubic nonlinearity happens at the boundary condition at $x = 1$. This is a van der Pol condition which is a well known self-regulating mechanism in automatic control. The above has become a useful model for studying chaotic vibration in distributed parameter systems. In [25], a rotation number is defined to obtain denseness of orbits and periodic points by either directly constructing a shift sequence or by applying results of M.I. Malkin [33] to determine the chaotic regime of α for the nonlinear reflection relation, thereby rigorously proving chaos. Nonchaotic cases for other values of α are also classified. Such cases correspond to limit cycles in nonlinear second order ODEs. It has been shown [26] that the interactions of these linear and nonlinear boundary conditions can cause chaos to the Riemann invariants (u, v) when the parameters enter a certain regime. Period-doubling routes to chaos and homoclinic orbits were established. When the initial data are smooth, satisfying certain compatibility conditions at the boundary points, the space-time trajectory or the state of the wave equation, which satisfies another type of the van der Pol boundary condition, can be chaotic. In [27], the nonlinear reflection curve due to the van der Pol boundary condition at the right end becomes a multivalued relation when one of the parameters (α) exceeds the characteristic impedance value ($\alpha = 1$). It is also shown that asymptotically there are two types of stable periodic solutions: (i) a single period-$2k$ orbit, or (ii) coexistence of a period-2k and a period-$2(k+1)$ orbit, where as the parameter α increases, k also increases and assumes all positive integral values. Even though unstable periodic solutions do appear, there is obviously no chaos. In [34], at exactly the midpoint of the interval I, energy is injected into the system through a pair of transmission conditions in the feedback form of anti-damping. A cause of chaos by snapback repellers has been identified. Those snapback repellers are repelling fixed points possessing homoclinic orbits of the non-invertible map in 2D corresponding to wave reflections and transmissions at, respectively, the boundary and the middle-of-the-span points.

The solution of (1.1) and (1.2) can be expressed by iterates of $G \circ F$ and $F \circ G$ [25-27]. Though quite explicit, such expressions are not informative as far as qualitative behavior is concerned. In order to characterize the possibly highly oscillatory behavior of u and v which is often observed in simulation, Chen et al. posed the following question [35]:

"[**Q**] *Assume that the composite reflection map $G \circ F$ is chaotic. Does there exist a large class of initial conditions (u_0, v_0) such that*

$$V_{[0,1]}(u_0) + V_{[0,1]}(v_0) < \infty, \tag{1.3}$$

but the solution $(u,\, v)$ [see 35] satisfies

$$\lim_{t\to\infty} [V_{[0,1]}(u(\cdot,t)) + V_{[0,1]}(v(\cdot,t))] = \infty?\text{"}$$

while (1.3) holds. Here $V_{[a,b]}(f)$ denotes the total variation of f on interval $[a, b]$ for a given function f.

The results in [35] showed that for a parameter range of η when $G_\eta \circ F$ is chaotic, e.g., for η either in $[\underline{\eta}_H, 1)$ or in $(1, \bar{\eta}_H]$, there exists a large class of initial conditions $u_0(\cdot)$ and $v_0(\cdot)$ satisfying (1.3)) such that

$$\lim_{t\to\infty} V_{[0,1]}(u(\cdot,t)) = \infty, \qquad \lim_{t\to\infty} V_{[0,1]}(v(\cdot,t)) = \infty.$$

Thus, the study of the highly oscillatory behavior of $u(x,t)$ and $v(x,t)$ for large t can be converted to the one of the growth rates of the total variations of the map $(G \circ F)^n$ on some intervals as $n \to \infty$, at least for certain initial conditions u_0 and v_0. For a general continuous map f on an interval I in $\mathbb{R}$, the relationships between the unbounded growth of the total variations of f^n as $n \to \infty$ and the complexity of the dynamics of f have been widely addressed recently, see [36], including the relations between the unbounded growth of total variation and chaos according to Devaney's definition (see [37, Definition 8.5]). In [36], it is also shown that if the interval map has sensitive dependence on initial data, then the total variations of the nth iterate f^n on each subinterval will grow unboundedly as $n \to \infty$. The converse theorem is also true, if, in addition, f has only infinitely many extremal points. Such interval maps will have infinitely many periodic points of prime periods 2^k, $k = 1, 2, 3, \cdots$. These results suggests that we can use the property of unbounded growth of total variations to study some chaotic dynamical systems, since sensitive dependence and infinitely many periodic points are some of the most important characteristics of chaos.

In this dissertation, we will study (1.1) with the boundary and initial conditions

$$\begin{cases} w_t(0,t) = -\eta w_x(0,t), \qquad \eta > 0, \quad \eta \neq 1, \quad t > 0, \\ w_x(1,t) = \gamma w_t(1,t) + \mu_1 w(1,t) - \mu_2 w^3(1,t), \quad \gamma, \quad \mu_1, \quad \mu_2 > 0, \\ w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), \quad 0 < x < 1. \end{cases} \tag{1.4}$$

Note that at $x = 1$, the boundary condition is a cubic nonlinearity, which is another van der Pol type condition. We will show later in Section B: Technical Approaches that such a van der Pol type condition can be converted to a Abel equation. For such boundary condition, so far, to my best acknowledge, almost no results or references are available now. The main reason is that the Abel equation is in general not integrable, so it seems not possible for us to express the solution $y(t)$ of (1.18) in an explicit functional form.

2. Model Problem 2: Korteweg-de Vries-Burgers (KdVB) Equation

The second model in this dissertation is KdVB equation

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0, \tag{1.5}$$

where α, β and s are real constants with $\alpha\beta s \neq 0$. The derivations of equation (1.5) from different physical phenomena can be seen from [28-32, 38, 39] and references therein. Equation (1.5) can also be regarded as a combination of the Burgers' equation and KdV equation, since the choices $\alpha \neq 0$, $\beta \neq 0$, and $s = 0$ lead equation (1.5) to the Burgers' equation

$$u_t + \alpha u u_x + \beta u_{xx} = 0, \tag{1.6}$$

and the choices $\alpha \neq 0$, $\beta = 0$, and $s \neq 0$ lead equation (1.5) to the KdV equation

$$u_t + \alpha u u_x + s u_{xxx} = 0. \tag{1.7}$$

It is well known that both (1.6) and (1.7) are exactly solvable, and have the traveling wave solutions as follows, respectively,

$$u(x,t) = \frac{2k}{\alpha} + \frac{2\beta k}{\alpha} \tanh k(x - 2kt),$$

and

$$u(x,t) = \frac{12sk^2}{\alpha}\text{sech}^2 k(x - 4sk^2 t).$$

A great number of theoretical issues concerning KdVB equation have received considerable attention. In particular, the traveling wave solution to KdVB equation has been studied extensively. Johnson examined the traveling wave solution to KdVB equation in the phase plane by means of a perturbation method in the regimes where $\beta \ll s$ and $s \ll \beta$, and developed formal asymptotic expansions for the solution [40]. Grad and Hu used a steady-state version of (1.5) to describe a weak shock profile in plasmas [41]. They studied the same problem using a similar method to that used by Johnson [40], and a related problem was studied by Jeffrey [42]. The related numerical investigation of the problem was carried out by Canosa and Gazdag et al. [43-45]. In [46], a numerical method is proposed mainly for solving KdVB equation by Zaki. The method based on the collocation method with quintic B-spline finite elements is set up to simulate the solutions of KdV, Burgers' and KdVB equations. A finite element solution of KdVB equation is established by means of Bubnov and Galerkin's method using cubic B-splines as element shape and weight functions [47]. Bona and Schonbeck studied the existence and uniqueness of bounded traveling wave solutions to (1.5) which tend to constant states at plus and minus infinity [48]. They also considered the limiting behavior of the traveling wave solution of (1.5) as $\beta \to 0$ with s of order 1, and also as $s \to 0$ with β of order 1. The case where both β and $s \to 0$ with β/s held fixed was also examined. The asymptotic behavior of the traveling wave solution to (1.5) in case $\alpha = 1$ and $\beta < 0$ was undertook by Guan and Gao, and the applications of the theory to diversified turbulent flow problems were described in details in [31, 49]. On using variable transformation and the theory of ordinary differential equation, the asymptotic behavior of the analytical solution to

(1.5) were presented by Shu [50]. Gibbon et al. showed that that equation (1.5) does not have the Painlevé property [51]. Qualitative results concerning the traveling wave solutions to KdVB equation in some special cases were also obtained by the above mentioned authors and others, but they did not find the exact functional form of the traveling wave solution, or any other exact solutions.

Since the late 1980s, many mathematicians and physicists have obtained explicit exact solutions to KdVB equation independently by various methods. Among them are Xiong who obtained an exact solution to (1.5) when $\alpha = 1$, $\beta = -c$ and $s = \beta$ by the analytic method [52], Liu et al. who obtained the same solution by the method of undetermined coefficients [53], Jeffrey and Xu et al. who obtained an exact solution to (1.5) by a direct method and a series method [54, 55], Halford and Vlieg-Hulstman who obtained the same result in [56] by using partial use of a Painlevé analysis, Wang who applied the homogeneous balance method to obtain an exact solution [14], which was verified by Parkes using the tanh-function method [57]. Kaya repeated the exactly same result by using the Adomian decomposition method [58]. However, except several minor errors in [14], the solutions obtained in the literature actually are equivalent to one another. That is, the traveling solitary wave solution to (1.5) can be expressed as a composition of a bell solitary wave and a kink solitary wave. A qualitative analysis and a more general traveling wave solution to equation (1.5) were presented by Feng by means of the first integral method [59].

Much attention also has been received to the following two-dimensional Burgers-KdV (2D-KdVB) equation

$$(u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx})_x + \gamma u_{yy} = 0, \tag{1.8}$$

where α, β, s, and γ are real constants. Barrera and Brugarino applied Lie group analysis to study the similarity solutions of (1.8) and examined some features of these invariant solutions, but explicit traveling wave solution to (1.8) was not shown [60]. Li and Wang use the Holf-Cole transformation and a computer algebra system to study (1.8) and obtained an exact traveling wave solution to (1.8) [61]. In the mean time, Ma proposed a bounded traveling wave solution to (1.8) by applying a special solution of square Holf-Cole type to an ordinary differential equation [62]. These two methods are compared with each other, and the solutions are proven to be equivalent by Parkes [63]. Fan reproduced the same result by using an extended tanh-function method for constructing multiple traveling wave solutions of nonlinear partial differential equations in a unified way [64]. Recently, Fan et al. [65] claimed that a new complex line soliton for the 2D-BKdV equation was obtained by making use of the same technique as described in [62]. By using the coordinate transformation method, Feng obtained a more general result [66], which includes all traveling solitary wave solutions in [65, 67].

The above statements motivate the following four main goals of this dissertation:

- To prove that the first order hyperbolic system derived from Model Problem 1 does not have period doubling.

- In order to find the invariant interval for the Mapping F, which involves the Abel equation, we first have to investigate an invariant interval for the Abel equation in Banach space.

- To propose two new methods to study KdVB equation for its traveling wave solutions, which appear to be more efficient than various approaches used in the literature.

- Some errors in the previous literature are corrected and a refined result is established.

B. Technical Approaches

1. Case 1: for Model Problem 1

We follow the method of characteristics as described in [26] to deal with system (1.1) and (1.4). By letting

$$\begin{cases} w_x(x,t) = u(x,t) + v(x,t) \\ w_t(x,t) = u(x,t) - v(x,t), \end{cases} \tag{1.9}$$

the PDE is diagonalized into a first-order symmetric hyperbolic system

$$\frac{\partial}{\partial t}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}, \qquad 0 < x < 1, \quad t > 0, \tag{1.10}$$

with the boundary conditions

$$u(0,t) - v(0,t) = -\eta[u(0,t) + v(0,t)] \tag{1.11}$$

$$u(1,t) + v(1,t) - \mu_1 w(1,t) + \mu_2 w^3(1,t) = \gamma[u(1,t) - v(1,t)]. \tag{1.12}$$

From (1.9), we define

$$\begin{aligned} y(t) &= w(1,t), \\ &= w(1,0) + \int_0^t [u(1,\tau) - v(1,\tau)]d\tau, \qquad t > 0. \end{aligned} \tag{1.13}$$

This gives

$$w_x(1,t) = y'(t) + 2v(1,t) \tag{1.14}$$

Thus, the boundary conditions (1.11) and (1.12) reduces to

$$v(0,t) = G_\eta(u(0,t)) \equiv \frac{1+\eta}{1-\eta}u(0,t), \qquad t > 0, \tag{1.15}$$

and

$$u(1,t) = F_{\mu_1,\mu_2}(v(1,t)), \qquad t > 0, \tag{1.16}$$

at, respectively, the left-end $x = 0$ and the right-end $x = 1$, where in (1.16), F_{μ_1,μ_2} is a nonlinear mapping such that for each given $v(1,t) \in \mathbb{R}$, $u = F_{\mu_1,\mu_2}(v)$ is uniquely determined by

$$u(1,t) = y'(t) + v(1,t), \qquad t > 0, \tag{1.17}$$

where $y(t)$ satisfies

$$\begin{cases} y'(t) = -\frac{\mu_2}{1-\gamma} y^3(t) + \frac{\mu_1}{1-\gamma} y(t) - \frac{2v(1,t)}{1-\gamma} \\ y(t_0) = w(1,t_0), \quad t_0 = 0, 1, 2, 3 \cdots. \end{cases} \quad t \in [n-1, n],\ n \in \mathbb{N}. \tag{1.18}$$

We may assume that the initial states w_0 and w_1 satisfy

$$w_0 \in \mathbf{C^1}([0,1]), \qquad w_1 \in \mathbf{C^1}([0,1]).$$

The initial conditions for u and v are

$$\begin{cases} u(x,0) = u_0(x) \equiv \frac{1}{2}[w_0'(x) + w_1(x)] \\ v(x,0) = v_0(x) \equiv \frac{1}{2}[w_0'(x) - w_1(x)], \end{cases} \quad 0 < x < 1. \tag{1.19}$$

From (1.10), one can see that

$$u_t - u_x = 0, \qquad v_t + v_x = 0,$$

respectively. Hence, along characteristics we have the constancy as follows:

$$\begin{aligned} u(x,t) &= \text{constant, along } x + t = \text{constant} \\ v(x,t) &= \text{constant, along } x - t = \text{constant.} \end{aligned}$$

For instance, along a characteristic $x - t = k$ (k is a constant) passing through the initial horizon $t = 0$, we get

$$v(x,t) = v_0(k), \quad \forall (x,t): \quad x - t = k, \quad 0 < k < 1.$$

When this characteristic intersects the right boundary $x = 1$ at time r, we get

$$v(1,r) = v_0(k), \qquad r = 1 - k.$$

At time $t = r$, a nonlinear reflection occurs according to (1.12), see Fig.1.

From time to time, we also need that the initial values u_0 and v_0 satisfy the compatibility conditions

$$v_0(0) = G_\eta(u_0(0)), \qquad u_0(1) = F_{\mu_1,\mu_2}(v_0(1)), \tag{1.20}$$

where $G_\eta = \frac{1+\eta}{1-\eta}$ and F_{μ_1,μ_2} is described as in (1.15) and (1.16).

For convenience, let us write F_{μ_1,μ_2} briefly as F, in case no ambiguity arises. Similarly, we will also write G_η briefly as G. Using the maps G_η, F_{μ_1,μ_2} in conjunction with the method of characteristics, the solution u and v of (1.10), (1.15) and (1.17)-(1.19) can be expressed explicitly as follows: for $t = 2k+\tau,\ k = 0,1,2,\cdots,\ 0 \le \tau \le 2$, and $0 \le x \le 1$,

$$\begin{cases} u(x,t) = \begin{cases} (F_{\mu_1,\mu_2} \circ G_\eta)^k (u_0(x+\tau)), & \tau \le 1 - x, \\ G_\eta^{-1} \circ (G_\eta \circ F_{\mu_1,\mu_2})^{k+1} (v_0(2 - x - \tau)), & 1 - x < \tau \le 2 - x, \\ (F_{\mu_1,\mu_2} \circ G_\eta)^{k+1} (u_0(\tau + x - 2)), & 2 - x < \tau < 2; \end{cases} \\ v(x,t) = \begin{cases} (G_\eta \circ F_{\mu_1,\mu_2})^k (v_0(x - \tau)), & \tau \le x, \\ G_\eta \circ (F_{\mu_1,\mu_2} \circ G_\eta)^k (u_0(\tau - x)), & x < \tau \le 1 + x, \\ (G_\eta \circ F_{\mu_1,\mu_2})^{k+1} (v_0(2 + x - \tau)), & 1 + x < \tau < 2, \end{cases} \end{cases} \tag{1.21}$$

Where $(G \circ F)^n = (G \circ F) \circ (G \circ F) \circ \cdots \circ (G \circ F)$, the n-times iterative composition of $G \circ F$.

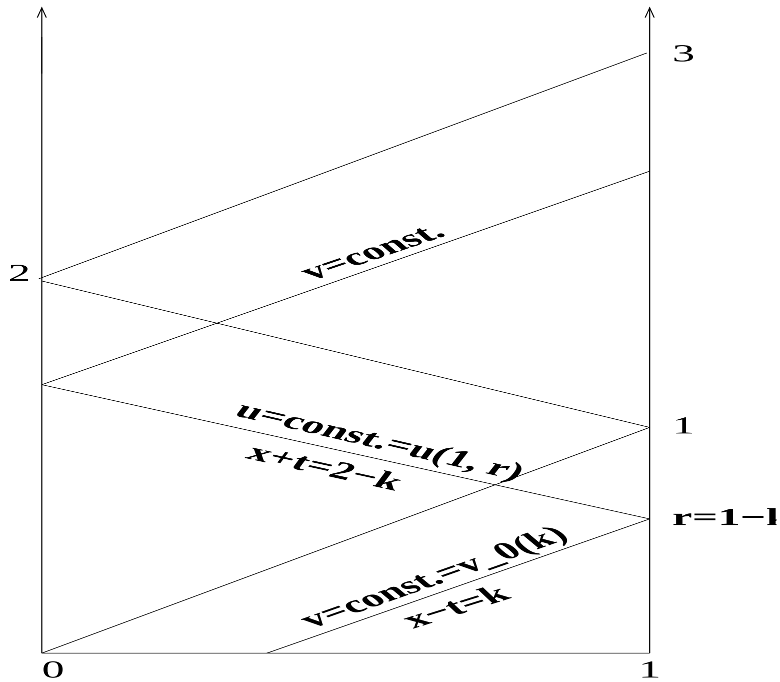

Fig. 1. Reflection of characteristics.

Note that in the sense of [25], (w_x, w_t) is topologically conjugate to (u, v), thus, in the case of (1.1) and (1.2), $G \circ F$ and $F \circ G$ constitute a natural Poincaré section of the solution of (1.10) by (1.21), the dynamical behavior of the gradient w of systems (1.1) can be decided completely by the dynamics of the maps $G \circ F$ and $F \circ G$. Furthermore, since $G \circ F$ and $F \circ G$ have topological conjugacy, we only need to consider the map $G \circ F$. Here, we follow from the definition in [24-27]: if $G \circ F$ is chaotic on some invariant interval, we say that the gradient w of the system (1.1) is chaotic.

To determine whether there is chaos for PDE system (1.1), let us look at the graphs of system (1.1) with

$$\begin{cases} w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t) - \gamma w(1,t), \\ t > 0, 0 < \alpha < 1, \beta > 0, \gamma > 0. \end{cases} \tag{1.22}$$

Choose

$$\begin{cases} w_0(x) = 0.5 - 0.95x + \frac{1}{2}x^2, \\ w_1(x) = 1.05 - x, \end{cases} \quad 0 \le x \le 1 \tag{1.23}$$

$$\alpha = 0.5,\ \beta = 1.95,\ \gamma = 0.01,$$

in (1.1), (1.15) and the first equation of (1.2). Then

$$u_0(x) = 0.05,\ v_0(x) = x - 1,$$

according to (1.19). We plot the graphics of $u(x,t)$ and $v(x,t)$ for $t = 30$ in Fig. 2. The reader may find that the snapshots of $u(x,t)$ and $v(x,t)$ display chaotic behavior.

In general, the treatment of PDEs requires more sophisticated mathematical techniques. while the presence of nonlinearities, basic issues such as existence and uniqueness of solutions oftentimes are already quite difficult to settle, not to say the determination of chaotic behavior. Also, since PDEs have many different types, most of us will agree that there is not yet available a *universally* accepted definition of chaos for time-dependent partial differential equations. Thus, chaos for PDEs may have to be studied on a case-by-case basis.

In the case where $\gamma = 0$ in (1.22), using the method of characteristics for hyperbolic systems one can extract clearly defined *interval maps* [6, 24, 25], which come from wave *reflection relations* totally characterizing the system, and use them as the natural Poincaré section for the system. Since the definition of chaos for interval maps is more or less standard (see, e.g., [37]), it is thus possible to classify whether the system is chaotic or not when $\gamma = 0$. Here we paraphrase it below.

Definition I.1 *[37, p. 50] Let X be a metric space with metric $d(\cdot,\cdot)$, and let $f\colon X \to X$ be continuous. We say that f is chaotic on X if*

(i) the set of all periodic points of f is dense in X;

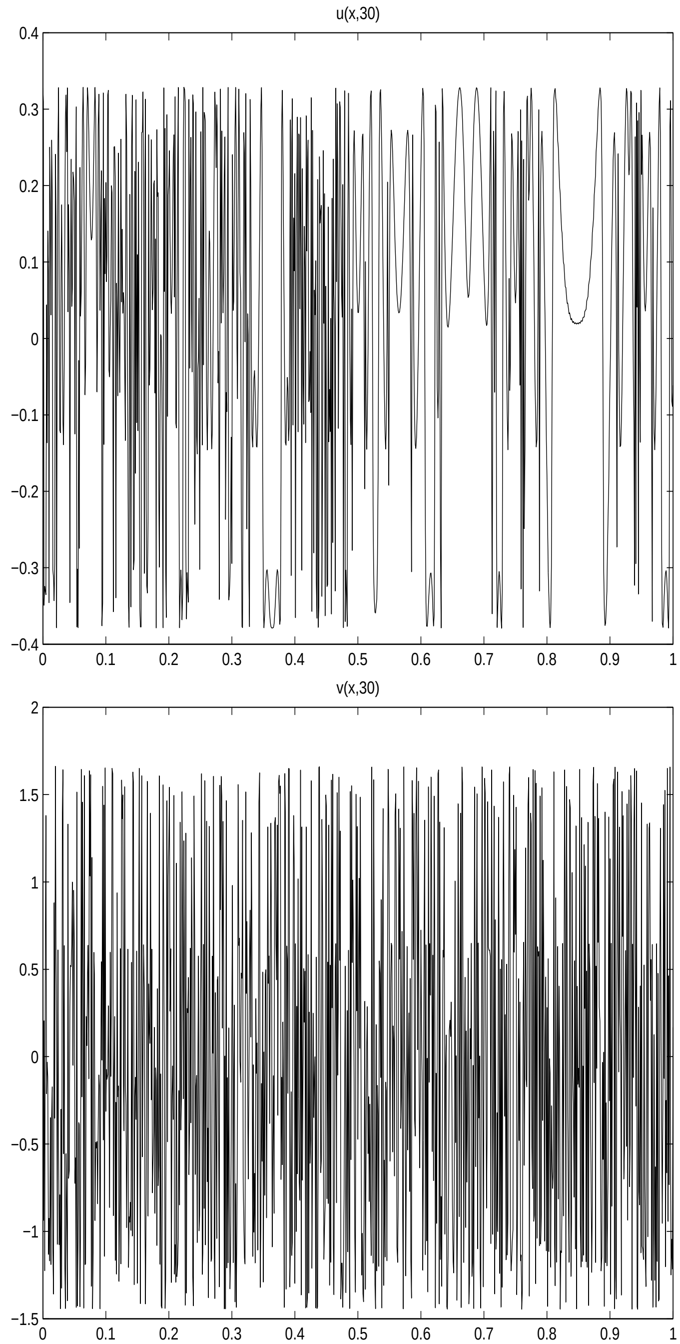

Fig. 2. Snapshots of $u(\cdot, t)$ and $v(\cdot, t)$ for $t = 30$ for (1.1), (1.22) and (1.23). The reader may observe quite chaotic oscillatory behavior of u and v.

(ii) f *is topologically transitive on* X*, i.e., for every pair of nonempty open sets* U *and* V *of* X*, there exists an* $n \in \mathbb{N}$ *such that* $f^n(U) \cap V \neq \emptyset$*;*

(iii) f *has sensitive dependence on initial data, i.e., there exists a* $\delta > 0$ *such that for every* $x_0 \in X$ *and for every open set* U *containing* x_0*, there exists a* $y \in U$ *and an* $n \in \mathbb{N}$ *such that* $d(f^n(y), f^n(x_0)) > \delta$*.*

□

We call δ in (iii) above a *sensitivity constant* of f. The reader may find more discussions on the definition of chaos in Robinson [68, pp. 83–85], and Wiggins [69, pp. 436–347], for example.

Conditions (i) and (ii) in Definition I.1 are independent of each other; however, condition (iii) is actually implicated by (i) and (ii), as pointed out by Banks et al. [70] in 1992.

Theorem I.1 *In Definition 1.1, conditions (i) and (ii) imply (iii).* □

One of the central ideas in this dissertation is to show that when we change the boundary condition on the right hand side as in (1.4), the chaotic behavior of system will be different from those described in [25-27, 36].

2. Case 2: for Model Problem 2

Without loss of generality, we assume that $s > 0$. Otherwise, using the transformations $s \to -s$, $u \to -u$, $x \to -x$, the coefficient of u_{xxx} in equation (1.5) can be transformed to the positive.

Assume that equation (1.5) has traveling wave solutions in the form $u(x,t) = u(\xi)$, $\xi = x - vt$, $(v \in \mathbb{R})$. Substituting it into equation (1.5) and performing one integration, then yield

$$u''(\xi) - ru'(\xi) - au^2(\xi) - bu(\xi) - d = 0, \tag{1.24}$$

where $r = -\frac{\beta}{s}$, $a = -\frac{\alpha}{2s}$, $b = \frac{v}{s}$ and d is an arbitrary integration constant. Equation (1.24) is a nonlinear second order ordinary differential equation. It is commonly believed that it is very difficult for us to find exact solutions to equation (1.24) by classical methods [49]. Let $x = u$, $y = u_\xi$, then equation (1.24) is equivalent to

$$\begin{cases} \dot{x} = y = P(x, y), \\ \dot{y} = ry + ax^2 + bx + d = Q(x, y). \end{cases} \tag{1.25}$$

Notice that (1.25) is a two-dimensional plane autonomous system, and $P(x, y)$, $Q(x, y)$ satisfy the conditions of the uniqueness and existence theorem ([71]). The integral orbits of autonomous system in the Poincaré phase plane depict graphically the types of motions determined by equation (1.24). Note that the orbits point to the right, to the left, or are vertical according as $\dot{x} > 0$ (upper half-plane), $\dot{x} < 0$ (lower half-plane), or $\dot{x} = 0$ (x-axis). This is because x is increasing, decreasing, or stationary in these three cases, respectively. For any solution $u = u(\xi)$ of equation (1.24), if $u(\xi) \equiv u_0$ for $\xi \in (-\infty, \infty)$ (u_0 is a constant), the corresponding integral curve to $u(\xi) = u_0$ in the (ξ, u)-plane is a line which is parallel to ξ-axis, and the associated orbit with $u(\xi) = u_0$ in the Poincaré phase plane is a point $u_0(x_0, y_0)$ which is usually called an equilibrium point (or critical point).

It is well-known that plane autonomous systems are particularly useful in physics and engineering. The equilibrium point in the Poincaré phase plane always corresponds to a static state. If (u_0, v_0) is an equilibrium point of (1.25), then any orbit

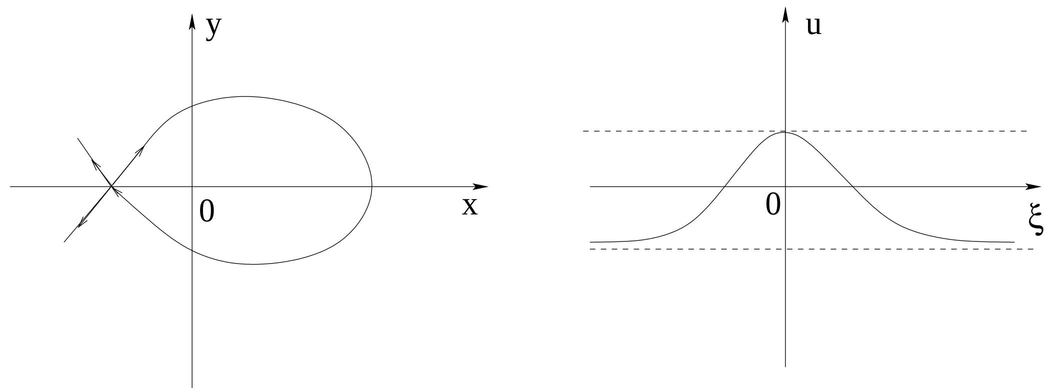

Fig. 3. An isolated and closed orbit with an equilibrium point in the Poincare phase plane represents a bell solitary wave solution in the (ξ, u)-plane.

except itself can not approach (u_0, v_0) within finite time. Vice versa, if an orbit of (1.25) approaches (u_0, v_0) as $\xi \to \infty$ (or $-\infty$), then (u_0, v_0) must be an equilibrium point of (1.25). In the Poincaré phase plane, an isolated and closed orbit which has no equilibrium point on itself represents a periodic oscillation to equation (1.24) in the (ξ, u)-plane. An orbit which emanates from an equilibrium point and terminates at a different equilibrium point as $\xi \to \infty$ (or $-\infty$) represents a kink-profile solitary wave solution in the (ξ, u)-plane to equation (1.24). An isolated and closed orbit which emanates from an equilibrium point and also terminates at the same equilibrium point as $\xi \to \infty$ (or $-\infty$), represents a bell-profile solitary wave solution to equation (1.24) in the (ξ, u)-plane (see Fig. 3).

The other vital idea in this dissertation is motivated by a natural question: "whether there is a more economical method to handle KdVB equation, which causes a more general result?". We will answer this question by introducing two methods: the first method—First Integral Method, based on the theory of commutative algebra and the second method— Coordinate Transformation Method, which converts KdVB equation to a simple form. In addition, by virtue of Hardy's Theorem, an asymptotic behavior of positive proper solutions to KdVB equation is demonstrated.

C. Outline

This dissertation contains 10 main components, each of which is discussed in different chapters. The 10 components (parts) are :

1. *Introductory part*
2. *Uniqueness of* $2n$*-periodic solution*
3. *An invariant interval for Abel equation*
4. *Numerical simulation results*
5. *Oscillatory asymptotic analysis to KdVB equation*
6. *First integral method for solving KdVB equation*
7. *Coordinate transformation method*
8. *Painlevé analysis*
9. *First integral method for the CKdV equation*
10. *Conclusion*

The *introductory part* is an introduction to two nonlinear models. It includes the reasons why we are interested in these two models, a brief review of previous results and an overview of our techniques. In the *uniqueness of* $2n$*-periodic solution* part, we first consider the uniqueness of 2- and 4-periodic solutions, respectively, then using the method of steps and mathematical induction, we extend the results to $2n$-periodic solution. We also provide a discussion for the stability of the periodic solution of Abel equation based on Liapounov's definition. Since the mapping F strongly depends on Abel equation, in Chapter III, we define a nonlinear operator and present an invariant interval for Abel equation in Banach space. An asymptotic expansion of solutions for Abel equation is also established accordingly. The *numerical results* are in Chapter IV. In this chapter, we illustrate some numerical simulations with different groups of experiments.

From Chapter V to Chapter VIII, we are concerned with KdVB equation. The *oscillatory asymptotic analysis* part is discussed in Chapter V. We begin our investigation by presenting a phase-plane analysis to a two-dimensional plane autonomous system, which is equivalent to KdVB equation; then KdVB equation is simplified to a steady-state form by considering waves traveling at a uniform speed. An oscillatory asymptotic analysis is presented and perturbation of the solitary wave is shown. From this analysis, one can see not only an overview of the asymptote of solution for the steady-state form of equation (1.5), but also some new insight into high dependence of traveling wave solutions of (1.5) on coefficients. In Chapter VI and Chapter VII, we introduce two new methods: First Integral Method and Coordinate Transformation Method, respectively, in details. The techniques for seeking traveling wave solutions described herein appear to be more efficient and less computational than those methods used in the literature. It is worthwhile to point out that the results presented in these two chapters do not depend on the particular example of BKdV equation. Some representative equations are listed in the last Chapter. Furthermore, comparisons with the existing results have been presented, which indicate that some previous results regarding traveling wave solutions of KdVB equation contain errors. We also clarify those errors. In Chapter VIII, we deal with KdVB equation by means of Painlevé analysis. The solutions obtained in this manner are in agreement with those described in Chapters VI and VII. In Chapter IX, we use the first integral method to study the traveling solitary wave solutions for the compound Burgers-Korteweg-de Vries Equation and derive some new several solitary wave and periodic wave solutions.

Finally, in Chapter X, we give a summary of the presented work, conclusions, possible extensions and future research plans.

CHAPTER II

UNIQUENESS OF PERIODIC SOLUTION

A. Motivation for Our Study

Periodic points or solutions are often closely related to the problems of stability and bifurcation in analyzing dynamical systems and period-doubling is an important route to chaos. This fact has been seen clearly in [26, 37] and provides us a helpful clue when we consider a given dynamical system, that is, we can not ignore the investigation of periodic points or solutions when we start to study the chaotic behavior of a dynamical system. To see this point, we would like to begin our study by demonstrating two examples. One is the quadratic family $F_\mu(x) = \mu x(1-x)$ and the other is system (1.1) with the boundary and initial conditions (1.2). Both of them illustrate many of the most important phenomena that occur in dynamical systems.

Throughout this dissertation, we denote the composition of two functions by $f \circ g(x) = f(g(x))$. The n-fold composition of f with itself recurs over and over again in the sequel. We denote this function by $f^n(x) = f \circ \cdots \circ f(x)$. The point x is a fixed point for f if $f(x) = x$. The point x is a periodic point of period n if $f^n(x) = x$. The least positive n for which $f^n(x) = x$ is called the prime period of x.

It is straightforward to verify that the map F_μ has following five propositions:

- $F_\mu(0) = F_\mu(1)$ and $F_\mu(p_\mu) = p_\mu$, where $p_\mu = \frac{\mu-1}{\mu}$.

- $0 < p_\mu < 1$ if $\mu > 1$.

- Suppose that $\mu > 1$. If $x < 0$, then $F_\mu^n(x) \to -\infty$ as $n \to \infty$. Similarly, if $x > 1$, then $F_\mu^n(x) \to -\infty$ as $n \to \infty$.

- Suppose that $1 < \mu < 3$. F_μ has two fixed points. One is an attracting fixed point at $p_\mu = \frac{\mu-1}{\mu}$ and the other is a repelling fixed point at 0.

- If $1 < \mu < 3$ and $0 < x < 1$, then $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$.

Now let μ begin from $\mu = 0$ and increase μ to $\mu = 4$ with increment $\Delta\mu = 0.01$, with μ as the horizontal axis. For each μ, choose:

$$x_0 = \frac{k}{100}, \quad k = 1, 2, \cdots, 99.$$

The plot $F_\mu^{400}(x)$ (i.e., a dot) for these values of x_0 on the vertical axis, see Fig.4.

In Fig.5, the graph of F_μ^{400} with $\mu = 3.55$ looks like a step function. The eight horizontal levels correspond to the period-8 bifurcation curves in Fig.4. From Fig.6, we can see apparently that the value of $\mu = 3.65$ is already in the chaotic regime. The curve has exhibited highly oscillatory behavior.

As we have seen from Fig. 4, the quadratic map $F_\mu(x) = \mu x(1-x)$ is simple dynamically for $0 \le \mu \le 3$ but chaotic when $\mu \ge 4$. Here the natural question is: how does F_μ become chaotic as μ increases? Where do the infinitely many periodic points which are present for large μ come from? In this section, we just give a geometric and intuitive answer to this question. In order to present our discussions in a straightforward way, here let recall a remarkable theorem due to Sarkovskii:

First, we introduce the Sarkovskii's ordering on the set of all positive integers. The ordering is arranged as follows

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \cdots$$
$$\rhd 2^3\cdot 3 \rhd 2^3\cdot 5 \rhd \cdots\cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

That is, first list all odd numbers except one, followed by 2 times the odds, 2^2 time the odds, 2^3 times the odds, etc. This exhausts all the natural numbers with the exception of the powers of two which we list last, in decreasing order. Sarkovskii's

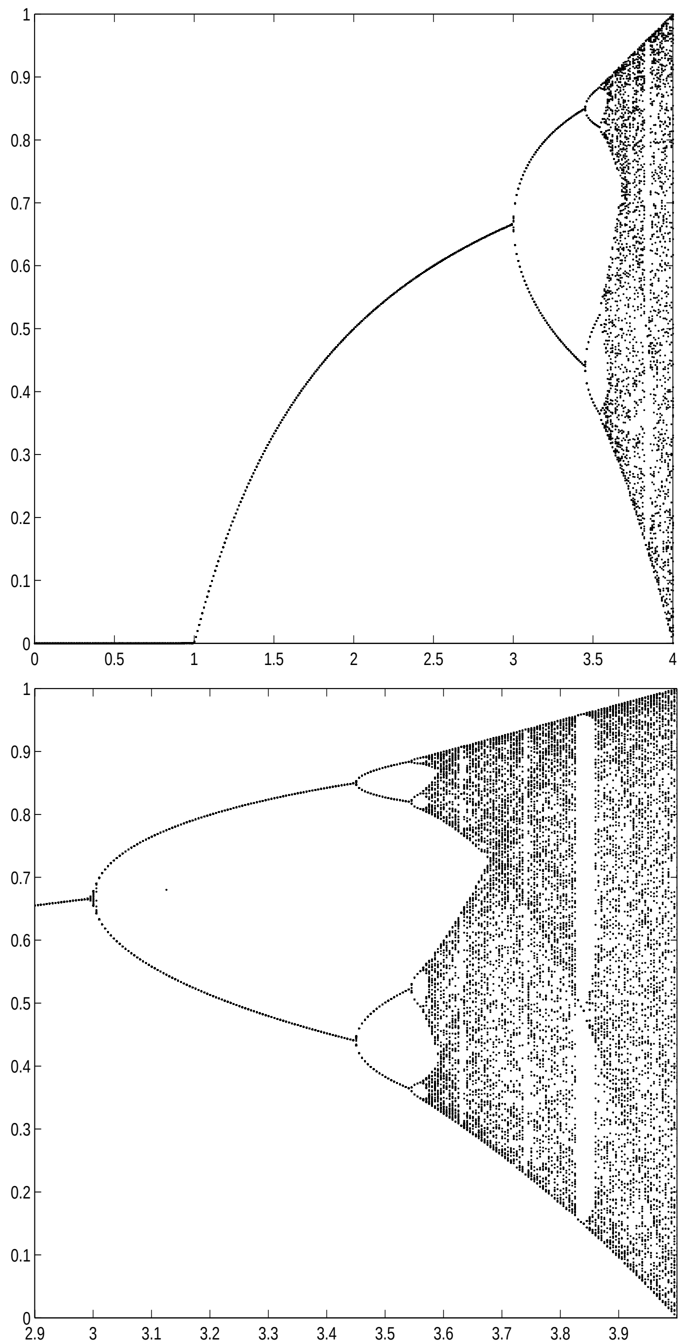

Fig. 4. The orbit diagrams of F_μ with $0 \le \mu \le 4$ and $2.9 \le \mu \le 4$, respectively. The reader may observe quite chaotic behavior when $\mu \ge 3.65$.

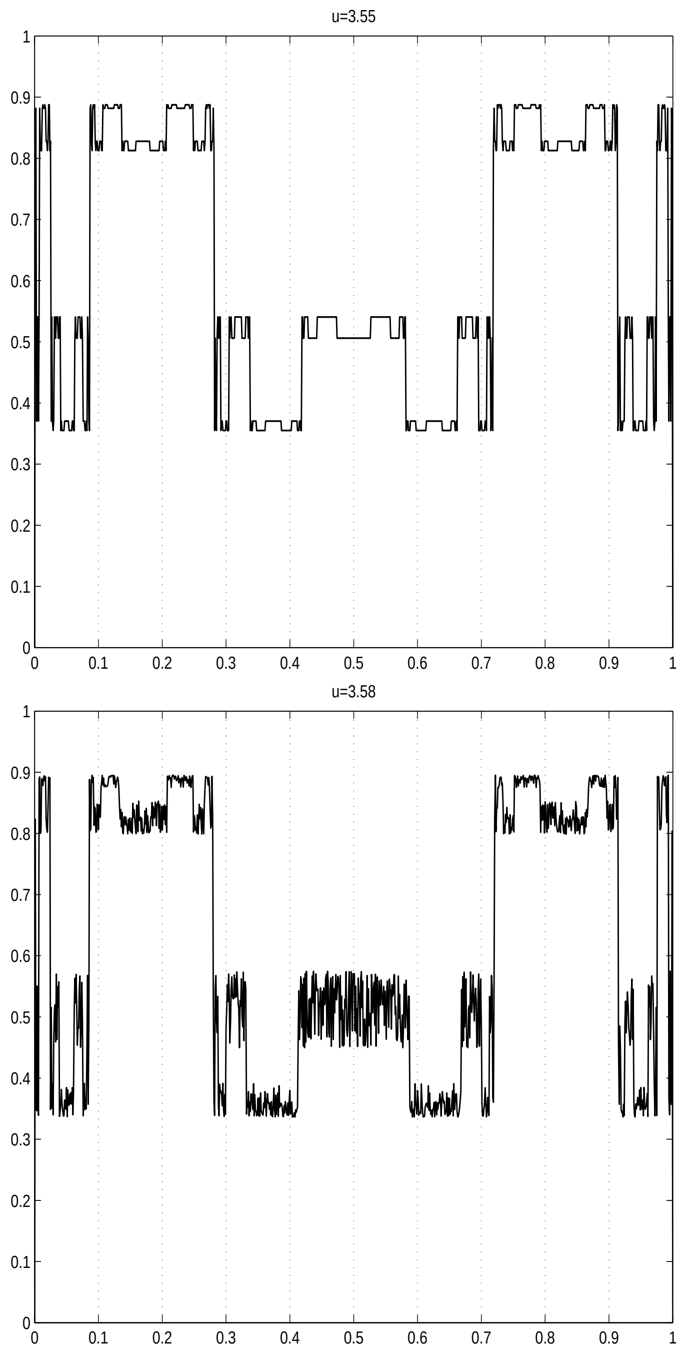

Fig. 5. The graphs of F_{μ}^{400} with $\mu = 3.55$ and $\mu = 3.58$, respectively.

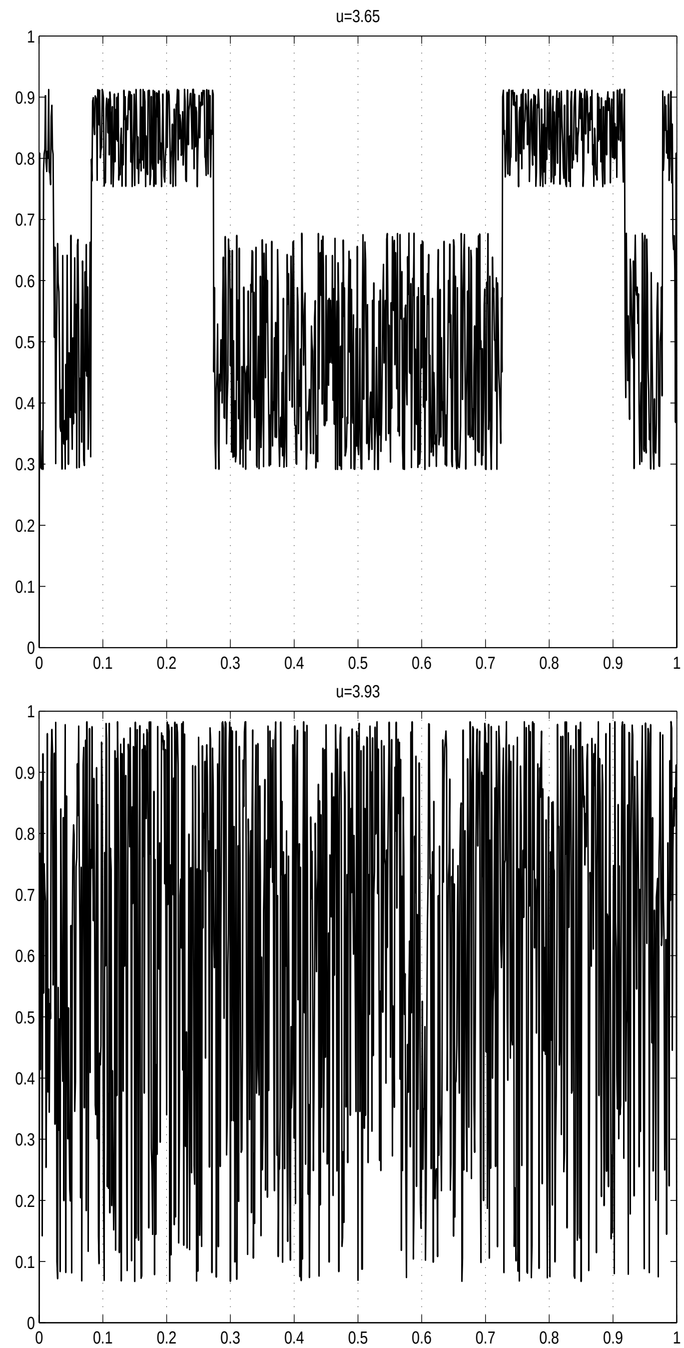

Fig. 6. The graphs of F_μ^{400} with $\mu = 3.65$ and $\mu = 3.93$, respectively.

Theorem is:

Theorem II.1 *Let I be a bounded closed interval and $f\colon I \to I$ be continuous. Let $n \rhd k$ be in Sarkovskii's ordering. If f has a (prime) period n orbit, then f also has a (prime) period k orbit.* □

Sarkovskii's Theorem provides a rough answer to the question of how infinitely many periodic points arise as the parameter is varied. Before F_μ can possibly have infinitely many periodic points with distinct periods, it must have periodic points with all periods of the forms 2^j. The local bifurcation theory provides two "typical" ways that these periodic points can arise: in saddle node bifurcations and via period-doublings. The question then becomes which type of bifurcations occur as F_μ becomes more chaotic.

As we shall see, the usual scenario for F_μ to become chaotic is for F_μ to undergo a series of period-doubling bifurcations. This is not always the case, but it is typical route to chaos. Recall that the graphs of F_μ for various values of μ are as depicted in Fig.7. For $1 < \mu < 3$, F_μ has a unique attracting fixed point at $p_\mu = (\mu - 1)/\mu$ so that $0 < p_\mu < 1$. Note that, as long as $F'_\mu(p_\mu) < 0$, there exists a "partner" $\hat{p_\mu}$ for p_μ in the sense that $F_\mu(\hat{p_\mu}) = p_\mu$ and $\hat{p_\mu} < p_\mu$.

Using graphical analysis of F_μ, we may also sketch the graphs of F^2_μ for various μ-values. These are depicted in Fig.8. Note in particular the portion of the graph of F^2_μ in the interval $[\hat{p_\mu}, p_\mu]$. We have enclosed this portion of the graph inside a box. Let us make three observations about this graph

- The graph of F^2_μ, although "upside-down", resembles the graph of the original quadratic map (for a different μ-value) in a sense to be made precise later.

- Indeed, inside the box, F^2_μ has one fixed point at an endpoint of the interval $[\hat{p_\mu}, p_\mu]$ and a unique critical point within this interval.

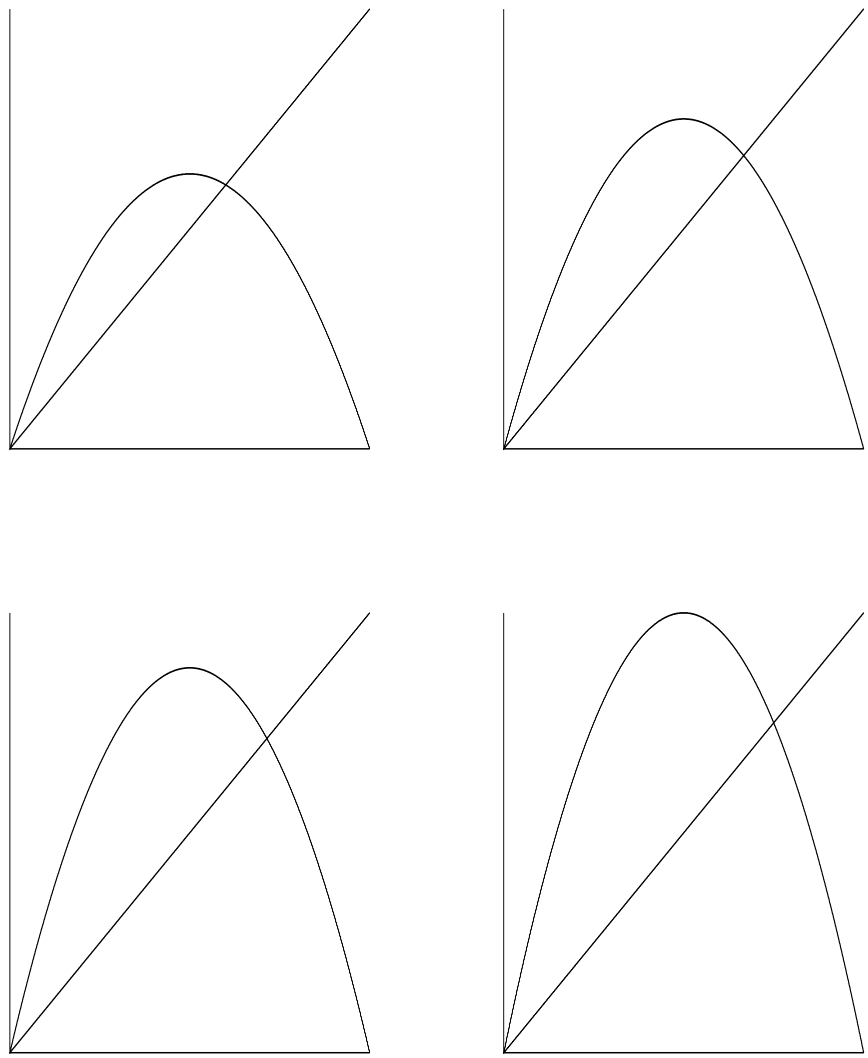

Fig. 7. The graphs of $F_\mu(x) = \mu x(1-x)$ for $\mu = 2.5$, 3, 3.5, 4 from left to right.

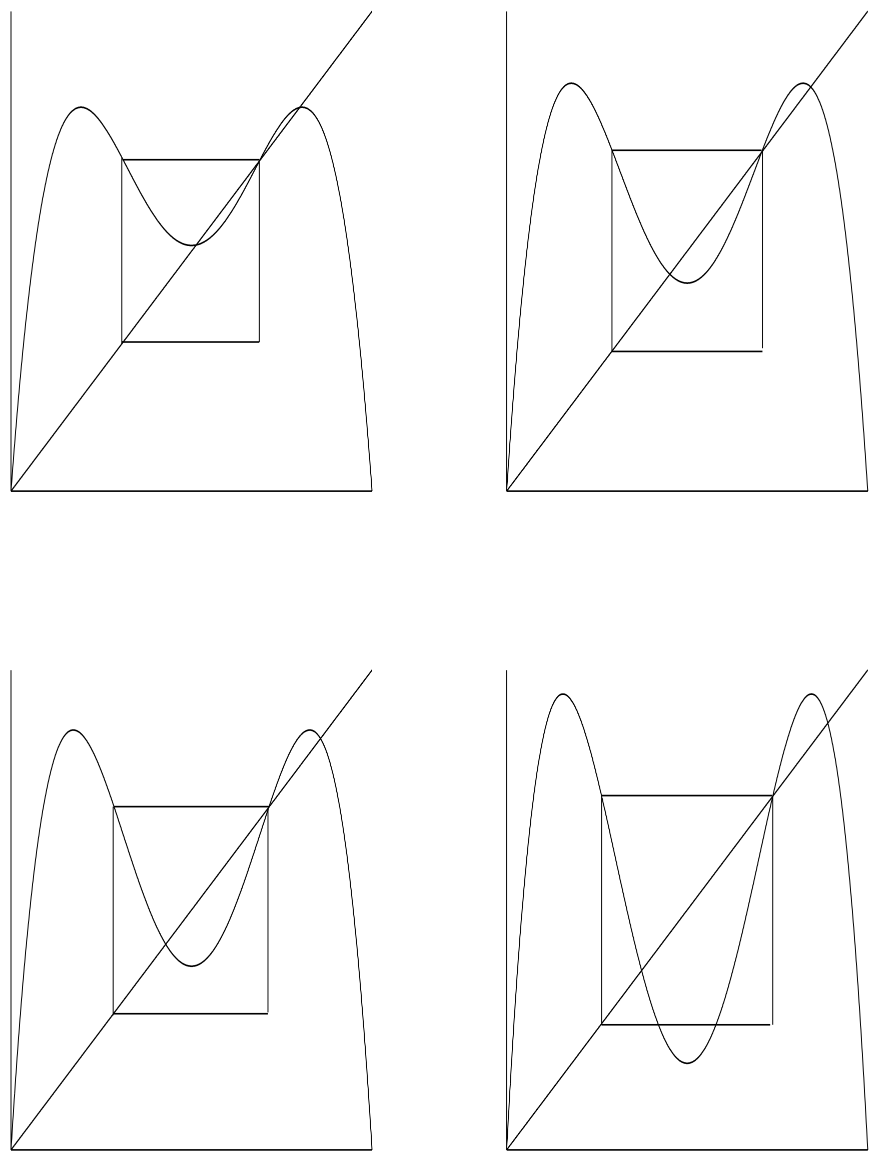

Fig. 8. The graphs of $F_\mu^2(x) = \mu x(1-x)$ for $\mu = 3.2$, 3.4, 3.5, 3.8 from left to right.

- As μ increases, the "hump" in this quadratic-like map grows until it eventually protrudes through the bottom of the box.

That is, the behavior of F_μ^2 on the interval $[\hat{p_\mu}, p_\mu]$ is similar to that of F_μ on its original domain [0,1]. In particular, as μ increases, we first expect a new fixed point in $[\hat{p_\mu}, p_\mu]$ for F_μ^2 (i.e., a period 2 point for F_μ) to be born. Eventually, this "fixed point" will itself period-double, just as p_μ did for F_μ, producing a period 4 point. Continuing this procedure, we may find a small box in which the graphs of F_μ^4, F_μ^8, ect., resemble the original quadratic function. Thus we are led to a succession of period-doubling bifurcations as μ increases. Hence we expect that the bifurcation digram for F_μ will include at least the complication shown in Fig.9.

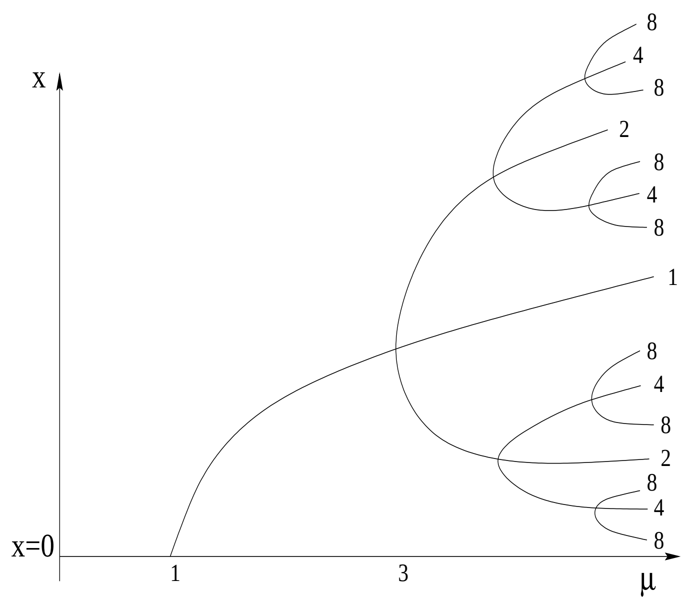

Fig. 9. The bifurcation diagram for F_μ showing the repeated period doubling. The integers represent the periods.

The computer allows us to verify these facts experimentally. Let us compute *the orbit digram* of F_μ. The orbit digram is a picture of the asymptotic behavior of the orbit of 1/2 for a variety of different $\mu-$values between 0 and 4. Note how the bifurcation diagram for F_μ is embedded in this picture.

We see clearly in Fig.4 many facts that we have discussed. For instance, when $0 \leq \mu \leq 1$, all orbits converge to the single attracting fixed point at 0. Note that this convergence is slow when $F'_\mu(0)$ is near 1 (when μ is close to 1). This accounts for the slight smear of points visible near this point in the orbit diagram. For $1 \leq \mu \leq 3$, all orbits are attracted to the fixed point $p_\mu \neq 0$, and this again is near from the orbit diagram. Thereafter, we see a succession of period-doubling bifurcation, confirming what we described above.

Notice that, for many μ-values beyond the period-doubling regime, it appears that the orbit of 1/2 fills out an interval. It is, of course, difficult to determine whether the orbit is really attracted to an attracting periodic orbit of very high period in this case, or whether it is in fact dense in an interval. The computer, with its limited precision, can not satisfactorily separate these two cases. Nevertheless, the orbit diagram gives experimental evidence that many of the μ-values after the period-doubling regime lead to chaotic dynamics.

This is shown more convincingly in Fig.4, where we have magnifield the portion of the orbit diagram corresponding to $3 \leq \mu \leq 4$. Note that the succession of period-doublings is plainly visible in this figure.

Consider the second example, system (1.1) with the boundary and initial conditions (1.2), which has been shown that when the initial data are smooth satisfying certain compatibility conditions at the boundary points, the space-time trajectory can be chaotic. Two period-doubling bifurcation theorems have been established perfectly [26]. If following immediately from Theorem 3.2 [26, p. 423], when fixing $\alpha = 0.5$ and $\beta = 1$, we consider the map $G_\alpha \circ F_{\alpha,\beta}$ (see [26], formula (9) and (10)) and let η vary in $(1, \infty)$, the orbit diagram of $G_\alpha \circ F_{\alpha,\beta}$ is depicted in Fig.10.

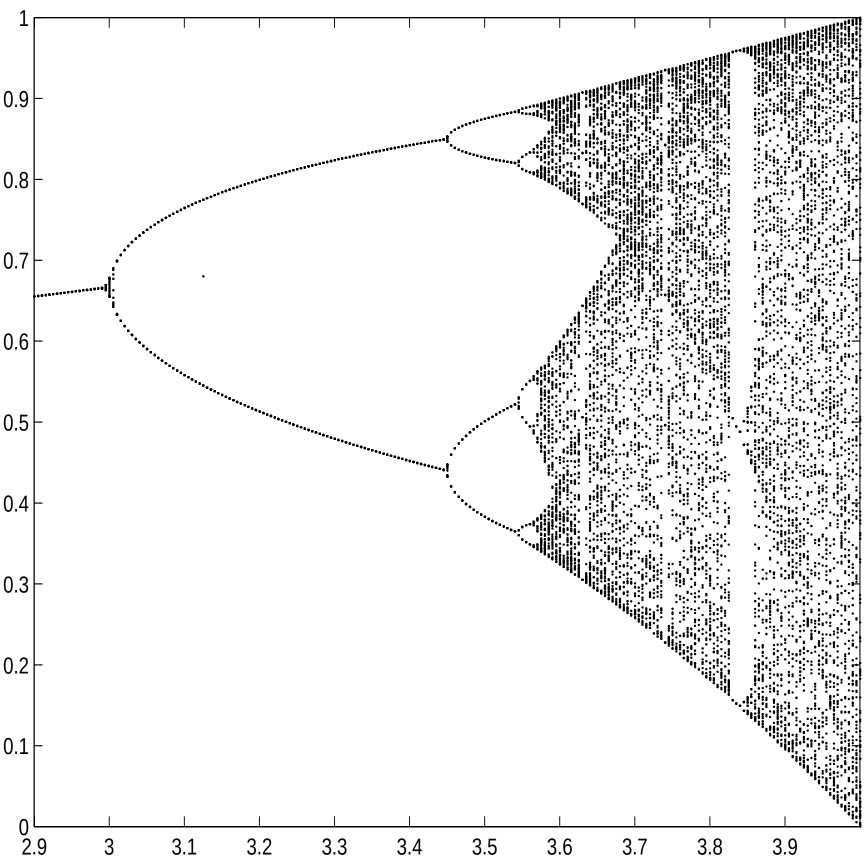

Fig. 10. The orbit diagram of $G_\alpha \circ F_{\alpha,\beta}$, where $\alpha = 0.5$, $\beta = 1$ and η varies in $[1.4, 2.5]$, for example 2. Note that the first period doubling occurs near $\eta_0 \approx 2.312$, agreeing with (62).

The above arguments indicate that periodic points or solutions play an important role in analyzing chaotic behavior of dynamical systems. This fact as well as Part (i) of Definition I.1 motivates us beginning our study with considering periodic solutions of system (1.10).

B. 2-Periodic Solution

Making use of (1.21), we obtain the following immediately

$$\begin{cases} v_1(x) = v(x,1) = \frac{1+\eta}{1-\eta} u_0(1-x) \\ u_1(x) = u(x,1) = y'(x) + v_0(1-x), \end{cases} \quad (0 \le x \le 1) \tag{2.1}$$

and

$$\begin{cases} v_2(x) = v(x,2) = \frac{1+\eta}{1-\eta} y'(1-x) + \frac{1+\eta}{1-\eta} v_0(1-x) \\ u_2(x) = u(x,2) = y'(x+1) + \frac{1+\eta}{1-\eta} u_0(x). \end{cases} \quad (0 \le x \le 1) \tag{2.2}$$

(2.1) and (2.2) can also be easily verified by utilizing the reflection of characteristics, see Fig.1.

First, we consider the 2-periodic solution of $(u,\ v)$ defined by (1.10). Define a map T: $\mathbb{B} \times \mathbb{B} \to \mathbb{B} \times \mathbb{B}$

$$T[u_0(x),\ v_0(x)] \to [u_2(x),\ v_2(x)], \tag{2.3}$$

where $\mathbb{B}$ denotes the space containing all continuous and bounded functions.

Suppose that $(U_0(x),\ V_0(x))$ is a 2-periodic solution of (1.10). Since a 2-periodic solution of (1.10) must be a fixed point of (2.3), that is,

$$T[U_0(x),\ V_0(x)] \to [U_2(x),\ V_2(x)] = [U_0(x),\ V_0(x)].$$

Using (2.1) and (2.2), we get

$$\begin{cases} U_2(x) = y'(x+1) + \frac{1+\eta}{1-\eta}U_0(x) = U_0(x) \\ V_2(x) = \frac{1+\eta}{1-\eta}y'(1-x) + \frac{1+\eta}{1-\eta}V_0(1-x) = V_0(x), \end{cases} \quad (0 \le x \le 1) \tag{2.4}$$

i.e.,

$$\begin{cases} y'(1+x) = -\frac{2\eta}{1-\eta}u_0(x) \\ y'(1-x) = -\frac{2\eta}{1-\eta}v_0(x) \end{cases} \quad (0 \le x \le 1). \tag{2.5}$$

Notice that from (1.18), for any t on the interval $[0, 2]$, we have

$$\begin{cases} y'(t) = -\frac{\mu_2}{1-\gamma}y^3(t) + \frac{\mu_1}{1-\gamma}y(t) - \frac{2v_0(1-t)}{1-\gamma} \\ y(0) = \phi_1(1) = w(1,0), \end{cases} \quad (0 \le t \le 1) \tag{2.6}$$

and

$$\begin{cases} y'(t) = -\frac{\mu_2}{1-\gamma}y^3(t) + \frac{\mu_1}{1-\gamma}y(t) - \frac{2v_1(2-t)}{1-\gamma} \\ y(1) = w(1,1), \end{cases} \quad (1 \le t \le 2) \tag{2.7}$$

respectively.

Using (2.1), (2.5) can be rewritten as

$$\begin{cases} y'(t) = -\frac{2\eta}{1-\eta}u_0(t-1), & (1 \le t \le 2) \\ y'(t) = -\frac{2\eta}{1-\eta}v_0(1-t), & (0 \le t \le 1). \end{cases} \tag{2.8}$$

Combining the second equation of (2.8) with (2.6), we thus have

$$\begin{cases} y'(t) = -\frac{\mu_2\eta}{1+\eta\gamma}y^3(t) - \frac{\mu_1\eta}{1+\eta\gamma}y(t) \\ y(0) = \phi_1(1). \end{cases} \quad (0 \le t \le 1) \tag{2.9}$$

This is the Bernoulli equation with constant coefficients. Making a transformation $zy^2 = 1$, we obtain the solutions to (2.9) as

$$y(t) = \pm\frac{1}{\sqrt{[\phi_1^{-2}(1) - \frac{\mu_2}{\mu_1}]e^{\frac{2\mu_1\eta}{1+\eta\gamma}t} + \frac{\mu_2}{\mu_1}}}. \tag{2.10}$$

According to (2.4), $V_0(x)$ is

$$\begin{aligned} V_0(x) &= -\frac{1+\eta}{2\eta}y'(1-x) \\ &= \pm\frac{(1+\eta)\mu_1[\phi_1^{-2}(1)-\frac{\mu_2}{\mu_1}]e^{\frac{2\mu_1\eta}{1+\eta\gamma}x}}{2(1+\eta r)\sqrt{\left([\phi_1^{-2}(1)-\frac{\mu_2}{\mu_1}]e^{\frac{2\mu_1\eta}{1+\eta\gamma}x}+\frac{\mu_2}{\mu_1}\right)^3}}. \end{aligned} \tag{2.11}$$

Similarly, combining the first equation of (2.8) and (2.7), we have

$$\begin{cases} y'(t) = -\frac{\mu_2}{1-\gamma}y^3(t) + \frac{\mu_1}{1-\gamma}y(t) + \frac{1+\eta}{\eta(1-\gamma)}y'(t) & (1 \le t \le 2) \\ y(1) = w(1,1), \end{cases}$$

that is,

$$\begin{cases} y'(t) = -\frac{\mu_2\eta}{1+\eta\gamma}y^3(t) - \frac{\mu_1\eta}{1+\eta\gamma}y(t) & (1 \le t \le 2) \\ y(1) = w(1,1). \end{cases} \tag{2.12}$$

Note that (2.12) is of the same form as (2.9). The differences between them are the initial conditions and the range of t. Recall that we define $y(t) = w(1,t) = \int_0^t w_t(1,t)dt$, hence the expression of the solution to (2.12) should be the same as (2.10) while t ranges from 1 to 2.

Making use of (2.4) again, we obtain

$$\begin{aligned} U_0(x) &= -\frac{1-\eta}{2\eta}y'(1+x) \\ &= \mp\frac{(1-\eta)\mu_1[\phi_1^{-2}(1)-\frac{\mu_2}{\mu_1}]e^{\frac{2\mu_1\eta}{1+\eta\gamma}x}}{2(1+\eta\gamma)\sqrt{\left([\phi_1^{-2}(1)-\frac{\mu_2}{\mu_1}]e^{\frac{2\mu_1\eta}{1+\eta\gamma}x}+\frac{\mu_2}{\mu_1}\right)^3}}. \end{aligned} \tag{2.13}$$

This means when $(U_0(x), V_0(x))$ satisfy (2.11) and (2.13), respectively, the map T defined by (2.3) does have an fixed point from $\mathbb{B}\times\mathbb{B} \to \mathbb{B}\times\mathbb{B}$. Note that if T has a 2-periodic solution from $\mathbb{B}\times\mathbb{B} \to \mathbb{B}\times\mathbb{B}$, $y(t)$ must satisfy another condition

$$y(0) = y(2).$$

Utilizing (1.13) and (2.10), we have

$$\begin{aligned} y(0) &= \phi_1(1) = w(1,0), && (2.14)\\ y(1) &= \pm\frac{1}{\sqrt{[y(0)^{-2}-\frac{\mu_2}{\mu_1}]e^{\frac{2\mu_1\eta}{1+\eta\gamma}}+\frac{\mu_2}{\mu_1}}}, && (2.15)\\ y(2) &= \pm\frac{1}{\sqrt{[y(1)^{-2}-\frac{\mu_2}{\mu_1}]e^{\frac{2\mu_1\eta}{1+\eta\gamma}}+\frac{\mu_2}{\mu_1}}}. && (2.16) \end{aligned}$$

Letting $y(0) = y(2)$, by (2.14)-(2.16), we get

$$y(0)^2 \cdot \frac{\mu_2}{\mu_1} \cdot [1 - e^{\frac{2\mu_1\eta}{1+\eta\gamma}}][1 + e^{\frac{2\mu_1\eta}{1+\eta\gamma}}] + e^{\frac{4\mu_1\eta}{1+\eta\gamma}} = 1. \quad (2.17)$$

After simplification, (2.17) reduces to

$$y(0) = \pm\sqrt{\frac{\mu_1}{\mu_2}}. \quad (2.18)$$

This implies that only in the case where $y(0) = \pm\sqrt{\frac{\mu_1}{\mu_2}}$, T has a unique 2-periodic solution, which is indeed constant.

Now, let us go back to (2.10), we only consider the case of "+" sign on the right hand side. The arguments for the case of "-" sign are closely similar.

Since the derivative of (2.10) is

$$y(t) = -\frac{1}{2}\frac{(\phi_1^{-2}(1)-\frac{\mu_2}{\mu_1})\frac{2\mu_1\eta}{1+\eta\gamma}e^{\frac{2\mu_1\eta}{1+\eta\gamma}t}}{\sqrt{\left([\phi_1^{-2}(1)-\frac{\mu_2}{\mu_1}]e^{\frac{2\mu_1\eta}{1+\eta\gamma}t}+\frac{\mu_2}{\mu_1}\right)^3}},$$

it is straightforward to obtain

(1). If $\phi_1^{-2}(1) - \frac{\mu_2}{\mu_1} < 0$, i.e., $y(0) > \frac{\mu_1}{\mu_2}$, then $y(t)$ is increasing.

(2). If $\phi_1^{-2}(1) - \frac{\mu_2}{\mu_1} > 0$, i.e., $y(0) < \frac{\mu_1}{\mu_2}$, then $y(t)$ is decreasing.

(3). If $\phi_1^{-2}(1) - \frac{\mu_2}{\mu_1} = 0$, i.e., $y(0) = \frac{\mu_1}{\mu_2}$, then $y(t)$ is constant.

The graphs of the solutions to (2.9) for arbitrary t are sketched as follows (Fig.11):

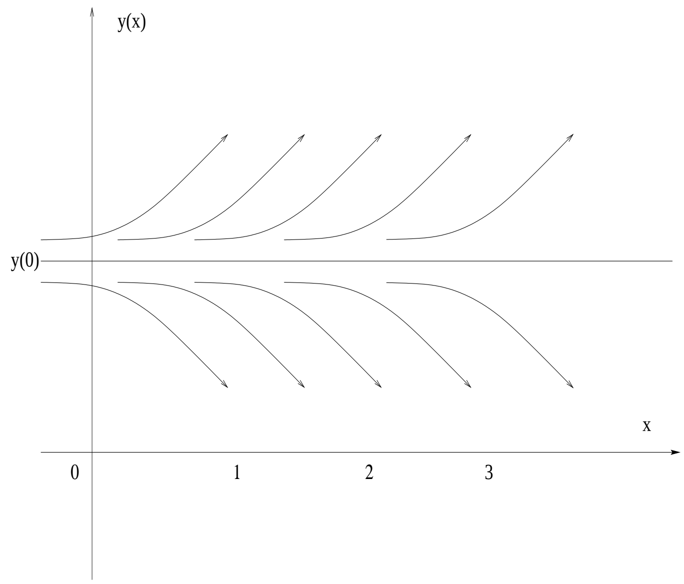

Fig. 11. The graphs of the solutions to (2.9) with initial condition $y(0) = \phi_1(1)$ for arbitrary t.

From the above picture, we can visually observe that the 2-periodic solution ($y(t) = \frac{\mu_1}{\mu_2}$) is unstable based on the sense of Liapunov Stability. This conclusion can be proven rigorously by starting to consider the graph of the cubic function $f(x) = \frac{\mu_2\eta}{1+\eta\gamma}x^3 - \frac{\mu_1\eta}{1+\eta\gamma}x$ (Fig.12):

Since $\frac{\mu_2\eta}{1+\eta\gamma} > 0$ and $\frac{\mu_1\eta}{1+\eta\gamma} > 0$, the graph of $f(x)$ intersects x-axis at three real points: $E(-\sqrt{\frac{\mu_1}{\mu_2}}, 0)$, $F(-\sqrt{\frac{\mu_1}{\mu_2}}, 0)$ and the origin $(0, 0)$. From Fig.12, there always exists a positive number $L > 0$, and such that

$$\frac{\mu_2\eta}{1+\eta\gamma}L^3 - \frac{\mu_1\eta}{1+\eta\gamma}L < 0,$$

where $L < \sqrt{\frac{\mu_1}{\mu_2}}$.

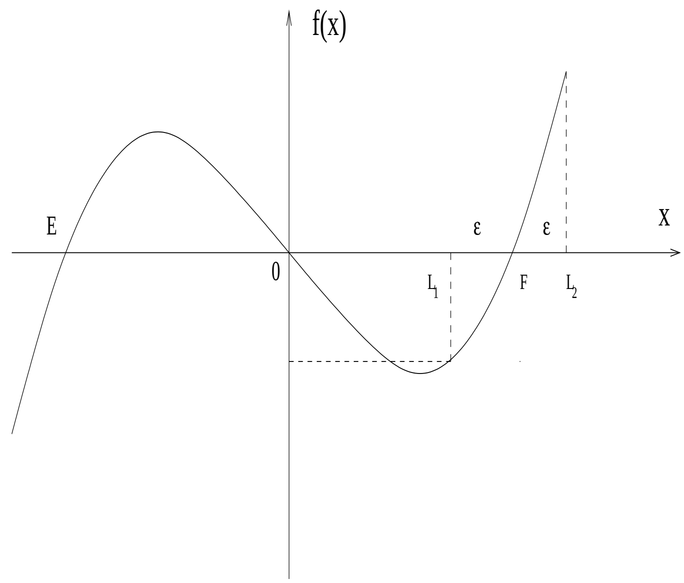

Fig. 12. The graph of the cubic function $f(x) = \frac{\mu_2\eta}{1+\eta\gamma}x^3 - \frac{\mu_1\eta}{1+\eta\gamma}x$.

Choosing $L_1 = \sqrt{\frac{\mu_1}{\mu_2}} - \varepsilon$ and $L_2 = \sqrt{\frac{\mu_1}{\mu_2}} + \varepsilon$ ($\varepsilon > 0$), then we have

$$\frac{dL_1}{dt} = 0 > \frac{\mu_2\eta}{1+\eta\gamma}L_1^3 - \frac{\mu_1\eta}{1+\eta\gamma}L_1,$$
$$\frac{dL_2}{dt} = 0 < \frac{\mu_2\eta}{1+\eta\gamma}L_2^3 - \frac{\mu_1\eta}{1+\eta\gamma}L_2.$$

Letting $\varepsilon \to 0^+$, the above are always true. According to the definition of Liapunov stability, we conclude that the 2-periodic solution $y(t) = \sqrt{\frac{\mu_1}{\mu_2}}$ to (2.9) for arbitrary t is unstable. The direction fields of (2.9) for all $t \geq 0$ are illustrated as in Fig.13 in the case where $\mu_1 = 2,\ \mu_2 = 1$.

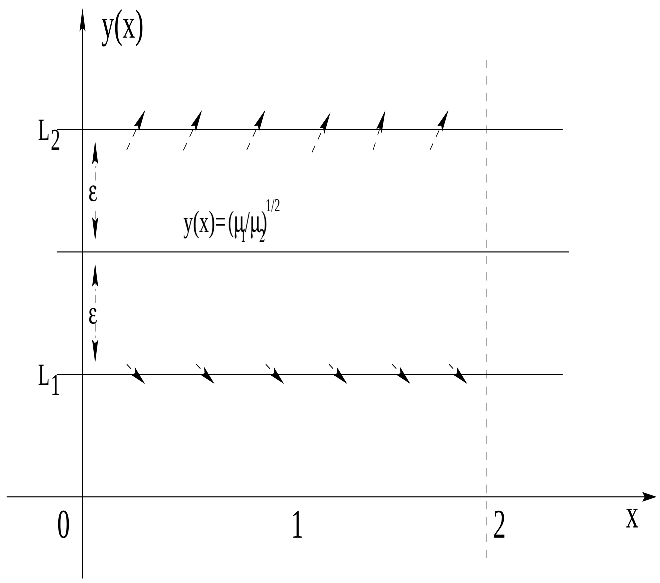

Fig. 13. The direction fields of (2.9) for $t \geq 0$.

C. 4-Periodic Solution

Now, we extend our study to the case of 4-periodic solution of the map T. Again utilizing (1.21), we can derive the following immediately

$$\begin{cases} u_3(x) = u(x,3) = y'(x+2) + v(1, x+2) \\ v_3(x) = v(x,3) = \frac{1+\eta}{1-\eta} u(1-x, 2), \end{cases} \qquad (0 \leq x \leq 1)$$

and

$$\begin{cases} u_4(x) = u(x,4) = y'(x+3) + \frac{1+\eta}{1-\eta} y'(x+1) + [\frac{1+\eta}{1-\eta}]^2 u_0(x) \\ v_4(x) = v(x,4) = \frac{1+\eta}{1-\eta} y'(3-x) + [\frac{1+\eta}{1-\eta}]^2 y'(1-x) + [\frac{1+\eta}{1-\eta}]^2 v_0(x), \end{cases} \qquad (0 \leq x \leq 1) \tag{2.19}$$

We first consider the particular case of $\eta = 0$. Thus (2.19) becomes

$$\begin{cases} u_4(x) = y'(x+3) + y'(x+1) + u_0(x) \\ v_4(x) = y'(3-x) + y'(1-x) + v_0(x), \end{cases} \quad (0 \le x \le 1) \tag{2.20}$$

Assume that $[U_0(x),\ V_0(x)]$ is a 4-periodic solution of T, i.e.,

$$T[U_0(x),\ V_0(x)] \to [U_4(x),\ V_4(x)] = [U_0(x),\ V_0(x)], \tag{2.21}$$

then, from (2.20), we have

$$\begin{cases} y(3-x) + y(1-x) = c_1 \\ y(3+x) + y(1+x) = c_2, \end{cases} \quad (0 \le x \le 1) \tag{2.22}$$

where c_1 and c_2 are constants. One can see that c_1 and c_2 in (2.22) are actually equal to each other. Hence we have c replace each of them. This is because using (2.22) itself and letting $x = 1$, we obtain

$$y(0) + y(2) = c_1, \tag{2.23}$$

$$y(2) + y(4) = c_1, \tag{2.24}$$

respectively. Due to the assumption that $[U_0(x),\ V_0(x)]$ is a 4-periodic solution of T, and thereby $y(0) = y(4)$, from (2.23) and (2.24) one can get $c_1 = c_2$ directly.

Recall that (see (1.18)):

(i). When $t \in [0, 1]$, we have

$$\begin{cases} y'(t) = -\frac{\mu_2}{1-\gamma} y^3(t) + \frac{\mu_1}{1-\gamma} y(t) - \frac{2v_0(1-t)}{1-\gamma} \\ y(0) = \phi_1(1) = w(1, 0). \end{cases} \tag{2.25}$$

(ii). When $t \in [1,2]$, we have

$$\begin{cases} y'(t) = -\frac{\mu_2}{1-\gamma}y^3(t) + \frac{\mu_1}{1-\gamma}y(t) - \frac{2v_1(2-t)}{1-\gamma} \\ y(1) = w(1,1). \end{cases} \tag{2.26}$$

(iii). When $t \in [2,3]$, from (2.22) we have

$$\begin{cases} y'(t) = -\frac{\mu_2}{1-\gamma}y^3(t) + \frac{\mu_1}{1-\gamma}y(t) - \frac{2v_2(3-t)}{1-\gamma} \\ y(2) = c - y(0). \end{cases} \tag{2.27}$$

(iv). When $t \in [3,4]$, from (2.22) we have

$$\begin{cases} y'(t) = -\frac{\mu_2}{1-\gamma}y^3(t) + \frac{\mu_1}{1-\gamma}y(t) - \frac{2v_3(4-t)}{1-\gamma} \\ y(3) = c - y(1). \end{cases} \tag{2.28}$$

Notice that the first equation of (2.22) can be re-expressed as

$$y(t+2) = c - y(t), \qquad 0 \le t \le 1.$$

Using this equality, the ordinary differential equation in (iii) can be re-written as

$$\begin{cases} -y'(t) = -\frac{\mu_2}{1-\gamma}[c - y(t)]^3 + \frac{\mu_1}{1-\gamma}[c - y(t)] - \frac{2}{1-\gamma}[y'(t) + v_0(1-t)] \\ y(0) = \phi_1(1). \end{cases} \quad (0 \le t \le 1) \tag{2.29}$$

Combining (2.29) with (i), we have

$$\frac{2r}{1-\gamma}y'(t) = \frac{\mu_2}{1-\gamma}(y^3(t) - [c - y(t)]^3) + \frac{\mu_1}{1-\gamma}[c - 2y(t)], \quad (0 \le t \le 1)$$

that is,

$$y'(t) = \frac{\mu_2}{2\gamma}(y^3(t) - [c - y(t)]^3) + \frac{\mu_1}{2\gamma}[c - 2y(t)]. \quad (0 \le t \le 1) \tag{2.30}$$

Similarly, using the fact that $y(t+3) = c - y(t+1)$ $(0 \leq t \leq 1)$ and the transformation $t = t+1$, the ordinary differential equation in (iv) can be re-written as

$$-y'(t) = -\frac{\mu_2}{1-\gamma}[c - y(t)]^3 + \frac{\mu_1}{1-\gamma}[c - y(t)] - \frac{2}{1-\gamma}[y'(t) + u_0(t-1)], \quad (1 \leq t \leq 2) \tag{2.31}$$

Combining (2.31) with (ii), we get

$$[\frac{2}{1-\gamma} - 1]y'(t) = -\frac{\mu_2}{1-\gamma}[c - y(t)]^3 + \frac{\mu_1}{1-\gamma}[c - y(t)] + [y'(t) + \frac{\mu_2}{1-\gamma}y^3(t) - \frac{\mu_1}{1-\gamma}y(t)], \quad (1 \leq t \leq 2)$$

that is,

$$y'(t) = \frac{\mu_2}{2\gamma}(y^3(t) - [c - y(t)]^3) + \frac{\mu_1}{2\gamma}[c - 2y(t)], \quad (1 \leq t \leq 2) \tag{2.32}$$

It is seen that (2.30) and (2.32) have the same form. The differences between them are: (a). (2.30) is with the initial condition $y(0) = \phi_1(1)$ and (2.32) is with the initial condition $y(1) = w(1,1)$, and (b). the range of t for (2.30) is $[0,1]$ and the the range of t for (2.32) is $[1,2]$. Therefore, (2.30) and (2.32) can be put together as follows:

$$\begin{cases} y'(t) = \frac{\mu_2}{2\gamma}(y^3(t) - [c - y(t)]^3) + \frac{\mu_1}{2\gamma}[c - 2y(t)] \\ y(0) = \phi_1(1), \end{cases} \quad (0 \leq t \leq 2) \tag{2.33}$$

From (2.33), by virtue of the theorem of existence and uniqueness of the solution, for a given $\phi_1(1)$, there is a unique value s for $y(2)$. According to (2.22), $y(0)+y(2)$ is fixed. Hence, if $y(0) = y(4)$, this equality holds only in situation $\phi_1(1) = c - s$. This indicates that system (1.10) with conditions (1.11) and (1.12) only has one 4-periodic solution at most. Since a 2-periodic solution must be a 4-periodic solution. Therefore, system (1.10) with conditions (1.11) and (1.12) has a unique 4-periodic solution in the case of $\eta = 0$.

Next, we are ready to consider 4-periodic solution in the case of $\eta > 0$.

Theorem II.2 *System (1.10) with conditions (1.11) and (1.12) has a unique 4-periodic solution in the case of* $\eta > 0$. □

Proof. Suppose that T has a 4-periodic solution $[U_0(x),\ V_0(x)]$ in the case of $\eta > 0$, the necessary and sufficient conditions are

$$T[U_0(x),\ V_0(x)] \to [U_4(x),\ V_4(x)] = [U_0(x),\ V_0(x)] \quad (\eta > 0), \tag{2.34}$$

and

$$y(0) = y(4). \tag{2.35}$$

(2.34) is equivalent to

$$y'(x+3) + \frac{1+\eta}{1-\eta} y'(1+x) + [\frac{1+\eta}{1-\eta} - 1]^2 u_0(x) = 0, \tag{2.36}$$

$$\frac{1+\eta}{1-\eta} y'(3-x) + [\frac{1+\eta}{1-\eta} - 1]^2 y'(1-x) + [\frac{1+\eta}{1-\eta} - 1]^2 v_0(x) = 0, \quad (0 \le x \le 1) \tag{2.37}$$

Using equation (ii), i.e.,

$$\begin{cases} y'(1+x) = -\frac{\mu_2}{1-\gamma} y^3(1+x) + \frac{\mu_1}{1-\gamma} y(1+x) - \frac{2v_1(1-x)}{1-\gamma} \\ y(1) = w(1,1), \end{cases} \quad (0 \le x \le 1)$$

and

$$v_1(1-x) = \frac{1+\eta}{1-\eta} u_0(x),$$

we have

$$\begin{aligned} u_0(x) = \quad & -\frac{1-\gamma}{2} \cdot \frac{1-\eta}{1+\eta} y'(1+x) - \frac{\mu_2}{2} \cdot \frac{1-\eta}{1+\eta} y^3(1+x) \\ & + \frac{\mu_1}{2} \cdot \frac{1-\eta}{1+\eta} y(1+x). \end{aligned} \tag{2.38}$$

Substituting (2.38) into (2.36), we obtain

$$\begin{aligned} y'(x+3) + \frac{1+\eta}{1-\eta} y'(1+x) - \frac{2\eta(1-\gamma)}{1-\eta^2} y'(1+x) - \frac{2\eta\mu_2}{1-\eta^2} y^3(1+x) \\ + \frac{2\eta\mu_1}{1-\eta^2} y(1+x) = 0, \qquad (0 \le x \le 1) \end{aligned} \tag{2.39}$$

Equation (2.39) is a first order delay differential equation, and the "delay" or "time lag" is 2. It can be solved using the method of steps [72, 73], which is a very useful approach to deal with the first-order delay equation.

Note that the initial function to (2.39): $\theta(x) = y(1+x)$ $(0 \le x \le 1)$ determined by (2.33) is unique. (Similarly, the initial function to (2.37): $\theta(x) = y(1-x)$ $(0 \le x \le 1)$ determined by (2.33) is also unique). Thus, we have

$$y'(x+3) = -\frac{1+\eta}{1-\eta}\theta'(x) + \frac{2\eta(1-\gamma)}{1-\eta^2}\theta'(x) + \frac{2\eta\mu_2}{1-\eta^2}\theta^3(x) - \frac{2\eta\mu_1}{1-\eta^2}\theta(x). \tag{2.40}$$

Denote that $G(x)$ is equal to the right hand side of (2.40). i.e.,

$$G(x) = -\frac{1+\eta}{1-\eta}\theta'(x) + \frac{2\eta(1-\gamma)}{1-\eta^2}\theta'(x) + \frac{2\eta\mu_2}{1-\eta^2}\theta^3(x) - \frac{2\eta\mu_1}{1-\eta^2}\theta(x).$$

Then, equation (2.40) reduces to

$$y(x+3) = \int G(x)dt + C, \qquad 0 \le x \le 1. \tag{2.41}$$

In order to make (2.35) hold, from (2.41), there is at most one constant $C = C_0$ such that

$$[\int G(x)dt + C_0]\Big|_{x=1} = y(0), \qquad 0 \le x \le 1.$$

This implies that T has at most one 4-periodic solution in the case of $\eta > 0$. Consider that a 2-periodic solution must be a 4-periodic solution. Therefore, the proof of the Theorem II.2 is completed. □

D. 2n-Periodic Solution

Continuing the discussions in the same manner in conjunction with making use of (1.21) and by mathematical induction, we have

$$\begin{aligned} y'(x+2n-1) = \quad & -\Big[\frac{1+\eta}{1-\eta}\Big]^{2n-3} y'(x+2n-3) - \Big[\frac{1+\eta}{1-\eta}\Big]^{2n-5} y'(x+2n-5) \\ & - \cdots - \frac{1+\eta}{1-\eta} y'(x+1) + \frac{2\eta(1-\gamma)}{1-\eta^2} y'(1+x) + \frac{2\eta\mu_2}{1-\eta^n} y^{2n-1}(1+x) \\ & - \cdots - \frac{2\eta\mu_1}{1-\eta^n} y(1+x) \quad (0 \le x \le 1). \end{aligned} \tag{2.42}$$

Similar to (2.25)-(2.28), we can list each first order ordinary differential equations in the intervals [0,1], [1,2], $\cdots$, [2n-1,2n], and in each interval, $y(x)$ is uniquely determined. Hence, using the method of steps, in the interval [2n-1,2n] there exists at most one constant c such that $y'(x+2n-1)$ $(0 \le x \le 1)$ satisfies (2.42) with condition $y(2n) = y(0)$. Therefore, we can conclude that system (1.10) with conditions (1.11) and (1.12) has a unique 2n-periodic solution in the case of $\eta > 0$, where n is arbitrary natural number.

CHAPTER III

INVARIANT INTERVAL FOR ABEL EQUATION

A. Preliminary Information

In the preceding chapter, we conclude that the map T does not have non-constant 2n-periodic solutions. Therefore, to study the occurrence of chaos of T, we can not use the chaotic theory associated with period doubling.

In this chapter, we consider the invariant interval of the Abel equation with variable coefficients

$$\begin{cases} y'(t) = \beta_3(t)y^3(t) + \beta_1(t)y(t) + \beta_0(t) \\ y(0) = y_0, \quad t \geq 0 \end{cases} \tag{3.1}$$

where $\beta_i(t) \ \ (i = 0, 1, 3)$ are continuous function and $y_0 \in \mathbb{R}$.

One of reasons why we are interested in Abel equation (3.1) is that we are trying to investigate the chaos of system (1.10) by means of the use of total variation as a measure of chaos. This requires us first to find an invariant interval for the map T which is strongly related to Abel equation. Another important reason is that, to find $u(1, t)$ of system (1.10) on the right end, each time we have to deal with Abel equation (1.18), which has the same form as (3.1).

Although equation (3.1) is only a first-order ordinary differential equation with a polynomial of order 3 on the right hand, seeking its exact solution is not a trivial problem, even in some very particular cases. Indeed, it is extremely interesting and somehow, more than challenging. The fact is that as we know, Abel equation is closely related to the Hilbert's sixteen problem which continues to attract widespread interest and is the source of a variety of questions on nonlinear differential equations. Information is sought on the number and possible configurations of limit cycles for systems of the form

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y), \end{cases} \tag{3.2}$$

where $P(x,y)$ and $Q(x,y)$ are polynomial functions. It is worthwhile to mention here that, a closely related question, and one which is of independent interest, is the derivation of necessary and sufficient conditions for a critical point of (3.2) to be a center. A center is a critical point in the neighborhood of which all orbits are closed; in contrast a limit cycle is an isolated closed orbit. When the origin is center, there is a first integral; thus the conditions for center can be interpreted as conditions for integrability. Some polynomial systems can be transformed to an equation of the form

$$\dot{z} = \alpha_1(t)z + \alpha_2(t)z^2 + \cdots + \alpha_n(t)z^n, \tag{3.3}$$

where the α_i are polynomials in $\sin t$ and $\cos t$ [74, 75]. It has been exploited in a number of previous papers [76-80 et al.] that when polynomial system (3.2) with a homogeneous nonlinearity can be transformed to equation (3.3) with $n = 3$.

On the other hand, studying Abel equation also has other meanings. For example, there are at least three classes of planar systems which are in some sense equivalent to Abel equations. The first planar polynomial systems of the form

$$\begin{cases} \dot{x} = -y + p(x,y) \\ \dot{y} = x + q(x,y), \end{cases} \tag{3.4}$$

with homogeneous polynomial $p(x,y)$ and $q(x,y)$ of degree m. There has been a longstanding problem, called the Painlevé center-focus problem, for the system (3.4): under what explicit conditions of $p(x,y)$ and $q(x,y)$, (3.4) has a center at the origin (0,0); i.e., all the orbits nearby are closed. This problem is equivalent to an analogue for a corresponding periodic Abel equation. To see this point, first let us note that the curves of (3.4) near the origin (0,0) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ are determined by

$$\frac{dr}{d\theta} = \frac{r^m \xi(\theta)}{1 + r^{m-1}\tau(\theta)}, \tag{3.5}$$

where ξ and τ are homogeneous polynomials in $\cos\theta$ and $\sin\theta$ of degree $m+1$, and can be easily expressed by $p(x, y)$ and $q(x, y)$. Then utilizing the transformation $\rho = r^{m-1}/(1 + r^{m-1}\tau(\theta))$ [81] to (3.5), we get a periodic Abel equation

$$\frac{d\rho}{d\theta} = a(\theta)\rho^2 + b(\theta)\rho^3, \tag{3.6}$$

where $a = (m-1)\xi + \tau$ and $b = (1-m)\xi\tau$. Hence the planar vector field (3.4) has a center at (0,0) if and only if the Abel equation (3.6) has a center at $\rho = 0$; that is, all the solutions nearly are closed: $\rho(0) = \rho(2\pi)$.

The second class is Liénard systems of the form [82,102]

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x). \end{cases}$$

Obviously, the orbits of this system are also determined through $y = 1/z$ by the Abel equation

$$\frac{dz}{dx} = f(x)z^2 + g(x)z^3.$$

The third one is several cubic systems which can be converted into Abel equations [83].

The periodic solutions of Abel equation have been studied extensively through using various methods, see [77, 84-86 et al.] for details. Other aspects on Abel equation such as recurrence relation for Bautin quantities, asymptotic behavior of the solutions, and center conditions have been investigated over the years. Typical references can be seen [87-91] and references therein.

B. Two Kinds of Abel equation

1. Abel Equation of the First Kind

In general, we call

$$y' = \sum_{i=0}^{3} f_i(x) y^i, \tag{3.7}$$

the Abel equation of the first kind. When f_i $(i = 1, 2, 3) \in \mathbf{C}^1$ and $f_3 \neq 0$, let

$$y(x) = w(x)\eta(\xi) - \frac{f_2(x)}{3f_3(x)}, \quad \xi = \int f_3(x) w^2(x) dx, \tag{3.8}$$

where

$$w(x) = \exp \int \Big(f_1(x) - \frac{f_2^2(x)}{3f_3(x)} \Big) dx. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), we are able to obtain a simple canonical form

$$\eta' = \eta^3 + R(x), \tag{3.10}$$

where $R(x)$ satisfies the equation

$$f_3(x) w^3(x) R(x) = f_0(x) + \frac{d}{dx}\Big(\frac{f_2(x)}{3f_3(x)}\Big) - \frac{f_1(x) f_2(x)}{3f_3(x)} + \frac{2f_2^3(x)}{27 f_3^2(x)}.$$

If there exists a constant α such that the function

$$\phi(x) = f_0(x) f_3^2(x) + \frac{1}{3}[f_2'(x) f_3(x) - f_2(x) f_3'(x) - f_1(x) f_2(x) f_3(x)] + \frac{2}{27} f_2^3(x),$$

satisfies the Bernoulli equation

$$f_3(x)\phi' + [f_2^2(x) - 3f_1(x) f_3(x) - 3f_3'(x)]\phi = 3\alpha\phi^{5/3},$$

then the solution to Abel equation (3.7) can be expressed as

$$y(x) = \frac{3\phi^{1/3} u(x) - f_2(x)}{3f_3(x)},$$

where $u(x)$ is determined by

$$\int \frac{du}{u^3 - \alpha u + 1} + c = \int \frac{\phi^{2/3}}{f_3(x)} dx.$$

2. Abel Equation of the Second Kind

The following two equations are called Abel equations of the second kind

$$(I). \quad [y + g(x)]y' = f_2(x)y^2 + f_1(x)y + f_0(x) \tag{3.11}$$

$$(II). \quad [g_1(x)y + g_0(x)]y' = \sum_{i=0}^{3} f_i(x)y^i. \tag{3.12}$$

If we make the transformation

$$u(x) = [y + g(x)]E(x), \qquad E(x) = \exp(-\int f_2(x)dx),$$

equation (3.11) can be reduced to

$$\begin{aligned} uu' &= [f_1(x) + g'(x) - 2g(x)f_2(x)]E(x)u + [f_0(x) - g(x)f_1(x) \\ &\quad + f_2(x)g^2(x)]E^2(x). \end{aligned} \tag{3.13}$$

If $y + g(x) \neq 0$, then make the transformation as

$$y + g(x) = \frac{1}{u(x)},$$

this will lead equation (3.13) to

$$\begin{aligned} &u' + [f_2(x)g^2(x) - g(x)f_1(x) + f_0(x)]u^3 + [f_1(x) - 2g(x)f_2(x) + g'(x)]u^2 \\ &+ f_2(x)u = 0 \end{aligned} \tag{3.14}$$

Note that (3.14) is a particular case of equation (3.7) and (3.13) is of the form

$$yy' = h_1(x)y + h_0(x). \tag{3.15}$$

Take the variable transformation as

$$y = v(x) + H(x), \qquad H(x) = \int h_1(x),$$

equation (3.15) reduces to

$$[v + H(x)]v' = h_0(x).$$

Then make the variable transformation:

$$v(x) = \eta(\xi), \qquad \xi = \int h_0(x)dx,$$

we obtain

$$[\eta + H(x)]\eta' = 1. \tag{3.16}$$

For equation (3.12), if we know a particular solution $y(x)$, and $g_1(x),\ g_0(x) \in \mathbf{C}^1$, and satisfy $g_1(x) \neq 0$, $g_1(x)y(x) + g_0(x) \neq 0$ for arbitrary x, then equation (3.12) can be changed to (3.7) by making the variable transformation as follows

$$\frac{1}{u(x)} = g_1(x)y(x) + g_0(x).$$

It is notable that both (3.10) and (3.16) have simple forms with a nonlinear term. Under some special cases, their solutions can be expressed explicitly. Unfortunately, in the general case, they are not integrable. This fact has been pointed out by Liouville exactly one century ago [92].

C. An Invariant Interval for Abel Equation in Banach Space

It is easy to see that for some particular $\beta_0(t)$ or if we know a particular solution of equation (3.1), then (3.1) can be converted into the following form by using a linear transformation $y(t) = y(t) + h(t)$

$$\begin{cases} y'(t) = f_3(t)y^3(t) + f_2(t)y^2(t) + f_1(t)y(t) \\ y(0) = y_1, \end{cases} \quad (t \geq 0) \tag{3.17}$$

Making the coordinate transformation as

$$y(t) = u(t) \cdot r(\xi), \quad \xi = \int u \cdot f_2 dt, \quad u(t) = \exp(\int f_1 dt) \tag{3.18}$$

(3.17) becomes

$$\begin{cases} r'(\xi) = a(\xi)r^3(\xi) + r^2(\xi) \\ r(\xi_0) = c \end{cases} \tag{3.19}$$

where $a(\xi) = u(t) \cdot \frac{f_3(t)}{f_2(t)}$.

Now, we restrict our attention to

$$r'(t) = a(t)r^2(t) + b(t)r^3(t), \quad t \in [t_0, t_1], \quad (t_1 > t_0 \geq 0) \tag{3.20}$$

with $r(t_0) = c$, since (3.19) is a particular case of it.

Let the solution $r(t, c)$ of (3.20) with $r(t_0, c) = c$ be expanded in power series

$$r(t, c) = c + r_2(t) \cdot c^2 + r_3(t) \cdot c^3 + \cdots. \tag{3.21}$$

Dividing (3.20) by $r^2(t)$ and integrating it from t_0 to t_1, we have

$$r(t, c) = \frac{c}{1 - c \cdot A(t) - c \cdot \int_{t_0}^{t_1} b(\tau)r(\tau)d\tau}, \tag{3.22}$$

where $A(t) = \int_{t_0}^{t_1} a(\tau)d\tau$ and c is a small integral constant.

The integral equation (3.22) can be re-expressed as

$$r(t, c) = c \cdot \Big[1 + A(t) \cdot r(t, c) + r(t, c) \cdot \int_{t_0}^{t_1} b(\tau)r(\tau, c)d\tau\Big]. \tag{3.23}$$

For convenience of statement, we may assume that $[t_0, t_1]$ be $[0, 1]$. Otherwise, use a fractional transformation. Assume that $\mathbb{B}[0,1]$ denotes the Banach space of all continuous functions on the interval $[0,1]$ with the norm $||f|| = max_{0\le t\le 1}|f(t)|$. It is easy to verify that if on $[0,1]$, $r(t,c)$ is continuous and satisfies (3.23), then the necessary and sufficient conditions are that it is continuously differentiable on $(0,1)$ and satisfies Abel equation (3.20) with the initial condition $r(0) = c$.

Define:

$$\begin{aligned} T_c : &\quad \mathbb{B}[0,1] \to \mathbb{B}[0,1] \\ T_c(f)(t) &= \frac{c}{1 - c\cdot A(t) - c\cdot \int_0^t b(\tau)f(\tau)d\tau}, \end{aligned}$$

for given $a, b \in \mathbb{B}[0,1]$ and $c \in \mathbb{R}$. One can see that T_c is well defined and differentiable on an arbitrary bounded set of $\mathbb{B}[0,1]$ if c is properly small.

Following a straightforward calculation, when $f, g \in \mathbb{B}[0,1]$ and $c \in \mathbb{R}$ with $||f|| \le M$, $||g|| \le M$ and $|c| < (||a|| + M\cdot ||b||)^{-1}$, we have

$$\begin{aligned} \frac{d}{dt}T_c(f)(t) &= \frac{c^2\cdot a(t) + c^2\cdot b(t)\cdot f(t)}{[1 - c\cdot A(t) - c\cdot \int_0^t b(\tau)f(\tau)d\tau]^2} \\ &= a(t)\cdot [T_c(f)(t)]^2 + b(t)[T_c(f)(t)]^3. \end{aligned}$$

and

$$\begin{aligned} T_c(f)(t) - T_c(g)(t) &= \frac{c}{1 - c\cdot A(t) - c\cdot \int_0^t b(\tau)f(\tau)d\tau} - \frac{c}{1 - c\cdot A(t) - c\cdot \int_0^t b(\tau)g(\tau)d\tau}, \\ &= \frac{c^2\cdot \int_0^t b(\tau)\cdot [f(\tau) - g(\tau)]d\tau}{[1 - c\cdot A(t) - c\cdot \int_0^t b(\tau)f(\tau)d\tau][1 - c\cdot A(t) - c\cdot \int_0^t b(\tau)g(\tau)d\tau]}, \\ &= T_c(f)(t)\cdot T_c(g)(t)\cdot \int_0^t b(\tau)[f(\tau) - g(\tau)]d\tau, \\ &\quad (0 \le t \le 1) \qquad (3.24) \end{aligned}$$

(I). If $c_1 = (||a|| + ||b|| + 1)^{-1}$, and let $||f|| \leq 1$ and $|c| \leq c_1$, then

$$\begin{aligned} |T_c(f)(t)| &= \Big|\frac{c}{1 - c\cdot A(t) - c\cdot \int_0^t b(\tau)f(\tau)d\tau}\Big| \\ &\leq \frac{|c|}{1 - |c|\cdot[||a|| + ||b||\cdot||f||]} \\ &\leq \frac{|c|}{1 - |c|\cdot[||a|| + ||b||]} \\ &\leq 1. \end{aligned}$$

That is,

$$||T_c(f)(t)|| \leq 1.$$

(II). If let $c_2 = (\sqrt{||b||} + ||a|| + ||b||)^{-1}$ and $|c| < c_2$, then T_c is a contract mapping on the closed unit ball $\mathbb{B}_1 = \{f \in \mathbb{B}[0,1], \quad ||f|| \leq 1\}$ of $\mathbb{B}[0,1]$.

This follows from (3.24) and (I), and

$$\begin{aligned} |T_c(f)(t) - T_c(g)(t)| &\leq ||T_c(f)(t)||\cdot||T_c(g)(t)||\cdot||b||\cdot||f - g|| \\ &\leq c_3\cdot||f - g||, \end{aligned}$$

where

$$c_3 = ||b||\cdot\Big[\frac{|c|}{1 - |c|\cdot(||a|| + ||b||)}\Big]^2.$$

Since $|c| < c_2$, one can see $c_3 < 1$. Consequently, T_c is contractive on $\mathbb{B}_1$.

Applying the Banach contraction principle, an iterated sequence $T_c^n(f)$ with $f \in \mathbb{B}_1$ converges uniformly to the fixed point of T_c in $\mathbb{B}_1$ for $t \in [0,1]$. This fixed point is unique and no other than the solution $r(t,c)$ (3.21) of Abel equation (3.20).

Furthermore, again using the Banach contraction principle, for any given $a, b \in \mathbb{B}[0,1]$ and $c \in \mathbb{R}$ with $|c| < (\sqrt{||b||} + ||a|| + ||b|| + 1)^{-1}$, and for any $f \in \mathbb{B}[0,1]$ with $||f|| \leq 1$, the solution $r(t,c)$ of Abel equation (3.20) with $r(0,c) = c$ can be uniformly approximated by an iterated sequence $\{T_c^n(f)(t)\}$, i.e.,

$$r(t,c) = \lim_{n\to\infty} T_c^n(f)(t), \quad 0 \le t \le 1. \tag{3.25}$$

In addition, the resulting error estimate is

$$r(t,c) - T_c^n(f)(t) = \mathrm{o}(c^{2n}).$$

This can be proven by using mathematical induction as well as (3.24).

Thus, (3.25) can be rewritten as follows

$$r(t,c) = \cfrac{c}{1 - c\cdot A(t) - c^2 \cdot \int_0^t \cfrac{b(t_1)dt_1}{1 - c\cdot A(t_1) - c^2 \cdot \int_0^{t_1} \cfrac{b(t_2)dt_2}{1 - c\cdot A(t_2) - c^2 \cdot \int_0^{t_2} \cdots}}}. \tag{3.26}$$

Making use of the inverse of (3.18) and changing to the original variable, we can obtain the asymptotic expansion of the solution $y(t)$ of (3.1) in terms of $\beta_i(t)$ $(i = 0, 1, 3)$ and y_0. Note that (3.26) can be applied for finding the numerical solution for the Abel equation (3.1) efficiently.

Although the above result to the Abel equation is actually useful, as we mentioned at the beginning, it is only available for some special functions $\beta_0(t)$. Due to (1.16) and (1.18), after several iterated compositions, $\frac{2v(1,t)}{1-\gamma}$ will be out of our control and lead equation (3.1) to a general case. Therefore, we can not apply this particular technique directly to the map $G_\eta \circ F_{\mu_1,\mu_2}$. However, we display it here since we believe that some day, somehow, it might be able to give us some helpful motivations or hints for finding an invariant interval for the map $G_\eta \circ F_{\mu_1,\mu_2}$.

CHAPTER IV

NUMERICAL SIMULATION RESULTS

In this chapter, we give a couple of examples, and illustrate a few computer graphics of the spatio-temporal profiles as well as snapshots of the vibrations of $u(x,t)$ and $v(x,t)$ of system (1.10)-(1.12). Throughout this chapter, we choose, for x: $0 \le x \le 1$.

Example 4.1: In this example, we set

$$\mu_1 = 1.2, \quad \mu_2 = 1.4, \quad \gamma = 0.01, \quad \eta = 40. \tag{4.1}$$

The initial conditions are

$$\begin{cases} w_0(x) = -\frac{1}{10\pi}\cos(\pi x) - \frac{1}{20}x^2 + \frac{1}{30}x^3 \\ w_1(x) = 0.1\sin(\pi x) + 0.1x - 0.1x^2, \end{cases}$$

resulting in

$$\begin{cases} u_0(x) = 0.1\sin(\pi x) \\ v_0(x) = 0.1x(x-1). \end{cases} \tag{4.2}$$

We note that the initial condition (4.2) satisfy the boundary conditions (1.11) and (1.12) at $t = 0$. In this sense, we may say that the initial conditions (1.11) and (1.12) are compatible.

Notice that once $u(x,t)$ and $v(x,t)$ have been computed from (1.21), we can recover $w(x,t)$ and $w_t(x,t)$ by

$$w(x,t) = \int_0^x [u(\xi,t) + v(\xi,t)]d\xi, \quad w_t(x,t) = u(x,t) - v(x,t).$$

Now we display the spatio-temporal profiles of $u(\cdot,\, t)$ and $v(\cdot,\, t)$ in Example 4.1, respectively, where $\mu_1 = 1.2$, $\mu_2 = 1.4$, $\gamma = 0.01$, $\eta = 40$ and $50 \le t \le 52$.

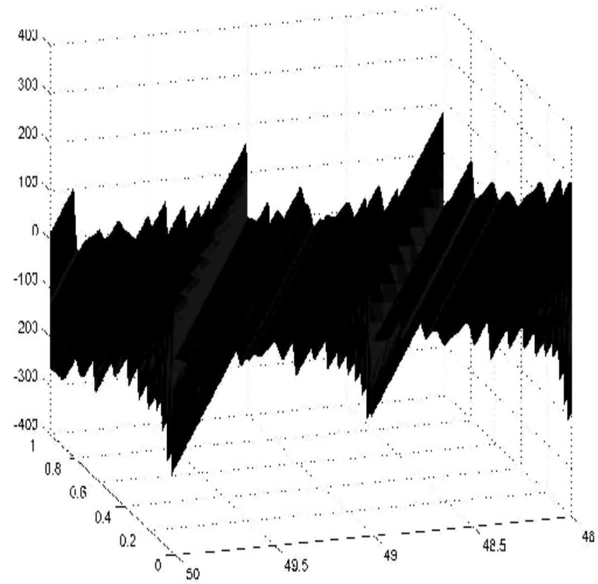

Fig. 14. Solution $u(x,t)$ of Example 4.1, $t \in [50,52]$; $\mu_1 = 3,\ \mu_2 = 4,\ \gamma = 0.01,\ \eta = 40.$

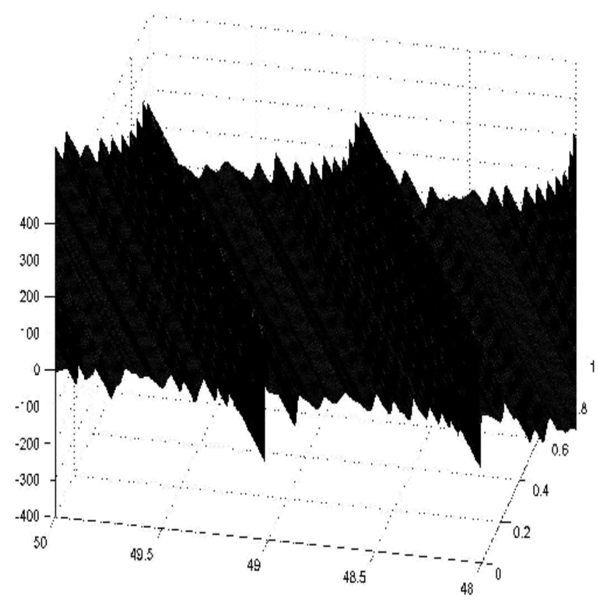

Fig. 15. Solution $v(x,t)$ of Example 4.1, $t \in [50,52]$; $\mu_1 = 3,\ \mu_2 = 4,\ \gamma = 0.01,$ and $\eta = 40$. Observe the disorderly vibration of $v(x,t)$.

From Fig.14 and Fig.15, the spatio-temporal profiles of $u(x,t)$ and $v(x,t)$ appear to have the same oscillatory behaviors as Fig.16 and Fig.17 illustrated in [26] for system (1.1) with conditions (1.2). One of the differences lies in the choices of parameters, especially in η. The other significant difference is that in [26], an invariant interval for the map F is provided, and two period-doubling routes to chaos: $\eta > 1$ and $0 < \eta < 1$, and homoclinic orbits are indicated clearly. But for system (1.1) with (1.4), we have proven that the period-doubling to the system does not exist. Since it is hard to find an invariant interval for the map F, there is no need to say the homoclinic orbit.

The following first five graphs are the snapshots of $u(x,\, t)$ and $v(x,\, t)$ using formula (1.21) with the same parameter values as in (4.1). It is worthwhile to mention a point that why we illustrate the snapshots of $u(x,\, t)$ (or $v(x,\, t)$) for different intervals of t at $x = 0.5$. Recall the formula (1.21) and Fig.1: reflection of characteristics, the snapshot of $u(0.5,\, t)$ where $t \in [n - 0.5,\, n + 0.5]$ (n is arbitrary natural number) is exactly the same as $u(x,\, n)$. In Fig.16, the solution profile for $u(x,\, t)$, can help us observe that the absolute values of $u(x,\, t)$ when $t = 44,\, 46,\, 48$, respectively, do not change too much, and the solution appears to become a little more oscillatory, even not very clearly. However, this phenomenon does not often take places for other choices of parameters μ_1, μ_2, γ and η. For example, when we set $\mu_1 = 2$ and $\mu_2 = 10$ in (4.1), the absolute values of $u(x,\, t)$ will increase significantly within finite time t. Furthermore, oscillaton of the solution $u(x,\, t)$ in Fig.17 and Fig.18 can be seen clearly if the step size is changed to sufficiently small.

When t ranges from 52 to 54, the snapshots of $u(x,t)$ and $v(x,t)$ are illustrated in Fig.19 and Fig.20, respectively. Fig.21 shows the snapshots of $u(x,t)$ and $v(x,t)$ at $t = 48$.

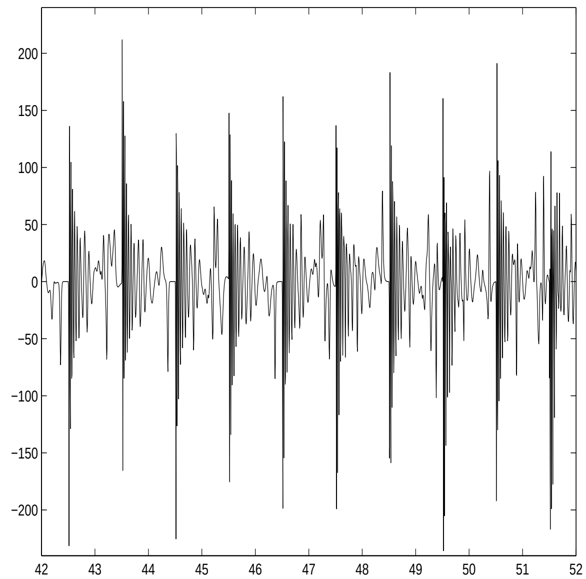

Fig. 16. The snapshot of $u(x,t)$ of Example 4.1, at $x = 0.5$ and $t \in [42, 52]$.

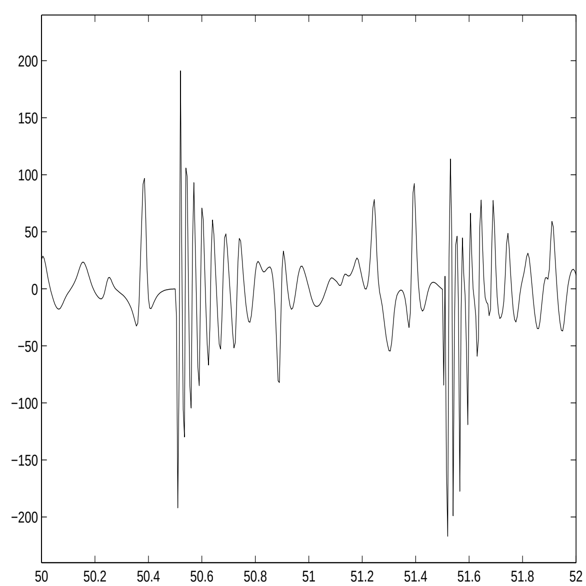

Fig. 17. The snapshot of $u(x,t)$ of Example 4.1, at $x = 0.5$ and $t \in [50, 52]$.

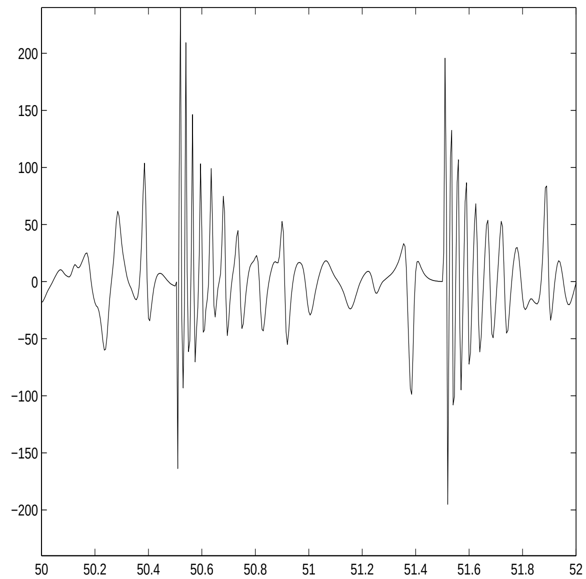

Fig. 18. The snapshot of $v(x,t)$ of Example 4.1, at $x = 0.5$ and $t \in [50, 52]$.

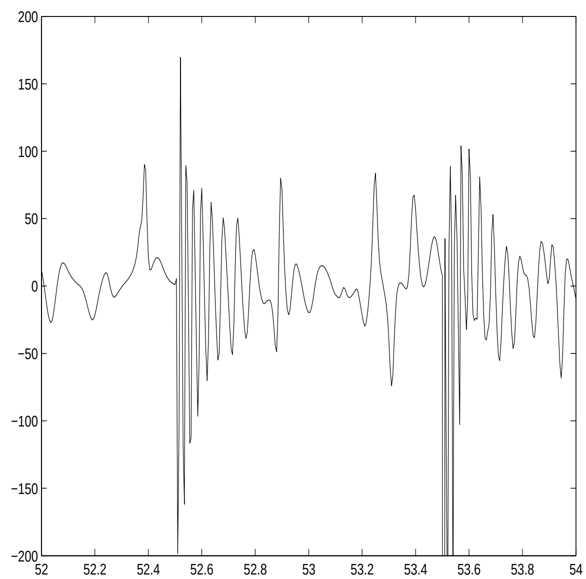

Fig. 19. The snapshot of $u(x,t)$ of Example 4.1, at $x = 0.5$ and $t \in [52, 54]$.

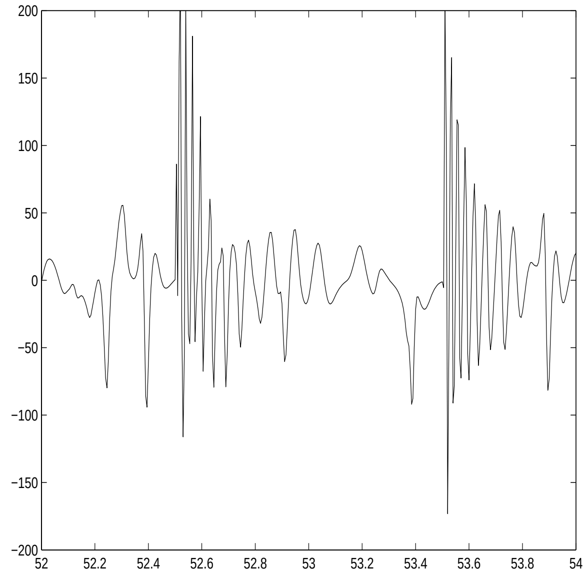

Fig. 20. The snapshot of $v(x,t)$ of Example 4.1, at $x = 0.5$ and $t \in [52, 54]$.

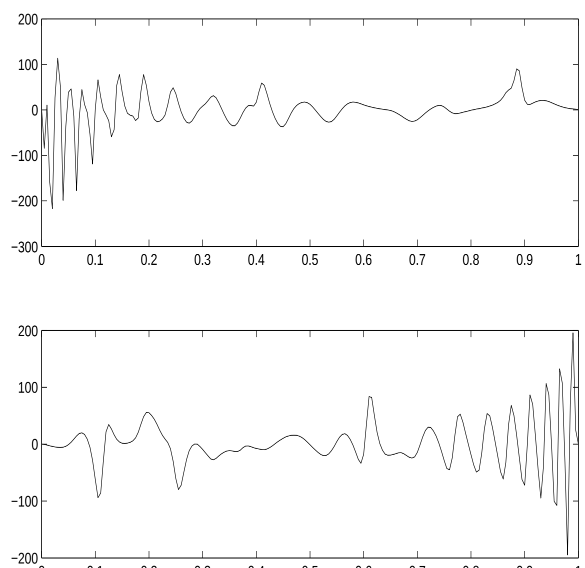

Fig. 21. The snapshots of $u(x,t)$ and $v(x,t)$ of Example 4.1, respectively, at $t = 48$.

Example 4.2: Throughout this example, we choose

$$\mu_1 = 1.2, \quad \mu_2 = 1.4, \quad \gamma = 0.01, \quad \eta = 40.$$

The initial conditions are

$$\begin{cases} w_0(x) = \frac{1}{30}x^3 - \frac{1}{20}x^2 - \frac{3}{4\pi}\cos(\pi x) + \frac{1}{12\pi}\cos(3\pi x) \\ w_1(x) = 0.1\sin^3(\pi x) + 0.1x - 0.1x^2, \end{cases}$$

resulting in

$$u_0(x) = 0.1\sin^3(\pi x) \tag{4.3}$$

$$v_0(x) = 0.1x(x-1). \tag{4.4}$$

For different ranges of t, the snapshots of $u(x,t)$ and $v(x,t)$ are illustrated in Figs.22-29, respectively .

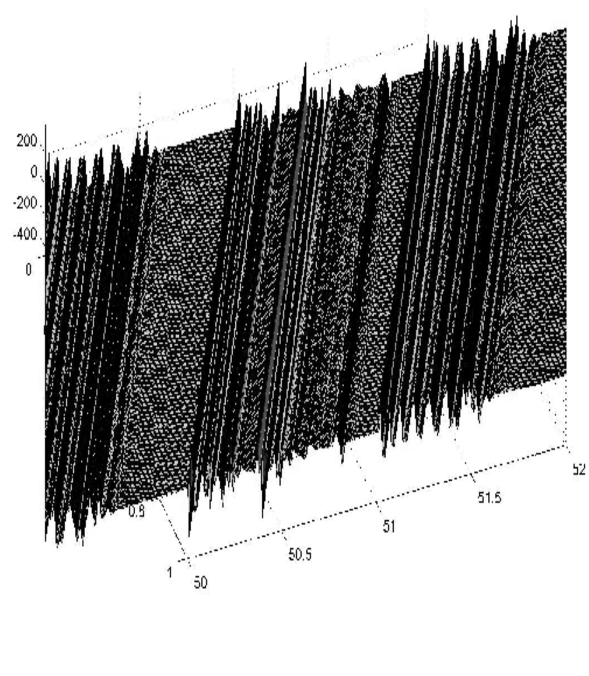

Fig. 22. Solution $u(x,t)$ of Example 4.2, $t \in [50,52]$; $\mu_1 = 3,\ \mu_2 = 4,\ \gamma = 0.01,\ \eta = 40.$

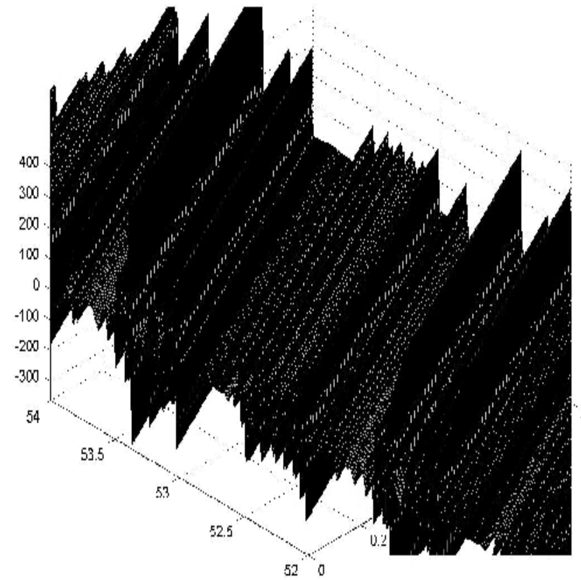

Fig. 23. Solution $v(x,t)$ of Example 4.2, $t \in [50,52]$; $\mu_1 = 3,\ \mu_2 = 4,\ \gamma = 0.01$, and $\eta = 40$. Observe the disorderly vibration of $v(x,t)$.

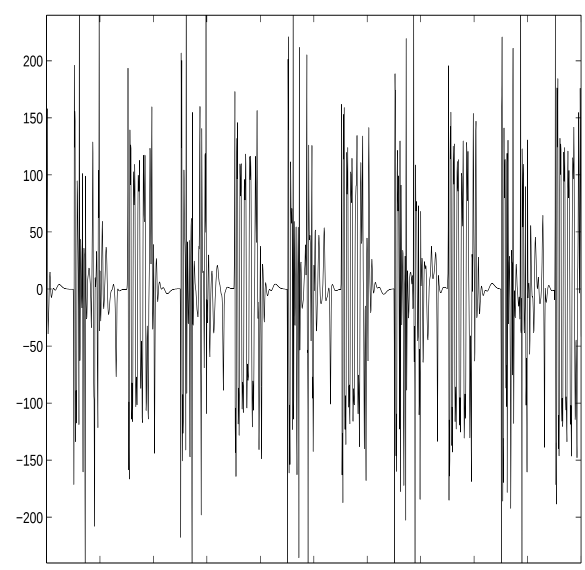

Fig. 24. The snapshot of $u(x,t)$ of Example 4.2, at $x = 0.5$ and $t \in [42,52]$.

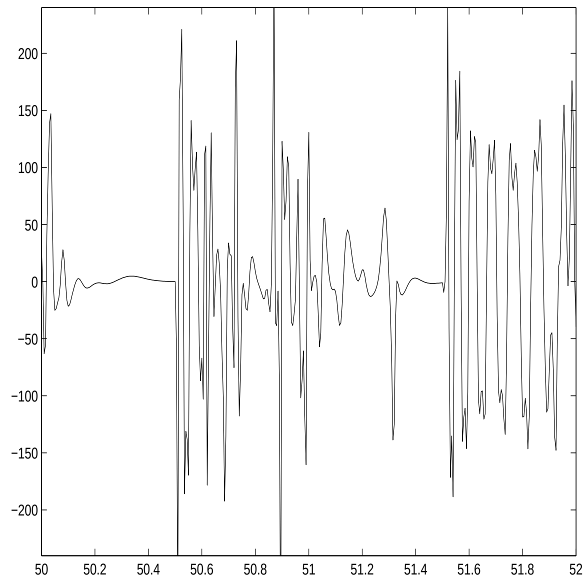

Fig. 25. The snapshot of $u(x,t)$ of Example 4.2, at $x = 0.5$ and $t \in [50, 52]$.

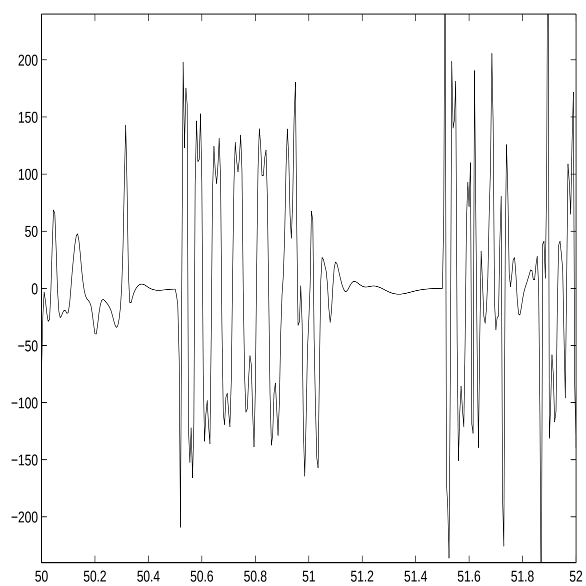

Fig. 26. The snapshot of $v(x,t)$ of Example 4.2, at $x = 0.5$ and $t \in [50, 52]$.

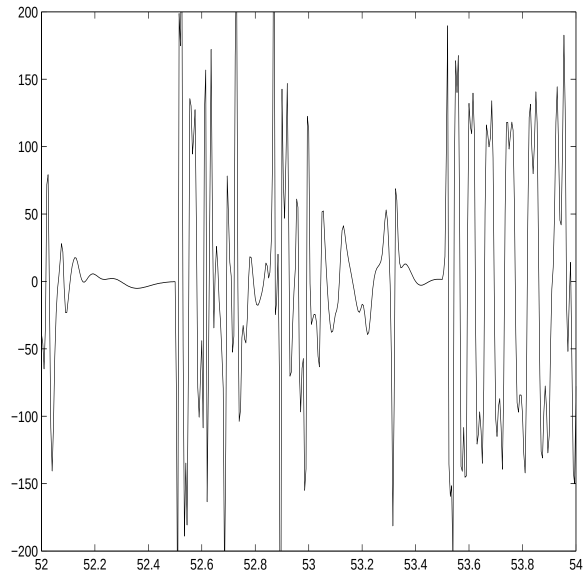

Fig. 27. The snapshot of $u(x,t)$ of Example 4.2, at $x = 0.5$ and $t \in [52, 54]$.

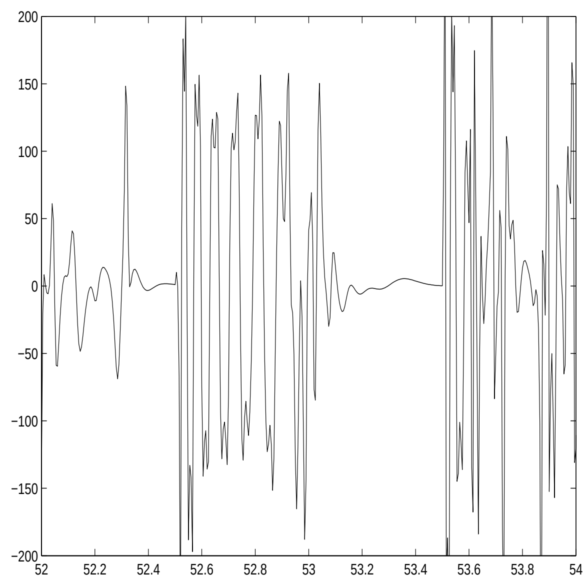

Fig. 28. The snapshot of $v(x,t)$ of Example 4.2, at $x = 0.5$ and $t \in [52, 54]$.

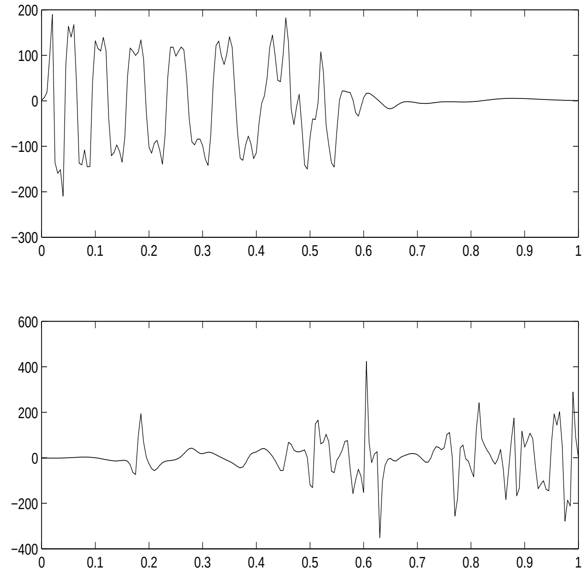

Fig. 29. The snapshots of $u(x,t)$ and $v(x,t)$ of Example 4.2, respectively, at $t = 44$.

Example 4.3: Throughout this example, we choose

$$\mu_1 = 1.2, \quad \mu_2 = 1.4, \quad \gamma = 0.01, \quad \eta = 40.$$

The initial conditions are as

$$w_0(x) = 0.2\sin(\frac{\pi}{2}x), \qquad w_1(x) = 0.2\sin(\pi x), \qquad x \in [0,\ 1].$$

resulting in

$$u_0(x) = 0.1[\frac{\pi}{2}\cos(\frac{\pi}{2}x) + \sin(\pi x)] \tag{4.5}$$

$$v_0(x) = 0.1[\frac{\pi}{2}\cos(\frac{\pi}{2}x) - \sin(\pi x)]. \tag{4.6}$$

The snapshots of $u(x,t)$ and $v(x,t)$ for Example 4.3 are indicated in Figs.30-37, respectively .

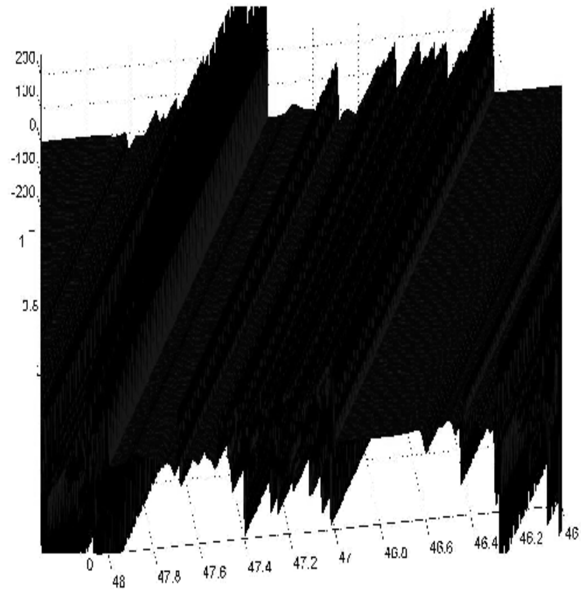

Fig. 30. Solution $u(x,t)$ of Example 4.3, $t \in [50,52]$; $\mu_1 = 3,\ \mu_2 = 4,\ \gamma = 0.01,\ \eta = 40.$

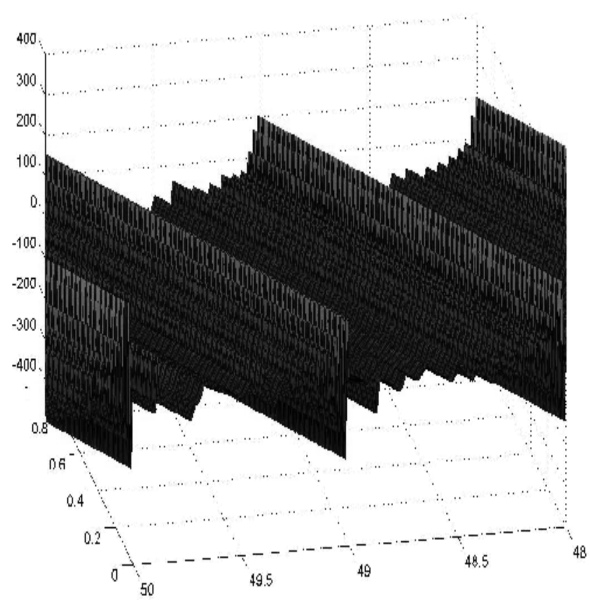

Fig. 31. Solution $v(x,t)$ of Example 4.3, $t \in [50,52]$; $\mu_1 = 3,\ \mu_2 = 4,\ \gamma = 0.01,$ and $\eta = 40.$ Observe the disorderly vibration of $v(x,t)$.

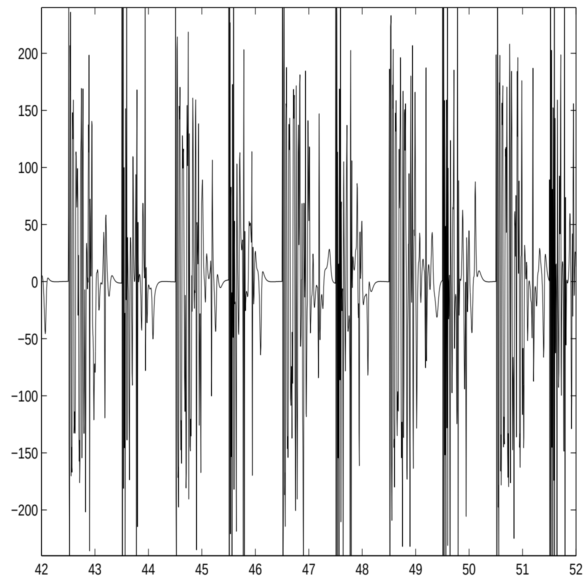

Fig. 32. The snapshot of $u(x, t)$ of Example 4.3, at $x = 0.5$ and $t \in [42, 52]$.

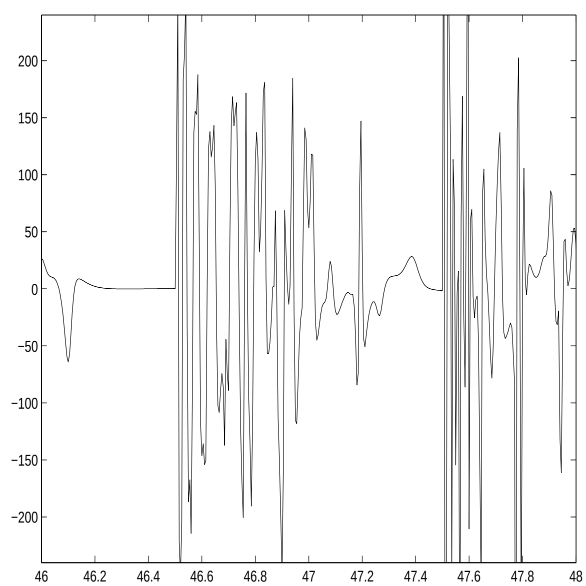

Fig. 33. The snapshot of $u(x, t)$ of Example 4.3, at $x = 0.5$ and $t \in [46, 48]$.

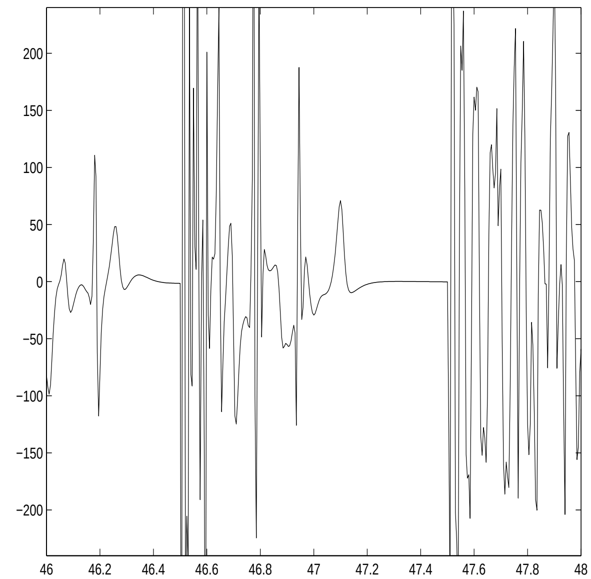

Fig. 34. The snapshot of $v(x,t)$ of Example 4.3, at $x = 0.5$ and $t \in [46, 48]$.

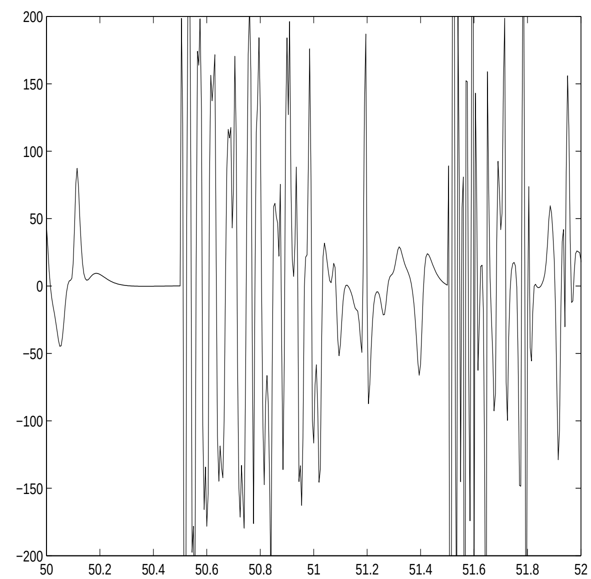

Fig. 35. The snapshot of $u(x,t)$ of Example 4.3, at $x = 0.5$ and $t \in [50, 52]$.

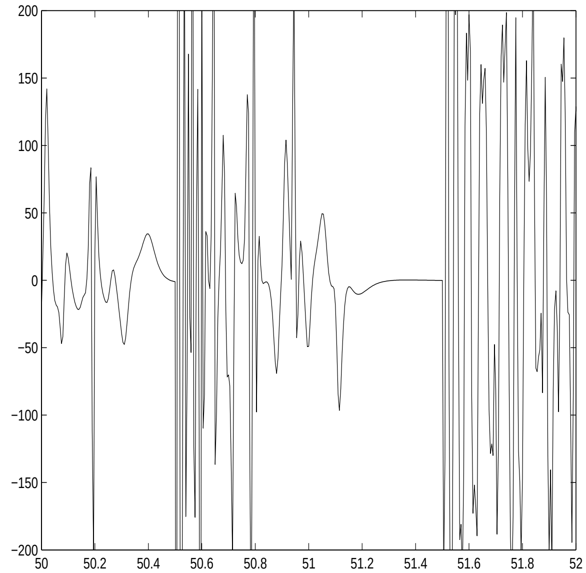

Fig. 36. The snapshot of $v(x,t)$ of Example 4.3, at $x = 0.5$ and $t \in [50, 52]$.

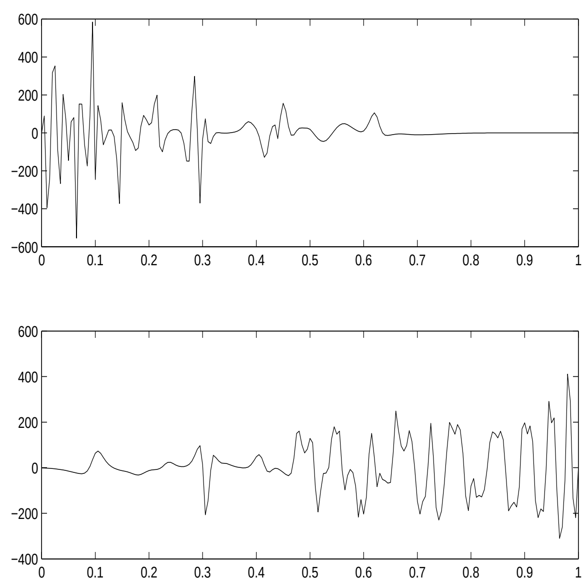

Fig. 37. The snapshots of $u(x,t)$ and $v(x,t)$ of Example 4.3, respectively, at $t = 46$.

CHAPTER V

KORTEWEG-DE VRIES-BURGERS EQUATION*

In this chapter, using the qualitative theory of ordinary differential equations, we examine the traveling wave solution to KdVB equation in the phase plane by analyzing a two-dimensional plane autonomous system which is equivalent to KdVB equation. We also demonstrate a perturbation method in the regimes where $\beta \ll s$ and $s \ll \beta$, and presented formal asymptotic expansions for the solution to the steady-state version of (1.5). These discussions will help us to understand the types of traveling wave solutions KdVB equation may have and motivate us to establish them in an explicit functional form in the following three chapters.

A. Introduction

Korteweg-de Vries equation and Burgers' equation have been studied for a long time, but detailed studies of BKdV equation only begins at the early of 1970s. Particularly, traveling wave solutions in functional forms to KdVB equation were not seen until the late 1980s. A lot of attention has been attracted on these three equations from a rather diverse group of scientists such as physicists and mathematicians, because these three equations not only arise from realistic physical phenomena, but also can be widely applied to many physically significant fields. Some references for applications of Korteweg-de Vries equation and Burgers' equation can been seen from [93-99].

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*Reprinted with permission from "Qualitative analysis and exact solutions to the Burgers-KdV equation" by Zhaosheng Feng, 2002. *Dynamics of Continuous, Discrete and Impulsive Systems*, Volume 9, 563-580. Copyright 2002 by Dynamics of Continuous, Discrete and Impulsive Systems.

Recall KdVB equation (1.5), i.e.,

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0,$$

where α, β and s are real constants with $\alpha\beta s \neq 0$. For different contexts, u stands for different physical meanings. In 1969, during a study of the propagation of waves on liquid-filled elastic tubes, Johnson [29] found that a particular limit of the problem led to (1.5), where $u(x,t)$ is proportional to the radial perturbation of the tube wall, and x and t are the characteristic and time variables, respectively. The equation (1.5) was valid in the far-field of an initially linear (small amplitude) near-field solution. One can see that this equation is the simplest form of wave equation in which nonlinearity $\alpha(uu_x)$, dispersion su_{xxx} and damping βu_{xx} all occur.

Examination of the steady-state form of (1.5) (especially in the phase plane) showed that the radial profile $u(x - C_0 t)$ was very similar to the observed surface profiles of bores. Indeed, as the damping parameter β is varied, the solution is altered from a monotonic profile to an oscillatory one headed by a near-solitary wave. For the bore, Benjamin and Lighthill [100] showed that if only some of the classical energy loss occurred at the bore, the excess could be carried away by a stationary wave train. In fact, they showed that the waves could be of the well-known 'cnoidal' form. Chester [101] also indicated that in some sense a perturbation of Poiseuille flow led to monotonic and oscillatory surface profiles. The steady-state version of (1.5) has been suggested by Grad and Hu [41] to describe the weak shock profile in plasmas.

We once study the existence of the first integral to the two-dimensional plane autonomous systems by the method of algebraic analysis [102]. Since through traveling wave transformation, KdVB equation can be converted to a two-dimensional plane autonomous system. This fact give us an impetus to extend the techniques used in [102] to KdVB equation. In order to understand KdVB equation more completely, we would like to present a qualitative analysis and an oscillatory asymptotic analysis to it in this chapter.

B. Phase-Plane Analysis

Without loss of generality, we assume that $s > 0$. Otherwise, using the transformations $s \to -s$, $u \to -u$, $x \to -x$, the coefficient of u_{xxx} in equation (1.5) can be transformed to the positive.

Assume that equation (1.5) has traveling wave solutions in the form

$$u(x,t) = u(\xi), \qquad \xi = x - vt, \qquad (v \in \mathbb{R}) \tag{5.1}$$

$(v \in \mathbb{R})$. Substituting (5.1) into equation (1.5) and integrating it once, then yield

$$u''(\xi) - ru'(\xi) - au^2(\xi) - bu(\xi) - d = 0.$$

This is equation (1.24), where $r = -\frac{\beta}{s}$, $a = -\frac{\alpha}{2s}$, $b = \frac{v}{s}$ and d is an arbitrary integration constant. Equation (5.1) is a nonlinear ordinary differential equation. It is commonly believed that it is very difficult for us to find exact solutions to equation (5.1) by usual ways [49]. Let $x = u$, $y = u_\xi$, then equation (5.1) is equivalent to (1.25), i.e.,

$$\begin{cases} \dot{x} = y = P(x,y), \\ \dot{y} = ry + ax^2 + bx + d = Q(x,y). \end{cases}$$

Notice that (1.25) is a two-dimensional plane autonomous system, and $P(x,y)$, $Q(x,y)$ satisfy the conditions of the uniqueness and existence theorem ([71]). When $\alpha = 1$ and $\beta < 0$, equation (1.5) is used as a standard equation featuring the interaction between dissipation and dispersion of turbulence [31], so in this section, our discussions are limited to the assumption

$$\alpha > 0, \qquad \beta < 0, \qquad v^2 + 2\alpha sd > 0. \tag{5.2}$$

The arguments for other cases are closely similar. Under the assumption (5.2), (1.25) has two equilibrium points

$$P(\frac{v}{\alpha} - \frac{1}{\alpha}\sqrt{v^2 + 2\alpha ds}, 0), \qquad Q(\frac{v}{\alpha} + \frac{1}{\alpha}\sqrt{v^2 + 2\alpha ds}, 0).$$

The coefficient matrices of the linearizing systems with respect to P and Q are as follows, respectively,

$$\begin{aligned} \mathbf{M_P} &= \begin{pmatrix} P'_x(x,y) & P'_y(x,y) \\ Q'_x(x,y) & Q'_y(x,y) \end{pmatrix}\Bigg|_{P}, \\ &= \begin{pmatrix} 0, & 1 \\ \frac{1}{s}\sqrt{v^2 + 2\alpha sd}, & -\frac{\beta}{s} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{M_Q} &= \begin{pmatrix} P'_x(x,y) & P'_y(x,y) \\ Q'_x(x,y) & Q'_y(x,y) \end{pmatrix}\Bigg|_{Q}, \\ &= \begin{pmatrix} 0, & 1 \\ -\frac{1}{s}\sqrt{v^2 + 2\alpha sd}, & -\frac{\beta}{s} \end{pmatrix}. \end{aligned}$$

Let λ_1, λ_2 designate two eigenvalues of $\mathbf{M_P}$, and μ_1, μ_2 designate two eigenvalues of M_Q, then we have

$$\begin{aligned} \lambda_1 &= -\frac{\beta}{2s} + \frac{1}{2}\sqrt{\frac{\beta^2}{s^2} + \frac{4}{s}\sqrt{v^2 + 2\alpha sd}}, \\ \lambda_2 &= -\frac{\beta}{2s} - \frac{1}{2}\sqrt{\frac{\beta^2}{s^2} + \frac{4}{s}\sqrt{v^2 + 2\alpha sd}}, \end{aligned}$$

and

$$\mu_1 = -\frac{\beta}{2s} + \frac{1}{2}\sqrt{\frac{\beta^2}{s^2} - \frac{4}{s}\sqrt{v^2 + 2\alpha sd}},$$
$$\mu_2 = -\frac{\beta}{2s} - \frac{1}{2}\sqrt{\frac{\beta^2}{s^2} - \frac{4}{s}\sqrt{v^2 + 2\alpha sd}}.$$

Note that when (5.2) holds, P is a saddle. Furthermore, if $\beta^2 > 4s\sqrt{v^2 + 2\alpha sd}$, then Q is an unstable node; if $\beta^2 < 4s\sqrt{v^2 + 2\alpha sd}$, then Q is an unstable focus.

Now, we consider the local behavior at infinite equilibrium point if it exists. Through the Poincaré transformation:

$$x = \frac{1}{z}, \qquad y = \frac{u}{z}, \qquad d\tau = \frac{dt}{z}, \qquad (z \neq 0),$$

(1.25) becomes

$$\begin{cases} \frac{du}{d\tau} = a + bz + dz^2 + ruz - u^2 z \\ \frac{dz}{d\tau} = -uz^2. \end{cases} \tag{5.3}$$

There is no equilibrium point on u-axis since $\alpha \neq 0$. This implies that u-axis is an orbit of (5.3).

On the other hand, through the Poincaré transformation:

$$x = \frac{w}{z}, \qquad y = \frac{1}{z}, \qquad d\tau = \frac{dt}{z}, \qquad (z \neq 0),$$

(1.25) becomes

$$\begin{cases} \frac{dw}{d\tau} = z - rwz - aw^3 - bw^2 z - dwz^2 \\ \frac{dz}{d\tau} = -rz^2 - aw^2 z - bwz^2 - dz^3. \end{cases} \tag{5.4}$$

The origin is the only equilibrium point of (5.4). This implies that (1.25) has two infinite equilibrium points E and F in y-axis. Furthermore, by analyzing the linearizing system of (5.4), E is a local source point, and F is a local sink.

Since $\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} = r$, using the Bendixson Theorem ([71]), (1.25) has no closed orbit in the Poincaré phase plane. That is, equation (1.24) neither has bell solitary wave solution, nor has periodic traveling wave solution. The global behavior of (1.25) is depicted in Fig.38.

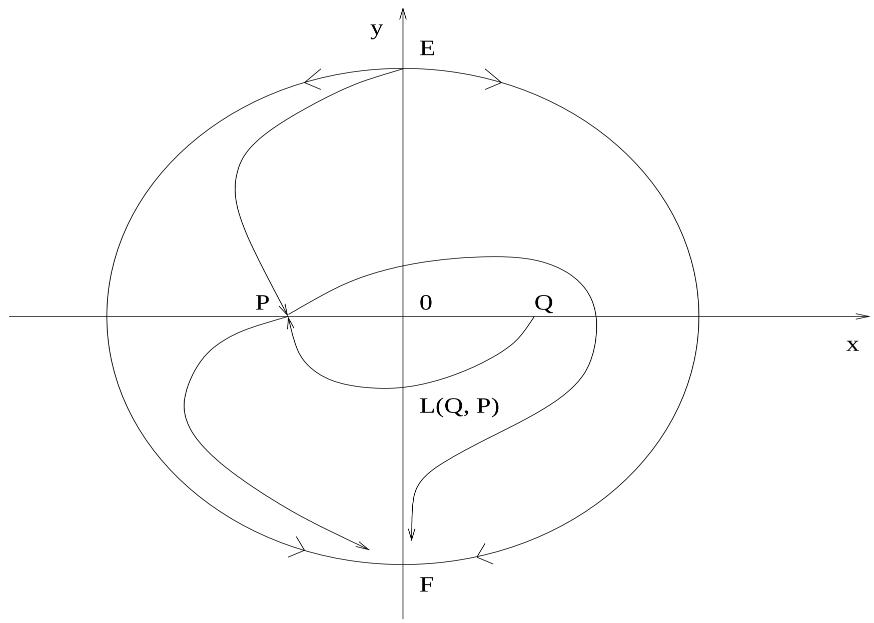

Fig. 38. The global behavior to the plane autonomous system (1.25) when (5.2) hold.

From Fig.38, we can conclude that both x and y-coordinates of the points which lie on all orbits except P, Q and $L(Q, P)$ are unbounded.

Notice that except the equilibrium points P, Q and the orbit $L(Q, P)$, all other orbits in the Poincaré phase plane either depart from the infinite equilibrium point or approach the infinite equilibrium point as $\xi \to +\infty$. This implies that the y-coordinates of the points which lie on the orbits except P, Q and $L(Q, P)$ must be unbounded. We now show that the x-coordinates of those points which lie on the same orbits must be unbounded too. By way of contradiction, assume that there exists a positive number δ such that $|x| < \delta$ as $y \to \infty$. By the Mean-value Theorem,

$\frac{dy}{dx}$ is unbounded.

On the other hand, since the slope of the tangent line to those orbits can be expressed

$$\frac{dy}{dx} = r + \frac{ax^2 + bx + d}{y}. \tag{5.5}$$

(5.5) implies that $\frac{dy}{dx} \to r$ as $y \to \infty$. This yields a contradiction. Consequently, the proof of the above conclusion is complete.

Since the plane autonomous system (1.25) is equivalent to equation (1.24), each nontrivial bounded traveling wave solution $u = u(\xi)$ of equation (1.24) in the (ξ, u)-plane corresponds to an orbit of (1.25) in the Poincaré phase plane, on which the x-coordinate of each point is bounded. By the above analysis, the unique orbit which satisfies this requirement under the assumption (5.2) is $L(Q, P)$. Namely, $u(\xi) \to x_P$ as $\xi \to +\infty$ and $u(\xi) \to x_Q$ as $\xi \to -\infty$, where x_P is the x-coordinate of P and x_Q is the x-coordinate of Q. Thus, we have

$$\begin{cases} u(-\infty) = \lim_{\xi \to -\infty} u(\xi) = \frac{v}{\alpha} + \frac{1}{\alpha}\sqrt{v^2 + 2\alpha ds} \\ u(+\infty) = \lim_{\xi \to +\infty} u(\xi) = \frac{v}{\alpha} - \frac{1}{\alpha}\sqrt{v^2 + 2\alpha ds}. \end{cases}$$

This implies that for given α and s, the limiting value of the traveling wave solution to equation (1.24) as $\xi \to +\infty$ is determined by the velocity of the wave v and the integration constant d.

Furthermore, by the qualitative theory of ordinary differential equations, we can conclude that when $\beta^2 > 4s\sqrt{v^2 + 2\alpha sd}$, the traveling wave solution $u(\xi)$ of equation (1.24) is strictly monotone decreasing with respect to ξ. To prove this, we only need to show that $u'(\xi) \neq 0$ for any $\xi \in \mathbb{R}$ because of the fact $u(-\infty) > u(+\infty)$. That is, except P and Q, $L(Q, P)$ does not intersect with x-axis.

Consider the autonomous system (1.25) in the Poincaré phase plane. Letting

$Q(x, y) = 0$, we obtain that the parabola

$$y = -\frac{1}{r}(ax^2 + bx + d),$$

which is the trajectory on which each orbit points to the left or right. Denote the vertex of the parabola by $V(-\frac{b}{2a}, \frac{b^2}{4ar})$. Construct two lines l_1 and l_2. l_1 is passing through V and parallel to the x-axis, and l_2 is passing through Q with the slope $K = -\frac{2}{\beta}\sqrt{v^2 + 2\alpha sd}$, i.e.,

$$l_2: \quad y = -\frac{2}{\beta}\sqrt{v^2 + 2\alpha sd}(x - \frac{v}{\alpha} - \frac{1}{\alpha}\sqrt{v^2 + 2\alpha sd}). \tag{5.6}$$

Suppose that T is the intersection point of l_1 and l_2. Denote x-coordinate of T by x_T. Immediately we have the following

$$\frac{v}{\alpha} = -\frac{b}{2a} < x_T < \frac{v}{\alpha} + \frac{1}{\alpha}\sqrt{v^2 + 2\alpha sd}. \tag{5.7}$$

Denote by Ω the domain bounded by line segments PQ, QT, TV and the curve VP. It suffices to prove that any integral orbit which starts from a point outside the domain Ω, can not enter the domain Ω as $\xi \to \pm\infty$. It is obvious that there is no integral orbit entering Ω through PQ and the curve VP, because all orbits at each point of x-axis between P and Q are orthogonally along the direction of positive y-axis and all orbits at each point of the curve VP point to the left. Next, we only need to prove that each orbit at the points of the line segments QT and TV points outward. By (5.5) and (5.6), the slope of tangent line to the integral orbit at the intersection point of the integral orbit and l_2 is as follows

$$\begin{aligned}\frac{dy}{dx}\Big|_{(x,y)\in l_2} &= r + a\frac{(x - \frac{v}{\alpha} - \frac{1}{\alpha}\sqrt{v^2 + 2\alpha ds})(x - \frac{v}{\alpha} + \frac{1}{\alpha}\sqrt{v^2 + 2\alpha ds})}{y}\Big|_{(x,y)\in l_2} \\ &= -\frac{\beta}{s} + \frac{\alpha\beta(x - \frac{v}{\alpha} + \frac{1}{\alpha}\sqrt{v^2 + 2\alpha ds})}{4s\sqrt{v^2 + 2\alpha ds}}.\end{aligned}$$

Since $x_T \leq x \leq \frac{v}{\alpha} + \frac{1}{\alpha}\sqrt{v^2 + 2\alpha sd}$, using (5.7), then

$$\frac{dy}{dx}|_{(x,y)\in QT} > -\frac{\beta}{s} + \frac{\beta}{2s} = -\frac{\beta}{2s}. \tag{5.8}$$

By (5.8) and our previous assumption $\beta^2 > 4s\sqrt{v^2 + 2\alpha sd}$, then

$$\frac{dy}{dx}|_{(x,y)\in QT} > -\frac{\beta}{2s} > -\frac{2}{\beta}\sqrt{v^2 + 2\alpha sd} = K.$$

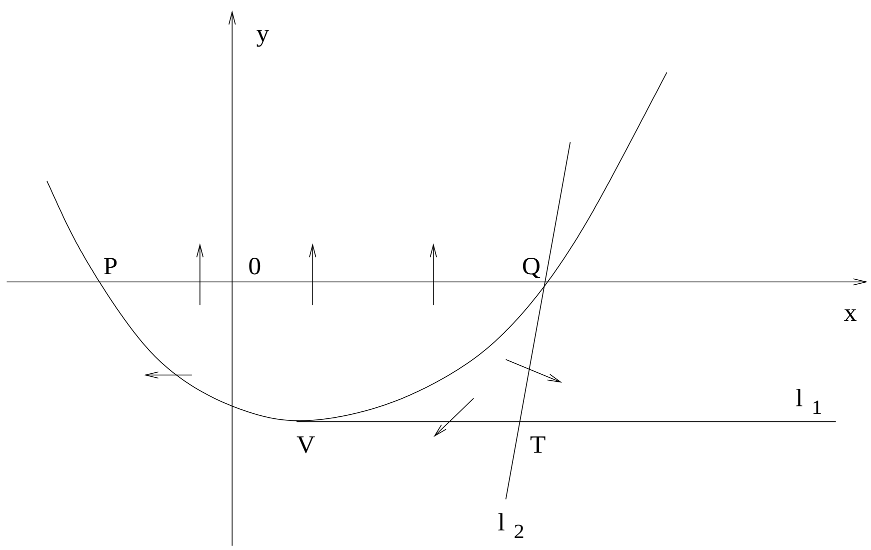

Fig. 39. The domain Ω bounded by $PQTVP$.

This implies that except at Q, each orbit at the points of line segment QT points outward. The argument for VT is closely similar to the above, so we omit it.

Therefore, the orbit $L(P, Q)$ must lie inside the domain Ω. In other words, except the end points P and Q, the orbit $L(P, Q)$ can not intersect with x-axis. Since $u(-\infty) > u(+\infty)$, we obtain that the traveling wave solution $u(\xi)$ of equation (1.24) corresponding to the orbit $L(P, Q)$ is strictly monotone decreasing with respect to ξ (see Fig.39).

C. Oscillatory Asymptotic Analysis

In this section, we illustrate an oscillatory asymptotic analysis to the steady-state version of equation (1.5). To simplify our arguments, we set $\alpha = s = 1$ and $\beta = -\delta$. From the oscillatory asymptotic analysis, one can see not only an overview of the asymptote of the steady-state form of equation (1.5), but also some new insight into highly dependence of traveling wave solutions of (1.5) on coefficients δ. This is also one of reasons which motivates us to try to propose new methods to seek traveling wave solutions to equation (1.5) in an explicit functional form. One can note that the results presented in the next couple chapters are in agreement with the analysis herein.

The steady-state form of equation (1.5) can be obtained by considering waves traveling at a uniform speed, so that

$$v(T) = \frac{1}{u_\infty}u(T), \quad T = \frac{x-(1/2)u_\infty t}{2(2/u_\infty)^{1/2}}, \quad u(-\infty) = h_\infty. \tag{5.9}$$

Also assuming that the upstream and downstream conditions remain undisturbed

$$v \to 0, \quad T \to \infty; \quad v \to 1, \quad T \to -\infty$$

and hence

$$\frac{d^2v}{dT^2} + 4v(v-1) = \epsilon\frac{dv}{dT}, \quad \epsilon = 2\delta(\frac{2}{u_\infty})^{\frac{1}{2}}. \tag{5.10}$$

It is well known that a linear oscillator with a small nonlinear term can be studied by the method of average [103]. An asymptotic solution as an expansion in ϵ is obtained by ensuring that successive terms are periodic.

For no damping (ϵ=0), ((5.10) is a nonlinear equation with solutions which per-

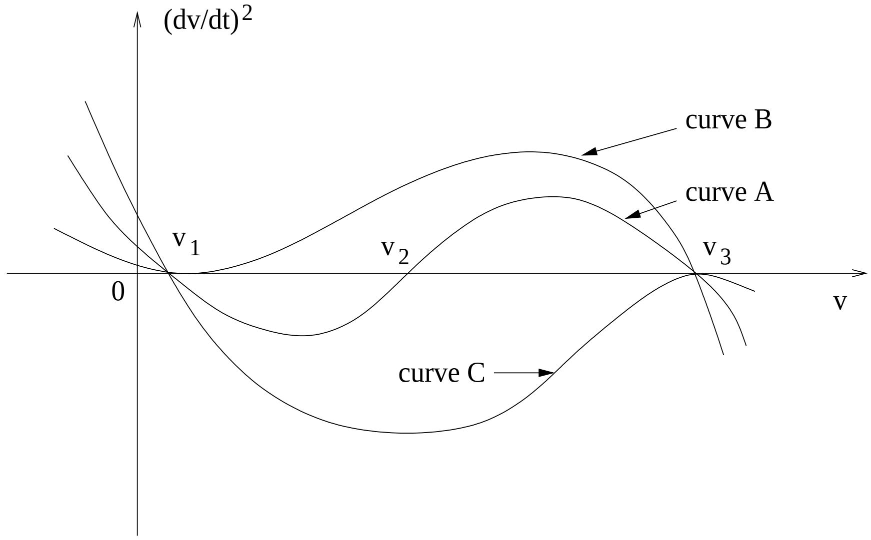

Fig. 40. Sketch of the cubic expression for the general (undamped) cnoidal wave.

form periodic oscillations—the so-called cnoidal waves of water wave theory. For small damping ($\epsilon \ll 1$), similar application of averaging and periodicity conditions are possible as explained by Kuzmak [104], but the analysis is considerably more complicated (usually resulting in the integration of Jacobian elliptic functions). The solutions of the damped equation extend over a large period range from the single solitary wave (infinite period) to the zero amplitude period cnoidal wave. Since averaging techniques rely on averaging over a large number of wavelengths, such methods break down as the solitary wave is approached. It is then necessary to march this solution to a straightforward perturbation of the solitary wave.

Note that (5.10) is a particular case of equation (1.24). The figure of (5.10) in the Poincaré phase plane shows that for small damping the solution starts very near to a solitary wave and steadily departs from it, causing the profile to oscillate about $v = 1$. It is evident that the small damping causes the period of the oscillation to the amplitude and period of the undamped solution to slowly vary. This idea together with the insistence on periodicity is the approach used by Kuzmak [104], and indeed any method involving averaged periodic solutions.

To utilize this method, we must find the undamped solution to equation (5.10) [105]. Thus considering the slightly more general equation without damping,

$$\frac{d^2v}{dT^2} + 4v(v-1) = A, \tag{5.11}$$

where A is a constant. Integrating (5.11) once yields

$$\begin{aligned} \frac{3}{8}(\frac{dv}{dT})^2 &= -v^3 + \frac{3}{2}v^2 + \frac{3}{4}Av + B, \\ &= (v-v_1)(v-v_2)(v_3-v), \end{aligned} \tag{5.12}$$

where $v_1 < v_2 < v_3$. The general form of the cubic expression on the right hand side of (5.12) is sketched in Fig.40 (Curve-A).

The only real solution of the equation occurs for $(dv/dT)^2 \geq 0$, and thus the solution is either at $v = v_1$ or a nonlinear oscillation between v_2 and v_3. Two special cases of Curve-A are $v_2 \to v_1$ (Curve-B) giving the solitary wave, and $v_2 \to v_3$ (Curve-C) giving a discontinuity between v_1 and v_3 (the hydraulic jump).

The solution can be written in terms of a Jacobian elliptic function $\mathrm{cn}(z, r)$ as

$$v = v_2 + (v_2 - v_3)\mathrm{cn}^2\{T[\frac{2}{3}(v_3 - v_1)]^{\frac{1}{2}}; r\}, \tag{5.13}$$

where $r = (v_3 - v_2)/(v_3 - v_1)$. See Abramowitz and Stegun [106]. In the case of $A = B = 0$, then $r = 1$ and the solution becomes $v = \frac{3}{2}\mathrm{sech}^2 T$, the solitary wave.

Now introducing a slow time scale

$$\tau = \epsilon T, \tag{5.14}$$

a two parameter (T, τ) expansion is sought,

$$v = v_0(T, \tau) + \epsilon v_1(T, \tau) + \cdots. \tag{5.15}$$

The form of (5.13) indicates that

$$v_0 = a(\tau) + b(\tau)\mathrm{cn}^2[T \cdot \alpha(\tau); r(\tau)], \tag{5.16}$$

where $\alpha(\tau) = (2b/3r)^{1/2}$, so that (5.16) describes the slowly-varying nature of the solution. Making use of (5.14), we have

$$\frac{d}{dT} = \frac{\partial}{\partial T}, \quad \frac{d^2}{dT^2} = \frac{\partial^2}{\partial T^2} + 2\epsilon\frac{\partial^2}{\partial T \partial \tau} + \epsilon^2\frac{\partial^2}{\partial \tau^2}$$

which together with (5.15) is substituted into (5.10) giving

$$O(1): \quad \frac{\partial^2 v_0}{\partial T^2} + 4v_0(v_0 - 1) = 0, \tag{5.17}$$

$$O(\epsilon): \quad \frac{\partial^2 v_1}{\partial T^2} + 4v_1(2v_0 - 1) = \frac{\partial v_0}{\partial T} - 2\frac{\partial^2 v_0}{\partial T \partial \tau} \tag{5.18}$$

Putting (5.16) into (5.17) and using the equality

$$-\mathrm{dn}^2 + 1 - r = -r\mathrm{cn}^2 = r(\mathrm{sn}^2 - 1)$$

gives

$$(4b^2/3r)[1 - r + 2(2r-1)\mathrm{cn}^2 - 3r\mathrm{cn}^4] + 4(a + b\mathrm{cn}^2)^2 - 4(a + b\mathrm{cn}^2) = 0,$$

and equating the coefficients of cn^2 and cn^4, respectively, we have

$$a = \frac{1}{2} - \frac{1}{3}(b/r)(2r-1), \quad b = \frac{3}{2}r(r^2 - r + 1)^{-1/2}. \tag{5.19}$$

The third equation is identically satisfied due to the choice of $\alpha(\tau)$.

We now need one more relation to enable a, b and r to be defined. This is obtained from the condition that $v_1(T, \tau)$ be periodic in T. To simplify (5.18), we put

$$v_1 = v_{0T} \cdot f(T, \tau), \tag{5.20}$$

giving

$$f_T = \frac{1}{v_{0T}^2} \int_{b_0}^{T} (v_{0T}^2 - 2v_{0T} \cdot v_{0T\tau}) dT, \tag{5.21}$$

where b_0 is arbitrary. Now v_1 is periodic if f_T is, thus if T_p is the period of f, then

$$\int_{a-\frac{1}{2}T_p}^{a+\frac{1}{2}T_p} (V_{0T}^2 - 2v_{0T} \cdot v_{0T\tau}) dT = 0,$$

$$C_1 e^{\tau} = \int_{a-\frac{1}{2}T_p}^{a+\frac{1}{2}T_p} (V_{0T})^2 dT, \tag{5.22}$$

where C_1 is constant. Using the definition of the Jacobian elliptic functions

$$L(r) = \int_0^{\frac{1}{2}\pi} \cos^2\phi \sin^2\phi (1 - r\sin^2\phi)^{\frac{1}{2}} d\phi, \tag{5.23}$$

with the well-known relations

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt \quad (c > b > 0),$$

and

$$F(\frac{1}{2}, \frac{1}{2}, 1; r) = \frac{1}{2}\pi K(r)$$
$$F(\frac{1}{2}, -\frac{1}{2}, 1; r) = \frac{1}{2}\pi E(r),$$

which are the complete elliptic integrals of the 1st and 2nd kind, respectively, relation (5.23) can be written as

$$r^2 l(r) \propto (r^2 - r + 1)E(r) - \frac{1}{2}(1-r)(2-r)K(r), \tag{5.24}$$

which together with

$$e^{\tau} \propto r^2(r^2 - r + 1)^{-\frac{5}{4}} L(r)$$

becomes the third relation defining $a(\tau)$, $b(\tau)$, $r(\tau)$. The constant of proportionality is to be found by putting $r = 1$ at $T = T_0$, so that

$$e^{\tau-\tau_0} = (r^2 - r + 1)^{-\frac{5}{4}}[(r^2 - r + 1)E(r) - \frac{1}{2}(1 - r)(2 - r)K(r)], \tag{5.25}$$

where T_0 is the apparent position where the damping is 'switched-on' for the oscillatory analysis. This can only be fixed by matching

The final solution given by (5.15), (5.20) and (5.21) is

$$v = v_0 + \epsilon v_{0T} \int_{a_0}^{T} \frac{1}{v_{0T}^2} \Big[\int_{b_0}^{T} (v_{0T}^2 - 2v_{0T} \cdot v_{0T\tau} dT \Big] dT + \cdots, \tag{5.26}$$

where a_0 is arbitrary. Kuzmak pointed out that a solution like (5.26) is only valid provided that $T_p < \infty$ (see (5.22)), i.e., if v_0 is truly oscillatory. Consequently (5.26) becomes invalid as $r \to 1$, since v_0 becomes nonlinear-periodic (see (5.13)).

D. Perturbation of the Solitary Wave

Since (5.26) becomes invalid, it is necessary to obtain a non-oscillatory solution of (5.10) by performing a straightforward expansion in ϵ. Actually we shall see that this amounts to a perturbation of the solitary wave.

For convenience (5.10) is integrated once, we get

$$\frac{1}{2}(\frac{dv}{dT})^2 = \epsilon \int_{\infty}^{T} (\frac{dv}{dT})^2 dT - \frac{4}{3}v^3 + 2v^2, \tag{5.27}$$

remembering that $v = \frac{dv}{dT} = 0$ at $T = +\infty$, so that the solution of (5.27) will be monotonic as $T \to +\infty$. Clearly, the solution (5.26) is valid as $T \to -\infty$ describing the oscillatory nature of the solution. A straightforward expansion is

$$v = V_0(T) + \epsilon V_1(T) + \cdots, \tag{5.28}$$

which gives

$$O(1): \quad \frac{1}{2}(\frac{dV_0}{dT})^2 = 2V_0^2 - \frac{4}{3}V_0^3, \tag{5.29}$$

$$O(\epsilon): \quad \frac{dV_0}{dT}\frac{dV_1}{dT} = \int_{\infty}^{T}(\frac{dV_0}{dT})^2 dT + 4V_0V_1(1-V_0). \tag{5.30}$$

The solution to (5.29) is the solitary wave

$$V_0 = \frac{3}{2}\text{sech}^2 T,$$

where the peak is fixed at $T = 0$. Substituting this into (5.30) and integrating the equation yields

$$\begin{aligned} V_1 &= \frac{2}{5}\text{sech}^2 T(\tanh T - 1) + \frac{1}{20}\tanh T(\tanh T - 6) - \frac{\tanh T}{10(1+\tanh T)} \\ &\quad + \frac{3}{4}T\tanh T\text{sech}^2 T, \end{aligned} \tag{5.31}$$

where the condition $\frac{dV_1}{dT}|_{T=0} = 0$ ensures that the peak occurs at $T = 0$. It is clear that the expansion (5.28) becomes invalid as

$$\tanh T \to -1, \quad \text{as} \;\; T \to -\infty.$$

Therefore, the expression (5.31) then approaches infinity.

CHAPTER VI

FIRST INTEGRAL METHOD

In this chapter, applying the ring theory of commutative algebra, we introduce a new approach which we currently call 'First Integral Method' to study KdVB equation for seeking its traveling wave solutions. In order to present our arguments in a straightforward manner, let us first introduce a technical theorem:

A. Divisor Theorem for Two Variables in the Complex Domain

Theorem VI.1 Divisor Theorem. *Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$, and $P(\omega, z)$ is irreducible in $\mathbb{C}[\omega, z]$. If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathbb{C}[\omega, z]$ such that*

$$Q(\omega, z) = P(\omega, z) \cdot G(\omega, z). \tag{6.1}$$

□

This Theorem follows immediately from Hilbert-Nullstellensatz Theorem [107]:

Theorem VI.2 Hilbert-Nullstellensatz. *Let k be a field and L an algebraic closure of k.*

(i). Every ideal γ of $k[X_1, \cdots, X_n]$ not containing 1 admits at least one zero in L^n.

(ii). Let $\mathbf{x} = (x_1, \cdots, x_n), \mathbf{y} = (y_1, \cdots, y_n)$ be two elements of L^n; for the set of polynomials of $k[X_1, \cdots, X_n]$ zero at $\mathbf{x}$ to be identical with the set of polynomials of $k[X_1, \cdots, X_n]$ zero at $\mathbf{y}$, it is necessary and sufficient that there exists a k-automorphism s of L such that $y_i = s(x_i)$ for $1 \leq i \leq n$.

(iii). For an ideal α of $k[X_1, \cdots, X_n]$ to be maximal, it is necessary and sufficient that there exists an $\mathbf{x}$ in L^n such that α is the set of polynomials of $k[X_1, \cdots, X_n]$

zero at $\mathbf{x}$.

(iv). For a polynomial Q *of* $k[X_1, \cdots, X_n]$ *to be zero on the set of zeros in* L^n *of an ideal* γ *of* $k[X_1, \cdots, X_n]$*, it is necessary and sufficient that there exist an integer* $m > 0$ *such that* $Q^m \in \gamma$. □

This is of some interest to ask whether the above Divisor Theorem can be proven by using the complex theory. The answer is 'Yes'. Next, we would like to offer an easier proof [108]:

Proof. For convenience, we give the following lemmas:

Lemma VI.1 *Suppose that* $U(\omega, z)$ *and* $V(\omega, z)$ *are polynomials in* $\mathbb{C}[\omega, z]$*, and* $U(\omega, z)$ *is irreducible in* $\mathbb{C}[\omega, z]$*. Suppose that* $R(\omega, z)$ *is a non-constant polynomial and a factor of* $U(\omega, z) \cdot V(\omega, z)$*, and* $\deg R(\omega, z) < \deg U(\omega, z)$ *with respect to* ω*. Then* $R(\omega, z)|V(\omega, z)$. □

Lemma VI.2 *Suppose that* $P(\omega, z)$ *is an irreducible polynomial in* $\mathbb{C}[\omega, z]$ *and that* $P_\omega(\omega, z)$ *is the partial derivative with respect to* ω*. Then there exist two polynomials* $A(\omega, z), B(\omega, z)$*, and a nonzero polynomial* $D(z)$ *in* $\mathbb{C}[\omega, z]$*, such that*

$$A(\omega, z) \cdot P(\omega, z) + B(\omega, z) \cdot P_\omega(\omega, z) = D(z). \tag{6.2}$$

□

The proofs of Lemma VI.1 and Lemma VI.2 can be seen in [109].

Notice that a polynomial $P(\omega, z)$ in $\mathbb{C}[\omega, z]$ can be written as

$$P(\omega, z) = \sum_{k=o}^{n} p_k(z)\omega^k, \tag{6.3}$$

where $p_k(z) \quad (k = 0, 1, \cdots, n)$ are polynomials in z and $p_n(z) \not\equiv 0$. If $P(\omega, z)$ is an irreducible polynomial in $\mathbb{C}[\omega, z]$, then $p_k(z), (k = 0, 1, \cdots, n)$ are all relatively prime.

For any fixed $z_0 \in \mathbb{C}$, $P(\omega, z_0)$ is a polynomial in ω. By the Fundamental Theorem of Algebra, it has n zeros in $\mathbb{C}$.

Definition VI.1 *If z_0 is a complex number such that the polynomial $P(\omega, z_0)$ does not have n distinct zeros in $\mathbb{C}$, then z_0 is called Special Zero Points of the polynomial $P(\omega, z)$.* □

Lemma VI.3 *If $P(\omega, z)$ is an irreducible polynomial in $\mathbb{C}[\omega, z]$, then $P(\omega, z)$ has at most finitely many Special Zero Points in $\mathbb{C}$.* □

Proof. Write $P(\omega, z)$ in (6.3) and consider the set

$$M = \{z | z \in C, \quad p_n(z) = 0, \quad or \quad D(z) = 0\}.$$

By (6.2), it is easily noted that M is a finite set. Suppose that $z^* \in \mathbb{C} \setminus M$, then the polynomial $P(\omega, z^*)$ with respect to ω must have n distinct zeros. Hence, the set of *Special Zero Points* of $P(\omega, z)$ is a subset of M. Therefore, $P(\omega, z)$ has at most finitely many Special Zero Points. □

Next, we prove Divisor Theorem using the above lemmas. For any $z \in \mathbb{C} \setminus M$, by Lemma **??**, the polynomial $P(\omega, z)$ with respect to ω must have n distinct roots $r_i (i = 1, 2, \cdots, n)$. By the hypothesis, r_i $(i = 1, 2, \cdots, n)$ are also the roots of $Q(\omega, z)$. Hence the degree for polynomial $Q(\omega, z)$ with respect to ω is greater than or equal to n.

Assume that

$$Q(\omega, z) = \sum_{k=0}^{m} q_k(z)\omega^k,$$

where $q_k(z)$ $(k = 0, 1, \cdots, m)$ are polynomials in z, $q_m(z) \not\equiv 0$, and $m \geq n$.

By the division theory for the polynomials in one variable, we have

$$Q(\omega, z) = h(\omega, z) \cdot P(\omega, z), \tag{6.4}$$

where

$$\begin{cases} h(\omega,z) = \sum_{k=0}^{m-n} h_k(z)\omega^k \\ h_{m-n}(z) = q_m(z)/p_n(z) \\ h_{m-n-1}(z) = \frac{1}{p_n(z)}[q_{m-1}(z) - \frac{q_m(z)p_{n-1}(z)}{p_n(z)}] = \frac{q_{m-1}*(z)}{p_n^2(z)} \\ \cdots\cdots \\ h_{m-n-i}(z) = \frac{q_{m-i}*(z)}{p_n^{i+1}(z)} \\ h_0(z) = \frac{q_n*(z)}{p_n^{m-n-1}(z)}. \end{cases} \tag{6.5}$$

Notice that the polynomials $q^*_{m-i}(z)$ could be obtained from $q_k(z)$ and $p_k(z)$ by applying the operations of addition, subtraction, multiplication and division. The denominators and numerators of (6.5) may have common factors.

Suppose that $u(z)$ is a polynomial with the least degree such that $u(z) \cdot h(\omega, z)$ is a polynomial in $\mathbb{C}[\omega, z]$. That is

$$u(z)h(\omega, z) = G_1(\omega, z), \tag{6.6}$$

where $u(z)$ and $G_1(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$. Note that there is no nontrivial common factor between $u(z)$ and $G_1(\omega, z)$. By (6.4), (6.6), we get

$$u(z) \cdot Q(\omega, z) = G_1(\omega, z) \cdot P(\omega, z). \tag{6.7}$$

If $u(z)$ is a nonzero constant, then we obtain the desired result. If $u(z)$ is a non-constant polynomial, since $P(\omega, z)$ is irreducible, by Lemma VI.1, $u(z)$ must divide $G_1(\omega, z)$. This yields a contradiction with the above assumption that $u(z)$ and $G_1(\omega, z)$ have no nontrivial common factor in $\mathbb{C}[\omega, z]$. Therefore, $u(z)$ must be nonzero constant. Letting $G(\omega, z) = [\frac{1}{u(z)}] \cdot G_1(\omega, z)$, from (6.7) we obtain (6.1). So the proof of Divisor Theorem is complete. □

B. Exact Solutions to KdVB Equation by First Integral Method

By the qualitative theory of ordinary differential equations [110], if we can find two first integrals to (1.25) under the same conditions, then the general solutions to (1.25) can be expressed explicitly. However, in general, it is really difficult for us to realize this, even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are.

In this section, we are applying first integral method to study exact solution to equation (1.5). That is, we will apply Divisor Theorem to obtain one first integral to system (1.25) which reduces equation (1.24) to a first-order integrable ordinary differential equation. An exact solution to equation (1.5) is thereby obtained by solving this equation.

Now, we are applying the Divisor Theorem to seek the first integral to (1.25). Suppose that $x = x(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to (1.25), and $p(x,y) = \sum_{i=0}^{m} a_i(x)y^i$ is an irreducible polynomial in $\mathbb{C}[x,y]$ such that

$$p[x(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(x)y^i = 0, \tag{6.8}$$

where $a_i(x)$ $(i = 0, 1, \cdots, m)$ are polynomials of x and all relatively prime in $\mathbb{C}[x,y]$, $a_m(x) \not\equiv 0$. (6.8) is also called the first integral to (1.25). We start our study by assuming $m = 2$ in (6.8). Note that $\frac{dp}{d\xi}$ is a polynomial in x and y, and $p[x(\xi), y(\xi)] = 0$ implies $\frac{dp}{d\xi}|_{(1.25)} = 0$. By Divisor Theorem, there exists a polynomial $H(x,y) = \alpha(x) + \beta(x)y$ in $\mathbb{C}[x,y]$ such that

$$\begin{aligned} \frac{dp}{d\xi}|_{(1.25)} &= (\frac{\partial p}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \xi})|_{(1.25)} \\ &= \sum_{i=0}^{2}[a_i'(x)y^i \cdot y] + \sum_{i=0}^{2}[ia_i(x)y^{i-1}(ry + ax^2 + bx + d)] \end{aligned}$$

$$= \quad [\alpha(x) + \beta(x)y][\sum_{i=0}^{2} a_i(x)y^i]. \tag{6.9}$$

On equating the coefficients of y^i (i=3, 2, 1, 0) on both sides of (6.9), we have

$$\mathbf{a}'(x) = \mathbf{A}(x) \cdot \mathbf{a}(x), \tag{6.10}$$

and

$$[0, ax^2 + bx + d, -\alpha(x)] \cdot \mathbf{a}(x) = 0, \tag{6.11}$$

where $\mathbf{a}(x) = (a_2(x), a_1(x), a_0(x))^t$, and

$$\mathbf{A(x)} = \begin{pmatrix} \beta(x), & 0, & 0 \\ \alpha(x) - 2r, & \beta(x), & 0 \\ -2(ax^2 + bx + d), & \alpha(x) - r, & \beta(x) \end{pmatrix}.$$

Since $a_i(x)$ $(i = 0, 1, 2)$, $\alpha(x)$ and $\beta(x)$ are polynomials, from the first equation of (6.10), i.e., $a_2'(x) = a_2(x) \cdot \beta(x)$, we deduce that $a_2(x)$ is a constant and $\beta(x) = 0$. For simplification, taking $a_2(x) = 1$ and solving (6.10), we have

$$\mathbf{a(x)} = \begin{pmatrix} 1 \\ \int [\alpha(x) - 2r] dx \\ \int [a_1(x)\alpha(x) - ra_1(x) - 2(ax^2 + bx + d)] dx \end{pmatrix}. \tag{6.12}$$

By (6.11) and (6.12), we conclude that $\deg \alpha(x) = 0$, i.e., $\deg a_1(x) = 1$. Otherwise, if $\deg \alpha(x) = k > 0$, then we deduce $\deg a_1(x) = k + 1$ and $\deg a_0(x) = 2k + 2$ from (6.12). This yields a contradiction with (6.11), since the degree of the polynomial $a_1(x) \cdot (ax^2 + bx + d)$ is $k + 3$, but the degree of the polynomial $a_0(x) \cdot \alpha(x)$ is $3k + 2$.

Assume that $a_1(x) = A_1 x + A_0$, $A_1, A_0 \in \mathbb{C}$ with $A_1 \neq 0$. By (6.12), we deduce that $A_1 = \alpha(x) - 2r$ and $a_0(x) = -\frac{2ax^3}{3} - bx^2 + \frac{A_1(A_1+r)}{2}x^2 - 2dx + A_0(A_1 + r)x + D$, here D is an arbitrary integration constant. Substituting $a_1(x)$ and $a_0(x)$ into (6.11) and setting all coefficients of x^i (i=3,2,1,0) to zero, then yield

$$\begin{cases} A_1 a = (-\frac{2a}{3}) \cdot (A_1 + 2r) \\ A_0 a + A_1 b = [\frac{A_1(A_1+r)}{2} - b] \cdot (A_1 + 2r) \\ A_1 d + A_0 b = [(A_1 + r)A_0 - 2d] \cdot (A_1 + 2r) \\ A_0 d = D \cdot (A_1 + 2r). \end{cases} \tag{6.13}$$

Solving (6.13), we obtain

$$\begin{cases} A_1 = -\frac{4r}{5}, \quad A_0 = -\frac{12r^3}{125a} - \frac{2br}{5a} \\ d = \frac{5}{8}(\frac{6r^2}{25} - b)(-\frac{12r^2}{125a} - \frac{2b}{5a}), \quad D = \frac{25}{48}(\frac{6r^2}{25} - b)(\frac{12r^2}{125a} + \frac{2b}{5a})^2. \end{cases} \tag{6.14}$$

Assume that

$$kb = \frac{6r^2}{25}, \quad k \in \mathbb{R} \text{ with } k \neq 0, \tag{6.15}$$

then d and D in (6.14) can be simplified as

$$d = -\frac{b^2}{4a}(k^2 - 1), \qquad D = \frac{b^3}{12a^2}(k+1)(k^2 - 1). \tag{6.16}$$

Substituting $a_0(x)$ and $a_1(x)$ into (6.8), then (6.8) becomes

$$y^2 - [\frac{4r}{5}x + \frac{2br}{5a}(k+1)]y - \frac{2a}{3}x^3 - bx^2 - \frac{2r^2}{25}x^2 - 2dx - \frac{2br^2}{25a}(k+1)x + D = 0. \tag{6.17}$$

From (6.17), y can be expressed in terms of x, i.e.,

$$\begin{aligned} y &= \frac{2r}{5}x + \frac{br}{5a}(k+1) \pm \sqrt{\frac{2a}{3}x^3 + (k+1)bx^2 + \frac{b^2}{2a}(k+1)^2 x + \frac{b^3}{12a^2}(k+1)^3} \\ &= \frac{2r}{5}x + \frac{br}{5a}(k+1) \pm \sqrt{\frac{2}{3a^2}[ax + \frac{(k+1)b}{2}]^3}. \end{aligned} \tag{6.18}$$

Combining (1.25) and (6.18), we have

$$\frac{dx}{\frac{2r}{5a}[ax+\frac{(k+1)b}{2}] \pm [ax+\frac{(k+1)b}{2}]\sqrt{\frac{2}{3a^2}[ax+\frac{(k+1)b}{2}]}} = d\xi. \tag{6.19}$$

Using a transformation $z = \sqrt{\frac{2}{3a^2}[ax+\frac{(k+1)b}{2}]}$, we obtain an exact solution to equation (1.5) as follows by solving (6.19) directly

$$\begin{aligned} u(\xi) &= \frac{3a}{2}\cdot\Big[\frac{\pm\frac{2r}{5a}e^{\frac{r}{5}\xi}}{e^{\frac{r}{5}\xi}+c}\Big]^2 - \frac{(k+1)b}{2a} \\ &= -\frac{12\beta^2}{25\alpha s}\Big[\frac{e^{-\frac{\beta}{5s}\xi}}{e^{-\frac{\beta}{5s}\xi}+c}\Big]^2 + \frac{(k+1)v}{\alpha}, \end{aligned} \tag{6.20}$$

where c is an arbitrary integration constant, $\xi = x - vt$ and $v = \frac{6\beta^2}{25sk}$ due to (6.15) and $b = \frac{v}{s}$ and $r = -\frac{\beta}{s}$. Note that (6.20) confirms the qualitative analysis presented in Section 2.

If assuming $m = 3, 4$ in (6.8), respectively, using the similar arguments as on p. 11-12, we obtain that (1.25) does not have any first integral in the form (6.8). We do not need to discuss the cases $m \geq 5$ due to the fact that in general, the polynomial equation with the degree greater or equal to 5 is not solvable.

In case $k = 1$ and $c = 1$ in (6.20), by using the equality $4A[\frac{e^{2t}}{1+e^{2t}}]^2 = -A\text{sech}^2 t + 2A\tanh t + 2A$, the explicit traveling solitary wave solution to equation (1.5) can be rewritten as

$$\begin{aligned} u(x,t) &= \frac{3\beta^2}{25\alpha s}\text{sech}^2\Big[\frac{1}{2}(-\frac{\beta}{5s}x+\frac{6\beta^3}{125s^2}t)\Big] \\ &\quad -\frac{6\beta^2}{25\alpha s}\tanh\Big[\frac{1}{2}(-\frac{\beta}{5s}x+\frac{6\beta^3}{125s^2}t)\Big] + \frac{6\beta^2}{25\alpha s}. \end{aligned} \tag{6.21}$$

In case $k = -1$ and $c = 1$ in (6.20), the explicit traveling solitary wave solution to equation (1.5) can be rewritten as

$$u(x,t) = \frac{3\beta^2}{25\alpha s}\text{sech}^2\Big[\frac{1}{2}(-\frac{\beta}{5s}x-\frac{6\beta^3}{125s^2}t)\Big]$$

$$-\frac{6\beta^2}{25\alpha s}\tanh\Big[\frac{1}{2}\big(-\frac{\beta}{5s}x-\frac{6\beta^3}{125s^2}t\big)\Big]-\frac{6\beta^2}{25\alpha s}. \tag{6.22}$$

(6.21) and (6.22) can be regarded as a composition of a bell solitary wave and a kink solitary wave.

Notice that the traveling wave solution given by (6.21) has the limit zero as $\delta_1 = -\frac{\beta}{5s}x + \frac{6\beta^3}{125s^2}t \to -\infty$ and the limit $\frac{12\beta^2}{25\alpha s}$ as $\delta_1 \to +\infty$, and the traveling wave solution given by (6.22) has the limit zero as $\delta_2 = -\frac{\beta}{5s}x - \frac{6\beta^3}{125s^2}t \to -\infty$ and the limit $-\frac{12\beta^2}{25\alpha s}$ as $\delta_2 \to +\infty$. All orders of derivatives of u with respect to δ_i tend to zero as $|\delta_i| \to +\infty$ $(i = 1, 2)$. This type of traveling wave solution to KdVB equation is the one studied by Bona et al. [10] who showed it is unique, but did not find its functional form. We remark that, as it stands, the traveling wave solution (6.21) and (6.22) can not be reduced to either the sech^2 type of traveling wave solution to the KdV equation in the limit $\beta \to 0$ or to the tanh type traveling wave solution to the Burgers' equation in the limit $s \to 0$. If such a reduction needs to be carried out, it is necessary to return to original KdV-Burgers equation and then to proceed to the limits $s \to 0$ or $\beta \to 0$, as a result of which the desired solutions will be obtained. The amplitudes, wave numbers, and frequencies of the traveling wave solutions given by (6.21) and (6.22) depend on the coefficients α, β, and s. Since the amplitudes are inversely proportional to α and s, the shock strengthens if either α or s becomes small. If the dissipative effect is weak, (6.21) and (6.22) give rise to a weak shock wave provided that α or s are $o(1.8)$ quantities, see Fig.41.

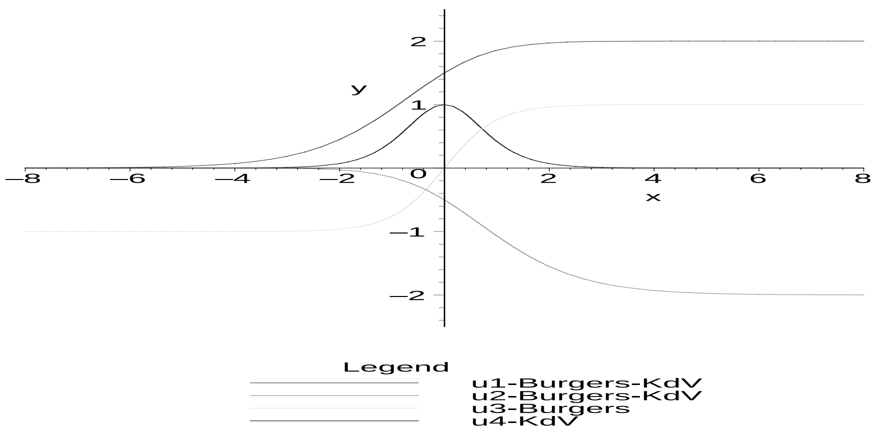

Fig. 41. Traveling wave solutions for case $\alpha = 2$, $\beta = 5$, $s = 3$. u1-KdV-Burgers is given by (6.21), u2-KdV-Burgers is given by (6.22), u3-Burgers is the solution for Burgers' equation (1.6) and u4-KdV is the solution for KdV equation (1.7).

C. Exact Solutions to 2D-KdVB Equation by First Integral Method

In this section, we extend the techniques shown in the preceding section to the general two-dimensional Burgers-KdV equation (1.8), i.e.,

$$(u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx})_x + \gamma u_{yy} = 0,$$

where α, β, s, and γ are real constants.

Assume that equation (1.8) has exact solution in the form

$$u(x, y, t) = u(\xi), \quad \xi = hx + ly - wt, \tag{6.23}$$

where h, l, w are real constants. Substitution of (6.23) into equation (1.8) yields

$$-whu_{\xi\xi} + \alpha h^2 (u u_\xi)_\xi + \beta h^3 u_{\xi\xi\xi} + s h^4 u_{\xi\xi\xi\xi} + \gamma l^2 u_{\xi\xi} = 0.$$

Integrating the above equation twice with respect to ξ, then we have

$$s h^4 u_{\xi\xi} + \beta h^3 u_\xi + \frac{\alpha}{2} h^2 u^2 + \gamma l^2 u - whu = R,$$

where R is the second integration constant and the first one is taken to zero.

Similar to (6.15), we assume that

$$25sk(wh - \gamma l^2) = 6\beta^2 h^2, \quad k \in \mathbb{R} \text{ with } k \neq 0, \tag{6.24}$$

then using the same arguments as in section 3, we obtain an exact solution to equation (1.8) immediately

$$u(x,y,t) = -\frac{12\beta^2}{25\alpha s} \cdot \Big[\frac{e^{-\frac{\beta}{5sh}(hx+ly-wt)}}{e^{-\frac{\beta}{5sh}(hx+ly-wt)} + c}\Big]^2 + \frac{wh - \gamma l^2}{\alpha h^2}(k+1), \tag{6.25}$$

where h, l, w are arbitrary and satisfy (6.24), and c is an arbitrary integration constant.

Rewrite (6.24) as

$$\frac{k(wh - \gamma l^2)}{\alpha h^2} = \frac{6\beta^2}{25s\alpha}, \quad k \in \mathbb{R} \text{ with } k \neq 0,$$

thus, (6.25) can be re-expressed as

$$u(x,y,t) = -\frac{12\beta^2}{25\alpha s} \cdot \Big[\frac{e^{-\frac{\beta}{5sh}(hx+ly-wt)}}{e^{-\frac{\beta}{5sh}(hx+ly-wt)} + c}\Big]^2 + \frac{wh - \gamma l^2}{\alpha h^2} + \frac{6\beta^2}{25\alpha s}. \tag{6.26}$$

Motivated by form of (6.25), now we assume that equation (1.8) has the solution in the following more general form

$$u(\xi) = \frac{24r^2}{25a} \cdot e^{\frac{2r}{5}\xi} \cdot \rho(-2e^{\frac{r}{5}\xi} + c) - \frac{1}{a}[\frac{3r^2}{25} + \frac{b}{2}], \tag{6.27}$$

where c is an arbitrary constant and ρ is a function to be determined. Then, we have

$$\begin{cases} u'(\xi) = \frac{48r^3}{125a} \cdot e^{\frac{2r}{5}\xi} \cdot \rho(-2e^{\frac{r}{5}\xi} + c) - \frac{48r^3}{125a} \cdot e^{\frac{3r}{5}\xi} \cdot \rho'(-2e^{\frac{r}{5}\xi} + c) \\ u''(\xi) = \frac{96r^4}{625a} \cdot e^{\frac{2r}{5}\xi} \cdot \rho(-2e^{\frac{r}{5}\xi} + c) - \frac{48r^4}{125a} \cdot e^{\frac{3r}{5}\xi} \cdot \rho'(-2e^{\frac{r}{5}\xi} + c) \\ \qquad + \frac{96r^4}{625a} \cdot e^{\frac{4r}{5}\xi} \cdot \rho''(-2e^{\frac{r}{5}\xi} + c). \end{cases} \tag{6.28}$$

Substituting (6.27) and (6.28) into equation (1.8), we obtain

$$\rho'' = \rho^2. \tag{6.29}$$

This implies that while ρ is the Weierstrass elliptic function with a condition $g_2 = 0$ satisfying (6.29), then (1.8) has the exact solution as follows

$$u(x,y,t) = -\frac{2\beta^2}{25\alpha s} \cdot e^{-\frac{2\beta}{5sh}(hx+ly-wt)} \cdot \rho(e^{-\frac{\beta}{5sh}(hx+ly-wt)} + c) + \frac{6\beta^2}{25\alpha s} + \frac{wh - \gamma l^2}{\alpha h^2}, \tag{6.30}$$

where c is an arbitrary constant.

Reverting to KdVB equation (1.5) and using the same discussions as the above, we have the following result immediately, that is, (1.5) has exact solution in the form

$$u(x,t) = -\frac{12\beta^2}{25\alpha s} \cdot e^{-\frac{2\beta}{5s}(x-vt)} \cdot \rho(e^{-\frac{\beta}{5s}(x-vt)} + k) + \frac{6\beta^2}{25\alpha s} + \frac{v}{\alpha}, \tag{6.31}$$

where k is arbitrary constant.

It is remarkable that (6.31) is a new type solution to equation (1.5) and (6.30) is a new type solution to (8) In fact, it is the general traveling wave solution for the case of $d = \frac{b^2}{4a} - \frac{9r^4}{625a}$. One can see that (6.30) is obtained only in the case of $d = \frac{b^2}{4a} - \frac{9r^4}{625a}$. In other words, if $d \neq \frac{b^2}{4a} - \frac{9r^4}{625a}$, we can not assume that equation (1.8) has the solution in the form of (6.27). It is apparent that (6.25) is the particular case of (6.30) because $\rho(y) = 6(y+c)^{-2}$ is a solution of (6.29). While we are writing this paper, it does not seem that the same results have been presented previously. There are many textbooks introducing and describing the Weierstrass elliptic function in details such as references 111-114 et al.. We refer the reader to one of those textbooks or any handbook of mathematics to get more information about (6.30). There is really a lot to say about the Weierstrass elliptic function, so we omit the detailed description here.

D. Comparisons with Previous Results

In this section, we give brief comparisons with the existing solutions presented in the literature.

It is remarkable the solutions obtained in the previous literature either are particular cases of (6.20), or contain errors. For example, Xiong et al. used the analytic method and the method of undetermined coefficients to study the KdV-Burgers equation

$$u_t + uu_x - cu_{xx} + \beta u_{xxx} = 0. \tag{6.32}$$

With the assumption $6c^2 = 25\beta v$, an exact solution to equation (6.32) is presented in [52, 53] as

$$u(\xi) = 2v - \frac{3c^2}{25\beta}(1 + \tanh\frac{c\xi}{10\beta})^2. \tag{6.33}$$

By using the equality $\frac{Ae^t}{1+e^t} = \frac{A}{2}\tanh(\frac{t}{2}) + \frac{A}{2}$, it is easy to see that (6.33) is in agreement with (6.21) as $\alpha = 1$, $\beta = -c$ and $s = \beta$.

By means of a direct method and a series method, Jeffrey and Xu et al. investigate the KdV-Burgers equation in [54, 55]

$$u_t + 2auu_x + 5bu_{xx} + cu_{xxx} = 0,$$

and find the solutions as follows provided that (2.9a,b) and (2.10a,b) hold on p. 561 in [55],

$$u_1 = \frac{3b^2}{2ac}[\mathrm{sech}^2(\frac{\theta}{2}) + 2\tanh(\frac{\theta}{2}) + 2], \tag{6.34}$$

where $\theta = \frac{b}{c}x - \frac{6b^3}{c^2}t + d$, and

$$u_2 = \frac{3b^2}{2ac}[\mathrm{sech}^2(\frac{\theta}{2}) - 2\tanh(\frac{\theta}{2}) - 2], \tag{6.35}$$

where $\theta = -\frac{b}{c}x - \frac{6b^3}{c^2}t + d$. Since $\tanh(-x) = -\tanh(x)$, it is easily seen that (6.34) and (6.35) are identical with (6.21) and (6.22), respectively, as $\alpha = 2a$, $\beta = 5b$, and $s = c$, and the conditions (2.9a,b) and (2.10a,b) agree with (6.15) in cases $k = \pm 1$. By partial use of a Painlevé analysis, the same results are presented by Halford and Vlieg-Hulstman in [56].

When $s = -s$, $\alpha = p$, $\beta = r$ in equation (1.5), Wang applied the homogeneous balance method to equation (1.5) in [14]. The corresponding result ((3.18) when taking $b = 0$) in [14] appears to coincide with (6.21) and (6.22). Notice that there are three errors on p. 286 in [14]: one is that the first term in (3.16) should be $-\frac{6r^3}{125s^2}$, as a result of which the denominator $125s$ in (3.18) should be replaced by $125s^2$. Parkes and Duffy also studied the Burgers-CKdV and Burgers-KdV equations by the tanh-function method, and obtained an equivalent result.

It is notable that the solution derived by Ma in [62] is identical to (6.26) while c is positive. The case where c is negative is not discussed in [62]. By using the Hopf-Cole transformation and a computer algebra system, Li and Wang also obtained a traveling wave solution to (1.8) in [62] simultaneously, which is proven by Parkes [63] to equivalent to one obtained in [62].

Recently, Fan et al. investigated the following 2D-BKdV equation by making use of an extended tanh-function method with symbolic computation

$$(u_t + uu_x - \alpha u_{xx} + \beta u_{xxx})_x + \gamma u_{yy} = 0, \tag{6.36}$$

and obtained a new complex line soliton to (6.36) as follows [65, p. 378]

$$u = -(d + c^2\gamma) \pm \frac{6\alpha^2}{25\beta}\tanh\xi + \frac{6\alpha^2}{25\beta}\mathrm{sech}^2\xi + i\frac{6\alpha^2}{25\beta}(1 \mp \tanh\xi)\cdot \mathrm{sech}\xi, \tag{6.37}$$

where $\xi = \mp\frac{\alpha}{5\beta}(x + cy + dt)$.

Indeed, (6.37) is the particular case of (6.25). Denote that $\xi_1 = -\frac{\alpha}{5\beta}(x + cy + dt)$ and $\xi_2 = \frac{\alpha}{5\beta}(x + cy + dt)$. Utilizing the equalities $\mathrm{sech}\xi = \frac{2}{e^{\xi}+e^{-\xi}}$, $\tanh\xi = \frac{e^{\xi}-e^{-\xi}}{e^{\xi}+e^{-\xi}}$ and

$4A[\frac{e^{2\xi}}{1+e^{2\xi}}]^2 = -A\text{sech}^2\xi + 2A\tanh\xi + 2A$, we have

$$\begin{aligned}
& -(d+c^2\gamma) + \frac{6\alpha^2}{25\beta}\tanh\xi_1 + \frac{6\alpha^2}{25\beta}\text{sech}^2\xi_1 + i\frac{6\alpha^2}{25\beta}(1-\tanh\xi_1)\cdot\text{sech}\xi_1 \\
= & -(d+c^2\gamma) + \frac{6\alpha^2}{25\beta} - [\frac{6\alpha^2}{25\beta}\tanh\xi_2 - \frac{3\alpha^2}{25\beta}\text{sech}^2\xi_2 + \frac{6\alpha^2}{25\beta}] + \frac{3\alpha^2}{25\beta}\text{sech}^2\xi_2 \\
& +i\frac{6\alpha^2}{25\beta}(1+\tanh\xi_2)\cdot\text{sech}\xi_2 \\
= & -\frac{48\alpha^2}{25\beta}\cdot e^{2\xi_2}\cdot\frac{1}{(-2e^{\xi_2}-2i)^2} + \frac{6\alpha^2}{25\beta} - (d+c^2\gamma). \qquad (6.38)
\end{aligned}$$

It is easy to see that (6.38) is identical to (6.26) while $c = -2i$.

Similarly, making use of the equalities $\text{sinh}y = \frac{1}{2}(e^y - e^{-y})$ and $\text{csch}y = \frac{1}{\text{sinh}}$, it is very easy to verify that the new solution claimed by Elwakil et al. [67, formula (31), p.183] is indeed the particular case of (6.26) while $c = -2$.

In view of (6.30) and (6.31), we find that they are more general than the previous results obtained by various methods used in the literature. The method of the solutions to (1.8) used in [14] is regarded to be complicated and tedious (see [63]). The methods to (1.8) used in [63] are limited to the equations which have the traveling wave solution in the form of a power of series in tanh-function (or sech-function), and need to solve at least five nonlinear algebraic equations when M is greater than 2 in (13) (see page L499 in [27]), so it is not efficient. The method used in [62] is only applicable when c is positive. It is worthwhile to mention here that the first integral method is not only more efficient but also has the merit of being widely applicable. We can definitely apply this technique to many nonlinear evolution equations, such as the nonlinear Schrödinger equation, the generalized Klein-Gordon equation, and the high-order KdV like equation, which can be converted to the following form through the traveling wave transformation,

$$u''(\xi) - \mu T[u, u'(\xi)] - R(u) = 0,$$

where μ is real, $R(u)$ is a polynomial with real coefficients and $T(u, v)$ is a polynomial in u, v with real coefficients.

CHAPTER VII

COORDINATE TRANSFORMATIONS METHOD AND PROPER SOLUTIONS TO 2D-KDVB EQUATION*

In this chapter, we introduce another new method, which is currently called Coordinate Transformations Method to the study of 2D-KdVB equation. In Section 1, we analyze the stability and bifurcation of system (7.4). In Section 2, a traveling wave solution to 2D-KdVB equation (1.8) is obtained applying the Coordinate Transformations Method. In Section 3, an asymptotic behavior of the proper solutions of 2D-KdVB equation is established by applying the qualitative theory of differential equations. At the end of this section, we list some other nonlinear differential equations, such as Fisher's equation can be handled in a similar manner. In Section 4, we point out that a statement of chaotic behavior of solutions of another Korteweg-de Vries-Burgers equation arising from ferroelectricity is not correct.

A. Analysis of Stability

Similar to one described in Chapter VI, we assume that equation (1.8) has an exact solution of the form (6.23), then integrate it twice with respect to ξ. Denote $r = -\frac{\beta}{sh}$, $a = -\frac{\alpha}{2sh^2}$, $b = \frac{wh-\gamma l^2}{sh^4}$ and $d = \frac{R}{sh^4}$. Then we have

$$U''(\xi) - rU'(\xi) - aU^2(\xi) - bU(\xi) - d = 0. \tag{7.1}$$

---

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When $\sqrt{b^2-4ad}>0$, to remove the constant term in equation (7.1), let

$$U=-\frac{1}{a}u(\xi)-\frac{b}{2a}-\frac{\sqrt{b^2-4ad}}{2a}, \tag{7.2}$$

and substituting (7.2) into equation (7.1) then yields

$$u''(\xi)+\delta u'(\xi)+u^2(\xi)-\mu u(\xi)=0, \tag{7.3}$$

where $\delta=-r$ and $\mu=-\sqrt{b^2-4ad}$. Letting $v=u_\xi$, equation (7.3) is equivalent to

$$\begin{cases} \dot{u}=v=P(u,v) \\ \dot{v}=-\delta v-u^2+\mu u=Q(u,v). \end{cases} \tag{7.4}$$

Assume that $\delta<0$ (the discussion for the case $\delta>0$ is closely similar). Using the same analysis in the Poincaré phase plane as in Chapter V, we obtain that (7.4) has two equilibrium points A(0,0) and B(u,0) and

(i). $\frac{\delta^2}{4}>\mu>0$, A(0,0) is a saddle point and B$(\mu,0)$ is an unstable nodal point.

(ii). $-\frac{\delta^2}{4}<\mu<0$, A(0,0) is an unstable nodal point and B$(\mu,0)$ is a saddle point.

(iii). $\frac{\delta^2}{4}<\mu$, A(0,0) is a saddle point and B$(\mu,0)$ is an unstable spiral point.

(iv). $-\frac{\delta^2}{4}>\mu$, A(0,0) is an unstable spiral point and B$(\mu,0)$ is a saddle point.

Now, we are considering the stability near the origin for system (7.4). It will help us understand that the stability of system (7.4) actually depends on the sign of μ. Making the similarity transformation

$$\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 0 & -\delta \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}, \tag{7.5}$$

and substituting (7.5) into (7.4), then yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{\mu}{\delta} \cdot \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{\delta} \cdot \begin{pmatrix} -(x+y)^2 \\ (x+y)^2 \end{pmatrix}. \tag{7.6}$$

Fix $\delta > 0$ and consider the corresponding extension system

$$\begin{cases} \dot{x} = \frac{\mu}{\delta}(x+y) - \frac{1}{\delta}(x+y)^2 \\ \dot{\mu} = 0 \\ \dot{y} = -\delta y - \frac{\mu}{\delta}(x+y) + \frac{1}{\delta}(x+y)^2. \end{cases} \tag{7.7}$$

By virtue of the stable manifold theorem for a hyperbolic equilibrium point, system (7.7) has a two-dimensional center manifold, which is tangent to the (x, μ)-plane at the equilibrium point $(x, \mu, y) = (0, 0, 0)$. Denote by $y = h(x, \mu)$ the center manifold of (7.7) near the origin, where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and satisfies

$$h(0,0) = 0, \quad \frac{\partial h(x,\mu)}{\partial x} = \frac{\partial h(x,\mu)}{\partial \mu}\Big|_{(0,0)} = 0.$$

By making use of the Taylor expansion, $y = h(x, \mu)$ can be expressed as

$$h(x,\mu) = a_1 x^2 + b_1 x\mu + c_1 \mu^2 + o(3), \tag{7.8}$$

where $o(3)$ contains terms of power 3 or higher. Thus, we have

$$\begin{aligned} \dot{y} &= \frac{\partial h(x,\mu)}{\partial x} \cdot \dot{x} + \frac{\partial h(x,\mu)}{\partial \mu} \cdot \dot{\mu} \\ &= -\delta h(x,\mu) - \frac{\mu}{\delta}[x + h(x,\mu)] + \frac{1}{\delta}[x + h(x,\mu)]^2. \end{aligned} \tag{7.9}$$

Substituting (7.8) and the third equation of (7.7) into (7.9), we get

$$(2a_1 x + b_1 \mu)[\frac{\mu}{\delta}(x + a_1 x^2 + bx\mu + c\mu^2 + o(3)] - \frac{1}{\delta}[x + a_1 x^2 + bx\mu + c\mu^2 + o(3)]^2$$

$$= -\delta[a_1x^2 + bx\mu + c\mu^2 + o(3)] - \frac{\mu}{\delta}[x + a_1x^2 + bx\mu + c\mu^2 + o(3)]$$
$$+\frac{1}{\delta}[x + a_1x^2 + bx\mu + c\mu^2 + o(3)]^2. \qquad (7.10)$$

Equating the coefficients of x^2, $x\mu$ and μ^2 on both sides of (7.10), respectively, then yields

$$a_1 = \frac{1}{\delta^2}, \quad b_1 = -\frac{1}{\delta^2}, \quad c_1 = 0.$$

Returning to the $x-$equation in (7.7), we have

$$\dot{x} = \frac{1}{\delta}(1 - \frac{\mu}{\delta^2})x(\mu - x) + o(3), \quad (\dot{\mu} = 0). \qquad (7.11)$$

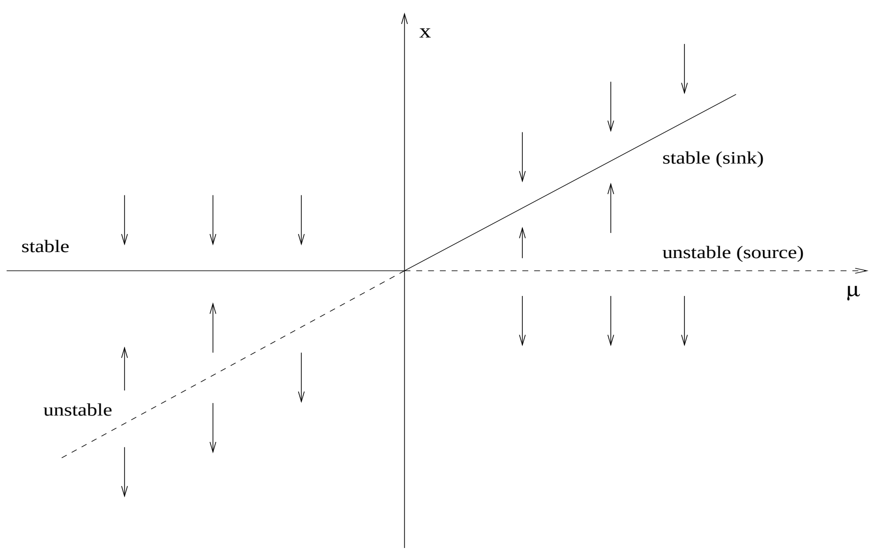

Fig. 42. Transcritical bifurcation.

The equilibria of (7.11) are $x = 0$ and $x = \mu$. When δ and $1 - \frac{\mu}{\delta^2}$ have the same sign, the stable branches are illustrated in Fig.42 by heavy lines and the unstable branches are indicated by broken lines. The trivial branch $x = 0$ loses stability at the bifurcation point $(x, \mu) = (0, 0)$. Simultaneously, there is an exchange of stability to the other branch. This implies that (7.4) loses its stability at $(u, \mu) = (0, 0)$. When $\delta > 0$ and $\mu < 0$ ($|\mu|$ is small), that is, when $\beta sh > 0$ and $(wh - \gamma l^2)^2 + 2\alpha Rh^2 > 0$, there exists a unique nontrivial bounded traveling wave solution to equation (1.8), which is stable and approaches to two bounded limits as $\xi \to +\infty$ and $\xi \to -\infty$, respectively. These two limits depend on the corresponding $u-$coordinates of equilibrium points A and B.

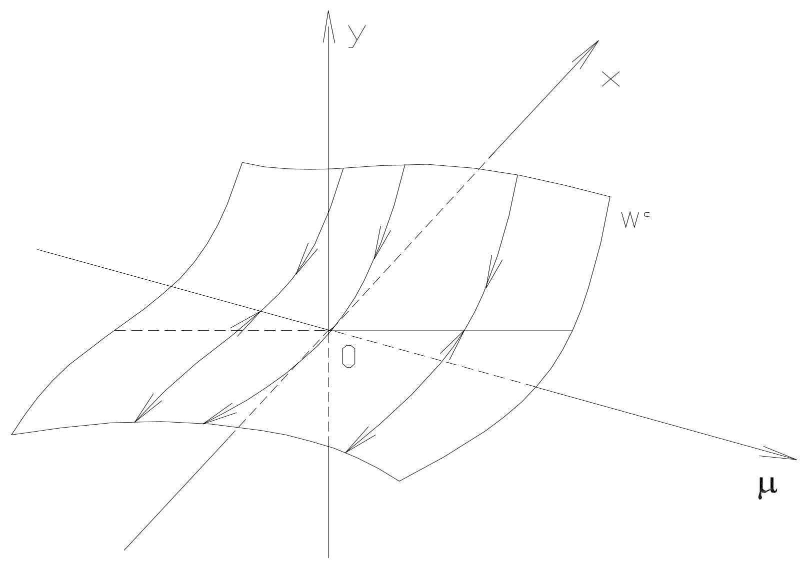

Fig. 43. The center manifold of system (7.7).

The center manifold W^c of system (7.7) is depicted in Fig.43. One can see that near the equilibrium point $(x, \mu, y) = (0, 0, 0)$, for each fixed μ, the stability of equilibrium points of system (7.6) is also determined by (7.11).

B. Coordinate Transformations Method

In this section, we restrict our attention to the study of seeking the exact traveling wave solution of equation (1.8) applying Coordinate Transformations Method. That is,

First, make the natural logarithm transformation

$$\xi = -\frac{1}{\delta}\ln\tau,$$

then equation (7.3) becomes

$$\delta^2\tau^2\frac{d^2u}{d\tau^2} + u^2 - \mu u = 0. \tag{7.12}$$

Take the variable transformation as described in [115]

$$q = \tau^k, \quad u = \tau^{-\frac{1}{2}(k-1)} \cdot H(q),$$

then equation (7.12) becomes

$$\frac{d^2H}{dq^2} = -\frac{1}{\delta^2k^2}q^{\frac{1-5k}{2k}}H^2, \tag{7.13}$$

where $k = \sqrt{1 - \frac{4\sqrt{b^2-4ad}}{\delta^2}}$.

To simplify the coefficient on the right hand side of (7.13) to be 1, we assume

$$\phi = q, \quad H = -\delta^2k^2\rho,$$

then equation (7.13) reduces to

$$\frac{d^2\rho}{d\phi^2} = \phi^m\rho^2, \tag{7.14}$$

where $m = \frac{1-5k}{2k}$.

Therefore, from equation (7.14) we can derive the following results immediately:

(I). When $k = \frac{1}{5}$, i.e., $\sqrt{b^2 - 4ad} = \frac{6r^2}{25}$, changing to our original variables, we obtain an exact solution to equation (1.8)

$$U(x, y, t) = -\frac{2\beta^2}{25\alpha s} \cdot e^{-\frac{2\beta}{5sh}(hx+ly-wt)} \cdot \rho(e^{-\frac{\beta}{5sh}(hx+ly-wt)} + c) + \frac{6\beta^2}{25\alpha s} + \frac{wh - \gamma l^2}{\alpha h^2}, \quad (7.15)$$

where c is arbitrary integration constant and $\rho(\phi, g_2, g_3)$ is the Weierstrass elliptic function with invariants g_2 and g_3 [112, 113] satisfying

$$\frac{d^2\rho}{d\phi^2} = \rho^2. \quad (7.16)$$

Since $\rho(\phi) = 6(\phi + C_0)^{-2}$ is a particular solution of equation (7.16), apparently we can obtain a particular traveling solitary wave solution to equation (1.8) from (7.15) directly

$$U(x, y, t) = -\frac{12\beta^2}{25\alpha s} \cdot \frac{e^{-\frac{2\beta}{5sh}(hx+ly-wt)}}{[e^{-\frac{\beta}{5sh}(hx+ly-wt)} + C_1]^2} + \frac{wh - \gamma l^2}{\alpha h^2} + \frac{6\beta^2}{25\alpha s}, \quad (7.17)$$

where C_1 is arbitrary constant.

(II). In general, equation (7.14) has a nontrivial solution in the polynomial form $\rho(\phi) = c_0\phi^{\omega}$, where

$$\omega = -m - 2, \qquad c_0 = (m + 2)(m + 3). \quad (7.18)$$

Reverting to system (7.4), we can see that this solution corresponds to the equilibrium point $B(\mu, 0)$ and the trivial solution $\rho = 0$ corresponds to the equilibrium point $A(0, 0)$.

It is easy to see that (7.15) is identical to (6.30) and (7.17) agrees with (6.26). Compared with various methods proposed for 2D-KdVB equation (1.8) in the previous references, the Coordinate Transformations Method introduced herein is more straightforward and less calculative.

C. Asymptotic Behavior of Proper Solutions

In Section B, we reduce 2D-KdVB equation (1.8) to (7.14). For certain value of m, it is possible to reduce (7.14) to a nonlinear equation with constant coefficients and thereby the way to the application of the Poincare-Liapounov theory to the study of (7.14). Nevertheless, by applying the qualitative theory of differential equations, the solutions to equation (7.14), in general, can not be expressed explicitly. Therefore, analyzing the asymptotic behavior of the solutions of equation (1.8) becomes sufficiently important and necessary. In order to isolate the large class tractable solutions, we employ the concept of proper solution, which is one that is real and nontrivial with continuous derivative for $\xi > \xi_0$. Since the arithmetic nature of m in equation (7.14) will have considerable influence upon the possible types of proper solutions, we only consider the positive proper solutions of equation (7.14). The arguments for the negative case can be handled similarly.

To present our discussion in a straightforward manner, we need the following technical theorem:

Theorem VII.1 *If both of $P(u,t)$ and $Q(u,t)$ are polynomials in u and t, then any solution of the equation*

$$\frac{du}{dt} = \frac{P(u,t)}{Q(u,t)}$$

continuous for $t > t_0$, is ultimately monotonic, together with all its derivative, and satisfies one or the other of the relations

$$u \sim At^j e^{P(t)}, \qquad u \sim At^j (\log t)^{1/l},$$

where $P(t)$ is a polynomial in t, A is constant and l is an integer. □

Applying Hardy's theorem, we can obtain the asymptotic behavior of proper solutions of the 2D-KdVB equation. That is, when $4\sqrt{b^2-4ad}<\delta^2$, i.e., $\sqrt{\frac{(wh-\gamma l^2)^2+2\alpha Rh^2}{s^2h^4}} < \frac{\beta^2}{4s^2}$, proper solutions of the 2D-KdVB equation have the asymptotic form as follows

$$U(x,y,t)\sim -\frac{\mathrm{sgn(s)}}{\alpha h^2}\sqrt{(wh-\gamma l^2)^2+2\alpha Rh^2}+\frac{wh-\gamma l^2}{\alpha h^2}. \tag{7.19}$$

Now, we prove (7.19). From equation (7.14), it is easy to see that $\rho(\phi)$ must be eventually monotone. Since if there is a point ϕ_0 such that $\rho'(\phi_0)=0$, $\rho(\phi)$ can only have a minimum at $\phi=\phi_0$ due to the fact that $\rho''=\phi^m\rho^2>0$. Hence ρ is eventually monotone decreasing or monotone increasing.

Now let us set $\rho=c_0\phi^\omega T$, here c_0 and ω have the same values as given in (7.18). The equation for T is

$$T''+(2\omega-1)T'+\omega(\omega-1)(T-T^2)=0. \tag{7.20}$$

Note that $T=0$ and $T=1$ are two trivial solutions to equation (7.20). Since $0<k<1$, we get $\omega=\frac{1}{2}-\frac{1}{2k}<0$ and $2\omega-1<0<\omega(\omega-1)$.

Consider the possible alternatives for T; we already know $T>0$. In the case of the solutions in the region $0<T<1$. From the ultimate monotonicity of the solutions, we have $T\to 0$ or $T\to 1$ as $\phi\to\infty$. One can easily rule out the possibility that $T\to 0$. The characteristic roots of the linearization of (7.20) are given by $-\omega$ and $1-\omega$. Since both roots are positive, it follows that $T=0$ is a throughly unstable solution and thus that no other solution of (7.20) can tend to this as $\phi\to\infty$. Thus the alternative is $T\to 1$, which implies

$$\rho\sim(\frac{1}{2k}-\frac{1}{2})(\frac{1}{2k}+\frac{1}{2})\phi^{\frac{1}{2}-\frac{1}{2k}}. \tag{7.21}$$

In the case that T crosses $T=1$, it must continue monotonically increasing, since any turning point must be a minimum. That T approaches a finite limit L greater than 1 is possible. In this case, $T' \to 0$ and $T'' \to 0$, so any finite limit including L must be a root of $T - T^2 = 0$. This yields a contradiction. Thus, we can deduce that $T \to \infty$. We now investigate this possibility by using Hardy's Theorem. Setting $F = T'$, equation (7.20) reduces to

$$F\frac{dF}{dT} + (2\omega - 1)F + \omega(\omega - 1) = 0. \tag{7.22}$$

As $T \to \infty$, we have either

$$F \sim e^{h(T)}T^{c_1}, \tag{7.23}$$

where $h(T)$ is a polynomial in T, or

$$F \sim T^{c_2}(\log T)^{c_3}, \tag{7.24}$$

where c_i $(i = 1, 2, 3)$ are constants.

Combine (7.22) with (7.23) or (7.24), respectively. Evaluation of the constants indicates that both cases lead to $F \geq T^{1+\epsilon}$ with $\epsilon > 0$ as $T \to \infty$. Going back to our assumption $F = \frac{dT}{d\phi}$, one can see that this is impossible if we are considering proper solutions to equation (7.20). Hence again if $T > 1$, we have $T \to 1$ as $\phi \to \infty$, which yields (7.21).

Using the inverse of transformations described in Section B, we have

$$H \sim -\delta^2 k^2 \rho(q) \sim -r^2 \cdot \frac{1 - k^2}{4} \cdot q^{\frac{1}{2} - \frac{1}{2k}},$$

and

$$u \sim \tau^{-\frac{1}{2}(k-1)} \cdot H(q) \sim -\sqrt{b^2 - 4ad}.$$

Making use of (7.2) and changing to the original variables, we obtain

$$
\begin{aligned}
U(x,y,t) &\sim \frac{\sqrt{b^2-4ad}}{2a}-\frac{b}{2a}\\
&\sim -\frac{\mathrm{sgn(s)}}{\alpha h^2}\sqrt{(wh-\gamma l^2)^2+2\alpha R h^2}+\frac{wh-\gamma l^2}{\alpha h^2}.
\end{aligned}
$$

This is the asymptotic behavior of proper solutions of the 2D-KdVB equation (1.8).

D. On Chaotic Behavior of Solutions of KdVB Equation

Recently, Zayko et al. investigated the Korteweg-de Vries-Burgers equation, which arises from ferroelectricity [40, 41] :

$$
\begin{cases}
P_{\tau}^{(1)}+AP^{(1)}P_{\xi}^{(1)}+DP_{\xi\xi}^{(1)}+BP_{3\xi}^{(1)}=0,\\
A=\frac{2\alpha}{K'(u)},\quad D=\frac{u}{K'(u)},\quad B=-\frac{u^2}{K'(u)},\\
K'(u)=\frac{dK(u)}{du},\\
K(u)=\frac{\omega_p^2u^2}{c^2-u^2}-\omega_0^2-2\alpha P_0=0,
\end{cases} \tag{7.25}
$$

where $P^{(1)}$ is the first term of the series expansion of the polarization P with respect to small attenuation coefficient δ; $\xi=\delta(z-ut)$ and $\tau=\delta^3 t$ are the scaled coordinate and time (both z and t are real), respectively; $K(u)=0$ is the dispersive equation for wave velocity u in long-wave limit; $P_0\simeq-\omega_0^2/\alpha$ is the equilibrium value of P; ω_p and ω_0 are the characteristic frequencies of the problem, which corresponds to the optical and the acoustic branches of the spectrum; c is the velocity of light in the system; α is a coefficient determined by the nonlinear properties of the system.

As shown in [116], to find $2\pi-$periodic traveling wave solutions $p=P^{(1)}(\xi-c\tau)$, one can convert equation (7.25) to Hammerstein's integral equation

$$
p(\theta)=-\int_0^{2\pi}G(\theta,\lambda)\big[(Av-1)p(\lambda)-0.5Ap^2(\lambda)\Big]d\lambda, \tag{7.26}
$$

where

$$G(\theta,\lambda)=\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos n(\theta-\lambda)}{n^2+1}$$

is the 2π-periodic Green function for the equation $p_{\theta\theta}-p=-\delta(\theta-\lambda)$ and $\theta=\frac{\xi-vt}{\sqrt{AB}}$. It has been shown that nontrivial solutions of (7.26) occur at bifurcation points, defined by the equation $Av=-n^2$, where the n^{th} harmonic of the solution grows abruptly [117]. The sequence of bifurcation points is infinite and approaches to T_c as $n\to\infty$, here T_c is the Curie temperature. These bifurcation points can be determined by the set of two equations $K(u)=0$ and $K'(u)=0$. Zayko and Nefedov [40] considered the behavior of the system beyond the point of phase transition $T>T_c$. They noticed that close to the Curie point, $u\ll c$ and $K(u)\simeq\omega_0^2+\omega_p^2u^2/c^2$. The equation $K(u)=0$ has pure imaginary solutions as $u=\pm ic\omega_0/\omega_p=\pm iu_i$ when $T>T_c$ and

$$\omega_0^2=\frac{\omega_p^2}{2\pi}\cdot\frac{T-T_c}{\chi},$$

where χ is the curie-Weiss constant [118]. Thus, the quantity $K'(u)=\pm 2i\omega_p\omega_0/c$ becomes imaginary too. Furthermore, the coordinate $\xi=\delta(z-ut)$ contains an imaginary contribution when $T>T_c$. To remove the corresponding imaginary part, authors used the following new coordinates

$$\tau'=\tau,\qquad \xi'=\xi+iu_i\tau/\delta^2,\qquad u_i=c\omega_0/\omega_p,$$

then equation (7.25) reduces to

$$U_\eta-iUU_\eta-i\mu U_{\eta\eta}-rU_{3\eta}=0, \tag{7.27}$$

where

$$\eta=\frac{\xi'-V\tau}{c},\qquad U=-\frac{\delta^2\omega_p}{c\omega_0}[V+i\frac{\alpha c}{\omega_0\omega_p}P^{(1)}],$$

$$(7.28)$$

$$r = \frac{\delta^2}{2\omega_p^2}, \qquad \mu = \frac{\delta^2}{2\omega_0\omega_p}.$$

Integrating (7.27) once and taking the integration constant to be d, then yield

$$U_{\eta\eta} + \frac{i\mu}{r}U_\eta + \frac{i}{2r}U^2 - \frac{1}{r}U - d = 0. \tag{7.29}$$

For (7.27), it was claimed in [40] that in the general case it has no analytical solution, thus a stationary points analysis for (7.27) was utilized in order to avoid seeking it. Meanwhile, it was concluded that the new unknown class of solutions of equation (7.25) demonstrates chaotic behavior due to the numerical results.

Next, we show that the exact solution to equation (7.25) with imaginary coefficients indeed can be derived by means of Coordinate Transformations Method.

Similar to the arguments as in Section B, we first make the coordinate transformation

$$U(\eta) = -i2rY(\eta) - i - ir\sqrt{\frac{1}{r^2} + \frac{2di}{r}}.$$

Substituting this into (7.29), gives

$$Y''(\eta) + \frac{i\mu}{r}Y'(\eta) + Y^2(\eta) + \sqrt{\frac{1}{r^2} + \frac{2di}{r}}Y(\eta) = 0. \tag{7.30}$$

Then, we make the natural logarithm transformation as follows

$$\eta = \frac{ir}{\mu}\ln x.$$

Equation (7.30) becomes

$$-\frac{\mu^2}{r^2}x^2\frac{d^2Y}{dx^2} + Y^2 + \sqrt{\frac{1}{r^2} + \frac{2di}{r}}Y = 0. \tag{7.31}$$

Taking the variable transformation

$$q = x^k, \quad Y = x^{-\frac{1}{2}(k-1)} \cdot H(q),$$

equation (7.31) can be written as

$$\frac{d^2 H}{dq^2} = \frac{r^2}{\mu^2 k^2} q^{\frac{1-5k}{2k}} H^2, \tag{7.32}$$

where $k = \sqrt{1 + \frac{4r^2\sqrt{(1/r^2)+(2di/r)}}{\mu^2}}$.

To simplify the coefficient on the right hand side of (7.32) to be 1, we assume

$$\phi = q, \quad H = \frac{\mu^2 k^2}{r^2} \rho,$$

then equation (7.32) reduces to

$$\frac{d^2 \rho}{d\phi^2} = \phi^{\frac{1-5k}{2k}} \rho^2. \tag{7.33}$$

Therefore, from equation (7.33) we can derive the following results to equation (7.25) immediately:

Case 1. When $d = -i\frac{36\mu^4 - 625r^2}{1250r^3}$, i.e., $k = \frac{1}{5}$, equation (7.29) has an exact solution

$$U(\eta) = -\frac{2i\mu^2}{25r} \cdot e^{-\frac{2ir}{5\mu}\eta} \cdot \rho(e^{-\frac{ir}{5\mu}\eta} + c_1) - i + \frac{6i\mu^2}{25r}.$$

Changing to the original variables, we obtain an exact solution to equation (7.25) as

$$\begin{aligned} P^{(1)} \quad = \quad & \frac{2\mu^2\omega_0^2}{25r\alpha\delta^2} \cdot e^{-\frac{2ir}{5\mu}(\frac{\xi}{c} + \frac{iu_i\tau}{\delta^2 c} - \frac{V\tau}{c})} \cdot \rho[e^{-\frac{ir}{5\mu}(\frac{\xi}{c} + \frac{iu_i\tau}{\delta^2 c} - \frac{V\tau}{c})} + c_2] \\ & + \frac{\omega_0^2}{\alpha\delta^2} - \frac{6\mu^2\omega_0^2}{25r\alpha\delta^2} + \frac{iV\omega_0\omega_p}{\alpha c}, \end{aligned} \tag{7.34}$$

where c_2 is arbitrary integration constant, $u_i = \frac{c\omega_0}{\omega_p}$, V, r, μ are the same as (7.28), and $\rho(\phi, g_2, g_3)$ is the Weierstrass elliptic function with invariants g_2 and g_3 [112, 113] satisfying

$$\frac{d^2\rho}{d\phi^2} = \rho^2. \tag{7.35}$$

Since $\rho(\phi) = 6(\phi + C_0)^{-2}$ is a particular solution of equation (7.35), apparently,from (7.34) we can obtain a particular solution to equation (7.25) directly

$$P^{(1)} = \frac{12\mu^2\omega_0^2}{25r\alpha\delta^2} \cdot \frac{e^{-\frac{2ir}{5\mu}(\frac{\xi}{c}+\frac{iu_j\tau}{\delta^2 c}-\frac{V\tau}{c})}}{[e^{-\frac{ir}{5\mu}(\frac{\xi}{c}+\frac{iu_j\tau}{\delta^2 c}-\frac{V\tau}{c})} + c_3]^2} + \frac{\omega_0^2}{\alpha\delta^2} - \frac{6\mu^2\omega_0^2}{25r\alpha\delta^2} + \frac{iV\omega_0\omega_p}{\alpha c}, \tag{7.36}$$

where c_3 is arbitrary complex constant.

Case 2. In general, equation (7.33) also has a nontrivial solution in the polynomial form

$$\rho(\phi) = (\frac{1}{4k^2} - \frac{1}{4})\phi^{\frac{1}{2}-\frac{1}{2k}}.$$

Reverting to equation (7.29), one can see that this solution corresponds to the trivial solution of (7.29), i.e., the constant solution.

It is notable that formula (7.34) is a new exact solution to equation (7.25), so far, to my best acknowledge, which is not illustrated in the literature. Look at (7.36) and consider a particular case of $c_3 = 1$. While $c \to 0$, each graph of the real part and imaginary part of (7.36) will become more oscillatory, which appears to coincide with the numerical results and pictures provided in [40,41]. Of course, we can not conclude that this demonstrates chaotic behavior, because every thing is predictable.

CHAPTER VIII

PAINLEVÉ ANALYSIS

A. Motivation

In this chapter, we are applying the Painlevé analysis to study 2D-KdVB equation. Our motivation is due to a fact that, so far, no one has used Painlevé analysis to deal with KdVB or 2D-KdVB equation and obtained a result in terms of elliptic functions.

B. Traveling Wave Solutions to 2D-KdVB Equation by Painlevé Analysis

We assume that equation (1.8) has an exact solution of the form (6.23), then integrate it twice with respect to ξ. We obtain a second-order ordinary differential equation, i.e., equation (7.1):

$$U''(\xi) - rU'(\xi) - aU^2(\xi) - bU(\xi) - d = 0,$$

where $r = -\frac{\beta}{sh}$, $a = -\frac{\alpha}{2sh^2}$, $b = \frac{wh-\gamma l^2}{sh^4}$ and $d = \frac{R}{sh^4}$.

When $\sqrt{b^2 - 4ad} > 0$, to reduce equation (7.1) to a simple form, we assume

$$U = v_1 u(z) + v_2, \qquad \xi = \gamma z, \tag{8.1}$$

where $v_1 = \frac{\sqrt{b^2-4ad}}{a}$, $v_2 = -\frac{b}{2a} - \frac{\sqrt{b^2-4ad}}{2a}$ and $\gamma = \frac{1}{\sqrt[4]{b^2-4ad}}$. Substituting (8.1) into equation (7.1), then yields

$$u''(z) + cu'(z) - u^2(z) + u(z) = 0, \tag{8.2}$$

where $c = -r\gamma$.

Painlevé and his colleagues considered the problem of classifying differential equations whose solutions, as functions of a complex variable, have only poles as movable (i.e., dependent upon initial conditions) singularities [13]. It has been shown that equations with this Painlevé property are more likely to be explicitly solvable.

To determine what kind of pole a solution of equation (8.2) may have, we assume $u(z) \sim k(z-z_0)^{-\tau}$, then the u'' and u^2 terms must balance. Thus

$$u'' \sim k(-\tau)(-\tau-1)(z-z_0)^{-\tau-2} = k^2(z-z_0)^{-2\tau} \sim u^2$$

This yields $\tau = 2$ and $k = 6$ for a nontrivial solution. If the solution is of Painlevé type, the necessary condition is that any Laurent series have coefficients well-defined by the equation. Hence, we are seeking a solution of the form:

$$u(z) = 6/z^2 + a_{-1}/z + a_0 a_1 z + \cdots \tag{8.3}$$

After making use of the translation invariance of (8.2), we drop the parameter z_0. When (8.3) is substituted into (8.2) and coefficients of z^{-1}, z^0, $\cdots$ are successively equated to zero, we obtain expressions for a_{-1}, a_0, a_1, a_2, a_3 in terms of c. A problem arises when attempting to find a_4 however. Upon setting the coefficient of z^2 to zero, we have the equation

$$0 \cdot a_4 + \frac{100}{40} \cdot [\frac{c}{5}]^2 - \frac{720}{8} \cdot [\frac{c}{5}]^6 = 0.$$

For the Laurent expansion to be valid, c must satisfy this equation, which yields $c = 0$ or $\pm\frac{5}{\sqrt{6}}$.

Next we only consider the case $c = -r\gamma = \frac{5}{\sqrt{6}}$, since negative c can be handled by changing x into $-x$. The Laurent series mentioned above is only a necessary condition for an equation to be of Painlevé type. Letting $u(z) = f(z) \cdot q(s) + g(z)$, $s = h(z)$, and substituting this into equation (8.2), we obtain the equation for q as follows

$$(h')^2 q'' \quad = \quad -q'h'\Big[c + \frac{2f'}{f} + \frac{h''}{h'}\Big] + fq^2$$

$$-\Big[\frac{cf'}{f}+1+\frac{f''}{f}-2g]q+\frac{1}{f}[-cg'+g^2-g-g''].\qquad(8.4)$$

Taking $g(z)=0$ and letting $f(z)$ and $h(z)$ satisfy

$$\frac{2f'}{f}+\frac{h''}{h'}+c=0,\qquad(8.5)$$

and

$$\frac{cf'}{f}+\frac{f''}{f}+1=0,\qquad(8.6)$$

then the equation for q is integrable. Since $c=\frac{5}{\sqrt{6}}$, from (8.6), we may take $f(z)=e^{-\frac{\sqrt{6}}{3}z}$ and from (8.5), we have

$$h(z)=pe^{-\frac{\sqrt{6}}{6}z},$$

where p is integration constant. Taking $p=1$, then (8.4) reduces to

$$q''=6q^2.\qquad(8.7)$$

The solution of this equation is $q=P(s-k,0,g_3)$, where $P(z,g_2,g_3)$ is the Weierstrass elliptic function with invariants g_2 and g_3 [112,113]. Here k and g_3 are arbitrary constants.

Therefore, when $c=\frac{5}{\sqrt{6}}$, an exact solution to equation (8.2) can be expressed as

$$u(z)=e^{-\frac{\sqrt{6}z}{3}}\cdot P(e^{-\frac{z}{\sqrt{6}}}+k),$$

where k is an arbitrary constant.

Changing to our original variables, we obtain an exact solution to equation (1.8) immediately

$$U(x,y,t)=-\frac{12\beta^2}{25\alpha s}\cdot e^{-\frac{2\beta}{5sh}(hx+ly-wt)}\cdot P(e^{-\frac{\beta}{5sh}(hx+ly-wt)}+c)+\frac{6\beta^2}{25\alpha s}+\frac{wh-\gamma l^2}{\alpha h^2},\quad(8.8)$$

where c is arbitrary constant.

Since $q(s) = s^{-2}$ is a particular solution of equation (8.7), apparently we can obtain a particular exact solution to equation (1.8) directly from (8.8) as follows

$$U(x,y,t) = -\frac{12\beta^2}{25\alpha s} \cdot \frac{e^{-\frac{2\beta}{5sh}(hx+ly-wt)}}{[e^{-\frac{\beta}{5sh}(hx+ly-wt)} + c]^2} + \frac{wh - \gamma l^2}{\alpha h^2} + \frac{6\beta^2}{25\alpha s}, \qquad (8.9)$$

where c is arbitrary constant.

Note that (8.8) and (8.9) are in agreement with the results presented in the preceding two chapters. Obviously, the above arguments are also applicable to KdVB equation (1.5). We omit the details.

CHAPTER IX

FIRST INTEGRAL METHOD FOR THE COMPOUND BURGERS-KORTEWEG-DE VRIES EQUATION

In this chapter, we apply the first integral method to the study of the compound Burgers-Korteweg-de Vries equation for seeking its solitary wave solutions. Several new kink-profile waves and periodic waves are established. The applications of these results to other nonlinear wave equations such as the modified Burgers-KdV equation and compound KdV equation are presented.

A. Compound Burgers-Korteweg-de Vries Equation

The equation

$$u_t + \alpha u u_x + \beta u^2 u_x + \mu u_{xx} - s u_{xxx} = 0, \quad \alpha,\ \beta,\ \mu,\ s \in \mathbb{R}, \tag{9.1}$$

is known as the compound Burgers-KdV (cBKdV) equation. It is a simple composition of the KdV, modified KdV and Burgers' equation.

We see that

(i) if $\mu = 0$ in (9.1), then

$$u_t + \alpha u u_x + \beta u^2 u_x - s u_{xxx} = 0. \tag{9.2}$$

So (9.1) becomes the compound KdV equation [119].

(ii) if $\alpha = 0$ in (9.1), then

$$u_t + \beta u^2 u_x + \mu u_x x - s u_{xxx} = 0, \tag{9.3}$$

and (9.3) incorporates the modified Burgers-KdV equation [120].

(iii) further, if $\mu = 0$ in (9.3), then

$$u_t + \beta u^2 u_x - s u_{xxx} = 0. \tag{9.4}$$

Eqaution (9.4) is the modified KdV equation [121].

(iv) by setting $\beta = 0$ and s to $-s$ in (9.1), then

$$u_t + \alpha u u_x + \mu u_{xx} + s u_{xxx} = 0, \tag{9.5}$$

which is the Buregers-KdV equation. Further, if we set $s = 0$ and $\mu = 0$ separatedly, then (9.5) yields, respectively

$$\text{the Burgers equation: } \; u_t + \alpha u u_x + \mu u_{xx} = 0, \tag{9.6}$$

$$\text{the KdV equation: } \; u_t + \alpha u u_x + s u_{xxx} = 0. \tag{9.7}$$

Thus, equation (9.5) is one of the simplest classical models containing nonlinear, dissipative, and dispersive effects. It is well known that both (9.6) and (9.7) are exactly solvable, and with the traveling wave solutions

$$u(x,\; t) = \frac{2k}{\alpha} + \frac{2\mu k}{\alpha} \tanh k(x - 2kt),$$

and

$$u(x,\; t) = \frac{12sk^2}{\alpha} \text{sech}^2 k(x - 4sk^2 t),$$

respectively.

Although there have been numerous papers for equations (9.2)-(9.7), containing some profound results [1, 3-4, 45, 94, 122-126 et al.], it seems that detailed studies for the cBKdV equation only began several years ago. For those who are not familiar with the physical meanings and significance of (9.1)-(9.7), we refer to references [3-4, 14, 30-31, 43, 93-95, 119 et al.]. In the past few years, the cBKdV equation has received considerable attention. In particular, the traveling wave solution to the cBKdV equation has been widely investigated. Wang [14] treated the cBKdV equation and Burgers-KdV equation using the homogeneous balance method and presented an exact solution. However, his approach is quite complicated. As a result, there are quite a few errors in [14]. Feng [127, 128] studied the cBKdV equation by applying the first integral method and the method of variation of parameters, respectively. Parkes and Duffy [59, 129] obtained a more general result by using the automated tanh-function method and adapting the method of variation of parameters used in [128]. Zhang and his co-workers were concerned with equation (9.1) with nonlinear terms of any order by the method of undetermined coefficients [10]. The methods and results for the cBKdV equation (9.1) mentioned above have contributed to our understanding of nonlinear physical phenomena and wave propagation.

In the present work, we explore traveling solitary wave solutions to the cBKdV equation by using the first integral method. This approach was originally introduced for the Burgers-KdV equation in an attempt to seek its traveling wave solutions in Chapter XI. Suppose that the cBKdV equation (9.1) has traveling wave solutions of the form

$$u(x,t) = u(\xi), \qquad \xi = x - vt, \tag{9.8}$$

where $v \in \mathbb{R}$ is wave velocity. Substituting (9.8) into equation (9.1) and performing one integration with respect to ξ, we obtain

$$u''(\xi) - ru'(\xi) - au^3(\xi) - bu^2(\xi) - cu(\xi) - d = 0, \tag{9.9}$$

where $r = \frac{\mu}{s}$, $a = \frac{\beta}{3s}$, $b = \frac{\alpha}{2s}$, $c = -\frac{v}{s}$, and d is an arbitrary integration constant.

Note that when all coefficients of (9.9) are non-zero, (9.9) does not satisfy the Painlevé conditions. It is commonly believed that it is very challenging to solve this second-order ordinary differential equation directly by utilizing classical methods such as the Laplace and Fourier transforms [11]. To find traveling wave solutions to the cBKdV equation, a crucial task is to handle the second-order nonlinear ordinary differential equation (9.9).

Let $z = u$, $y = u_\xi$ in (9.9). Then we obtain an equivalent first order system

$$\begin{cases} \dot{z} = y \\ \dot{y} = ry + az^3 + bz^2 + cz + d. \end{cases} \tag{9.10}$$

On the Poincaré phase plane, the phase orbits defined by the vector fields corresponding to system (9.10) give useful information about all traveling wave solutions of (9.9). Hence, in order to study traveling wave solutions of the cBKdV equation and their properties, we may begin our study by considering the two-dimensional autonomous system (9.10). We know that an equilibrium point on the Poincaré phase plane always corresponds to a steady state. If (u_0, v_0) is an equilibrium point of (9.10), then any orbit except the point orbit itself can not approach (u_0, v_0) within finite time. Conversely, if an orbit of (9.10) approaches (u_0, v_0) as $\xi \to \infty$ (or $-\infty$), then (u_0, v_0) must be an equilibrium point of (9.10). On the Poincaré phase plane, an isolated and closed orbit which has no equilibrium point on itself represents a periodic oscillation to (9.9) on the (ξ, u)-plane. An orbit which emanates from a regular equilibrium point and terminates at a different regular equilibrium point as $\xi \to \infty$ or $-\infty$ represents a kink-profile wave in the (ξ, u)-plane to (9.9). An isolated and closed orbit which emanates from a regular equilibrium point and also terminates at the same equilibrium point as $\xi \to \infty$ or $-\infty$, represents a bell-profile wave to (9.9) on the (ξ, u)-plane.

B. Traveling Solitary Wave Solutions

A key idea of our approach to find first integral is to utilize the Divisor Theorem from ring theory of commutative algebra. Then making use of this first integral, the second-order differential equation (9.9) is reduced to a first-order integrable differential equation. Finally, the traveling wave solutions to the cBKdV equation are determined directly through solving the resulting first-order integrable differential equations.

Now, we apply the Divisor Theorem to seek a first integral to system (9.10). Suppose that $z = z(\xi)$ and $y = y(\xi)$ are nontrivial solutions to (9.10), and $q(z, y) = \sum_{i=0}^{m} a_i(z)y^i$ is an irreducible polynomial in $\mathbb{C}[z, y]$ such that

$$q[z(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(z)y^i = 0, \tag{9.11}$$

where $a_i(z)$ $(i = 0, 1, \cdots, m)$ are polynomials of z and $a_m(z) \not\equiv 0$. We call (9.11) a first integral in polynomial form to system (9.10). We start by assuming $m = 1$ in (9.11). Note that $\frac{dq}{d\xi}$ is a polynomial in z and y, and $q[z(\xi), y(\xi)] = 0$ implies $\frac{dq}{d\xi}|_{(9.10)} = 0$. Using the Divisor Theorem, we know that there exists a polynomial $H(x, y) = g(z) + h(z)y$ in $\mathbb{C}[z, y]$ such that

$$\begin{aligned} \frac{dq}{d\xi}|_{(9.10)} &= (\frac{\partial q}{\partial z}\frac{\partial z}{\partial \xi} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial \xi})|_{(9.10)}, \\ &= \sum_{i=0}^{1}[a_i'(z)y^i \cdot y] + \sum_{i=0}^{1}[ia_i(z)y^{i-1} \cdot (ry + az^3 + bz^2 + cz + d)], \\ &= [g(z) + h(z)y][\sum_{i=0}^{1} a_i(z)y^i]. \end{aligned} \tag{9.12}$$

On equating the coefficients of y^i (i=2, 1, 0) on both sides of (9.12), we have

$$\mathbf{a}'(z) = \mathbf{A}(z) \cdot \mathbf{a}(z), \tag{9.13}$$

and

$$[az^3 + bz^2 + cz + d, -g(z)] \cdot \mathbf{a}(z) = 0, \tag{9.14}$$

where $\mathbf{a}(z) = (a_1(z), a_0(z))^t$, and

$$\mathbf{A(z)} = \begin{pmatrix} h(z), & 0 \\ g(z) - r, & h(z) \end{pmatrix}.$$

Since $a_i(z)$ $(i = 0, 1)$ are polynomials of z, from $(9.13)_1$, we deduce that $a_1(z)$ is a constant and $h(z) = 0$. For simplification, taking $a_1(z) = 1$ and solving (9.10), we have

$$\mathbf{a(z)} = \begin{pmatrix} 1 \\ \int [g(z) - r] dz \end{pmatrix}. \tag{9.15}$$

From (9.14) and (9.15), we conclude $\deg g(z) = 1$ and $\deg a_0(z) = 2$. Suppose that $g(z) = g_1 z + g_0$ and $a_0(z) = A_2 z^2 + A_1 z + A_0$. From $(9.13)_2$, we deduce that $2A_2 z + A_1 + r = g_1 z + g_0$. Substituting this into (9.14) and setting all coefficients of z^i (i=3,2,1,0) to zero, we have

$$\begin{cases} a = 2A_2^2 \\ b = A_2(A_1 + r) + 2A_1 A_2 \\ c = A_1(A_1 + r) + 2A_0 A_2 \\ d = A_0(A_1 + r) \end{cases} \tag{9.16}$$

Since d is an arbitrary integration constant, from $(9.16)_4$, one can see that A_0 is arbitrary. When $\frac{b^2}{3a} - c - \frac{r^2}{6} \geq 0$, set

$$2k^2 = \frac{b^2}{3a} - c - \frac{r^2}{6},$$

where k is an arbitrary real constant. Thus, A_0 in (9.16) can be re-expressed as

$A_0 = \pm\frac{1}{\sqrt{2a}}(\frac{r^2}{18} + \frac{b^2}{9a} - 2k^2) - \frac{br}{9a}$. When $a > 0$, solving (9.16) gives

$$\begin{cases} A_2 = \pm\sqrt{\frac{a}{2}} \\ A_1 = \frac{1}{3}(\pm\sqrt{\frac{2}{a}}b - r) \end{cases} \tag{9.17}$$

Hence, we obtain a first integral to (9.10):

$$\begin{aligned} y &= -A_2 z^2 - A_1 z - A_0 \\ &= \pm\sqrt{\frac{a}{2}}[(z + \frac{b}{3a} \mp \frac{r}{6}\sqrt{\frac{2}{a}})^2 - \frac{2k^2}{a}]. \end{aligned} \tag{9.18}$$

Combining (9.18) with (9.10), we can reduce (9.9) to

$$\frac{du}{d\xi} = \mp\sqrt{\frac{a}{2}}[(u + \frac{b}{3a} \mp \frac{r}{6}\sqrt{\frac{2}{a}})^2 - \frac{2k^2}{a}]. \tag{9.19}$$

Making a transformation $u + \frac{b}{3a} \mp \frac{r}{6}\sqrt{2/a} = \sqrt{\frac{2k^2}{a}} \cdot \sin\theta$, one can derive $\theta = 2\arctan(e^{\pm k\xi}) - \frac{\pi}{2}$ and from where solve (9.19) directly to obtain

$$u(x,t) = \pm\sqrt{\frac{6s}{\beta}}[k \cdot \tanh(k\theta) + \frac{\mu}{6s}] - \frac{\alpha}{2\beta}, \tag{9.20}$$

where $\theta = x - vt + x_0$, $v = \frac{\mu^2}{6s} + 2sk^2 - \frac{\alpha^2}{4\beta}$, and k and x_0 are arbitrary real constants.

Similarly, making a transformation $u + \frac{b}{3a} \mp \frac{r}{6}\sqrt{2/a} = \sqrt{\frac{2k^2}{a}} \cdot \sec\theta$, one can derive $\theta = 2\arctan(e^{\mp k\xi})$ and an exact solution to the cBKdV equation (9.1)

$$u(x,t) = \pm\sqrt{\frac{6s}{\beta}}[k \cdot \coth(k\theta) + \frac{\mu}{6s}] - \frac{\alpha}{2\beta}, \tag{9.21}$$

where $\theta = x - vt + x_0$, $v = \frac{\mu^2}{6s} + 2sk^2 - \frac{\alpha^2}{4\beta}$, and k and x_0 are arbitrary real constants.

When $\frac{b^2}{3a} - c - \frac{r^2}{6} < 0$, assume that $\frac{b^2}{3a} - c - \frac{r^2}{6} = -2k^2$. Making use of the identities $\tanh(ik) = i\tan k$ and $\coth(ik) = -i\cot k$, we obtain periodic wave solutions to the cBKdV equation (9.1) as

$$u(x,t) = \pm\sqrt{\frac{6s}{\beta}}[k\cdot\tan(k\theta)+\frac{\mu}{6s}]-\frac{\alpha}{2\beta}, \tag{9.22}$$

$$u(x,t) = \pm\sqrt{\frac{6s}{\beta}}[k\cdot\cot(k\theta)+\frac{\mu}{6s}]-\frac{\alpha}{2\beta}, \tag{9.23}$$

where $\theta = x - vt + x_0$, $v = \frac{\mu^2}{6s} + 2sk^2 - \frac{\alpha^2}{4\beta}$, and k and x_0 are arbitrary real constants. Apparently, (9.22) is equivalent to (9.23) by using a phase shift.

It is notable that (9.20) is the same as the one obtained by Parkes and Duffy through their use of the automated tanh-function method and the adaptation of the method of variation of parameters [59, 129]. However, periodic wave solutions in the form (9.21), (9.22) or (9.23) have not been found in the previous literature until now. Note that (9.21)-(9.23) are unbounded and have singularities at infinitely many space-time points. Such solutions are of the formation of the so-called "hot spots" (or "blow up") of solutions [130]. It is not hard to verify that the corresponding results in Zhang [131] using the method of undetermined coefficients, are particular cases of (9.20)-(9.23). Formulas described in [127, P.62, (i), (ii)] are in agreement with our (9.20) when $k = \frac{1}{2}B_2$ (B_2 is the same as given in [127] and the formulas in [128, (9.14)-(9.17)] agree with our (9.20)-(9.23) when $k = \frac{\mu}{6s}$. Wang utilized the homogeneous balance method to find traveling solitary wave solutions to the cBKdV equation (9.1). The corrected versions of (2.16) and (2.17) in [14] are proven to be equivalent merely to (9.20) by Parkes and Duffy [59].

Note that the choice $\alpha = 0$ changes (9.1) into the modified Burgers-KdV equation (9.3). Making use of (9.20), when $c+\frac{r^2}{6} < 0$, we obtain that the modified Burgers-KdV equation (9.3) has explicit traveling solitary wave solutions of the form

$$u(x,t) = \pm\sqrt{\frac{6s}{\beta}}[k\cdot\tanh(k\theta)+\frac{\mu}{6s}], \tag{9.24}$$

$$u(x,t) \;=\; \pm\sqrt{\frac{6s}{\beta}}[k\cdot\coth(k\theta)+\frac{\mu}{6s}], \tag{9.25}$$

where $\theta = x - vt + x_0$, $v = \frac{\mu^2}{6s} + 2sk^2$, and k and x_0 are arbitrary real constants.

When $c + \frac{r^2}{6} > 0$, we obtain periodic wave solutions to the modified Burgers-KdV equation (9.3) as

$$u(x,t) \;=\; \pm\sqrt{\frac{6s}{\beta}}[k\cdot\tan(k\theta)+\frac{\mu}{6s}], \tag{9.26}$$

$$u(x,t) \;=\; \pm\sqrt{\frac{6s}{\beta}}[k\cdot\cot(k\theta)+\frac{\mu}{6s}], \tag{9.27}$$

where $\theta = x - vt + x_0$, $v = \frac{\mu^2}{6s} + 2sk^2$, and k and x_0 are arbitrary real constants. Clearly, (9.26) is equivalent to (9.27) by using a phase shift. It is easy to verify that the rational solutions obtained in [120, 127, 131, 133-134] are in agreement with (9.24) when k takes some particular values as in these given literature. A recent result—formula (9.25) described in Feng's recent work [132] is identical to (9.24) and formulas (9.12)-(9.14) are particular cases of (9.25)-(9.27).

The choice $\mu = 0$ changes (9.1) to the compound KdV equation (9.2). According to (9.20) again, when $\frac{b^2}{3a} - c > 0$, we obtain that the compound KdV equation (9.2) has explicit traveling solitary wave solutions

$$u(x,t) \;=\; \pm\sqrt{\frac{6s}{\beta}}k\cdot\tanh(k\theta)-\frac{\alpha}{2\beta}, \tag{9.28}$$

$$u(x,t) \;=\; \pm\sqrt{\frac{6s}{\beta}}k\cdot\coth(k\theta)-\frac{\alpha}{2\beta}, \tag{9.29}$$

where $\theta = x - vt + x_0$, $v = 2sk^2 - \frac{\alpha^2}{4\beta}$, and k and x_0 are arbitrary real constants.

When $\frac{b^2}{3a} - c < 0$, we obtain periodic wave solutions to the compound KdV equation (9.2) immediately

$$u(x,t) \;=\; \pm\sqrt{\frac{6s}{\beta}}k\cdot\tan(k\theta)-\frac{\alpha}{2\beta}, \tag{9.30}$$

$$u(x,t) = \pm\sqrt{\frac{6s}{\beta}}k\cdot\cot(k\theta) - \frac{\alpha}{2\beta}, \tag{9.31}$$

where $\theta = x - vt + x_0$, $v = 2sk^2 - \frac{\alpha^2}{4\beta}$, and k and x_0 are arbitrary real constants. Note that (9.30) is equivalent to (9.31) by using a phase shift. The solutions presented by Zhang [131] agree with (9.28) only for certain special values of k. Formula (9.23) in [127] is in agreement with (9.28) when $k = \sqrt{-\frac{v}{2} - \frac{\alpha^2}{8\beta}}$, but (9.25)-(9.27) and (9.29)-(9.31) were not presented in either [127] or [132].

It should be noted that Pan concluded in [119] that the compound KdV equation (9.2) has no bounded traveling solitary wave solution in the form (9.28) due to the following relation

$$\int_{-\infty}^{\infty}[u'(\xi)]^2 d\xi = \lim_{r\to 0}\frac{1}{12r}(c_+ - c_-)^3[a + b(c_+ + c_-)].$$

Unfortunately, this statement is incorrect. This is because of the fact that the existence for bounded traveling solitary wave solutions corresponding to the condition $a + b(c_+ + c_-) = 0$ is neglected. Our (9.28) coincides with this case. We need to remark that the above arguments are strictly based on the condition $a \neq 0$ (i.e., $\frac{\beta}{3s} \neq 0$); otherwise, $\deg g(z) = 1$ can not be concluded. Hence, the traveling wave solution for the Burgers-KdV equation can not be obtained simply from (9.20)-(9.23) by assuming $\beta = 0$.

Since when β is equal to zero, the third term in the cBKdV equation disappears which leads eq. (9.1) to be the Burgers-KdV equation (9.2). In our previous work [128], we pointed out that the Burgers-KdV equation has bounded traveling wave solutions, which is actually a combination of a bell-profile wave and a kink-profile wave:

$$u(x,t) = \frac{3\beta^2}{25\alpha s}\text{sech}^2\Big[\frac{1}{2}(-\frac{\beta}{5s}x \pm \frac{6\beta^3}{125s^2}t)\Big]$$

$$-\frac{6\beta^2}{25\alpha s}\tanh\Big[\frac{1}{2}\big(-\frac{\beta}{5s}x \pm \frac{6\beta^3}{125s^2}t\big)\Big] \pm \frac{6\beta^2}{25\alpha s}.$$

Motivated by this fact, we may assume that the cBKdV equation (9.1) also has a composition of bell-profile waves and kink-profile waves in the form

$$u(x,t) = \sum_{i=1}^{n}\Big(B_i\tanh^i[C_i(x - vt + x_0)] + D_i\mathrm{sech}^i[C_i(x - vt + x_0)]\Big) + B_0, \quad (9.32)$$

where B_i, C_i, D_i, v and B_0 are constants to be determined.

Suppose $n = 1$ in (9.32). Then we have

$$\begin{aligned} u'' - ru' \;=\; & -2B_1C_1^2\mathrm{sech}^2[C_i(x - vt + x_0)] \cdot \tanh[C_i(x - vt + x_0)] \\ & +D_1C_1^2\mathrm{sech}[C_i(x - vt + x_0)] \cdot \tanh^2[C_i(x - vt + x_0)] \\ & -D_1C_1^2\mathrm{sech}^3[C_i(x - vt + x_0)] - rB_1C_1\mathrm{sech}^2[C_i(x - vt + x_0)] \\ & +rD_1C_1\tanh[C_i(x - vt + x_0)] \cdot \mathrm{sech}[C_i(x - vt + x_0)]. \end{aligned} \quad (9.33)$$

Split the first term in (9.33) into two parts. Namely, let $p_1 + p_2 = -2B_1C_1^2$ and use the equality $1 - \tanh^2\theta = \mathrm{sech}^2\theta$. Then the first term can be re-written as $p_1\mathrm{sech}^2[C_i(x-vt+x_0)]\cdot\tanh[C_i(x-vt+x_0)] - p_2\tanh^3[C_i(x-vt+x_0)] + p_2\tanh[C_i(x-vt+x_0)]$. We do the same thing for the second, third and fourth terms in (9.33). Substituting (9.32) into (9.9), letting $d = 0$ and equating the corresponding coefficients to zero, after simplification, we have

$$\begin{cases} \alpha D_1^2 + 2\beta D_1^2 B_0 - 2\mu C_1 B_1 = 0 \\ \alpha B_1 D_1 + 2\beta B_1 D_1 B_0 + \mu C_1 B_1 = 0 \\ \alpha B_1 B_0 + \beta(B_1 D_1^2 + B_1 B_0^2) + 2sC_1^2 B_1 + vB_1 = 0 \\ \alpha D_1 B_0 + \beta(D_1 B_0^2 + D_1 B_1^2) - sC_1^2 D_1 + vD1 = 0 \\ \beta(D_1^3 - 3D_1 B_1^2) + 6sD_1 C_1^2 = 0 \\ \beta(B_1^3 - 3D_1^2 B_1) - 6sB_1 C_1^2 = 0 \end{cases}$$

Solving the above algebraic system with the aid of mathematical software such as Mathematica or Maple, we have

$$(I). \qquad B_0 = \frac{\alpha}{2\beta}, \quad B_1 = \pm\frac{1}{2\beta} \cdot \sqrt{\frac{21\alpha^2 + \beta\alpha^2}{36 - 3\beta}},$$
$$C_1 = \pm\frac{11\alpha}{72s - 6\beta s}, \quad D_1 = \pm\frac{1}{2\beta} \cdot \sqrt{\frac{14\alpha^2 + \beta\alpha^2}{36 - 3\beta}},$$
$$v = -\frac{2\alpha^2}{3\beta}.$$

$$(II). \qquad B_0 = \frac{\alpha}{2\beta}, \quad B_1 = \pm\frac{1}{2} \cdot \sqrt{\frac{\mu^2}{3\beta^2\mu^2 - 12\beta s}},$$
$$C_1 = \pm\frac{\alpha}{\sqrt{3\beta^2\mu^2 - 12\beta s}},$$
$$D_1 = \pm\frac{1}{2\beta} \cdot \sqrt{\frac{\alpha^2}{3} - \frac{8\alpha^2 s}{3\beta^2\mu^2 - 12\beta s}}, \quad v = -\frac{2\alpha^2}{3\beta}.$$

$$(III). \qquad B_0 = \frac{\alpha}{4\beta}\cdot(-3 \pm i\sqrt{39}), \quad B_1 = \pm\frac{1}{2\beta}\cdot\sqrt{4sC_1^2 - \frac{\alpha^2}{3}},$$
$$C_1 = \pm\frac{1}{4}\cdot\sqrt{\frac{260\alpha^2 + 27\alpha \mp 12\alpha^2\sqrt{39}i \mp 3\alpha\sqrt{39}i}{3}},$$
$$D_1 = \pm\frac{1}{\beta}\cdot\sqrt{36sC_1^2 - 2s\beta^2C_1^2 - 3\alpha^2}, \quad v = -\frac{2\alpha^2}{3\beta}.$$

Thus, the cBKdV equation (9.1) has the traveling solitary wave solutions

$$\begin{aligned} u_1(x,\ t) &= \pm\frac{1}{2\beta}\cdot\sqrt{\frac{21\alpha^2+\beta\alpha^2}{36-3\beta}}\cdot\tanh\Big(\pm\frac{11\alpha}{72s-6\beta s}\cdot[x-\frac{2\alpha^2}{3\beta}t+x_0]\Big) \\ &\pm\frac{1}{2\beta}\cdot\sqrt{\frac{14\alpha^2+\beta\alpha^2}{36-3\beta}}\cdot\text{sech}\Big(\pm\frac{11\alpha}{72s-6\beta s}\cdot[x-\frac{2\alpha^2}{3\beta}t+x_0]\Big)+\frac{\alpha}{2\beta}; \quad (9.34) \\ u_2(x,\ t) &= \pm\frac{1}{2}\cdot\sqrt{\frac{\mu^2}{3\beta^2\mu^2-12\beta s}}\cdot\tanh\Big(\pm\frac{\alpha}{\sqrt{3\beta^2\mu^2-12\beta s}}\cdot[x-\frac{2\alpha^2}{3\beta}t+x_0]\Big) \\ &\pm\frac{1}{2\beta}\cdot\sqrt{\frac{\alpha^2}{3}-\frac{8\alpha^2 s}{3\beta^2\mu^2-12\beta s}}\cdot\text{sech}\Big(\pm\frac{\alpha}{\sqrt{3\beta^2\mu^2-12\beta s}}\cdot[x-\frac{2\alpha^2}{3\beta}t+x_0]\Big) \\ &+\frac{\alpha}{2\beta}; \quad (9.35) \end{aligned}$$

and

$$\begin{aligned} u_3(x,\ t) &= \pm\frac{1}{2\beta}\cdot\sqrt{4sC_1^2-\frac{\alpha^2}{3}}\cdot\tanh\Big(C_1[x-\frac{2\alpha^2}{3\beta}t+x_0]\Big) \\ &\pm\frac{1}{\beta}\cdot\sqrt{36sC_1^2-2s\beta^2C_1^2-3\alpha^2}\cdot\text{sech}\Big(C_1[x-\frac{2\alpha^2}{3\beta}t+x_0]\Big) \\ &+\frac{\alpha}{4\beta}\cdot(-3\pm i\sqrt{39}), \quad (9.36) \end{aligned}$$

where C_1 is given as in (III).

Formulas (9.34)-(9.36) are new traveling solitary wave solutions for the cBKdV equation. It seems that these results have not been found by other researchers, to the best of our knowledge. Note that the traveling wave solution given by (9.34) has the limit $\frac{\alpha}{2\beta}$ as $\delta_1 = \pm\frac{11\alpha}{72s-6\beta s}\cdot[x-\frac{2\alpha^2}{3\beta}t] \to +\infty$ (or $-\infty$); and the traveling wave solution given by (9.35) has the limit $\frac{\alpha}{2\beta}$ as $\delta_2 = \pm\frac{\alpha}{\sqrt{3\beta^2\mu^2-12\beta s}} \cdot [x - \frac{2\alpha^2}{3\beta}t] \to \infty$ (or $-\infty$). All orders of derivatives of u with respect to δ_i tend to zero as $|\delta_i| \to +\infty$ $(i = 1, 2)$. This type of traveling wave solution to the cBKdV equation is very similar to those of the Burgers-KdV equation [27], containing a bell-profile wave and a kink-profile wave, and also approaching to a constant as $|\delta_i| \to +\infty$ $(i = 1, 2)$. We remark that, as it stands, the traveling solitary wave solutions (9.34)-(9.36) can not be reduced to the sech^i type traveling wave solution of the KdV equation in the limit $\beta \to 0$ and $\mu \to 0$. At this stage, it is unclear to us whether the cBKdV equation has other types of bounded traveling wave solutions except those in (9.20) and (9.32).

C. Further Discussions

When each coefficient of the cBKdV equation is nonzero, the cubic $au^3 + bu^2 + cu + d$ can be factorized into two cases, i.e., either

$$a(u - u_1)[(u - u_2)^2 + u_3^2]$$

or

$$a(u - u_1)(u - u_2)(u - u_3).$$

For the first case, the phase plane analysis shows that the cBKdV equation (9.1) does not have the bounded traveling solitary wave solution. Hence, to seek bounded traveling wave solutions to the cBKdV equation (9.1), in this section we focus our attention merely on the latter case, namely

$$f(u) = au^3 + bu^2 + cu + d = a(u - u_1)(u - u_2)(u - u_3), \quad u_1 \leq u_2 \leq u_3.$$

We assume that the cBKdV equation (9.1) has traveling solitary wave solutions of the form

$$u(x,t) = U(\xi), \quad \xi = x - vt, \quad U(-\infty) = u_3, \quad U(\infty) = u_1, \tag{9.37}$$

which upon substituting into equation (9.1) gives

$$L(U) = U''(\xi) - rU'(\xi) - a(U - u_1)(U - u_2)(U - u_3) = 0. \tag{9.38}$$

The equivalent phase plane system for the above is

$$\begin{cases} \dot{U} = V \\ \dot{V} = rV + f(U). \end{cases}$$

So

$$\frac{dV}{dU} = \frac{rV + f(U)}{V}, \tag{9.39}$$

which has three regular equilibrium points

$$(u_1,\ 0), \quad (u_2,\ 0), \quad (u_3,\ 0). \tag{9.40}$$

To classify the type of equilibrium points given by (9.40), we need to analyze the eigenvalue problems for (9.39) and determine the sign of r, such that traveling waves, of the kind we are seeking, exist.

Linearizing (9.39) at each equilibrium point $U = u_i,\ (i = 1, 2, 3)$, we have

$$\frac{dV}{dU} = \frac{rV + f'(u_i)(U - u_i)}{V}, \quad (i = 1, 2, 3),$$

which, applying standard linear phase plane analysis, has eigenvalues $\lambda_1,\ \lambda_2$:

$$\lambda_1,\ \lambda_2 = \frac{r \pm \sqrt{r^2 - 4f'(u_i)}}{2}, \quad (i = 1, 2, 3).$$

Under assumption $a > 0$, we have the following equilibrium point classification:

(a) for the equilibrium point $(u_1,\ 0)$, $f'(u_1) > 0$, it is a stable spiral point (or node) if $r < 0$ and $r^2 < 4f'(u_1)$ (or $r^2 > 4f'(u_1)$).

(b) for the equilibrium point $(u_2,\ 0)$, it is a saddle point for all r.

(c) for the equilibrium point $(u_3,\ 0)$, $f'(u_3) > 0$, it is a stable spiral point (or node) if $r < 0$ and $r^2 < 4f'(u_1)$ (or $r^2 > 4f'(u_1)$).

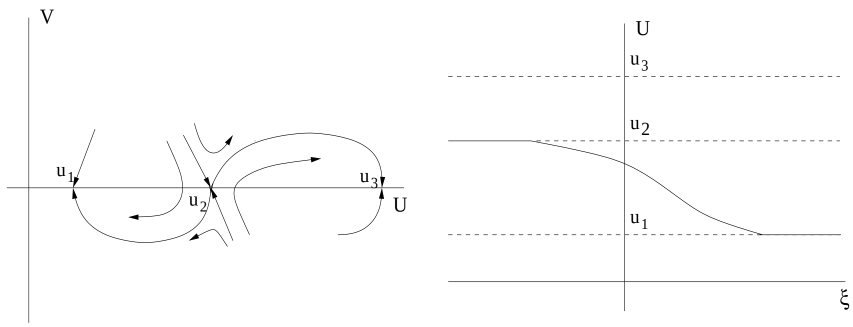

Fig. 44. The left figure: schematic phase plane portrait for a wave connecting the static states $(u_1,\ 0)$ and $(u_2,\ 0)$. The right figure: a kink-profile wave solution from u_2 to u_1.

If $a > 0$ and $r > 0$, $(u_1,\ 0)$ and $(u_3,\ 0)$ become unstable, but the type of equilibria is the same. There are a few phase plane trajectories depending on the size of $f'(u_i)$ $(i = 1, 2, 3)$ and the sign of r. Rather than give a complete catalogue of all the possibilities, we give only one example: $r^2 > 4 \cdot \max[f'(u_1),\ f'(u_3)]$ and $f'(u_2) < 0$, in which case $(u_1,\ 0)$ and $(u_3,\ 0)$ are stable nodes, and $(u_2,\ 0)$ is a saddle point. The arguments for other cases are mostly similar. A phase portrait and the traveling wave which corresponds to the phase orbit connecting $(u_1,\ 0)$ and $(u_2,\ 0)$ is sketched in Fig.44.

The sign of r can be determined by multiplying equation (9.38) by U' and integrating from $-\infty$ to ∞. This yields

$$\int_{-\infty}^{\infty} [U'U'' - r(U')^2 - U'f(U)]dz = 0.$$

Since $U'(\pm\infty) = 0$, $U(-\infty) = u_3$ and $U(\infty) = u_1$, the above is integrated to

$$r\int_{-\infty}^{\infty} (U')^2 dz = -\int_{-\infty}^{\infty} f(U)U'dz = -\int_{u_3}^{u_1} f(U)dU,$$

and so, since the multiplier of r is always positive, we have

$$\begin{cases} r > 0 \ \text{ if } \ \int_{u_1}^{u_3} f(\tau)d\tau > 0, \\ r = 0 \ \text{ if } \ \int_{u_1}^{u_3} f(\tau)d\tau = 0, \\ r < 0 \ \text{ if } \ \int_{u_1}^{u_3} f(\tau)d\tau < 0. \end{cases} \tag{9.41}$$

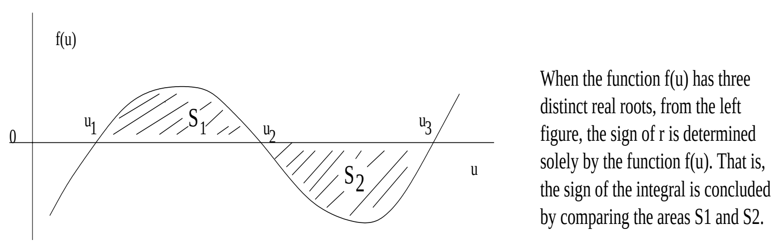

Fig. 45. Areas S_1 and S_2.

In the preceding section, we have illustrated that under some given conditions, the cBKdV equation has bounded kink-profile traveling solitary wave solutions. We then use the first integral method to reduce the second-order ordinary differential equation (9.9) to the first-order integrable ordinary differential equation (9.19). Sometimes it is useful to imagine that the problem under consideration is solved and worked backward, step and step, until one arrives at the given data. Then one may be able to reverse the steps and thereby constructing a solution to the original problem and analyzing its property such as stability. Let us proceed with this point of view below.

Note that (9.19) is actually a Riccati equation. Hence, we have a right clue to assume that $U(\xi)$ satisfy

$$U' = k_0 U^2 - k_0(u_1 + u_3)U + k_0 u_1 u_3, \tag{9.42}$$

whose solutions tend exponentially to u_1 and u_3 as $\xi \to \infty$, which is the desirable appropriate behavior. Substituting this equation into (9.38), we have

$$\begin{aligned} L(U) &= (U - u_1)(U - u_3)[k_0^2(2U - u_1 - u_3) - rk_0 - a(U - u_2)] \\ &= (U - u_1)(U - u_3)\{[(2k_0^2 - a)U - [k_0^2(u_1 + u_3) + rk_0 - au_2)]\}; \end{aligned}$$

so, in order for $L(U)$ to be zero we must have

$$2k_0^2 - a = 0, \quad k_0^2(u_1 + u_3) + rk_0 - au_2 = 0.$$

Solving this equation gives

$$r = -\sqrt{\frac{a}{2}}(u_1 - 2u_2 + u_3), \qquad k_0 = \sqrt{\frac{a}{2}}. \tag{9.43}$$

Hence, by using the differential equation (9.42) it is easily seen that its solutions can satisfy the full equation if a and r are as given by (9.43). The exact solution U is then obtained by solving (9.42). That is

$$U(\xi) = \frac{u_3 + Cu_1 \exp[k_0(u_3 - u_1)\xi]}{1 + C\exp[k_0(u_3 - u_1)\xi]}, \tag{9.44}$$

where C is an arbitrary integration constant. Apparently, this solution satisfies (9.37), and (9.44) are in agreement with formula (9.20) according to the equality $\tanh\theta = 1 - \frac{2}{\exp(2\theta)+1}$. Notice that this kink-profile wave is stable. It approaches to u_1 and u_2 steadily as ξ goes to ∞ and $-\infty$, respectively. The sign of r, from (9.43), is determined by the relative sizes of the u_i $(i = 1, 2, 3)$. If the average of u_1 and u_3 is less than u_2, then $r > 0$, and negative otherwise. This is of course the same result we have got from (9.41) [see Fig.45]. We may use the same manner to discuss the stability of (9.44) as we have described in Chapter VII.

CHAPTER X

CONCLUSIONS

This dissertation presents the recent results on a research in two models: the 1D linear wave equation with van der Pol nonlinear boundary conditions and Korteweg-de Vries-Burgers equation.

For the first model, our approach is to use the method of characteristics and formulate the problem into an equivalent first order hyperbolic system, to derive a nonlinear reflection relation caused by the nonlinear boundary conditions. Since the solution of the first order hyperbolic system depends completely on this nonlinear relation and its iterates, the problems is reduced to discrete iteration problem of the type $u_{n+1} = F(u_n)$, where F is the nonlinear reflection relation. We follow the definition that the PDE system is chaotic if the mapping F is chaotic as an interval map. In order to study whether the mapping F has the same chaotic behavior as those shown in Chen's previous works, we first study periodic solutions of a first order hyperbolic system; then we obtain an invariant interval for a class of Abel equation with which the mapping F involves; finally numerical simulations and visualizations with different coefficients are illustrated. We would like to point out that the computer graphics depend crucially on values of parameters μ_1, μ_2, γ and η.

In a subsequent work, we plan to address an invariant interval for a more general Abel equation and determine the chaotic regime for mapping F. Since the equivalent first order hyperbolic system (1.10) does not have period-doubling, we will focus our attention on the use of total variation as a measure of chaos. We need to apply the well-known Sharkovski's Theorem and the following technical theorem:

Theorem X.1 Theorem. *Let I be a bounded closed interval and $F\colon I \to I$ be continuous. Assume that I_1, I_2, $\cdots$, I_n are closed subintervals of I which overlap at* *most at endpoints, and the covering relation*

$$I_1 \to I_2 \to I_3 \cdots \to I_n \to I_1 \cup I_j, \text{ for some } j \neq 1.$$

Then

$$\lim_{n \to +\infty} V_I(f^n) = +\infty.$$

□

The first integral method, which we introduce to deal with the second model, is a new approach and appears to be more efficient to handle some nonlinear wave equations. According to qualitative theory of differential equations, if one can find two first integrals to a two-dimensional autonomous system under the same conditions, then the general solutions to this system can be expressed explicitly. However, in general, it is really difficult for us to realize this, even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first-integrals, nor is there a logical way for telling us what these first-integrals are.

The key idea of this method is to apply the Hilbert-Nullstellensatz to obtain one first integral to (1.25), then using this first integral, we reduce equation (1.25) to a first-order integrable ordinary differential equation, finally an exact solution to (1.5) is then obtained by solving this resultant first order equation. From Chapter X, we see that the traveling solitary wave solutions we obtain to the Korteweg-de Vries-Burgers equation, are more general than the existing ones. In Chapter VII, we introduce another new method currently called Coordinate Transformations Method to the study of 2D-KdVB equation. The stability and bifurcation of system (7.4) are analyzed. In Section 2, an asymptotic behavior of the proper solutions of 2D-KdVB equation is established. It is worthwhile to mention here that these two new methods are not only more efficient and less computational but also have the merit of being widely applicable. We give an example in Chapter IX, that is, we use the first integral method to study the compound Korteweg-de Vries-Burgers equation and generate several new kink-profile and periodic wave solutions. It is notable that when all coefficients of the compound Korteweg-de Vries-Burgers equation are not zero, the second-order ordinary differential equation (9.9) does not satisfy Painlevé property condition. One can definitely apply the above two new techniques to many nonlinear evolution equations, which can be converted to the following form through the traveling wave transformation,

$$u''(\xi) - DT[u, u'(\xi)] - R(u) = 0,$$

where D is real constant, $R(u)$ is a polynomial with real coefficients and $T(u, v)$ is a polynomial in u, v with real coefficients.

Some representative equations are listed below:

(1). Fisher's equation [135]: $u_t = vu_{xx} + su(1-u)$

(2). Modified Burgers-KdV equation [10]: $u_t + \beta u^p u_x + \mu u_{xx} - su_{xxx} = 0$

(3). Generalized compound KdV equation [10]: $u_t + \alpha uu_x + \beta u^p u_x + su_{xxx} = 0$

(4). Generalized Klein-Gordon equation [1, 136]: $u_{tt} - (u_{xx} + u_{yy}) + \alpha^2 u_t + g(uu^*)u = 0$ if g is a polynomial function.

(5). Nonlinear Schrödiner equation [1, 136]: $iu_t + u_{xx} - u_{yy} + g(uu^*)u = 0$ if g is a polynomial function.

(6). Emden equation [115]: $vu'' + 2u' + \alpha v^m u^n = 0$, $\alpha > 0$

(7). Emden-Fowler equation [115]: $\frac{d}{dt}(t^q \frac{du}{dt}) \pm t^\delta u^n = 0$

(8). An approximate sine-Gordon equation [137]: $u_{tt} + r_1 u_t - r_2 \Delta u_{(n)} + u - \frac{1}{6}u^3 = d_3 u^3 + d_2 u^2 + d_1 u + d_0$

(9). Combined dissipative double-dispersive equation [138] : $u_{tt} - \alpha_1 u_{xx} - \alpha_2 u_{xxt} -$

$\alpha_3(u)^2_{xx} - \alpha_4 u_{xxxx} + \alpha_5 u_{xxtt} = 0.$

In summary, this dissertation addresses a wide range of issues concerning the linear and nonlinear models and therefore leaves some open problems and study directions. Some of the derived results have to be extended, others further studied. In the near future, we plan to develop and extend the above mathematical techniques to some high-order nonlinear wave equations such as the modified KdVB equation equation, and continue considering the problem of classifying chaotic behaviors for the 1-dimensional wave equation $w_{tt} - w_{xx} = 0$ on the unit interval $x \in (0, 1)$ with different nonlinear boundary conditions.

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Selected Publications:

[1] Z. Feng, Traveling wave solutions and proper solutions to the two-dimensional Burgers-Korteweg-de Vries equation, J. Phys. A (Math. Gen.) 36 (2003) 8817-8829.

[2] Z. Feng, Qualitative analysis and exact solutions to the Burgers-KdV equation, Dynamics of Continuous, Discrete and Impulsive Systems, 9 (2002) 563-580.

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