STUDIES IN INTERPOLATION AND APPROXIMATION OF MULTIVARIATE BANDLIMITED FUNCTIONS

A Dissertation

by

BENJAMIN AARON BAILEY

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2011

Major Subject: Mathematics

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ABSTRACT

Studies in Interpolation and Approximation of Multivariate Bandlimited Functions.

(August 2011)

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The focus of this dissertation is the interpolation and approximation of multivariate bandlimited functions via sampled (function) values. The first set of results investigates polynomial interpolation in connection with multivariate bandlimited functions. To this end, the concept of a uniformly invertible Riesz basis is developed (with examples), and is used to construct Lagrangian polynomial interpolants for particular classes of sampled square-summable data. These interpolants are used to derive two asymptotic recovery and approximation formulas. The first recovery formula is theoretically straightforward, with global convergence in the appropriate metrics; however, it becomes computationally complicated in the limit. This complexity is sidestepped in the second recovery formula, at the cost of requiring a more local form of convergence. The second set of results uses oversampling of data to establish a multivariate recovery formula. Under additional restrictions on the sampling sites and the frequency band, this formula demonstrates a certain stability with respect to sampling errors. Computational simplifications of this formula are also given.

To Mom and Dad

ACKNOWLEDGMENTS

This dissertation is dedicated to my family and friends, whose support, encouragement, and stability have proven invaluable over the last 6 years. I am also grateful to the plethora of exceptional teachers under whom I have studied in the course of my education. Additional thanks also goes to my committee, in particular to my advisors Thomas and Siva for the patience that they have extended towards me, and their level-headedness in the face of my foibles. Also, thank you Thomas, for providing such ample financial support over the last several years.

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CHAPTER I

INTRODUCTION

A. An Overview

This dissertation explores interpolation and approximation in the space of multivariate bandlimited functions $PW_{[-\pi,\pi]^d}$ (see Definition II.1) from the point of view of sampling. That is, given a function $f \in PW_{[-\pi,\pi]^d}$, data sites $(t_n)_{n\in\mathbb{Z}^d}$ arising from some exponential frame or Riesz basis condition, and the sequence of sampled values $(f(t_n))_{n\in\mathbb{Z}^d}$ (typically square-summable), how can one exactly recover f in some concrete fashion? Once the requisite theory and background are presented in Chapter II, two broad approaches are then utilized. In Chapter III, polynomial interpolants are introduced which allow for the construction of approximants for arbitrary bandlimited functions (Theorems III.26 and III.30) which demonstrate desirable convergence. In Chapter IV, oversampling of data (sampling at points $(\frac{t_n}{\lambda})_n$ where $\lambda > 1$) is employed so that the derived approximants are stable with respect to certain systematic errors in $\left(f\left(\frac{t_n}{\lambda}\right)\right)_n \in \ell_2(\mathbb{Z}^d)$ (see Theorems IV.3 and IV.7).

B. Introduction to Chapter II

The basic notions and theory necessary for this dissertation are presented in Chapter II. The definition and fundamental properties of bandlimited functions are developed in Section A. In Section B, frames and Riesz bases are introduced, with emphasis placed upon those which consist of complex exponential functions. Of particular interest is Lemma II.17, (the Bessel space Lemma) which is used repeatedly throughout

This dissertation follows the style of the Journal of Approximation Theory.

this dissertation. Concrete examples of exponential frames and Riesz bases are given in Section C, notably in Theorems II.18, II.19, II.20, and II.21. An explicit relationship between constants appearing in Theorems II.20 and II.21 is given in Section D.

C. Introduction to Chapter III

Approximation of univariate bandlimited functions as limits of polynomials has a rich pedigree, which is illustrated by historical answers to the following question: If $(\mathrm{sinc}\pi(\cdot - t_n))_{n\in\mathbb{Z}}$ is a Riesz basis for $PW_{[-\pi,\pi]}$, what are the canonical product expansions of the biorthogonal functions for this Riesz basis? The first results along these lines were given by Paley and Wiener in [1], and improved upon by Levinson in [2, pages 47-67]). Subsequently Levin extended these results to different classes of Riesz bases in [3]. A complete solution is given by Lyubarskii and Seip in [4] and Pavlov in [5]. In particular, they prove the following theorem, which is the starting point of Chapter III.

Theorem I.1. *Let $(t_n)_n \subset \mathbb{R}$, (where $t_n \neq 0$ when $n \neq 0$), be a sequence such that the family of functions $(\mathrm{sinc}\pi(\cdot - t_n))_n$ is a Riesz basis for $PW_{[-\pi,\pi]}$. Then the function*

$$S(z) = \lim_{r\to\infty}(z - t_0) \prod_{\{t_n \ : \ |t_n|<r \ , \ n\neq 0\}} \left(1 - \frac{z}{t_n}\right)$$

is entire, where convergence is uniform on compacta, and the biorthogonal functions $(G_n)_n$ of $(\mathrm{sinc}\pi(\cdot - t_n))_n$ are given by

$$G_n(z) = \frac{S(z)}{(z - t_n)S'(t_n)}.$$

The following is a readily proven corollary of Theorem I.1:

Corollary I.2. *Let $(t_n)_n \subset \mathbb{R}$ and $(G_k)_k$ be defined as in Theorem I.1. Then for each*

k, there exists a sequence of polynomials $(\Phi_{N,k})_N$ such that

1) $\Phi_{N,k}(t_n) = G_k(t_n)$ when $|t_n| < N$.

2) $\lim_{N\to\infty} \Phi_{N,k} = G_k$ uniformly on compacta.

Corollary I.2 motivates two questions:

1) Let $(t_n)_n \subset \mathbb{R}^d$ be chosen such that $\left(e^{i\langle\cdot,t_n\rangle}\right)_n$ is a Riesz basis for $L_2([-\pi,\pi]^d)$. What are sufficient conditions on $\left(e^{i\langle\cdot,t_n\rangle}\right)_n$ such that every multivariate bandlimited function f, (not just biorthogonal functions associated with a particular exponential Riesz basis), has a corresponding sequence of polynomials which interpolates f on increasingly large subsets of $(t_n)_n$?

2) If polynomial interpolants (of the type described above) for a multivariate bandlimited function exist, can these interpolants be used to approximate the function in some simple and straightforward way?

Let $(t_n)_{n\in\mathbb{Z}^d} \subset \mathbb{R}^d$ be a sequence such that the family of exponentials $\left(e^{i\langle\cdot,t_n\rangle}\right)_{n\in\mathbb{Z}^d}$ is a uniformly invertible Riesz basis for $L_2([-\pi,\pi]^d)$ (defined in Chapter III, Section B). Under this condition, Theorem III.26 shows that polynomial interpolants of the type described in question 1) exist, along with bounds on the coordinate degree (not just the total degree) of each polynomial. This theorem also addresses question 2), by demonstrating that multivariate bandlimited functions can be approximated globally, in both uniform and L_2 metrics, by a rational function times a multivariate sinc function. Stated informally,

$$f(t) \simeq \Psi_\ell(t)\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)}, \quad \ell > 0, \tag{1.1}$$

where $(\Psi_\ell)_{\ell\in\mathbb{N}}$ is the desired sequence of interpolating polynomials and $(Q_{d,\ell})_\ell$ is a sequence of polynomials which eventually removes all the zeros of the SINC function. The fraction in expression (1.1) becomes more computationally complicated as

ℓ increases. Theorem III.30 gives a more satisfactory answer to question 2) by using

$$\exp\Big(-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}\Big),\quad \ell>0,$$

in lieu of the fraction in expression (1.1). The exponent in the above expression is now a rational function of ℓ. This simplicity necessitates replacing global L_2 and uniform convergence with a more local (though not totally local) convergence.

The author is unaware of any other multivariate theorem addressing questions 1) and 2) above, and which satisfies the following:

a) The exponential Riesz bases under consideration are not necessarily tensor products of single-variable Riesz bases.

b) Convergence stronger than "uniform convergence on compacta" is proven.

It should be noted that Theorems III.26, III.30, and Corollary III.31 do not, at this point, recover Corollary I.2 or Theorem I.1 in its generality of allowable sequences $(t_n)_n \subset \mathbb{R}$, though the comments above show that their value is due primarily to their multidimensional nature and convergence properties. This being said, Theorem III.44 (in Section G) presents an alternative proof of Theorem I.1 in the case that $\left(e^{i\langle\cdot,t_n\rangle}\right)_{n\in\mathbb{Z}^d}$ is a uniformly invertible Riesz basis for $L_2[-\pi,\pi]$.

D. Introduction to Chapter IV

The subject of recovery of bandlimited signals from discrete data has its origins in the Whittaker-Kotel'nikov-Shannon Sampling Theorem (1.2), historically the first and simplest such recovery formula, presented below. Without loss of generality the bandwidth is restricted to $[-\pi,\pi]$.

$$f(t)=\sum_{n\in\mathbb{Z}} f(n)\mathrm{sinc}\pi(t-n),\quad t\in\mathbb{R},\quad f\in PW_{[-\pi,\pi]}. \tag{1.2}$$

Convergence in (1.2) is global with respect to L_2 and L_∞ metrics. Equation (1.2) has drawbacks. Foremost, the recovery formula does not converge given certain types of error in the sampled data. Suppose sampled data corresponding to a bandlimited function f has noise, say perhaps $(f(n)+\epsilon_n)_{n\in\mathbb{Z}}$ where $\epsilon_n = \epsilon\ \mathrm{sign}(\mathrm{sinc}\pi(1/2-n))$. If we try to estimate f by substituting $(f(n)+\epsilon_n)_{n\in\mathbb{Z}}$ in place of $(f(n))_{n\in\mathbb{Z}}$ in (1.2), we obtain:

$$\tilde{f}(t) := f(t) + \epsilon \sum_{n\in\mathbb{Z}} (\mathrm{sign}(\mathrm{sinc}\pi(1/2-n)))\mathrm{sinc}\pi(t-n),$$

which yields

$$\tilde{f}(1/2) = f(1/2) + \epsilon \sum_{n\in\mathbb{Z}} |\mathrm{sinc}\pi(1/2-n)| = \infty.$$

This demonstrates that (1.2) is unstable under ℓ_∞ perturbations of the sampled data. One way to remedy this deficiency in (1.2) is to introduce oversampling, by which we mean the following process: given data sites $(t_n)_n$, increase the density of this sequence by a factor of $\lambda > 1$, and obtain samples $\left(f\left(\frac{t_n}{\lambda}\right)\right)_n$. If we have a sequence of data sites $(t_n)_n$ and corresponding samples $(f(t_n))_n$ (with no noise) for which a bandlimited function $f \in PW_{[-\pi,\pi]}$ can be perfectly recovered, what information does an increase in the density of data sites provide?

Equation (1.3) below, (proven in [6] by Daubechies and DeVore) uses oversampling of equally spaced data sites to expand f via translates of a Schwartz function g rather than the slowly decaying sinc function:

$$f(t) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} f\Big(\frac{n}{\lambda}\Big) g\Big(t-\frac{n}{\lambda}\Big), \quad t\in\mathbb{R}. \tag{1.3}$$

Convergence of (1.3) is global with respect to L_2 and L_∞ metrics. The following theorem in [6] illustrates a certain stability of the recovery formula (1.3) in contrast to (1.2). Suppose we have sample values $\tilde{f}_n = f\left(\frac{n}{\lambda}\right) + \epsilon_n$ where $\sup_n |\epsilon_n| = \epsilon$. If, in (1.3), we replace $f\left(\frac{n}{\lambda}\right)$ by $\tilde{f}_n$, and call the resulting expression $\tilde{f}$, then we have the following error bounds in recovery.

Theorem I.3 (Daubechies, DeVore)**.**

$$\sup_{t\in\mathbb{R}} |f(t) - \tilde{f}(t)| \le \epsilon\Big(\|g\|_{L_1} + \frac{1}{\lambda}\|g'\|_{L_1}\Big). \tag{1.4}$$

As a comment, it is unnecessary for g to be a Schwartz function for (1.3) and (1.4) to hold; it is enough for g to be continuously differentiable where $g, g' \in L_1(\mathbb{R})$. The true reason for requiring g to decay rapidly becomes apparent in Daubechies' and DeVore's treatment of quantization in [6], a topic which is not addressed here.

In Chapter IV, (1.3) and (1.4) are generalized in Theorems IV.3 and IV.7 in Sections B and C respectively. The setting is described below.

1) The underlying space is PW_E, a space of multivariate functions. The frequency domain $E \subset \mathbb{R}^d$ is a set satisfying natural geometric conditions, as described in Proposition IV.1. An important example will be $PW_{[-\pi,\pi]^d}$.

2) The sampling nodes $(t_n)_n \subset \mathbb{R}^d$ are such that the set of functions $\big(e^{i\langle\cdot,t_n\rangle}\big)_n$ is a frame for $L_2(E)$ (defined in Chapter II, Section B). This generalizes (1.3), which uses the crucial fact that $(e^{in(\cdot)})_{n\in\mathbb{Z}}$ is an orthogonal basis for $L_2[-\pi,\pi]$.

The sampling formula in Theorem IV.3 is of the form

$$f(t) = \frac{1}{\lambda^d}\sum_k (Bf_{\mathcal{T}/\lambda})_k g\Big(t - \frac{t_k}{\lambda}\Big), \quad t \in \mathbb{R}^d, \quad f \in PW_E,$$

where convergence is global with respect to both L_2 and L_∞ metrics. In the equality above, g is a Schwartz function, B is an infinite matrix relating to the frame operator for $\big(e^{i\langle\cdot,t_n\rangle}\big)_n$, and $f_{\mathcal{T}/\lambda} = \big(f\big(\frac{t_n}{\lambda}\big)\big)_n$. Theorem IV.3 focuses on resolving two questions.

1) If we restrict to $E = [-\pi,\pi]^d$, when can a measure of stability (in the manner of (1.4)) be achieved for the recovery formula in Theorem IV.3?

2) When can the matrix B be explicitly computed, and when can its properties

as an operator from one sequence space to another be determined? Even if we restrict to $E = [-\pi, \pi]^d$, then under the full generality of Theorem IV.3, the entries of B are difficult to ascertain.

Regarding the first question, a criterion for the recovery formula to be stable given ℓ_p error ($1 \leq p < \infty$) in sampled data is given in Theorem IV.7. This criterion is satisfied if $\left(e^{i\langle \cdot, t_n \rangle}\right)_n$ is a tight frame for $L_2([-\pi, \pi]^d)$ (Theorem IV.18 in Section C), or in the univariate case, if $\left(e^{it_n(\cdot)}\right)_n$ can be made into an orthogonal basis for $L_2[-\pi, \pi]$ after replacement of finitely many complex exponential functions (Theorem IV.16 in Section C).

Regarding the second question, a reasonable degree of explicitness of the entries of B and an understanding of its behavior as an operator can be achieved if either $\left(e^{i\langle \cdot, t_n \rangle}\right)_n$ is a tight frame or a Riesz for $L_2([-\pi, \pi]^d)$. If $\left(e^{i\langle \cdot, t_n \rangle}\right)_n$ is a tight frame, then B can be explicitly determined (Theorem IV.18 in Section C). If $\left(e^{i\langle \cdot, t_n \rangle}\right)_n$ is a Riesz basis, then we have a sequence of approximants for functions in $PW_{[-\pi,\pi]^d}$ in which the infinite matrix B can be replaced by a sequence of finite matrices each of whose entries is computable by linear-algebraic means. This is the content of Theorem IV.22 in Section D.

CHAPTER II

PRELIMINARY MATERIAL*

This chapter introduces the basic notions and theory that will be used throughout this dissertation.*

A. Introduction to bandlimited functions

In this dissertation, an isomorphism $T : X \to Y$ between two normed spaces is a linear map such that for some $m, M > 0$,

$$m\|x\|_X \leq \|Tx\|_Y \leq M\|x\|_X, \quad x \in X.$$

If an isomorphism is onto, it will be explicitly stated.

The following is the d-dimensional L_2 unitary Fourier transform:

$$\mathcal{F}(f)(\cdot) = \text{P.V.}\ \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{-i\langle \cdot, \xi \rangle} d\xi, \quad f \in L_2(\mathbb{R}^d),$$

where the inverse transform is given by

$$\mathcal{F}^{-1}(f)(\cdot) = \text{P.V.}\ \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{i\langle \cdot, \xi \rangle} d\xi, \quad f \in L_2(\mathbb{R}^d).$$

Definition II.1. Given a bounded set $E \subset \mathbb{R}^d$ with positive Lebesgue measure, we define

$$PW_E := \{f \in L_2(\mathbb{R}^d) \mid \operatorname{supp}(\mathcal{F}^{-1}(f)) \subset E\}.$$

Functions in PW_E are said to be *bandlimited.*

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*Part of this chapter is reprinted with permission from An asymptotic equivalence between two frame perturbation theorems, by B. A. Bailey, in: M. Neamtu, L. Schumaker (Eds.), Proceedings of Approximation Theory XIII: San Antonio 2010, Springer (in press) pp. 1-7, Copyright 2011 by Springer-Verlag.

Definition II.2. Define the function SINC : $\mathbb{R}^d \to \mathbb{R}$ by

$$\mathrm{SINC}(x) := \mathrm{sinc}(x(1)) \cdot \ldots \cdot \mathrm{sinc}(x(d))$$

where $\mathrm{sinc}(t) := \frac{\sin(t)}{t}$ for $t \in \mathbb{R}$.

The following are facts concerning PW_E which will be used frequently.

1) PW_E is isometric to $L_2(E)$ by way of the unitary Fourier transform.

2) $PW_{[-\pi,\pi]^d}$ consists of functions from $\mathbb{R}^d$ to $\mathbb{C}$, though it is easily verified that they naturally extend to entire functions from $\mathbb{C}^d$ to $\mathbb{C}$. In this dissertation we restrict the domain to $\mathbb{R}^d$.

3) We have

$$\mathcal{F}\Big(\frac{1}{(2\pi)^{d/2}} e^{i\langle\cdot,\tau\rangle}\chi_{[-\pi,\pi]^d}(\cdot)\Big)(t) = \mathrm{SINC}\pi(t-\tau)$$

by direct computation.

4) From 1) and 3) above, and the fact that $\Big(\frac{1}{(2\pi)^{d/2}} e^{i\langle\cdot,n\rangle}\Big)_{n\in\mathbb{Z}^d}$ is an orthonormal basis for $L_2([-\pi,\pi]^d)$, we see that $\big(\mathrm{SINC}\pi(\cdot-n)\big)_{n\in\mathbb{Z}^d}$ is an orthonormal basis for $PW_{[-\pi,\pi]^d}$.

5) If $f \in PW_{[-\pi,\pi]^d}$ and $t \in \mathbb{R}^d$, then (since $\mathcal{F}$ is unitary),

$$\begin{aligned} f(t) &= \mathcal{F}(\mathcal{F}^{-1}f)(t) = \Big\langle (\mathcal{F}^{-1}f)(\cdot), \frac{1}{(2\pi)^{d/2}} e^{i\langle t,\cdot\rangle}\chi_{[-\pi,\pi]^d}(\cdot)\Big\rangle \\ &= \big\langle f(\cdot), \mathrm{SINC}\pi(\cdot - t)\big\rangle, \end{aligned} \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $L_2(\mathbb{R}^d)$.

6) In PW_E, L_2 convergence implies uniform convergence:

$$\begin{aligned} \|f\|_\infty &= \sup_{t\in\mathbb{R}^d}\left|\frac{1}{(2\pi)^{d/2}}\int_E (\mathcal{F}^{-1}f)(\xi)e^{-i\langle t,\xi\rangle}d\xi\right| \le \frac{1}{(2\pi)^{d/2}}\int_E |(\mathcal{F}^{-1}f)(\xi)|d\xi \\ &\le \frac{\mu(E)^{1/2}}{(2\pi)^{d/2}}\Big(\int_E |(\mathcal{F}^{-1}f)(\xi)|^2 d\xi\Big)^{1/2} \\ &= \frac{\mu(E)^{1/2}}{(2\pi)^{d/2}}\|\mathcal{F}^{-1}f\|_2 = \frac{\mu(E)^{1/2}}{(2\pi)^{d/2}}\|f\|_2, \end{aligned} \tag{2.2}$$

where the second inequality above follows from the Cauchy-Schwarz inequality.

7) The d-dimensional Riemann-Lebesgue Lemma, [7, Theorem 8.22, page 249] implies

$$\lim_{\|x\|_\infty \to \infty} f(x) = 0, \quad f \in PW_{[-\pi,\pi]^d}. \tag{2.3}$$

8) We have the multivariate Whittaker-Kotel'nikov-Shannon Sampling Theorem (2.4) [8, page 57]: If $f \in PW_{[-\pi,\pi]^d}$, then

$$f(t) = \sum_{n \in \mathbb{Z}^d} f(n)\mathrm{SINC}(\pi(t-n)), \quad t \in \mathbb{R}^d, \tag{2.4}$$

where the sum converges in $PW_{[-\pi,\pi]^d}$, and hence uniformly.

9) The following celebrated result due to Paley and Wiener (see [9, Theorem 19.3]) characterizes bandlimited functions of a single variable.

Theorem II.3. *A function f is in $PW_{[-\pi,\pi]}$ if and only if each of the following statements holds:*

1) f is entire.

2) There exists $M \geq 0$ such that $|f(z)| \leq Me^{\pi|z|}$ for $z \in \mathbb{C}$.

3) $f\big|_{\mathbb{R}} \in L_2(\mathbb{R})$.

B. Introduction to frames and Riesz bases

The following information concerning frames may be found in [10, Section 4].

Definition II.4. A *frame* for a separable Hilbert space H is a sequence $(f_n)_n \subset H$ such that for some $0 < A < B$,

$$A\|f\|^2 \leq \sum_n |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{2.5}$$

The optimal numbers A and B in (2.5) are called the *lower* and *upper frame bounds*, respectively.

Proposition II.5. *Let H be a Hilbert space with orthonormal basis $(e_n)_n$. The following conditions are equivalent.*

1) The sequence $(f_n)_n \subset H$ is a frame for H.

2) The synthesis operator *$L : H \to H$ defined by $Le_n = f_n$ is bounded linear and onto.*

3) The analysis operator *$L^* : H \to H$ given by $f \mapsto \sum_n \langle f, f_n \rangle e_n$ is an isomorphism.*

Proof. 1) $\iff$ 2): This follows from the basic theory of adjoint operators.
2) $\iff$ 3): This follows immediately from the computation of $L^* f = \sum_n \langle f, f_n \rangle e_n$. $\square$

Definition II.6. Given a frame $(f_n)_n$ with synthesis operator L, the map $S = LL^*$ given by

$$Sf = \sum_n \langle f, f_n \rangle f_n$$

is an onto isomorphism. S is called the *frame operator* associated to the frame. We note that S is positive and self-adjoint.

Definition II.7. A *tight frame* for a Hilbert space is a frame such that the upper and lower frame bounds are equal. Equivalently, a tight frame is a frame such that the frame operator is a scalar multiple of the identity.

Definition II.8. A sequence $(f_n)_n \subset H$ satisfying

$$\sum_n |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H$$

is a called a *Bessel sequence.* The smallest number B such that the inequality above holds is called the *upper frame bound.*

The following proposition characterizes Bessel sequences.

Proposition II.9. *$(f_n)_n \subset H$ is a Bessel sequence if and only if the synthesis operator is bounded.*

Proof. $(f_n)_n \subset H$ is a Bessel sequence if and only if L^* has norm $\sqrt{B}$, which holds if and only if L has norm $\sqrt{B}$. □

Definition II.10. Let H be a Hilbert space with orthonormal basis $(e_n)_n$. A sequence $(f_n)_n \subset H$ is called a *Riesz basis* for H if the map $Le_n = f_n$ is an onto isomorphism.

If $(f_n)_n \subset H$ is a Riesz (resp. Schauder) basis for H, then there exists an associated Riesz (resp. Schauder) basis of functions $(f_n^*)_n \subset H$ such that $\langle f_n, f_m^* \rangle = \delta_{mn}$. This basis is called the biorthogonal basis associated with $(f_n)_n$. Expressed in the terminology of frames,

$$f_n^* = S^{-1} f_n.$$

The basic connection between frames and the sampling theory of bandlimited functions (more generally in a reproducing kernel Hilbert space) is straightforward. Let $(f_n)_n = \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, t_n \rangle}\right)_n$ be a frame for $PW_{[-\pi,\pi]^d}$ with frame operator S. If $\mathcal{F}$ is the unitary Fourier transform and $f \in PW_{[-\pi,\pi]^d}$, then

$$S(\mathcal{F}^{-1}(f)) = \sum_n \langle \mathcal{F}^{-1}(f), f_n \rangle f_n = \sum_n \mathcal{F}(\mathcal{F}^{-1}(f))(t_n) f_n = \sum_n f(t_n) f_n,$$

implying that

$$\mathcal{F}^{-1}(f) = \sum_n f(t_n) S^{-1} f_n,$$

so that

$$f = \sum_n f(t_n) \mathcal{F}(S^{-1} f_n).$$

If we restrict to the Riesz basis case, we have the following corollary.

Corollary II.11. *Let* $\left(\frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, t_n \rangle}\right)_{n\in\mathbb{N}}$ *be a Riesz basis for* $L_2([-\pi,\pi]^d)$*, with biorthogonal functions* $(f_n^*)_{n\in\mathbb{N}}$*. If* $G_n := \mathcal{F} f_n^*$ *for* $n \in \mathbb{N}$*, then*

$$G_n(t_m) = \delta_{nm} \tag{2.6}$$

and

$$f = \sum_{n=1}^{\infty} f(t_n) G_n(t). \tag{2.7}$$

Note that for general d, we recover (2.4) when $(t_n)_n$ is an enumeration of $\mathbb{Z}^d$.

The following theorem (see [11]) illustrates another natural link between exponential Riesz bases and sampling. The proof of Theorem II.12 when $d = 1$ appears in [11, Theorem 9, page 143], and the proof for general d (from a functional analytic point of view) is identical.

Theorem II.12. *Let $(t_n)_{n\in\mathbb{Z}^d} \subset \mathbb{R}$. The following are equivalent:*

1) The sequence of functions $\left(e^{i\langle(\cdot),t_n\rangle}\right)_{n\in\mathbb{Z}^d}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$.

2) The map $f \mapsto (f(t_n))_{n\in\mathbb{Z}^d}$ is a bijection from $PW_{[-\pi,\pi]^d}$ to $\ell_2(\mathbb{Z}^d)$.

Definition II.13. A subset S of $\mathbb{R}^d$ is *uniformly separated* if

$$\inf_{x,y\in S\ ,\ x\neq y} \|x-y\|_2 > 0.$$

Definition II.14. A subset S of $\mathbb{R}^d$ is *relatively uniformly separated* if it is the union of finitely many uniformly separated sets.

The following is an immediate corollary of Theorem 3.1 in [12]:

Proposition II.15. *If $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is chosen such that $(e^{i\langle t_n,(\cdot)\rangle})_{n\in\mathbb{N}}$ is a Bessel sequence for $L_2([-\pi,\pi]^d)$, then $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is relatively uniformly separated.*

The following statement is quickly derived from basic definitions: If $\left(e^{i\langle t_n,(\cdot)\rangle}\right)_{n\in\mathbb{Z}^d}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$, then $(t_n)_{n\in\mathbb{Z}^d}$ is uniformly separated.

Definition II.16. An *exponential Riesz basis (resp. frame)* is a sequence of functions $(e^{i\langle\cdot,t_n\rangle})_n$ which is a Riesz basis (resp. frame).

As a note, there exists a great body of research on the separation properties of exponential frames and Riesz bases. Here we have only discussed what is necessary for the purposes at hand.

The Bessel sequence Lemma (BSL) (see [13, Lemma 1]), is central to many results in this dissertation.

Lemma II.17 (BSL)**.** *Choose* $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ *such that* $(h_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle(\cdot),t_k\rangle}\right)_{k\in\mathbb{N}}$ *is a Bessel sequence in* $L_2([-\pi,\pi]^d)$ *with upper frame bound* B*.*
If $(\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ *and* $(f_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle(\cdot),\tau_k\rangle}\right)_{k\in\mathbb{N}}$*, then for all* $r, s \geq 1$ *and any finite sequence* $(a_k)_k$*, we have*

$$\Big\| \sum_{k=r}^{s} a_k(h_k - f_k) \Big\|_{L_2([-\pi,\pi]^d)} \leq \sqrt{B}\Big(e^{\pi d\Big(\sup\limits_{r\leq k\leq s} \|t_k-\tau_k\|_\infty\Big)} - 1\Big)\Big(\sum_{k=r}^{s} |a_k|^2\Big)^{\frac{1}{2}}.$$

Proof. By Proposition II.9,

$$\Big\|\sum_{k=1}^{n} a_k h_k\Big\|_{L_2([-\pi,\pi]^d)} \leq \sqrt{B}\Big(\sum_{k=1}^{n} |a_k|^2\Big)^{1/2}, \quad \text{for all} \quad (a_k)_{k=1}^n \subset \mathbb{C}.$$

Let $\delta_k = \tau_k - t_k$ where $\delta_k = (\delta_{k1}, \cdots, \delta_{kd})$, then

$$\phi_{r,s}(x) := \sum_{k=r}^{s} \frac{a_k}{(2\pi)^{d/2}}\big[e^{i\langle t_k,x\rangle} - e^{i\langle\tau_k,x\rangle}\big] = \sum_{k=r}^{s} \frac{a_k}{(2\pi)^{d/2}} e^{i\langle t_k,x\rangle}\big[1 - e^{i\langle\delta_k,x\rangle}\big]. \tag{2.8}$$

Define $J = \{(j_1, \cdots, j_d) \in \mathbb{Z}^d \mid j_i \geq 0, (j_1, \cdots, j_d) \neq 0\}$. For any δ_k,

$$\begin{aligned}
1 - e^{i\langle\delta_k,x\rangle} &= 1 - e^{i\delta_{k1}x_1} \cdot \ldots \cdot e^{i\delta_{kd}x_d} \\
&= 1 - \Big(\sum_{j_1=0}^{\infty} \frac{(i\delta_{k1}x_1)^{j_1}}{j_1!}\Big) \cdot \ldots \cdot \Big(\sum_{j_d=0}^{\infty} \frac{(i\delta_{kd}x_d)^{j_d}}{j_d!}\Big) \\
&= 1 - \sum_{(j_1,\cdots,j_d),\ j_i\geq 0} \frac{(i\delta_{k1}x_1)^{j_1} \cdot \ldots \cdot (i\delta_{kd}x_d)^{j_d}}{j_1! \cdot \ldots \cdot j_d!} \\
&= -\sum_{(j_1,\cdots,j_d)\in J} i^{j_1+\ldots+j_d} \frac{(\delta_{k1}x_1)^{j_1} \cdot \ldots \cdot (\delta_{kd}x_d)^{j_d}}{j_1! \cdot \ldots \cdot j_d!},
\end{aligned}$$

From (2.8), we obtain

$$\begin{aligned}
|\phi_{r,s}(x)| &= \left|\sum_{k=r}^{s}\frac{a_k}{(2\pi)^{d/2}}e^{i\langle t_k,x\rangle}\Big[\sum_{(j_1,\cdots,j_d)\in J} i^{j_1+\ldots+j_d}\frac{(\delta_{k1}x_1)^{j_1}\cdot\ldots\cdot(\delta_{kd}x_d)^{j_d}}{j_1!\cdot\ldots\cdot j_d!}\Big]\right| \\
&= \left|\sum_{(j_1,\cdots,j_d)\in J}\frac{x_1^{j_1}\cdot\ldots\cdot x_d^{j_d}}{j_1!\cdot\ldots\cdot j_d!}i^{j_1+\ldots+j_d}\sum_{k=r}^{s}\frac{a_k}{(2\pi)^{d/2}}\delta_{k1}^{j_1}\cdot\ldots\cdot\delta_{kd}^{j_d}e^{i\langle t_k,x\rangle}\right| \\
&\leq \sum_{(j_1,\cdots,j_d)\in J}\frac{\pi^{j_1+\ldots+j_d}}{j_1!\cdot\ldots\cdot j_d!}\Big|\sum_{k=r}^{s}a_k\delta_{k1}^{j_1}\cdot\ldots\cdot\delta_{kd}^{j_d}\frac{e^{i\langle t_k,x\rangle}}{(2\pi)^{d/2}}\Big|.
\end{aligned}$$

For brevity denote the outer summand above by $h_{j_1,\ldots,j_d}(x)$. Then

$$\begin{aligned}
\|\phi_{r,s}\|_2 &\leq \Big(\int_{[-\pi,\pi]^d}\Big|\sum_{(j_1,\cdots,j_d)\in J}h_{j_1,\ldots,j_d}(x)\Big|^2dx\Big)^{\frac{1}{2}} \\
&\leq \sum_{(j_1,\cdots,j_d)\in J}\Big(\int_{[-\pi,\pi]^d}\Big|h_{j_1,\ldots,j_d}(x)\Big|^2dx\Big)^{\frac{1}{2}} \\
&= \sum_{(j_1,\cdots,j_d)\in J}\frac{\pi^{j_1+\cdot\ldots\cdot+j_d}}{j_1!\cdot\ldots\cdot j_d!}\Big(\int_{[-\pi,\pi]^d}\Big|\sum_{k=r}^{s}a_k\delta_{k1}^{j_1}\cdot\ldots\cdot\delta_{kd}^{j_d}\frac{e^{i\langle t_k,x\rangle}}{(2\pi)^{d/2}}\Big|^2dx\Big)^{\frac{1}{2}} \\
&\leq \sqrt{B}\sum_{(j_1,\cdots,j_d)\in J}\frac{\pi^{j_1+\cdot\ldots\cdot+j_d}}{j_1!\cdot\ldots\cdot j_d!}\Big(\sum_{k=r}^{s}|a_k|^2|\delta_{k1}^{j_1}|^2\cdot\ldots\cdot|\delta_{kd}^{j_d}|^2\Big)^{\frac{1}{2}} \\
&\leq \sqrt{B}\sum_{(j_1,\cdots,j_d)\in J}\frac{\pi^{j_1+\cdot\ldots\cdot+j_d}}{j_1!\cdot\ldots\cdot j_d!}\left(\sum_{k=r}^{s}|a_k|^2\Big(\sup_{r\leq k\leq s}\|\tau_k-t_k\|_\infty\Big)^{2(j_1+\ldots+j_d)}\right)^{\frac{1}{2}} \\
&= \sqrt{B}\sum_{(j_1,\cdots,j_d)\in J}\frac{\Big(\pi\sup\limits_{r\leq k\leq s}\|\tau_k-t_k\|_\infty\Big)^{j_1+\cdot\ldots\cdot+j_d}}{j_1!\cdot\ldots\cdot j_d!}\Big(\sum_{k=r}^{s}|a_k|^2\Big)^{\frac{1}{2}} \\
&= \sqrt{B}\Big[\prod_{l=1}^{d}\Big(\sum_{j_\ell=0}^{\infty}\frac{(\pi\sup\limits_{r\leq k\leq s}\|\tau_k-t_k\|_\infty)^{j_\ell}}{j_\ell!}\Big)-1\Big]\Big(\sum_{k=r}^{s}|a_k|^2\Big)^{\frac{1}{2}} \\
&= \sqrt{B}\Big(e^{\pi d\big(\sup\limits_{r\leq k\leq s}\|\tau_k-t_k\|_\infty\big)}-1\Big)\Big(\sum_{k=r}^{s}|a_k|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

□

C. Examples of exponential Riesz bases and frames

The following deep result due to Beurling ([14, see Theorem 1, Theorem 2, and (38)]) provides a multitude of exponential frames.

Theorem II.18 (Beurling)**.** *Let $(t_n)_n \subset \mathbb{R}^d$ be a sequence such that*

$$\begin{aligned} \inf_{t_n \neq t_m} \|t_n - t_m\|_{\ell_2} &> 0, \quad \textit{and} \\ \sup_{\xi \in \mathbb{R}^d} \inf_n \|t_n - \xi\|_{\ell_2} &< \frac{\pi}{2}. \end{aligned}$$

If E is a subset of the closed unit ball in $\mathbb{R}^d$ and E has positive Lebesgue measure, then $(e^{i\langle \cdot, t_n\rangle})_n$ is a frame for $L_2(E)$.

Other examples exponential Riesz bases and frames are shown here.

Theorem II.19. *(Kadec's "1/4" Theorem) Let $(t_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence of real numbers such that*

$$\sup_{n\in\mathbb{Z}} |n - t_n| < 1/4.$$

Then the sequence of functions $\left(e^{it_n(\cdot)}\right)$ is a Riesz basis for $L_2[-\pi,\pi]$. Furthermore, if C is any constant such that $\sup_{n\in\mathbb{Z}} |n - t_n| < C$ implies that $\left(e^{it_n(\cdot)}\right)$ is a Riesz basis for $L_2[-\pi,\pi]$, then $C \leq 1/4$.

The following is the scheme that Kadec used to prove Theorem II.19, first proven in [15] (see [11] for a nice exposition). Define the operator T on $L_2[-\pi,\pi]$ by

$$T\left(e^{in(\cdot)}\right) = e^{in(\cdot)} - e^{it_n(\cdot)} = e^{in(\cdot)}(1 - e^{i(t_n - n)(\cdot)}), \quad n \in \mathbb{Z}.$$

Expand $e^{i(t_n-n)(\cdot)}$ with respect to the orthogonal basis

$$\mathcal{B} = \left\{1, \cos(nx), \sin\left(n - \frac{1}{2}\right)x\right\}_{n\in\mathbb{N}}$$

for $L_2[-\pi,\pi]$, and use this expansion to estimate the norm of T. Inspired calculation shows that $\|T\| < 1$, so that by usual Neumann series manipulation, the map

$$I - T \; : \; e^{in(\cdot)} \mapsto e^{it_n(\cdot)}$$

is an onto isomorphism. To prove optimality of $C = 1/4$, consider the sequence of exponentials $(f_n)_{n\in\mathbb{Z}}$ where

$$f_n(x) = \begin{cases} e^{i\left(n+\frac{1}{4}\right)x}, & n < 0; \\ 1, & n = 0; \\ e^{i\left(n-\frac{1}{4}\right)x}, & n > 0 \end{cases} .$$

It can be shown (with much effort) that $(f_n)_{n\in\mathbb{Z}\setminus\{0\}}$ has dense linear span in $L_2[-\pi,\pi]$, so that $(f_n)_{n\in\mathbb{Z}}$ cannot be a Riesz basis. See [11, Chapter 3] for an exposition of this proof.

An impressive generalization of Kadec's "1/4" Theorem when $d = 1$ is Avdonin's "1/4 in the mean" Theorem, [16]. In [17], Sun and Zhou have proven the following multidimensional version of Kadec's "1/4" Theorem through a direct generalization of Kadec's original proof. In this case, optimality (i.e., the counterpoint of the second part of Theorem II.19) is not addressed.

Theorem II.20 (Sun, Zhou)**.** *Let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that*

$$(h_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle(\cdot),t_k\rangle}\right)_{k\in\mathbb{N}}$$

is a frame (resp. Riesz basis) for $L_2([-\pi,\pi]^d)$ with frame bounds A^2 and B^2. For $d \geq 1$, define

$$D_d(x) := \Big(1 - \cos\pi x + \sin\pi x + \operatorname{sinc}(\pi x)\Big)^d - (\operatorname{sinc}(\pi x))^d,$$

and let x_d be the unique number such that $0 < x_d \leq 1/4$ and $D_d(x_d) = \frac{A}{B}$. If

$(\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ and $(f_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle (\cdot), \tau_k\rangle}\right)_{k\in\mathbb{N}}$ is a sequence such that

$$\sup_{k\in\mathbb{N}} \|\tau_k - t_k\|_\infty < x_d, \tag{2.9}$$

then the sequence $(f_k)_{k\in\mathbb{N}}$ is also a frame (resp. Riesz basis) for $L_2([-\pi,\pi]^d)$.

The scheme of the proof of Theorem II.20 is as follows. Define the operator T on $L_2([-\pi,\pi]^d)$ by

$$T\big(e^{i\langle n,(\cdot)\rangle}\big) = e^{i\langle n,(\cdot)\rangle} - e^{i\langle t_n,(\cdot)\rangle} = e^{i\langle n,(\cdot)\rangle}\big(1 - e^{i\langle t_n - n,(\cdot)\rangle}\big), \quad n \in \mathbb{Z}^d.$$

Let $\mathcal{B}$ be the basis from the proof of Kadec's Theorem. Expand $e^{i\langle t_n - n,(\cdot)\rangle}$ with respect to the orthogonal basis $\mathcal{B}\otimes\cdots\otimes\mathcal{B}$ for $L_2([-\pi,\pi]^d)$. This expansion leads to the estimate $\|T\| < 1$, and bounded invertibility of $I - T$ follows as before.

Theorem II.21 below, (see [13]), is another generalization of Kadec's "1/4" theorem whose proof, though conceptually similar to that of Theorem II.20, is technically simpler. The univariate case of this result was proven by Duffin and Eachus in [18].

Theorem II.21. *Let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that*

$$(h_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle (\cdot), t_k\rangle}\right)_{k\in\mathbb{N}}$$

is a frame (resp. Riesz basis) for $L_2([-\pi,\pi]^d)$ with frame bounds A^2 and B^2. If $(\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ and $(f_k)_{k\in\mathbb{N}} := \left(\frac{1}{(2\pi)^{d/2}} e^{i\langle (\cdot), \tau_k\rangle}\right)_{k\in\mathbb{N}}$ is a sequence such that

$$\sup_{k\in\mathbb{N}} \|\tau_k - t_k\|_\infty < \frac{1}{\pi d} \ln\left(1 + \frac{A}{B}\right), \tag{2.10}$$

then the sequence $(f_k)_{k\in\mathbb{N}}$ is also a frame (resp. Riesz basis) for $L_2([-\pi,\pi]^d)$.

If we let $h_k(x) = \frac{1}{(2\pi)^{d/2}} e^{i\langle (\cdot), t_k\rangle}$ where $(t_k)_{k\in\mathbb{N}} = \mathbb{Z}^d$, then $(h_k)_{k\in\mathbb{N}}$ has frame bounds $A^2 = B^2 = 1$, and we have the following corollary.

Corollary II.22. *Let $(n_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$, and let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ such that*

$$\sup_{k\in\mathbb{N}} \|n_k - t_k\|_\infty = L < \frac{\ln(2)}{\pi d}. \tag{2.11}$$

Then the sequence $(f_k)_{k\in\mathbb{N}}$, defined by $f_k(x) = \frac{1}{(2\pi)^{d/2}} e^{i\langle x,t_k\rangle}$, is a Riesz basis for $L_2([-\pi,\pi]^d)$.

Corollary II.22 is useful as it gives a simple and concrete criterion for a sequence of exponential functions to be a Riesz basis for $L_2([-\pi,\pi]^d)$.

Proof of Theorem II.21. Let $(e_k)_{k=1}^\infty$ be an orthonormal basis for $L_2([-\pi,\pi]^d)$. Define linear maps L and $\tilde{L}$ from $\text{span}(e_k)_{k=1}^\infty$ to $L_2([-\pi,\pi]^d)$ by $Le_n = h_n$ and $\tilde{L}e_n = f_n$. L extends boundedly to $L_2([-\pi,\pi]^d)$. Define $\delta = \sup_{k\in\mathbb{N}} \|\tau_k - t_k\|_\infty$. Applying Lemma II.17, we see that $\tilde{L}$ also extends boundedly to $L_2([-\pi,\pi]^d)$, and that

$$\|L - \tilde{L}\| \le B\big(e^{\pi d\delta} - 1\big) := \beta A$$

for some $0 \le \beta < 1$. This implies $\|L^* f - \tilde{L}^* f\| \le \beta A$, when $\|f\| = 1$. Rearranging, we have

$$A(1-\beta) \le \|\tilde{L}^* f\|, \quad \text{when} \quad \|f\| = 1,$$

so $\tilde{L}^*$ is an isomorphism. By Proposition II.5, $(f_k)_{k\in\mathbb{N}}$ is a frame for $L_2([-\pi,\pi]^d)$. □

D. A comparison between Theorems II.20 and II.21

It is natural to ask how the constants x_d and $\frac{1}{\pi d}\ln\left(1 + \frac{A}{B}\right)$ from Theorems II.20 and II.21 are related. A relationship is given in the following theorem proven in [13].

Theorem II.23. *If x_d is the unique number satisfying $0 < x_d \le 1/4$ and $D_d(x_d) = \frac{A}{B}$,* *then*

$$\lim_{d\to\infty} \frac{x_d - \frac{1}{\pi d}\ln\left(1+\frac{A}{B}\right)}{\frac{\left[\ln\left(1+\frac{A}{B}\right)\right]^2}{6\pi\left(1+\frac{B}{A}\right)d^2}} = 1.$$

We prove Theorem II.23 via a sequence of propositions.

Proposition II.24. *Let d be a positive integer. If $f(x) := 1-\cos(x)+\sin(x)+\operatorname{sinc}(x)$ and $g(x) := \operatorname{sinc}(x)$, then*

$$\begin{aligned}
&1) \qquad f'(x)+g'(x) > 0, \quad x \in (0,\pi/4),\\
&2) \qquad g'(x) < 0, \quad x \in (0,\pi/4),\\
&3) \qquad f''(x) > 0, \quad x \in (0,\Delta) \quad \text{for some} \quad 0<\Delta<1/4.
\end{aligned}$$

Proof. For 1), let

$$\phi(x) := x^2(f'(x)+g'(x)) = x^2\sin(x)+x^2\cos(x)+2x\cos(x)-2\sin(x).$$

Noting that $\phi(0)=0$, it suffices to show that $\phi' > 0$ on $(0,\pi/4)$. Now

$$\phi'(x) = x(x\cos(x)-x\sin(x)+2\cos(x)) = \frac{x}{\cos(x)}(x+2-x\tan(x)),$$

so it suffices to show that $\psi(x) := x+2-x\tan(x) > 0$ on $(0,\pi/4)$. Now

$$\psi'(x) = 1 - x\sec^2(x) - \tan(x)$$

is decreasing on $(0,\pi/4)$, and $\psi'(0)=1$ and $\psi'(\pi/4)<0$, so there exists unique $c \in (0,\pi/4)$ such that $\psi'(c)=0$. We conclude that ψ is increasing on $(0,c)$, and decreasing on $(c,\pi/4)$, but $\psi(0)=\psi(\pi/4)=2$, so $\psi(x)>2$ on $(0,\pi/4)$.

For 2),

$$g'(x) = \frac{x\cos(x)-\sin(x)}{x^2} = \frac{x-\tan(x)}{x^2\cos(x)},$$

but $x-\tan(x)<0$ on $(0,\pi/4)$ as $0-\tan(0)=0$ and $(x-\tan(x))' = 1-\sec^2(x) < 0$

on $(0, \pi/4)$.

For 3), by standard Taylor series expansions we have $f(x) = 1 + x + \frac{x^2}{3} + O(x^3)$, so that $f''(0) = 2/3$. Continuity of f'' gives the desired result. $\square$

Proposition II.25. *The following statements hold:*

1) For $d > 0$, $D_d(x)$ and $D_d'(x)$ are positive on $(0, 1/4)$.

2) For all $d > 0$, $D_d''(x)$ is positive on $(0, \Delta)$.

Proof. Note $D_d(x) = f(\pi x)^d - g(\pi x)^d$ is positive. This expression yields

$$D_d'(x)/(d\pi) = f(\pi x)^{d-1} f'(\pi x) - g(\pi x)^{d-1} g'(\pi x) > 0 \quad \text{on} \quad (0, 1/4),$$

by Proposition II.24. Differentiating again, we obtain

$$\begin{aligned} D_d''(x)/(d\pi^2) &= (d-1)\big[f(\pi x)^{d-2}(f'(\pi x))^2 - g(\pi x)^{d-2}(g'(\pi x))^2\big] + \\ &+ [f(\pi x)^{d-1} f''(\pi x) - g(\pi x)^{d-1} g''(\pi x)] \quad \text{on} \quad (0, 1/4). \end{aligned}$$

If $g''(\pi x) \leq 0$ for some $x \in (0, 1/4)$, then the second bracketed term is positive. If $g''(\pi x) > 0$ for some $x \in (0, 1/4)$, then the second bracketed term is positive if $f''(\pi x) - g''(\pi x) > 0$, but

$$f''(\pi x) - g''(\pi x) = \pi^2(\cos(\pi x) - \sin(\pi x))$$

is positive on $(0, 1/4)$.

To show the first bracketed term is positive, it suffices to show that

$$f'(\pi x)^2 > g'(\pi x)^2 = (f'(\pi x) + g'(\pi x))(f'(\pi x) - g'(\pi x)) > 0$$

on $(0, \Delta)$. Noting $f'(\pi x) - g'(\pi x) = \pi(\cos(\pi x) + \sin(\pi x)) > 0$, it suffices to show that $f'(\pi x) + g'(\pi x) > 0$, but this is true by Proposition II.24. $\square$

Note that Proposition II.25 implies that x_d is unique.

Corollary II.26. *We have* $\lim_{d\to\infty} x_d = 0$.

Proof. Fix $n > 0$ with $1/n < \Delta$, then $\lim_{d\to\infty} D_d(1/n) = \infty$ (since f is increasing, implying that $0 < -\cos(\pi/n) + \sin(\pi/n) + \operatorname{sinc}(\pi/n)$). For sufficiently large d, $D_d(1/n) > \frac{A}{B}$. But $\frac{A}{B} = D_d(x_d) < D_d(1/n)$, so $x_d < 1/n$ by Proposition II.25. □

Proposition II.27. *Define* $\omega_d = \frac{1}{\pi d} \ln\left(1 + \frac{A}{B}\right)$. *We have*

$$\begin{aligned}
\lim_{d\to\infty} d\Big(\frac{A}{B} - D_d(\omega_d)\Big) &= \frac{A}{6B}\Big[\ln\Big(1+\frac{A}{B}\Big)\Big]^2, \\
\lim_{d\to\infty} \frac{1}{d} D_d'(\omega_d) &= \pi\Big(1+\frac{A}{B}\Big), \\
\lim_{d\to\infty} \frac{1}{d} D_d'(x_d) &= \pi\Big(1+\frac{A}{B}\Big).
\end{aligned}$$

Proof. 1) For the first equality, note that

$$D_d(\omega_d) = \Big[(1+h(x))^{\ln(c)/x} - g(x)^{\ln(c)/x}\Big]\Big|_{x=\frac{\ln(c)}{d}} \tag{2.12}$$

where $h(x) = -\cos(x) + \sin(x) + \operatorname{sinc}(x)$, $g(x) = \operatorname{sinc}(x)$, and $c = 1 + \frac{A}{B}$. L'Hospital's Rule implies that

$$\lim_{x\to 0}(1+h(x))^{\ln(c)/x} = c \quad \text{and} \quad \lim_{x\to 0} g(x)^{\ln(c)/x} = 1.$$

Looking at the first equality in the line above, another application of L'Hospital's Rule yields

$$\lim_{x\to 0} \frac{(1+h(x))^{\ln(c)/x} - c}{x} = c\ln(c)\left[\frac{\frac{h'(x)}{1+h(x)} - 1}{x} - \frac{\ln(1+h(x)) - x}{x^2}\right]. \tag{2.13}$$

Observing that $h(x) = x + x^2/3 + O(x^3))$, we see that

$$\lim_{x\to 0} \frac{\frac{h'(x)}{1+h(x)} - 1}{x} = -\frac{1}{3}.$$

L'Hospital's Rule applied to the second term on the right-hand side of (2.13) gives

$$\lim_{x\to 0} \frac{(1+h(x))^{\ln(c)/x} - c}{x} = \frac{-c\ln(c)}{6}. \tag{2.14}$$

In a similar fashion,

$$\lim_{x\to 0} \frac{g(x)^{\ln(c)/x} - 1}{x} = \ln(c) \lim_{x\to 0} \left[\frac{\frac{g'(x)}{g(x)}}{x} - \frac{\ln(g(x))}{x^2} \right]. \tag{2.15}$$

Observing that $g(x) = 1 - x^2/6 + O(x^4)$, we see that

$$\lim_{x\to 0} \frac{\frac{g'(x)}{g(x)}}{x} = -\frac{1}{3}.$$

L'Hospital's Rule applied to the second term on the right-hand side of (2.15) gives

$$\lim_{x\to 0} \frac{g(x)^{\ln(c)/x} - 1}{x} = -\frac{\ln(c)}{6}. \tag{2.16}$$

Combining (2.12), (2.14), and (2.16), we obtain

$$\lim_{d\to\infty} d\Big(\frac{A}{B} - D_d(\omega_d)\Big) = \frac{A}{6B}\Big[\ln\Big(1 + \frac{A}{B}\Big)\Big]^2.$$

2) For the second limit we have, (after simplification),

$$\frac{1}{d} D_d'(\omega_d) = \pi \left[\frac{\Big(1 + h\big(\frac{\ln(c)}{d}\big)\Big)^{\big(\ln(c)\big)/\big(\frac{\ln(c)}{d}\big)}}{1 + h\Big(\frac{\ln(c)}{d}\Big)} - \frac{g\Big(\frac{\ln(c)}{d}\Big)^{\big(\ln(c)\big)/\big(\frac{\ln(c)}{d}\big)}}{g\Big(\frac{\ln(c)}{d}\Big)} g'\Big(\frac{\ln(c)}{d}\Big) \right].$$

In light of the previous work, this yields

$$\lim_{d\to\infty} \frac{1}{d} D_d'(\omega_d) = \pi\Big(1 + \frac{A}{B}\Big).$$

3) To prove the third assertion, note that $(1 + h(\pi x_d))^d = \frac{A}{B} + g(\pi x_d)^d$ gives

$$\frac{1}{d} D_d'(x_d) = \pi \left[\frac{\frac{A}{B} + g(\pi x_d)^d}{1 + h(\pi x_d)} h'(\pi x_d) - \frac{g(\pi x_d)^d}{g(\pi x)} g'(\pi x_d) \right]. \tag{2.17}$$

Also, the first inequality in proposition II.27 shows that, for sufficiently large d (also large enough so that $x_d < \Delta$ and $\omega_d < \Delta$), $D_d(\omega_d) < \frac{A}{B} = D_d(x_d)$. This implies $\omega_d < x_d$ since D_d is increasing on $(0, 1/4)$. But D_d is also convex on $(0, \Delta)$, so we can conclude that

$$D_d'(\omega_d) < D_d'(x_d). \tag{2.18}$$

Combining this with (2.17), we obtain

$$\left[\frac{1}{d}D_d'(\omega_d) + \frac{\pi g(\pi x_d)^d}{g(\pi x_d)} g'(\pi x_d)\right]\left(\frac{1 + h(\pi x_d)}{h'(\pi x_d)}\right) < \pi\left(\frac{A}{B} + g(\pi x_d)^d\right) < \pi\left(1 + \frac{A}{B}\right).$$

The limit as $d \to \infty$ of the first term in the chain of inequalities above is $\pi\left(1 + \frac{A}{B}\right)$, so

$$\lim_{d\to\infty} \pi\left(\frac{A}{B} + g(\pi x_d)^d\right) = \pi\left(1 + \frac{A}{B}\right).$$

Combining this with (2.17), we obtain $\lim_{d\to\infty} \frac{1}{d} D_d'(x_d) = \pi\left(1 + \frac{A}{B}\right)$. □

Proof of Theorem II.23. For large d, the mean value theorem implies

$$\frac{D_d(x_d) - D_d(\omega_d)}{x_d - \omega_d} = D_d'(\xi), \quad \xi \in (\omega_d, x_d),$$

so that

$$x_d - \omega_d = \frac{\frac{A}{B} - D_d(\omega_d)}{D_d'(\xi)}.$$

For large d, convexity of D_d on $(0, \Delta)$ implies

$$\frac{d\left(\frac{A}{B} - D_d(\omega_d)\right)}{\frac{1}{d}D_d'(x_d)} < d^2(x_d - \omega_d) < \frac{d\left(\frac{A}{B} - D_d(\omega_d)\right)}{\frac{1}{d}D_d'(\omega_d)}.$$

Applying Proposition II.27 proves the theorem. □

CHAPTER III

MULTIVARIATE POLYNOMIAL INTERPOLATION AND BANDLIMITED FUNCTIONS

A. Introduction

This chapter is outlined as follows. Sections B and C introduce and develop the basic properties of uniformly invertible operators and Riesz bases, and give examples of such objects. Theorems III.26 and III.30 (the main results pertaining to polynomial interpolation and approximation) are established in Sections D and E, along with pertinent corollaries. Section F addresses the optimality of the growth rates appearing in Theorem III.30. The notion of uniform invertibility also leads to an alternative proof of a significant special case of Theorem I.1; this is the content of Section G.

B. Uniform invertibility of operators and Riesz bases

Given an exponential Riesz basis $(f_n)_{n\in\mathbb{Z}^d}$ for $L_2([-\pi,\pi]^d)$, Theorem I.1 and (2.7) clearly demonstrate the need to approximate $(f_n^*)_{n\in\mathbb{Z}^d}$ in a concrete manner. This motivates the concept of uniform invertibility; it is introduced in Section B, and plays a central role in subsequent sections. Informally speaking, a uniformly invertible Riesz basis is a Riesz basis $(f_n)_{n\in\mathbb{Z}^d}$ such that:

1) It can be obtained as a "limit" of a sequence of simpler Riesz bases, each one of which (except for finitely many terms) is an orthonormal basis.

2) The set of biorthogonal functions $(f_n^*)_{n\in\mathbb{Z}^d}$ of $(f_n)_{n\in\mathbb{Z}^d}$ is also a "limit" of the sets of biorthogonal functions of the simpler Riesz bases in 1). This is the most important feature of uniformly invertible Riesz bases, because the biorthogonal functions of the Riesz bases in 1) which we will examine are simply products of rational functions

and the SINC function. These notions are formalized in this section.

Definition III.1. Let H be a Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. If $(k_\ell)_{\ell\in\mathbb{N}}$ is a strictly increasing sequence of positive integers, define P_ℓ to be the orthogonal projection onto $\operatorname{span}\big(e_n\big)_{n\le k_\ell}$ for $\ell \in \mathbb{N}$.

Definition III.2. Let $L : H \to H$ be a bounded linear map. If $P_\ell L P_\ell : P_\ell H \to P_\ell H$ is invertible with inverse mapping $(P_\ell L P_\ell)^{-1}$, then extend $(P_\ell L P_\ell)^{-1}$ to H by defining

$$(P_\ell L P_\ell)^{-1}x := (P_\ell L P_\ell)^{-1}P_\ell x.$$

We note that this is a convenient abuse of notation, as $P_\ell L P_\ell$ is also a map from H to itself, and is certainly not invertible with that choice of domain and range.

Definition III.3. Let $L : H \to H$ be an onto isomorphism. L is *uniformly invertible with respect to the projections* $(P_\ell)_{\ell\in\mathbb{N}}$ if

1) $P_\ell L P_\ell : P_\ell H \to P_\ell H$ is invertible for $\ell \in \mathbb{N}$, and

2) $\sup_{\ell\in\mathbb{N}} \|(P_\ell L P_\ell)^{-1}\| < \infty$.

Definition III.4. A Riesz basis $(f_n)_{n\in\mathbb{N}^d}$ for H is a *uniformly invertible Riesz basis (UIRB) with respect to the projections* $(P_\ell)_{\ell\in\mathbb{N}}$ if the onto isomorphism defined by $Le_n = f_n$ is uniformly invertible with respect to the projections $(P_\ell)_{\ell\in\mathbb{N}}$.

Definition III.5. Given an operator L on H, we define the operator L_ℓ, $\ell \in \mathbb{N}$ by

$$L_\ell = LP_\ell + I - P_\ell.$$

We can now state and prove the following lemmas:

Lemma III.6. *Let* $(f_n)_{n\in\mathbb{N}} \subset H$, $(e_n)_{n\in\mathbb{N}}$ *be an orthonormal basis for* H, *and*

$$\operatorname{span}(e_n)_{n\in\mathbb{N}}$$

be the linear-algebraic span of $(e_n)_{n\in\mathbb{N}}$. *Define* $L : \mathrm{span}(e_n)_{n\in\mathbb{N}} \to H$ *by* $Le_n = f_n$. *For each* $\ell > 0$, *the following statements are equivalent:*

1) $(f_n)_{n\le k_\ell} \cup (e_n)_{n>k_\ell}$ *is a Riesz basis for* H.

2) $P_\ell L P_\ell : P_\ell H \to P_\ell H$ *is invertible.*

3) L_ℓ *is an onto isomorphism.*

Proof. 1) $\Longleftrightarrow$ 3) is immediate.

1) $\Longrightarrow$ 2): From the definition of L_ℓ we see that it extends to an onto isomorphism on H. This yields $P_\ell L_\ell = P_\ell L P_\ell$, which implies $P_\ell = P_\ell L P_\ell L_\ell^{-1}$, so that

$$P_\ell = (P_\ell L P_\ell)(P_\ell L_\ell^{-1} P_\ell),$$

whence $P_\ell L P_\ell$ is invertible, and

$$(P_\ell L P_\ell)^{-1} = P_\ell L_\ell^{-1} P_\ell. \tag{3.1}$$

2) $\Longrightarrow$ 1): It suffices to show that L_ℓ is an onto isomorphism.

Note that L_ℓ extends to a continuous map on H.

First we show that L_ℓ is one to one. Say $0 = L_\ell x = LP_\ell x + (I - P_\ell)x$, then $0 = P_\ell L P_\ell x$, so that $0 = (P_\ell L P_\ell)^{-1} P_\ell L P_\ell x = P_\ell x$. We conclude that $x = (I - P_\ell)x$. This implies

$$0 = L_\ell x = L_\ell (I - P_\ell)x = (I - P_\ell)x = x.$$

Next we show that L_ℓ is onto. Note $L_\ell(I - P_\ell)x = (I - P_\ell)x$, so we only need to show that for all x, $P_\ell x$ is in the range of L_ℓ. Given $x \in H$, define

$$y = (P_\ell L P_\ell)^{-1} x + P_\ell x - L(P_\ell L P_\ell)^{-1} x.$$

Then

$$\begin{aligned}
L_\ell y &= (LP_\ell + I - P_\ell)((P_\ell LP_\ell)^{-1}x + P_\ell x - L(P_\ell LP_\ell)^{-1}x) \\
&= (LP_\ell + I - P_\ell)(P_\ell LP_\ell)^{-1}x + (LP_\ell + I - P_\ell)P_\ell x \\
&\quad -(LP_\ell + I - P_\ell)L(P_\ell LP_\ell)^{-1}x \\
&= LP_\ell(P_\ell LP_\ell)^{-1}x + LP_\ell x - L(P_\ell LP_\ell)(P_\ell LP_\ell)^{-1}x - LP_\ell(P_\ell LP_\ell)^{-1}x \\
&\quad +(P_\ell LP_\ell)(P_\ell LP_\ell)^{-1}x, \\
&= P_\ell x
\end{aligned}$$

where we have used the following in the second and third lines:

$$P_\ell(P_\ell LP_\ell)^{-1} = (P_\ell LP_\ell)^{-1}.$$

Thus L_ℓ is a continuous bijection between Hilbert spaces. An application of the Banach Open Mapping Theorem shows that L_ℓ is an onto isomorphism. □

Lemma III.7. *Define L as in Lemma III.6. For each $\ell \in \mathbb{N}$, L_ℓ extends to an onto isomorphism on H if and only if it is one to one.*

Proof. One direction is immediate. Suppose that L_ℓ is one to one. It immediately extends to a bounded linear operator on H. By Lemma III.6, we only need to show that $P_\ell LP_\ell : P_\ell H \to P_\ell H$ is invertible. Finite dimensionality of $P_\ell H$ further reduces the problem to showing that $P_\ell LP_\ell : P_\ell H \to P_\ell H$ is one to one. Let $(P_\ell LP_\ell)P_\ell x = 0$. We have

$$\begin{aligned}
L_\ell(P_\ell x - (I - P_\ell)LP_\ell x) &= L_\ell P_\ell x - L_\ell(I - P_\ell)LP_\ell x \\
&= L_\ell P_\ell x - (LP_\ell + I - P_\ell)(I - P_\ell)LP_\ell x \\
&= L_\ell P_\ell x - (I - P_\ell)LP_\ell x.
\end{aligned}$$

Since L_ℓ is one to one, we have that $P_\ell x = (I-P_\ell)LP_\ell x$, so that $P_\ell x = 0$. $\square$

Lemma III.8. *Let $(f_n)_{n\in\mathbb{N}}$ be a Riesz basis for H, where $Le_n = f_n$. The following are equivalent:*

1) $(f_n)_{n\in\mathbb{N}}$ is a UIRB with respect to the projections $(P_\ell)_{\ell\in\mathbb{N}}$.

2) L_ℓ is an onto isomorphism for $\ell \in \mathbb{N}$, and

$$\sup_{\ell\in\mathbb{N}} \|L_\ell^{-1}\| < \infty.$$

Proof. 1) $\Longrightarrow$ 2): By Lemma III.6, we only need to show that $\sup_{\ell\in\mathbb{N}} \|L_\ell^{-1}\| < \infty$. This follows from the identity

$$L_\ell^{-1} = [I-(I-P_\ell)L](P_\ell L P_\ell)^{-1} + I - P_\ell, \tag{3.2}$$

which can be seen as follows:

$$\begin{aligned} & [I-(I-P_\ell)L](P_\ell L P_\ell)^{-1} + I - P_\ell \\ = \; & [I-(I-P_\ell)L]P_\ell L_\ell^{-1} P_\ell + I - P_\ell \quad (\textit{by eq. } (3.1)) \\ = \; & P_\ell L_\ell^{-1} P_\ell - (I-P_\ell)LP_\ell L_\ell^{-1} P_\ell + I - P_\ell \\ = \; & P_\ell L_\ell^{-1} P_\ell - LP_\ell L_\ell^{-1} P_\ell + (P_\ell L P_\ell)(P_\ell L_\ell^{-1} P_\ell) + I - P_\ell \\ = \; & P_\ell L_\ell^{-1} P_\ell - LP_\ell L_\ell^{-1} P_\ell + I \\ = \; & (I-L)P_\ell L_\ell^{-1} P_\ell + I. \end{aligned} \tag{3.3}$$

We have $(I-L)P_\ell = I - L_\ell$, so

$$\begin{aligned} [I-(I-P_\ell)L](P_\ell L P_\ell)^{-1} + I - P_\ell \; &= \; (I-L_\ell)L_\ell^{-1}P_\ell + I \qquad (3.4) \\ &= \; L_\ell^{-1}P_\ell - P_\ell + I. \end{aligned}$$

From the definition of L_ℓ, we see that $L_\ell(I-P_\ell) = I - P_\ell$. Composing from the left by L_ℓ^{-1} yields $I - P_\ell = L_\ell^{-1}(I - P_\ell)$. Rearranging, we have $L_\ell^{-1}P_\ell - P_\ell + I = L_\ell^{-1}$, which proves the identity.

2) $\Longrightarrow$ 1): This follows from (3.1). □

Lemma III.9 is the formal statement of 2) from the beginning of this section.

Lemma III.9. *If $(f_n)_{n\in\mathbb{N}}$ is a UIRB with respect to the projections $(P_\ell)_{\ell\in\mathbb{N}}$, where $Le_n = f_n$, then*

$$\lim_{\ell\to\infty}(L_\ell^*)^{-1}f = (L^*)^{-1}f, \quad \textit{for all } f \in H. \tag{3.5}$$

Proof. Note that

$$(L_\ell^*)^{-1} - (L^*)^{-1} = (L_\ell^*)^{-1}(L^* - L_\ell^*)(L^*)^{-1}$$

and

$$\lim_{\ell\to\infty} L_\ell^* f = L^* f, \quad \textit{for all } f \in H.$$

Applying Lemma III.8, we have (3.5). □

Lemma III.10. *Let $L : H \to H$, given by $Le_n = f_n$, be an onto isomorphism. The following are equivalent.*

1) $(f_n)_{n\in\mathbb{N}}$ is a UIRB with respect to the projections $(P_\ell)_{\ell\in\mathbb{N}}$.

2) For all $f \in H$, $\lim_{\ell\to\infty}(L_\ell^*)^{-1}(I - P_\ell)f = 0$.

Proof. Applying Lemma III.8, it is clear that 1) implies 2). For the other direction, note that the equality $L_\ell^* = P_\ell L^* + I - P_\ell$ implies that

$$I = (L_\ell^*)^{-1}P_\ell L^* + (L_\ell^*)^{-1}(I - P_\ell), \tag{3.6}$$

from which $((L_\ell^*)^{-1}P_\ell)_{\ell\in\mathbb{N}}$ is pointwise bounded. Together with the assumption in 2), this implies $((L_\ell^*)^{-1})_{\ell\in\mathbb{N}}$ is pointwise bounded, hence norm bounded by the Uniform Boundedness Principle. Noting that $\|(L_\ell^*)^{-1}\| = \|L_\ell^{-1}\|$ yields uniform invertibility of L. □

Lemma III.11. *Let $L : H \to H$, given by $Le_n = f_n$, be an onto isomorphism. The following are equivalent:*

1) For all $f \in H$, we have

$$f = \lim_{\ell\to\infty} (L_\ell^*)^{-1} P_\ell L^* f. \tag{3.7}$$

2) $(f_n)_{n\in\mathbb{N}}$ is a UIRB with respect to $(P_\ell)_{\ell\in\mathbb{N}}$.

Proof. Recall (3.6) and apply Lemma III.10. □

The next pair of propositions shows that uniform invertibility of an operator is preserved under appropriate small-norm or compact perturbations.

Proposition III.12. *Let $L : H \to H$ be a uniformly invertible operator with respect to $(P_\ell)_{\ell\in\mathbb{N}}$.*

1) If $0 \neq \liminf_{\ell\to\infty} \|(P_\ell L P_\ell)^{-1}\| =: M < \infty$, and A is an operator such that

$$\|L - A\| < \frac{1}{M},$$

then there exists a subsequence $(k_\ell)_{\ell\in\mathbb{N}}$ such that A is uniformly invertible with respect to $(P_{k_\ell})_{\ell\in\mathbb{N}}$.

2) If $\sup_{\ell\in\mathbb{N}} \|(P_\ell L P_\ell)^{-1}\| =: M < \infty$, and A is an operator such that

$$\|L - A\| < \frac{1}{M},$$

then A is uniformly invertible with respect to $(P_\ell)_{\ell\in\mathbb{N}}$.

Proof. Proof of 1). We first show that A is invertible for large ℓ. Let ℓ be large enough so that L_ℓ and $P_\ell L P_\ell$ are invertible. Equation (3.2) implies that

$$(L_\ell^*)^{-1} - (P_\ell L^* P_\ell)^{-1} = [I - (P_\ell L^* P_\ell)^{-1} L^*](I - P_\ell),$$

yielding

$$\lim_{\ell\to\infty}(L_\ell^*)^{-1}f-(P_\ell L^*P_\ell)^{-1}f=0,\quad \text{for all } f\in H. \tag{3.8}$$

Equations (3.5) and (3.8) show

$$\lim_{\ell\to\infty}(P_\ell L^*P_\ell)^{-1}f=(L^*)^{-1}f,\quad \text{for all } f\in H.$$

The equality $\|L^{-1}\|=\|(L^*)^{-1}\|$ implies

$$\|L^{-1}\|\le\liminf_{\ell\to\infty}\|(P_\ell LP_\ell)^{-1}\|. \tag{3.9}$$

There exists $\gamma<1$ such that

$$\|L-A\|\le\frac{\gamma}{M}, \tag{3.10}$$

Equations (3.9) and (3.10) yield $\|L-A\|\le\frac{\gamma}{\|L^{-1}\|}$, implying $\|I-L^{-1}A\|\le\gamma$, so that

$$A^{-1}=\sum_{k=0}^{\infty}(I-L^{-1}A)^kL^{-1}$$

by standard Neumann series manipulation.

We now verify that $(P_{k_\ell}AP_{k_\ell})^{-1}$ is well-defined for some sequence $(k_\ell)_{\ell\in\mathbb{N}}$, and that the norms are bounded. Equation (3.9) implies

$$\|P_\ell LP_\ell-P_\ell AP_\ell\|\le\frac{\gamma}{M},$$

so that

$$\|P_\ell-(P_\ell LP_\ell)^{-1}(P_\ell AP_\ell)\|\le\frac{\gamma}{M}\|(P_\ell LP_\ell)^{-1}\|.$$

This yields

$$\liminf_{\ell\to\infty}\|P_\ell-(P_\ell LP_\ell)^{-1}(P_\ell AP_\ell)\|\le\gamma.$$

Therefore there exists a sequence $(k_\ell)_{\ell\in\mathbb{N}}$ such that

$$\|P_{k_\ell}-(P_{k_\ell}LP_{k_\ell})^{-1}(P_{k_\ell}AP_{k_\ell})\|\le\frac{\gamma+1}{2}<1.$$

Again, Neumann series manipulation shows that

$$(P_{k_\ell} A P_{k_\ell})^{-1} = \sum_{j=0}^{\infty} [P_{k_\ell} - (P_{k_\ell} L P_{k_\ell})^{-1} (P_{k_\ell} A P_{k_\ell})]^j (P_{k_\ell} L P_{k_\ell})^{-1},$$

and

$$\sup_{\ell \in \mathbb{N}} \|(P_{k_\ell} A P_{k_\ell})^{-1}\| \leq \frac{2}{1-\gamma} \sup_{\ell \in \mathbb{N}} \|(P_{k_\ell} L P_{k_\ell})^{-1}\| < \infty.$$

Proof of 2). Modify the proof above in the obvious way. □

Proposition III.13. *Let $L : H \to H$ be uniformly invertible with respect to the projections $(P_\ell)_{\ell \in \mathbb{N}}$. If $\Delta : H \to H$ is a compact operator such that $\tilde{L} := L + \Delta$ is an onto isomorphism, then there exists $N > 0$ such that $\tilde{L}$ is uniformly invertible with respect to the projections $(P_\ell)_{\ell \geq N}$.*

Proof. From the definition of L_ℓ, we have

$$I = (I - P_\ell) L_\ell^{-1} + L P_\ell L_\ell^{-1},$$

so that

$$L^{-1}(P_\ell - I) L_\ell^{-1} = P_\ell L_\ell^{-1} - L^{-1}$$

for sufficiently large ℓ. This implies

$$(L_\ell^*)^{-1} P_\ell - (L^*)^{-1} = (L_\ell^*)^{-1} (P_\ell - I)(L^*)^{-1}. \tag{3.11}$$

As $\ell \to \infty$, the right-hand side of (3.11) has 0 limit pointwise. Combined with the compactness of Δ^*, we obtain

$$\lim_{\ell \to \infty} (L_\ell^*)^{-1} P_\ell \Delta^* = (L^*)^{-1} \Delta^* \tag{3.12}$$

where the limit is in the operator norm topology. Taking the adjoint of each term in

(3.12) and adding the identity yields

$$\lim_{\ell\to\infty}\left(I+\Delta P_\ell L_\ell^{-1}\right)=I+\Delta L^{-1}=(L+\Delta)L^{-1}, \tag{3.13}$$

where the limit is also in the operator norm topology. The right-hand side of (3.13) is an onto isomorphism, so there exists N such that $\ell \geq N$ implies $I+\Delta P_\ell L_\ell^{-1}$ is an onto isomorphism, and that

$$\lim_{\ell\to\infty}(I+\Delta P_\ell L_\ell^{-1})^{-1}=L(L+\Delta)^{-1}.$$

This yields

$$\sup_{\ell\geq N}\|(I+\Delta P_\ell L_\ell^{-1})^{-1}\|<\infty. \tag{3.14}$$

Defining $\tilde{L}_\ell=\tilde{L}P_\ell+I-P_\ell$, we obtain

$$\tilde{L}_\ell=L_\ell+\Delta P_\ell=(I+\Delta P_\ell L_\ell^{-1})L_\ell.$$

When $\ell \geq N$, we have

$$\tilde{L}_\ell^{-1}=L_\ell^{-1}(I+\Delta P_\ell L_\ell^{-1})^{-1},$$

and (3.14) implies

$$\sup_{\ell\geq N}\|\tilde{L}_\ell^{-1}\|\leq \sup_{\ell\geq N}\|L_\ell^{-1}\|\sup_{\ell\geq N}\|(I+\Delta P_\ell L_\ell^{-1})^{-1}\|<\infty,$$

from which uniform invertibility of $\tilde{L}$ follows. □

C. Examples of uniformly invertible exponential Riesz bases

Our main results, to wit, Theorems III.26 and III.30 to follow, are stated in terms of UIRBs. We demonstrate here that this is indeed a fairly wide class.

Definition III.14. Define $C_{\ell,d}=\{-\ell,\cdots,\ell\}^d$.

Definition III.15. For $\ell \in \mathbb{N}$, define $P_\ell : L_2([-\pi,\pi]^d) \to L_2([-\pi,\pi]^d)$ to be the orthogonal projection onto $\operatorname{span}(e_n)_{n\in C_{\ell,d}}$.

Theorems III.16 and III.17 show that some earlier examples of exponential Riesz bases (and simple modifications thereof) are UIRBs.

Theorem III.16. *The Riesz bases given in Theorems II.20 and II.22 are UIRBs with respect to the projections $(P_\ell)_{\ell\in\mathbb{N}}$ from Definition III.15.*

Proof. The proofs of Theorems II.20 and II.22 in [17] and [13] rely on the fact that the map $Ae_n = f_n$ satisfies $\|I - A\| = \delta < 1$. Apply Theorem III.12 for $L = I$. □

Theorem III.17. *Let D_d and x_d be as in Theorem II.20. Let $(t_n)_{n\in\mathbb{Z}^d} \subset \mathbb{R}^d$ be a sequence satisfying either*

$$1) \qquad \limsup_{\|n\|_\infty\to\infty} \|\tau_n - n\|_\infty < x_d, \quad D_d(x_d) = 1, \quad 0 < x_d \le 1/4, \quad \text{or}$$

$$2) \qquad \limsup_{\|n\|_\infty\to\infty} \|\tau_n - n\|_\infty < \frac{\ln(2)}{\pi d}.$$

If $\left(\frac{1}{(2\pi)^{d/2}} e^{i\langle(\cdot),\tau_n\rangle}\right)_{n\in\mathbb{Z}^d}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$, then there exists $N > 0$ such that it is a UIRB with respect to $(P_\ell)_{\ell\ge N}$ (a subset of the projections from Definition III.15).

The proof of Theorem III.17 relies on Corollary III.18 and Corollary III.19.

Corollary III.18. *Given two sequences $(t_n)_{n\in\mathbb{Z}^d} \subset \mathbb{R}^d$ and $(\tau_n)_{n\in\mathbb{Z}^d} \subset \mathbb{R}^d$, define $(f_n)_{n\in\mathbb{Z}^d}$ and $(g_n)_{n\in\mathbb{Z}^d}$ by $f_n(\cdot) = \frac{1}{(2\pi)^{d/2}} e^{i\langle\cdot,t_n\rangle}$ and $g_n(\cdot) = \frac{1}{(2\pi)^{d/2}} e^{i\langle\cdot,\tau_n\rangle}$. If $(f_n)_{n\in\mathbb{Z}^d}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$, and*

$$\lim_{\|n\|_\infty\to\infty} \|t_n - \tau_n\|_\infty = 0,$$

then the operator K defined by $Ke_n = f_n - g_n$ is compact.

Proof. If $Le_n = f_n$, then certainly

$$\Big\| \sum_{n\in\mathbb{Z}^d} a_n f_n \Big\| \le \|L\| \Big(\sum_{n\in\mathbb{Z}^d} |a_n|^2 \Big)^{1/2}, \quad \text{for all} \ \ (a_n)_{n\in\mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d).$$

Let $f = \sum_{n\in\mathbb{Z}^d}^{\infty} a_n e_n$, where $\sum_{n\in\mathbb{Z}^d} |a_n|^2 = 1$. Then by Lemma II.17,

$$\begin{aligned} & \|(K - KP_\ell)f\| \\ = \ & \Big\| \sum_{\|n\|_\infty > \ell+1} a_n(f_n - g_n) \Big\| \le \|L\| \Big(e^{\left(\sup_{\|n\|_\infty \ge \ell+1} \|t_n - \tau_n\|_\infty\right)} - 1 \Big) \|(I - P_\ell)f\| \\ \le \ & \|L\| \Big(e^{\left(\sup_{\|n\|_\infty \ge \ell+1} \|t_n - \tau_n\|_\infty\right)} - 1 \Big) \to_{\ell\to\infty} 0. \end{aligned}$$

As K is the limit of finite rank operators in the operator norm topology it is compact.

□

Corollary III.19. *Let $(t_n)_{n\in\mathbb{Z}^d}$, $(\tau_n)_{n\in\mathbb{Z}^d}$, $(f_n)_{n\in\mathbb{Z}^d}$, and $(g_n)_{n\in\mathbb{Z}^d}$ be defined as in Corollary III.18. If $(f_n)_{n\in\mathbb{Z}^d}$ is a UIRB with respect to a set of projections $(P_\ell)_{\ell\in\mathbb{N}}$, and $(g_n)_{n\in\mathbb{Z}^d}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$, then there exists $N > 0$ such that $(g_n)_{n\in\mathbb{Z}^d}$ is a UIRB with respect to $(P_\ell)_{\ell\ge N}$.*

Proof. Apply Proposition III.13 and Corollary III.18. □

Proof of Theorem III.17. Apply Theorem III.16 and Corollary III.19. □

Simple examples show that in Theorem III.17, the assumption that

$$\Big(\frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, \tau_n \rangle} \Big)_{n\in\mathbb{Z}^d}$$

is a Riesz basis for $L_2([-\pi,\pi]^d)$ cannot be dropped when $d \ge 2$. Example: The standard exponential orthonormal basis $(e_n)_{n\in\mathbb{Z}^d}$ is of course uniformly invertible, but the set

$$\Big(\frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, (1,1/2,0,\cdots,0) \rangle} \Big) \cup (e_n)_{n\ne 0}$$

is not a Riesz basis, as

$$e^{i\langle\cdot,(1,1/2,0,\cdots,0)\rangle} \in \overline{\operatorname{span}\left(e^{i\langle\cdot,(1,n,0,\cdots,0)\rangle}\right)_{n\in\mathbb{Z}}}.$$

However, this condition can be dropped when $d = 1$. This follows from the following theorem.

Theorem III.20. *Let $(t_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence such that $(f_n)_{n\in\mathbb{Z}} = \left(\frac{1}{\sqrt{2\pi}}e^{it_n(\cdot)}\right)_{n\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi,\pi]$. If $(\tau_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ is a sequence of distinct points such that*

$$\lim_{|n|\to\infty} |t_n - \tau_n| = 0,$$

then $(g_n)_{n\in\mathbb{Z}} = \left(\frac{1}{\sqrt{2\pi}}e^{i\tau_n(\cdot)}\right)_{n\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi,\pi]$.

The proof of Theorem III.20 relies on Lemma III.21 below, which appears as Lemma 3.1 in [19]. The proof of Lemma III.21 found in [19] itself relies on a citation, so for the sake of completeness Lemma III.21 is presented here with a self-contained proof.

Lemma III.21. *Let $(f_n)_{n\in\mathbb{Z}}$ be an exponential Riesz basis for $L_2[-\pi,\pi]$. If finitely many terms in $(f_n)_{n\in\mathbb{Z}}$ are replaced by arbitrary complex exponential functions, then the resulting sequence (provided it consists of distinct functions) is a Riesz basis for $L_2[-\pi,\pi]$.*

Proof. If we can prove the case when we make only one replacement, the general result follows inductively. Let $f_n(\cdot) = \frac{1}{\sqrt{2\pi}}e^{it_n(\cdot)}$ for $n \neq 0$, and $g_0(\cdot) = \frac{1}{\sqrt{2\pi}}e^{i\tau_0(\cdot)}$ where $\tau_0 \in \mathbb{R}$ and $\tau_0 \neq t_n$ for $n \neq 0$. We will prove that $(g_0) \cup (f_n)_{n\neq 0}$ is a Riesz basis by proving a) and b) below:

a) Let $(f_n^*)_{n\in\mathbb{Z}}$ be the biorthogonal basis for $(f_n)_{n\in\mathbb{Z}}$. Then $(g_0) \cup (f_n)_{n\neq 0}$ is a Riesz basis if $\langle g_0, f_0^*\rangle \neq 0$.

b) The inequality $\langle g_0, f_0^* \rangle \neq 0$ holds.

Proof of a). Let $(e_n)_{n\in\mathbb{Z}}$ be an orthonormal basis for $L_2[-\pi,\pi]$. If we show that the bounded linear map $T : L_2[-\pi,\pi] \to L_2[-\pi,\pi]$

$$Te_n = \begin{cases} g_0, & n = 0; \\ f_n, & n \neq 0 \end{cases}$$

is one to one and onto, then the Banach Open Mapping Theorem asserts that T is an onto isomorphism, and we are done. The relation $\langle g_0, f_0^* \rangle \neq 0$ quickly implies that T is one to one, so we only need to show that T is onto. Since $(f_n)_{n\in\mathbb{Z}}$ is a Riesz basis and $(f_n)_{n\neq 0} = (Te_n)_{n\neq 0}$, it suffices to show that $f_0 = T\phi$ for some $\phi \in L_2[-\pi,\pi]$. Rearrangement of

$$g_0 = \sum_{n\in\mathbb{Z}} \langle g_0, f_n^* \rangle f_n = \langle g_0, f_0^* \rangle f_0 + \sum_{n\neq 0} \langle g_0, f_n^* \rangle f_n$$

yields

$$f_0 = \frac{1}{\langle g_0, f_0^* \rangle} g_0 - \sum_{n\neq 0} \frac{\langle g_0, f_n^* \rangle}{\langle g_0, f_0^* \rangle} f_n = T\Big(\frac{1}{\langle g_0, f_0^* \rangle} e_0 - \sum_{n\neq 0} \frac{\langle g_0, f_n^* \rangle}{\langle g_0, f_0^* \rangle} e_n \Big).$$

Proof of b). After passing to the Fourier transform and recalling that $G_0 = \mathcal{F} f_0^*$, we note that $\langle g_0, f_0^* \rangle \neq 0$ is equivalent to $G_0(\tau_0) \neq 0$. If we can show that the only zeros of G_0 in $\mathbb{R}$ are $(t_n)_{n\neq 0}$, we are done. Suppose there exists $\lambda \in \mathbb{R}$, $\lambda \notin (t_n)_{n\neq 0}$ such that $G_0(\lambda) = 0$ with multiplicity m. Define the entire function

$$H(t) = \frac{(t_0 - \lambda)^m}{(t-\lambda)^m} G_0(t).$$

Note that $H|_{\mathbb{R}} \in L_2(\mathbb{R})$, and H is of exponential type π, so $H \in PW_{[-\pi,\pi]}$ by Theorem II.3. The expansion

$$H(t) = \sum_{n\in\mathbb{Z}} H(t_n) G_n(t),$$

combined with $H(t_n) = \delta_{n,0}$, shows that $H(t) = G_0(t)$ for all $t \in \mathbb{R}$, an immediate contradiction. We conclude that $G_0(\lambda) \neq 0$. $\square$

Proof of Theorem III.20. Define $Le_n = f_n$ and $\tilde{L}e_n = g_n$. By Corollary III.18, $\tilde{L}$ is bounded linear and $\tilde{L} = L + \Delta$ for some compact operator Δ. Define the operator

$$R_\ell e_n = \begin{cases} f_n, & |n| \leq \ell; \\ g_n, & |n| > \ell \end{cases}.$$

Rewritten, we have

$$R_\ell = LP_\ell + (L + \Delta)(I - P_\ell) = L + \Delta(I - P_\ell).$$

Compactness of Δ implies that $\lim_{\ell\to\infty} R_\ell = L$ in the operator norm topology. We conclude that R_{ℓ_0} is an onto isomorphism for some ℓ_0 sufficiently large; that is, the set

$$(f_n)_{|n|\leq\ell_0} \cup (g_n)_{|n|>\ell_0} \tag{3.15}$$

is a Riesz basis for $L_2[-\pi, \pi]$. If we apply Lemma III.21, by replacing $(f_n)_{|n|\leq\ell_0}$ with $(g_n)_{|n|\leq\ell_0}$ in expression (3.15), we have that $(g_n)_{n\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi, \pi]$. $\square$

D. The first main result

For the remainder of this chapter we use the unitary d-dimensional L_2 Fourier transform

$$\mathcal{F}(f)(\cdot) = \text{P.V. } \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{-i\langle\cdot,\xi\rangle} d\xi, \quad f \in L_2(\mathbb{R}^d),$$

where the inverse transform is given by

$$\mathcal{F}^{-1}(f)(\cdot) = \text{P.V. } \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{i\langle\cdot,\xi\rangle} d\xi, \quad f \in L_2(\mathbb{R}^d).$$

To avoid confusion of indices, we write $t \in \mathbb{R}^d$ as $t = (t(1), \cdots, t(d))$.

From here to the end of this chapter, if a sequence of points $(t_n)_{n\in\mathbb{Z}^d}$ is specified, the sequence $(f_n)_{n\in\mathbb{Z}^d}$ is given by $\left(\frac{1}{(2\pi)^{d/2}} e^{i\langle\cdot,t_n\rangle}\right)_{n\in\mathbb{Z}^d}$.

Definition III.22. If $\ell > 0$, and $(t_n)_{n\in\mathbb{Z}^d}$ is specified, the sequence $(f_{\ell,n})_{n\in\mathbb{Z}^d}$ refers to

$$(f_n)_{n\in C_{\ell,d}} \cup (e_n)_{n\notin C_{\ell,d}}.$$

Definition III.23. If any Riesz basis $(f_n)_n$ for $L_2([-\pi,\pi]^d)$ is specified with biorthogonal functions $(f_n^*)_n$, the sequence $(G_n)_n$ is defined by $G_n = \mathcal{F} f_n^*$ (see Corollary II.11).

Definition III.24. For $\ell, d \in \mathbb{N}$, define the multivariate polynomial

$$Q_{d,\ell}(t) = \prod_{k_1=1}^{\ell} \left(1 - \frac{t(1)^2}{k_1^2}\right) \cdot \ldots \cdot \prod_{k_d=1}^{\ell} \left(1 - \frac{t(d)^2}{k_d^2}\right), \quad t = (t(1), \cdots, t(d)).$$

We note that the function $t \mapsto \frac{\text{SINC}(\pi t)}{Q_{d,\ell}(t)}$ has removable discontinuities which can be computed according to the formula

$$\lim_{t\to n} \frac{\text{sinc}(\pi t)}{Q_{1,\ell}(t)} = \frac{(\ell!)^2}{(\ell+n)!(\ell-n)!}, \quad n \in \{-\ell, \ldots \ell\}.$$

For all t, $\frac{\text{SINC}(\pi t)}{Q_{d,\ell}(t)}$ is taken to mean $\lim_{\tau\to t} \frac{\text{SINC}(\pi\tau)}{Q_{d,\ell}(\tau)}$. The same is true for the reciprocal.

Definition III.25. If $p(x_1, \cdots, x_d)$ is a multivariate polynomial, the *coordinate degree* of p is the maximum degree of p in x_i for any index i.

Hereafter, $(P_\ell)_{\ell\in\mathbb{N}}$ will refer to the projections from Definition III.15. An analogous version of Theorem III.26 (in contrast to its current statement) holds if $(f_n)_{n\in\mathbb{Z}^d}$ is a UIRB for any subsequence of $(P_\ell)_{\ell\in\mathbb{N}}$. The proof (up to a trivial re-indexing) is identical, and the examples of UIRBs from the previous section do not warrant such generality. For the sake of simplicity, we choose not to pass to a subsequence. Since there is no ambiguity, "$(f_n)_{n\in\mathbb{Z}^d}$ is a UIRB for $(P_\ell)_{\ell\in\mathbb{N}}$" will be abbreviated by

"$(f_n)_{n\in\mathbb{Z}^d}$ is a UIRB". Inner products are all denoted by $\langle\cdot,\cdot\rangle$. The underlying Hilbert space, be it $\mathbb{R}^d$, $L_2([-\pi,\pi]^d)$, ℓ_2 or $PW_{[-\pi,\pi]^d}$ will be clear from context. Unless it is explicitly stated otherwise, all norms are Hilbert space norms.

The following is the first main result of this chapter.

Theorem III.26. *Let $(t_n)_{n\in\mathbb{Z}^d} \subset \mathbb{R}^d$, and suppose that $(f_n)_{n\in\mathbb{Z}^d}$ is a UIRB. Given $f \in PW_{[-\pi,\pi]^d}$, there exists a unique sequence of polynomials $(\Psi_\ell)_{\ell\in\mathbb{N}}$, $\Psi_\ell : \mathbb{R}^d \to \mathbb{R}$, such that*

(a) Ψ_ℓ has coordinate degree at most 2ℓ.

(b) $\Psi_\ell(t_n) = f(t_n)$ for all $n \in C_{\ell,d}$.

(c) $f(t) = \lim_{\ell\to\infty} \Psi_\ell(t)\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)}$, where the limit is in both L_2 and uniform senses.

This paragraph outlines the broad strokes in the proof of Theorem III.26. As $(f_n)_{n\in\mathbb{Z}^d}$ is a UIRB for $L_2([-\pi,\pi]^d)$, $(f_{\ell,n})_{n\in\mathbb{Z}^d}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$, as in Definition III.22. Using (2.4) to expand each function in the biorthogonal system $(G_{\ell,n})_{n\in\mathbb{Z}^d}$, (see Definition III.23), we find that $G_{\ell,n}$ is a rational function times a SINC function. Examination of this rational function shows the existence of polynomials $p_{\ell,n}(t)$, where the coordinate degree of each polynomial $p_{\ell,n}$ is at most 2ℓ, and $p_{\ell,n}(t_m) = \delta_{nm}$ for $n,m \in C_{\ell,d}$. The existence of polynomials satisfying (a) and (b) follows. Simple estimates show that for large ℓ,

$$G_{\ell,n}(t) \simeq p_{\ell,n}(t)\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)}. \tag{3.16}$$

If we expand $f \in PW_{[-\pi,\pi]^d}$ against $(G_{\ell,n})_n$, we have

$$f(t) = \sum_{n\in C_{\ell,d}} f(t_n)G_{\ell,n}(t) + \sum_{n\notin C_{\ell,d}} f(n)G_{\ell,n}(t).$$

Uniform invertibility shows that the second sum can always be neglected for large ℓ. For statement (c) combine the expression above with (3.16):

$$f(t) \simeq \sum_{n\in C_{\ell,d}} f(t_n)G_{\ell,n}(t) \simeq \Big(\sum_{n\in C_{\ell,d}} f(t_n)p_{\ell,n}(t)\Big)\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)}.$$

The proof of Theorem III.26 requires several lemmas, beginning with the following equivalence between the existence of particular Riesz bases and a polynomial interpolation condition:

Lemma III.27. *Let $(t_n)_{n\in\mathbb{Z}^d} \subset \mathbb{R}^d$. The sequence $(f_{\ell,n})_{n\in\mathbb{Z}^d}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$ if and only if both of the following conditions hold:*

1) For all $n \in C_{\ell,d}$, $t_n \in \big(\mathbb{R}\setminus(\mathbb{Z}\setminus\{-\ell,\cdots,\ell\})\big)^d$.

2) For any sequence $(c_k)_{k\in C_{\ell,d}}$, there exists a unique polynomial Ψ_ℓ with coordinate degree at most 2ℓ such that $\Psi_\ell(t_k) = c_k$ for $k \in C_{\ell,d}$.

Proof. Suppose that the sequence $(f_{\ell,n})_{n\in\mathbb{Z}^d}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$. We compute the functions $G_{\ell,n}$, when $n \in C_{\ell,d}$, by (2.6) and (2.4):

$$\begin{aligned} G_{\ell,n}(t) &= \sum_{k\in C_{\ell,d}} G_{\ell,n}(k)\mathrm{SINC}\pi(t-k) \qquad (3.17)\\ &= \Bigg(\sum_{k\in C_{\ell,d}} \frac{G_{\ell,n}(k)(-1)^{k(1)+\ldots+k(d)}t(1)\cdot\ldots\cdot t(d)}{(t(1)-k(1))\cdot\ldots\cdot(t(d)-k(d))}\Bigg)\mathrm{SINC}(\pi t), \quad t\in\mathbb{R}^d. \end{aligned}$$

Denote the k^{th} summand in (3.17) by $A_{\ell,n,k}$, then

$$\begin{aligned} A_{\ell,n,k} &= A_{\ell,n,k}\frac{\prod_{1\le i\le d}\Big(\prod_{j_i\in\{-\ell,\cdots,\ell\}\setminus\{k(i)\}}(t(i)-j_i)\Big)}{\prod_{1\le i\le d}\Big(\prod_{j_i\in\{-\ell,\cdots,\ell\}\setminus\{k(i)\}}(t(i)-j_i)\Big)}\\ &= \frac{G_{\ell,n}(k)(-1)^{k(1)+\ldots+k(d)}t(1)\cdot\ldots\cdot t(d)\prod_{1\le i\le d}\Big(\prod_{j_i\in\{-\ell,\cdots,\ell\}\setminus\{k(i)\}}(t(i)-j_i)\Big)}{\prod_{1\le i\le d}\Big(\prod_{j_i\in\{-\ell,\cdots,\ell\}}(t(i)-j_i)\Big)}\\ &= \frac{G_{\ell,n}(k)\frac{1}{(\ell!)^{2d}}(-1)^{k(1)+\ldots+k(d)+\ell d}\prod_{1\le i\le d}\Big(\prod_{j_i\in\{-\ell,\cdots,\ell\}\setminus\{k(i)\}}(t(i)-j_i)\Big)}{\prod_{j_1=1}^{\ell}\Big(1-\frac{t(1)^2}{j_1^2}\Big)\cdot\ldots\cdot\prod_{j_d=1}^{\ell}\Big(1-\frac{t(d)^2}{k_d^2}\Big)} = \frac{p_{\ell,n,k}(t)}{Q_{d,\ell}(t)}, \end{aligned}$$

where $p_{\ell,n,k}$ is some polynomial with coordinate degree at most 2ℓ. Substituting into equation (3.17), we obtain

$$G_{\ell,n}(t) = \Big(\sum_{k\in C_{\ell,d}} p_{\ell,n,k}(t)\Big)\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)} := \phi_{\ell,n}(t)\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)},$$

where $\phi_{\ell,n}$ is a polynomial having coordinate degree at most 2ℓ. This yields the equation

$$1 = \phi_{\ell,n}(t_n)\Big(\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)}\Big)\Big|_{t_n},$$

which shows that

$$\phi_{\ell,n}(t_n) \neq 0 \quad \text{and} \quad \frac{\mathrm{SINC}(\pi t_n)}{Q_{d,\ell}(t_n)} \neq 0. \tag{3.18}$$

The fact that

$$\frac{\mathrm{sinc}(\pi t)}{Q_{1,\ell}(t)} = 0 \quad \text{if and only if} \quad t \in \mathbb{Z}\setminus\{-\ell,\cdots,\ell\}$$

implies that

$$\frac{\mathrm{SINC}(\pi t_n)}{Q_{d,\ell}(t_n)} \neq 0 \quad \text{if and only if} \quad t_n \in \big(\mathbb{R}\setminus(\mathbb{Z}\setminus\{-\ell,\cdots,\ell\})\big)^d,$$

which proves the first assertion.

For $n, m \in C_{\ell,d}, n \neq m$,

$$0 = G_{\ell,n}(t_m) = \phi_{\ell,n}(t_m)\frac{\mathrm{SINC}(\pi t_m)}{Q_{d,\ell}(t_m)}. \tag{3.19}$$

From (3.18) and (3.19), we conclude that

$$\phi_{\ell,n}(t_m) = \begin{cases} \frac{Q_{d,\ell}(t_n)}{\mathrm{SINC}\pi t_n} \neq 0, & n = m; \\ 0, & n \neq m \end{cases},$$

for $n, m \in C_{\ell,d}$. From this, the "existence" part of assertion 2) readily follows. Restated, the evaluation map taking the space of all polynomials of coordinate degree at most 2ℓ to $\mathbb{R}^{(2\ell+1)^d}$ is onto. These spaces have the same dimension, hence the evaluation map is a bijection, and this completes the proof of 2).

Suppose that 1) and 2) hold. For $n \in C_{\ell,d}$, let $p_{\ell,n}$ be the unique polynomial of coordinate degree at most 2ℓ such that $p_{\ell,n}(t_m) = \delta_{nm}$ for $m \in C_{\ell,d}$. Define

$$\begin{aligned}\Phi_{\ell,n}(t) &:= \frac{Q_{d,\ell}(t_n)\mathrm{SINC}\pi t}{Q_{d,\ell}(t)\mathrm{SINC}\pi t_n} p_{\ell,n}(t) \\ &= \frac{\left(\frac{Q_{d,\ell}(t_n)}{\mathrm{SINC}\pi t_n}\right) p_{\ell,n}(t(1),\cdots,t(d)) \sin(\pi t(1)) \cdot \ldots \cdot \sin(\pi t(d))}{\pi t(1) \prod_{j_1=1}^{\ell}\left(1 - \frac{t(1)^2}{j_1^2}\right) \cdot \ldots \cdot \pi t(d) \prod_{j_d=1}^{\ell}\left(1 - \frac{t(d)^2}{k_d^2}\right)}. \end{aligned} \tag{3.20}$$

If, in (3.20), we sequentially apply partial fraction decomposition in each real variable $t(1), \cdots, t(d)$, we see that $\Phi_{\ell,n}(t)$ is of the form

$$\Phi_{\ell,n}(\cdot) = \sum_{n \in C_{\ell,d}} a_n \mathrm{SINC}\pi(\cdot - n) \in PW_{[-\pi,\pi]^d}.$$

Therefore, by (2.1),

$$\delta_{n,m} = \Phi_{\ell,n}(t_m) = \langle \Phi_{\ell,n}(\cdot), \mathrm{SINC}\pi(\cdot - t_m)\rangle = \langle \mathcal{F}^{-1}(\Phi_{\ell,n}), f_m\rangle, \quad n, m \in C_{\ell,d},$$

and $\Phi_{\ell,n}(m) = 0$ when $m \notin C_{\ell,d}$. Define the map L_ℓ by $L_\ell e_n = f_{\ell,n}$. Let $f = \sum_{n\in\mathbb{Z}^d} c_n e_n$ such that $L_\ell f = 0$. Then

$$0 = \sum_{n \in C_{\ell,d}} c_n f_n + \sum_{n \notin C_{\ell,d}} c_n e_n.$$

If, for each $n \in C_{\ell,d}$ we integrate the above equation against $\mathcal{F}^{-1}(\Phi_{\ell,n})$, we see that $c_n = 0$ for $n \in C_{\ell,d}$, so that $c_n = 0$ for all $n \in \mathbb{Z}^d$. Thus L_ℓ is one to one, so by Lemma III.7, it is an onto isomorphism from $L_2([-\pi,\pi]^d)$ to itself. □

Proof of (a) and (b) of Theorem III.26. Lemmas III.8 and III.27 imply the existence of a unique sequence of polynomials satisfying requirements (a) and (b) of Theorem

III.26, namely,

$$\Psi_\ell(t) = \sum_{n \in C_{\ell,d}} f(t_n) p_{\ell,n}(t),$$

where $p_{\ell,n}$ is defined as in the proof of Lemma III.27. $\square$

It remains to show that this sequence of polynomials satisfies condition (c) of Theorem III.26; this is accomplished with the aid of the following propositions.

Proposition III.28. *Let $(t_n)_{n\in\mathbb{Z}^d}$ be any sequence in $\mathbb{R}^d$. The following are true:*
1) $\sup_{x\in\mathbb{R}} \sup_{\ell\in\mathbb{N}} \left|\frac{\operatorname{sinc}(\pi x)}{Q_{1,\ell}(x)}\right| = 1.$
2) If $\Delta_{\ell,d} = \left\{ n \in \mathbb{Z}^d \;\middle|\; \left\|\frac{t_n}{\ell+1}\right\|_\infty < \frac{1}{\ell^{2/3}} \right\}$ *for* $\ell \in \mathbb{N}$, *then*

$$0 \leq 1 - \frac{\operatorname{SINC}(\pi t_n)}{Q_{d,\ell}(t_n)} < 1 - e^{\frac{-d(\ell+2)}{\ell^{4/3}-1}}, \quad n \in \Delta_{\ell,d}. \tag{3.21}$$

Proof. For 1), the identity

$$\operatorname{sinc}(\pi t) = \prod_{k=1}^{\infty} \left(1 - \frac{t^2}{k^2}\right)$$

implies

$$\frac{\operatorname{sinc}(\pi t)}{Q_{1,\ell}(t)} = \prod_{k=\ell+1}^{\infty} \left(1 - \frac{t^2}{k^2}\right), \tag{3.22}$$

where convergence is uniform on compact subsets of $\mathbb{C}$. Fix $\ell \in \mathbb{N}$. If $t \in [0, \ell+1]$, then $\left|\frac{\operatorname{sinc}(\pi t)}{Q_{1,\ell}(t)}\right| \leq 1$. Note that $|Q_{1,\ell}(t)| = \prod_{k=1}^{\ell} \left(\frac{t^2}{k^2} - 1\right)$ is increasing on $(\ell+1, \infty)$. If $t \in (\ell+1, \infty)$, then

$$\left|\frac{\operatorname{sinc}(\pi t)}{Q_{1,\ell}(t)}\right| = \left|\frac{\sin(\pi t)}{\pi t Q_{1,\ell}(t)}\right| < \frac{1}{\pi(\ell+1)|Q_{1,\ell}(\ell+1)|}.$$

Computation yields

$$|Q_{1,\ell}(\ell+1)| = \frac{(2\ell+1)!}{\ell!(\ell+1)!},$$

so

$$\left|\frac{\operatorname{sinc}(\pi t)}{Q_{1,\ell}(t)}\right| < \frac{(\ell!)^2}{\pi(2\ell+1)!} < 1.$$

Observing that $\frac{\mathrm{sinc}(\pi t)}{Q_{1,\ell}(t)}$ is even proves 1).

For 2), let $t \in \mathbb{R}$ such that $\left|\frac{t}{\ell+1}\right| < \frac{1}{\ell^{2/3}}$, then $0 < \frac{\mathrm{sinc}(\pi t)}{Q_{1,\ell}(t)}$, and

$$\begin{aligned} -\log\Big(\frac{\mathrm{sinc}(\pi t)}{Q_{1,\ell}(t)}\Big) &= -\sum_{k=\ell+1}^{\infty} \log\Big(1-\frac{t^2}{k^2}\Big) = \sum_{k=\ell+1}^{\infty}\sum_{j=1}^{\infty} \frac{t^2}{jk^{2j}} \\ &= \sum_{j=1}^{\infty} \frac{1}{j}\Big(\sum_{k=\ell+1}^{\infty}\frac{1}{k^{2j}}\Big)t^{2j}. \end{aligned} \tag{3.23}$$

The function $1/x^{2j}$ is decreasing, so basic calculus shows that

$$\sum_{k=\ell+1}^{\infty} \frac{1}{k^{2j}} < \frac{1}{(\ell+1)^{2j}} + \frac{1}{(2j-1)(\ell+1)^{2j-1}}.$$

Equation (3.23) implies

$$\begin{aligned} -\log\Big(\frac{\mathrm{sinc}(\pi t)}{Q_{1,\ell}(t)}\Big) &< \sum_{j=1}^{\infty}\frac{1}{j}\Big(\frac{t}{\ell+1}\Big)^{2j} + (\ell+1)\sum_{j=1}^{\infty}\frac{1}{j(2j-1)}\Big(\frac{t}{\ell+1}\Big)^{2j} \\ &< (\ell+2)\sum_{j=1}^{\infty}\Big(\frac{t}{\ell+1}\Big)^{2j} < \frac{\ell+2}{\ell^{4/3}-1}. \end{aligned} \tag{3.24}$$

If $n \in \Delta_{\ell,d}$, then for each $1 \le k \le d$, $\left|\frac{t_n(k)}{\ell+1}\right| > \frac{\ell+2}{\ell^{4/3}-1}$, so that

$$\log\Big(\frac{\mathrm{SINC}(\pi t_n)}{Q_{d,\ell}(t_n)}\Big) = \sum_{k=1}^{d} \log\Big(\frac{\mathrm{sinc}(\pi t_n(k))}{Q_{1,\ell}(t_n(k))}\Big) > -\frac{d(\ell+2)}{\ell^{4/3}-1}.$$

Statement 2) of Proposition III.28 follows readily. □

Proposition III.29. *Statement (c) of Theorem III.26 is true if and only if*

$$0 = \lim_{\ell\to\infty} \sum_{n\in C_{\ell,d}} |f(t_n)|^2 \Big[1 - \frac{\mathrm{SINC}\pi t_n}{Q_{d,\ell}(t_n)}\Big]^2 := \lim_{\ell\to\infty} S_{\ell,d}, \quad f \in PW_{[-\pi,\pi]^d}. \tag{3.25}$$

Proof. Note that $Le_n = f_n$ implies that $f_n^* = (L^*)^{-1}e_n$. Similarly, $f_{\ell,n}^* = (L_\ell^*)^{-1}e_n$.

Given $f \in PW_{[-\pi,\pi]^d}$, let $g = \mathcal{F}^{-1}(f)$. Equation (3.7) shows:

$$\begin{aligned}\mathcal{F}^{-1}(f) &= \lim_{\ell\to\infty} (L_\ell^*)^{-1} \sum_{n\in C_{\ell,d}} \langle L^* g, e_n\rangle e_n = \lim_{\ell\to\infty} (L_\ell^*)^{-1} \sum_{n\in C_{\ell,d}} \langle g, f_n\rangle e_n \\ &= \lim_{\ell\to\infty} \sum_{n\in C_{\ell,d}} \langle g, f_n\rangle f_{\ell,n}^* = \lim_{\ell\to\infty} \sum_{n\in C_{\ell,d}} f(t_n) f_{\ell,n}^*.\end{aligned}$$

Passing to the Fourier transform, we have

$$f = \lim_{\ell\to\infty} \sum_{n\in C_{\ell,d}} f(t_n)\mathcal{F}(f_{\ell,n}^*), \quad f \in PW_{[-\pi,\pi]^d}, \tag{3.26}$$

where the limit exists in both L_2 and uniform senses. Equation (2.7) shows that the values of a function in $PW_{[-\pi,\pi]^d}$ on the set $(t_n)_{n\in C_{\ell,d}} \cup (n)_{n\notin C_{\ell,d}}$ uniquely determine the function. This and (3.20) show that

$$\mathcal{F}(f_{\ell,n}^*)(t) = G_{\ell,n}(t) = \frac{Q_{d,\ell}(t_n)\mathrm{SINC}\pi t}{Q_{d,\ell}(t)\mathrm{SINC}\pi t_n} p_{\ell,n}(t), \quad n \in C_{\ell,d}.$$

This implies that

$$\begin{aligned}\Psi_\ell(t)\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)} &= \Big(\sum_{n\in C_{\ell,d}} f(t_n)p_{\ell,n}(t)\Big)\frac{\mathrm{SINC}(\pi t)}{Q_{d,\ell}(t)} \\ &= \sum_{n\in C_{\ell,d}} f(t_n)\frac{\mathrm{SINC}\pi t_n}{Q_{d,\ell}(t_n)}\mathcal{F}(f_{\ell,n}^*)(t).\end{aligned}$$

Combined with (3.26), we see that statement (c) of Theorem III.26 holds if and only if

$$0 = \lim_{\ell\to\infty} \sum_{n\in C_{\ell,d}} f(t_n)\left[1 - \frac{\mathrm{SINC}\pi t_n}{Q_{d,\ell}(t_n)}\right]\mathcal{F}(f_{\ell,n}^*), \quad f \in PW_{[-\pi,\pi]^d},$$

where the limit is in the L_2 sense. Passing to the inverse Fourier transform, the above equality holds if and only if

$$0 = \lim_{\ell\to\infty} (L_\ell^*)^{-1}\left(\sum_{n\in C_{\ell,d}} f(t_n)\left[1 - \frac{\mathrm{SINC}\pi t_n}{Q_{d,\ell}(t_n)}\right]e_n\right), \quad f \in PW_{[-\pi,\pi]^d}. \tag{3.27}$$

As $(L_\ell)_{\ell>0}$ is pointwise bounded, the Uniform Boundedness Principle proclaims that

$$0 < \sup_\ell \|L_\ell^*\| = \sup_\ell \|L_\ell\| := C < \infty.$$

Uniform invertibility of L implies

$$0 < \sup_{\ell\geq 0} \|(L_\ell^*)^{-1}\| = \sup_{\ell\geq 0} \|(L_\ell^*)^{-1}\| := c < \infty.$$

Together we have

$$\frac{1}{C}\|g\| \leq \|(L_\ell^*)^{-1}g\| < c\|g\|, \quad g \in L_2([-\pi,\pi]^d).$$

The inequalities above, combined with (3.27), proves the proposition. $\square$

Proof of statement (c) in Theorem III.26. Let $S_{\ell,d}$ be as in (3.25). Proposition III.28 gives the following:

$$\begin{aligned} S_{\ell,d} &\leq \Big(\sum_{n\in\Delta_{\ell,d}} + \sum_{n\in\mathbb{Z}^d\setminus\Delta_{\ell,d}}\Big)|f(t_n)|^2\Big[1-\frac{\text{SINC}\pi t_n}{Q_{d,\ell}(t_n)}\Big]^2 \\ &\leq \Big(1-e^{\frac{-d(\ell+2)}{\ell^{4/3}-1}}\Big)^2 \sum_{n\in\mathbb{Z}^d}|f(t_n)|^2 + \sum_{n:\ \frac{\ell+1}{\ell^{2/3}}\leq\|t_n\|_\infty} 4|f(t_n)|^2. \end{aligned} \tag{3.28}$$

Now $(f(t_n))_{n\in\mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d)$ implies that $\lim_{\ell\to\infty} S_{\ell,d} = 0$, whence the result by Proposition III.29. $\square$

E. The second main result

Theorem III.26 can be simplified. The function

$$t \mapsto \frac{\text{SINC}(\pi t)}{Q_{d,\ell}(t)}$$

becomes more computationally complicated for large values of ℓ. If, at the cost of global L_2 and uniform convergence, we adopt an approximation

$$\mathrm{SINC}(\pi t) \simeq Q_{d,\ell}(t) \exp\Big(- \sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}} \Big), \quad |t| \ll \ell, \tag{3.29}$$

we bypass this difficulty as the exponent of the above quantity is simply a rational function of $\ell > 0$. This is stated precisely in the upcoming theorem, which is the second main result of this chapter.

Theorem III.30. *Let $(t_n)_{\mathbb{Z}^d} \subset \mathbb{R}^d$ be a sequence such that $(f_n)_{n\in\mathbb{Z}^d}$ is a UIRB. If N is a non-negative integer and $A > 0$, define*

$$E_{\ell,N,A} = \Big[- A(\ell+1/2)^{\frac{2N+1}{2N+2}}, A(\ell+1/2)^{\frac{2N+1}{2N+2}} \Big].$$

Let $f \in PW_{[-\pi,\pi]^d}$ where $(\Psi_\ell)_\ell$ is the sequence of interpolating polynomials from Theorem III.26. Define

$$I_{f,\ell}(t) = \Psi_\ell(t) \exp\Big(- \sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}} \Big).$$

Then

$$\lim_{\ell\to\infty} \Big\| f(t) - I_{f,\ell}(t) \Big\|_{L_2((E_{\ell,N,A})^d)} = 0, \tag{3.30}$$

and

$$\lim_{\ell\to\infty} \Big\| f(t) - I_{f,\ell}(t) \Big\|_{L_\infty((E_{\ell,N,A})^d)} = 0. \tag{3.31}$$

If $N = 0$ in Theorem III.30, we have the following analogue of Corollary I.2 to arbitrary multivariate bandlimited functions (at the expense of introducing uniform invertibility):

Corollary III.31. *For all $f \in PW_{[-\pi,\pi]^d}$, we have*

$$\lim_{\ell\to\infty} \Big\| f(t) - \Psi_\ell(t) \Big\|_{L_2([-A(\ell+1/2)^{1/2}, A(\ell+1/2)^{1/2}]^d)} = 0, \tag{3.32}$$

and

$$\lim_{\ell\to\infty}\Big\|f(t)-\Psi_\ell(t)\Big\|_{L_\infty([-A(\ell+1/2)^{1/2},A(\ell+1/2)^{1/2}]^d)}=0. \tag{3.33}$$

Theorem II.12 helps provide a nice interpretation of Corollary III.31. Consider a sequence $(t_n)_{n\in\mathbb{Z}^d}\subset\mathbb{R}^d$ (subject to the hypotheses of Theorem III.30), and sampled data $\big((t_n,c_n)\big)_{n\in\mathbb{Z}^d}$ where $(c_n)_{n\in\mathbb{Z}^d}\in\ell_2(\mathbb{Z}^d)$. A unique sequence of Lagrangian polynomial interpolants exists, and in global L_2 and uniform senses, converges to the unique bandlimited interpolant of the same data.

When $N=1$, we have a sampling theorem with a Gaussian multiplier:

$$f(t)\simeq\Psi_\ell(t)\exp\Big(-\frac{\|t\|_2^2}{(\ell+1/2)}\Big),\quad f\in PW_{[-\pi,\pi]^d}.$$

Compare Theorem III.30 with Theorem 2.6 in [20], which is a multivariate sampling theorem with a Gaussian multiplier with global L_2 and uniform convergence. Also compare Theorem III.30 with Theorem 2.1 in [21], which, when $d=1$ and the data sites are equally spaced, gives another recovery formula involving a Gaussian multiplier in the context of oversampling.

The proof of Theorem III.30 relies on two lemmas, whose proofs will be deferred until the end of the section.

Lemma III.32. *Let $d>0$, N be a non-negative integer, and $A>0$. There exists $M>0$ such that for sufficiently large ℓ, and any $t\in(E_{\ell,N,A})^d$,*

$$\begin{aligned}&\Big|Q_{d,\ell}(t)e^{-\sum_{k=1}^N\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}-e^{\Big(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\Big)}\mathrm{SINC}(\pi t)\Big|\\&\quad\le M(\ell+1/2)^{-\frac{1}{N+1}}|\mathrm{SINC}(\pi t)|.\end{aligned}$$

Lemma III.33. *For all $f\in PW_{[-\pi,\pi]^d}$ and any non-negative integer N, we have*

$$\lim_{\ell\to\infty}\sup_{t\in(E_{\ell,N,A})^d}\Big|\Big(e^{\Big(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\Big)}-1\Big)f(t)\Big|=0.$$

Proof of Theorem III.30. If $f \in PW_{[-\pi,\pi]^d}$, Theorem III.26 states that

$$f(t) = \frac{\Psi_\ell(t)}{Q_{d,\ell}(t)}\mathrm{SINC}(\pi \mathrm{t}) + \xi_\ell(t)$$

where $\xi_\ell \to 0$ on $\mathbb{R}^d$ in both L_2 and L_∞ senses. By Lemma III.32, we have

$$\begin{aligned}
&\sup_{t\in(E_{\ell,N,A})^d}\left|\Psi_\ell(t)e^{-\sum_{k=1}^N \frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}} - e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}\frac{\Psi_\ell(t)}{Q_{d,\ell}(t)}\mathrm{SINC}(\pi \mathrm{t})\right| \\
&\quad \le M(\ell+1/2)^{-\frac{1}{N+1}}\sup_{t\in(E_{\ell,N,A})^d}(|f(t)|-|\xi_\ell(t)|),
\end{aligned} \tag{3.34}$$

the right side of which has zero limit. Also,

$$\begin{aligned}
&\sup_{t\in(E_{\ell,N,A})^d}\left|\left(e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}-1\right)\frac{\Psi_\ell(t)}{Q_{d,\ell}(t)}\mathrm{SINC}(\pi \mathrm{t})\right| \qquad (3.35)\\
&\le \sup_{t\in(E_{\ell,N,A})^d}\left|\left(e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}-1\right)f(t)\right| + \\
&\quad \left(e^{\left(\frac{dA^{2(N+1)}}{(N+1)(2N+1)}\right)}-1\right)\sup_{t\in(E_{\ell,N,A})^d}|\xi_\ell(t)|,
\end{aligned}$$

whose right-hand side, by Lemma III.33, also has zero limit. Combining (3.34) and (3.35), we obtain

$$\lim_{\ell\to\infty}\left\|\Psi_\ell(t)e^{-\sum_{k=1}^N \frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}} - \frac{\Psi_\ell(t)}{Q_{d,\ell}(t)}\mathrm{SINC}(\pi \mathrm{t})\right\|_{L_\infty((E_{\ell,N,A})^d)} = 0.$$

Equation (3.31) follows by a final application of Theorem III.26.

Now we prove (3.30). Lemma III.32 and Theorem III.26 imply

$$\begin{aligned}
&\left\|\Psi_\ell(t)e^{-\sum_{k=1}^N \frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}} - e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}\frac{\Psi_\ell(t)}{Q_{d,\ell}(t)}\mathrm{SINC}(\pi \mathrm{t})\right\|_{L_2((E_{\ell,N,A})^d)} \\
&\quad \le M(\ell+1/2)^{-\frac{1}{N+1}}\|f-\xi_\ell\|_{L_2((E_{\ell,N,A})^d)},
\end{aligned} \tag{3.36}$$

the right-hand side of which has zero limit. Also,

$$\begin{aligned}
&\left\|\left(e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}-1\right)\frac{\Psi_\ell(t)}{Q_{d,\ell}(t)}\mathrm{SINC}(\pi \mathrm{t})\right\|_{L_2((E_{\ell,N,A})^d)} \qquad (3.37)\\
&\le \left\|\left(e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}-1\right)f(t)\right\|_{L_2((E_{\ell,N,A})^d)}+\\
&\left\|\left(e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}-1\right)\xi_\ell(t)\right\|_{L_2((E_{\ell,N,A})^d)}.
\end{aligned}$$

The second term in the right-hand side of (3.37) is bounded above by

$$\left(e^{\left(\frac{dA^{2(N+1)}}{(N+1)(2N+1)}\right)}-1\right)\|\xi_\ell\|_{L_2((E_{\ell,N,A})^d)},$$

which has zero limit. The integrand of the first term in the right-hand side of (3.37) (as a function over $\mathbb{R}^d$), converges uniformly to zero by Lemma III.33, and is bounded above by

$$\left(e^{\left(\frac{dA^{2(N+1)}}{(N+1)(2N+1)}\right)}-1\right)|f(t)|^2 \in L_1(\mathbb{R}^d),$$

so this term has zero limit by the Dominated Convergence Theorem. Combining (3.36) and (3.37) yields

$$\lim_{\ell\to\infty}\left\|\Psi_\ell(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}-\frac{\Psi_\ell(t)}{Q_{d,\ell}(t)}\mathrm{SINC}(\pi \mathrm{t})\right\|_{L_2((E_{\ell,N,A})^d)}=0.$$

Equation (3.30) follows by a final application of Theorem III.26. □

The proof of Lemma III.32 relies on the following proposition.

Proposition III.34. *If $f:(0,\infty)\to(0,\infty)$ is convex, decreasing, differentiable, and integrable away from 0, then*

$$\frac{1}{4}f'(\ell+1/2)\le\sum_{k=\ell+1}^{\infty}f(k)-\int_{\ell+1/2}^{\infty}f(x)dx\le 0,\quad \ell\ge 0. \qquad (3.38)$$

Proof. Geometric considerations show that

$$
\begin{aligned}
&1)\quad f(k) \le \int_{k-1/2}^{k+1/2} f(x)dx, \quad k \ge 1, \quad \text{and}\\
&2)\quad \int_{k}^{k+1} f(x)dx \le \frac{1}{2}[f(k)+f(k+1)], \quad k \ge 1.
\end{aligned}
$$

The rightmost inequality in (3.38) follows from 1) by summing over k. From 2) we obtain

$$
\begin{aligned}
&\int_{\ell+1}^{\infty} f(x)dx \le \frac{1}{2}\sum_{k=\ell+1}^{\infty} f(k) + \frac{1}{2}\sum_{k=\ell+1}^{\infty} f(k+1),\\
&\frac{1}{2}f(\ell+1) + \int_{\ell+1}^{\infty} f(x)dx \le \sum_{k=\ell+1}^{\infty} f(k),\\
&\frac{1}{2}f(\ell+1) - \int_{\ell+\frac{1}{2}}^{\ell+1} f(x)dx \le \sum_{k=\ell+1}^{\infty} f(k) - \int_{\ell+\frac{1}{2}}^{\infty} f(x)dx.
\end{aligned}
\tag{3.39}
$$

There exists $\ell+1/2 < \xi < \ell+1$ such that

$$
\frac{1}{4}f'(\ell+1/2) \le \frac{1}{4}f'(\xi) = \frac{1}{2}f(\ell+1) - \frac{1}{2}f(\ell+1/2) \le \frac{1}{2}f(\ell+1) - \int_{\ell+\frac{1}{2}}^{\ell+1} f(x)dx.
$$

Combining the inequality above with (3.39) proves the proposition. $\square$

Proof of Lemma III.32. Letting $|t| < \ell + 1/2$ and recalling (3.23), we see that

$$
\begin{aligned}
&-\log\left(\frac{\operatorname{sinc}(\pi t)}{Q_{1,\ell}(t)}\right) - \sum_{k=1}^{\infty} \frac{1}{k(2k-1)} \frac{t^{2k}}{(\ell+1/2)^{2k-1}}\\
&= \sum_{k=1}^{\infty} \Big[\sum_{j=\ell+1}^{\infty} \frac{1}{j^{2k}} - \frac{1}{(2k-1)(\ell+1/2)^{2k-1}} \Big] \frac{t^{2k}}{k}.
\end{aligned}
\tag{3.40}
$$

Applying Proposition III.34 to the function $f(t) = \frac{1}{t^{2k}}$ when $k \ge 1$, we obtain

$$
\frac{-k}{2(\ell+1/2)^{2k+1}} \le \sum_{j=\ell+1}^{\infty} \frac{1}{j^{2k}} - \frac{1}{(2k-1)(\ell+1/2)^{2k-1}} \le 0.
$$

Equation (3.40) becomes

$$\frac{-1}{2(\ell+1/2)}\sum_{k=1}^{\infty}\left(\frac{t}{\ell+1/2}\right)^{2k} \le -\log\left(\frac{\operatorname{sinc}(\pi t)}{Q_{1,\ell}(t)}\right) - \sum_{k=1}^{\infty}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}} \le 0.$$

Restated,

$$\begin{aligned}
&-\frac{1}{2(\ell+1/2)}\frac{\left(\frac{t}{\ell+1/2}\right)^2}{1-\left(\frac{t}{\ell+1/2}\right)^2} + \sum_{k=N+1}^{\infty}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}} \\
&\le -\log\left(\frac{\operatorname{sinc}(\pi t)}{Q_{1,\ell}(t)}\right) - \sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}} \\
&\le \sum_{k=N+1}^{\infty}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}.
\end{aligned} \tag{3.41}$$

Exponentiating,

$$\begin{aligned}
&e^{\left(-\frac{1}{2(\ell+1/2)}\frac{\left(\frac{t}{\ell+1/2}\right)^2}{1-\left(\frac{t}{\ell+1/2}\right)^2}\right)} e^{\sum_{k=N+1}^{\infty}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}} \\
&\quad\le \frac{Q_{1,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}}}{\operatorname{sinc}(\pi t)} \le e^{\sum_{k=N+1}^{\infty}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}}.
\end{aligned} \tag{3.42}$$

Let ℓ be chosen large enough so that $A(\ell+1/2)^{\frac{2N+1}{2N+2}} < \ell+1/2$. If ℓ is large enough, then for any $t \in E_{\ell,N,A}$, $t = c(\ell+1/2)^{\frac{2N+1}{2N+2}}$ for some $c \in [-A, A]$. For such t, (3.42) implies

$$\begin{aligned}
&e^{\left(-\frac{1}{2(\ell+1/2)^{\frac{N+2}{N+1}}}\frac{c^2}{1-c^2(\ell+1/2))^{\frac{-1}{N+1}}}\right)} e^{\sum_{k=N+1}^{\infty}\frac{c^{2k}}{k(2k-1)}(\ell+1/2)^{(1-\frac{k}{N+1})}} \\
&\quad\le \frac{Q_{1,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}}}{\operatorname{sinc}(\pi t)} \le e^{\sum_{k=N+1}^{\infty}\frac{c^{2k}}{k(2k-1)}(\ell+1/2)^{(1-\frac{k}{N+1})}}.
\end{aligned}$$

If $t \in (E_{\ell,N,A})^d$, then $t = c(\ell+1/2)^{\frac{2N+1}{2N+2}}$ for some $c \in [-A, A]^d$. For any such t, we have

$$e^{\left(-\frac{d}{2(\ell+1/2)^{\frac{N+2}{N+1}}}\frac{A^2}{1-A^2(\ell+1/2))^{\frac{-1}{N+1}}}\right)}e^{\sum_{k=N+1}^{\infty}\frac{\|c\|_{2k}^{2k}}{k(2k-1)}(\ell+1/2)^{(1-\frac{k}{N+1})}} \tag{3.43}$$
$$\leq \frac{Q_{d,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}}{\mathrm{SINC}(\pi t)} \leq e^{\sum_{k=N+1}^{\infty}\frac{\|c\|_{2k}^{2k}}{k(2k-1)}(\ell+1/2)^{(1-\frac{k}{N+1})}}.$$

On one hand,

$$e^{\sum_{k=N+1}^{\infty}\frac{\|c\|_{2k}^{2k}}{k(2k-1)}(\ell+1/2)^{(1-\frac{k}{N+1})}} \leq e^{\left(\frac{\|c\|_{2(N+1)}^{2(N+1)}}{(N+1)(2N+1)}+O\left((\ell+1/2)^{\frac{-1}{N+1}}\right)\right)} \tag{3.44}$$

where the "big O" constant is independent of $c \in [-A, A]^d$. On the other hand,

$$e^{\left(\frac{\|c\|_{2(N+1)}^{2(N+1)}}{(N+1)(2N+1)}\right)} \leq e^{\sum_{k=N+1}^{\infty}\frac{\|c\|_{2k}^{2k}}{k(2k-1)}(\ell+1/2)^{(1-\frac{k}{N+1})}}. \tag{3.45}$$

Inequalities (3.43), (3.44), and (3.45) yield

$$\left(e^{\left(-\frac{d}{2(\ell+1/2)^{\frac{N+2}{N+1}}}\frac{A^2}{1-A^2(\ell+1/2))^{\frac{-1}{N+1}}}\right)}-1\right)e^{\left(\frac{\|c\|_{2(N+1)}^{2(N+1)}}{(N+1)(2N+1)}\right)} \tag{3.46}$$
$$\leq \frac{Q_{d,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}}{\mathrm{SINC}(\pi t)} - e^{\left(\frac{\|c\|_{2(N+1)}^{2(N+1)}}{(N+1)(2N+1)}\right)}$$
$$\leq e^{\frac{dA^{2(N+1)}}{(N+1)(2N+1)}}\left(e^{O\left(\frac{1}{(\ell+1/2)^{\frac{1}{N+1}}}\right)}-1\right).$$

The leftmost side of (3.46) is of the order $O((\ell+1/2)^{-\frac{N+2}{N+1}})$, and the rightmost side of (3.46) is of the order $O((\ell+1/2)^{-\frac{1}{N+1}})$, where the "big O" constants are independent of $c \in [-A, A]^d$. The lemma follows readily. □

Proof of Lemma III.33. Equivalently, we need to show

$$\lim_{\ell\to\infty}\sup_{c\in[-A,A]^d}\left|\left(e^{\left(\frac{\|c\|_{2(N+1)}^{2(N+1)}}{(N+1)(2N+1)}\right)}-1\right)f\left(c(\ell+1/2)^{\frac{2N+1}{2N+2}}\right)\right| = 0.$$

Suppose the contrary. Let $c_\ell \in [-A, A]^d$ be a value that maximizes the ℓ-th term in the above limit. There exists $(\ell_k)_{k\in\mathbb{N}}$, and $\epsilon > 0$ such that for all $k \in \mathbb{N}$,

$$\begin{aligned}\epsilon &\leq \sup_{c\in[-A,A]^d} \left|\left(e^{\left(\frac{\|c\|_{2(N+1)}^{2(N+1)}}{(N+1)(2N+1)}\right)} - 1\right) f\left(c(\ell_k + 1/2)^{\frac{2N+1}{2N+2}}\right)\right| \\ &\leq \left(e^{\left(\frac{dA^{2(N+1)}}{(N+1)(2N+1)}\right)} - 1\right)\left|f\left(c_{\ell_k}(\ell_k + 1/2)^{\frac{2N+1}{2N+2}}\right)\right|,\end{aligned}$$

so that the sequence $\left(f\left(c_{\ell_k}(\ell_k + 1/2)^{\frac{2N+1}{2N+2}}\right)\right)_{k\in\mathbb{N}}$ is bounded away from 0. By the d-dimensional Riemann-Lebesgue lemma, this implies there exists $\delta > 0$ such that $\left\|c_{\ell_k}(\ell_k + 1/2)^{\frac{2N+1}{2N+2}}\right\|_{2(N+1)} \leq \delta$ for $k \in \mathbb{N}$, that is,

$$\|c_{\ell_k}\|_{2(N+1)} \leq \delta(\ell_k + 1/2)^{-\frac{2N+1}{2N+2}}.$$

This forces

$$\begin{aligned}\epsilon &\leq \sup_{c\in[-A,A]^d} \left|\left(e^{\left(\frac{\|c\|_{2(N+1)}^{2(N+1)}}{(N+1)(2N+1)}\right)} - 1\right) f\left(c(\ell_k + 1/2)^{\frac{2N+1}{2N+2}}\right)\right| \\ &\leq \left(e^{\left(\frac{\delta^{2(N+1)}}{(\ell_k+1/2)^{2N+1}(N+1)(2N+1)}\right)} - 1\right)\|f\|_\infty.\end{aligned}$$

The last term in the above inequality has limit 0 as $\ell \to \infty$, which is a contradiction.

$\square$

F. Comments regarding the optimality of Theorem III.30

In the statement of Theorem III.30, it is not apparent whether or not $(E_{\ell,N,A})_\ell$ can be replaced with a more rapidly growing sequence of intervals; however, Proposition III.35 shows that if $f(t) = \mathrm{SINC}(\pi t)$, (3.31) and (3.30) can hold for a sequence of intervals $(E_{\ell,N})_\ell$ which grow faster than $(E_{\ell,N,A})_\ell$. Propositions III.40 and III.42 show that growth bounds of the intervals in Proposition III.35 are optimal for this function. Thus, the bounds in Proposition III.35 provide upper bounds for the growth of *any* sequence $(E_{\ell,N})_\ell$ such that either (3.31) or (3.30) hold for general multivariate bandlimited functions.

Proposition III.35. *Define*

$$\begin{aligned} C_{\ell,N} &= \Big(\frac{1}{4}(2N+1)^2(\ell+1/2)^{2N+1}\log(\ell+1/2)\Big)^{\frac{1}{2(N+1)}}, \text{ and} \\ D_{\ell,N} &= \Big(\frac{1}{2}(2N+1)^2(\ell+1/2)^{2N+1}\log(\ell+1/2)\Big)^{\frac{1}{2(N+1)}}, \end{aligned}$$

and let $I_{\mathrm{SINC}\pi(\cdot),\ell}$ *be the approximant from Theorem III.30 corresponding to* $f(\cdot) = \mathrm{SINC}\pi(\cdot)$. *Then the following hold:*

$$\lim_{\ell\to\infty} \Big\| \mathrm{SINC}(\pi t) - I_{\mathrm{SINC}\pi(\cdot),\ell}(t) \Big\|_{L_2([-C_{\ell,N},C_{\ell,N}]^d)} = 0, \tag{3.47}$$

and

$$\lim_{\ell\to\infty} \Big\| \mathrm{SINC}(\pi t) - I_{\mathrm{SINC}\pi(\cdot),\ell}(t) \Big\|_{L_\infty([-D_{\ell,N},D_{\ell,N}]^d)} = 0. \tag{3.48}$$

The proof of (3.47) requires the following two propositions.

Proposition III.36.

$$\lim_{\ell\to\infty} \Big\| \Big(e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)} - 1 \Big) \mathrm{SINC}(\pi t) \Big\|_{L_2([-C_{\ell,N},C_{\ell,N}]^d)} = 0. \tag{3.49}$$

Proof. Let $t = \alpha C_{\ell,N}$ where $\alpha \in [-1,1]^d$. Noting that

$$e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)} = \Big(\ell + \frac{1}{2}\Big)^{\frac{2N+1}{4(N+1)}\|\alpha\|_{2(N+1)}^{2(N+1)}},$$

the quantity in (3.49) becomes

$$\begin{aligned} &\Big(\int_{[-C_{\ell,N},C_{\ell,N}]^d} \Big| \Big(\Big(\ell+\frac{1}{2}\Big)^{\frac{2N+1}{4(N+1)}\|\alpha\|_{2(N+1)}^{2(N+1)}} - 1 \Big) \mathrm{SINC}(\pi t) \Big|^2 dt \Big)^{1/2} \\ &\quad \le \frac{1}{C_{\ell,N}^{d/2}} \Big(\int_{[-1,1]^d} \Big| \Big(\ell+\frac{1}{2}\Big)^{\frac{2N+1}{4(N+1)}\|\alpha\|_{2(N+1)}^{2(N+1)}} - 1 \Big|^2 d\alpha \Big)^{1/2} \\ &\quad \le \frac{2^{d/2}\Big(\ell+\frac{1}{2}\Big)^{d\frac{2N+1}{4(N+1)}} + 2^{d/2}}{\big(\frac{1}{4}(2N+1)^2\big)^{\frac{d}{4(N+1)}} \big(\log(\ell+1/2)\big)^{\frac{d}{4(N+1)}} (\ell+1/2)^{d\frac{2N+1}{4(N+1)}}} \to_{\ell\to\infty} 0. \end{aligned}$$

This proves the proposition. $\square$

Proposition III.37.

$$\lim_{\ell\to\infty} \left\| Q_{d,\ell}(t) e^{-\sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}} - e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)} \mathrm{SINC}(\pi t) \right\|_{L_2([-C_{\ell,N}, C_{\ell,N}]^d)}$$
$$= \ 0 \tag{3.50}$$

Proof. If $t \in \mathbb{R}^d$ and $\|t\|_\infty < \ell + 1/2$, then (3.42) implies

$$\begin{aligned} &\left(e^{\sum_{k=N+1}^{\infty} \frac{1}{k(2k-1)} \frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}\right) \prod_{k=1}^{d} e^{\left(-\frac{1}{2(\ell+1/2)} \frac{\left(\frac{t}{\ell+1/2}\right)^2}{1-\left(\frac{t}{\ell+1/2}\right)^2}\right)} \\ &\leq \frac{Q_{d,\ell}(t) e^{-\sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}}{\mathrm{SINC}(\pi t)} \leq e^{\sum_{k=N+1}^{\infty} \frac{1}{k(2k-1)} \frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}. \end{aligned} \tag{3.51}$$

Let $t \in [-C_{\ell,N}, C_{\ell,N}]^d$ where $t = \alpha C_{\ell,N}$, $\alpha \in [-1,1]$. Consider the right-hand side of (3.51) for such t.

$$\begin{aligned} e^{\sum_{k=N+1}^{\infty} \frac{1}{k(2k-1)} \frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}} &\leq \left(\ell + \frac{1}{2}\right)^{\frac{2N+1}{4(N+1)} \|\alpha\|_{2(N+1)}^{2(N+1)}} e^{(\ell+1/2) O\left(\left\| \frac{t}{\ell+1/2} \right\|_{2(N+2)}^{2(N+2)}\right)} \\ &\leq \left(\ell + \frac{1}{2}\right)^{\frac{2N+1}{4(N+1)} \|\alpha\|_{2(N+1)}^{2(N+1)}} e^{\left(M (\ell+1/2)^{-\frac{1}{N+1}} (\log(\ell+1/2))^{\frac{N+2}{N+1}} \|\alpha\|_{2(N+2)}^{2(N+2)}\right)}. \end{aligned} \tag{3.52}$$

for some constant M. Noting that

$$\frac{t^2}{(\ell+1/2)^3} = \frac{\|\alpha\|_2^2 \left(\frac{1}{4}(2N+1)^2\right)^{\frac{1}{N+1}} \left(\log(\ell+1/2)\right)^{\frac{1}{N+1}}}{(\ell+1/2)^{\frac{N+2}{N+1}}},$$

we can bound the left-hand side of (3.51) from below as follows:

$$\begin{aligned} &e^{\left(-m \frac{\|\alpha\|_2^2 \left(\log(\ell+1/2)\right)^{\frac{1}{N+1}}}{(\ell+1/2)^{\frac{N+2}{N+1}}}\right)} \left(\ell + \frac{1}{2}\right)^{\frac{2N+1}{4(N+1)} \|\alpha\|_{2(N+1)}^{2(N+1)}} \\ &\leq \left(e^{\sum_{k=N+1}^{\infty} \frac{1}{k(2k-1)} \frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}\right) \prod_{k=1}^{d} e^{\left(-\frac{1}{2(\ell+1/2)} \frac{\left(\frac{t}{\ell+1/2}\right)^2}{1-\left(\frac{t}{\ell+1/2}\right)^2}\right)}, \end{aligned} \tag{3.53}$$

where $m > 0$ is chosen independently of ℓ. Relations (3.51) through (3.53) imply

$$\begin{aligned}
&\left(e^{\left(-m\frac{(\log(\ell+1/2))^{\frac{1}{N+1}}\|\alpha\|_2^2}{(\ell+1/2)^{\frac{N+2}{N+1}}}\right)}-1\right)\left(\ell+\frac{1}{2}\right)^{\frac{2N+1}{4(N+1)}\|\alpha\|_{2(N+1)}^{2(N+1)}}|\mathrm{SINC}(\pi t)| \\
&\leq \left|Q_{d,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}-e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}\mathrm{SINC}(\pi t)\right| \\
&\leq \left(e^{\left(M\frac{(\log(\ell+1/2))^{\frac{N+2}{N+1}}\|\alpha\|_{2(N+2)}^{2(N+2)}}{(\ell+1/2)^{\frac{1}{N+1}}}\right)}-1\right)\left(\ell+\frac{1}{2}\right)^{\frac{2N+1}{4(N+1)}\|\alpha\|_{2(N+1)}^{2(N+1)}}|\mathrm{SINC}(\pi t)|.
\end{aligned}$$

Further simplification implies (for appropriate constants C, C', and C'') that

$$\begin{aligned}
&\left\|Q_{d,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}-e^{\left(\frac{\|t\|_{2(N+1)}^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}\mathrm{SINC}(\pi t)\right\|_{L_2([-C_{\ell,N},C_{\ell,N}]^d)} \\
\leq\;& C\frac{(\log(\ell+1/2))^{\frac{N+2}{N+1}}}{(\ell+1/2)^{\frac{1}{N+1}}}\left(\int_{[-C_{\ell,N},C_{\ell,N}]^d}\left|\left(\ell+\frac{1}{2}\right)^{\frac{2N+1}{4(N+1)}\|\alpha\|_{2(N+1)}^{2(N+1)}}\|\alpha\|_2^2\mathrm{SINC}(\pi t)\right|^2dt\right)^{1/2} \\
=\;& C'\frac{(\log(\ell+1/2))^{\frac{N+2}{N+1}}}{(\ell+1/2)^{\frac{1}{N+1}}}\left(\int_{[-1,1]^d}\frac{\left|\left(\ell+\frac{1}{2}\right)^{\frac{2N+1}{4(N+1)}\|\alpha\|_{2(N+1)}^{2(N+1)}}\|\alpha\|_2^2\mathrm{SINC}(\pi t)\right|^2}{(\log(\ell+1/2))^{\frac{d}{2(N+1)}}\left(\ell+\frac{1}{2}\right)^{\frac{2N+1}{2(N+1)}d}}d\alpha\right)^{1/2} \\
\leq\;& C''\frac{(\log(\ell+1/2))^{\frac{N+2}{N+1}}}{(\ell+1/2)^{\frac{1}{N+1}}(\log(\ell+1/2))^{\frac{d}{4(N+1)}}},
\end{aligned}$$

after the change in variable $t=\alpha C_{\ell,N}$ and simple estimates for the integrand. This proves the proposition. □

Proof of (3.47). This follows immediately from Propositions III.36 and III.37. □

The proof of (3.48) requires the following two propositions.

Proposition III.38.

$$\lim_{\ell\to\infty}\left\|\left(e^{\left(\frac{t^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}-1\right)\mathrm{sinc}(\pi t)\right\|_{L_\infty[-D_{\ell,N},D_{\ell,N}]}=0. \tag{3.54}$$

Proof. Let $t\in[-D_{\ell,N},D_{\ell,N}]$; then $t=\alpha D_{\ell,N}$ for $\alpha\in[-1,1]$. Simplification shows that (3.54) holds if

$$\lim_{\ell\to\infty}\sup_{\alpha\in[0,1]}\left|\frac{(\ell+1/2)^{\alpha^{2(N+1)}\frac{2N+1}{2(N+1)}}-1}{\alpha\big(\log(\ell+1/2)\big)^{\frac{1}{2(N+1)}}(\ell+1/2)^{\frac{2N+1}{2(N+1)}}}\right|=0. \tag{3.55}$$

Note that for large ℓ,

$$\sup_{\alpha\in[1/2,1]}\left|\frac{(\ell+1/2)^{\alpha^{2(N+1)}\frac{2N+1}{2(N+1)}}-1}{\alpha\big(\log(\ell+1/2)\big)^{\frac{1}{2(N+1)}}(\ell+1/2)^{\frac{2N+1}{2(N+1)}}}\right|\leq\frac{2}{\big(\log(\ell+1/2)\big)^{\frac{1}{2(N+1)}}}. \tag{3.56}$$

Let $0<\alpha\leq 1/2$. The Mean Value Theorem implies

$$\left|\frac{(\ell+1/2)^{\alpha^{2(N+1)}\frac{2N+1}{2(N+1)}}-1}{\alpha}\right|\leq(2N+1)(\ell+1/2)^{\alpha^{2(N+1)}\frac{2N+1}{2(N+1)}}\alpha^{2N+1}\log(\ell+1/2). \tag{3.57}$$

This yields

$$\sup_{\alpha\in[0,1/2]}\left|\frac{(\ell+1/2)^{\alpha^{2(N+1)}\frac{2N+1}{2(N+1)}}-1}{\alpha\big(\log(\ell+1/2)\big)^{\frac{1}{2(N+1)}}(\ell+1/2)^{\frac{2N+1}{2(N+1)}}}\right|\leq M\frac{\big(\log(\ell+1/2)\big)^{\frac{2N+1}{2(N+1)}}}{(\ell+1/2)^{\frac{2N+1}{2(N+1)}(1-(1/2)^{2(N+1)})}}$$

for some constant M. Combined with (3.56), we have (3.55), which proves the proposition. $\square$

Proposition III.39.

$$0=\lim_{\ell\to\infty}\left\|Q_{1,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}}-e^{\left(\frac{t^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)}\mathrm{sinc}(\pi t)\right\|_{L_\infty[-D_{\ell,N},D_{\ell,N}]}.$$

Proof. Let $t\in[-C_{\ell,N},C_{\ell,N}]$ where $t=\alpha C_{\ell,N}$, $\alpha\in[-1,1]$. Proceeding in the same manner as in the proof of Proposition III.37, we see (for appropriate constants C and

C') that

$$\left| Q_{1,\ell}(t) e^{-\sum_{k=1}^{N} \frac{1}{k(2k-1)} \frac{t^{2k}}{(\ell+1/2)^{2k-1}}} - e^{\left(\frac{t^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1)}\right)} \operatorname{sinc}(\pi t) \right|_{L_\infty([-C_{\ell,N}, C_{\ell,N}])}$$
$$\leq \frac{C(\ell+1/2)^{\alpha^{2(N+1)} \frac{2N+1}{2(N+1)}} \alpha^2 (\log(\ell+1/2))^{\frac{N+2}{N+1}} |\sin(\pi t)|}{\alpha(\ell+1/2)^{\frac{1}{N+1}} (\log(\ell+1/2))^{\frac{1}{2(N+1)}} (\ell+1/2)^{\frac{2N+1}{2(N+1)}}}$$
$$\leq \frac{C'(\log(\ell+1/2))^{\frac{2N+3}{2(N+1)}}}{(\ell+1/2)^{\frac{1}{N+1}}}.$$

This proves the proposition. □

Proof of (3.48). The previous two propositions prove (3.48) when $d = 1$. The multi-dimensional case follows inductively. □

Proposition III.40. *Let $N \geq 0$. If $(M_{\ell,N})_\ell$ is a sequence of positive numbers such that* (3.47) *holds when $(C_{\ell,N})_\ell$ is replaced by $(M_{\ell,N})_\ell$, then*

$$\limsup_{\ell \to \infty} \frac{M_{\ell,N}}{C_{\ell,N}} \leq 1. \tag{3.58}$$

The proof of Proposition III.40 requires the following simple estimate:

Proposition III.41. *Let $a > 1/2$, $\epsilon > 0$, $0 < \omega < 1$, then*

$$\int_a^{(1+\epsilon)a} \frac{\sin^2 \pi x}{x^{1+\omega}} dx > \frac{\epsilon}{2a^\omega (1+\epsilon)^\omega} - \frac{a}{2(a-1/2)^{1+\omega}}.$$

Proof. Let $b = (1+\epsilon)a$. We have

$$\int_a^b \frac{\sin^2 \pi x}{x^{1+\omega}} dx + \int_a^b \frac{\cos^2 \pi x}{x^{1+\omega}} dx = \frac{1}{\omega}\Big(\frac{1}{a^\omega} - \frac{1}{b^\omega}\Big)$$

and

$$\int_a^b \frac{\cos^2 \pi x}{x^{1+\omega}} dx = \int_{a-1/2}^{b-1/2} \frac{\sin^2 \pi x}{(x+1/2)^{1+\omega}} dx < \int_{a-1/2}^{b-1/2} \frac{\sin^2 \pi x}{x^{1+\omega}} dx.$$

This yields

$$2\int_a^b \frac{\sin^2 \pi x}{x^{1+\omega}}dx - \int_{b-1/2}^b \frac{\sin^2 \pi x}{x^{1+\omega}}dx + \int_{a-1/2}^a \frac{\sin^2 \pi x}{x^{1+\omega}}dx > \frac{1}{\omega}\Big(\frac{1}{a^\omega} - \frac{1}{b^\omega}\Big),$$

so that

$$\int_a^b \frac{\sin^2 \pi x}{x^{1+\omega}}dx > \frac{1}{2\omega}\Big(\frac{1}{a^\omega} - \frac{1}{b^\omega}\Big) - \frac{1}{2(a-1/2)^{1+\omega}}.$$

Noting that

$$\frac{1}{2\omega}\Big(\frac{1}{a^\omega} - \frac{1}{b^\omega}\Big) = \frac{\epsilon}{2\omega a^\omega (1+\epsilon)^\omega}\frac{(1+\epsilon)^\omega - 1}{\epsilon} > \frac{\epsilon}{2a^\omega(1+\epsilon)^\omega}$$

proves the proposition. $\square$

Proof of Proposition III.40. Fix $N \geq 0$, and define $c = \frac{2N+1}{2N+4} + \delta/2$ where $0 < \delta$ is small enough so that $c < 1/2$. Define

$$A_\ell = (c(N+1)(2N+1)\log(\ell+1/2))^{\frac{1}{2(N+1)}}(\ell+1/2)^{\frac{-1}{2(N+1)}}$$

and

$$\epsilon_\ell = (\ell+1/2)^{1-2c}A_\ell.$$

Algebra shows that $\lim_{\ell\to\infty}\epsilon_\ell = 0$. Let $t \in [A_\ell(\ell+1/2), (1+\epsilon_\ell)A_\ell(\ell+1/2)]$, then $t = \alpha(\ell+1/2)$ for some $\alpha \in [A_\ell, (1+\epsilon_\ell)A_\ell]$. For large ℓ, note that (3.42) implies

$$\frac{1}{2\pi}e^{\left(\frac{(\ell+1/2)\alpha^{2(N+1)}}{(N+1)(2N+1)}\right)}\frac{|\sin\pi\alpha(\ell+1/2)|}{\alpha(\ell+1/2)} \leq \left|Q_{1,\ell}(t)e^{-\sum_{k=1}^N \frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}}\right|. \quad (3.59)$$

Moving to the multivariate case, if $t \in [A_\ell(\ell+1/2), (1+\epsilon_\ell)A_\ell(\ell+1/2)]^d$, then $t = \alpha(\ell+1/2)$ for some $\alpha \in [A_\ell, (1+\epsilon_\ell)A_\ell]^d$. This yields

$$\prod_{i=1}^d \frac{1}{2\pi\alpha_i^c}\frac{|\sin\pi\alpha_i(\ell+1/2)|}{(\alpha_i(\ell+1/2))^{1-c}} \leq \left|Q_{d,\ell}(t)e^{-\sum_{k=1}^N \frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}\right|.$$

For sufficiently large ℓ, we can conclude that

$$\begin{aligned}&\left[\frac{1}{9\pi^2 A_\ell^{2c}}\int_{A_\ell(\ell+1/2)}^{(1+\epsilon_\ell)A_\ell(\ell+1/2)}\frac{\sin^2\pi x}{x^{2-2c}}dx\right]^d\\&\quad\le\int_{[A_\ell(\ell+1/2),(1+\epsilon_\ell)A_\ell(\ell+1/2)]^d}\left|Q_{d,\ell}(t)e^{-\sum_{k=1}^N\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}\right|^2dt.\end{aligned}$$

Applying Proposition III.41 for $a = A_\ell(\ell+1/2)$, $\epsilon = \epsilon_\ell$, and $\omega = 1-2c$, and using the definition of ϵ_ℓ, we obtain

$$\begin{aligned}&\left[\frac{1}{9\pi^2}\Big[\frac{1}{2(1+\epsilon_\ell)^{1-2c}}-\frac{1}{2A_\ell^{2c}(A_\ell(\ell+1/2)-1)^{2-2c}}\Big]\right]^d\\&\quad\le\int_{[A_\ell(\ell+1/2),(1+\epsilon_\ell)A_\ell(\ell+1/2)]^d}\left|Q_{d,\ell}(t)e^{-\sum_{k=1}^N\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}\right|^2dt.\end{aligned}$$

The first term in the brackets in the left-hand side of the foregoing inequality converges to $1/2$ as $\ell\to\infty$, while the second term has limit 0. We conclude there exists a constant $\beta>0$ such that

$$\beta\le\int_{[A_\ell(\ell+1/2),(1+\epsilon_\ell)A_\ell(\ell+1/2)]^d}\left|Q_{d,\ell}(t)e^{-\sum_{k=1}^N\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}\right|^2dt,\quad \ell>0. \tag{3.60}$$

If $M_{\ell,N}\ge(\ell+1/2)(1+\epsilon_\ell)A_\ell$ for infinitely many ℓ, there exists a subsequence $(\ell_k)_{k\in\mathbb{N}}$ such that (in particular),

$$\lim_{\ell_k\to\infty}\left\|\mathrm{SINC}(\pi t)-Q_{d,\ell_k}(t)e^{-\sum_{k=1}^N\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell_k+1/2)^{2k-1}}}\right\|_{L_2([A_{\ell_k}(\ell_k+1/2)),A_{\ell_k}(\ell_k+1/2)(1+\epsilon_{\ell_k})]^d)}=0.$$

This contradicts (3.60). This yields that for sufficiently large ℓ,

$$\begin{aligned}M_{\ell,N}&<(\ell+1/2)(1+\epsilon_\ell)A_\ell\\&=(1+\epsilon_\ell)\Big(\Big(\frac{2N+1}{4N+4}+\delta/2\Big)(N+1)(2N+1)(\ell+1/2)^{2N+1}\log(\ell+1/2)\Big)^{\frac{1}{2(N+1)}}.\end{aligned}$$

Note that since $\epsilon_\ell\to0$, for large ℓ, the quantity

$$(1+\epsilon_\ell)\Big(\frac{2N+1}{4N+4}+\delta/2\Big)^{\frac{1}{2(N+1)}}$$

is less than, (and bounded away from) the quantity $\left(\frac{2N+1}{4N+4}+\delta\right)^{\frac{1}{2(N+1)}}$. We conclude that for any $\delta > 0$, there exists $\ell_{N,\delta}$ such that

$$\sup_{\ell > \ell_{N,\delta}} \frac{M_{\ell,N}}{((N+1)(2N+1)\log(\ell+1/2))^{\frac{1}{2(N+1)}}(\ell+1/2)^{\frac{2N+1}{2(N+1)}}} < \left(\frac{2N+1}{4N+4}+\delta\right)^{\frac{1}{2(N+1)}}.$$

Proposition III.40 follows. □

Proposition III.42. *Let $N \geq 0$. If $(M_{\ell,N})_\ell$ is a sequence of positive numbers such that* (3.48) *holds when $(D_{\ell,N})_\ell$ is replaced by $(M_{\ell,N})_\ell$, then*

$$\limsup_{\ell\to\infty} \frac{M_{\ell,N}}{D_{\ell,N}} \leq 1. \tag{3.61}$$

The proof of Proposition III.42 requires the following fact:

Proposition III.43. *Let $0 < \epsilon \leq 1$. If I is a closed interval with length ϵ, then there exists $t \in I$ such that $|\sin(\pi t)| \geq \sin(\pi\epsilon/2)$.*

Proof. The function $f(x) = |\sin \pi x|$ is 1-periodic, so it suffices to prove the proposition for intervals satisfying one of the two following conditions: either 1) $0 \in I$, or 2) $I \subset (0,1)$.

Case 1). Let $I = [-c_1, c_2]$ where $c_1, c_2 \geq 0$, and $c_1 + c_2 = \epsilon$. Then $c_i \geq \epsilon/2$ for some $i = 1, 2$. If $c_i \leq 1/2$, then $\epsilon/2 \leq c_i \leq 1/2$ implies $\sin(\pi\epsilon/2) \leq \sin(\pi c_i)$, so that $\sin(\pi\epsilon/2) \leq |\sin(\pi(\pm c_i))|$ where either c_i or $-c_i$ is in I. If $1/2 < c_i$, then $-c_i < -1/2$. From this, either $t = 1/2$ or $t = -1/2$ is in I.

Case 2) Let $I = [c_1, c_1 + \epsilon]$ where $0 < c_1 < c_1 + \epsilon < 1$. If $1/2 \in I$, we are done. Let $1/2 < c_1 < c_1 + \epsilon < 1$, so that $0 < \epsilon < 1 - c_1 < 1/2$. This yields $\sin(\pi\epsilon/2) < \sin(\pi\epsilon) < \sin\pi(1 - c_1) = \sin(\pi c_1)$. Let $c_1 < c_1 + \epsilon < 1/2$, then $\epsilon/2 < c_1 + \epsilon < 1/2$ implies $\sin(\pi\epsilon/2) < \sin\pi(c_1 + \epsilon)$. □

Proof of Proposition III.42. Let $N \geq 0$. Choose $\delta > 0$ such that $c := \frac{2N+1}{2N+2} + \delta/2 < 1$.

Define

$$A_\ell = (c(N+1)(2N+1)\log(\ell+1/2))^{\frac{1}{2(N+1)}}(\ell+1/2)^{\frac{-1}{2(N+1)}}$$

and

$$\epsilon_\ell = A_\ell(\ell+1/2)^{1-c}.$$

Algebra shows that $\lim_{\ell\to\infty}\epsilon_\ell = 0$. Let $t \in [A_\ell(\ell+1/2), A_\ell(\ell+1/2)+\epsilon_\ell]$. Proceeding as before, for sufficiently large ℓ, we have

$$\frac{1}{2\pi}e^{\left(\frac{t^{2(N+1)}}{(\ell+1/2)^{2N+1}(N+1)(2N+1))}\right)}\frac{|\sin(\pi t)|}{t} \le \left|Q_{1,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}}\right|.$$

Now for all $t \in [A_\ell(\ell+1/2), A_\ell(\ell+1/2)+\epsilon_\ell]$,

$$\frac{1}{2\pi}\frac{(\ell+1/2)^c}{A_\ell(\ell+1/2)+\epsilon_\ell}|\sin(\pi t)| \le \left|Q_{1,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{t^{2k}}{(\ell+1/2)^{2k-1}}}\right|.$$

In the multivariate case, if $t \in [A_\ell(\ell+1/2), A_\ell(\ell+1/2)+\epsilon_\ell]^d$, we obtain

$$\frac{1}{(2\pi)^d}\frac{(\ell+1/2)^{cd}}{(A_\ell(\ell+1/2)+\epsilon_\ell)^d}\prod_{i=1}^{d}|\sin(\pi t_i)| \le \left|Q_{d,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}\right|.$$

For large ℓ, an application of Proposition III.43 yields

$$\frac{1}{(3\pi)^d}\frac{|\sin(\pi\epsilon_\ell/2)|^d}{A_\ell^d(\ell+1/2)^{(1-c)d}} \le \left\|Q_{d,\ell}(t)e^{-\sum_{k=1}^{N}\frac{1}{k(2k-1)}\frac{\|t\|_{2k}^{2k}}{(\ell+1/2)^{2k-1}}}\right\|_{L_\infty([A_\ell(\ell+1/2),A_\ell(\ell+1/2)+\epsilon_\ell]^d)}.$$

By the definition of ϵ_ℓ, the right-hand side of the above equation tends to a positive constant. The remainder of the proof is almost identical to that of Proposition III.40.

□

G. An alternative proof of a special case of Theorem I.1

The main importance of the Theorems III.26 and III.30 is their multidimensional nature; however, we can use them to present an alternative proof of the following special case of Theorem I.1.

Theorem III.44. *Let* $(t_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ *be a sequence such that* $t_k = 0$ *for at most one index* k*. Let*

$$\big(\mathrm{sinc}\pi((\cdot) - t_n)\big)_{n\in\mathbb{Z}}$$

be a UIRB for $PW_{[-\pi,\pi]}$*. The biorthogonal functions* $(G_n)_{n\in\mathbb{Z}}$ *of* $\big(\mathrm{sinc}\pi((\cdot) - t_n)\big)_{n\in\mathbb{Z}}$ *are given by*

$$G_n(t) = \frac{H(t)}{(t-t_n)H'(t_n)},$$

where

$$H(t) = (t - t_0)\prod_{n=1}^{\infty}\Big(1 - \frac{t}{t_n}\Big)\Big(1 - \frac{t}{t_{-n}}\Big).$$

We begin by recalling the following fundamental theorem from complex analysis.

Theorem III.45 (Weierstrass' Factorization Theorem)**.** *Define*

$$E_n(z) = \begin{cases} 1 - z, & n = 0; \\ (1-z)\exp\Big(\frac{z}{1} + \ldots + \frac{z^n}{n}\Big), & n > 0 \end{cases}.$$

Let $(a_n)_{n=1}^{\infty} \subset \mathbb{C}$ *be a sequence such that* $0 < |a_n| \to \infty$*. Let* $(p_n)_{n=1}^{\infty}$ *be a sequence of positive integers which satisfies* $\sum_{n=1}^{\infty}\Big(\frac{r}{|a_n|}\Big)^{p_n+1} < \infty$*, for all* $r > 0$*. If*

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}\Big(\frac{z}{a_n}\Big),$$

then

1) The product above converges uniformly on compacta.

2) f *is an entire function.*

3.) The zero set of f *is* $(a_n)_{n=1}^{\infty}$*, and the multiplicity of each zero is the number of times it occurs in the list* $(a_n)_{n=1}^{\infty}$*.*

Corollary III.46 (Corollary of Theorem III.45)**.** *Let* f *be a entire function not identically zero. Let* 0 *be a root of* f *with multiplicity* m*, and let* $(a_n)_{n=1}^{\infty}$ *be the set* *of non-zero roots of f repeated by multiplicity. If $(p_n)_{n=1}^{\infty}$ is a sequence of positive integers that satisfies*

$$\sum_{n=1}^{\infty}\Big(\frac{r}{|a_n|}\Big)^{p_n+1} < \infty, \quad \forall r > 0,$$

then there exists an entire, non-vanishing, function h such that

$$f(z) = z^m h(z)\prod_{n=1}^{\infty} E_{p_n}\Big(\frac{z}{a_n}\Big),$$

where the product converges uniformly on compacta.

Proof. Note that $|a_n| \to \infty$, otherwise $(a_n)_{n=1}^{\infty}$ would have an accumulation point in the plane, forcing f to be the zero function. Applying Theorem III.45, we see that the function

$$h(z) := \frac{f(z)}{z^m \prod_{n=1}^{\infty} E_{p_n}\Big(\frac{z}{a_n}\Big)}$$

is non-vanishing. $\square$

Proof of Theorem III.44. Fix $n \in \mathbb{Z}$. From the proof of Lemma III.21, the only zeros of G_n are $(t_k)_{k\neq n}$, and they form a uniformly separated set because $(f_n)_{n\in\mathbb{Z}}$ is a Riesz basis. Rearrange $(t_k)_{k\in\mathbb{Z}}$ to $(t_{k(j)})_{j\in\mathbb{Z}}$ such that $j_1 < j_2$ implies $t_{k(j_1)} < t_{k(j_2)}$. Define

$$\delta = \inf_{j_1,j_2\in\mathbb{Z}} |t_{k(j_2)} - t_{k(j_1)}|.$$

For $|j|$ sufficiently large, we have $|t_{k(j)}| > \frac{|j|\delta}{2}$. Let

$$(a_n)_{n=1}^{\infty} = (t_1, t_{-1}, \ldots, t_\ell, t_{-\ell}, \ldots).$$

If $r > 0$, then

$$\sum_{n=1}^{\infty}\Big(\frac{r}{|a_n|}\Big)^2 = \sum_{j:k(j)\neq 0}\Big(\frac{r}{t_{k(j)}}\Big)^2 < \infty,$$

since the j^{th} term of the 2^{nd} sum is of the order $\frac{1}{j^2}$. Letting $p_n = 1$ for all $n > 0$, and applying Corollary III.46, we conclude (after consolidating the cases $n = 0$, $n \neq 0$, $t_0 \neq 0$, $t_0 = 0$), that there exists a non-vanishing entire function h_n such that

$$G_n(t) = h_n(t) \lim_{\ell\to\infty} \Big[\frac{H_\ell(t)}{t-t_n} \exp\Big(t \sum_{k=1}^{\ell} \Big(\frac{1}{t_k} + \frac{1}{t_{-k}}\Big)\Big)\Big], \quad t \in \mathbb{R},$$

where

$$H_\ell(t) = (t-t_0) \prod_{k=1}^{\ell} \Big(1 - \frac{t}{t_k}\Big)\Big(1 - \frac{t}{t_{-k}}\Big)$$

and convergence is uniform on compacta. If, in the notation of Corollary III.31, we let $f = G_n$ and note that $\Psi_\ell(t) = \frac{H_\ell(t)}{(t-t_n)H'_\ell(t_n)}$, then we have

$$G_n(t) = h_n(t) \lim_{\ell\to\infty} \Big[\Psi_\ell(t) H'_\ell(t_n) \exp\Big(t \sum_{k=1}^{\ell} \Big(\frac{1}{t_k} + \frac{1}{t_{-k}}\Big)\Big)\Big], \quad t \in \mathbb{R}. \tag{3.62}$$

Fix $\tau \notin (t_k)_{k\neq n} \cup (-t_k)_{k\neq n}$, that is, $G_n(\tau) \neq 0$ and $G_n(-\tau) \neq 0$; then

$$\frac{G_n(\tau)G_n(-\tau)}{h_n(\tau)h_n(-\tau)} = \lim_{\ell\to\infty} \Psi_\ell(\tau)\Psi_\ell(-\tau)|H'_\ell(t_n)|^2.$$

Recalling that $\lim_{\ell\to\infty} 1/\Psi_\ell(\pm\tau) = 1/G_n(\pm\tau)$, we find that

$$\frac{1}{\sqrt{h_n(\tau)h_n(-\tau)}} = \lim_{\ell\to\infty} |H'_\ell(t_n)|.$$

The equality above holds for $\tau \in \mathbb{R} \setminus ((t_k)_{k\neq n} \cup (-t_k)_{k\neq n})$, so by continuity of h_n, the equality holds for all $\tau \in \mathbb{R}$, hence

$$\frac{1}{|h_n(0)|} = \lim_{\ell\to\infty} |H'_\ell(t_n)|.$$

Let $t \notin (t_k)_{k\neq n}$, that is, $G_n(t) \neq 0$. Noting that

$$|G_n(t)| = \lim_{\ell\to\infty} |\Psi_\ell(t)|||H'_\ell(t)| \exp\Big(t \sum_{k=1}^{\ell} \Big(\frac{1}{t_k} + \frac{1}{t_{-k}}\Big)\Big),$$

we obtain

$$|h_n(0)| = |h_n(t)| \lim_{\ell\to\infty} \exp\left(t \sum_{k=1}^{\ell} \left(\frac{1}{t_k} + \frac{1}{t_{-k}}\right)\right), \quad t \notin (t_k)_{k\neq n}.$$

Now h_n is real-valued and non-vanishing, so

$$0 \neq L_n := \frac{1}{h_n(0)} = \lim_{\ell\to\infty} \frac{1}{h_n(t) \exp\left(t \sum_{k=1}^{\ell} \left(\frac{1}{t_k} + \frac{1}{t_{-k}}\right)\right)}, \quad t \notin (t_k)_{k\neq n}.$$

Combined with (3.62), we have

$$0 \neq L_n = \lim_{\ell\to\infty} H_\ell'(t_n). \tag{3.63}$$

From

$$G_n(t) = \lim_{\ell\to\infty} \frac{H_\ell(t)}{(t-t_n)H_\ell'(t_n)}$$

and (3.63), we obtain

$$(t-t_n)L_n G_n(t) = \lim_{\ell\to\infty} (t-t_0) \prod_{k=1}^{\ell} \left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right) := H(t),$$

so that

$$G_n(t) = \frac{H(t)}{(t-t_n)L_n} = \frac{H(t) - H(t_n)}{(t-t_n)L_n}.$$

Letting $t \to t_n$ in the above equation shows that $L_n = H'(t_n)$, and the proof is complete. $\square$

CHAPTER IV

OVERSAMPLING AND MULTIVARIATE BANDLIMITED FUNCTIONS*

A. Introduction

This chapter is outlined as follows.* In Section B, we derive an oversampling formula for multivariate functions whose frequency domain is a fairly general set E, (see Proposition IV.1), when the sampling sites are $(t_n)_{n\in\mathbb{N}}$, where $(e^{i\langle(\cdot),t_n\rangle})_{n\in\mathbb{N}}$ forms a frame for $L_2(E)$. Section C investigates the stability of (4.1) under perturbation of the sampled data along with concrete examples. Section D presents a computationally feasible version of (4.1) in the case where the set $(e^{i\langle(\cdot),t_n\rangle})_{n\in\mathbb{N}}$ is a Riesz basis.

B. The multidimensional oversampling theorem

In this section we derive a multidimensional version of (1.3), (Theorem IV.3) for unequally spaced sample points, and the corresponding non-oversampling formula is given in Theorem IV.6.

In their proof of (1.3), Daubechies and DeVore regard $\mathcal{F}^{-1}(f)$ as an element of $L_2[-\lambda\pi,\lambda\pi]$ for some $\lambda>1$. The fact that $[-\pi,\pi]\subset[-\lambda\pi,\lambda\pi]$ allows for the construction of the bump function $\mathcal{F}^{-1}(g)\in C^\infty(\mathbb{R})$ which is 1 on $[-\pi,\pi]$ and 0 off $[-\lambda\pi,\lambda\pi]$. If their result is to be generalized to a sampling theorem for PW_E in higher dimensions, a suitable condition for E allowing the existence of a bump function is necessary. If $0\in E\subset\mathbb{R}^d$ is chosen to be compact such that for all $\lambda>1$, $E\subset\operatorname{int}(\lambda E)$, then Lemma 8.18 in [7, page 245], a C^∞-version of the Urysohn lemma, implies the existence of a smooth bump function which is 1 on E and 0 off λE. It is

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to such regions that we generalize (1.3).

There is a geometric characterization of compact sets $E \subset \mathbb{R}^d$ containing 0 such that $E \subset \operatorname{int}(\lambda E)$ for all $\lambda > 1$. Intuitively, E must be a continuous radial stretching of the closed unit ball. This is formulated precisely in the following proposition.

Proposition IV.1. *If $0 \in E \subset \mathbb{R}^d$ is compact, then the following are equivalent:*

1) $E \subset \operatorname{int}(\lambda E)$ for all $\lambda > 1$.

2) There exists a continuous map $\phi : S^{d-1} \to (0, \infty)$ such that

$$E = \{ty\phi(y) \mid y \in S^{d-1}, t \in [0, 1]\}.$$

The proof needs the following definition:

Definition IV.2. A subset $S \in \mathbb{R}^d$ is said to be *starshaped* about 0 if

$$[0, x] := \{tx \mid t \in [0, 1]\} \subset S$$

whenever $x \in S$.

Proof of Proposition IV.1. 1) $\Rightarrow$ 2): E is starshaped about 0: If not, there is $x_0 \in E$, $0 < t_0 < 1$, such that $t_0x_0 \notin E$. Let $\lambda = \frac{1}{t_0} > 1$. Now $x_0 \in \lambda E$, so $t_0x_0 = \frac{1}{\lambda}x_0 \in E$. Define $\phi : S^{d-1} \to (0, \infty)$ by $x \mapsto \sup\{\lambda \geq 0 \mid \lambda x \in E\}$.

ϕ is well defined: Certainly $\phi : S^{d-1} \to [0, \infty)$ is well-defined since E is bounded and $0 \in E$. We need to show that $0 \notin \phi(S^{d-1})$. Now $0 \in E$ implies $0 \in \operatorname{int}(2E)$. There exists an ϵ-ball B_ϵ about 0 such that $0 \in B_\epsilon \subset 2\operatorname{int}(E)$, so $0 \in B_{\epsilon/2} \subset \operatorname{int}(E)$. So for all $x \in S^{d-1}$, we have $\frac{\epsilon}{3}x \in B_{\epsilon/2} \subset \operatorname{int}(E)$. So $\phi(x) \geq \epsilon/3$.

Note that $x\phi(x) \in E$ for all $x \in S^{d-1}$: There exists $\lambda_i \nearrow \phi(x)$ such that $\lambda_i x \in E$, so that $\lambda_i x \to x\phi(x)$. As E is closed it follows that $x\phi(x) \in E$.

ϕ is continuous: Suppose not; then there exists $y \in S^{d-1}, \epsilon > 0, (x_n)_{n=1}^{\infty} \subset S^{d-1}$ such that $x_n \to y$ and $|\phi(x_n) - \phi(y)| \geq \epsilon$. Now $\phi(S^{d-1})$ is bounded, so there exists a subsequence $(x_{n_k})_k \to y$ (relabeled as $(x_k)_k$) and $c \in \mathbb{R}^d$ such that $\phi(x_k) \to c$. Now $x_k\phi(x_k) \in E \to cy$ implies $cy \in E$ since E is closed. This yields $c \leq \phi(y)$. Now $|\phi(x_n) - \phi(y)| \geq \epsilon$ implies $c < \phi(y)$. Choose any $t \in (c, \phi(y))$. As E is starshaped, $ty \in E$. If ty is an interior point of E, then $tx_k \in E$ for sufficiently large k. This implies that $t \leq \phi(x_k)$ for large k, which implies $t \leq c$. We conclude that ty isn't an interior point. So any $ty \in [cy, y\phi(y)]$ is a boundary point of E. Choose $\lambda > 1$ such that $\lambda c < \phi(y)$, then $y\phi(y) \in E$ is in $[\lambda cy, \lambda y\phi(y)]$ which consists of boundary points of λE, but 1) implies $y\phi(y) \in \text{int}(\lambda E)$, so ty isn't a boundary point of E either. We conclude that ϕ must be continuous.

Observing that $\{ty\phi(y) | y \in S^{d-1}, t \in [0,1]\}$ is also starshaped, it is almost immediate that it coincides with E.

2) $\Rightarrow$ 1): Given that $\phi : S^{d-1} \to (0, \infty)$ continuous, define

$$E = \{ty\phi(y) | y \in S^{d-1}, t \in [0,1]\}.$$

Let $\mathbb{B}^d$ be the closed unit ball in $\mathbb{R}^d$. Note that each point in $\mathbb{B}^d \setminus 0$ can be written uniquely in the form ty where $t \in (0,1]$ and $y \in S^{d-1}$. Define $\psi : \mathbb{B}^d \to E$ by $0 \mapsto 0$, and $ty \mapsto ty\phi(y)$. ψ is clearly a continuous and onto. To verify that ψ is one to one, note that $t_1y_1\phi(y_1) = t_2y_2\phi(y_2)$ implies $t_1\phi(y_1) = t_2\phi(y_2)$, so that $y_1 = y_2$. ψ is a continuous bijection from $\mathbb{B}^d$ to E. Standard topology implies that ψ is a homeomorphism since $\mathbb{B}^d$ is compact and E is Hausdorff. In particular we have $\partial E = \psi(S^{d-1})$. Note that as λE is starshaped, $E \subset \lambda E$. Suppose $E \not\subseteq \text{int}(\lambda E)$ for some λ, then there is some $x_0 \in E$ such that $x_0 \in \lambda(\partial E)$, so that x_0 can be written in the form $\lambda\phi(y)y$ for some $y \in S^{d-1}$, but $\lambda\phi(y)y \notin E$. We conclude $E \subset \text{int}(\lambda E)$ for $\lambda > 1$. $\square$

We are now ready to state Theorem IV.3, which is a slight modification of Theorem 3.1 in [22]. To ease calculation, in this chapter we use the isomorphic Fourier transform

$$\mathcal{F}(f)(\cdot) = \text{P.V.} \int_{\mathbb{R}^d} f(\xi)e^{-i\langle\cdot,\xi\rangle}d\xi, \quad f \in L_2(\mathbb{R}^d),$$

with inverse transform

$$\mathcal{F}^{-1}(f)(\cdot) = \text{P.V.} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi)e^{i\langle\cdot,\xi\rangle}d\xi, \quad f \in L_2(\mathbb{R}^d).$$

Theorem IV.3. *Let $0 \in E \subset \mathbb{R}^d$ be compact such that for all $\lambda > 1$, $E \subset \text{int}(\lambda E)$. Choose $\mathcal{T} = (t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $(f_n)_{n\in\mathbb{N}}$, defined by $f_n(\cdot) = \alpha e^{i\langle\cdot,t_n\rangle}$, $\alpha > 0$, is a frame for $L_2(E)$ with frame operator S. Let $\lambda_0 > 1$ with $\mathcal{F}^{-1}(g) : \mathbb{R}^d \to \mathbb{R}$, $\mathcal{F}^{-1}(g) \in C^\infty$, where $0 \le F^{-1}(g) \le 1$ on $\mathbb{R}^d$, $\mathcal{F}^{-1}(g)|_E = 1$ and $\mathcal{F}^{-1}(g)|_{(\lambda_0 E)^c}$=0. If $\lambda \ge \lambda_0$ and $f \in PW_E$, then*

$$f(t) = \frac{\alpha^2}{\lambda^d} \sum_{k\in\mathbb{N}} \Big(\sum_{n\in\mathbb{N}} B_{kn} f\Big(\frac{t_n}{\lambda}\Big)\Big) g\Big(t - \frac{t_k}{\lambda}\Big), \quad t \in \mathbb{R}^d, \tag{4.1}$$

where $B_{kn} = \langle S^{-1}f_n, S^{-1}f_k\rangle_E$. Convergence of the sum is in $L_2(\mathbb{R}^d)$, hence also in $L_\infty(\mathbb{R}^d)$. Furthermore, the map $B : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ defined by

$$(y_k)_{k\in\mathbb{N}} \mapsto \Big(\sum_{n\in\mathbb{N}} B_{kn}y_n\Big)_{k\in\mathbb{N}}$$

is bounded linear, and is an onto isomorphism if and only if $(f_n)_{n\in\mathbb{N}}$ is a Riesz basis for $L_2(E)$.

Before embarking on the proof, we need two definitions.

Definition IV.4. If $\mathcal{T} = (x_k)_k$ is a sequence in $\mathbb{R}^d$ and f is a function with $\mathcal{T}$ in its domain, then $f_{\mathcal{T}}$ denotes the sequence $(f(x_k))_k$.

Definition IV.5. Define $f_{\lambda,n}(\cdot) = f_n\big(\frac{\cdot}{\lambda}\big)$. Note that $(f_{\lambda,n})_n$ is a frame for $L_2(\lambda E)$. Denote its frame operator by S_λ.

Proof of Theorem IV.3. Step 1: We show that

$$f = \alpha \sum_n f\Big(\frac{t_n}{\lambda}\Big)\mathcal{F}[(S_\lambda^{-1} f_{\lambda,n})\mathcal{F}^{-1}(g)], \quad f \in PW_E. \tag{4.2}$$

We know $\mathrm{supp}(\mathcal{F}^{-1}(f)) \subset E \subset \lambda E$, so we may work with $\mathcal{F}^{-1}(f)$ via its frame decomposition. We have

$$\mathcal{F}^{-1}(f) = S_\lambda^{-1} S_\lambda(\mathcal{F}^{-1}(f)) = \sum_n \langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E} S_\lambda^{-1} f_{\lambda,n}, \quad \text{on} \quad \lambda E.$$

This yields

$$\mathcal{F}^{-1}(f) = \sum_n \langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E} (S_\lambda^{-1} f_{\lambda,n})\mathcal{F}^{-1}(g), \quad \text{on} \quad \mathbb{R}^d,$$

since $\mathcal{F}(g) = 1$ on the support of $\mathcal{F}^{-1}(f)$. Taking Fourier transforms we obtain

$$f = \sum_n \langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E} \mathcal{F}[(S_\lambda^{-1} f_{\lambda,n})\mathcal{F}^{-1}(g)], \quad \text{on} \quad \mathbb{R}^d. \tag{4.3}$$

Now

$$\langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E} = \int_{\lambda E} \mathcal{F}^{-1}(f)(\xi)\alpha e^{-i\langle \xi, \frac{t_n}{\lambda}\rangle} d\xi = \alpha f\Big(\frac{t_n}{\lambda}\Big)$$

which, when substituted into (4.3), yields (4.2).

Step 2: We show that

$$f(\cdot\,) = \alpha^2 \sum_n f\Big(\frac{t_n}{\lambda}\Big)\Big[\sum_k \langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E}\, g\Big(\cdot - \frac{t_k}{\lambda}\Big)\Big], \tag{4.4}$$

where convergence is in L_2: We compute $\mathcal{F}[(S_\lambda^{-1} f_{\lambda,n})\mathcal{F}^{-1}(g)]$. For $h \in L_2(\lambda E)$ we have

$$h = S_\lambda(S_\lambda^{-1} h) = \sum_k \langle S_\lambda^{-1} h, f_{\lambda,k}\rangle_{\lambda E} f_{\lambda,k} = \sum_k \langle h, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E} f_{\lambda,k}.$$

Letting $h = S_\lambda^{-1} f_{\lambda,n}$, we have

$$S_\lambda^{-1} f_{\lambda,n} = \sum_k \langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E} f_{\lambda,k}.$$

This gives

$$\begin{aligned}
\mathcal{F}[(S_\lambda^{-1} f_{\lambda,n})\mathcal{F}^{-1}(g)](\cdot) &= \sum_k \langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E} \mathcal{F}[f_{\lambda,k}\mathcal{F}^{-1}(g)](\cdot) \\
&= \sum_k \langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E} \int_{\lambda E} \alpha e^{i\langle \xi, \frac{t_k}{\lambda}\rangle} \mathcal{F}^{-1}(g)(\xi) e^{-i\langle \xi, \cdot\rangle} d\xi \\
&= \sum_k \langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E} \int_{\lambda E} \alpha \mathcal{F}^{-1}(g)(\xi) e^{-i\langle \cdot - \frac{t_k}{\lambda}, \xi\rangle} d\xi \\
&= \alpha \sum_k \langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E}\ [\mathcal{F}\mathcal{F}^{-1} g]\Big(\cdot - \frac{t_k}{\lambda}\Big) \\
&= \alpha \sum_k \langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E}\ g\Big(\cdot - \frac{t_k}{\lambda}\Big),
\end{aligned}$$

so (4.4) follows from (4.2).

Step 3: We show that

$$\langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E} = \frac{1}{\lambda^d}\langle S^{-1} f_n, S^{-1} f_k\rangle_E, \quad \text{for} \quad n,k \in \mathbb{N}. \tag{4.5}$$

First we show $(S_\lambda^{-1} f_{\lambda,n})(\cdot) = \frac{1}{\lambda^d}(S^{-1} f_n)(\frac{\cdot}{\lambda})$, or equivalently that

$$f_{\lambda,n} = \frac{1}{\lambda^d} S_\lambda\big((S^{-1} f_n)(\frac{\cdot}{\lambda})\big).$$

We have for any $g \in L_2(\lambda E)$,

$$\langle g, f_{\lambda,k}\rangle_{\lambda E} = \int_{\lambda E} g(\xi)\alpha e^{-i\langle \frac{\xi}{\lambda}, t_k\rangle} d\xi = \lambda^d \int_E g(\lambda x)\alpha e^{-i\langle x, t_k\rangle} dx = \lambda^d \langle g(\lambda(\cdot)), f_k\rangle_E.$$

By definition of the frame operator S_λ,

$$S_\lambda g = \sum_{k\in\mathbb{N}} \langle g, f_{\lambda,k}\rangle_{\lambda E} f_{\lambda,k},$$

which then becomes

$$S_\lambda g = \lambda^d \sum_k \langle g(\lambda(\cdot)), f_k\rangle_E f_{\lambda,k}.$$

Substituting $g = \frac{1}{\lambda^d}(S^{-1}f_n)(\frac{\cdot}{\lambda})$ into the equation above we obtain

$$\frac{1}{\lambda^d} S_\lambda\big((S^{-1}f_n)(\frac{\cdot}{\lambda})\big) \;=\; \sum_k \langle S^{-1}f_n, f_k\rangle_E f_{\lambda,k} = \big(S(S^{-1}f_n)\big)\Big(\frac{\cdot}{\lambda}\Big) = f_{\lambda,n}.$$

We now compute the desired inner product:

$$\begin{aligned}\langle S_\lambda^{-1} f_{\lambda,n}, S_\lambda^{-1} f_{\lambda,k}\rangle_{\lambda E} &= \frac{1}{\lambda^{2d}} \int_{\lambda E} (S^{-1}f_n)\Big(\frac{x}{\lambda}\Big)\overline{(S^{-1}f_k)\Big(\frac{x}{\lambda}\Big)}dx \\ &= \frac{\lambda^d}{\lambda^{2d}} \int_E (S^{-1}f_n)(x)\overline{(S^{-1}f_k)(x)}dx = \frac{1}{\lambda^d}\langle S^{-1}f_n, S^{-1}f_k\rangle_E.\end{aligned}$$

Note that (4.4) becomes

$$f(\cdot\,) = \frac{\alpha^2}{\lambda^d} \sum_n f\Big(\frac{t_n}{\lambda}\Big)\Big[\sum_k \langle S^{-1}f_n, S^{-1}f_k\rangle g\Big(\cdot - \frac{t_k}{\lambda}\Big)\Big]. \tag{4.6}$$

Step 4: The map $B : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ given by $(x_k)_{k\in\mathbb{N}} \mapsto \big(\sum_n B_{kn}x_n\big)_{k\in\mathbb{N}}$ is bounded linear and self-adjoint: Let $(d_k)_{k\in\mathbb{N}}$ be the standard basis for $\ell_2(\mathbb{N})$, and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis for $L_2(E)$. If $Le_n = f_n$ is the synthesis operator, then $S = LL^*$, and we have

$$B_{kj} = \langle S^{-1}f_j, S^{-1}f_k\rangle = \langle L^*(S^{-1})^2 Le_j, e_k\rangle.$$

It follows that the map $B : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ is (after the change of basis $d_n \mapsto e_n$), the map

$$L^*(S^{-1})^2 L : L_2(E) \to L_2(E),$$

which is bounded linear and self-adjoint. Clearly B is an onto isomorphism if and only if L and L^* are both onto, i.e., if and only if the map $Le_n = f_n$ is an onto isomorphism.

Step 5: Verification of (4.1). Note that $f\left(\frac{\cdot}{\lambda}\right), g\left(t-\frac{\cdot}{\lambda}\right) \in L_2(\lambda E)$, and recall that $(f_{\lambda,n})_n$ is a frame for $L_2(\lambda E)$, say with upper frame bound B_λ. We have

$$\sum_n \left|f\left(\frac{t_n}{\lambda}\right)\right|^2 = \sum |\langle \mathcal{F}^{-1}(f), f_{\lambda,n}\rangle_{\lambda E}|^2 \leq B_\lambda \|\mathcal{F}^{-1}(f)\|^2, \tag{4.7}$$

and

$$\sum_n \left|g\left(t-\frac{t_n}{\lambda}\right)\right|^2 = \sum \left|\left\langle \mathcal{F}^{-1}\left(g\left(t-\frac{\cdot}{\lambda}\right)\right), f_{\lambda,n}\right\rangle_{\lambda E}\right|^2 \leq B_\lambda \left\|\mathcal{F}^{-1}\left(g\left(t-\frac{\cdot}{\lambda}\right)\right)\right\|^2.$$

For each $t \in \mathbb{R}^d$, let $g_\lambda(t) = \left(g\left(t-\frac{t_n}{\lambda}\right)\right)_{n\in\mathbb{N}}$, and recall Definition IV.4. Note that (4.6) becomes

$$\begin{aligned}
f(t) &= \frac{\alpha^2}{\lambda^d}\sum_n f\left(\frac{t_n}{\lambda}\right)\Big[\sum_k B_{kn} g\left(t-\frac{t_k}{\lambda}\right)\Big] = \frac{\alpha^2}{\lambda^d}\sum_n f\left(\frac{t_n}{\lambda}\right)\overline{\Big[\sum_k B_{nk}\overline{g\left(t-\frac{t_k}{\lambda}\right)}\Big]} \\
&= \frac{\alpha^2}{\lambda^d}\sum_n (f_{\mathcal{T}/\lambda})_n(B\overline{g_\lambda(t)})_n = \frac{\alpha^2}{\lambda^d}\langle f_{\mathcal{T}/\lambda}, B\overline{g_\lambda(t)}\rangle = \frac{\alpha^2}{\lambda^d}\langle Bf_{\mathcal{T}/\lambda}, \overline{g_\lambda(t)}\rangle \\
&= \frac{\alpha^2}{\lambda^d}\sum_k (Bf_{\mathcal{T}/\lambda})_k g\left(t-\frac{t_k}{\lambda}\right) = \frac{\alpha^2}{\lambda^d}\sum_{k\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}} B_{kn} f\left(\frac{t_n}{\lambda}\right)\right) g\left(t-\frac{t_k}{\lambda}\right),
\end{aligned}$$

which proves (4.1).

Step 6: We verify that (4.1) converges in $L_2(\mathbb{R}^d)$ (and hence uniformly). Define

$$f_n(t) = \frac{\alpha^2}{\lambda^d}\sum_{1\leq k\leq n}(Bf_{\mathcal{T}/\lambda})_k g\left(t-\frac{t_k}{\lambda}\right)$$

and

$$f_{m,n}(t) = \frac{\alpha^2}{\lambda^d}\sum_{m\leq k\leq n}(Bf_{\mathcal{T}/\lambda})_k g\left(t-\frac{t_k}{\lambda}\right).$$

Then

$$\begin{aligned}
[\mathcal{F}^{-1}(f_{m,n})](\xi) &= \frac{\alpha^2}{\lambda^d}\sum_{m\leq k\leq n}(Bf_{\mathcal{T}/\lambda})_k \mathcal{F}^{-1}\Big[g\left(\xi-\frac{t_n}{\lambda}\right)\Big] \\
&= \frac{\alpha^2}{\lambda^d}\sum_{m\leq k\leq n}(Bf_{\mathcal{T}/\lambda})_k \mathcal{F}^{-1}(g)(\xi)e^{i\langle \xi, \frac{t_k}{\lambda}\rangle},
\end{aligned}$$

so

$$\begin{aligned}\|[\mathcal{F}^{-1}(f_{m,n})]\|_2^2 &= \frac{\alpha^2}{\lambda^d}\int_{\lambda E}|\mathcal{F}^{-1}(g)(\xi)|^2\Big|\sum_{m\le k\le n}(Bf_{\mathcal{T}/\lambda})_k e^{i\langle \xi,\frac{t_k}{\lambda}\rangle}\Big|^2 d\xi\\ &\le \frac{\alpha^2}{\lambda^d}\Big\|\sum_{m\le k\le n}(Bf_{\mathcal{T}/\lambda})_k f_{\lambda,k}\Big\|_2^2.\end{aligned}$$

If $(h_n)_n$ is a orthonormal basis for $L_2(\lambda E)$, then the map $Th_k = f_{\lambda,k}$ (the synthesis operator) is bounded linear, so

$$\|[\mathcal{F}^{-1}(f_{m,n})]\|_2^2 \le \frac{\alpha^2}{\lambda^d}\Big\|T\Big(\sum_{m\le k\le n}(Bf_{\mathcal{T}/\lambda})_k h_k\Big)\Big\|_2^2 \le \frac{\alpha^2}{\lambda^d}\|T\|^2\sum_{m\le k\le n}|(Bf_{\mathcal{T}/\lambda})_k|^2.$$

But $Bf_{\mathcal{T}/\lambda}\in\ell^2(\mathbb{N})$, so $\|[\mathcal{F}^{-1}(f_{m,n})]\|_2\to 0$ as $m,n\to\infty$. As $\mathcal{F}^{-1}$ is an onto isomorphism, we have $\|f_{m,n}\|\to 0$, implying that $\|f-f_n\|\to 0$ as $n\to\infty$. □

Note that (4.1) is conveniently written as

$$f(t)=\frac{\alpha^2}{\lambda^d}\sum_k (Bf_{\mathcal{T}/\lambda})_k g\Big(t-\frac{t_k}{\lambda}\Big),\quad t\in\mathbb{R}^d. \tag{4.8}$$

The following is a version of Theorem IV.3 corresponding to $\lambda = 1$.

Theorem IV.6. *Choose $(t_n)_{n\in\mathbb{N}}\subset\mathbb{R}^d$ such that $(f_n)_{n\in\mathbb{N}}$, defined by*

$$f_n(\cdot\,)=\frac{1}{(2\pi)^{d/2}}e^{i\langle\cdot,t_n\rangle},$$

is a frame for $L_2([-\pi,\pi]^d)$. If $f\in PW_E$, then

$$f(t)=\sum_{k\in\mathbb{N}}\Big(\sum_{n\in\mathbb{N}}B_{kn}f(t_n)\Big)\mathrm{SINC}(\pi(t-t_k)),\quad t\in\mathbb{R}^d. \tag{4.9}$$

The matrix B and the convergence of the sum are as in Theorem IV.3.

The proof of Theorem IV.6 is a simplification of the proof of Theorem IV.3, and is omitted. We can write (4.9) as

$$f(t) = \sum_{k\in\mathbb{N}} (Bf_{\mathcal{T}})_k \mathrm{SINC}(\pi(t-t_k)). \tag{4.10}$$

Theorem IV.6 is similar in spirit to Theorem 1.9 in [23, page 19].

C. Comments regarding the stability of Theorem IV.3

A desirable trait in a recovery formula is stability given error in the sampled data. Theorem IV.7 given below is an analogue of Theorem I.3 which applies to (4.1) under an additional assumption about the symmetry of E about 0.

Theorem IV.7. *Let the domain E be symmetric about 0, and let $(t_n)_{n\in\mathbb{N}}$, λ, and g (taken to be real valued) satisfy the hypotheses of Theorem IV.3. Additionally, assume that the map $x \mapsto Bx$ (interpreted as matrix multiplication) is bounded from ℓ_p to ℓ_∞ for some $1 \le p < \infty$. If $\epsilon = (\epsilon_n)_{n\in\mathbb{N}} \in \ell_p$, and*

$$\tilde{f}_{\lambda,\epsilon}(t) := \frac{1}{\lambda^d}\sum_{k\in\mathbb{N}}\Big(\sum_{n\in\mathbb{N}} B_{kn}\Big[f\big(\frac{t_n}{\lambda}\big)+\epsilon_n\Big]\Big)g\Big(t-\frac{t_k}{\lambda}\Big), \quad t\in\mathbb{R}^d,$$

then

$$\|f-\tilde{f}_{\lambda,\epsilon}\|_{L_\infty} \le \tag{4.11}$$
$$\|\epsilon\|_{\ell_p}\|B\|\Big[\Big(\sum_{i=1}^{\ell}\frac{1}{\Delta_i^d}\Big)\|g\|_{L_1} + \Big(\sum_{i=1}^{\ell}\frac{1}{\Delta_i^{d-1}}\Big)\frac{1}{\lambda}\int_{\mathbb{R}^d}\|(\nabla g)(x)\|_{\ell_1^d}dx + o\Big(\frac{1}{\lambda}\Big)\Big],$$

where

$$(t_k)_{k\in\mathbb{N}} = \bigcup_{i=1}^{\ell}(\tau_k^i)_{k\in S_i}, \quad S_i\subset\mathbb{N}, \quad \Delta_i = \inf_{k\neq j}\|\tau_k^i-\tau_j^i\|_{\ell_\infty} > 0.$$

Before we prove Theorem IV.7, we note that the assumption that B be continuous can be formally weakened:

Proposition IV.8. *If $(b_{nm})_{n,m\in\mathbb{N}}$ is an infinite matrix, and the map*

$$x := (x_k)_k \mapsto \Big(\sum_j b_{kj}x_j\Big)_k := Bx$$

is well defined as a linear function from $\ell_p(\mathbb{N})$ to $\ell_\infty(\mathbb{N})$ for some $1 \le p \le \infty$, then it is bounded.

Proof. Observe that the map B is $\ell_p \to \ell_\infty$ continuous if and only if

$$\sup_n \|(b_{nm})_m\|_{\ell_q} < \infty.$$

1) First, if $(c_k)_{k=1}^\infty$ is a sequence in $\mathbb{C}$ such that

$$F : \ell_p(\mathbb{N}) \to \mathbb{C}, \quad Fx = \sum_{k=1}^\infty c_k x_k,$$

is well defined as a linear function, then F is continuous: Define $F_n : \ell_p(\mathbb{N}) \to \mathbb{C}$ by $F_n x = \sum_{k=1}^n c_k x_k$ for $n \ge 1$. Given any $x = (x_k)_{k=1}^\infty \in \ell_p(\mathbb{Z})$, note that

$$\begin{aligned}\sup_{n\ge1} |F_n x| &= \sup_{n\ge1}\Big|\sum_{k=1}^n c_k x_k\Big| \le \sup_{n\ge1}\sum_{k=1}^n |c_k||x_k| = \sum_{k=1}^\infty |c_k||x_k| \\ &= F\big((|x_k|\mathrm{sign}(c_k))_{k=1}^\infty\big) < \infty\end{aligned}$$

because F is well defined. Let $1/p + 1/q = 1$. By the Uniform Boundedness Principle, we have

$$\sup_{n\ge1}\|F_n\| = \sup_{n\ge1}\big(\sum_{k=1}^n |c_k|^q\big)^{\frac{1}{q}} = \|(c_k)_{k\in\mathbb{N}}\|_q < \infty,$$

so $|Fx| \le \|(c_k)_{k\in\mathbb{N}}\|_q\|x\|_p$.

2) As B is well defined, we have $\|Bx\|_\infty = \sup_{n\ge1}\big|\sum_{k=1}^\infty b_{nk}x_k\big| < \infty$ for any $x \in \ell_p(\mathbb{N})$, so that for all $n \ge 1$, the map $B_n : \ell_p(\mathbb{N}) \to \mathbb{C}$ given by $B_n x = \sum_{k=1}^\infty b_{nk}x_k$ is well defined. Applying part 1), we conclude that $\|B_n\| < \infty$. Thus $\sup_{n\ge1}|B_n x| < \infty$ for all $x \in \ell_p(\mathbb{N})$, so by the Uniform Boundedness Principle, $\sup_{n\in\mathbb{Z}}\|B_n\| < \infty$. We conclude $\|Bx\|_\infty \le \Big(\sup_n \|B_n\|\Big)\|x\|_p$. ∎

The proof of Theorem IV.7 requires the following lemma.

Lemma IV.9. *Let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ be a relatively uniformly separated set, and let $g : \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function. If $\lambda > 0$, then*

$$\sup_{t\in\mathbb{R}^d} \frac{1}{\lambda^d} \sum_{k\in\mathbb{N}} \Big| g\Big(t - \frac{t_k}{\lambda}\Big)\Big| \leq$$
$$\Big(\sum_{i=1}^{\ell} \frac{1}{\Delta_i^d}\Big)\|g\|_{L_1} + \Big(\sum_{i=1}^{\ell} \frac{1}{\Delta_i^{d-1}}\Big)\frac{1}{\lambda}\int_{\mathbb{R}^d} \|(\nabla g)(x)\|_{\ell_1^d} dx + o\Big(\frac{1}{\lambda}\Big), \tag{4.12}$$

where

$$(t_k)_{k\in\mathbb{N}} = \bigcup_{i=1}^{\ell} (\tau_k^i)_{k\in S_i}, \quad S_i \subset \mathbb{N}, \quad \Delta_i = \inf_{k\neq j} \|\tau_k^i - \tau_j^i\|_{\ell_\infty} > 0.$$

The proof of Lemma IV.9 requires several propositions.

Proposition IV.10. *Let $\Omega \subset \mathbb{R}^d$ be convex, and $g : \mathbb{R}^d \to \mathbb{R}$ be continuously differentiable. Then*

$$|g(x) - g(y)| \leq \max_{\omega\in\Omega} \|(\nabla g)(\omega)\|_{\ell_1^d} \|x - y\|_{\ell_\infty^d}, \quad x, y \in \Omega.$$

Proof. Let $u \in \mathbb{R}^d$. If $x, x + u \in \Omega$, then

$$g(x + u) - g(x) = (\nabla g)(\tilde{x}) \cdot u, \ \ \text{some } \tilde{x} \in [x, x + u].$$

By Hölder's inequality and convexity,

$$|g(x + u) - g(x)| \leq \max_{\omega\in\Omega} \|(\nabla g)(\omega)\|_{\ell_1^d} \|u\|_{\ell_\infty^d}.$$

□

Definition IV.11. For $n \in \mathbb{Z}^d$, define

$$C_n = [n_1 - 1/2, n_1 + 1/2) \times \cdots \times [n_d - 1/2, n_d + 1/2).$$

The following observations are immediate:

1) If $\alpha > 0$, then $(\alpha C_n)_{n\in\mathbb{Z}^d}$ is a disjoint cover for $\mathbb{R}^d$.

2) If $x, y \in \alpha C_n$, $\alpha > 0$, then $\|x-y\|_{\ell_\infty^d} < \alpha$ and $\|x-\alpha n\|_{\ell_\infty^d} \leq \alpha/2$.

Proposition IV.12. *If $(t_k)_{k\in S} \subset \mathbb{R}^d$ is a sequence (possibly finite) satisfying*

$$\inf_{k\neq j} \|t_k - t_j\|_{\ell_\infty^d} = \Delta,$$

and $g : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function, then

$$\frac{\Delta^d}{\lambda^d}\sum_{k\in S}\left|g\Big(\frac{t_k}{\lambda}\Big)\right| \leq \|g\|_{L_1} + \frac{\Delta}{\lambda}\sum_{n\in\mathbb{Z}^d}\max_{\omega\in\frac{\Delta}{\lambda}C_n}\|(\nabla g)(\omega)\|_{\ell_1^d}\Big(\frac{\Delta}{\lambda}\Big)^d. \tag{4.13}$$

Proof. Part 1): Note there exists unique $n(k)$ such that $\frac{t_k}{\lambda} \in \frac{\Delta}{\lambda}C_{n(k)}$. If $k \neq j$, then $\|\frac{t_k}{\lambda} - \frac{t_j}{\lambda}\| \geq \frac{\Delta}{\lambda}$, so by observation 2), $n(k) \neq n(j)$. By Proposition IV.10 and observation 2),

$$\left|g\Big(\frac{t_k}{\lambda}\Big) - g\Big(\frac{\Delta n(k)}{\lambda}\Big)\right| \leq \max_{\omega\in\frac{\Delta}{\lambda}C_n}\|(\nabla g)(\omega)\|_{\ell_1^d}\frac{\Delta}{2\lambda},$$

yielding

$$\begin{aligned}
\frac{\Delta^d}{\lambda^d}\sum_{k\in S}\left|g\Big(\frac{t_k}{\lambda}\Big)\right| &\leq \frac{\Delta^d}{\lambda^d}\sum_{k\in S}\left|g\Big(\frac{\Delta n(k)}{\lambda}\Big)\right| + \frac{\Delta}{2\lambda}\sum_{k\in\mathbb{N}}\max_{\omega\in\frac{\Delta}{\lambda}C_{n(k)}}\|(\nabla g)(\omega)\|_{\ell_1^d}\Big(\frac{\Delta}{\lambda}\Big)^d \\
&\leq \frac{\Delta^d}{\lambda^d}\sum_{n\in\mathbb{Z}^d}\left|g\Big(\frac{\Delta n}{\lambda}\Big)\right| + \frac{\Delta}{2\lambda}\sum_{n\in\mathbb{Z}^d}\max_{\omega\in\frac{\Delta}{\lambda}C_n}\|(\nabla g)(\omega)\|_{\ell_1^d}\Big(\frac{\Delta}{\lambda}\Big)^d.
\end{aligned}$$

In the 2^{nd} inequality above we used that $(n(k))_{k\in S}$ consists of distinct lattice points.

Part 2):

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^d}\left|g\Big(\frac{\Delta n}{\lambda}\Big)\right|\Big(\frac{\Delta}{\lambda}\Big)^d - \|g\|_{L_1} &\leq \int_{\mathbb{R}^d}\sum_{n\in\mathbb{R}^d}\Big[\left|g\Big(\frac{\Delta n}{\lambda}\Big)\right| - \left|g(x)\right|\Big]\chi_{\frac{\Delta}{\lambda}C_n}(x)dx \\
&\leq \int_{\mathbb{R}^d}\sum_{n\in\mathbb{R}^d}\left|g\Big(\frac{\Delta n}{\lambda}\Big) - g(x)\right|\chi_{\frac{\Delta}{\lambda}C_n}(x)dx.
\end{aligned}$$

By Proposition IV.10,

$$\Big|g\Big(\frac{\Delta n}{\lambda}\Big) - g(x)\Big| \le \max_{\omega\in\frac{\Delta}{\lambda}C_n} \|(\nabla g)(\omega)\|_{\ell_1^d}\frac{\Delta}{2\lambda},$$

so

$$\sum_{n\in\mathbb{Z}^d}\Big|g\Big(\frac{\Delta n}{\lambda}\Big)\Big|\Big(\frac{\Delta}{\lambda}\Big)^d \le \|g\|_{L_1} + \frac{\Delta}{2\lambda}\sum_{n\in\mathbb{Z}^d}\max_{\omega\in\frac{\Delta}{\lambda}C_n}\|(\nabla g)(\omega)\|_{\ell_1^d}\Big(\frac{\Delta}{\lambda}\Big)^d.$$

Parts 1) and 2) together prove the proposition. □

Proposition IV.13. *The following holds.*

$$\lim_{\lambda\to\infty}\sum_{n\in\mathbb{Z}^d}\max_{\omega\in\frac{\Delta}{\lambda}C_n}\|(\nabla g)(\omega)\|_{\ell_1^d}\Big(\frac{\Delta}{\lambda}\Big)^d = \int_{\mathbb{R}^d}\|(\nabla g)(x)\|_{\ell_1^d}dx.$$

Proof. Define

$$f_\lambda(x) = \sum_{n\in\mathbb{Z}^d}\max_{\omega\in\frac{\Delta}{\lambda}C_n}\|(\nabla g)(\omega)\|_{\ell_1^d}\chi_{\frac{\Delta}{\lambda}C_n}(x), \quad x\in\mathbb{R}^d.$$

We need to show that

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^d} f_\lambda(x)dx = \int_{\mathbb{R}^d}\|(\nabla g)(x)\|_{\ell_1^d}dx.$$

Let $\lambda_i \to \infty$. Given any $x\in\mathbb{R}^d$, there exists n_i such that $x\in\frac{\Delta}{\lambda_i}C_{n_i}$. Note that $\mathrm{diam}\Big(\frac{\Delta}{\lambda_i}C_{n_i}\Big)\to 0$. Using this and the continuity of ∇g, we have

$$\lim_{i\to\infty} f_{\lambda_i}(x) = \lim_{i\to\infty}\max_{\omega\in\frac{\Delta}{\lambda}C_{n_i}}\|(\nabla g)(\omega)\|_{\ell_1^d} = \|(\nabla g)(x)\|_{\ell_1^d}.$$

As ∇g decays super-algebraically, elementary manipulation shows the following: There exists a positive integer m and a constant $C>0$ such that if $\lambda>\Delta$, and

$$H(x) = \begin{cases} C, & \|x\|_{\ell_\infty} < 1; \\ \frac{C}{1+(\|x\|_{\ell_\infty}-1)^m}, & \|x\|_{\ell_\infty}\ge 1 \end{cases},$$

then $H\in L_1$ and $0\le f_\lambda(x)\le H(x)$ for all $x\in\mathbb{R}^d$. Applying the Dominated

Convergence Theorem proves the proposition. □

Proof of Lemma IV.9. For all $t \in \mathbb{R}^d$ $\lambda > 0$, $1 \leq i \leq \ell$, we have

$$\inf_{n \neq m} \|(\lambda t - \tau_n^i) - (\lambda t - \tau_m^i)\|_{\ell_\infty^d} = \Delta_i.$$

Propositions IV.12 and IV.13 imply the relations

$$\begin{aligned} \frac{1}{\lambda^d} \sum_{k \in S_i} \Big| g\Big(t - \frac{\tau_k^i}{\lambda}\Big)\Big| &\leq \frac{1}{\Delta_i^d} \|g\|_{L_1} + \frac{1}{\lambda \Delta_i^{d-1}} \sum_{n \in \mathbb{Z}^d} \max_{\omega \in \frac{\Delta_i}{\lambda} C_n} \|(\nabla g)(\omega)\|_{\ell_1^d} \Big(\frac{\Delta_i}{\lambda}\Big)^d \\ &= \frac{1}{\Delta_i^d} \|g\|_{L_1} + \frac{1}{\lambda \Delta_i^{d-1}} \int_{\mathbb{R}^d} \|(\nabla g)(x)\|_{\ell_1^d} dx + o\Big(\frac{1}{\lambda}\Big). \end{aligned}$$

Summing over i finishes the proof. □

Proposition IV.14. *If the domain E in Theorem IV.3 is symmetric about the origin, then g can be taken to be real valued.*

Proof. If $\mathcal{F}^{-1}(g)$ is a function satisfying the requirements of Theorem IV.3, then

$$h(\xi) = \frac{(\mathcal{F}^{-1}(g))(\xi) + (\mathcal{F}^{-1}(g))(-\xi)}{2}$$

satisfies them also and is even. Let $\tilde{g} = \mathcal{F}(h)$, then

$$\mathrm{Im}\big(\tilde{g}(t)\big) = -\int_E h(\xi) \sin\langle t, \xi \rangle d\xi = 0, \quad t \in \mathbb{R}^d, \tag{4.14}$$

because E is symmetric and the integrand is odd. □

Proof of Theorem IV.7. By (4.1) we know

$$f(t) - \tilde{f}_{\lambda,\epsilon}(t) = \frac{1}{\lambda^d} \sum_{k=1}^{\infty} (B\epsilon)_k g\Big(t - \frac{t_k}{\lambda}\Big),$$

so

$$\|f - \tilde{f}_{\lambda,\epsilon}\|_{L_\infty} \leq \|B\| \|\epsilon\|_{\ell_p} \sup_{t \in \mathbb{R}^d} \frac{1}{\lambda^d} \sum_{k \in \mathbb{N}} \Big| g\Big(t - \frac{t_k}{\lambda}\Big)\Big|.$$

The definition of E implies that E has non-empty interior and contains a closed cube D. Therefore $(f_k)_{k\in\mathbb{N}}$ is a frame for $L_2(D)$, implying $(t_k)_{k\in\mathbb{N}}$ is relatively uniformly separated. An application of Lemma IV.9 completes the proof. $\square$

When $d=1$, (4.11) can be simplified to

$$\|f-\tilde{f}_{\lambda,\epsilon}\|_{L_\infty} \leq \|\epsilon\|_{\ell_p}\|B\|\Big[\Big(\sum_{i=1}^{\ell}\frac{1}{\Delta_i}\Big)\|g\|_{L_1}+\frac{\ell}{\lambda}\|g'\|\Big]. \tag{4.15}$$

In this case, Propositions IV.12 and IV.13 can be replaced by the following statement which is easy to prove.

Proposition IV.15. *If $(t_k)_{k\in S}\subset\mathbb{R}$ is a sequence (possibly finite) satisfying*

$$\inf_{k\neq j}|t_k-t_j|=\Delta,$$

and $g:\mathbb{R}\to\mathbb{R}$ is a function such that $g,g'\in L_1(\mathbb{R})\cap C(\mathbb{R})$, then

$$\frac{1}{\lambda}\sum_{k\in S}\Big|g\Big(\frac{t_k}{\lambda}\Big)\Big|\leq\frac{1}{\Delta}\|g\|_{L_1}+\frac{1}{\lambda}\|g'\|_{L_1}.$$

Assertion (4.15) follows quickly from this.

We now turn our attention to concrete examples of matrices B which are $\ell_p\to\ell_\infty$ continuous. If $(f_n)_n$ is an exponential frame or Riesz basis for $L_2([-\pi,\pi]^d)$, $(f_n)_n$ can be indexed by any countable set, say by $\mathbb{Z}^d$, which is in fact the natural indexing set for all of the concrete examples of Riesz bases which have been presented. If we index $(f_n)_n$ by this set, then the proofs of Theorems IV.3 and IV.7 can be modified so that (4.1) and (4.11) hold with the index set $\mathbb{Z}^d$ replacing $\mathbb{N}$. In this case, (4.1) takes the form

$$f(t)=\frac{\alpha^2}{\lambda^d}\sum_{k\in\mathbb{Z}^d}\Big(\sum_{n\in\mathbb{Z}^d}B_{kn}f\Big(\frac{t_n}{\lambda}\Big)\Big)g\Big(t-\frac{t_k}{\lambda}\Big),\quad t\in\mathbb{R}^d. \tag{4.16}$$

Now B acts as a bounded linear operator on $\ell_2(\mathbb{Z}^d)$ as follows:

$$(x_k)_{k\in\mathbb{Z}^d} \to \Big(\sum_{j\in\mathbb{Z}^d} b_{kj}x_j\Big)_{k\in\mathbb{Z}^d}.$$

In this setting, the next result (which pertains only to the univariate case) provides examples of sequences $(t_n)_{n\in\mathbb{Z}}$ for which B is $\ell_p \to \ell_\infty$ continuous for all $1 \le p < \infty$.

Theorem IV.16. *Let $(t_n)_{|n|\le\ell} \cup (n)_{|n|>\ell} \subset \mathbb{R}$ be a sequence of distinct points indexed by $\mathbb{Z}$ such that $t_k = 0$ for at most one index k. The sequence of exponentials $\left(\frac{1}{\sqrt{2\pi}}e^{it_n(\cdot)}\right)_{n\in\mathbb{Z}}$ is a UIRB for $L_2[-\pi,\pi]$ (Theorems III.13, III.18 and III.21), and the matrix B from* (4.16) *can be written in the following form*

$$B = I + C, \quad |C_{nm}| \le \frac{M}{(|n|+1)(|m|+1)}. \tag{4.17}$$

Sketch of proof. Theorem I.1 shows that if

$$H(t) = (t-t_0)\prod_{k=1}^{\ell}\Big(1-\frac{t}{t_k}\Big)\Big(1-\frac{t}{t_k}\Big)\frac{\mathrm{sinc}(\pi t)}{Q_{1,\ell}(t)},$$

then

$$G_n(t) = \frac{H(t)}{(t-t_n)H'(t_n)}, \quad \text{and}$$

$$\begin{aligned} B_{nm} &= \langle G_n, G_m\rangle = \Big\langle \sum_{k\in\mathbb{Z}} G_n(k)\mathrm{sinc}\pi((\cdot)-k), \sum_{k\in\mathbb{Z}} G_m(k)\mathrm{sinc}\pi((\cdot)-k)\Big\rangle \\ &= \sum_{k\in\mathbb{Z}} G_n(k)G_m(k). \end{aligned} \tag{4.18}$$

Computation and estimation is facilitated by noticing that all but finitely many terms in the equation above are 0 when $n \ne m$. □

The estimates in Theorem IV.16 are sharp: If $t_0 = D \notin \mathbb{Z}\setminus\{0\}$, $t_n = n$ for $n \ne 0$, then direct calculation (as described in the outlined proof above) yields

$$\begin{aligned} &i) \quad B_{0m} = \frac{D(-1)^m}{\operatorname{sinc}(\pi D)(m-D)}, \quad m \neq 0, \qquad ii) \quad B_{00} = \frac{1}{\operatorname{sinc}^2(\pi D)}, \\ &iii) \quad B_{nm} = \delta_{nm} + \frac{D^2(-1)^{n+m}}{(n-D)(m-D)}, \quad 0 \neq n, \quad 0 \neq m. \end{aligned}$$

If $1 \leq p \leq \infty$, we see that the maps given by Theorem IV.16 are $\ell_p \to \ell_\infty$ continuous for $1 \leq p < \infty$. Note that the specific example above illustrates that in this case, B is not $\ell_\infty \to \ell_\infty$ continuous.

An aside: for general B (when it is invertible), how does the rate of decay of the entries of B relate to the rate of decay of the entries of $(B^{-1})_{nm} = \mathrm{SINC}\pi(t_n - t_m)$? Even when $d = 1$, simple cases can be difficult to resolve. If $0 < \delta < 1/4$, $t_0 = 0$, and $t_n = n - \operatorname{sign}(n)\delta$ when $n \neq 0$, then $|B^{-1}_{nm}| = |\operatorname{sinc}\pi(t_n - t_m)|$ is exactly $O(|n-m|^{-1})$, and the deep theorem below (see [24]), doesn't apply. This suggests that this approach to determining stability is inherently difficult.

Theorem IV.17 (Jaffard). *If $A = (a_{k\ell})_{k,\ell \in \mathbb{Z}^d}$ is boundedly invertible on $\ell_2(\mathbb{Z}^d)$ and $|a_{kl}| = O(\|k-l\|_\infty^{-s})$ for some $s > d$, then its inverse $B = A^{-1}$ has the same polynomial-type off-diagonal decay $|b_{kl}| = O(\|k-l\|_\infty^{-s})$.*

Theorem IV.18 below shows a direct connection between stability and tight frames of exponentials. Its proof does not require knowledge of the rate of decay of entries of B.

Theorem IV.18. *If $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is a sequence such that $\left(e^{i\langle \cdot, t_n\rangle}\right)_{n\in\mathbb{N}}$ is a tight frame for $L_2([-\pi,\pi]^d)$, then the matrix B from Theorem IV.3 is $\ell_p(\mathbb{N}) \to \ell_\infty(\mathbb{N})$ continuous for all $1 \leq p < \infty$.*

Proof. As $\left(e^{i\langle \cdot, t_n\rangle}\right)_{n\in\mathbb{N}}$ is a tight frame, there is a scalar ν such that $S^{-1} = \nu I$, so that for $k, n \in \mathbb{N}$,

$$B_{kn} = \langle S^{-1} f_n, S^{-1} f_k \rangle = |\nu|^2 \langle f_k, f_n \rangle = |\nu|^2 \mathrm{SINC}\pi(t_k - t_n).$$

Continuity is trivial when $p = 1$. Let $1 < q < \infty$ be the conjugate exponent to p. To verify continuity we need to show that

$$\sup_{n\in\mathbb{N}} \left\| \big(\mathrm{SINC}\pi(t_n - t_m)\big)_{m\in\mathbb{N}} \right\|_{\ell_q} < \infty.$$

It suffices to show that if $(\tau_n)_{n\in\mathbb{N}}$ is any relatively uniformly separated sequence, say

$$(\tau_n)_{n\in\mathbb{N}} = \bigcup_{i=1}^{\ell} (\tau_k^i)_{k\in S_i}, \quad S_i \subset \mathbb{N}, \quad \Delta_i = \inf_{k\neq j} \|\tau_k^i - \tau_j^i\|_{\ell_\infty^d} > 0,$$

then $\left\| \big(\mathrm{SINC}(\pi\tau_m)\big)_{m\in\mathbb{N}} \right\|_q \leq M$ where M depends only on q, and $\Delta_1, \cdots, \Delta_\ell$. Reducing further, it suffices to show that if $(\tau_n)_{n\in\mathbb{N}}$ is uniformly separated with

$$\Delta = \inf_{k\neq j} \|\tau_k - \tau_j\|_{\ell_\infty^d} > 0,$$

then there exists M depending only on q and Δ such that

$$\left\| \big(\mathrm{SINC}(\pi\tau_m)\big)_{m\in\mathbb{N}} \right\|_{\ell_q} \leq M.$$

For $n \in \mathbb{Z}^d$, let C_n be as in Definition IV.11. Then

$$\left\| \big(\mathrm{SINC}(\pi\tau_m)\big)_{m\in\mathbb{N}} \right\|_{\ell_q}^q \leq \sum_{n\in\mathbb{Z}^d} \sum_{\{m \,:\, t_m \in \Delta C_n\}} |\mathrm{SINC}(\pi\tau_m)|^q. \tag{4.19}$$

There exists $M > 0$ such that for all $t \in \mathbb{R}$, $|\mathrm{sinc}(\pi t)| \leq M/(\Delta + |t|)$, so that

$$|\mathrm{SINC}(\pi t)|^q \leq \frac{M^{qd}}{(\Delta + |t(1)|)^q \cdot \ldots \cdot (\Delta + |t(d)|)^q}. \tag{4.20}$$

If $t_m \in \Delta C_n$ then $\Delta(n_i - 1/2) \leq t_m(i) < \Delta(n_i + 1/2)$ hence

$$\frac{1}{(\Delta + |t_m(i|)} \leq \frac{1}{\Delta(|n_i| + 1/2)}. \tag{4.21}$$

Combining (4.19), (4.20), and (4.21) with $\#\{m : t_m \in \Delta C_n\} \leq 1$, we find that

$$\begin{aligned}\big\|\big(\mathrm{SINC}(\pi\tau_m)\big)_{m\in\mathbb{N}}\big\|_q^q &\leq \frac{M^{qd}}{\Delta^{qd}}\Big[\sum_{n\in\mathbb{Z}^d}\frac{1}{(|n_1|+1/2)^q}\cdot\ldots\cdot\frac{1}{(|n_d|+1/2)^q}\Big]\\ &= \frac{M^{qd}}{\Delta^{qd}}\Big[\sum_{n\in\mathbb{Z}}\frac{1}{(|n|+1/2)^q}\Big]^d < \infty,\end{aligned}$$

which proves the proposition. $\square$

The following example shows that the conclusion of Theorem IV.18 can be false if $p = \infty$. The set $\big(\frac{1}{\sqrt{2\pi}}e^{i(n/2)(\cdot)}\big)_{n\in\mathbb{Z}}$ is a tight frame for $L_2[-\pi,\pi]$, as it is the union of the orthonormal bases

$$\Big(\frac{1}{\sqrt{2\pi}}e^{in(\cdot)}\Big)_{n\in\mathbb{Z}} \quad \text{and} \quad \Big(\frac{1}{\sqrt{2\pi}}e^{i(n+1/2)(\cdot)}\Big)_{n\in\mathbb{Z}}.$$

In this case $S^{-1} = \frac{1}{2}I$, and by direct computation,

$$\|(B_{0,n})_{n\in\mathbb{Z}}\|_{\ell_1} = \Big\|\Big(\frac{1}{4}\mathrm{sinc}\big(\frac{\pi n}{2}\big)\Big)_{n\in\mathbb{Z}}\Big\|_{\ell_1} = \infty.$$

While Theorem IV.18 does hold for arbitrary tight frames, it is clear that it should not be applied in a cavalier fashion. The example above shows that the matrix B can unnecessarily complicate a fundamentally simple configuration of sampling sites, and render itself useless. In this case, Theorem I.3 can be trivially extended to apply to the previous example (and other finite unions of shifted equally-spaced sampling sites) to show stability given ℓ_∞ perturbations in data. However, if it is known that $(t_n)_n$ yields an exponential tight frame, and no natural decomposition of $(t_n)_n$ is apparent, then the usage of Theorem IV.18 is justified.

D. Restriction of Theorem IV.3 to the Riesz basis case

In this section we consider Theorem IV.3 when $E = [-\pi,\pi]^d$ and $\alpha = \frac{1}{(2\pi)^{d/2}}$ (because then $\|f_n\| = 1$). In this case, defining $g_0 = \frac{1}{(2\pi)^d} g$, (which can be assumed to be real by Proposition IV.14), we have

$$f(t) = \frac{1}{\lambda^d} \sum_{k\in\mathbb{N}} \Big(\sum_{n\in\mathbb{N}} B_{kn} f\Big(\frac{t_n}{\lambda}\Big)\Big) g_0\Big(t - \frac{t_k}{\lambda}\Big), \quad t \in \mathbb{R}^d. \tag{4.22}$$

Note that $\|\mathcal{F}^{-1}(g_0)\|_\infty = 1/(2\pi)^d$.

The summands in (4.22) involve an infinite invertible matrix B; however, considerable simplification can be achieved if we consider sequences $(t_n)_{n\in\mathbb{N}}$ such that $(f_n)_{n\in\mathbb{Z}}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$ rather than a general frame. Let $(b_k)_k$ be the standard basis for $\ell_2(\mathbb{N})$, and let $P_\ell : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be the orthogonal projection onto $\operatorname{span}\{h_1, \cdots, h_\ell\}$. Let $(f_n)_{n\in\mathbb{Z}}$ be a Riesz basis for $L_2([-\pi,\pi]^d)$. Define the operator

$$B_\ell := (P_\ell B^{-1} P_\ell)^{-1} + (I - P_\ell).$$

In the definition above, the operator $P_\ell B^{-1} P_\ell$ is certainly not invertible on $\ell_2(\mathbb{N})$, but it will be shown that it is invertible as an $\ell \times \ell$ matrix and has the following entries:

$$(P_\ell B^{-1} P_\ell)_{nm} = \mathrm{SINC}\pi(t_n - t_m), \quad 0 \le n, m \le \ell.$$

Define

$$f_\lambda^\ell(t) = \frac{1}{\lambda^d} \sum_{k=1}^{\ell} [(P_\ell B^{-1} P_\ell)^{-1} f_{\mathcal{T}/\lambda}]_k g_0\Big(t - \frac{t_k}{\lambda}\Big) + \frac{1}{\lambda^d} \sum_{k=\ell+1}^{\infty} f\Big(\frac{t_k}{\lambda}\Big) g_0\Big(t - \frac{t_k}{\lambda}\Big).$$

Theorem IV.22 states the exact relationship between f and f_λ^ℓ. Before we embark on it, we need to establish several lemmas.

Lemma IV.19. *If $Q : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ is self-adjoint, positive, and boundedly invert-*

ible, then

$$\begin{aligned}
&a) \qquad \sup_{\ell} \|(P_\ell Q P_\ell)^{-1}\| = \|Q^{-1}\|,\\
&b) \qquad Q^{-1}x = \lim_{\ell\to\infty} (P_\ell Q P_\ell)^{-1}x, \quad \forall x \in \ell_2(\mathbb{N}), \text{ and}\\
&c) \qquad \lim_{\ell\to\infty} \|(P_\ell Q P_\ell)^{-1}\| = \|Q^{-1}\|.
\end{aligned}$$

Proof. a): If $P_\ell x \neq 0$, then

$$0 < \langle QP_\ell x, P_\ell x\rangle = \langle (P_\ell Q P_\ell)P_\ell x, P_\ell x\rangle,$$

so $P_\ell Q P_\ell : P_\ell \ell_2(\mathbb{N}) \to P_\ell \ell_2(\mathbb{N})$ is positive-definite, and self-adjoint as an $\ell \times \ell$ matrix operator. There exists a self-adjoint boundedly invertible operator A such that $Q = A^2$. Now $Q_{kj} = \langle A_k, A_k\rangle$ where A_k and A_j are the k^{th} and j^{th} columns of A. Also, $Ab_k = A_k$. For any $\ell > 0$,

$$\frac{1}{\|(P_\ell Q P_\ell)^{-1}\|} \sum_{k=1}^{\ell} |c_k|^2 \leq \sum_{k,j=1}^{\ell} c_k \overline{c_j} Q_{kj} = \sum_{k,j=1}^{\ell} c_k \overline{c_j} \langle A_k, A_k\rangle = \Big\|A\Big(\sum_{k=1}^{\ell} c_k b_k\Big)\Big\|^2, \tag{4.23}$$

so that

$$\frac{1}{\sup_\ell \|(P_\ell Q P_\ell)^{-1}\|} \sum_{k=1}^{\infty} |c_k|^2 \leq \Big\|A\Big(\sum_{k=1}^{\infty} c_k b_k\Big)\Big\|^2. \tag{4.24}$$

In (4.23), equality is always attained for some $(c_k)_{k=1}^{\ell}$, so (4.24) implies the equalities

$$\frac{1}{\sup_\ell \|(P_\ell Q P_\ell)^{-1}\|} = \frac{1}{\|A^{-1}\|^2} = \frac{1}{\|Q^{-1}\|}.$$

Proof of b): General principles imply that

$$\lim_{\ell\to\infty} P_\ell Q^{-1} P_\ell x = Q^{-1}x, \quad \forall x \in \ell_2(\mathbb{N}),$$

so it suffices to show that

$$\lim_{\ell\to\infty} (P_\ell Q P_\ell)^{-1}x - P_\ell Q^{-1} P_\ell x = 0, \quad \forall x \in \ell_2(\mathbb{N}).$$

Now

$$\begin{aligned}
& (P_\ell Q P_\ell)^{-1} - P_\ell Q^{-1} P_\ell \\
= \; & (P_\ell Q P_\ell)^{-1}[P_\ell - P_\ell Q P_\ell Q^{-1} P_\ell] = (P_\ell Q P_\ell)^{-1} P_\ell Q (I - P_\ell) Q^{-1} P_\ell \\
= \; & (P_\ell Q P_\ell)^{-1} P_\ell Q [(I - P_\ell) Q^{-1} - (I - P_\ell) Q^{-1} (I - P_\ell)].
\end{aligned}$$

This implies

$$\begin{aligned}
& \|(P_\ell Q P_\ell)^{-1} x - P_\ell Q^{-1} P_\ell x\| \\
\leq \; & \|(P_\ell Q P_\ell)^{-1}\| \|P_\ell Q\| \|(I - P_\ell) Q^{-1} x - (I - P_\ell) Q^{-1} (I - P_\ell) x\| \\
\leq \; & \|Q^{-1}\| \|Q\| \Big(\|(I - P_\ell) Q^{-1} x\| + \|(I - P_\ell) Q^{-1} (I - P_\ell) x\| \Big) \\
\leq \; & \|Q^{-1}\| \|Q\| \big(\|(I - P_\ell) Q^{-1} x\| + \|Q^{-1} (I - P_\ell) x\| \big) \to 0
\end{aligned}$$

by part a), which proves b).

Proof of part c): From b) we conclude that

$$\|Q^{-1}\| \leq \liminf_{\ell \to \infty} \|(P_\ell Q P_\ell)^{-1}\|.$$

Combining this with a) finishes the proof. $\square$

Lemma IV.20. *If L is a boundedly invertible operator on $\ell_2(\mathbb{N})$ (over $\mathbb{C}$), and $B := (L^* L)^{-1}$, then*

1) For all $x \in \ell_2(\mathbb{N})$, $Bx = \lim_{\ell \to \infty} \big[(P_\ell B^{-1} P_\ell)^{-1} x + (I - P_\ell) x \big]$.

2) The following are equivalent:

a) $B = \lim_{\ell \to \infty} \big[(P_\ell B^{-1} P_\ell)^{-1} + (I - P_\ell) \big]$ in the operator norm topology,

b) $B = I + K$, for some compact operator K.

c) $L = U + C$ where U is onto unitary and C is compact.

Proof. 1) follows immediately by Lemma IV.19.

Proof of 2): i) First we show that a) holds if and only if $B^{-1} = L^* L = I + \tilde{C}$ for some compact operator $\tilde{C}$ (which is clearly equivalent to $B = I + K$, K compact). Define

$$B_\ell = (P_\ell B^{-1} P_\ell)^{-1} + (I - P_\ell).$$

Then $\lim_{\ell\to\infty} B_\ell = B$ implies that

$$\begin{aligned} 0 &= \lim_{\ell\to\infty}(B^{-1} - B_\ell^{-1}) = \lim_{\ell\to\infty}(B^{-1} - (I - P_\ell + P_\ell B^{-1} P_\ell)) \\ &= B^{-1} - I - \lim_{\ell\to\infty} P_\ell (B^{-1} - I) P_\ell. \end{aligned}$$

$B^{-1} - I$ is the limit of finite rank operators in the operator norm topology and is therefore compact. For the converse, if $B^{-1} = I + \tilde{C}$, then

$$\begin{aligned} B^{-1} - B_\ell^{-1} &= P_\ell B^{-1}(I - P_\ell) + (I - P_\ell)(B^{-1} - B_\ell^{-1}) \\ &= P_\ell(I + \tilde{C})(I - P_\ell) + (I - P_\ell)\tilde{C} \\ &= P_\ell \tilde{C}(I - P_\ell) + (I - P_\ell)\tilde{C}. \end{aligned}$$

The quantity above has zero limit in the operator norm topology, so $\lim_{\ell\to\infty} B_\ell = B$.
ii) We now show $L^*L - I$ is compact if and only if $L = U + C$. Suppose $\tilde{C} = L^*L - I$ is compact. $\tilde{C}$ is also a self-adjoint operator on $\ell_2(\mathbb{N})$. By the spectral theorem, there exists a diagonal matrix D consisting of the eigenvalues $(d_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ of $\tilde{C}$, ($\lim_{k\to\infty} d_{kk} = 0$), and an invertible unitary matrix V whose columns are the eigenvectors of $\tilde{C}$, such that $L^*L - I = VDV^*$. This implies

$$L^*L = V(I + D)V^*.$$

The statement

$$0 < [(LV^*)^*(LV^*)]_{kk} = (I + D)_{kk}$$

shows that diagonal matrix $\tilde{D}$ with entries $\tilde{D}_{kk} = \sqrt{1 + d_{kk}}$ is a real, boundedly invertible matrix such that $\tilde{D}^2 = I + D$. Note that $\tilde{D} = I + K$ where K is compact.

Define $W = \tilde{D}V^*$, so that

$$L^*L = V\tilde{D}\tilde{D}V^* = W^*W.$$

Consequently, $\langle L^*Lx, x\rangle = \langle W^*Wx, x\rangle$ for $x \in \ell_2(\mathbb{N})$, hence

$$\|Lx\| = \|Wx\|, \quad x \in \ell_2(\mathbb{N}).$$

We conclude that there exists an invertible unitary matrix $\tilde{U}$ such that

$$L = \tilde{U}W = \tilde{U}(I+K)V^* = \tilde{U}V^* + \tilde{U}KV^*,$$

which gives the desired decomposition of L. The converse follows by direct calculation.

$\square$

Lemma IV.21. *Let $(n_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$ and define*

$$e_k(\cdot) = \frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, n_k\rangle}, \quad k \geq 1.$$

Let $\mathcal{T} = (t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that $(f_k)_{k\in\mathbb{N}}$, defined by

$$f_k(\cdot) = \frac{1}{(2\pi)^{d/2}} e^{i\langle \cdot, t_k\rangle}, k \in \mathbb{N},$$

is a Bessel sequence for $L_2([-\pi,\pi]^d)$. Let L be defined by $Le_k = f_k$. Then $L = I + C$ for some compact operator C if and only if $\lim_{k\to\infty} \|n_k - t_k\|_\infty = 0$.

Proof. Let $L = I + C$ for some compact operator C. Consider L and C as operators on ℓ_2 under the change of basis $e_k \mapsto b_k$ (the standard basis for ℓ_2), and regard L and C as infinite matrices. Then

$$\lim_{k\to\infty} L_{kk} = \lim_{k\to\infty} \mathrm{SINC}\pi(t_k - n_k) = \lim_{\ell\to\infty}(1 + C_{kk}) = 1.$$

This can happen if and only if

$$\lim_{k\to\infty} \operatorname{sinc}\pi(t_k(i) - n_k(i)) = 1, \quad 1 \le i \le d,$$

that is, if and only if $\lim_{k\to\infty} \|n_k - t_k\|_\infty = 0$.

For the converse, apply Lemma II.17:

$$\begin{aligned} \Big\|(I-L)(I-P_\ell)\Big\| &= \sup_{\sum_{k=\ell+1}^{\infty}|a_k|^2=1} \Big\|(I-L)\sum_{k=\ell+1}^{\infty} a_k e_k\Big\| \\ &= e^{\left(\sup\limits_{k\ge\ell+1}\|n_k-t_k\|_\infty\right)} - 1 \to_{\ell\to\infty} 0. \end{aligned}$$

$I - L$ is the limit of finite rank operators in the operator norm and is therefore compact. □

We note that if $(t_n)_n$ is a sequence of points such that $L = I + C$, then if $\alpha \in \mathbb{R}$ and $(\tau_n)_n = (t_n + \alpha)_n$, then the associated isomorphism $\tilde{L}$ is of the form $\tilde{L} = U + C$ where U is unitary.

We are now ready for the theorem that relates f and f_λ^ℓ.

Theorem IV.22. *Let $(n_k)_{k\in\mathbb{N}}$, $(e_k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}}$, and $(f_k)_{k\in\mathbb{N}}$ be as in Lemma IV.21. Additionally, suppose that $(f_k)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$ with upper frame bound M. The following hold.*

$$\begin{aligned} &a) \quad \|f - f_\lambda^\ell\|_{L_2} \le \sqrt{M}\Big\|(B-B_\ell)\Big(\frac{1}{\lambda^{d/2}} f\Big(\frac{t_k}{\lambda}\Big)\Big)_{k\in\mathbb{N}}\Big\|_{L_2} \to_{\ell\to\infty} 0, \quad \text{and} \\ &b) \quad \|f - f_\lambda^\ell\|_{L_\infty} \le \lambda_0^{d/2}\sqrt{M}\Big\|(B-B_\ell)\Big(\frac{1}{\lambda^{d/2}} f\Big(\frac{t_k}{\lambda}\Big)\Big)_{k\in\mathbb{N}}\Big\|_{L_2} \to_{\ell\to\infty} 0. \end{aligned}$$

If, in addition, $B = I + C$ for some compact C, then

$$\begin{aligned} &c) \quad \sup_{\|f\|_{L_2}=1} \|f - f_\lambda^\ell\|_{L_2} \le M\|(B-B_\ell)\| \to_{\ell\to\infty} 0, \quad \text{and} \\ &d) \quad \sup_{\|f\|_{L_2}=1} \|f - f_\lambda^\ell\|_{L_\infty} \le \lambda_0^{d/2} M\|(B-B_\ell)\| \to_{\ell\to\infty} 0. \end{aligned}$$

Proof. For a), note that $f \mapsto (2\pi)^{d/2}\mathcal{F}^{-1}(f)$ is an onto isometry on $L_2(\mathbb{R}^d)$, so

$$\begin{aligned}\|f - f_\lambda^\ell\|_{L_2} &= \Big\|\frac{1}{\lambda^d}\sum_{k=1}^{\infty}[(B-B_\ell)f_{\mathcal{T}/\lambda}]_k g_0\Big(\cdot - \frac{t_k}{\lambda}\Big)\Big\|_{L_2} \\ &= \Big\|\frac{1}{\lambda^d}\sum_{k=1}^{\infty}[(B-B_\ell)f_{\mathcal{T}/\lambda}]_k (2\pi)^{d/2}\mathcal{F}^{-1}\Big[g_0\Big(\cdot - \frac{t_k}{\lambda}\Big)\Big]\Big\|_{L_2([-\lambda_0\pi,\lambda_0\pi]^d)} \\ &\le \frac{1}{\lambda^{d/2}}\Big\|\sum_{k=1}^{\infty}[(B-B_\ell)f_{\mathcal{T}/\lambda}]_k \frac{1}{\lambda^{d/2}} f_k\Big(\frac{\cdot}{\lambda}\Big)\Big\|_{L_2([-\lambda\pi,\lambda\pi]^d)}.\end{aligned}$$

The map $f(\cdot) \mapsto \frac{1}{\lambda^{d/2}}f\big(\frac{\cdot}{\lambda}\big)$ is an onto isometry from $L_2([-\pi,\pi]^d)$ to $L_2([-\lambda\pi,\lambda\pi]^d)$, so $\Big(\frac{1}{\lambda^{d/2}}f_k\big(\frac{\cdot}{\lambda}\big)\Big)_{k\in\mathbb{N}}$ is a frame for $L_2([-\lambda\pi,\lambda\pi]^d)$ with frame constant M. This implies

$$\begin{aligned}\|f - f_\lambda^\ell\|_{L_2} &\le \frac{1}{\lambda^{d/2}}\sqrt{M}\Big(\sum_{k=1}^{\infty}|[(B-B_\ell)f_{\mathcal{T}/\lambda}]_k|^2\Big)^{1/2} \\ &= \sqrt{M}\Big\|[(B-B_\ell)\Big(\frac{1}{\lambda^{d/2}}f\Big(\frac{t_k}{\lambda}\Big)\Big)_{k\in\mathbb{N}}\Big\|. \qquad (4.25)\end{aligned}$$

An application of (2.2) shows b). For c), (4.25) implies

$$\|f - f_\lambda^\ell\|_{L_2} \le \sqrt{M}\|[(B-B_\ell)\|\Big(\sum_{k=1}^{\infty}\frac{1}{\lambda^d}\Big|f\Big(\frac{t_k}{\lambda}\Big)\Big|^2\Big)^{1/2}. \qquad (4.26)$$

Furthermore,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{\lambda^d}\Big|f\Big(\frac{t_k}{\lambda}\Big)\Big|^2 &= \sum_{k=1}^{\infty}\Big|\Big\langle(2\pi)^{d/2}\mathcal{F}^{-1}(f)(\cdot), \frac{e^{i\langle\cdot,t_k/\lambda\rangle}}{(2\pi)^{d/2}\lambda^{d/2})}\Big\rangle\Big|^2 \\ &= \sum_{k=1}^{\infty}\Big|\Big\langle(2\pi)^{d/2}\mathcal{F}^{-1}(f)(\cdot), \frac{1}{\lambda^{d/2}}f_k\Big(\frac{\cdot}{\lambda}\Big)\Big\rangle\Big|^2 \\ &\le M\|(2\pi)^{d/2}\mathcal{F}^{-1}(f)\|^2 = B\|f\|^2.\end{aligned}$$

Combining the above inequality with (4.26) proves c), and another application of (2.2) yields d). □

The impact of λ on the rate of convergence is not apparent in Theorem IV.22, and is almost certainly due to the method of proof. Theorem IV.23, an analogue of

Theorem IV.6, presents a similar approximation without the aid of oversampling.

Theorem IV.23. *Define*

$$f^\ell(t) = \sum_{k=1}^{\ell} [(P_\ell B^{-1} P_\ell)^{-1} f_{\mathcal{T}}]_k \mathrm{SINC}\pi(t-t_k) + \sum_{k=\ell+1}^{\infty} f(t_k)\mathrm{SINC}\pi(t-t_k).$$

Under the hypotheses of Theorem IV.22,

$$\begin{aligned} \|f - f^\ell\|_{L_2} &\leq \sqrt{M}\|(B-B_\ell)(f(t_k))_{k\in\mathbb{N}}\| \to_{\ell\to\infty} 0. \\ \|f - f^\ell\|_{L_\infty} &\leq \sqrt{M}\|(B-B_\ell)(f(t_k))_{k\in\mathbb{N}}\| \to_{\ell\to\infty} 0. \end{aligned}$$

If, in addition, $B = I + C$ for some compact C, then

$$\begin{aligned} \sup_{\|f\|_{L_2}=1} \|f - f^\ell\|_{L_2} &\leq M\|(B-B_\ell)\| \to_{\ell\to\infty} 0. \\ \sup_{\|f\|_{L_2}=1} \|f - f^\ell\|_{L_\infty} &\leq M\|(B-B_\ell)\| \to_{\ell\to\infty} 0. \end{aligned}$$

The proof of Theorem IV.23 is similar to the proof of Theorem IV.22 and is omitted. It is worth stating the following corollary, which provides a direct generalization of (2.4).

Corollary IV.24. *Let $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence such that $\left(\frac{1}{(2\pi)^{d/2}} e^{i\langle\cdot,t_k\rangle}\right)_{k\in\mathbb{N}}$ is a Riesz basis for $L_2([-\pi,\pi]^d)$. Define the $\ell\times\ell$ matrix A_ℓ by $(A_\ell)_{nm} = \mathrm{SINC}\pi(t_n - t_m)$. For all $f \in PW_{[-\pi,\pi]^d}$, we have*

$$f(t) = \lim_{\ell\to\infty} \sum_{k=1}^{\ell} \Big(\sum_{n=1}^{\ell} (A_\ell^{-1})_{kn} f(t_n) \Big) \mathrm{SINC}\pi(t-t_k), \quad t \in \mathbb{R}^d.$$

The sum converges with respect to both L_2 and L_∞ metrics.

Proof. Note that the 2^{nd} term of f^ℓ from Theorem IV.23 has 0 limit with respect to both L_2 and L_∞ metrics. □

CHAPTER V

CONCLUSION

In Chapter III, we investigated polynomial interpolation in relation to approximation of multivariate bandlimited functions. Given a sequence $(t_n)_{n\in\mathbb{Z}^d}$ such that $\left(e^{i\langle\cdot,t_n\rangle}\right)_{n\in\mathbb{Z}^d}$ is a UIRB for $L_2([-\pi,\pi]^d)$, strong connections were established between the following.

a) The existence of Lagrangian polynomial interpolants (with manageable coordinate degrees) which (in the limit) interpolate arbitrary ℓ_2 data at $(t_n)_{n\in\mathbb{Z}^d}$, and

b) the existence of exponential Riesz bases for $L_2([-\pi,\pi]^d)$, each of which, after replacement of finitely many elements, is an orthonormal basis.

Given a set of ℓ_2 data and the corresponding polynomial interpolants, we produced (in Theorems III.26 and III.30) asymptotic recovery and approximation formulas for multivariate bandlimited functions. While the approximants in Theorem III.26 demonstrate global L_2 and L_∞ convergence on $\mathbb{R}^d$ and are simply expressed in theory, they become computationally complicated in the limit. This deficiency was remedied in Theorem III.30, where computational manageability was obtained at the price of introducing a more local convergence on increasingly large subsets of $\mathbb{R}^d$. Near-optimality of the growth rates of these subsets was addressed by Propositions III.42 and III.40.

In Chapter IV, oversampling of data at sites associated with an exponential frame condition was used to derive a multivariate recovery formula ((4.1) in Theorem IV.3). Given minor restraints on the sampling sites and the frequency domain, Theorem IV.7 demonstrates a certain stability in (4.1) with respect to ℓ_p errors in otherwise ideal ℓ_2 data. Computational simplifications of (4.1) were given in Theorems IV.22 and IV.23 in the case where the sampling sites arise from an exponential Riesz basis.

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