SCHEDULING SCREENING INSPECTIONS FOR REPLACEABLE AND NON-REPLACEABLE SYSTEMS

A Dissertation

by

BAHADIR ARAL

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2007

Major Subject: Industrial Engineering

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Approved by:

Chair of Committee,	Georgia-Ann Klutke
Committee Members,	Martin A. Wortman
	Illya V. Hicks
	Daren B.H. Cline
Head of Department,	Brett A. Peters

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ABSTRACT

Scheduling Screening Inspections
for Replaceable and Non-Replaceable Systems. (August 2007)
Bahadır Aral, B.S., Boğaziçi University, Turkey;
M.S., Texas A&M University
Chair of Advisory Committee: Dr. Georgia-Ann Klutke

This dissertation focuses on developing inspection schedules to detect *non-self-announcing* events which can only detected by inspections. Failures of protective systems ,such as electronic equipments, alarms and stand-by systems, incipient failures and the emergence of certain medical diseases are examples of such events. Inspections are performed at pre-determined times to detect whether or not the event has occurred, and necessary actions are taken upon the detection. In this research, my interest is in developing effective inspection schedules to detect non-self-announcing events that balance system downtime and inspection effort.

To evaluate the quality of an inspection schedule, I use the availability (for replaceable) and the detection delay (for non-replaceable systems) as performance measures. When the monetary cost of inspection and the cost of delay are difficult to determine, non-monetary performance measures become more meaningful. In this research, the focus is on maximizing availability or minimizing detection delay given a limited number of inspections or a limited inspection rate. I show that for replaceable and non-replaceable systems, it is possible to construct inspection schedules that perform better than periodic inspection with respect to our performance measures.

The occurrence of the event I would like to detect may be influenced by certain individual characteristics. For instance, the risk of developing a certain type of disease might be different for different subgroups within the population. In this case,

because of the non-homogeneity in the population, benefits of performing screening tests may not be fully achieved for each sub-group by using an inspection strategy developed for the entire population. Thus, it may be of value for an individual to learn more information about his/her likehood to have the disease. To address this issue, I analyze the change in the expected delay if schedules are based on the whole population information or the individual information and provide numerical results.

To my mother, Neslihan Aral

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I am grateful to my parents for their constant support and sacrifices they made for me. My parents' unconditional love inspired me to worker harder and do better in the life. I also would like to thank my beautiful and loving sisters for taking care of me since I was a little kid and for helping whenever I needed them. Without my family's love and support, I would not be able to finish my research.

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CHAPTER I

INTRODUCTION

Developing inspection schedules for detecting a particular event has been of the interest to the reliability community for many years. For instance, in industrial studies concerned with maintaining machines or equipment exposed to a hazardous environment, a particular event of interest is the failure time of the equipment. For such studies, the objective is to design effective inspection strategies in order to detect failures as soon as possible so as to minimize disruptions caused by down times. Similarly, in health care studies, timing medical tests to detect the appearance of certain pre-clinical conditions is crucial to the well-being of the population at risk. In this case, the event we would like to detect is the occurrence of such conditions and the objective could be to reduce the detection delay or to increase the chance of detecting the disease before it reaches a clinical level. In both cases, there is a non-self-announcing event (hereafter we will refer to it as a *failure* to be consistent with the literature), which can only be detected by an inspection, and the goal of inspections is to identify this event as soon as possible. In this research, we investigate issues regarding effective inspection schedules for systems with non-self-announcing failures.

For industrial equipment, any kind of unplanned disruptions in production can cause serious financial problems. For instance, the estimated cost of an unplanned shutdown of BP's Alaska refinery in August, 2006 has been blamed for a 2% drop in BP's stock value (See (Isidore, 2006)). Besides the monetary cost of lost production, excessive downtime can affect competitiveness and the market share. Since many

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This dissertation follows the style of *IIE Transactions*.

plants and facilities are composed of complex devices, which are also inter-connected among themselves, it is important to carefully identify the likehood of failures or malfunctions of devices in order to design effective inspections strategies. For industrial studies, in this dissertation, we will consider replaceable systems, where the failed part or equipment will be replaced as soon as a failure is detected.

For replaceable systems, it is important to know that how our system performs in ideal conditions (i.e, the laboratory or nominal life) and how the operating environment affects its performance. There are two approaches to describe this situation. The first approach is to describe the operating environment and the nominal life separately. This approach provides a general framework but generally increases the complexity of the problem. On the other hand, a simpler approach is to assume that the lifetime distribution under the operating conditions is known. The specific properties of the lifetime distribution can then be used to build the maintenance policy. In the first category, the system has a known nominal life distribution under ideal conditions (i.e, L_i in Figure 1 is known) and it deteriorates as a response to its environment. The deterioration caused by its operating environment is modeled as a stochastic process and called as *damage process*. By using the nominal life distribution and the properties of the damage process, the lifetime (L_i^* in Figure 1) is characterized (if possible) or they are together used to calculate desired performance measures for specific inspection policies.

Both monetary and non-monetary measures can be used to evaluate maintenance strategies. Monetary measures can be meaningful if it is possible to quantify the monetary cost of downtime and inspection. However, it may not be possible to quantify the monetary trade-off between the downtime and inspection costs. Additionally, in some cases, the maintenance budget may be fixed, so that only certain number of inspections are allowed. In such situations, the goal is to maximize availability given

a fixed number of inspections. In Chapter III, we address how to design inspections for maximizing availability using a constant inspection rate. Although non-monetary performance measures are used in Chapter III, when the costs of inspections and downtimes are known, with slight modifications, they can be incorporated into our model.

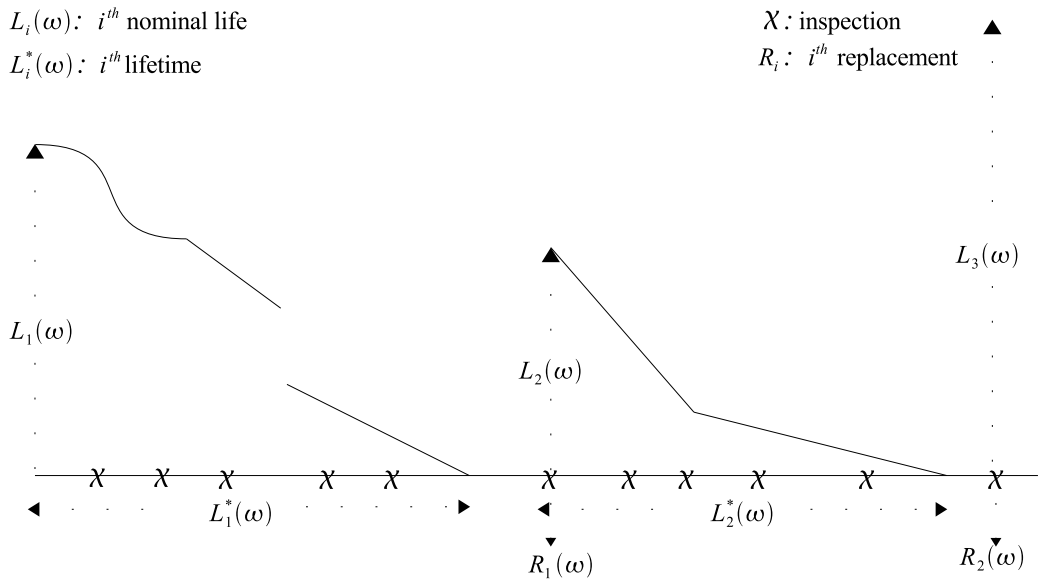

Fig. 1. Sample path for replaceable systems

Some medical diseases do not become apparent as soon as they start. So, screening is an important way to detect these kinds of diseases at early stages. The benefit of screening is that if a disease is identified at an early stage, the treatment is usually more successful and the chance of survival may be increased. There are different screening tests and schedules for different types of diseases. For instance, the American Cancer Society recommends three tests to detect signs of breast cancer: mammography, clinical breast exam and breast self-examination. Additionally, the American Medical Womens Association recommends annual mammography and clinical breast exam for normal-risk women 40 years of age and older. For colorectal

cancer, it is suggested that people over 50 should be screened using a combination of fecal occult blood testing, sigmoidoscopy and colonoscopy. The American Cancer Society suggest different guides for colorectal cancer such as

- a fecal occult blood test (FOBT) or fecal immunochemical test (FIT) every year OR
- flexible sigmoidoscopy every 5 years, OR
- an FOBT or FIT every year plus flexible sigmoidoscopy every 5 years, OR
- colonoscopy every 10 years.

The recommended frequency of tests differs for different types of cancers as well as for different types of test. This can be explained by having different risks at certain ages and the quality of a particular test for a certain cancer. For instance, mammography's ability to detect a breast cancer when it is already started is estimated to 77.6% in (Houssami, Ciatto, Irwig, Simpson and Macaskill, 2002). Therefore, when designing screening inspection schedules, the fallibility of tests should also taken into account.

Most current recommended screening schedules are periodic. Clinical studies have, however, suggested that the incidence rate of having a certain disease changes over an individual's lifetime. These facts motivate us to consider unequally spaced inspections that may be able to outperform periodic inspections. In Chapter IV, we consider a general non-replaceable system. In this system, as soon as the failure is detected, inspections are halted and treatment begins. We propose inspection schedules which will have lower detection delays than periodic inspections does, using the same number of inspections as periodic inspections. Our approach can be used to evaluate how detection delay is affected by performing different types of tests and by following different schedules since we also take the fallibility of tests into account.

Another issue while designing screening schedules is how valuable it is to design customized schedules if more information about the failure intensity for a particular person or system is available. For instance, advances in genomic medicine have allowed for the identification of certain genetic markers that are believed to be associated with increased risk of disease. For example, mutations in the BRCA1 and BRCA2 genes have associated with increased risk of developing breast cancer. If a person has these genes, it might be a beneficial for her to follow a different screening schedule that other patients. In other words, the possibility of having extra information leads us to the question of how to quantify the value of information if the population at risk is formed by mixtures of sub-populations with different susceptibilities to a certain disease. In Chapter V, we look at how much improvement can be gained by using customized schedules and how valuable customized schedules are to the whole population.

This dissertation is organized as follows. In Chapter II, we review the literature for maintenance strategies for general replaceable systems and non-replaceable systems. In Chapter III, we analyze a simple failure prone system which fails due to shocks, whose appearance are modeled by a non-homogeneous Poisson point process with a known intensity function. For a such system, we express the stationary time averaged availability for periodic inspections, develop the concept of intensity-based inspections, in which inter-inspection times are matched to the intensity function. Finally, we develop and analyze an improved inspection strategy which provides a higher availability than periodic inspections without changing the limiting inspection rate. In Chapter IV, we describe a general framework to model non-replaceable systems, which includes non-monotonic hazard rate functions, fallible inspections and a pre-clinical duration. We provide an algorithm for improving a given schedule to reduce the expected delay without changing the number of inspections if the pre-clinical

duration is infinite. Chapter V focuses on how to quantify the value of information for a mixture population and we provide numerical examples for different mixtures. Finally, in Chapter VI, contributions of our research and future directions are discussed.

CHAPTER II

RELATED LITERATURE

Many probabilistic models have been studied in the operations research literature to plan maintenance strategies to detect failures. In this research, we are interested in developing inspection strategies for systems with non-self-announcing failures, which can only be detected by inspections. In this dissertation, we distinguish between replaceable and non-replaceable systems. For replaceable systems, a replacement can be performed either as a result of detecting failures or preventively to avoid the delay caused by failures. On the other hand, for non-replaceable systems, inspections are only performed until the detection of the first failure and after that, no more inspections are made.

1. Industrial Studies: Replaceable Systems

Studies focused on systems with non-self-announcing failures date back to Barlow, Hunter and Proschan (1963). Since inspections are required to detect failures, developing inspection schedules and assessing their performance have been the main objective. Many of the papers in the literature used the expected cost of maintenance as their performance measure to assess inspection schedules (Kolesar (1966), Menipaz (1979) and Munford (1981)). In such studies, costs of inspections and downtimes are taken into the account to describe the maintenance cost. When it is difficult to quantify the relationship between the cost of inspections and downtimes, non-monetary performance measures such as the availability are used to assess inspection schedules (Wortman and Klutke (1994) and Parmigiani (1994)). An excellent literature review for studies on inspection strategies for systems with non-self-announcing failures using different types of performance measures can be found in Valdez-Flores

and Feldman (1989). In this section, studies focused on replaceable systems with non-self-announcing failures will be discussed.

Replaceable systems with non-self-announcing failures have been studied in the literature using two different approaches. In the first one, it is assumed that a system has a nominal life and exposes to a hazardous environment, which is described by a stochastic process. Main objectives for such models are being able to characterize the lifetime distribution and to construct effective inspection strategies to detect failures if possible. In the second approach, the lifetime distribution under current operating conditions is known (i.e., no need to separately model exogenous environment and the nominal life) and the goal is to construct effective inspection strategies.

In the first category, different assumptions about properties of exogenous operating environments and nominal life distributions can provide different insights. Properties of operating environment and nominal life distributions can be used to characterize properties of lifetime distributions. Esary, Marshall and Proschan (1973) showed that properties of nominal life distributions are inherited by the lifetime distribution if the damage process is described by a homogeneous Poisson point process. More specifically, they studied the conditions under which nominal life distribution properties such as having log-concave densities (which is also called Pòlya frequency functions of order 2), increasing (decreasing) hazard rate function, an increasing (decreasing) failure rate function on average or a new better (worse) than used distribution function are inherited by the lifetime distribution. Abdel-Hameed and Proschan (1973) extended these results when the damage process is described by a non-homogeneous Poisson point process. For more general damage processes such as pure jump processes (Abdel-Hameed, 1984b) and Lévy processes (Abdel-Hameed, 1984a), similar results were also obtained. Yang and Klutke (2000b) studied how inspection schedules will change if parameters of damage processes change. Specifically, they studied

a quantile based inspection schedule change when the damage process is an increasing Lévy process and the damage process's Lévy measure changes.

Since developing optimum inspection schedules is quite challenging when explicitly describing the nominal life distribution and the damage process, calculating performance measures for more common inspection schedules such as periodic inspections is more appealing. Wortman, Klutke and Ayhan (1994) expressed the stationary availability for systems with non-self-announcing failure if the nominal life distribution is known and the damage process is a compound Poisson process. They showed that deterministic inspections are the best inspection policy among a renewal types of inspection schedules with a constant inspection rate. Kiessler, Klutke and Yang (2002) expressed the time averaged stationary availability for periodic inspections if the nominal life distribution is known and the hazard rate function of the damage process is described by a discrete Markov chain. Çınlar and Özekici (1987) worked on a similar problem if the nominal life distribution is exponential and the damage process is an increasing semi-Markov process. Many different damage processes such as Hunt, Itô, semi-Markov, Lévy processes are studied in the literature. An extended literature review for damage processes used for replaceable systems can be founded in Yang (1999).

In the case where the lifetime distribution is known, it becomes easier to construct and assess inspections schedules. Parmigiani (1993) studied the optimum inspection schedules to minimize the long run cost per unit time, which includes the cost of different types of inspections, the number of inspections, the cost of the down time, for systems with non-self-announcing failures. He showed that if the lifetime density function is PF_2 (i.e., the density is a log-concave function), then the optimum inter-inspection times should be non-increasing. Yang and Klutke (2000a) studied more general densities than Parmigiani using the stationary time averaged availability and

the limiting inspection rate as their performance measures. Their model schedules inspections such that the conditional survival probability is constant between inspections. Additionally, in the case that lifetime distributions have monotone failure rates, they proposed two hybrid policies, which are mixtures of periodic and their original inspection strategy so that inter-inspection times will not be a death-watch.

Earlier studies for replaceable systems either calculated performance measures for specific inspection schedules or constructed optimum inspection schedules for a specific class of densities with a increasing failure rates (i.e., PF_2 densities). In this research, our focus is to schedule screening inspection effectively for systems with lifetime distributions which may have non-monotone failure rates. We explicitly construct an improved inspection schedule over periodic inspection such that our schedule provides a higher time averaged availability than periodic inspections do but has the same limiting inspection rate with them.

2. Health Care Studies: Non-replaceable Systems

For non-replaceable systems with non-self-announcing failures, the primary goal is to characterize the lifetime distribution so that inspections schedules can be constructed according to its properties. Disease screening is a good example for non-replaceable systems with non-self-announcing failures. In order to evaluate the benefit of screening trials, several large randomized screening trials such as HIP (Health Insurance Plan of Greater New York (1963-1966)), CNBSS (Canadian National Breast Screening Study (1980-1985)) and PLCO (Prostate, Lung, Colorectal and Ovarian Cancer Screening Trial (1991-2001)) were performed. The primary benefit of screening trials is that if diseases are detected at an early stage, it will provide a substantial reduction in mortality. We would like to point out that though most studies suggest screening

trials to save lives, in some cases, benefits of screening trials are questioned by some studies. For instance, Gøtzsche and Olsen (2000) and Olsen and Gøtzsche (2001) suggested that mammography can expose women to unnecessary surgical procedures. But, later, Freedman, Petitti and Robins (2004) counterpointed these two studies by stating that Gøtzsche and Olsen judgements are due to misreadings of the data and the literature. Especially, Freedman et al. provide excellent references for studies focus on quantifying benefits of screening trials.

Though disease screening or incipient failures are non-self-announcing, after a certain duration, the disease moves from a pre-clinical stage to a clinical (i.e., externally observable) phase. In other words, the failure is non-self-announcing during only a certain period of time, (the pre-clinical duration) and after that period, it becomes self-announcing. Different objectives can arise for maintaining non-replaceable systems depending on the fact that the pre-clinical duration is finite or infinite. In this section, we review both non-replaceable systems with finite pre-clinical duration and those with infinite pre-clinical duration.

2.1. Finite Pre-clinical Duration

In the literature, it is assumed that certain medical diseases evolve according to *the natural disease history* (Figure 2) as time goes by. It is assumed that there is a certain time duration which patients are free from the disease, (the disease-free duration), and then the disease starts and only can be detected by screening tests or inspections in our words. This duration is called pre-clinical duration, which can have multiple sub-time intervals. At the end of the pre-clinical duration, obvious symptoms of the disease are present. Zelen and Feinleib (1969) consider such a disease model under the assumption that the disease-free duration is uniformly distributed. Their goal is to develop ways to estimate the expected pre-clinical duration, which is used to determine the expected

lead time gained as a result of performing screening tests. Zelen and Feinleib used The Health Insurance Plan for Greater New York (HIP) data for the breast cancer screening using mammography and concluded that the pre-clinical distribution was exponential. To assess the value of screening, Swartz (1978) developed a probabilistic model based on the change in the life expectancy as a result of performing screening inspections. Later, Albert, Gertman and Louis (1978) introduced the mathematical framework for the natural history of a disease. Their work is the first one which mathematically defines traditional epidemiologic descriptors as age specific incidence and prevalence, lifetime attack rate, mean duration of the disease and in this sense, it provides the foundations for statistical problems related to disease screening. In a similar model, Louis *et al* worked how to estimate the joint distribution of the disease-free duration and the pre-clinical duration using a non-parametric model in Louis, Albert and Heghinian (1978) and Albert, Gertman, Louis and Liu (1978) studied the effects of changes in population at risk to the estimation of the expected pre-clinical duration. Estimation of the expected pre-clinical duration and/or the estimation of sensitivity of screening tests are studied for breast cancer and colorectal cancer in Day and Walter (1984), Walter and Day (1983) and Prevost, Launoy, Duffy and Chen (1998). Recently, Dinse and Hoel (1992) investigated time trends in incidence rates for various cancers.

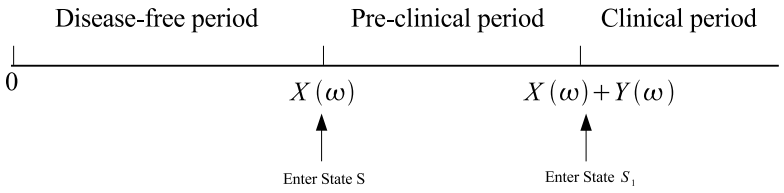

Fig. 2. Sample path for a disease history

Although studies mentioned so far used clinical data to estimate the expected pre-clinical duration and the lead time gained due to the screening for a specific schedule, they did not consider inspection strategies to reduce the expected delay between the detection and the disease start time. Zelen (1993) considered the problem of better schedules for medical disease screening under the assumptions that the disease-free duration has a uniform distribution and the pre-clinical duration has a exponential distribution. He used the probability of detection during the screening duration as his performance measure. This measure is especially suitable for non-invasive diseases since being detected before reaching the clinical level is more important than the expected delay. He showed that periodic inspection schedule would be optimum only if screening inspections are error-free. Gustafsson and Adami (1992) developed simulation based inspection schedules using age-specific incidence rate for cervical cancer in Sweden and age-specific sensitivity of inspections. Later, Parmigiani (1997) developed inspections strategies to minimize approximate expected cost, which have three parts: the cost per inspection, the cost of medical care due to the detection in pre-clinical or clinical state and the cost of being detected at a certain health status. In his study, he assumed a disease natural history as in Albert, Gertman and Louis (1978) and used an inspection intensity model, which was originally proposed by Keller (1974), to approximate the expected cost.

2.2. Infinite Pre-clinical Duration

Infinite pre-clinical durations can be considered for systems with purely non-self-announcing failures. Although this is a simplification for the case of finite pre-clinical duration, it provides insightful information to examine the effects of properties of the lifetime distribution on screening schedules. In this category, Barlow et al. (1963) can be considered as the milestone paper for developing inspection schemes when

the lifetime distribution is known. In their model, the lifetime distribution F and its density f are assumed to be known, and there are two types of cost, the cost of each inspections performed c_1 and the cost of unit downtime c_2. If inspections are performed at $\{x_k\}_{k=0}^{\infty}$, the expected cost can be expressed as

$$C = \mathbb{E}[Cost] = \sum_{k=1}^{\infty} \int_{x_{k-1}}^{x_k} k \cdot c_1 + c_2 \cdot (x_k - t) F(dt) \tag{2.1}$$

$$\frac{\partial C}{\partial x_k} = 0 \quad k = 1, 2, \ldots \tag{2.2}$$

and the optimal cost can be determined by solving (2.2). Inter-inspection times satisfying (2.2) can be expressed in terms of c_1, c_2 and x_1 (the first inspection time). They showed that inter-inspection times should be nonincreasing if the failure rate $\alpha(t) = f(t)/(1 - F(t))$ is increasing. Therefore, the problem becomes finding the first inspection time such that inter-inspection times will be nonincreasing.

Sengupta (1982) extended Barlow et al. to the case where inspections are fallible (an inspection reports the correct state with probability $1 - \gamma$ when the system is failed) and the lifetime distribution is exponential. Sengupta shows that the optimum inspection should be in form of $x_{k+1} = x + k \cdot y,\ k = 0, 1, \ldots$, which says that it is optimum to follow periodic inspections after the first inspection. Earlier, Parmigiani (1993; 1996) extended results in Barlow et al. and Sengupta by allowing fallible-inspections for lifetime distributions with increasing failure rate. Recently, Jiang and Jardine (2005) addressed the high sensitivity of inter-inspection times in Barlow et al. (1963) to the accuracy of the first inspection time and developed easily computable methods to approximate inter-inspection times in Barlow et al..

Trade-offs between the detection delay and the number of inspections performed are modeled as an unconstrained cost problem in the papers of Barlow et al., Sengupta, Parmigiani and Jiang and Jardine. Kirch and Klein (1974) developed an

inspection strategy that has the same expected number of inspection until detection with a given periodic inspection. Their strategy assumed that every 5 years, periodic inspection with different inter-inspection times will applied. Their assumptions about the distribution of the failure and the time inspections cease are somewhat restrictive in order to apply convex constrained optimization.

Since inspection strategies based on Barlow's approach require cumbersome calculations, Keller (1974; 1982) proposed an approximation to this model by introducing the concept of the inspection intensity function $n(t)$. He assumed that inspection will be performed at $x_{k+1} = x_k + (n(x_k))^{-1}$ and used an approximation to the original cost function in Barlow et al. Using calculus of variations, he determined the optimum intensity and showed that his approximation method is accurate. Later, Kaio and Osaki (1984) used the same method to described the optimum inspection rate function when fallible inspections are allowed. More recently, Leung (2001) studied the optimal inspection rate function for systems having multiple types of failures and systems with unknown lifetime distribution, respectively.

Millioni and Pliska (1988) developed an optimum inspection strategy for a system in which there are three stages (good, defective and bad). In their model, during the defective stage, corrective actions (i.e., treatments) can be taken in order to avoid entering the bad stage and transitions between stages are described by a increasing Markov renewal process. In their model, they assumed that inspections were fallible. Özekici and Papazyan (1988) also developed a cost model under transitions governed by a increasing Markov renewal process, presented a numerical results in which inspections could display both false positive and false negative. Later, Özekici and Pliska (1991) extended their previous results by allowing both types of fallible inspection but assuming that once the disease leaves the healthy state and transitions between other states are described by a continuous increasing Markov chain.

For non-replaceable systems, studies focused on designing the effective inspection schedules assumed that the disease-free duration (the lifetime) has an increasing hazard rate function. However, clinical studies suggest that the hazard rate function for disease occurrence may be non-monotonic. Many papers discuss the medical importance of screening, but we review here by only those papers that discuss an analytical approach to scheduling screening exams. In our research, for non-replaceable systems, we focus on how to schedule screening inspections using a limited number of inspections if the disease-free duration has a non-monotone hazard rate.

CHAPTER III

IMPROVED INSPECTIONS FOR REPLACEABLE SYSTEMS

In this chapter, we consider scheduling inspections in a simple replaceable failure prone system when the hazard rate function for time to failure is known. Our results show that it is possible to construct effective inspection schedules for such systems using hazard rate information when the number of inspections performed are considered as a limited resource. The chapter is organized as followed. The first section describes our model and defines the performance measures used. In the second section, we present closed form expressions for performance measures of interest for periodic inspections and construct inspection strategies with unequal times between inspections based on the hazard rate function. We prove that one of these strategies outperforms periodic inspections. Finally, we investigate performance of different inspection policies numerically in the last section.

1. Assumptions and Notation

We consider a simple failure prone system subject to shock degradation where our interest is to detect the first of occurrence of a particular event (we call it a failure to be consistent with the reliability literature). The detection of a security breach in a network or the appearance of a tumor are examples of these types of events. Suppose that the system has a nominal life of 1 unit under operating conditions and is subject to shocks such that each shock removes 1 unit of the life. Assume further that a stochastic point process describes the time of shocks. In this scenario, the first shock will cause the system to fail. Although assumptions about the lifetime and the size of shocks are restrictive when considering failures of a machine, which will usually degrade gradually, it is reasonable for us since we are interested in detecting shocks

(in this case, shocks represent events we would like to detect) as early as possible rather than to model how much damage each shock causes.

In the following, we assume that inspections are performed at times $\{\tau_k : k \geq 1\}$, and that inspections can detect whether the system is operational or failed. If an inspection finds the system operational, the system is left undisturbed and checked again at the next scheduled inspection time. If an inspection finds the system failed, the system is restored to a good-as-new state (that is, perfect repair or replacement by an identical system) and the inspection process continues, taking the replacement time as a new time origin for the inspection strategy. Shocks occur according to a non-homogeneous Poisson point process $\{N(t) : t \geq 0\}$ (i.e., $N(t)$ represents the number of shocks by the time t). Additionally, we assume that $\{N(t) : t \geq 0\}$ has a periodic piecewise constant intensity function $\alpha(t)$. More specifically, we will assume that the intensity function $\alpha(\cdot)$ of the point process satisfies the following conditions. For some $c_i \in \mathbf{Q}^+$ for $i = 1, \ldots, n_\alpha$ and $\{\alpha_i\}_{i=1}^{n_\alpha}$,

$$\alpha(t) = \alpha(t + i \cdot c_{n_\alpha}) \; \forall \; t > 0 \text{ and } \; \forall \; i \in \mathbf{N} \tag{3.1}$$

$$\alpha(t) = \alpha_i > 0 \text{ for } t \in [c_i, c_{i+1}) \; i = 0, 1, \ldots, n_\alpha \tag{3.2}$$

$$m(t) = \int_0^t \alpha(s) ds \tag{3.3}$$

Finally, let $V(t)$ be the status of the system at time t. That is,

$$V(t) = \begin{cases} 1, & \text{If the system is working} \\ 0, & \text{Otherwise} \end{cases}$$

A typical sample path is shown in Figure 3 where $L_i, i = 1, 2, \ldots$ denote successive failure-free times (*uptimes*) or successive lifetimes and R_i, $i = 1, 2, \ldots$ denote the replacement epochs.

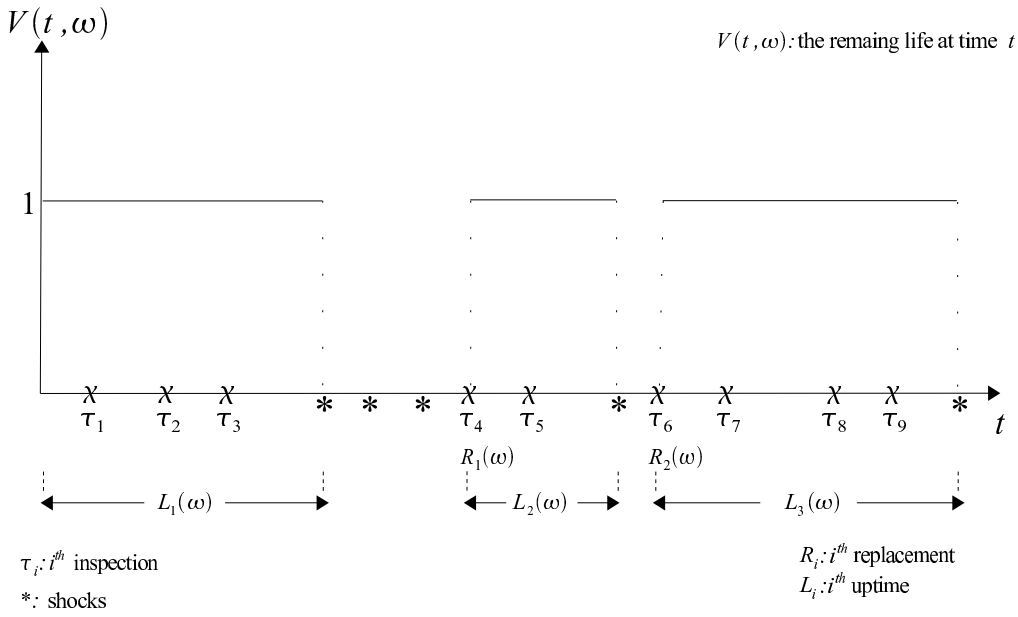

Fig. 3. Sample path for failure prone replaceable systems

The objective is to schedule inspections to increase the limiting uptime duration for a given limiting inspection rate (so, inspections are considered a limited resource). In order to this, we use two performance measures, the stationary time-averaged availability and the limiting inspection rate (hereafter, we refer them as *the availability* and *the inspection rate*, respectively).

Definition 1 *The limiting average availability, A_{av}, and the limiting inspection rate, I_r are defined as*

$$A_{av} = \lim_{t\to\infty} \frac{\int_0^t P\{V(s) > 0\}\, ds}{t} \tag{3.4}$$

$$I_r = \lim_{t\to\infty} \frac{M(t)}{t} \tag{3.5}$$

where $M(t) = \sup\{k \geq 1 : \tau_k \leq t\}$ is the number inspections performed by the time t.

Before continuing to the next section, we consider the relationship between the hazard

rate function of L_i and the intensity function $\alpha(t)$ of the shock process $\{N(t) : t \geq 0\}$. First, let us define the conditional hazard rate function.

Definition 2 *The random variable $H_i(t)$ is called the conditional hazard rate function for $L_i, i = 1, \ldots$ if*

$$H_i(t) = \lim_{s \to 0} \frac{\mathbb{E}\left[1_{\{t < L_i \leq t+s\}} | R_{i-1}\right]}{s \cdot \mathbb{E}\left[1_{\{L_i > t\}} | R_{i-1}\right]} \tag{3.6}$$

$$= \lim_{s \to 0} \frac{P\left\{t < L_i \leq t+s | R_{i-1}\right\}}{s \cdot P\left\{L_i > t | R_{i-1}\right\}} \tag{3.7}$$

It is easy to see that

$$H_i(t) = \lim_{s \to 0} \frac{e^{-(m(R_{i-1}+t)-m(R_{i-1}))} \cdot \left(1 - e^{-(m(R_{i-1}+t+s)-m(R_{i-1}+t))}\right)}{s \cdot e^{-(m(R_{i-1}+t)-m(R_{i-1}))}} \tag{3.8}$$

$$= \alpha(R_{i-1} + t) \tag{3.9}$$

Equation (3.9) states that by knowing the intensity function and the replacement time for a particular cycle, we can determine the hazard rate function of the lifetime distribution for that particular cycle. This was a main motivation for us to look at this simple failure prone system since scheduling inspection using information about the intensity function is the same as scheduling them using the information about the hazard rate function of each cycle. In the next sections, we will express performance measures for different inspection schedules and compare them.

2. Inspection Strategies

2.1. Periodic Inspections

For a given $\tau > 0$, the periodic inspection schedule is one where $\tau_k = k \cdot \tau$, $k \geq 0$. The periodic inspection policy is widely used in practice since it is easily implemented, but it has the disadvantage of not taking the failure intensity function into account

explicitly. Many authors have studied periodic maintenance policies ((Shahani and Crease, 1977), (Zuckerman, 1980), (Nakagawa, 1984), (Abdel-Hameed, 1987) and (Wortman et al., 1994)) under different assumptions. We will denote periodic inspection schedules with inter-arrival times τ as $PI(\tau)$. The next theorem gives the availability and the inspection rate for $PI(\tau)$.

Theorem 1 *The availability and the inspection rate for $PI(\tau)$ where $\tau \in \mathbf{Q}^+$ are*

$$A_{av} = \frac{\sum_{i=1}^{k_\tau^*} \int_{(i-1)\cdot\tau}^{i\cdot\tau} e^{-(m(s)-m(\tau\cdot(i-1)))}\, ds}{k_\tau^* \cdot \tau} \tag{3.10}$$

$$I_r = \frac{1}{\tau} \tag{3.11}$$

where k_τ^* [1] is defined as

$$k_\tau^* = \inf\{k \geq 1 : k \cdot \tau \text{ is divisible by } c_{n_\alpha}\}$$

Proof: Define $\zeta_0 = \tau$ and $\zeta_i = \tau + i \cdot (k_\tau^* \cdot \tau),\ i = 1, 2, \ldots$. Note that since $m(\zeta_i + t) - m(\zeta_i) = m(\zeta_0 + t) - m(\zeta_0),\ i = 1, 2, \ldots,\ N(\zeta_i + t) - N(\zeta_i) \stackrel{d}{=} N(\zeta_0 + t) - N(\zeta_0)$. So, $\{\zeta_i : i \geq 0\}$ are regeneration points for $\{N(t) : t \geq 0\}$. Therefore, by using Smith's theorem (see page 263 in Resnick (1992)),

$$A_{av} = \frac{\mathbb{E}[\psi]}{k_\tau^* \cdot \tau} \tag{3.12}$$

where $\psi = \int_{\zeta_0}^{\zeta_1} 1_{\{V(s)=1\}}\, ds$ is the total up-time between ζ_0 and ζ_1. Now, we will write ψ in terms $\psi_i = \inf\{t > 0 : N(t + i \cdot \tau) - N(i \cdot \tau) > 0\},\ i = 1, 2 \ldots, k_\tau^*$.

$$\psi = \sum_{i=1}^{k_\tau^*} \psi_i \wedge \tau \tag{3.13}$$

[1] *When τ and c_{n_α} are rational numbers, k_τ^* will certainly be a finite number. However, if either τ or c_{n_α} is irrational, k_τ^* may not exist.*

Note each term in the equation (3.13) represents the up-time between i^{th} inspection and $(i+1)^{th}$ inspection.

$$A_{av} = \frac{\mathbb{E}\big[\sum_{i=1}^{k_\tau^*} \psi_i \wedge \tau\big]}{k_\tau^* \cdot \tau} \tag{3.14}$$

$$= \frac{\sum_{i=1}^{k_\tau^*} \int_0^\tau P(\psi_i > s)\, ds}{k_\tau^* \cdot \tau} \tag{3.15}$$

$$= \frac{\sum_{i=1}^{k_\tau^*} \int_{i\cdot\tau}^{(i+1)\cdot\tau} e^{-(m(s)-m(i\cdot\tau))}\, ds}{k_\tau^* \cdot \tau} \tag{3.16}$$

The number of inspection by time t, is $M(t) = \lfloor \frac{t}{\tau} \rfloor$. So,

$$\frac{1}{\tau} \cdot \left(1 - \frac{\tau}{t}\right) \leq \frac{M(t)}{t} \leq \frac{1}{\tau} \tag{3.17}$$

By taking the limit of the equation (3.17), we get the result. ■

2.2. Intensity-Based Inspections

A natural approach for scheduling inspections is to choose inspections such that inter-inspection times will increase (or decrease) if the hazard rate is decreasing (increasing) so that we are likely to perform more inspections when there is a higher chance of failure. We consider inspections such that the expected number of shocks between two inspection is constant (see Figure 4). In this case, we set

$$\mathbb{E}[N(\tau_1)] = m(\tau_1) = \beta \tag{3.18}$$

$$\mathbb{E}[N(\tau_i) - N(\tau_{i-1})] = m(\tau_i) - m(\tau_{i-1}) = \beta,\ i = 2, 3, \ldots \tag{3.19}$$

By equations (3.18) and (3.19), $\tau_i = m^{-1}(i \cdot \beta)$ $i = 1, 2, \ldots$. Fixing the number of shocks between inspections is the same as fixing the probability of having no shocks

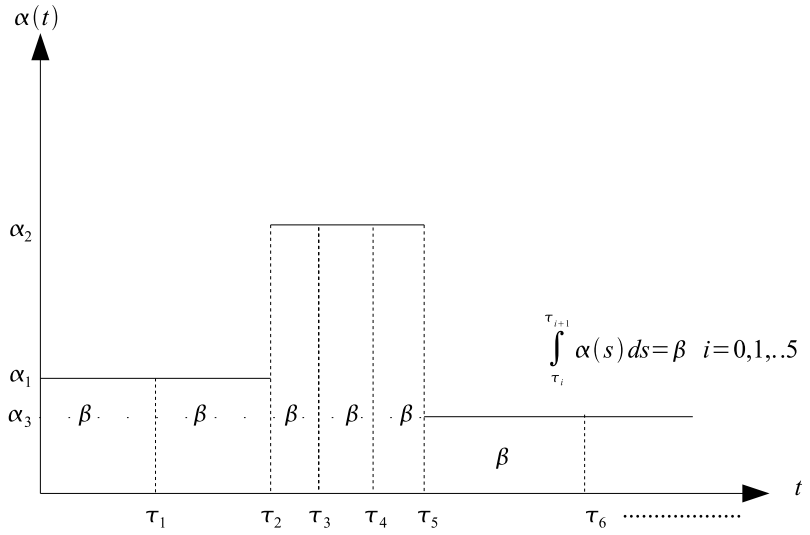

Fig. 4. $IBI(\beta)$ inspection times

between two inspections since one shock causes the failure.

$$P\{\text{No failure between } \tau_i \text{ and } \tau_{i-1}\} = P\{N(\tau_i) - N(\tau_{i-1}) = 0\} \tag{3.20}$$

$$= e^{-(m(\tau_i)-m(\tau_{i-1}))} \tag{3.21}$$

$$= e^{-\beta} \tag{3.22}$$

A similar policy, the quantile-based-inspection which assumes that the survival probability between inspections are fixed, was studied in Yang and Klutke (2000a) and Shahani and Crease (1977) for systems whose the lifetime distribution is known. In our context, we call this the *intensity based inspection policy* with β ($IBI(\beta)$) since inspection times can be calculated using the intensity function $\alpha(t)$.

Theorem 2 *If inspections are performed at* $\{m^{-1}(i \cdot \beta)\}_{i=1}^{\infty}$, *then*

$$A_{av} = \frac{1 - e^{-\beta}}{\beta} \tag{3.23}$$

$$I_r = \frac{\overline{\alpha}}{\beta} \tag{3.24}$$

where $\{c_i\}_{i=0}^{\infty}$ *are discontinuity points of* $\alpha(\cdot)$, c_{n_α} *is the period of* $\alpha(\cdot)$ *and* $\overline{\alpha} =$

$\sum_{i=1}^{n_\alpha} \alpha_i \cdot \frac{(c_i - c_{i-1})}{c_{n_\alpha}}$

Note: Theorem 2 says that applying $IBI(\beta)$ when shocks are governed by a non-homogeneous Poisson point process gives the same availability as if we followed $PI(\beta)$ when shocks are governed by a homogeneous Poisson point process with $\alpha(t) = 1$. This result is intuitively clear because epochs of a non-homogeneous Poisson point process (let's denote them $\{X_n : n \geq 1\}$) are a transformation of those of a homogeneous Poisson process with $\alpha(t) = 1$ (let's denote them $\{\Gamma_n : n \geq 1\}$) under the inverse transformation $X_n = m^{-1}(\Gamma_n)$ (See page 312 in Resnick (1992) for further details). Since the $IBI(\beta)$ is obtained from $PI(\beta)$ using this same transformation, (3.23) is expected.

Proof: By definition, $M(t) = \sup\{n \geq 1 : m^{-1}(i \cdot \beta) \leq t\}$.

$$I_r = \lim_{t\to\infty} \frac{M(t)}{t} \tag{3.25}$$

$$= \lim_{t\to\infty} \frac{M(t)}{m(t)} \cdot \lim_{t\to\infty} \frac{m(t)}{t} \tag{3.26}$$

$$= \frac{1}{\beta} \cdot \overline{\alpha} \tag{3.27}$$

Let R_n, L_n be the the n^{th} replacement time and the up-time between R_{n-1} and R_n as in Figure 3, respectively. Additionally, let NI_n denote the number of inspection performed during the n^{th} replacement cycle. Then, it is easy to see that $\{NI_i\}_{i=1}^{\infty}$ is an i.i.d sequence with $P(NI_1 = k) = (1 - e^{-\beta}) \cdot e^{-\beta\cdot(k-1)}$. So, $\{m(R_i)\}_{i=0}^{n}$ (i.e., $m(R_n) = \beta \sum_{i=1}^{n} NI_i$) is a renewal sequence and let's define

$$U(t) = \inf\{n \geq 1 : m(R_n) > t\} \tag{3.28}$$

$$\tilde{U}(t) = \mathbb{E}\left[U(t)\right] \tag{3.29}$$

$$A_{av} = \lim_{t\to\infty} \frac{\int_0^t P(V(s) > 0)\, ds}{t} \tag{3.30}$$

$$= \lim_{t\to\infty} \frac{m(t)}{t} \cdot \frac{\sum_{k=1}^{\infty} \mathbb{E}\left[L_k \cdot 1_{\{R_{k-1}\le t\}}\right]}{m(t)} - \frac{\mathbb{E}\left[\left(L_{U(m(t))} + R_{U(m(t))-1} - t\right)_+\right]}{t} \tag{3.31}$$

Note that since $m(t)$ is not necessarily a linear function, $(L_n, R_n - R_{n-1} : n \ge 1)$ are not identically distributed, and we can not apply the Renewal Reward Theorem directly. We will use a similar approach by showing

$$\frac{\sum_{k=1}^{\infty} \mathbb{E}\left[L_k \cdot 1_{\{R_{k-1}\le t\}}\right]}{m(t)} \to \frac{1 - e^{-\beta}}{\overline{\alpha} \cdot \beta} \tag{3.32}$$

$$\frac{\mathbb{E}\left[\left(L_{U(m(t))} + R_{U(m(t))-1} - t\right)_+\right]}{t} \to 0 \tag{3.33}$$

Before proving (3.32) and (3.33) , let $A_1(t) = \int_0^\infty \int_y^\infty \frac{\left(\frac{1}{\alpha(m^{-1}(y))} - \frac{1}{\overline{\alpha}}\right) \cdot e^{y-z}}{m(t)}\, dz\, \tilde{U}(dy)$ and $\alpha_{min} = \min_{i=1,\ldots,n_\alpha} \alpha_i$. Then,

$$\lim_{t\to\infty} A_1(t) = 0 \tag{3.34}$$

This follows from

$$\limsup_{t\to\infty} A_1(t) = \left(\sup_{t<\infty} \int_0^t e^{t-y}\, \tilde{U}(dy)\right) \cdot \limsup_{t\to\infty} \left|\frac{m^{-1}(t)}{t} - \frac{1}{\overline{\alpha}}\right|$$
$$+ \left(\sup_{t<\infty} \int_0^t e^{t-y}\, \tilde{U}(dy)\right) \cdot \frac{\overline{\alpha} + \alpha_{min}}{\overline{\alpha} \cdot \alpha_{min}} \cdot \limsup_{t\to\infty} \frac{1}{t} \tag{3.35}$$

$$= 0 \tag{3.36}$$

Note that $\sup_{t<\infty} \int_0^t e^{t-y}\, \tilde{U}(dy) < \infty$ because e^{-x} is a directly Riemann integrable function (see page 231 in Resnick (1992)).

$$\lim_{t\to\infty} \frac{\sum_{k=1}^{\infty} \mathbb{E}\left[L_k \cdot 1_{\{R_{k-1}\le t\}}\right]}{m(t)} = \lim_{t\to\infty} \frac{\tilde{U}(m(t))}{m(t)} + \lim_{t\to\infty} A_1(t) \tag{3.37}$$

$$= \frac{1}{\overline{\alpha} \cdot \mathbb{E}\left[m(R_1)\right]} \text{ by the Elementary Renewal Theorem} \tag{3.38}$$

$$= \frac{1 - e^{-\beta}}{\overline{\alpha} \cdot \beta} \tag{3.39}$$

Let $A_2(t) = \frac{\mathbb{E}\left[(L_{U(m(t))} + R_{U(m(t))-1} - t)_+\right]}{t}$.

$$\lim_{t\to\infty} A_2(t) \le \lim_{t\to\infty} \frac{\mathbb{E}\left[R_{U(t)} - R_{U(t)-1}\right]}{m^{-1}(t)} \tag{3.40}$$

$$= \lim_{t\to\infty} \frac{\mathbb{E}\left[\int_{m(R_{U(t)-1})}^{m(R_{U(t)})} \frac{1}{\alpha(m^{-1}(z))}\, dz\right]}{m^{-1}(t)} \tag{3.41}$$

$$\le \lim_{t\to\infty} \frac{\mathbb{E}\left[m(R_{U(t)}) - m(R_{U(t)-1})\right]}{m^{-1}(t) \cdot \alpha_{min}} \tag{3.42}$$

$$= \lim_{t\to\infty} \frac{t}{m^{-1}(t) \cdot \alpha_{min}} \cdot \lim_{t\to\infty} \frac{\mathbb{E}\left[m(R_{U(t)}) - m(R_{U(t)-1})\right]}{t} \tag{3.43}$$

$$= \frac{\overline{\alpha}}{\alpha_{min}} \cdot 0 = 0 \tag{3.44}$$

Since $\{m(R_i)\}_{i=1}^{\infty}$ is a renewal sequence and $U(t)$ is its renewal function,

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[m(R_{U(t)}) - m(R_{U(t)-1})\right]}{t} \to 0 \tag{3.45}$$

follows from page 134 in Ross (1996). ■

We expected that IBI will outperform PI since inter-inspection times in IBI are matched to the intensity function. However, we found that in most cases, it does not perform better than PI. For instance, consider the intensity function $\alpha^{(1)}(\cdot)$ in

Figure 5. In this case, the intensity changes every one unit of time and changes in the intensity function are not drastic.

$$\alpha^{(1)}(t) = \begin{cases} 0.9, & t \in \bigcup_{n=0}^{\infty} [3n, 3n+1) \\ 1.2, & t \in \bigcup_{n=0}^{\infty} [3n+1, 3n+2) \\ 0.6, & t \in \bigcup_{n=0}^{\infty} [3n+2, 3n+3) \end{cases}$$

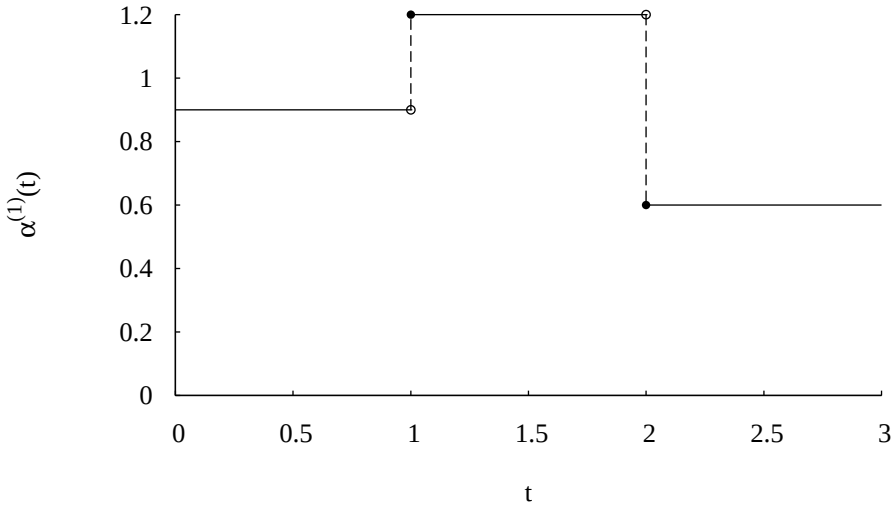

Fig. 5. The intensity function $\alpha^{(1)}(t)$ versus t

Table I shows that periodic inspections usually perform better than intensity-based inspections though the difference between them is not significant . In few cases, IBI slightly outperforms PI ($I_r = 3.3$, 0.33 and 0.17) but our numerical studies show this situation happens rarely.

Table I. Comparison of the availability of IBI and PI for the intensity function $\alpha^{(1)}(\cdot)$

I_r	$A_{av}(IBI)$	$A_{av}(PI)$	$100 \cdot \frac{A_{av}(PI)-A_{av}(IBI)}{A_{av}(IBI)}$
10	0.9563	0.9564	0.0098%
5	0.9152	0.9155	0.038%
3.3	0.8764	0.8762	−0.021%
2	0.8053	0.8071	0.22%
1	0.6594	0.6646	0.79%
0.5	0.4637	0.4698	1.3%
0.33	0.3455	0.3294	−4.7%
0.25	0.2702	0.2737	1.3%
0.17	0.1843	0.1758	−4.7%
0.1	0.1111	0.1125	1.3%

We also look at a case where the intensity function drastically changes by considering the intensity function $\alpha^{(2)}(\cdot)$ in Figure 6. The intensity function $\alpha^{(2)}(\cdot)$ has the same period with $\alpha^{(1)}(\cdot)$ and also changes every one unit of time as $\alpha^{(1)}(\cdot)$ does but there are considerable fluctuations in values of $\alpha^{(2)}(\cdot)$. In contrast to the first case, IBI performs poorly when it is compared with PI (see Table II). We analyzed different types of intensity functions to compare the performance of IBI and PI. Our results showed that IBI generally can not provide an improvement over PI or if it does, the improvement is insignificant. To this end, we will not consider IBI in the remaining of this chapter when comparing inspection strategies.

$$\alpha^{(2)}(t) = \begin{cases} 5, & t \in \bigcup_{n=0}^{\infty} [3n, 3n+1) \\ 30, & t \in \bigcup_{n=0}^{\infty} [3n+1, 3n+2) \\ 10, & t \in \bigcup_{n=0}^{\infty} [3n+2, 3n+3) \end{cases}$$

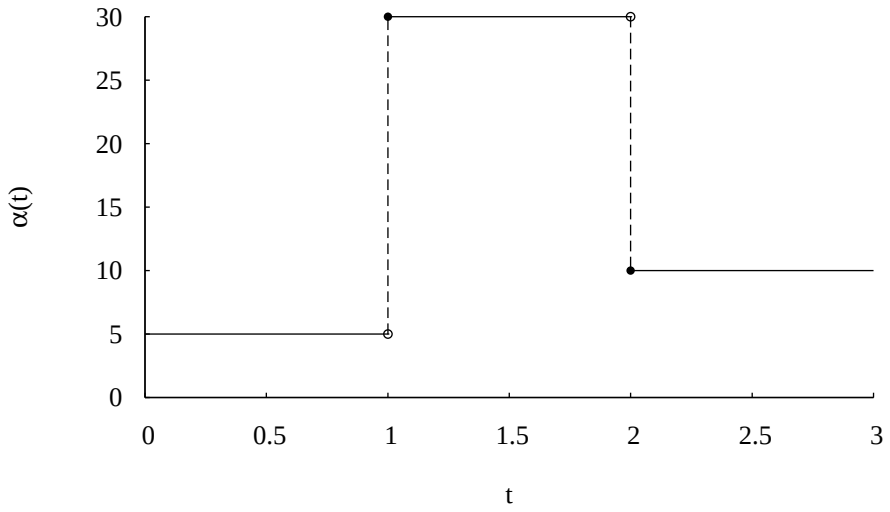

Fig. 6. The intensity function $\alpha^{(2)}(t)$ versus t

2.3. Improved Inspections

Though intensity based inspections seems to adapt to changes in the hazard rate function, they do not necessarily perform better than periodic inspections (there are some cases in which IBI slightly outperforms periodic inspections and these examples are presented in Section 3.1). Thus, they do not help us to achieve the primary goal which is to construct inspection schedules better than periodic inspections. In this section, we present an iterative approach which *does* provide a better inspection schedule over periodic inspections in terms of our performance measures.

When we examine Theorem 1 closely, we observe that it uses the fact that $\{i \cdot k_{\tau}^{*} \cdot$

Table II. Comparison of the availability of IBI and PI for the intensity function $\alpha^{(2)}(\cdot)$

I_r	$A_{av}(IBI)$	$A_{av}(PI)$	$100 \cdot \frac{A_{av}(PI)-A_{av}(IBI)}{A_{av}(IBI)}$
10	0.5179	0.5786	12%
5	0.3167	0.4102	30%
3.3	0.2198	0.3167	44%
2	0.1333	0.2108	58%
1	0.06667	0.1107	66%
0.5	0.03333	0.05537	66%
0.33	0.02222	0.06629	198%
0.25	0.01667	0.02768	66%
0.17	0.01111	0.03315	198%
0.1	0.006667	0.01107	66%

$\tau\}_{i=1}^{\infty}$ are regeneration points for $\{N(t) : t \geq 0\}$ so that the availability is the expected up-time between τ and $(k_\tau^* + 1) \cdot \tau$ divided by $k_\tau^* \cdot \tau$. So, calculating (and increasing) the availability for inspections which repeat themselves every $k_\tau^* \cdot \tau$ interval is the same as calculating (and increasing) the expected uptime between 0 and $k_\tau^* \cdot \tau$ when k_τ^* many inspections are scheduled up to $k_\tau^* \cdot \tau$. To use the same analytical approach, we will consider a class of inspection schedules, which satisfy the following conditions,

1. There exists $\{\tilde{\tau}_i\}_{i=1}^{k_\tau^*}$ such that $0 = \tilde{\tau}_0 < \tilde{\tau}_1 < \tilde{\tau}_2 < \ldots < \tilde{\tau}_{k_\tau^*} \leq k_\tau^* \cdot \tau$

2. Inspection will be performed at $\{\tau_k : k \geq 1\}$ such that

$$\tau_k = \tau \cdot k_\tau^* \cdot \lfloor \frac{k}{k_\tau^*} \rfloor + \tilde{\tau}_{k - \lfloor \frac{k}{k_\tau^*} \rfloor \cdot k_\tau^*} \; k = 1, \ldots \tag{3.46}$$

Therefore, $\tau_1 = \tilde{\tau}_1$, $\tau_2 = \tilde{\tau}_2$, …, $\tau_{k_\tau^*} = k_\tau^* \cdot \tau$, $\tau_{k_\tau^*+1} = k_\tau^* \cdot \tau + \tilde{\tau}_1$, $\tau_{k_\tau^*+2} = k_\tau^* \cdot \tau + \tilde{\tau}_2$ and so on. By choosing inspection times in this way, we guarantee that the inspection rate

of this schedule is the same as $PI(\tau)$ because in both inspections strategies, k_τ^* many inspections are performed on every interval in the form of $((i-1)\cdot k_\tau^*,\, i\cdot k_\tau^*)\,, i = 1, 2, \ldots$. But, the crucial question is how to choose these $\{\tilde{\tau}_i\}_{i=1}^{k_\tau^*}$, depending on the inter-inspection time of periodic inspections, to improve the availability. The following theorem provides a simple way to choose them.

Theorem 3 *For fixed $\tau \in \mathbf{Q}^+$,*

1. *if $k_\tau^* = 1$,*

$$\tilde{\tau}_1 = \text{argmax}\{g(z, \tau + z) : 0 \le z \le \tau\} \tag{3.47}$$

if $k_\tau^* \ge 2$, then $i = 1, \ldots, k_\tau^* - 1$

$$\tilde{\tau}_i = \text{argmax}\{g(\tilde{\tau}_{i-1}, z) + g(z, (i+1)\cdot\tau) : \tilde{\tau}_{i-1} \le z \le (i+1)\cdot\tau\}, \tag{3.48}$$

$$\tilde{\tau}_{k_\tau^*} = \text{argmax}\{g(\tilde{\tau}_{k_\tau^*-1}, z) + g(z, k_\tau^*\cdot\tau + \tilde{\tau}_1) : \tilde{\tau}_{k_\tau^*-1} \le z \le k_\tau^*\cdot\tau\} \tag{3.49}$$

where $g(a, z) = \int_a^z e^{-(m(s)-m(a))}\, ds$.

2. *If inspections are performed at $\tau_k = \tau\cdot k_\tau^*\cdot\lfloor\frac{k}{k_\tau^*}\rfloor + \tilde{\tau}_{k-\lfloor\frac{k}{k_\tau^*}\rfloor\cdot k_\tau^*}$ $k = 1, \ldots,$ then the availability $\tilde{A}_{av}$ and the inspection rate $\tilde{I}_r$ are*

$$\tilde{A}_{av} = \frac{\sum_{i=1}^{k_\tau^*} \int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} e^{-(m(s)-m(\tilde{\tau}_i)}\, ds}{k_\tau^*\cdot\tau} \tag{3.50}$$

$$\ge A_{av}\left(PI(\tau)\right) \tag{3.51}$$

$$\tilde{I}_r = \frac{1}{\tau} \tag{3.52}$$

Proof: By definition, $M(t) = \sup\{k \ge 1 : \tau_k \le t\}$. So,

$$\frac{k_\tau^*\cdot\lfloor\frac{t}{k_\tau^*\cdot\tau}\rfloor}{t} \le \frac{M(t)}{t} \le \frac{k_\tau^*\cdot\lceil\frac{t}{k_\tau^*\cdot\tau}\rceil}{t} \tag{3.53}$$

By taking the limit of (3.53), we get the result.

$$\tilde{I}_r = \frac{1}{\tau} \tag{3.54}$$

Let $U(I_1, I_2, \ldots, I_{k_\tau^*})$ denote the expected up-time between I_1 and $I_1 + k_\tau^* \cdot \tau$ provided that $I_i \le I_{i+1} \le k_\tau^* \cdot \tau,\ i = 1, 2, \ldots, k_\tau^* - 1$ and inspections are performed at $\tau_k = k_\tau^* \cdot \tau \cdot \lfloor \frac{k}{k_\tau^*} \rfloor + I_{k - \lfloor \frac{k}{k_\tau^*} \rfloor \cdot k_\tau^*},\ k = 1, \ldots$. By the proof of Theorem 1, it is easy to see that the average availability will be $\frac{U(I_1, I_2, \ldots, I_{k_\tau^*})}{k_\tau^* \cdot \tau}$. Therefore, it is enough to prove that $U(\tilde{\tau}_1, \ldots, \tilde{\tau}_{k_\tau^*}) \ge U(\tau, 2\tau, \ldots, k_\tau^* \cdot \tau)$.

Let $\Delta U_1 = U(\tilde{\tau}_1, 2\tau, 3\tau, \ldots, k_\tau^* \cdot \tau) - U(\tau, 2\tau, \ldots, k_\tau^* \cdot \tau)$.

$$\Delta U_1 = \big(g(0, \tilde{\tau}_1) - g(\tilde{\tau}_1, 2\tau)\big) - \big(g(0, \tau) - g(\tau, 2\tau)\big) \tag{3.55}$$

$$\ge 0 \ \text{ by the definition of } \tilde{\tau}_1 \tag{3.56}$$

A similar result holds for $l = 2, 3, \ldots, k_\tau^* - 1$. Let

$$\Delta U_l = U(\tilde{\tau}_1, \ldots, \tilde{\tau}_l, (l+1) \cdot \tau, \ldots, k_\tau^* \cdot \tau) - U(\tilde{\tau}_1, \ldots, \tilde{\tau}_{l-1}, l \cdot \tau, \ldots, k_\tau^* \cdot \tau) \tag{3.57}$$

$$\Delta U_l = \big(g(\tilde{\tau}_{l-1}, \tilde{\tau}_l) + g(\tilde{\tau}_l, (l+1) \cdot \tau)\big) - \big(g(\tilde{\tau}_{l-1}, l \cdot \tau) + g(l \cdot \tau, (l+1) \cdot \tau)\big) \tag{3.58}$$

$$\ge 0 \text{ by the definition of } \tilde{\tau}_l \tag{3.59}$$

Finally, let $\Delta U_{k_\tau^*} = U(\tilde{\tau}_1, \ldots, \tilde{\tau}_{k_\tau^*}) - U(\tilde{\tau}_1, \ldots, \tilde{\tau}_{k_\tau^* - 1}, k_\tau^* \cdot \tau)$.

$$\Delta U_{k_\tau^*} = \big(g(\tilde{\tau}_{k_\tau^* - 1}, \tilde{\tau}_{k_\tau^*}) + g(\tilde{\tau}_{k_\tau^*}, \tilde{\tau}_1 + k_\tau^* \cdot \tau)\big) - \big(g(\tilde{\tau}_{k_\tau^* - 1}, k_\tau^* \cdot \tau) + g(0, \tilde{\tau}_1)\big) \tag{3.60}$$

$$\ge 0 \tag{3.61}$$

By equations (3.56), (3.59) and (3.61),

$$U(\tau_1, \ldots, \tau_{k_\tau^*}) \geq U(\tau, 2\tau, \ldots, k_\tau^* \cdot \tau) \tag{3.62}$$

■

The idea behind Theorem 3 is the following:

1. At the first step, we assume that the first inspection time $\tilde{\tau}_1$ is unknown and the rest of inspection times are known (they're at $\tau_j = j \cdot \tau,\ j = 2, \ldots, k_\tau^*$). We choose the best possible inspection between 0 and 2τ using Equation (3.48) (see Figure 7).

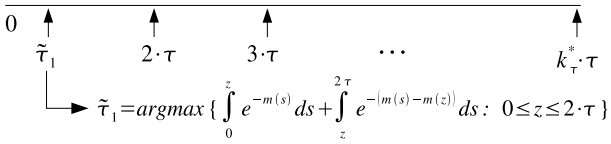

Fig. 7. The first iteration in Theorem 3

2. At the j^{th} step ($j = 2, \ldots, k_\tau^* - 1$), we have already fixed the first $j-1$ inspection times and we assume that only the j^{th} inspection time is unknown and the rest of inspections are performed at $\tau_i = i \cdot \tau,\ i = j+1, \ldots, k_\tau^*$. Equation (3.48) gives us the best possible j^{th} inspection time between $\tilde{\tau}_{j-1}$ and $(j+1) \cdot \tau$ (see Figure 8).

3. Finally, at the k_τ^{*th} step, we find the k_τ^{*th} inspection using (3.49).

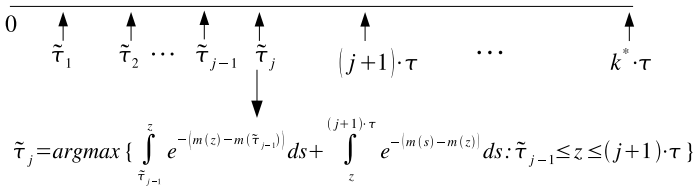

Fig. 8. The j^{th} iteration in Theorem 3

Although Theorem 3 suggests constructing an improved inspection schedule over periodic inspection by iteratively solving one dimensional optimization problems (Equations (3.47),(3.48) and (3.49)), it does not specify how to solve them. Before explaining how to solve those equations, consider a simple case where the intensity function is $\alpha^{(3)}(t)$ as below and $\tau = 0.4$, which is the same period of the intensity function.

$$\alpha^{(3)}(t) = \begin{cases} 2, & t \in \bigcup_{n=0}^{\infty} [0.4 \cdot n, 0.4 \cdot n + 0.1) \\ 3.4, & t \in \bigcup_{n=0}^{\infty} [0.4 \cdot n + 0.1, 0.4 \cdot n + 0.2) \\ 1.6, & t \in \bigcup_{n=0}^{\infty} [0.4 \cdot n + 0.3, 0.4 \cdot (n+1)) \end{cases}$$

Therefore, $k_\tau^* = 1$ and we want to solve the following equation

$$\tilde{\tau}_1 = \text{argmax}\{ \int_{z}^{0.4+z} e^{-(m(s)-m(z))}\, ds : \ 0 \le z \le 0.4\} \tag{3.63}$$

Figure 9 shows how the availability changes as a function of the value of the first inspection time. The figure suggests that in order to maximize this function, since $\int_{z}^{0.4+z} e^{-(m(s)-m(z))}\, ds$ is not differentiable at the discontinuity points of the intensity function, we can find its maximum by finding local maximums in each interval where the intensity function is continuous and then comparing these local maximums to get the global maximum over the whole interval.

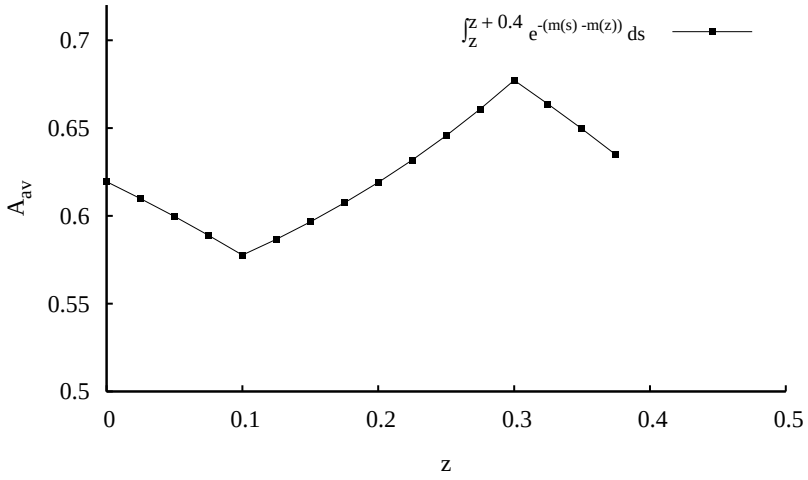

Fig. 9. A_{av} for $\alpha^{(3)}(t)$ versus the first inspection time when $\tau = 0.4$

Lemma 1 gives details of how to find the local maximum in each sub-interval where the intensity function is constant.

Lemma 1 *Let $f(\cdot)$ be a differentiable function on $[d, e]$ and let $\{j_k\}_{k=1}^n$ be discontinuity points of $\alpha(\cdot)$ on (a, b), $j_0 = a$ and $j_{n+1} = b$.*

1. *If*

$$f'(z) = C_1 \cdot e^{-\alpha\cdot(z-d)} + C_2 \cdot e^{-\alpha\cdot(e-z)} \tag{3.64}$$

and $C_1 \geq 0$ and $\alpha \geq 0$, then

$$\max\{f(z) : z \in [d, e]\} = \max\{f(z) : z \in \{d, e, s_{(d,e)}\}\} \tag{3.65}$$

where $s_{(d,e)} = \frac{\alpha\cdot(d+e)-2\cdot\ln(-\frac{C_2}{C_1})}{2\alpha}$

2. *For any $a < b$,*

$$\max_{z\in[a,b]}\{g(a, z) + g(z, b)\} = \max_{0\leq k\leq n} \max\{g(a, z) + g(z, b) : z \in \{j_k, j_{k+1}, s_{(j_k,j_{k+1})}\}\} \tag{3.66}$$

where $s_{(j_k,j_{k+1})} = \frac{\alpha(j_1)\cdot(j_k+j_{k+1})-2\cdot\ln(-\frac{C_2}{C_1})}{2\alpha(j_k)}$, $C_1 = e^{-\left(m(j_1)-m(a)\right)}$ and

$$C_2 = \int_{j_2}^{b} e^{-\left(m(s)-m(j_2)\right)} - 1.$$

Proof: If $C_2 \geq 0$, then f is an increasing function and the maximum of f is achieved at e. If $C_2 < 0$, then

$$f''(z) = -\alpha \cdot C_1 \cdot e^{-\alpha\cdot(z-d)} + \alpha \cdot C_2 \cdot e^{-\alpha\cdot(e-z)} < 0 \tag{3.67}$$

Therefore, $f(z)$ is concave if $C_2 < 0$ and

$$f'(z) = 0 \Leftrightarrow z = \frac{\alpha \cdot (d+e) - 2 \cdot \ln(-\frac{C_2}{C_1})}{2\alpha}.$$

Note that if there is no root for $f'(z) = 0$ on the domain of f, f has to be a monotone function. Therefore, the maximum of f can be achieved at either endpoints or at the stationary point if it exists.

In order to do the maximization in the second part of Lemma 1, let's look at the $g(a,z) + g(z,b)$ on (j_1, j_2) where j_1 and j_2 be consecutive jump points for $\alpha(\cdot)$ (i.e., $m(s) - m(j_1) = \alpha(j_1) \cdot (s - j_1),\ j_1 \leq s \leq j_2$). For $z \in (j_1, j_2)$,

$$\begin{aligned}\frac{d}{dz}\left(g(a,z) + g(z,b)\right) &= e^{-\left(m(z)-m(a)\right)} - e^{-\left(m(j_2)-m(z)\right)} \\ &\quad + \int_{j_2}^{b} \alpha(j_1) \cdot e^{-\left(m(s)-m(z)\right)} ds \qquad (3.68) \\ &= C_1 \cdot e^{-\alpha(j_1)\cdot(z-j_1)} + C_2 \cdot e^{-\alpha(j_1)\cdot(j_2-z)} \qquad (3.69)\end{aligned}$$

where $C_1 = e^{-\left(m(j_1)-m(a)\right)}$ and $C_2 = \int_{j_2}^{b} e^{-\left(m(s)-m(j_2)\right)} - 1$.

$$\begin{aligned}\max_{z\in[a,b]} \{g(a,z) + g(z,b)\} &= \max_{0\leq k\leq n} \max \{g(a,z) + g(z,b) : z \in [j_k, j_{k+1}]\} \qquad (3.70) \\ &= \max_{0\leq k\leq n} \max\{g(a,z) + g(z,b) : z \in \{j_k, j_{k+1}, s_{(j_k,j_{k+1})}\}\} \\ &\qquad (3.71)\end{aligned}$$

The equation (3.71) follows from the fact that $g(a,z)+g(z,b)$ behaves as the function f in the part 1 when $z \in [j_1, j_2]$, ■

Corollary 1 *If the intensity function is constant between two inspection points and one more inspection is to be scheduled between these inspections, then the midpoint of the original inspections is optimum for the extra inspection time. In other words, if* $\alpha(t)=\alpha,\ t \in (a,b)$ *for some* $\alpha > 0$, *then*

$$\text{argmax}\{g(a,z)+g(z,b) : z \in (a,b)\} = \frac{a+b}{2} \tag{3.72}$$

Proof: Note if $\alpha(t)=\alpha,\ t \in (a,b)$, $C_1 = 1$ and $C_2 = -1 < 0$. By the first part of Lemma 1, $g(a,z)+g(z,b)$ is concave on this interval and its stationary point is

$$s_{(a,b)} = \frac{\alpha \cdot (a+b) - 2 \cdot \ln(1)}{2 \cdot \alpha} = \frac{a+b}{2} \tag{3.73}$$

■

Our iterative improvement approach considers periodic inspections as a starting point and finds the local maximum for $\tilde{\tau}_i,\ i = 1, \ldots, k_\tau^*$. However, for each point, the right endpoint of its domain changes in the next iteration (i.e., the domain for $\tilde{\tau}_1$ is $[0, 2 \cdot \tau]$ but after $\tilde{\tau}_2$ is calculated, it becomes $[0, \tilde{\tau}_2]$ and so on). This change in domains of the inspection times suggests that using our iterative approach repeatedly by choosing the last updated version of $\tilde{\tau}_i,\ i = 1, \ldots, k_\tau^*$. To this end, our iterative algorithm will work repeatedly by assuming that the previous solution as its starting point and a descriptive pseudocode is given below.

Input: τ, k_τ^*, ϵ and N_{max}

Set $\tilde{\tau}_i^{(0)} \leftarrow i \cdot \tau,\ i = 0, \ldots, k_\tau^*$, $max_{distance} \leftarrow \infty$ *and* $l \leftarrow 1$;

while $l < N_{max}$ *or* $max_{distance} < \epsilon$ **do**

$\tilde{\tau}_i^{(l)} \leftarrow \tilde{\tau}_i^{(l-1)},\ i = 0, \ldots, k_\tau^*$;

for $j \leftarrow 1$ **to** k_τ^* **do**

$\tilde{\tau}_j^{(l)} \leftarrow \underset{z \in [\tilde{\tau}_{j-1}^{(l)},\, \tilde{\tau}_{j+1}^{(l-1)}]}{\operatorname{argmax}} \{g(\tilde{\tau}_{j-1}^{(l)}, z) + g(z, \tilde{\tau}_{j+1}^{(l-1)})\}$ using Lemma 1

end

$max_{distance} \leftarrow \max\left\{\left|\tilde{\tau}_j^{(l)} - \tilde{\tau}_j^{(l-1)}\right| : j = 1, \ldots, k_\tau^*\right\}$;

$l \leftarrow l + 1$

end

Examples demonstrating how inspection points are moved by the algorithm can found in Appendix A.

3. Numerical Studies

In this section, we present numerical results for our inspections schedules for replaceable systems with non-self-announcing failures. First, we analyze the performance of the improved inspection scheme against periodic inspections. Later, we discuss the basic properties of the improved inspection algorithm. Specifically, we examine numerically the convergence of the improved inspection scheme and if convergence is demonstrated, the scheme converges to the the optimum inspection schedule.

3.1. Performance of The Improved Inspection Scheme

In this section, we present the performance of the improved inspection scheme and periodic inspections. Consider the intensity function $\alpha^{(4)}(t)$ with The period 5 given in Figure 10.

Table III. The availability for $PI(\tau)$ and the improved inspections when $\alpha^{(4)}(\cdot)$, $N_{max} = 50$ and $\epsilon = 0.0001$

τ	I_r	$A_{av}(PI(\tau))$	$\tilde{A}_{av}$	$\frac{\tilde{A}_{av} - A_{av}(PI(\tau))}{A_{av}(PI(\tau))}$
0.2	5	0.938	0.941	0.24%
0.4	2.5	0.883	0.888	0.63%
0.6	1.67	0.832	0.842	1.2%
0.8	1.25	0.786	0.797	1.5%
1	1	0.743	0.754	1.4%
1.2	0.8	0.686	0.711	3.8%
1.8	0.571	0.61	0.644	5.5%
2	0.5	0.574	0.605	5.4%
2.5	0.4	0.484	0.551	14%
3	0.333	0.456	0.497	9.2%
3.5	0.286	0.408	0.459	13%
4	0.25	0.37	0.43	16%
5	0.2	0.301	0.389	29%
7.5	0.133	0.201	0.265	32%
10	0.1	0.157	0.202	29%
15	0.0667	0.105	0.135	29%

$$\alpha^{(4)}(t) = \begin{cases} 0.5, & t \in \bigcup_{n=0}^{\infty} [5n,\ 5n+1) \\ 0.75, & t \in \bigcup_{n=0}^{\infty} [5n+1,\ 5n+2.25) \\ 1.5, & t \in \bigcup_{n=0}^{\infty} [5n+2.25,\ 5n+2.75) \\ 0.6, & t \in \bigcup_{n=0}^{\infty} [5n+2.75,\ 5n+4) \\ 0.3, & t \in \bigcup_{n=0}^{\infty} [5n+4,\ 5n+5) \end{cases}$$

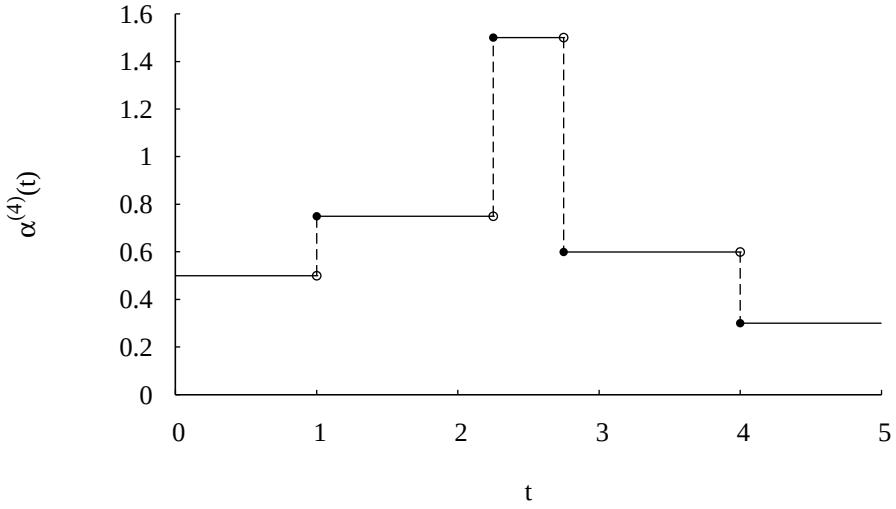

Fig. 10. The intensity function $\alpha^{(4)}(t)$ versus t

Table III shows the availability for periodic inspection and the improved inspections. As expected, when the inspection rate is high (in other words, inter-inspection for periodic inspections is small), the improvement algorithm does not give a substantial increase in the availability. Although the algorithm can not improve the availability of periodic inspections more than 1.5%, for $\tau \leq 1$, the improvement becomes more visible as the inspection rate (and hence the availability) gets smaller. Since the inspection rate can be considered a limited resource for the maintenance provider's point of view, the improved inspections are beneficial when the available resource are scarce. Furthermore, in some cases, the difference between their performance ($\tau = 2$

and $\tau = 4$) are relatively substantial. More examples about performance of these strategies can be found in Appendix B

3.2. Convergence of the Improved Inspection Scheme

Let's consider the same intensity function $\alpha^{(4)}(t)$ in the previous section. Assume that $\tau = 0.625$ (so, $k_\tau^* = 8$), $N_{max} = 30$ and $\epsilon = 0.0001$. As a result of applying our improvement algorithm, the availability increases 0.823 to 0.837 (i.e., 1.7% improvement in the availability). The question we would like to ask is how $\tilde{\tau}_i^{(n)}, i = 1, \ldots, 8$ changes (or more specifically, do they converge?). Figure 11 shows how $\tilde{\tau}_i^{(n)}, i = 1, \ldots, 8$ changes through several iterations. It suggests that all $\tilde{\tau}_i^{(n)}, i = 1, \ldots, 8$ converge but the speed of the convergence is not always same. For instance, $\tilde{\tau}_7^{(n)}$ does not change after 5^{th} iteration but on the other side, $\tilde{\tau}_3^{(n)}$ requires 14 iterations to converge. Although the convergence could not be proven analytically, when we look at different types of intensity functions, they all suggest that the convergence occurs (see Appendix C).

Numerical evidences for the convergence raise the issue whether or not the algorithm converges to the optimum schedule. Checking the optimality when $k_\tau^* = 2$ is possible by an exhaustive search of the three dimensional graph of A_{av} versus $(\tilde{\tau}_1, \tilde{\tau}_2)$ (the first two inspection times). For higher dimensions, this would not work and the non-differentiability at discontinuity points of the intensity function makes it hard to analyze derivatives of the expression for the availability.

In this part, we present numerical result that suggests that when $k_\tau^* = 2$ (i.e., $\tau = 2.5$), the algorithm finds the optimum solution. For the case $k_\tau^* = 2$ and $0 \leq \tilde{\tau}_1 < \tilde{\tau}_2 \leq 2 \cdot \tau$

$$A_{av} = \frac{\int_{\tilde{\tau}_1}^{\tilde{\tau}_2} e^{-(m(s)-m(\tilde{\tau}_1))} ds + \int_{\tilde{\tau}_2}^{2\cdot\tau+\tilde{\tau}_1} e^{-(m(s)-m(\tilde{\tau}_2))} ds}{2 \cdot \tau} \tag{3.74}$$

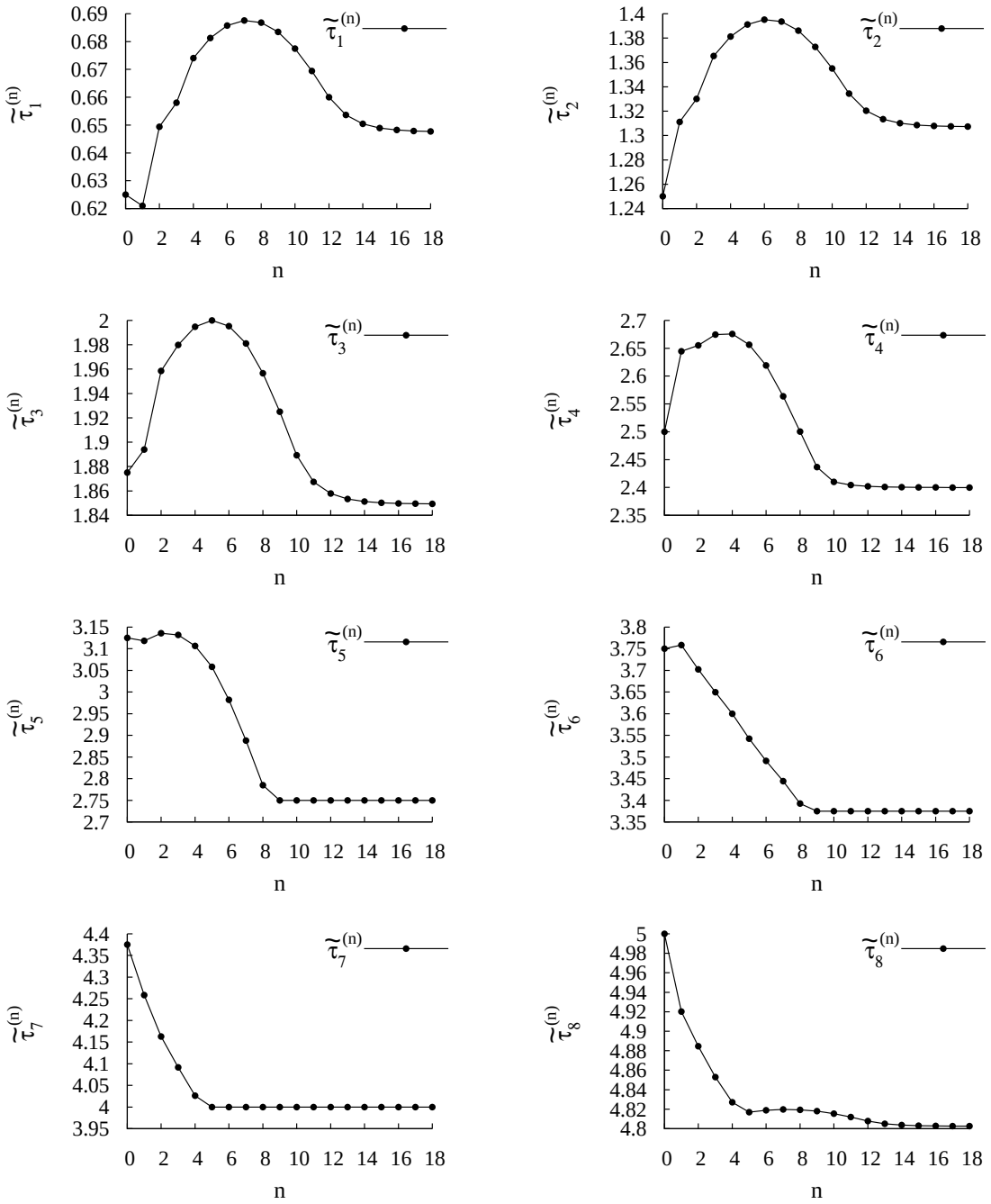

Fig. 11. $\tilde{\tau}_i^{(n)}, i = 1, \ldots, 8$ for $\alpha^{(4)}(\cdot)$ versus n

When A_{av} is plotted against $(\tilde{\tau}_1, \tilde{\tau}_2)$ for $\alpha^{(4)}(t)$ (see Figure 10 for $\alpha^{(4)}(t)$), in two iterations, the optimum schedule is obtained. Figure 12 and 13 shows the optimality and the movement of $\tilde{\tau}_i$, $i = 1, 2$. Although this particular case suggests that the convergence occurs fast for the case $k_\tau^* = 2$, Appendix D provides additional examples where the convergence is slower.

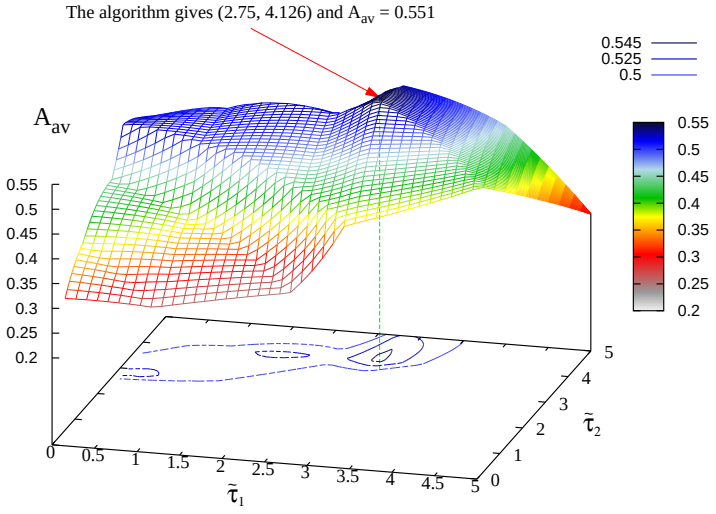

Fig. 12. A_{av} versus $(\tilde{\tau}_1, \tilde{\tau}_2)$ for $\alpha^{(4)}(t)$ when $k_\tau^* = 2$

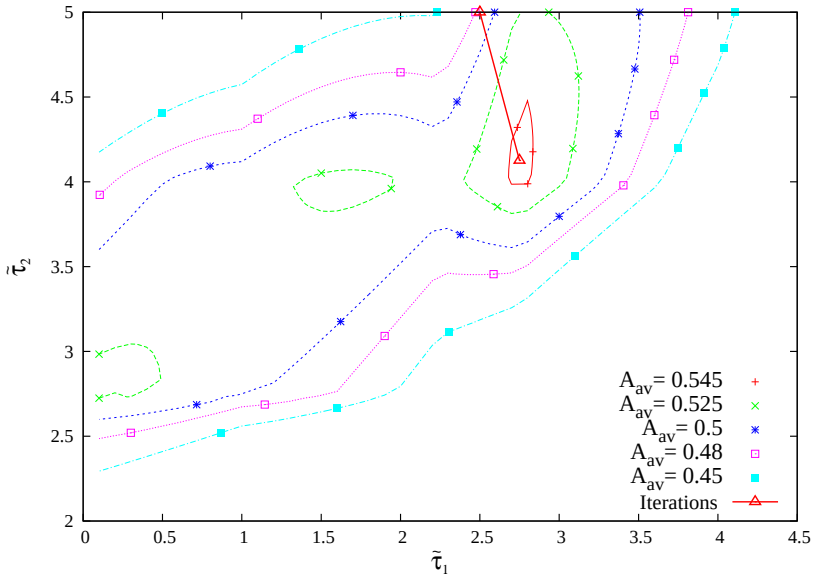

Fig. 13. The improvement algorithm iterations for $\alpha^{(4)}(t)$ when $k_\tau^* = 2$

CHAPTER IV

AN IMPROVED INSPECTION STRATEGY FOR NON-REPLACEABLE SYSTEMS

This chapter discusses how to schedule screening inspections for non-replaceable systems when the lifetime distribution is known. For such systems, our goal is to reduce the expected delay between the detection and the failure by using scheduling a fixed number of inspections on a finite interval. The results show that for if the pre-clinical duration is infinite, then a similar iterative approach in Chapter III can be used to reduce the expected delay.

This chapter is organized as followed. The first section describes assumptions about the model and defines performance measures which will be used in this chapter. In the second section, we discuss the assumption for the lifetime distribution to be a Pòlya frequency function of order 2 (PF_2), which has been used in the literature, and compare it with our assumption. In the next section, the performance measure is described for the general natural disease history when pre-clinical duration is finite or infinite, and improved inspection schedules are constructed for the case where pre-clinical duration is infinite. In the last section, numerical results are discussed.

1. Assumptions and Notation

Consider a system with non-self-announcing failure such that the failure can be detected by inspection during a certain period, after which failure becomes self-apparent. The occurrence of certain medical diseases, surveillance and computer network security are good examples for such systems. In the health care literature, the lifetime (the failure-free duration) is referred to as *the disease free duration* and the time from the disease start to the first time it becomes apparent is called *the pre-clinical*

duration. Our goal in this chapter is to schedule a fixed number of inspections over a finite horizon to reduce the expected delay between the time of failure and the time of detection.

We assume that the time to failure, or onset of disease in the health-care setting (the lifetime), is represented by a random variable L. After the failure occurs, it is detectable only by inspections for a period of time denoted by Y (the *pre-clinical period*), after which the failure is outwardly observable, and the inspection process ends. Thus failure is detectable only by inspection during the interval $[L, L + Y)$. The hazard rate function of L (hereafter, denoted by $\alpha(t)$) is used to describe the distribution of L and is assumed to be piecewise linear right continuous. Specifically, $\alpha(t)$ is assumed to satisfy the following conditions:

1. for any given t, there exits a_t, b_t, c_t and d_t such that $t \in [c_t, d_t)$ and

$$\alpha(s) = a_t + b_t \cdot (s - c_t) \geq 0, \ \forall s \in [c_t, d_t) \tag{4.1}$$

2. There is no $s_1 < s_2$ such that

$$\alpha(s) = 0 \ \forall s \in (s_1, s_2) \tag{4.2}$$

In this chapter, we consider both error-free and fallible inspections. Fallible inspections are performed at $\{\tau_i\}_{i=0}^{k^*+1}$ over a finite horizon $[a, b]$ such that $\tau_0 = a$ and $\tau_{k^*+1} = b$ and an error-free inspection is performed only if a fallible inspection reports that the failure occurred. This two stage inspection policy reflects the situation that error-free inspections (for instance, a biopsy) can be too harmful or expensive to perform as a first screening course. Since a fallible inspection that reports a failure is followed by an error-free inspection, the specificity of fallible inspections (false positive probability) will not affect the expected delay between the occurrence and identifi-

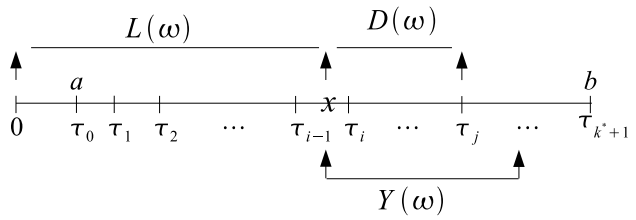

Fig. 14. Sample path for non-replaceable systems

cation of failure, which serves as our performance measure. Parmigiani (1996; 1993) considered the effect of the specificity of fallible inspections by assigning different costs for fallible and error-free inspections. However, the sensitivity of fallible inspections (false negative probability), must also be considered, since for any given schedule, as the sensitivity decreases, the expected detection delay will decrease as well. As Figure 14 illustrates, it is possible to to perform more than one inspections after failure because inspections may report that the system is working when in fact it has failed.

The inspection sensitivity will be denoted by γ; i.e.,

$$P\{\text{Inspection reports no failure} \mid \text{Failure occurred}\} = \gamma \tag{4.3}$$

Additionally, we assume that the results of successive inspections are independent from each other given L and Y. In other words, if Z_i represents the result of the i^{th} inspection (with $Z_i = F\,(NF)$ denoting that the i^{th} inspection reports failure (no failure)), then for $i = 1, \ldots, k^* + 1$ and $r_j \in \{F, NF\}$, $j = 1, \ldots, k^* + 1$,

$$P\{Z_1 = r_1, \ldots, Z_{k^*+1} = r_{k^*+1} | \tau_{i-1} < L \leq \tau_i,\, Y\} = \prod_{j=1}^{k^*+1} P\{Z_j = r_j | \tau_{i-1} < L \leq \tau_i,\, Y\} \tag{4.4}$$

As our performance measure, we will use the expected detection delay (simply called the expected delay), where the delay is defined as follows.

Definition 3 *The delay, D, for a schedule $\{\tau_i\}_{i=1}^{k^*}$ with $\tau_0 = a$ and $\tau_{k^*+1} = b$, is defined as, $i = 1, \ldots, k^* + 1$*

$$D(\omega) = \sum_{j=i}^{k^*+1} (\tau_j - L(\omega)) \cdot 1_{\{\tau_j \le L(\omega)+Y(\omega)\}} \cdot 1_{\{Z_j(\omega)=F\}} \cdot \prod_{l=i}^{j-1} 1_{\{Z_l(\omega)=NF\}}$$
$$+ Y(\omega) \cdot \sum_{j=i}^{k^*+1} 1_{\{\tau_j > L(\omega)+Y(\omega) > \tau_{j-1}\}} \cdot \prod_{l=i}^{j-1} 1_{\{Z_l(\omega)=NF\}},\ \omega \in \tau_{i-1} < L(\omega) \le \tau_i \quad (4.5)$$

As mentioned earlier, our goal is to reduce the expected delay by scheduling a fixed number of inspections in the interval $[a, b]$. The objective in this chapter resembles ones in the previous chapter when we compare decreasing the expected delay for a fixed number of inspections in this chapter with increasing the availability for a fixed inspection rate in the previous chapter.

2. Properties of PF_2 Densities

There are numerous studies in the reliability literature focused on timing screening tests for detecting failures. Most of them assume the lifetime density has a Pòlya frequency function of order 2 (PF_2). In this section, we address some shortcomings for using the PF_2 assumption and present some clinical data which does not support the PF_2 assumption.

The following four statemens are equivalent definitions for a Pòlya frequency function of order 2 (PF_2) and are taken from Barlow et al. (1963).

Definition 4 *f is PF_2 if one of four statement below holds*

1. *for $t_1 < t_2$ and $y_1 < y_2$*

$$\begin{vmatrix} f(t_1 - y_1) & f(t_1 - y_2) \\ f(t_2 - y_1) & f(t_2 - y_2) \end{vmatrix} \ge 0 \quad (4.6)$$

2.

$$\frac{f(x-\delta)}{f(x)} \tag{4.7}$$

is increasing in x for all $\delta \geq 0$.

3.

$$\frac{f(x+\delta)}{f(x)} \tag{4.8}$$

is decreasing in x for all $\delta \geq 0$.

4.

$$f(t) = e^{-\psi(t)} \tag{4.9}$$

where $\psi(t)$ is convex in an open interval $u < t < v$ $(-\infty \leq u < v \leq +\infty)$, $\psi(t) = +\infty$, $t < u$ and $t > v$ and

$$\lim_{s \downarrow u} \psi(s) \leq \psi(u) \leq +\infty, \quad \lim_{s \uparrow v} \psi(s) \leq \psi(v) \leq +\infty \tag{4.10}$$

Because of the characterization of equation (4.9), PF_2 functions are also called as log-concave functions. Many common distributions, including uniform, normal, exponential, logistics, extreme-value and Laplace, have log-concave densities for any value in their parameters' range. On the other hand, Weibull, power function, gamma and beta distribution families have log-concave densities when their parameters satisfy certain conditions. If a distribution has a log-concave density, then its hazard rate function is non-decreasing. Therefore, distributions with log-concave densities form a subset of distributions with non-decreasing hazard rate functions. Although it is reasonable to assume that a device or system gets more fragile as it is used so that the increasing hazard rate assumption could be valid, it is easy to construct non-log-

concave densities with an increasing hazard rate function. For instance, consider a density $f(t) = \alpha_{13}(t) \cdot e^{-\int_0^t \alpha_{13}(s)\, ds}$, $t \geq 0$ where $\alpha_{13}(t)$ is the hazard rate function for f and is a piecewise linear increasing function as below.

$$\alpha_{13}(t) = \begin{cases} 0.2, & t \in [0, 1) \\ 0.2 + 0.1 \cdot (t - 1], & t \in [1, 2] \end{cases}$$

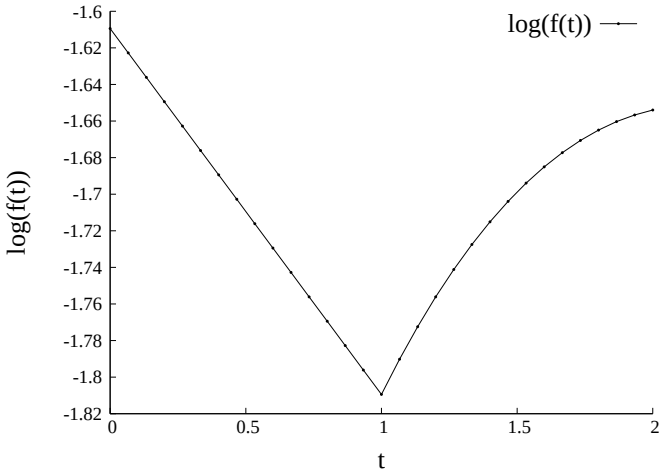

Fig. 15. The $\log(f(t))$ versus t for the hazard rate function $\alpha_{13}(t)$

As illustrated in Figure 15, f does not have a log-concave density although its hazard rate function is increasing. Thus the class of log-concave densities is not rich enough to model all systems with increasing hazard rate functions. Moreover, several clinical studies suggest that the assumption of a log-concave hazard rate, or even an increasing hazard rate, may not be appropriate for certain diseases. For instance, data from Albert, Gertman and Louis (1978), page 23, suggest that the density function for the disease free duration for carcinoma in situ of the cervix, does not satisfy the PF_2 assumption (see Figure 16).

To this end, our assumption that the hazard rate for the lifetime distribution is a piecewise linear function can be considered a reasonable assumption since it can help

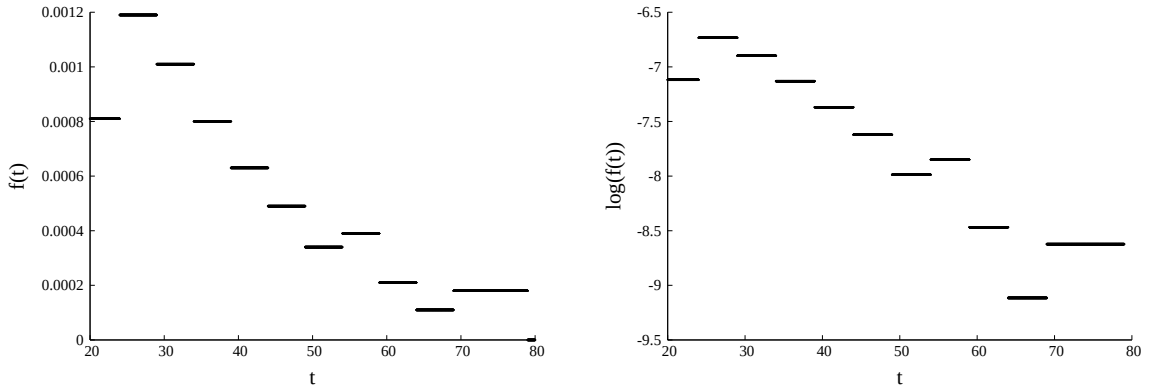

Fig. 16. The density and the log-density for the disease free duration of carcinoma in situ of the cervix in 1960-1966 British Columbia

us to develop inspections strategies for more general class of the lifetime distributions (i.e., ones with monotone or non-monotome hazard rate functions, which can not be described by PF_2 densities).

3. A General Approach Using the Natural Disease History

As described in Definition 3, the delay for an inspections schedule is defined as the time between the detection and the occurrence of the failure if it happend during the screening interval (i.e., it is defined only if the failure occurred between $\tau_0 = a$ and $\tau_{k^*+1} = b$). Three cases can be considered for the general disease history.

1. The pre-clinical duration is infinite.

2. The pre-clinical duration is an almost surely finite random variable and is independent of the disease-free duration.

3. The pre-clinical duration is an almost surely finite random variable and the pre-clinical and the disease-free durations are dependent (i.e., the age might be an affect on the duration which the disease becomes self-apparent).

For the first case, it is possible to construct an iterative algorithm as in Chapter III, which can provide a lower expected delay from any given starting schedule. Although we are not able to provide an explicit method to construct improved inspection strategies for the second case, the performance measure for this case can be described along with the first case and it will be one of the future directions of this research to investigate this case in details. We would like to investigate the third case in the future since it will be the most general model which can capture the true nature of the disease history. Theorem 4 describes the expected delay for the first two cases.

Theorem 4 *Given that the sensitivity of fallible inspections is* γ.

1. *If the disease-free duration and the pre-clinical durations are independent, their densities are* $f(\cdot)$ *(with a know hazard rate function* $\alpha(\cdot)$*) and* $g(\cdot)$*, respectively then the expected delay* $\mathbb{E}[D]$ *for a given inspection schedule* $\{\tau_i\}_{i=0}^{k+1}$ *(*$\tau_0 = a$ *and* $\tau_{k+1} = b$*),*

$$\mathbb{E}[D] = \sum_{i=1}^{k^*+1} \sum_{j=i}^{k^*+1} \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_j - s}^{\infty} g(x)\, d(x) \cdot (\tau_j - s) \cdot \gamma^{j-i} \cdot (1-\gamma) \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds$$
$$+ \sum_{i=1}^{k^*+1} \sum_{j=i}^{k^*+1} \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{j-1} - s}^{\tau_j - s} \gamma^{j-i} \cdot x \cdot g(x)\, dx \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds \qquad (4.11)$$

2. *If the pre-clinical duration is infinite and the distribution of the disease-free duration has a hazard rate function* $\alpha(\cdot)$*, then the expected delay* $\mathbb{E}[D]$ *for a given schedule* $\{\tau_i\}_{i=0}^{k+1}$ *(*$\tau_0 = a$ *and* $\tau_{k+1} = b$*),*

$$\mathbb{E}[D] = \sum_{i=1}^{k^*+1} \sum_{j=i}^{k^*+1} \int_{\tau_{i-1}}^{\tau_i} (\tau_j - s) \cdot \gamma^{j-i} \cdot (1-\gamma) \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds \qquad (4.12)$$

Proof: Note that since inspections are applied systems with a lifetime larger that a, the conditional lifetime density for such systems is $\alpha(s) \cdot e^{-(m(s)-m(a))}$, $s \geq a$.

1. for $i = 1, \ldots, k^* + 1$.

$$\mathbb{E}\left[D|\tau_{i-1} < L \leq \tau_i,\, Y\right] = \sum_{j=i}^{k^*+1} (\tau_j - L) \cdot 1_{\{\tau_j \leq L+Y\}} \cdot \gamma^{j-i} \cdot (1-\gamma) + \sum_{j=i}^{k^*+1} Y \cdot 1_{\{\tau_j > L+Y > \tau_{j-1}\}} \cdot \gamma^{j-i} \tag{4.13}$$

Since L and Y are independent with probability density functions $f(\cdot)$ (its hazard rate function is $\alpha(\cdot)$) and $g(\cdot)$, respectively

$$\mathbb{E}[D] = \sum_{i=1}^{k^*+1} \int_{\tau_{i-1}}^{\tau_i} \left\{ \int_0^{\infty} \mathbb{E}\left[D|L = s,\, Y = x\right] \cdot g(x)\, dx \right\} \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds \tag{4.14}$$

By combining (4.13) and (4.14), the equation (4.11) is obtained.

2. For this case, the detection can only occur as a result of a positive test (because pre-clinical duration is infinite). Therefore, the second term in the delay expression is almost surely 0. So,

$$\mathbb{E}\left[D|\tau_{i-1} < L \leq \tau_i,\right] = \sum_{j=i}^{k^*+1} (\tau_j - L) \cdot \gamma^{j-i} \cdot (1-\gamma) \tag{4.15}$$

$$\mathbb{E}[D] = \sum_{i=1}^{k^*+1} \int_{\tau_{i-1}}^{\tau_i} \left\{\mathbb{E}\left[D|L = s\right]\right\} \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds \tag{4.16}$$

(4.12) follows from (4.15) and (4.16) ■

3.1. An Improved Inspection Strategy for Infinite Pre-clinical Duration

In this section, we show that it is possible to reduce the expected delay by iteratively changing a given inspection schedule when pre-clinical duration is infinite. The development is similar to Chapter III, which optimizes the performance measure for a single variable in each step. Before giving details of the algorithm, we consider some

special cases that provide insight into improving the scheduling algorithm. First, we consider a simple case in which $k^* = 1$, $a = 0$, $b = 5$ and the hazard rate function is $\alpha_{14}(t)$. Additionally, we assume that all of inspections are error-free (i.e., $\gamma = 0$). Since $k^* = 1$, the goal is to find τ_1 between $\tau_0 = a$ and $\tau_2 = b$ that minimizes $\mathbb{E}[D]$.

$$\alpha_{14}(t) = \begin{cases} 0.2 - 0.05 \cdot t, & t \in [0, 2) \\ 0.1 + 0.2 \cdot (t - 2), & t \in [2, 4) \\ 0.5 - 0.2 \cdot (t - 4), & t \in [4, 5] \end{cases}$$

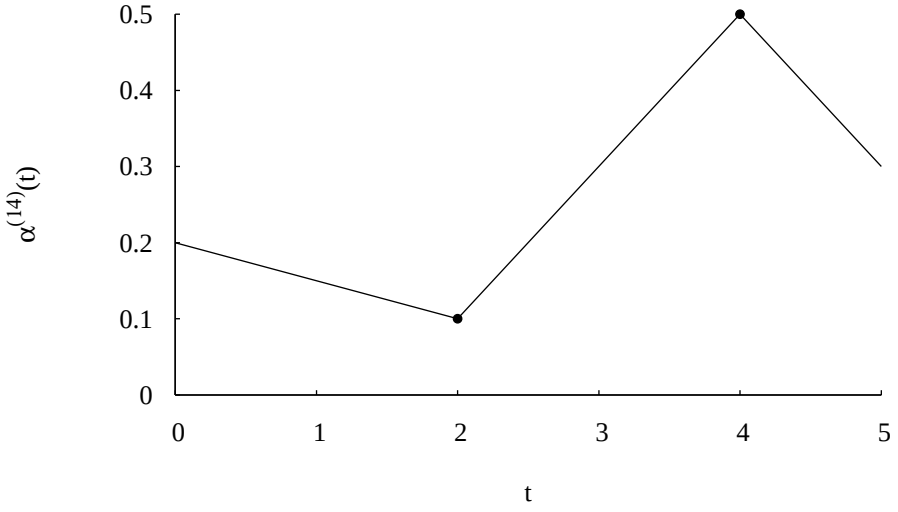

Fig. 17. The hazard rate function $\alpha_{14}(t)$ versus t

In other words, τ_1 can be written as

$$\tau_1 = \underset{z \in [\tau_0, \tau_2]}{\operatorname{argmin}} \left\{ \int_{\tau_0}^{z} (z - s)\, \alpha^{(14)}(s) e^{-(m(s) - m(a))}\, ds + \int_{z}^{\tau_2} (\tau_2 - s)\, \alpha^{(14)}(s) e^{-(m(s) - m(a))}\, ds \right\} \tag{4.17}$$

Figure 18 show how $\mathbb{E}[D]$ changes versus τ_1 and suggests that on each interval where the hazard rate function is linear, $\mathbb{E}[D]$ is either unimodal, or the interval can be divided into two pieces such that in the first part, $\mathbb{E}[D]$ is concave and in the remaining part, $\mathbb{E}[D]$ is convex. We will show that the second situation arises if the

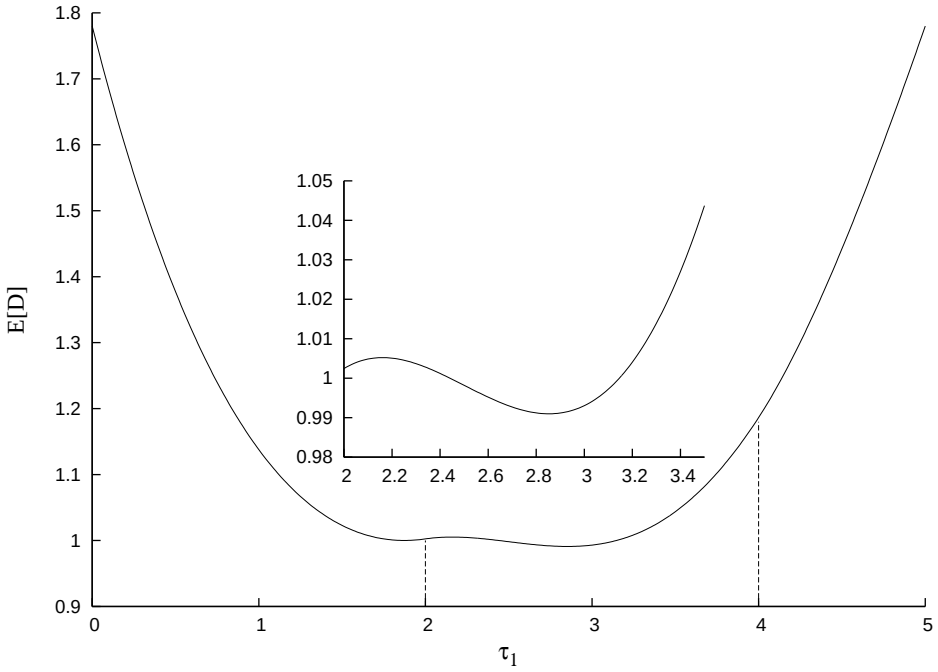

Fig. 18. Expected delay versus τ_1 for the hazard rate function $\alpha_{14}(\cdot)$

hazard function is *"sharply"* increasing in the interval. This suggests that when the hazard rate is relatively high, it is beneficial to postpone the inspection for a while to increase the chance of finding the failure as soon as it happens.

Next, we consider a slightly more complicated case in which all of the inspection times except the l^{th} inspection time are known (assume that they are performed at $x_i, i = 0, \ldots, l-1, l, \ldots, k^*+1$), and the goal is how to choose it between x_{l-1} and x_{l+1} so that it will minimize the expected delay. In other words, we want to minimize $\mathbb{E}[D(x_l)]$, which is the function of x_l, l^{th} inspection

$$\mathbb{E}[D(x_l)] = \sum_{i=1}^{l+1} \int_{x_{i-1}}^{x_i} \sum_{j=i}^{k^*+1} (x_j - s) \cdot \gamma^{j-i} \cdot (1-\gamma) \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds + \underbrace{\sum_{i=l+2}^{k^*+1} \int_{x_{i-1}}^{x_i} \sum_{j=i}^{k^*+1} (x_j - s) \cdot \gamma^{j-i} \cdot (1-\gamma) \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds}_{Constant} \tag{4.18}$$

Lemma 2 will show that equation (4.18) can be divided into disjoint intervals in which (4.18) is either unimodal or it is concave and then becomes convex.

Lemma 2

1. *For any given* $x_0, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{k^*+1}$ *and* $x_l \in [x_{l-1}, x_{l+1}]$*, let's define*

$$g_l(x_l) := \sum_{i=1}^{l+1} \int_{x_{i-1}}^{x_i} \sum_{j=i}^{k^*+1} (x_j - s) \cdot (1-\gamma) \cdot \gamma^{j-i} \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds \tag{4.19}$$

If $[j_1, j_2] \subseteq [x_{l-1}, x_l]$ *such that* $\alpha(s) = \lambda \cdot (s - j_1) + \delta$ *for some* λ *and* δ*, then* $g_l(x_l)$ *is either unimodal on* $[j_1, j_2]$ *or there is a point* $j_1 < j^* < j_2$ *such that it is concave on* $[j_1, j^*]$ *and convex on* $[j^*, j_2]$*. Therefore,*

$$\underset{x_{l-1} \le u \le x_{l+1}}{\operatorname{argmin}} \; g(u) = \underset{i=1,\ldots,n_l}{\operatorname{argmin}} \left\{ \min_{\{j_{i-1}^l, s_i^l, j_i^l\}} g(u) \right\} \tag{4.20}$$

where $\{[j_{i-1}^l j_i^l]\}_{i=1}^{n_l}$ *are subintervals such that* $\alpha(s) = \lambda_i \cdot (s - j_{i-1}^l) + \delta_i$ *for some* λ_i *and* δ_i*,* $[x_{l-1}, x_{l+1}] = \bigcup_{i=1}^{n_l} [j_{i-1}^l j_i^l]$ *and* s_i^l *is the stationary point for the convex part of* $g(x_l)$ *on* $[j_{i-1}^l j_i^l]$ *(if it exists).*

Proof: Consider an arbitrary subinterval $[j_{i-1}^l, j_i^l]$ of $[x_{l-1}, x_l]$ such $s \in [j_{i-1}^l, j_i^l)$, $\alpha(s) = \lambda_i \cdot (s - j_{i-1}^l) + \delta_i,\ s \in [j_{i-1}^l . j_i^l)$. If $x_l \in (j_{i-1}^l, j_i^l)$, then

$$\begin{aligned} g_l^{'}(x_l) = & \sum_{i=1}^{l} \int_{x_{i-1}}^{x_i} (1-\gamma) \cdot \gamma^{l-i} \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds \\ & - \left(\sum_{j=l+1}^{k^*+1} (x_j - x_l) \cdot \gamma^{j-(l+1)} \right) (1-\gamma)^2 \cdot \alpha(x_l) \cdot e^{-(m(x_l)-m(a))} \end{aligned} \tag{4.21}$$

$$g_l^{''}(x_l) = k(x_l) \cdot e^{-(m(x_l)-m(a))} \tag{4.22}$$

where

$$k(x_l) = (1-\gamma)\cdot\left(2-\gamma^{k^*+1-l}\right)\cdot\alpha(x_l) + (1-\gamma)^2\left(-\lambda_i+\alpha(x_l)^2\right)\cdot\sum_{j=l+1}^{k^*+1}(x_j-x_l)\cdot\gamma^{j-(l+1)}$$

There are two case for sign changes for $g_l^{''}(x_l)$, $x_l \in (j_{i-1}^l, j_i^l)$:

Case 1: If $\lambda_i \leq 0$ or $\lambda_i > 0$ and $\delta_i^2 \geq \lambda_i$, then $g_l^{''}(x_l) \geq 0$, $x_l \in (j_{i-1}^l, j_i^l)$.

Case 2: If $\lambda_i > 0$ and $\delta_i^2 < \lambda_i$, then there is a $j_{i-1}^l < j^* \leq j_i^l$ such that $-\lambda_i + \alpha(s)^2 < 0, s \in (j_{i-1}^l . j^*)$ and $-\lambda_i + \alpha(s)^2 \geq 0$, $s \in (j^*, j_i^l)$. So $g_l^{''}(x_l) \geq 0$, $x_l \in (j^*, j_i^l)$ and also, note that $k^{'}(x_l) \geq 0, x_l \in (j_{i-1}^l, j^*)$. Therefore, $g_l^{''}(x_l) < 0$ if and only if $x_l \in (j_{i-1}^l, j^*)$ and $k(j_{i-1}^l) < 0$.

So, on each $[j_{i-1}^l, j_i^l], i = 1, \ldots, n_l$, the minimum can obtained either endpoints or the stationary point for the convex part, which can be obtained by numerically solving (4.21). Therefore, the equation (4.20) follows by finding the minimum on each $[j_{i-1}^l, j_i^l], i = 1, \ldots, n_l$. ■

Using the same approach in Chapter III, we can improve any initial inspection schedule, $\{\tau_i\}_{i=0}^{k^*+1}$, by rescheduling inspections one by one in the following way:

- Consider the expected delay as a function first inspection point τ_1 under the assumption that the rest of inspections are performed according to the original schedule and minimize this function on $[\tau_0, \tau_2]$. Let's call the new first inspection time as $\tilde{\tau}_1$.

- Similarly, consider the expected delay as a function i^{th} inspection point under the assumption that $\tau_l = \tilde{\tau}_l$, $l = 1, \ldots, i-1$ and the rest of inspections are performed according to the original schedule. Then, minimize this function on the $[\tilde{\tau}_{i-1}, \tau_{i+1}]$.

Theorem 5 *For a given inspection schedule* $\{\tau_i\}_{i=0}^{k^*+1}$*, define* $\tilde{\tau}_0 = a$*,* $\tilde{\tau}_{k+1} = b$ *and* $l = 1, \ldots, k^*$*:*

$$\tilde{\tau}_l = \underset{\tilde{\tau}_{i-1} \le u \le \tau_{i+1}}{\operatorname{argmin}} \left\{ \sum_{i=1}^{l+1} \int_{x_{i-1}}^{x_i} \sum_{j=i}^{k^*+1} (x_j - s) \cdot (1-\gamma) \cdot \gamma^{j-i} \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds \right\} \quad (4.23)$$

where $x_i = \tilde{\tau}_i,\ i = 0, 1, \ldots, l-1$*,* $x_l = u$ *and* $x_i = \tau_i,\ i = l+1, \ldots, k^*+1$*.*

If $\mathbb{E}[D]$ *(*$\mathbb{E}[\tilde{D}]$*) denotes the expected delay if inspections are performed at* $\{\tau_i\}_{i=0}^{k^*+1}$ *(*$\{\tilde{\tau}_i\}_{i=0}^{k^*+1}$*), then*

$$\mathbb{E}[D] \ge \mathbb{E}[\tilde{D}] \quad (4.24)$$

Proof: The first part uses Lemma 2 to calculate $\tilde{\tau}_i, i = 1, \ldots, k^*$ and the proof of the second part is the same as the proof of Theorem 3 in Chapter III. ■

Our iterative approach takes a starting schedule and updates each inspection point one by one assuming that inspections after the current inspection point are made according to the original schedule and produces a new inspection schedule. Since our approach does not require a new specific assumption about the initial schedule other than being a valid schedule ($\tau_0 = a < \tau_1 < \ldots < \tau_{k^*} < \tau_{k^*+1} = b$), we can use the new schedule as the new starting schedule and try to improve that one too. So, hereafter we assume that our iterative approach will work repeatedly by assuming that the previous solution as its starting point and a descriptive pseudocode is below.

Input: a, b, k^*, ϵ and N_{max}

Set $\tau = \frac{b-a}{k^*+1}$, $\tilde{\tau}_i^{(0)} \leftarrow a + i \cdot \tau,\ i = 0, \ldots, k^* + 1$, $max_{distance} \leftarrow \infty$ *and* $n \leftarrow 1$

while $n < N_{max}$ *or* $max_{distance} < \epsilon$ **do**

$\tilde{\tau}_i^{(n)} \leftarrow \tilde{\tau}_i^{(n-1)},\ i = 0, \ldots, k_\tau^*$;

for $l \leftarrow 1$ **to** k^* **do**

$$\tilde{\tau}_l^{(n)} = \underset{\tilde{\tau}_{l-1}^{(n)} \le u \le \tilde{\tau}_{l+1}^{(n)}}{\operatorname{argmin}} \left\{ \sum_{i=1}^{l+1} \int_{x_{i-1}}^{x_i} \sum_{j=i}^{k^*+1} (x_j - s) \cdot (1-\gamma) \cdot \gamma^{j-i} \cdot \alpha(s) \cdot e^{-(m(s)-m(a))}\, ds \right\}$$

using Lemma 2 where $x_i = \tilde{\tau}_i^{(n)},\ i = 0, 1, \ldots, l-1$, $x_l = u$ and $x_i = \tau_i^{(n)},\ i = l+1, \ldots, k^*+1$.

end

$max_{distance} \leftarrow \max \left\{ \left| \tilde{\tau}_i^{(n)} - \tilde{\tau}_i^{(n-1)} \right| : i = 1, \ldots, k^* \right\}$;

$n \leftarrow n + 1$

end

Examples for improved schedules can found in Appendix E.

4. Numerical Examples for the Infinite Pre-clinical Duration

In this section, we will present numerical results for our inspections schedules for non-replaceable systems with infinite pre-clinical duration. We are interested in three questions,

- Does the iterative algorithm converge?
- If it does, is there any numerical evidence that it convergences to the optimum schedule?
- How does it perform against periodic inspections?

Although, we do not have a proof, numerical studies suggest that the answer for first two questions is positive. Later, we compare improved inspections and periodic

inspections for different k^* values.

As an example, consider the hazard rate function in Figure 19. We apply our iterative approach to $\alpha_{19}(\cdot)$ with $a = 0$, $b = 5$, $k^* = 6$, $\epsilon = 0.0001$, $\gamma = 0$ and $N_{max} = 30$. As seen in Figure 20, $\tau_i^{(n)}, i = 1, \ldots, 6$ converges but the speed of the convergence is different for each inspection. Additional numerical studies of the convergence can be found in Appendix F.

$$\alpha_{19}(t) = \begin{cases} 0.05, & [0,1) \\ 0.05 + 0.05 \cdot (t-1), & [1,2) \\ 0.2, & [2,3) \\ 0.1 - 0.05 \cdot (t-3), & [3,4) \\ 0.025, & [4,5] \end{cases}$$

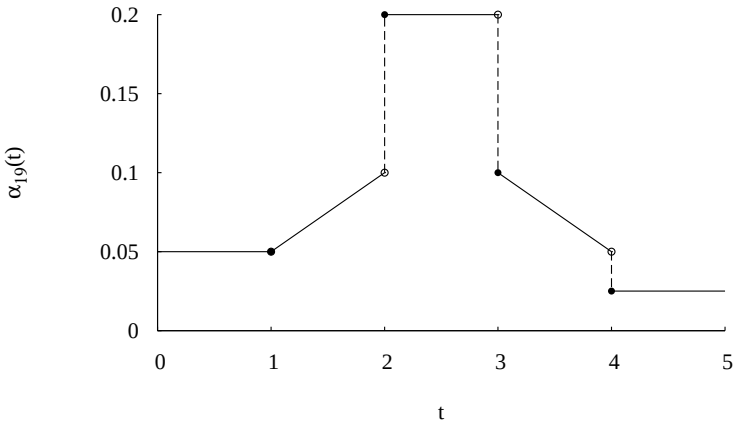

Fig. 19. The hazard rate function $\alpha_{19}(t)$ versus t

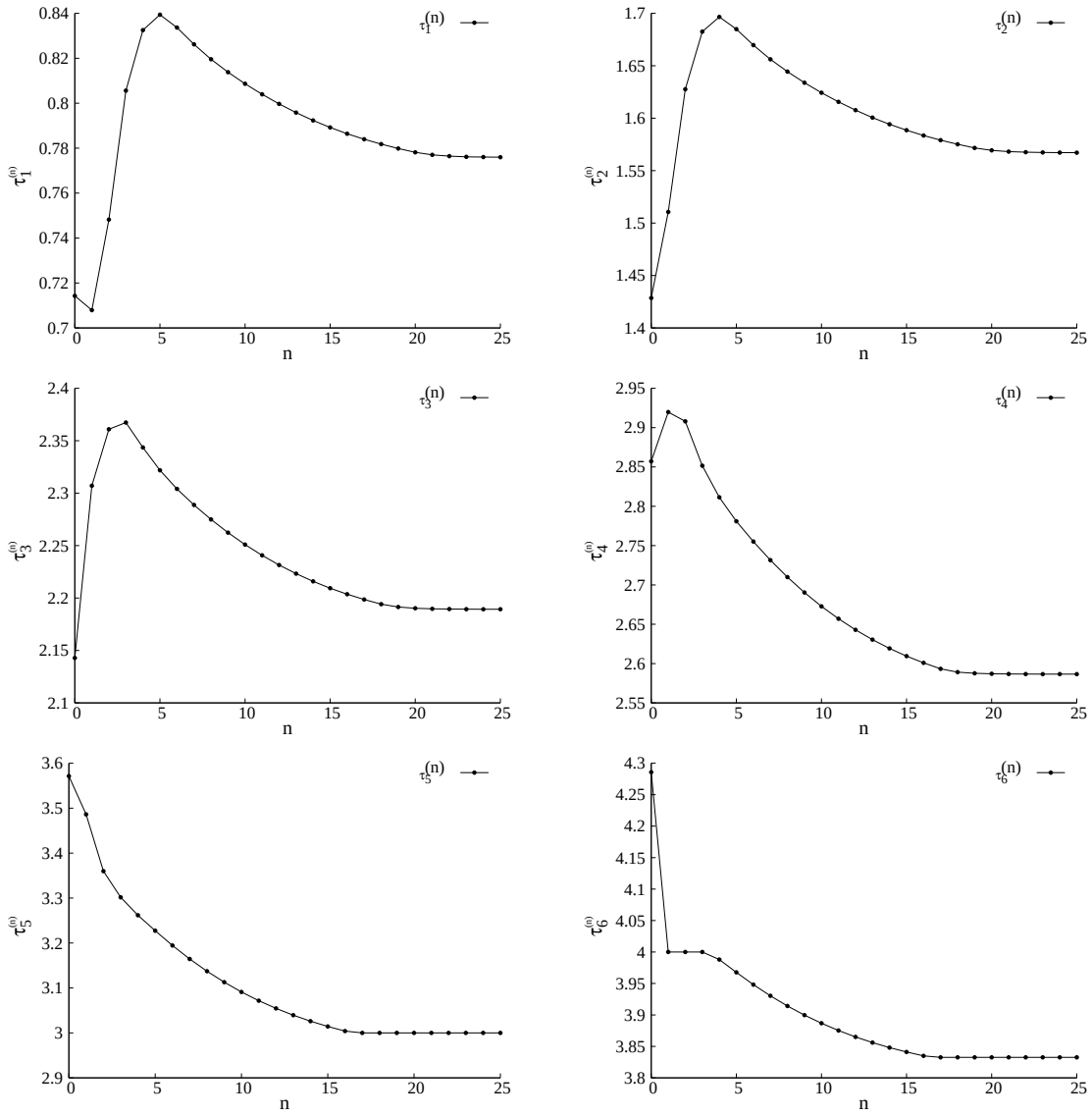

Fig. 20. $\tau_i^{(n)}$, $i = 1, \ldots, 6$ versus n for $\alpha_{19}(t)$

As in Chapter III, the convergence of our improvement algorithm raises the question whether or not it converges to the optimum schedule. Although we are not able to prove the optimality analytically, we investigate the graph of $\mathbb{E}[D]$ versus (τ_1, τ_2) for various hazard rate functions when $k^* = 2$. For instance, for $\alpha_{19}(\cdot)$, if $\gamma = 0.1$, $N_{max} = 30$, $\epsilon = 0.0001$, our algorithm converges to $\tau_1 = 2.192$, $\tau_2 = 3.079$ with $\mathbb{E}[D] = 0.272$. Figure 21 shows that this is the true optimum. More examples

can be found in Appendix G.

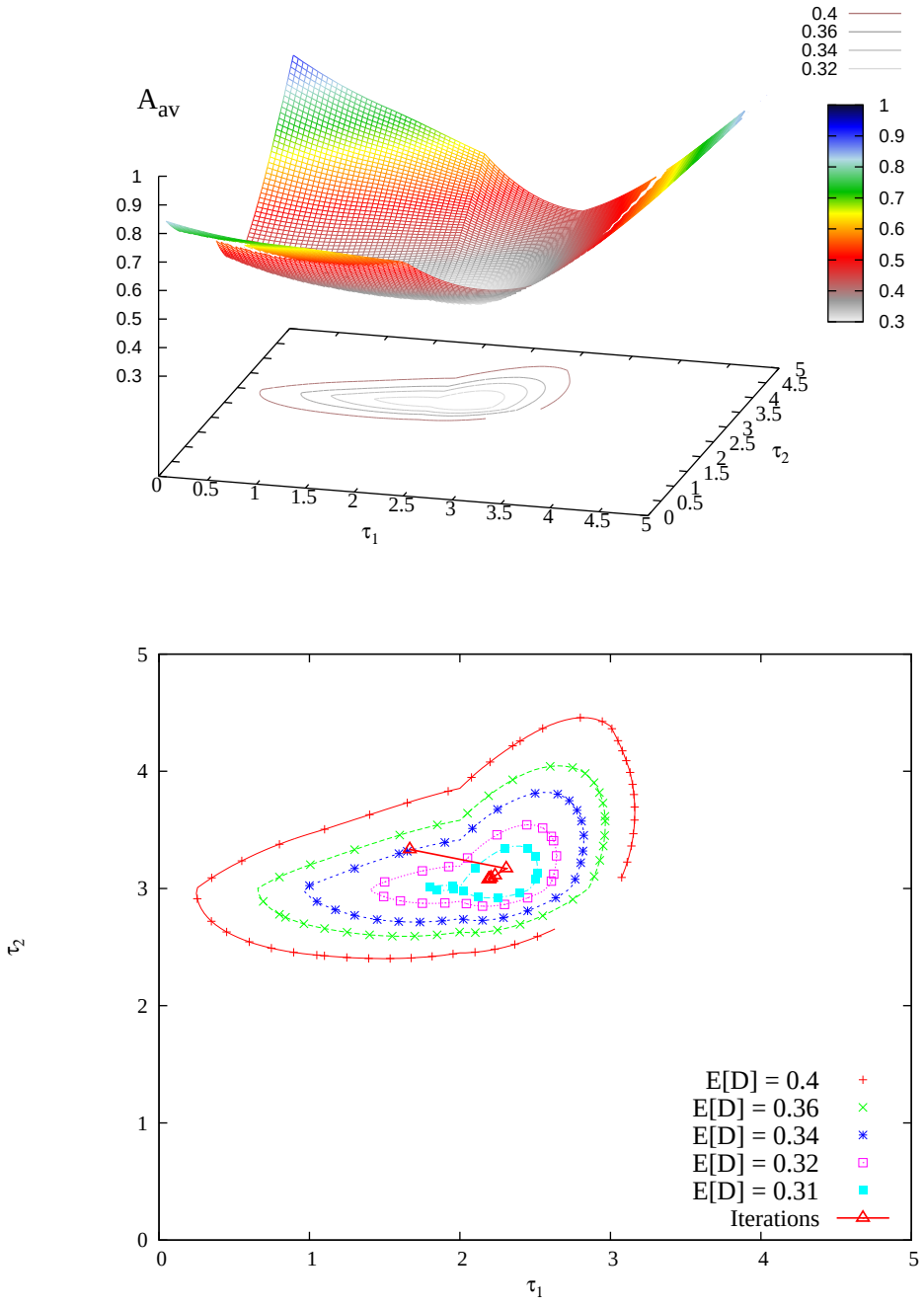

Fig. 21. $\mathbb{E}[D]$ versus (τ_1, τ_2) and iterations of the improvement algorithm for $\alpha_{19}(\cdot)$

Since periodic inspections are widely used for screening inspections, we would like to compare our improved inspections with periodic inspections. Again, for the hazard rate $\alpha_{19}(\cdot)$, if improved inspections are constructed using $N_{max} = 30$, $\epsilon = 0.0001$ and $\gamma = 0.1$, the expected delay for improved inspections and periodic inspections are presented in Table IV. More examples can be found in Appendix H.

Table IV. Comparison of periodic and improved inspections for $\alpha_{15}(\cdot)$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	0.463	0.432	-6.72%
2	0.341	0.3	-12%
3	0.257	0.233	-9.54%
4	0.21	0.184	-12.2%
5	0.175	0.154	-11.8%
6	0.151	0.131	-13.1%
7	0.132	0.115	-13.3%
8	0.117	0.102	-13.3%
10	0.096	0.0843	-12.1%
20	0.0503	0.0444	-11.9%

CHAPTER V

QUANTIFYING THE VALUE OF INFORMATION FOR THE MIXTURE OF POPULATIONS

In Chapter IV, we were able to develop effective inspection schedules for non-replaceable systems with infinite pre-clinical duration. One of the critical assumptions in Chapter IV was that the population at risk is homogeneous so that knowing the hazard rate function is enough to describe the likehood of failure for each member of the population at risk. However, it is possible to have a non-homogeneous population such that each sub-population shows different susceptibility to the failure. For instance, medical evidence suggests that carrying BRCA1 and BRCA2 genes (or their mutated versions) increases the chance of developing breast cancer, or having a certain family background such as multiple occurrence of a certain disease among other family members can change susceptibility to the diseases.

Therefore, when inspection schedules are planned according to aggregate population information, the benefits of screening inspections may not be fully utilized by each sub-group because of their difference from the rest of populations. For example, the mixture of populations whose sub-populations have increasing hazard rate functions does not necessarily have an increasing hazard rate function. A good schedule for a non-monotone hazard rate function is not necessary a good one for increasing hazard rate functions. This observation raises the question of how we can quantify the value of information that identifies a particular sub-class of the population in terms of benefits to individual populations and the whole population. In this chapter, we provide some numerical insights and measurements when sub-populations can completely be identified.

This chapter is organized as followed. In the first section, assumptions are stated

and performance measurements are defined for the mixture of populations. In the next section, we describe the mixture of populations in terms of its sub-populations and discuss how to approximate its hazard rate function. In the last session, we present a numerical study

1. Assumptions and Notation

Consider a collection of positive random variables $\{L_i, i = 1, \ldots, n_{pop}\}$ such that the probability density function and the hazard rate function for L_i are $f_i(\cdot)$ and $\alpha_i(\cdot)$, respectively. We assume that $\alpha_i(\cdot)$ is a piecewise linear function for each i satisfying the following conditions:

1. for any given t, there exits a_t, b_t, c_t and d_t such that $t \in [c_t, d_t)$ and

$$\alpha_i(s) = a_t + b_t \cdot (s - c_t) \geq 0, \ \forall s \in [c_t, d_t) \tag{5.1}$$

2. There is no $s_1 < s_2$ such that

$$\alpha_i(s) = 0 \ \forall s \in (s_1, s_2) \tag{5.2}$$

L_i represents the lifetime for each sub-population, and the lifetime for the mixture of populations is denoted by L_Λ where Λ is a discrete random variable independent of L_i, $i = 1, \ldots, n_{pop}$ and has the probability density function

$$P\{\Lambda = i\} = p_i, \ i = 1, \ldots, n_{pop} \tag{5.3}$$

$$\sum_{i=1}^{n_{pop}} p_i = 1 \tag{5.4}$$

In this chapter, we consider a non-replaceable system such that its disease-free duration is L_Λ. We assume that the pre-clinical duration is infinite and inspections are scheduled to detect the failure. The setup in Chapter IV (two stage inspection policy,

the fixed number of inspections on a finite interval and etc) is assumed to hold in this chapter as well. We can not apply the results of Chapter IV directly because the hazard rate function for L_Λ is not necessarily a piecewise linear function. But, if we can approximate the hazard rate function of L_Λ by piecewise linear functions, we can construct an effective inspection schedules and use them to evaluate the expected delay for each population. In the next section, a simple approximation approach will be discussed and the schedule constructed. The approximation to the mixture hazard rate function will be denoted as $\{\tau_j(pop)\}_{j=1}^{k^*+1}$, where k^* is the number of additional screening inspections during the screening horizon $[a, b]$ as in Chapter IV.

We would like to answer two questions:

- How does the expected delay change for sub-populations if we know the hazard rate function for each sub-population?
- How can one quantify the value of identifying sub-populations to the whole population?

For the first question, it is obvious that by identifying sub-populations, we can construct better schedules, which provide a lower expected delay for each population. However, depending on how different from each other the sub-populations are and the mixture probability of each group, the improvement may or may not be significant. In order to quantify the value of information, we consider two different expected delays for each sub-population: the expected delay without the information and the expected delay with the information. We use $\mathbb{E}\left[D_i\left(pop\right)\right]$ to denote the expected delay for the i^{th} population if inspections are performed at $\{\tau_j(pop)\}_{j=1}^{k^*+1}$ (i.e., inspections are not customized). If the i^{th} population is identified, then a customized schedule, $\{\tau_j(i)\}_{j=1}^{k^*+1}$, can be constructed using the iterative approach in Chapter IV. Thus, $\mathbb{E}\left[D_i\right]$ will denote the expected delay for the i^{th} sub-population if inspections are

applied at $\{\tau_j(i)\}_{j=1}^{k^*+1}$.

The next question is how the benefit of customized schedules can be measured for the whole population. In order to quantify the value of identification of sub-populations to the whole population, we consider the following measure

$$\sum_{i=1}^{n_{pop}} p_i \cdot \frac{\mathbb{E}\left[D_i\left(pop\right)\right]}{\mathbb{E}\left[D_i\right]} \tag{5.5}$$

If we consider $\frac{\mathbb{E}[D_i(pop)]}{\mathbb{E}[D_i]}$ as the benefit gained by the i^{th} population by using a customized schedule, equation (5.5) can be interpreted as the weighted average of the benefit. We analyze this measure for different mixture rates in Section 2.

1.1. The Hazard Rate for the Mixture of Populations

The following proposition shows how to describe the density of the lifetime and the hazard rate functions for mixtures of populations.

Proposition 1 *For a mixture lifetime random variable L_Λ,*

1. *The density of L_Λ, $f_{mix}(\cdot)$ is*

$$f_{mix}(s) = \sum_{i=1}^{n_{pop}} p_i \cdot f_i(s) \tag{5.6}$$

$$= \sum_{i=1}^{n_{pop}} p_i \cdot \alpha_i(s) \cdot e^{-\int_0^s \alpha_i(u)\, du} \tag{5.7}$$

2. The hazard rate of L_Λ, $\alpha_{mix}(\cdot)$ is

$$\alpha_{mix}(s) = \frac{\sum_{i=1}^{n_{pop}} p_i \cdot f_i(s)}{\int_s^\infty \sum_{i=1}^{n_{pop}} p_i \cdot f_i(u)\, du} \tag{5.8}$$

$$= \frac{\sum_{i=1}^{n_{pop}} p_i \cdot \alpha_i(s) \cdot e^{-\int_0^s \alpha_i(u)\, du}}{\int_s^\infty \sum_{i=1}^{n_{pop}} p_i \cdot \alpha_i(u) \cdot e^{-\int_0^u \alpha_i(v)\, dv}\, du} \tag{5.9}$$

Proof:

1. First, note that we can write $f_i(\cdot)$, $i = 1, \ldots, n_{pop}$ as

$$f_i(s) = \alpha_i(s) \cdot e^{-\int_0^s \alpha_i(v)\, dv} \tag{5.10}$$

So,

$$P\{L_\Lambda \leq s\} = \sum_{i=1}^{n_{(pop)}} p_i \cdot P\{L_i \leq s | \Lambda = i\} \tag{5.11}$$

$$= \sum_{i=1}^{n_{(pop)}} p_i \cdot \int_0^s f_i(s)\, ds \tag{5.12}$$

Equations (5.6) and (5.6) follows by taking the derivative of (5.12).

2. Since $\alpha_{mix}(s) = \frac{f_{mix}(s)}{P\{L_\Lambda > s\}}$, equations (5.8) and (5.9) follows easily follows (5.6), (5.7) and (5.12).

■

Before stating how we can simply approximate to $\alpha_{mix}(\cdot)$, consider the following examples to better understand how the mixture hazard rate may not reflect individual populations. Assume that there are two populations with the following hazard rate

functions,

$$\alpha_1(t) = 0.25 \tag{5.13}$$

$$\alpha_2(t) = 7.5 \cdot t^2 \tag{5.14}$$

Further, we assume that $\alpha_{mix}(\cdot, p)$ represents the mixture hazard rate function when $P\{\Lambda = 1\} = p$ and $P\{\Lambda = 2\} = 1 - p$. Although both populations have a non-decreasing hazard rate function, Figure 22 shows that $\alpha_{mix}(\cdot, p)$ is non-monotone. This situation arises when one of populations has a relatively shorter lifetime than other (i.e., the second population is a lot more susceptible to the failure).

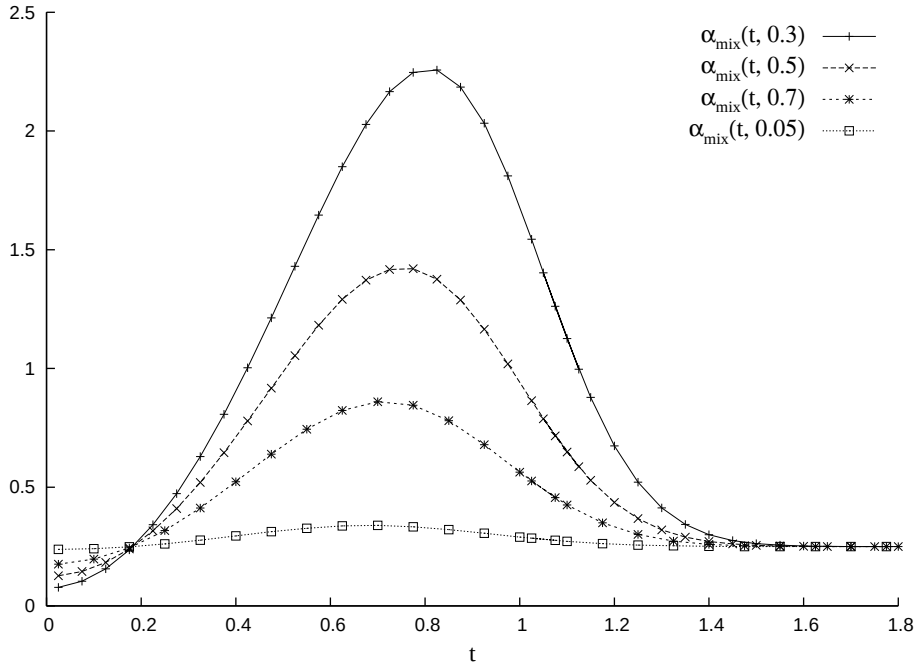

Fig. 22. Hazard rate functions for the mixture of two populations

As mentioned earlier, the lifetime distribution of the mixture of two (or more) populations with piecewise linear hazard rate functions does not necessarily have a piecewise linear hazard rate function. For instance, consider two populations with hazard rate functions $\alpha_1(t) = 0.5$ and $\alpha_2(t) = 0.1$, respectively. Additionally, assume

that the first population forms 90% of the whole population. Figure 23 shows that although hazard rate functions are simply constant, the mixture hazard rate is not even a linear function.

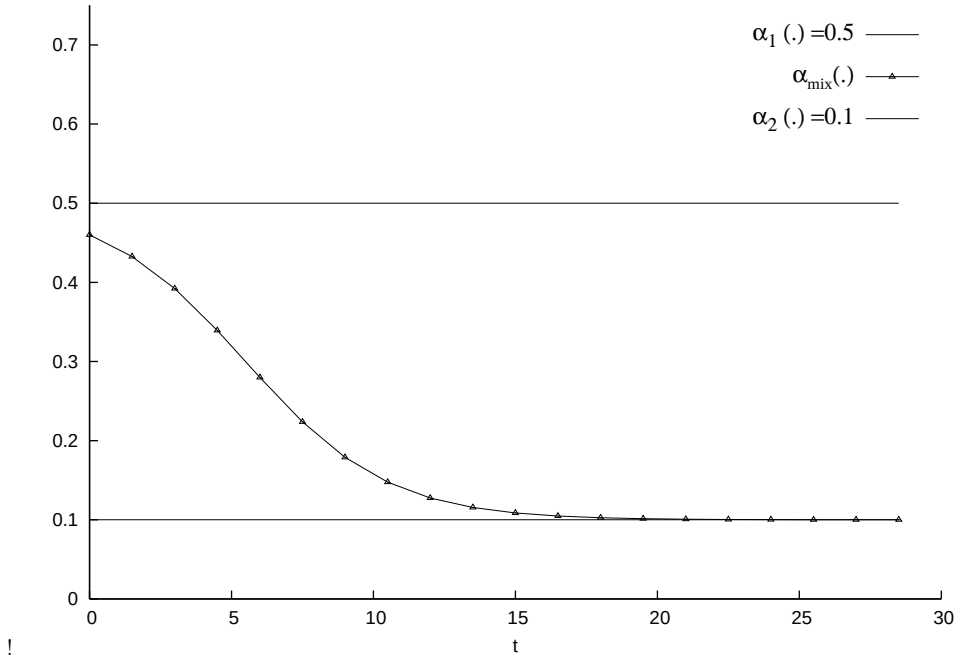

Fig. 23. The hazard rate function for the mixture of two populations with constant hazard rate functions

So, in order to apply our approach in Chapter IV, the mixture hazard rate function should be approximated by a piecewise linear function. There can be many ways to approximate the mixture hazard rate function but in this research, our focus is to describe the mixture hazard rate by a close enough piecewise linear function using a simple approach. We use a simple approach, which is called linear spline. In short, this approach takes a finite number points on an interval, connects those points by linear functions, this piecewise linear function becomes the approximate function on the interval. The accuracy of the approximation can be controlled by changing

the number of intervals. The next definition gives the details of this approach.

Definition 5 *Let $\alpha_{mix}(\cdot)$ be a mixture hazard rate for some L_Λ. On a finite interval $[a, b]$ with knots $x_0 = a < x_1 < \ldots < x_{n-1} < x_n = b$, the approximate hazard rate, $\tilde{\alpha}_{mix}(\cdot)$, is*

$$\tilde{\alpha}_{mix}(s) = \alpha_{mix}(x_{i-1}) + \frac{\alpha_{mix}(x_i) - \alpha_{mix}(x_{i-1})}{x_i - x_{i-1}} \cdot (s - x_{i-1}),\ s \in [x_{i-1}, x_i) \qquad (5.15)$$

for $i = 1, \ldots, n$.

2. Numerical Examples

In this section, we present how each sub-population is affected by customization of inspection schedules and how the whole population is affected by customizations. We consider two cases:

- Case 1: we assume that there are two populations with hazard rate functions $\alpha_1(t) = 0.5$ and $\alpha_2(t) = 0.1$. Inspections are applied between $a = 0$ and $b = 30$.
- Case 2: we assume that there are two populations with hazard rate functions $\alpha_1(t) = 0.05 + 0.005 * t$ and $\alpha_2(t) = 0.05 + 0.0075 * t$. Inspections are applied between $a = 0$ and $b = 30$.

We looked that $\mathbb{E}\left[D_i\left(pop\right)\right]$, $i = 1, 2$ and $\mathbb{E}\left[D_i\right]$, $i = 1, 2$ values for two values $k^* = 6$ and $k^* = 15$, and different values for p. In order to approximate the mixture hazard rate, a linear spline with 5 equal knots is used.

In case 1, two populations are substantially different from each other (i.e., the first population can be considered five times more susceptible to the failure than the second population). As seen in Table V and VI, there is a considerable individual benefit in most mixture percentages. In Table VII, if we look at the whole population benefit, which also takes into the account the rate of subpopulations, the overall benefit is the highest when the population at risk is evenly formed by sub-populations (in other words, the rate of sub-populations in the mixture are close to each other). Even for relatively small mixture rates, there is a slight increase in the benefit due to a considerable difference between characteristics of two populations.

In Case 2, two populations have a same hazard rate at the beginning and populations become different as t gets larger (i.e., their hazard rate is not close each other anymore). However, even at the end of the screening period, they do not differ from each other as two populations in Case 1 do. Therefore, customizing schedules do not reduce the expected delay as much as in Case 1 (see Table VIII and IX) and the benefit of customizing schedules does not provide an substantial benefit for the population at risk (see Table X).

Table V. Case 1: $\mathbb{E}\left[D_i\left(pop\right)\right]$ and $\mathbb{E}\left[D_i\right]$ comparisons for $k^* = 6$

p	$\mathbb{E}[D_1(pop)]$	$\mathbb{E}[D_2(pop)]$	$\mathbb{E}[D_1]$	$\mathbb{E}[D_2]$	$\frac{\mathbb{E}[D_1]-E[D_1(pop)]}{\mathbb{E}[D_1(pop)]}$	$\frac{\mathbb{E}[D_1]-\mathbb{E}[D_1(pop)]}{\mathbb{E}[D_1(pop)]}$
0.05	1.45	1.86	0.749	1.85	-48.3%	-0.174%
0.1	1.37	1.86	0.749	1.85	-45.3%	-0.585%
0.2	1.25	1.89	0.749	1.85	-40.3%	-1.82%
0.3	1.16	1.92	0.749	1.85	-35.5%	-3.62%
0.4	1.08	1.97	0.749	1.85	-30.9%	-5.97%
0.5	1.02	2.03	0.749	1.85	-26.5%	-8.83%
0.6	0.967	2.11	0.749	1.85	-22.6%	-12%
0.7	0.925	2.19	0.749	1.85	-19%	-15.4%
0.8	0.865	2.38	0.749	1.85	-13.5%	-22.1%
0.9	0.806	2.74	0.749	1.85	-7.13%	-32.3%
0.95	0.774	3.14	0.749	1.85	-3.25%	-41%

Table VI. Case 1: $\mathbb{E}\left[D_i\left(pop\right)\right]$ and $\mathbb{E}\left[D_i\right]$ comparisons for $k^* = 15$

p	$\mathbb{E}[D_1(pop)]$	$\mathbb{E}[D_2(pop)]$	$\mathbb{E}[D_1]$	$\mathbb{E}[D_2]$	$\frac{\mathbb{E}[D_1]-E[D_1(pop)]}{\mathbb{E}[D_1(pop)]}$	$\frac{\mathbb{E}[D_1]-\mathbb{E}[D_1(pop)]}{\mathbb{E}[D_1(pop)]}$
0.05	0.671	0.787	0.317	0.791	-52.7%	0.526%
0.1	0.638	0.785	0.317	0.791	-50.3%	0.832%
0.2	0.589	0.783	0.317	0.791	-46.1%	1.03%
0.3	0.55	0.785	0.317	0.791	-42.4%	0.863%
0.4	0.518	0.788	0.317	0.791	-38.7%	0.372%
0.5	0.491	0.794	0.317	0.791	-35.3%	-0.309%
0.6	0.466	0.801	0.317	0.791	-31.9%	-1.26%
0.7	0.441	0.814	0.317	0.791	-28%	-2.79%
0.8	0.414	0.835	0.317	0.791	-23.4%	-5.23%
0.9	0.381	0.886	0.317	0.791	-16.8%	-10.7%
0.95	0.358	0.961	0.317	0.791	-11.5%	-17.7%

Table VII. Case 1: Benefit for the whole population for $k^* = 6$ and $k^* = 15$

$k^* = 6$	
p	$\sum_{i=1}^{2} p_i \cdot \frac{\mathbb{E}[D_i(pop)]}{\mathbb{E}[D_i]}$
0.05	1.0484
0.1	1.0882
0.2	1.1497
0.3	1.1915
0.4	1.2172
0.5	1.2291
0.6	1.2297
0.7	1.2192
0.8	1.1811
0.9	1.1168
0.95	1.0666

$k^* = 15$	
p	$\sum_{i=1}^{2} p_i \cdot \frac{\mathbb{E}[D_i(pop)]}{\mathbb{E}[D_i]}$
0.05	1.0508
0.1	1.0937
0.2	1.1629
0.3	1.2145
0.4	1.2502
0.5	1.2748
0.6	1.2865
0.7	1.281
0.8	1.2559
0.9	1.1935
0.95	1.134

Table VIII. Case 2: $\mathbb{E}\left[D_i\left(pop\right)\right]$ and $\mathbb{E}\left[D_i\right]$ comparisons for $k^* = 6$

p	$\mathbb{E}[D_1(pop)]$	$\mathbb{E}[D_2(pop)]$	$\mathbb{E}[D_1]$	$\mathbb{E}[D_2]$	$\frac{\mathbb{E}[D_1]-E[D_1(pop)]}{\mathbb{E}[D_1(pop)]}$	$\frac{\mathbb{E}[D_1]-\mathbb{E}[D_1(pop)]}{\mathbb{E}[D_1(pop)]}$
0.05	2.02	1.96	2.01	1.96	−0.59%	−0.00207%
0.1	2.02	1.96	2.01	1.96	−0.526%	−0.00769%
0.2	2.01	1.96	2.01	1.96	−0.41%	−0.0294%
0.3	2.01	1.96	2.01	1.96	−0.309%	−0.0648%
0.4	2.01	1.96	2.01	1.96	−0.223%	−0.113%
0.5	2.01	1.96	2.01	1.96	−0.152%	−0.174%
0.6	2.01	1.96	2.01	1.96	−0.095%	−0.247%
0.7	2.01	1.96	2.01	1.96	−0.0521%	−0.33%
0.8	2.01	1.97	2.01	1.96	−0.0224%	−0.424%
0.9	2.01	1.97	2.01	1.96	−0.00525%	−0.527%
0.95	2.01	1.97	2.01	1.96	−0.00118%	−0.583%

Table IX. Case 2: $\mathbb{E}\left[D_i\left(pop\right)\right]$ and $\mathbb{E}\left[D_i\right]$ comparisons for $k^* = 15$

p	$\mathbb{E}[D_1(pop)]$	$\mathbb{E}[D_2(pop)]$	$\mathbb{E}[D_1]$	$\mathbb{E}[D_2]$	$\frac{\mathbb{E}[D_1]-E[D_1(pop)]}{\mathbb{E}[D_1(pop)]}$	$\frac{\mathbb{E}[D_1]-\mathbb{E}[D_1(pop)]}{\mathbb{E}[D_1(pop)]}$
0.05	0.856	0.835	0.859	0.835	0.301%	−0.0621%
0.1	0.856	0.836	0.859	0.835	0.319%	−0.127%
0.2	0.856	0.837	0.859	0.835	0.341%	−0.266%
0.3	0.856	0.838	0.859	0.835	0.344%	−0.414%
0.4	0.856	0.84	0.859	0.835	0.33%	−0.57%
0.5	0.856	0.841	0.859	0.835	0.303%	−0.733%
0.6	0.856	0.842	0.859	0.835	0.262%	−0.901%
0.7	0.857	0.844	0.859	0.835	0.211%	−1.07%
0.8	0.857	0.845	0.859	0.835	0.149%	−1.25%
0.9	0.858	0.847	0.859	0.835	0.0785%	−1.43%
0.95	0.858	0.848	0.859	0.835	0.0402%	−1.53%

Table X. Case 2: Benefit for the whole population for $k^* = 6$ and $k^* = 15$

$k^* = 6$	
p	$\sum_{i=1}^{2} p_i \cdot \frac{\mathbb{E}[D_i(pop)]}{\mathbb{E}[D_i]}$
0.05	1.0003
0.1	1.0006
0.2	1.0011
0.3	1.0014
0.4	1.0016
0.5	1.0016
0.6	1.0016
0.7	1.0014
0.8	1.001
0.9	1.0006
0.95	1.0003

$k^* = 15$	
p	$\sum_{i=1}^{2} p_i \cdot \frac{\mathbb{E}[D_i(pop)]}{\mathbb{E}[D_i]}$
0.05	1.0004
0.1	1.0008
0.2	1.0015
0.3	1.0019
0.4	1.0021
0.5	1.0022
0.6	1.0021
0.7	1.0018
0.8	1.0013
0.9	1.0007
0.95	1.0004

CHAPTER VI

CONCLUSION

In this chapter, we present a summary of the research and the contributions of our results. We also provide a discussion of possible future directions.

1. Summary and Contributions

It is crucial to use available information as effectively as possible while making decisions. In this research, our motivation was to use information such as the knowledge of the hazard rate function when scheduling screening inspections for systems with non-self-announcing failures. Since inspections are resources for such systems, the better allocation of resources (i.e, the better schedules) is of great interest. We looked at two different systems: replaceable (industrial studies) and non-replaceable systems (healthcare studies). Depending on whether or not we are allowed to make a replacement, our performance measures and mathematical models change but in both cases, we use the hazard rate function to design better schedules.

In Chapter III, we consider scheduling inspections in a simple replaceable failure prone system when the hazard rate function for time to failure is known. The availability and the inspection rate are used as our performance measures. For such systems, we derived expressions periodic inspections, intensity based inspections and improved inspections, which are iteratively constructed from periodic inspections. Our results showed that when the inspection rate is kept constant, it is possible to construct improved inspections schedules that a higher availability than periodic inspections. Additionally, our numerical result showed that for lower inspection rates, the difference between improved inspections and periodic inspections increases. As opposed to earlier studies, which consider inspections as a limited resource, our ap-

proach provide an *analytical* assessment tool when the hazard rate function is not described by monotone functions.

For non-replaceable systems in Chapter IV, we considered a general natural history model with two stage inspection policy in which fallible inspections are performed first and error-free inspections are performed only if a fallible inspection report the failure. For a two-stage inspection policy, we derived the expression for the expected detection delay when pre-clinical and disease-free durations are independent. Later, our result showed an iterative approach can be used to reduce the expected detection delay without changing the number of inspection performed during the screening horizon when pre-clinical duration is infinite. Though there are some earlier studies which focuses on scheduling inspections for non-replaceable systems with non-self-announcing failures, most of them are restricted to the special class of hazard rate functions. In this research, our results holds for piecewise linear hazard rate functions, which can be used to approximate more general hazard rate functions than ones in earlier studies. To our best knowledge, this study is the first study which studied a more general class of hazard rate functions and developed analytical proven improved two stage inspection schedules using a fixed number of inspections.

When screening inspections are applied to a mixture of populations, the benefit of screening may not be fully utilized by each sub-populations. For instance, people with certain family background or certain types of genes may be more or less likely to have a certain disease than the rest of population. In such case, customizing schedules for these people might be really beneficial. In Chapter V, we developed measures to quantify the value of benefits of customized schedules to individuals and the whole population at risk. Numerical studies showed that customization can be really beneficial when there are minorities which has a lot more susceptible to the failure than the rest of the population at risk.

2. Directions for Future Research

For replaceable systems, we developed inspection schedules that will always give higher availability than periodic inspections under the constant inspection rate. In this case, our assumption about $\alpha(t)$ is that it is a periodic piecewise constant function. Although this assumptions simplified our analytical solution, it is restrictive in the sense that it cannot capture continuous changes in $\alpha(t)$ and, more importantly, it only considers deterministic failure rates. To address these issues, we plan to consider following cases in which

1. $\alpha(t)$ is a periodic piecewise non-constant function
2. $\alpha(t)$ is described by a continuous time Markov chain.

So far, we assumed that $\alpha(t)$ is a periodic deterministic function. In this case, both the time spent in the each failure rate state and the change of states of failure rates are deterministic. However, it is reasonable to expect that there will be uncertainty in both how failure rate states changes and how much time spent in each one of them. So, in order to allow random failure rate state times and randomly changing failure rate states, we plan to allow $\{\alpha(t) : t \geq 0\}$ to be described by a stochastic process, particularly, a continuous Markov chain.

Our results in Chapter IV are based on the fact that pre-clinical duration is infinite. Although we derived an expression for the expected delay when pre-clinical duration is finite and independent from the disease-free duration, we did not analyze how to construct better schedules for this case. In the medical literature, there is ongoing work focused on the estimation of pre-clinical duration for certain types of screenings. Incorporating the finite pre-clinical duration to our model will be valuable to assess cases studies for medical screening problems. We plan to use

certain classes of distribution families (such as exponential and uniform families) as the pre-clinical duration and build better schedules for them. The ultimate goal is to develop screening schedules which also consider the dependency between disease-free and pre-clinical durations.

We quantified the value of customizing schedules for individual and the whole population at risk in Chapter V. We assumed that it is possible to correctly separate the population at risk into sub-populations using some identification tests. However, these tests may not be error-free and, so certain follow-up tests may need to be applied for the correct identification. We plan to make a cost-benefit analysis of fallible identification tests to better understand the value of customizing schedules.

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APPENDIX A

COMPARISON OF DIFFERENT INSPECTION SCHEMES FOR REPLACEABLE SYSTEMS

In this appendix, how inspection points are moved by our improvement algorithm in Chapter III are presented. We compare periodic inspection points and our improved inspection points for simple failure prone replaceable systems. Throughout this part, we use $N_{max} = 30$ and $\epsilon = 0.0001$.

$$\alpha^{(5)}(t) = \begin{cases} 0.2, & t \in \bigcup_{n=0}^{\infty} [15n,\, 15n+5) \\ 0.4, & t \in \bigcup_{n=0}^{\infty} [15n+5,\, 15n+10) \\ 0.6, & t \in \bigcup_{n=0}^{\infty} [15n+10,\, 15\cdot(n+1)) \end{cases}$$

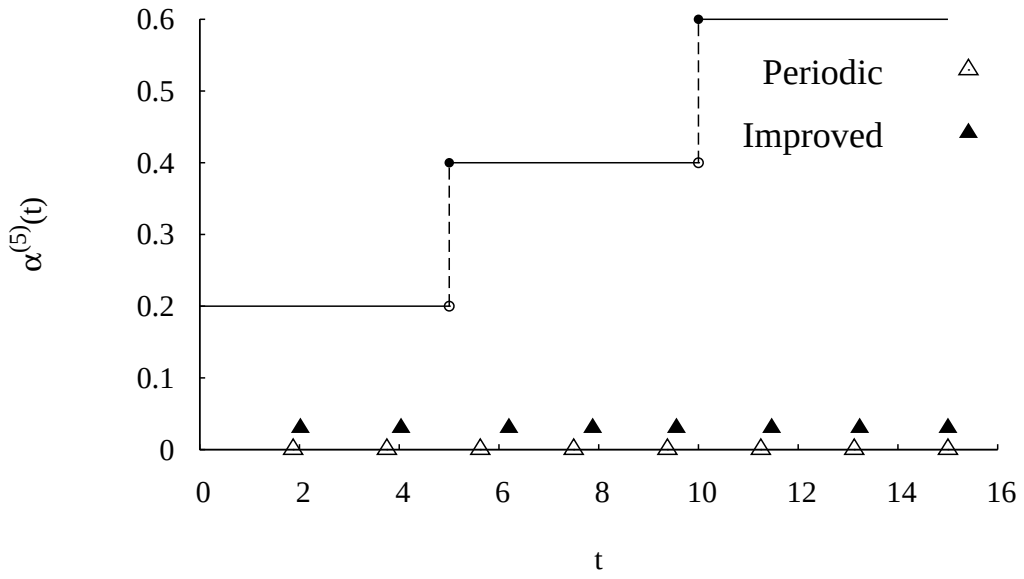

Fig. 24. Improved inspections and periodic inspections for $\alpha^{(5)}(\cdot)$ when $\tau = 1.875$

Table XI. The improved inspection and periodic inspection schedules for $\alpha^{(5)}(\cdot)$ when $\tau = 1.875$

i	$PI(\tau)$	$\tilde{\tau}_i$	$\Delta\tilde{\tau}_i$
1	1.875	2.019	2.019
2	3.75	4.038	2.019
3	5.625	6.199	2.161
4	7.5	7.879	1.68
5	9.375	9.559	1.68
6	11.25	11.47	1.912
7	13.12	13.24	1.765
8	15	15	1.765

$$\alpha^{(6)}(t) = \begin{cases} 0.6, & t \in \bigcup_{n=0}^{\infty} [15n,\, 15n+5) \\ 0.4, & t \in \bigcup_{n=0}^{\infty} [15n+5,\, 15n+10) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [15n+10,\, 15\cdot(n+1)) \end{cases}$$

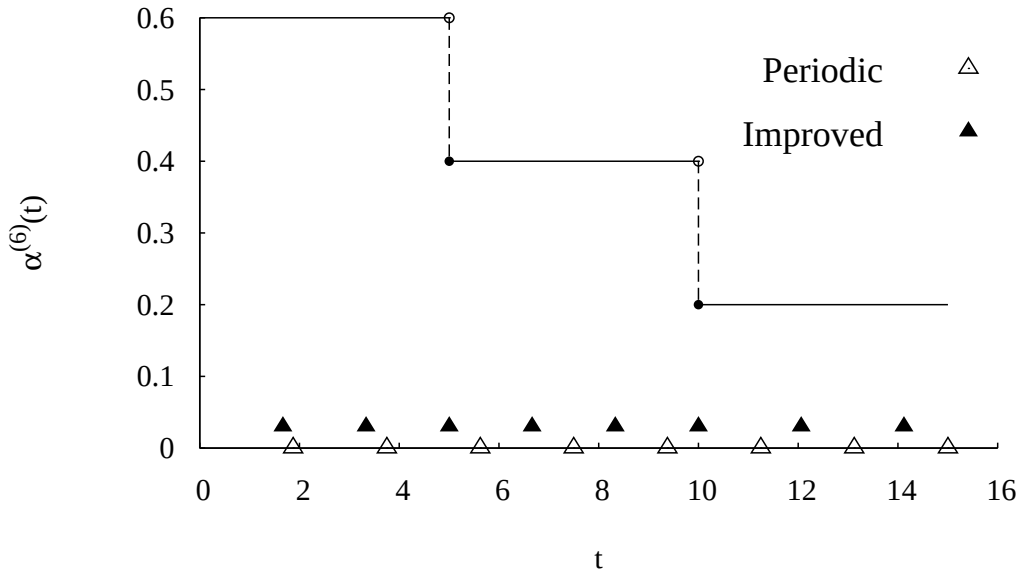

Fig. 25. Improved inspections and periodic inspections for $\alpha^{(6)}(\cdot)$ when $\tau = 1.875$

Table XII. The improved inspection and periodic inspection schedules for $\alpha^{(6)}(\cdot)$ when $\tau = 1.875$

i	$PI(\tau)$	$\tilde{\tau}_i$	$\Delta\tilde{\tau}_i$
1	1.875	1.667	1.667
2	3.75	3.333	1.667
3	5.625	5	1.667
4	7.5	6.667	1.667
5	9.375	8.333	1.667
6	11.25	10	1.667
7	13.12	12.06	2.061
8	15	14.12	2.061

$$\alpha^{(7)}(t) = \begin{cases} 0.2, & t \in \bigcup_{n=0}^{\infty} [15n,\ 15n+1) \\ 0.3, & t \in \bigcup_{n=0}^{\infty} [15n+1,\ 15n+2) \\ 0.4, & t \in \bigcup_{n=0}^{\infty} [15n+2,\ 15n+3) \\ 0.5, & t \in \bigcup_{n=0}^{\infty} [15n+3,\ 15n+6) \\ 0.4, & t \in \bigcup_{n=0}^{\infty} [15n+6,\ 15n+7) \\ 0.3, & t \in \bigcup_{n=0}^{\infty} [15n+7,\ 15n+8) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [15n+8,\ 15n+9) \\ 0.1, & t \in \bigcup_{n=0}^{\infty} [15n+9,\ 15n+12) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [15n+12,\ 15n+13) \\ 0.3, & t \in \bigcup_{n=0}^{\infty} [15n+13,\ 15n+14) \\ 0.4, & t \in \bigcup_{n=0}^{\infty} [15n+14,\ 15\cdot(n+1)) \end{cases}$$

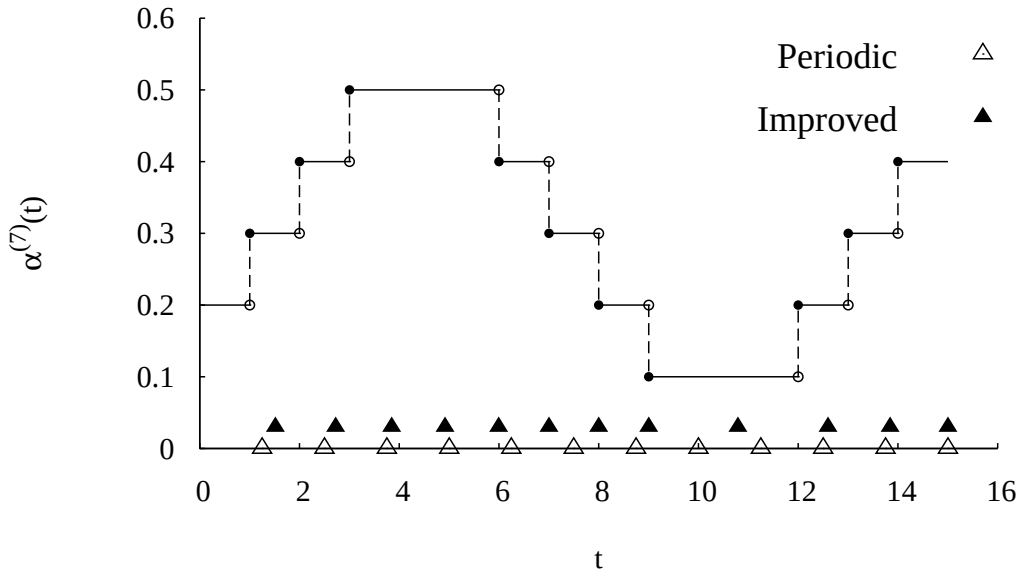

Fig. 26. Improved inspections and periodic inspections for $\alpha^{(7)}(\cdot)$ when $\tau = 1.25$

Table XIII. The improved inspection and periodic inspection schedules for $\alpha^{(7)}(\cdot)$ when $\tau = 1.25$

i	$PI(\tau)$	$\tilde{\tau}_i$	$\Delta\tilde{\tau}_i$
1	1.25	1.515	1.515
2	2.5	2.723	1.209
3	3.75	3.85	1.126
4	5	4.921	1.071
5	6.25	5.991	1.071
6	7.5	7	1.009
7	8.75	8	1
8	10	9	1
9	11.25	10.79	1.791
10	12.5	12.6	1.809
11	13.75	13.84	1.24
12	15	15	1.16

$$\alpha^{(8)}(t) = \begin{cases} 0.4, & t \in \bigcup_{n=0}^{\infty} [15n,\ 15n+1) \\ 0.3, & t \in \bigcup_{n=0}^{\infty} [15n+1,\ 15n+2) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [15n+2,\ 15n+3) \\ 0.1, & t \in \bigcup_{n=0}^{\infty} [15n+3,\ 15n+6) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [15n+6,\ 15n+7) \\ 0.3, & t \in \bigcup_{n=0}^{\infty} [15n+7,\ 15n+8) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [15n+8,\ 15n+9) \\ 0.1, & t \in \bigcup_{n=0}^{\infty} [15n+9,\ 15n+12) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [15n+12,\ 15n+13) \\ 0.3, & t \in \bigcup_{n=0}^{\infty} [15n+13,\ 15n+14) \\ 0.4, & t \in \bigcup_{n=0}^{\infty} [15n+14,\ 15\cdot(n+1)) \end{cases}$$

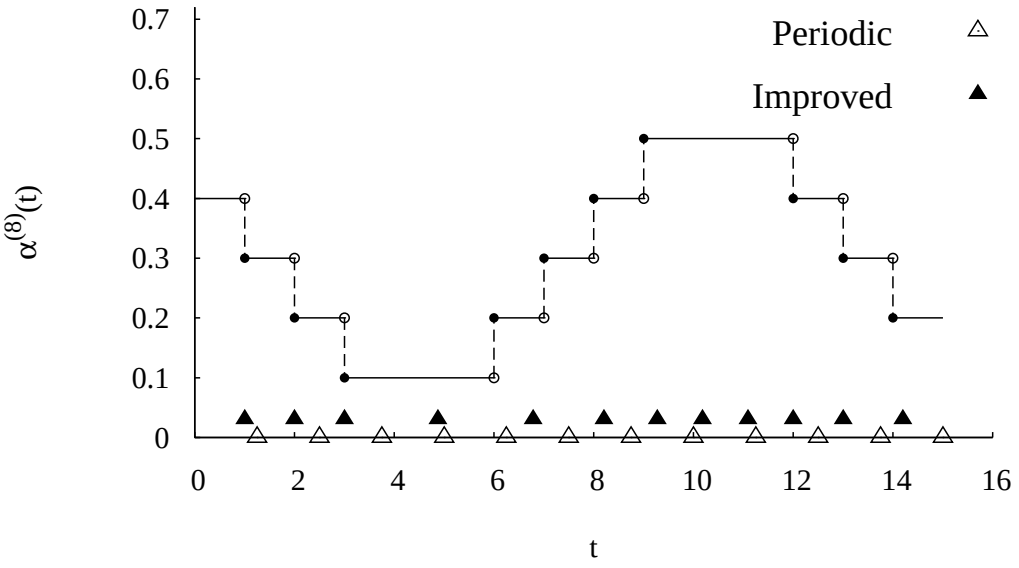

Fig. 27. Improved inspections and periodic inspections for $\alpha^{(8)}(\cdot)$ when $\tau = 1.25$

Table XIV. The improved inspection and periodic inspection schedules for $\alpha^{(8)}(\cdot)$ when $\tau = 1.25$

i	$PI(\tau)$	$\tilde{\tau}_i$	$\Delta\tilde{\tau}_i$
1	1.25	1	1
2	2.5	2	1
3	3.75	3	1
4	5	4.876	1.876
5	6.25	6.783	1.907
6	7.5	8.206	1.422
7	8.75	9.273	1.068
8	10	10.18	0.9089
9	11.25	11.09	0.9088
10	12.5	12	0.9088
11	13.75	13	1
12	15	14.2	1.2

APPENDIX B

NUMERICAL RESULTS FOR PERFORMANCE OF DIFFERENT INSPECTIONS STRATEGIES FOR REPLACEABLE SYSTEMS

The availability for periodic inspections and the improved inspections are presented in this appendix. $N_{max} = 50$ and $\epsilon = 0.0001$ are used for the improved inspections.

$$\alpha^{(9)}(t) = \begin{cases} 0.1, & t \in \bigcup_{n=0}^{\infty} [6n,\, 6n + 1.5) \\ 0.15, & t \in \bigcup_{n=0}^{\infty} [6n + 1.5,\, 6n + 3) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [6n + 3,\, 6n + 4.5) \\ 0.25, & t \in \bigcup_{n=0}^{\infty} [6n + 4.5,\, 6n + 6) \end{cases}$$

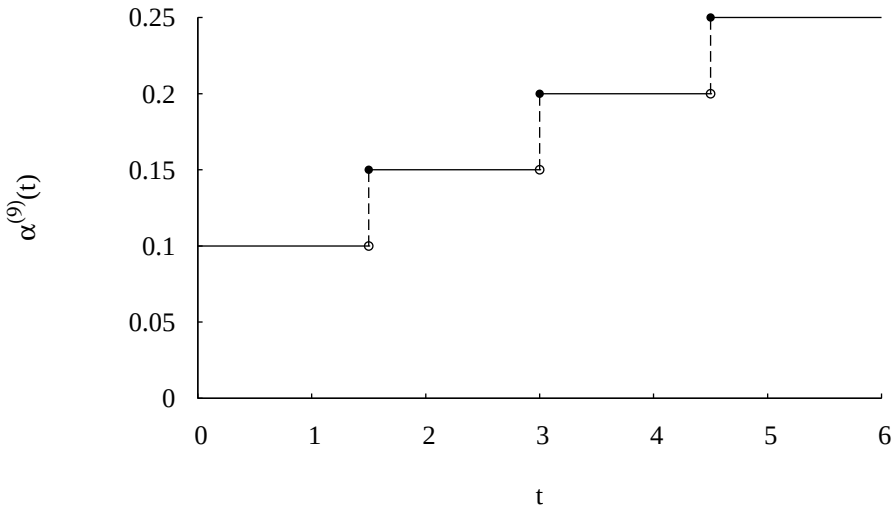

Fig. 28. The intensity function $\alpha^{(9)}(t)$ versus t where $\tau = 1$ and $k_\tau^* = 6$

Table XV. The availability for $PI(\tau)$ and the improved inspections when $\alpha^{(9)}(\cdot)$, $N_{max} = 50$ and $\epsilon = 0.0001$

τ	I_r	$A_{av}(PI(\tau))$	$\tilde{A}_{av}$	$\frac{\tilde{A}_{av} - A_{av}(PI(\tau))}{A_{av}(PI(\tau))}$
0.2	5	0.983	0.983	0.019%
0.4	2.5	0.966	0.967	0.073%
0.6	1.67	0.95	0.951	0.11%
0.8	1.25	0.934	0.936	0.24%
1	1	0.92	0.922	0.23%
1.2	0.8	0.899	0.903	0.47%
1.8	0.571	0.862	0.87	0.86%
2	0.5	0.854	0.856	0.24%
2.5	0.4	0.811	0.823	1.5%
3	0.333	0.795	0.801	0.73%
3.5	0.286	0.749	0.766	2.3%
4	0.25	0.729	0.74	1.6%
5	0.2	0.669	0.704	5.1%
7.5	0.133	0.558	0.591	5.9%
10	0.1	0.478	0.503	5.1%
15	0.0667	0.361	0.384	6.1%

Table XVI. The availability for $PI(\tau)$ and the improved inspections when $\alpha^{(10)}(\cdot)$, $N_{max} = 50$ and $\epsilon = 0.0001$

τ	I_r	$A_{av}(PI(\tau))$	$\tilde{A}_{av}$	$\frac{\tilde{A}_{av} - A_{av}(PI(\tau))}{A_{av}(PI(\tau))}$
0.2	5	0.983	0.983	0.033%
0.4	2.5	0.965	0.967	0.11%
0.6	1.67	0.949	0.951	0.21%
0.8	1.25	0.933	0.936	0.28%
1	1	0.916	0.92	0.49%
1.2	0.8	0.899	0.902	0.36%
1.8	0.571	0.862	0.868	0.69%
2	0.5	0.836	0.853	1.9%
2.5	0.4	0.811	0.819	0.94%
3	0.333	0.766	0.792	3.3%
3.5	0.286	0.749	0.761	1.6%
4	0.25	0.714	0.737	3.2%
5	0.2	0.667	0.687	3.1%
7.5	0.133	0.558	0.574	2.9%
10	0.1	0.469	0.489	4.3%
15	0.0667	0.348	0.369	6.1%

$$\alpha^{(10)}(t) = \begin{cases} 0.25, & t \in \bigcup_{n=0}^{\infty} [6n,\, 6n+1.5) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [6n+1.5,\, 6n+3) \\ 0.15, & t \in \bigcup_{n=0}^{\infty} [6n+3,\, 6n+4.5) \\ 0.1, & t \in \bigcup_{n=0}^{\infty} [6n+4.5,\, 6n+6) \end{cases}$$

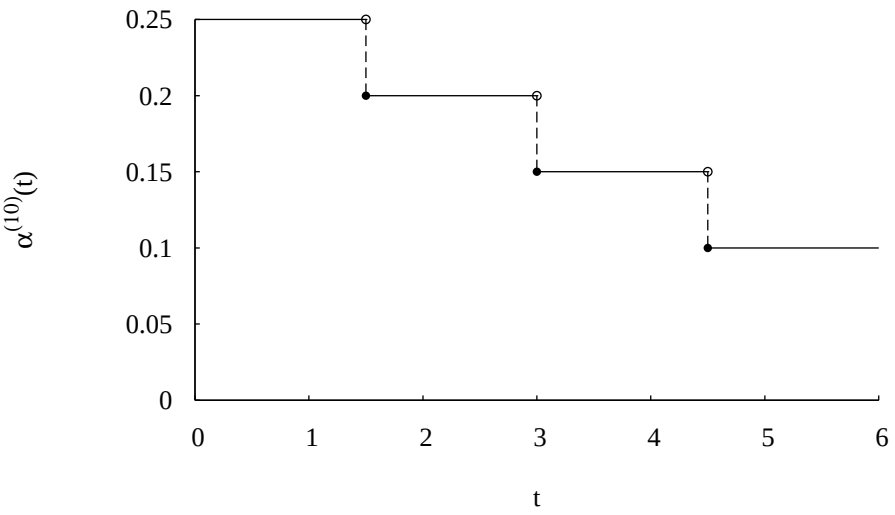

Fig. 29. The intensity function $\alpha^{(10)}(t)$ versus t where $\tau = 1$ and $k_\tau^* = 6$

$$\alpha^{(11)}(t) = \begin{cases} 0.1, & t \in \bigcup_{n=0}^{\infty} [3n,\, 3n+0.5) \\ 0.15, & t \in \bigcup_{n=0}^{\infty} [3n+0.5,\, 3n+1.0) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [3n+1.0,\, 3n+2.0) \\ 0.15, & t \in \bigcup_{n=0}^{\infty} [3n+2.0,\, 3n+2.5) \\ 0.1, & t \in \bigcup_{n=0}^{\infty} [3n+2.5,\, 3n+3) \end{cases}$$

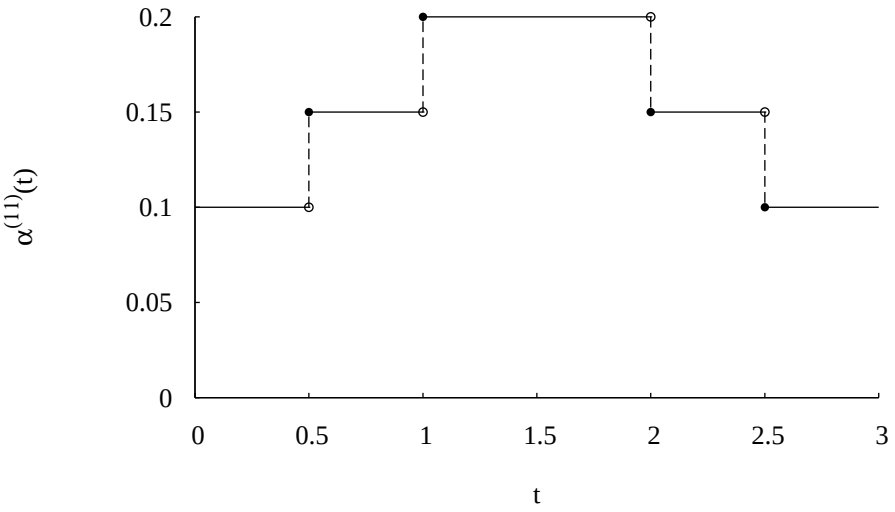

Fig. 30. The intensity function $\alpha^{(11)}(t)$ versus t where $\tau = 0.5$ and $k_\tau^* = 6$

Table XVII. The availability for $PI(\tau)$ and the improved inspections when $\alpha^{(11)}(\cdot)$, $N_{max} = 50$ and $\epsilon = 0.0001$

τ	I_r	$A_{av}(PI(\tau))$	$\tilde{A}_{av}$	$\frac{\tilde{A}_{av} - A_{av}(PI(\tau))}{A_{av}(PI(\tau))}$
0.1	10	0.993	0.993	0.012%
0.2	5	0.985	0.985	0.033%
0.4	2.5	0.971	0.971	0.075%
0.5	2	0.963	0.965	0.13%
0.75	1.33	0.946	0.947	0.12%
1	1	0.929	0.931	0.22%
1.2	0.8	0.912	0.915	0.28%
1.5	0.667	0.896	0.9	0.49%
1.7	0.575	0.88	0.884	0.45%
2	0.5	0.864	0.87	0.66%
2.2	0.444	0.849	0.856	0.84%
2.5	0.4	0.834	0.845	1.2%
3	0.333	0.807	0.826	2.4%
4	0.25	0.752	0.76	0.99%
6	0.167	0.661	0.677	2.4%
9	0.111	0.55	0.563	2.4%
12	0.0833	0.465	0.476	2.4%

$$\alpha^{(12)}(t) = \begin{cases} 0.2, & t \in \bigcup_{n=0}^{\infty} [3n,\ 3n+0.5) \\ 0.15, & t \in \bigcup_{n=0}^{\infty} [3n+0.5,\ 3n+1.0) \\ 0.1, & t \in \bigcup_{n=0}^{\infty} [3n+1.0,\ 3n+2.0) \\ 0.15, & t \in \bigcup_{n=0}^{\infty} [3n+2.0,\ 3n+2.5) \\ 0.2, & t \in \bigcup_{n=0}^{\infty} [3n+2.5,\ 3n+3) \end{cases}$$

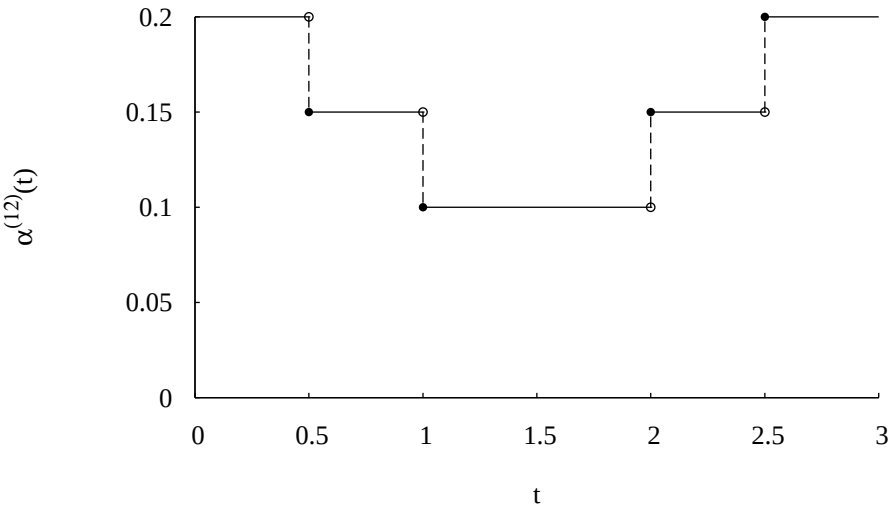

Fig. 31. The intensity function $\alpha^{(12)}(t)$ versus t where $\tau = 1$ and $k_\tau^* = 6$

Table XVIII. The availability for $PI(\tau)$ and the improved inspections when $\alpha^{(12)}(\cdot)$, $N_{max} = 50$ and $\epsilon = 0.0001$

τ	I_r	$A_{av}(PI(\tau))$	$\tilde{A}_{av}$	$\frac{\tilde{A}_{av} - A_{av}(PI(\tau))}{A_{av}(PI(\tau))}$
0.1	10	0.993	0.993	0.011%
0.2	5	0.985	0.985	0.031%
0.4	2.5	0.971	0.971	0.068%
0.5	2	0.963	0.965	0.12%
0.75	1.33	0.946	0.947	0.14%
1	1	0.929	0.93	0.13%
1.2	0.8	0.912	0.915	0.28%
1.5	0.667	0.896	0.901	0.6%
1.7	0.575	0.88	0.884	0.45%
2	0.5	0.864	0.869	0.54%
2.2	0.444	0.849	0.856	0.82%
2.5	0.4	0.834	0.844	1.2%
3	0.333	0.804	0.826	2.8%
4	0.25	0.752	0.76	0.99%
6	0.167	0.658	0.677	2.8%
9	0.111	0.548	0.563	2.8%
12	0.0833	0.463	0.476	2.8%

APPENDIX C

NUMERICAL RESULTS FOR CONVERGENCE OF IMPROVEMENT ALGORITHM FOR REPLACEABLE SYSTEMS

This appendix provides more numerical results for the convergence of $\tilde{\tau}_i^{(n)}, i = 1, \ldots, k_\tau^*$ for improved inspection schedules in Chapter III. The same intensity functions in Appendix B are used ($\alpha^{(9)}(t)$, $\alpha^{(10)}(t)$, $\alpha^{(11)}(t)$ and $\alpha^{(12)}(t)$). We use $N_{max} = 30$ and $\epsilon = 0.0001$, τ and k_τ^* will be stated for each intensity function.

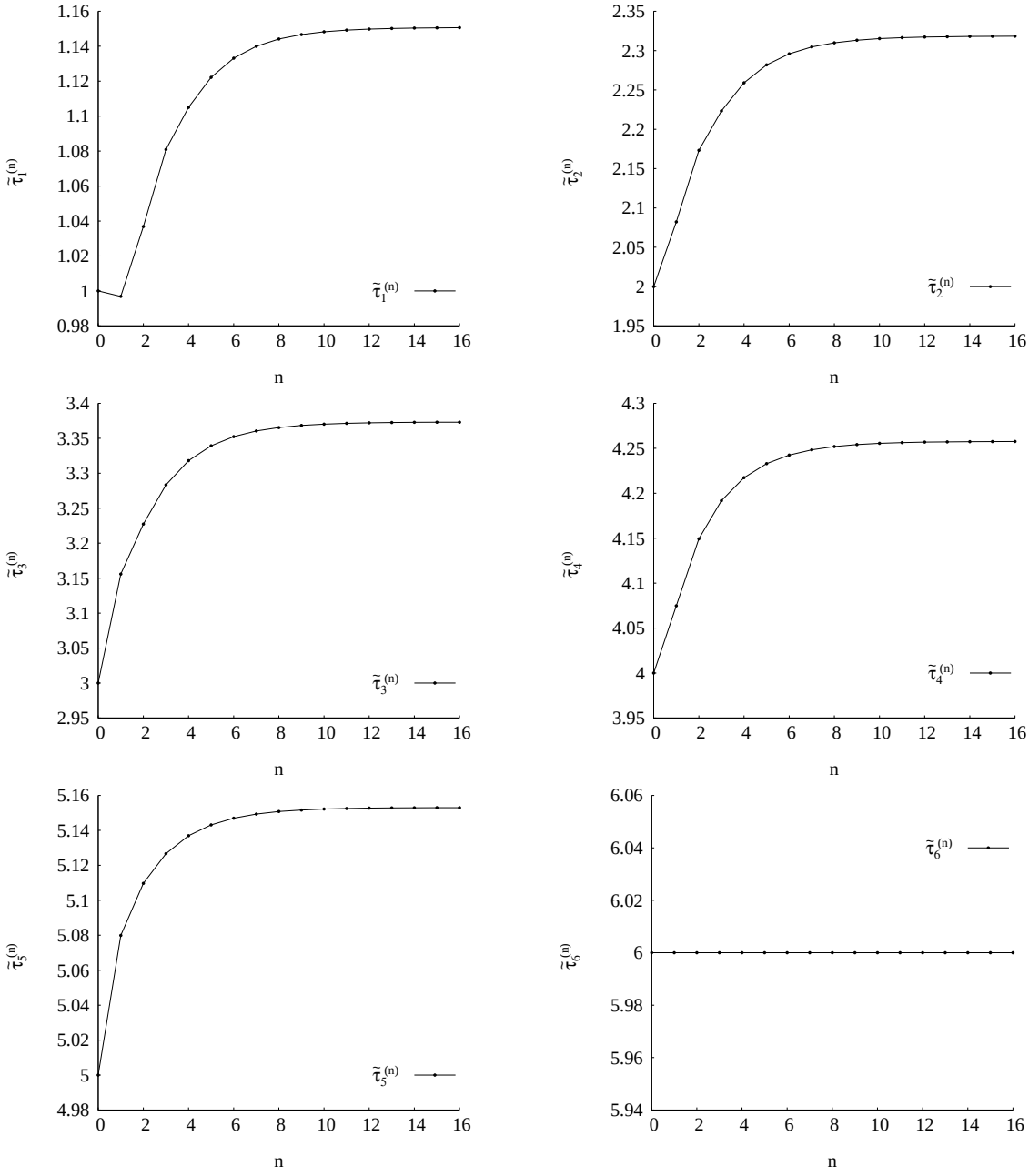

Fig. 32. $\tilde{\tau}_i^{(n)}, i = 1, \ldots, 6$ for $\alpha^{(9)}(\cdot)$ versus n

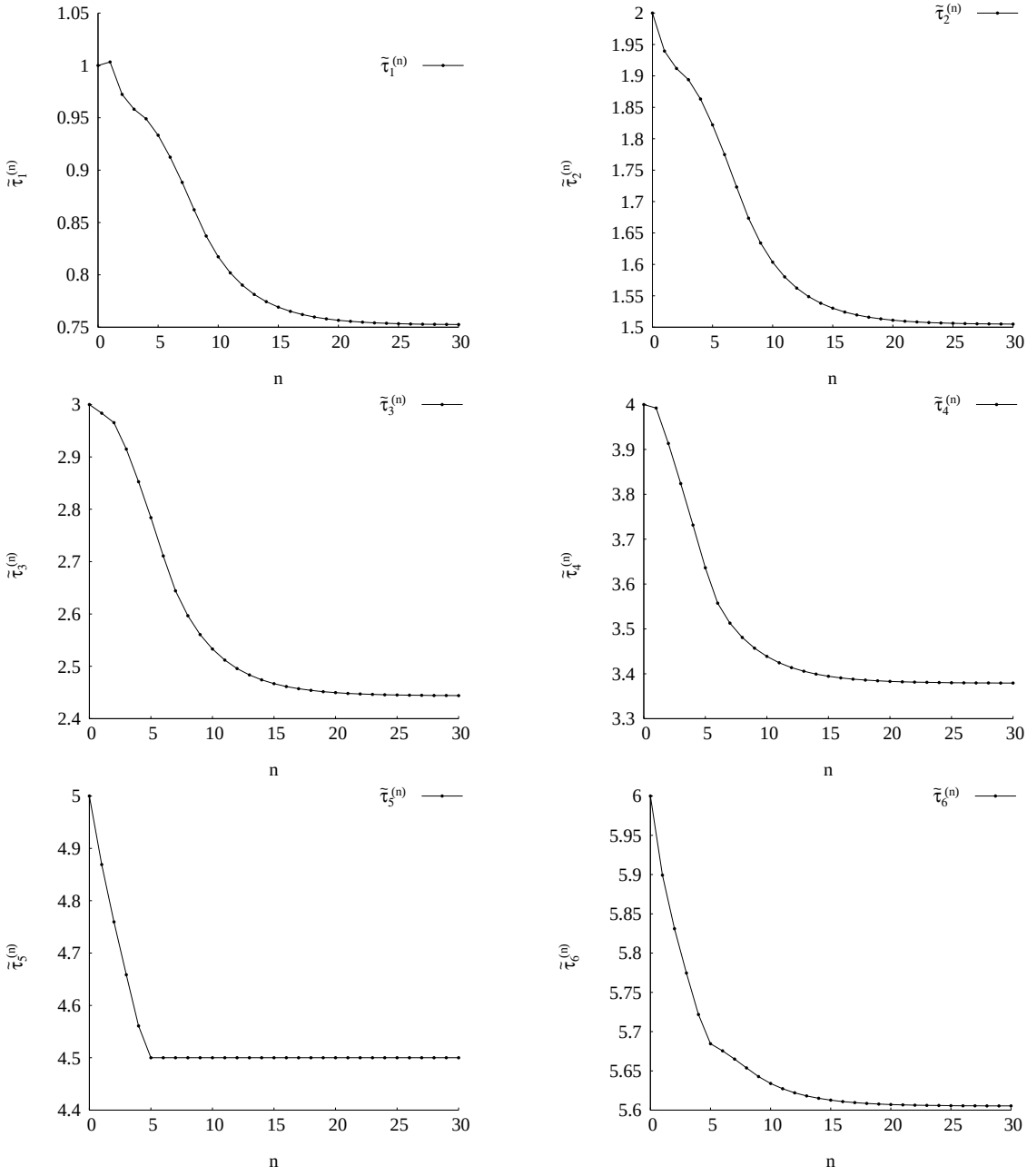

Fig. 33. $\tilde{\tau}_i^{(n)}, i = 1, \ldots, 6$ for $\alpha^{(10)}(\cdot)$ versus n

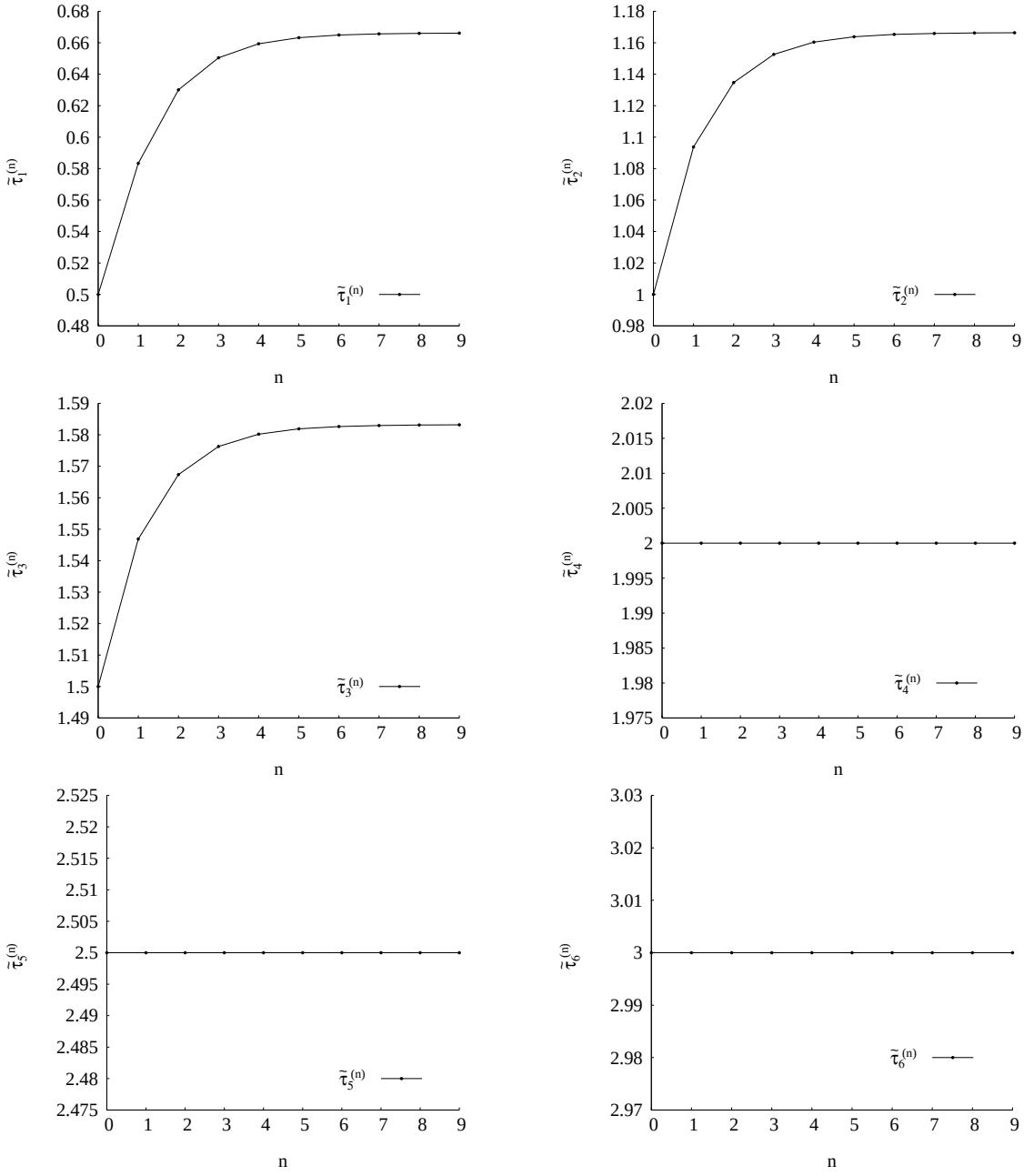

Fig. 34. $\tilde{\tau}_i^{(n)}, i = 1, \ldots, 6$ for $\alpha^{(11)}(\cdot)$ versus n

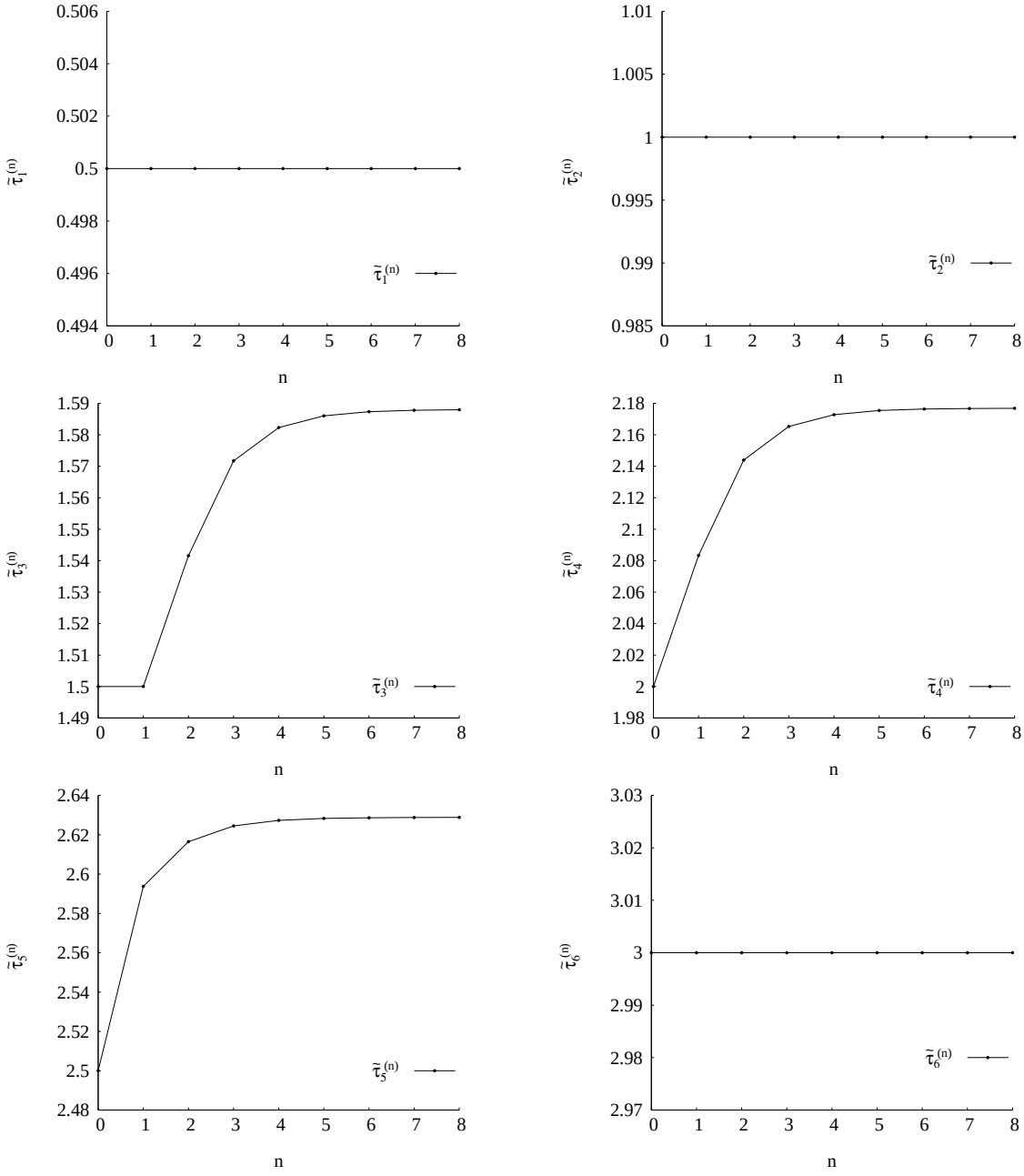

Fig. 35. $\tilde{\tau}_i^{(n)}, i = 1, \ldots, 6$ for $\alpha^{(12)}(\cdot)$ versus n

APPENDIX D

NUMERICAL RESULTS FOR CONVERGENCE TO OPTIMAL SCHEDULE OF IMPROVEMENT ALGORITHM FOR REPLACEABLE SYSTEMS WHEN $K_\tau^* = 2$

In this appendix, the availability plot versus the first two inspection times are presented for the case $k_\tau^* = 2$. The same intensity functions in Appendix B are used ($\alpha^{(9)}(t)$, $\alpha^{(10)}(t)$, $\alpha^{(11)}(t)$ and $\alpha^{(12)}(t)$). Additionally, $N_{max} = 30$ and $\epsilon = 0.0001$ are used.

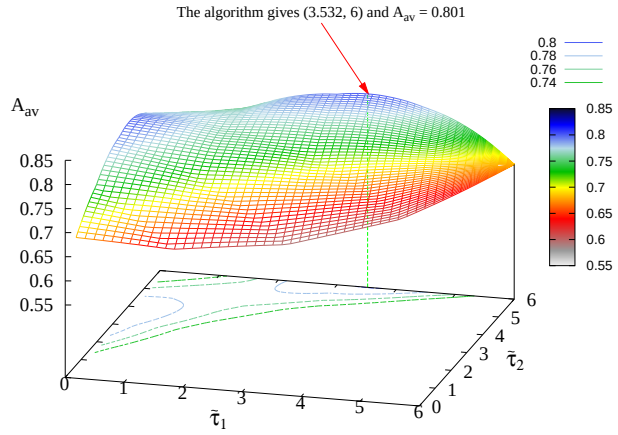

Fig. 36. A_{av} versus $(\tilde{\tau}_1, \tilde{\tau}_2)$ for $\alpha^{(9)}(\cdot)$ when $k_\tau^* = 2$

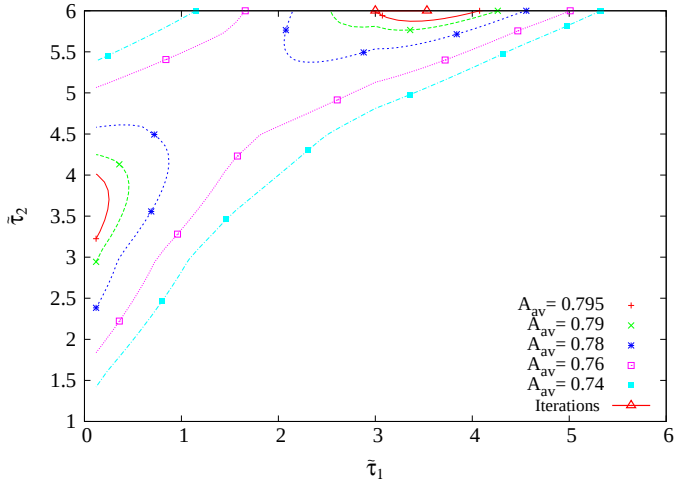

Fig. 37. The improvement algorithm iterations for $\alpha^{(9)}(\cdot)$ when $k_\tau^* = 2$

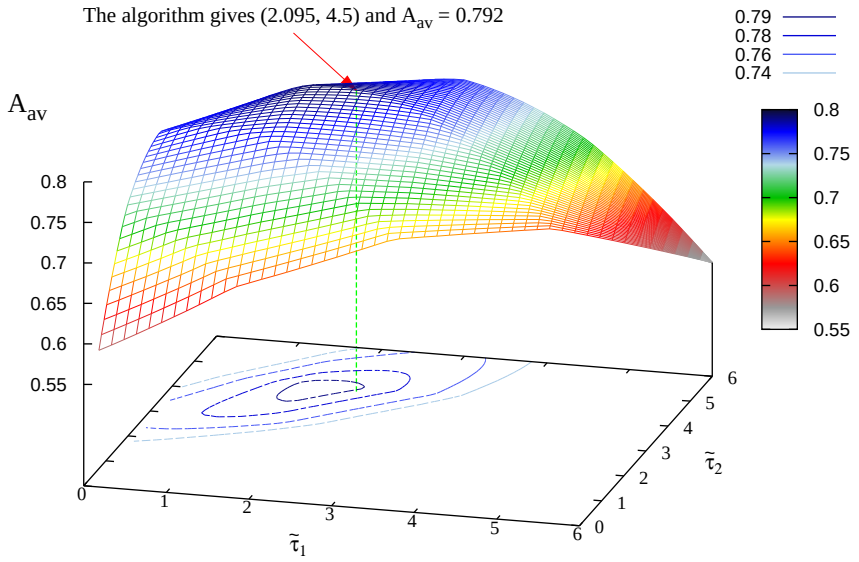

Fig. 38. A_{av} versus $(\tilde{\tau}_1, \tilde{\tau}_2)$ for $\alpha^{(10)}(\cdot)$ when $k_\tau^* = 2$

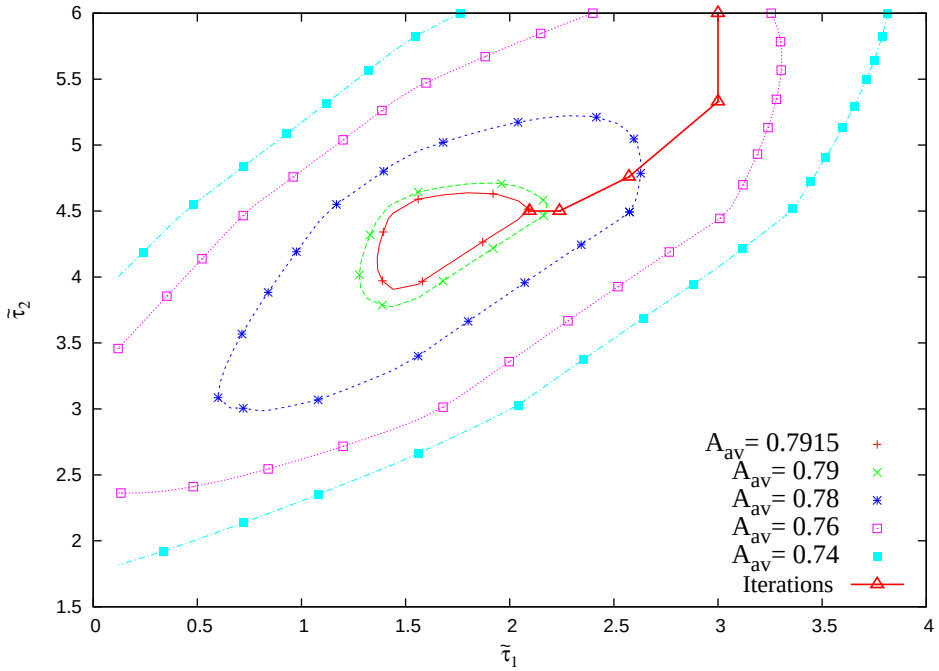

Fig. 39. The improvement algorithm iterations for $\alpha^{(10)}(\cdot)$ when $k_\tau^* = 2$

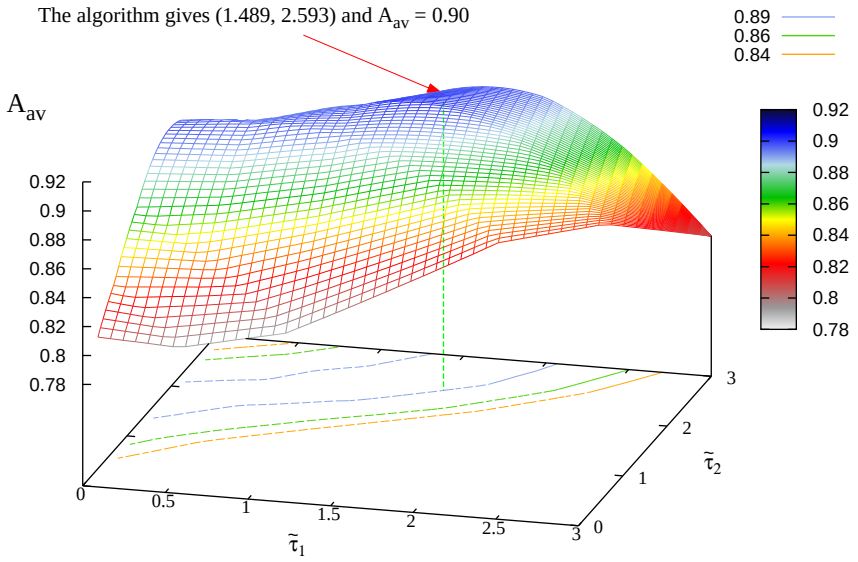

Fig. 40. A_{av} versus $(\tilde{\tau}_1, \tilde{\tau}_2)$ for $\alpha^{(11)}(\cdot)$ when $k_\tau^* = 2$

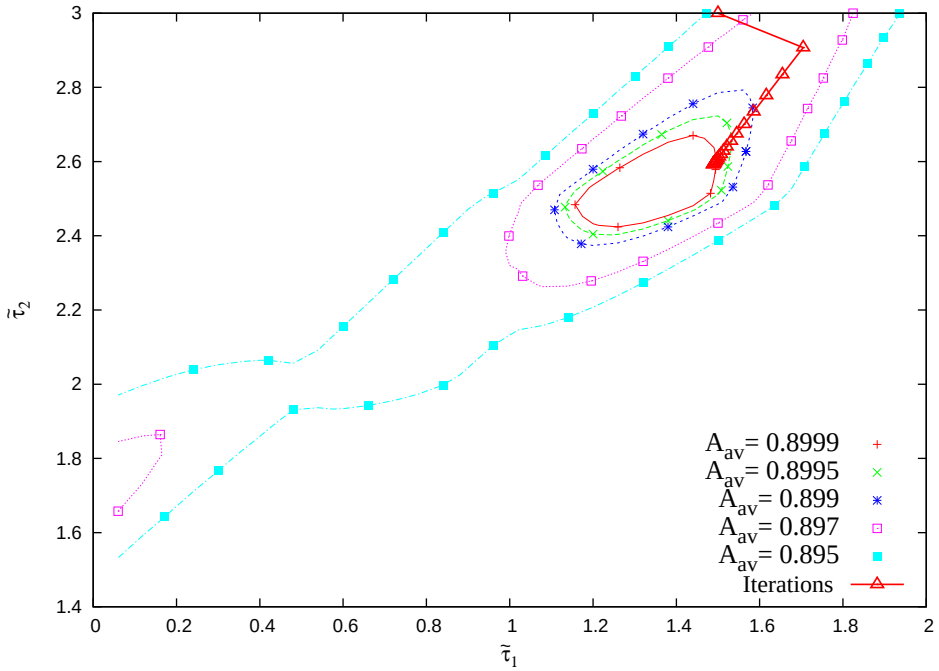

Fig. 41. The improvement algorithm iterations for $\alpha^{(11)}(\cdot)$ when $k_\tau^* = 2$

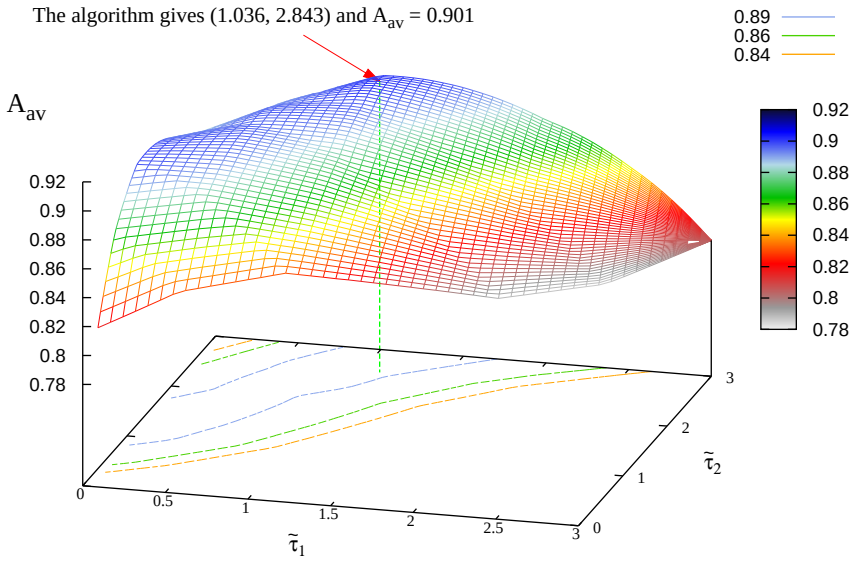

Fig. 42. A_{av} versus $(\tilde{\tau}_1, \tilde{\tau}_2)$ for $\alpha^{(12)}(\cdot)$ when $k_\tau^* = 2$

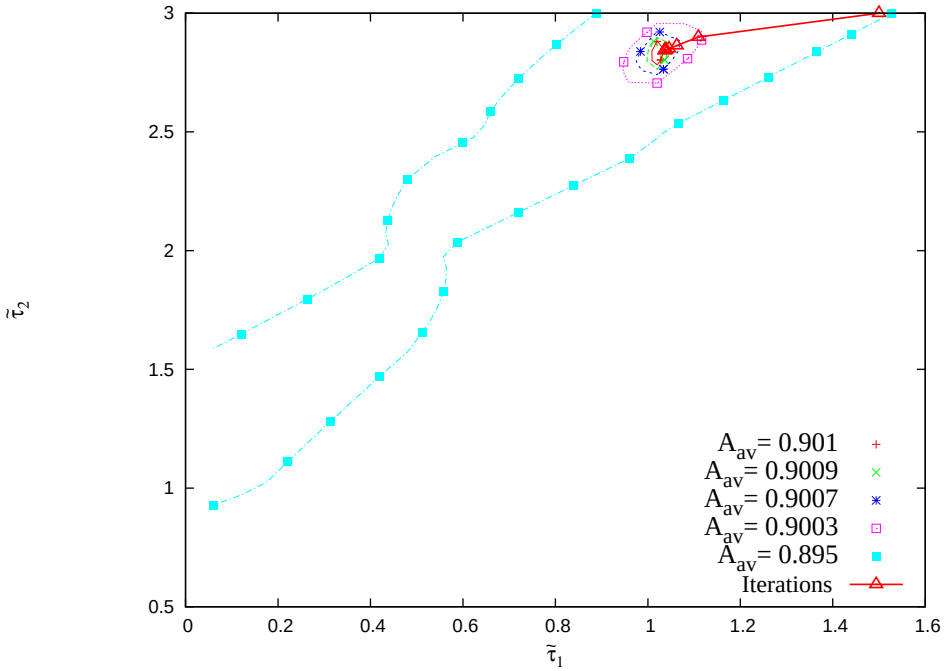

Fig. 43. The improvement algorithm iterations for $\alpha^{(12)}(\cdot)$ when $k_\tau^* = 2$

APPENDIX E

IMPROVED INSPECTION SCHEDULES FOR NON-REPLACEABLE SYSTEMS

In this appendix, improved inspection points generated by our improvement algorithm in Chapter IV are presented. Hazard rate functions and the k^* values are given for cases. We assume that periodic inspections are used as a starting schedule, $a = 0$, $b = 15$, $N_{max} = 50$, $\gamma = 0.1$ and $\epsilon = 0.0001$.

$$\alpha_{15}(t) = \begin{cases} 0.05 \cdot t, & t \in [0, 5) \\ 0.25 + 0.075 \cdot (t - 5), & t \in [5, 10) \\ 0.625 + 0.1 \cdot (t - 10), & t \in [10, 15] \end{cases}$$

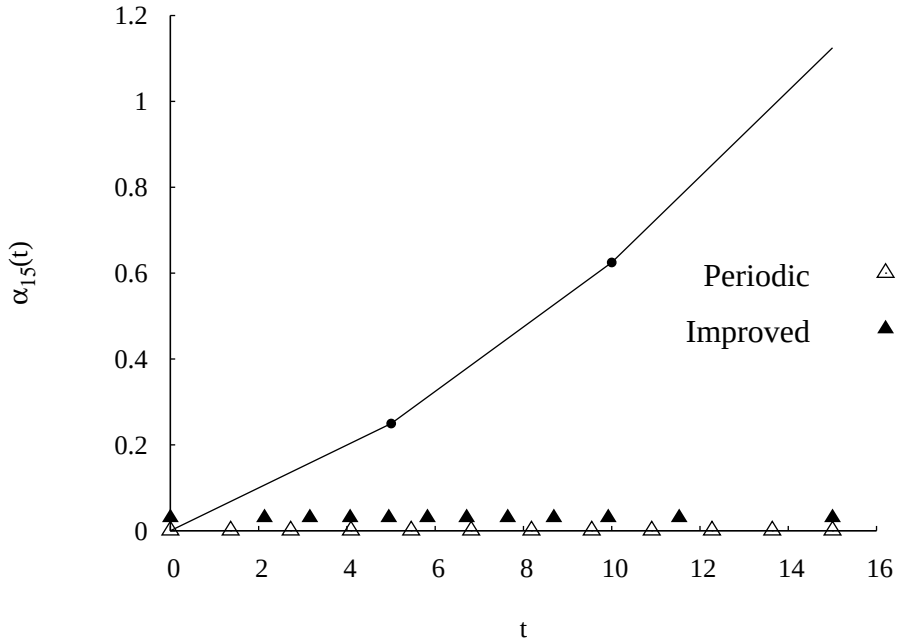

Fig. 44. Improved and periodic inspections for $\alpha_{15}(\cdot)$ when $k^* = 10$

Table XIX. Improved and periodic inspection schedules for $\alpha_{15}(\cdot)$ when $k^* = 10$

i	$PI(\tau)$	$\tilde{\tau}_i$	$\Delta\tilde{\tau}_i$
1	1.364	2.133	2.133
2	2.727	3.161	1.028
3	4.091	4.073	0.9118
4	5.455	4.947	0.8743
5	6.818	5.826	0.879
6	8.182	6.71	0.8843
7	9.545	7.646	0.9355
8	10.91	8.687	1.041
9	12.27	9.92	1.234
10	13.64	11.53	1.608
11	15	15	3.472

$$\alpha_{16}(t) = \begin{cases} 1.125 - 0.1 \cdot t, & t \in [0, 5) \\ 0.625 - 0.075 \cdot (t-5), & t \in [5, 10) \\ 0.625 - 0.05 \cdot (t-10), & t \in [10, 15] \end{cases}$$

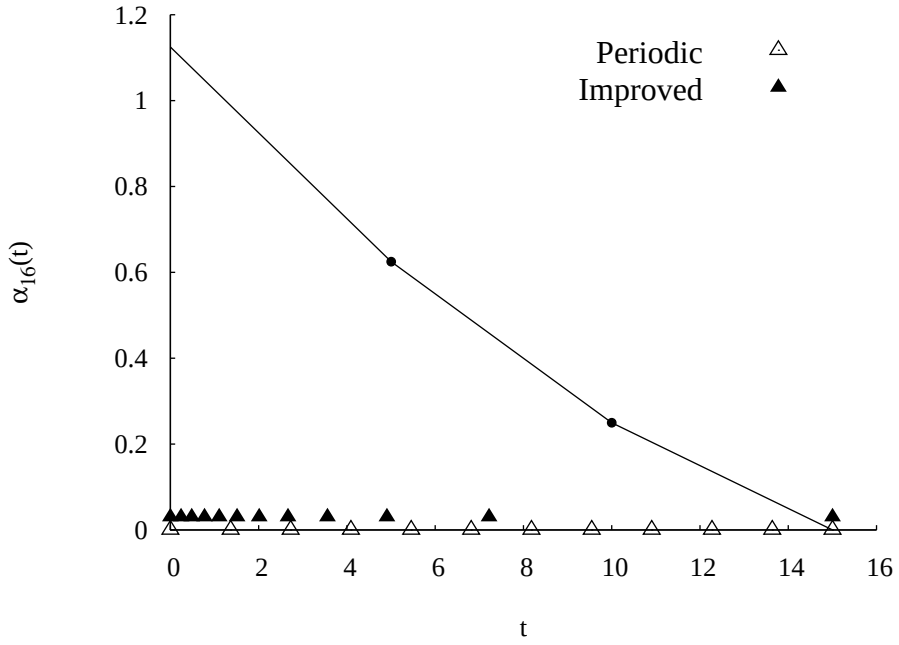

Fig. 45. Improved and periodic inspections for $\alpha_{16}(\cdot)$ when $k^* = 10$

Table XX. Improved and periodic inspection schedules for $\alpha_{16}(\cdot)$ when $k^* = 10$

i	$PI(\tau)$	$\tilde{\tau}_i$	$\Delta\tilde{\tau}_i$
1	1.364	0.2403	0.2403
2	2.727	0.4864	0.2461
3	4.091	0.7706	0.2842
4	5.455	1.105	0.3345
5	6.818	1.508	0.4033
6	8.182	2.01	0.5019
7	9.545	2.662	0.6516
8	10.91	3.559	0.8976
9	12.27	4.907	1.347
10	13.64	7.223	2.317
11	15	15	7.777

$$\alpha_{17}(t) = \begin{cases} 0.3 + 0.1 \cdot t, & t \in [0, 2) \\ 0.5 + 0.075 \cdot (t-2), & t \in [2, 4) \\ 0.65 + 0.05 \cdot (t-4), & t \in [4, 6] \\ 0.75, & t \in [6, 9) \\ 0.75 - 0.05 \cdot (t-9), & t \in [9, 11) \\ 0.65 - 0.075 \cdot (t-11), & t \in [11, 13) \\ 0.5 - 0.1 \cdot (t-13), & t \in [13, 15] \end{cases}$$

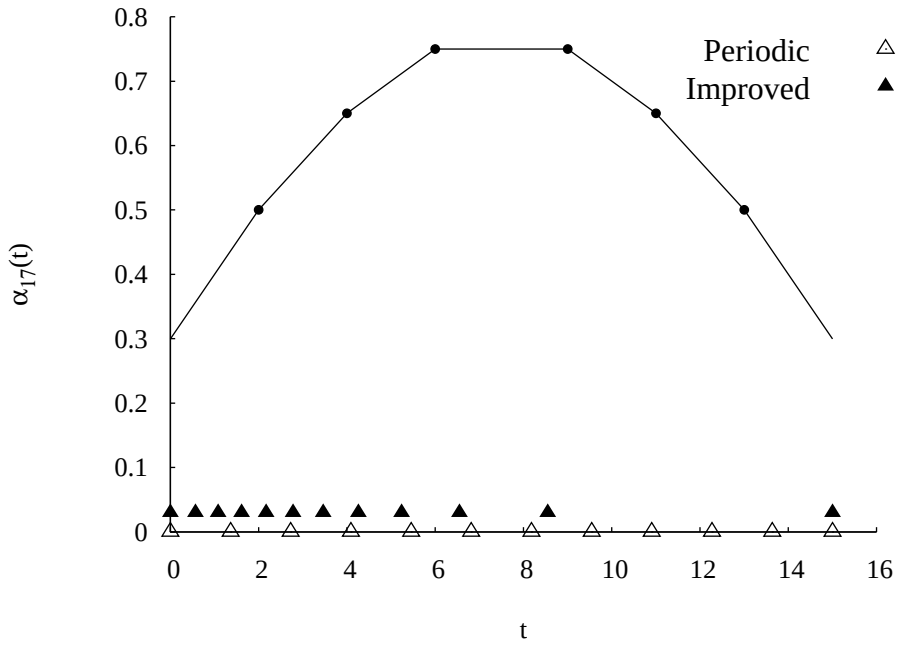

Fig. 46. Improved and periodic inspections for $\alpha_{17}(\cdot)$ when $k^* = 10$

Table XXI. Improved and periodic inspection schedules for $\alpha_{17}(\cdot)$ when $k^* = 10$

i	$PI(\tau)$	$\tilde{\tau}_i$	$\Delta\tilde{\tau}_i$
1	1.364	0.5669	0.5669
2	2.727	1.081	0.5139
3	4.091	1.611	0.5299
4	5.455	2.169	0.5585
5	6.818	2.777	0.6076
6	8.182	3.461	0.6841
7	9.545	4.258	0.7968
8	10.91	5.24	0.9827
9	12.27	6.549	1.308
10	13.64	8.551	2.002
11	15	15	6.449

$$\alpha_{18}(t) = \begin{cases} 0.6 - 0.15 \cdot t, & t \in [0, 2) \\ 0.3 - 0.075 \cdot (t-2), & t \in [2, 4) \\ 0.15 - 0.05 \cdot (t-4), & t \in [4, 6] \\ 0.05, & t \in [6, 9) \\ 0.05 + 0.05 \cdot (t-9), & t \in [9, 11) \\ 0.15 + 0.075 \cdot (t-11), & t \in [11, 13) \\ 0.3 + 0.15 \cdot (t-13), & t \in [13, 15] \end{cases}$$

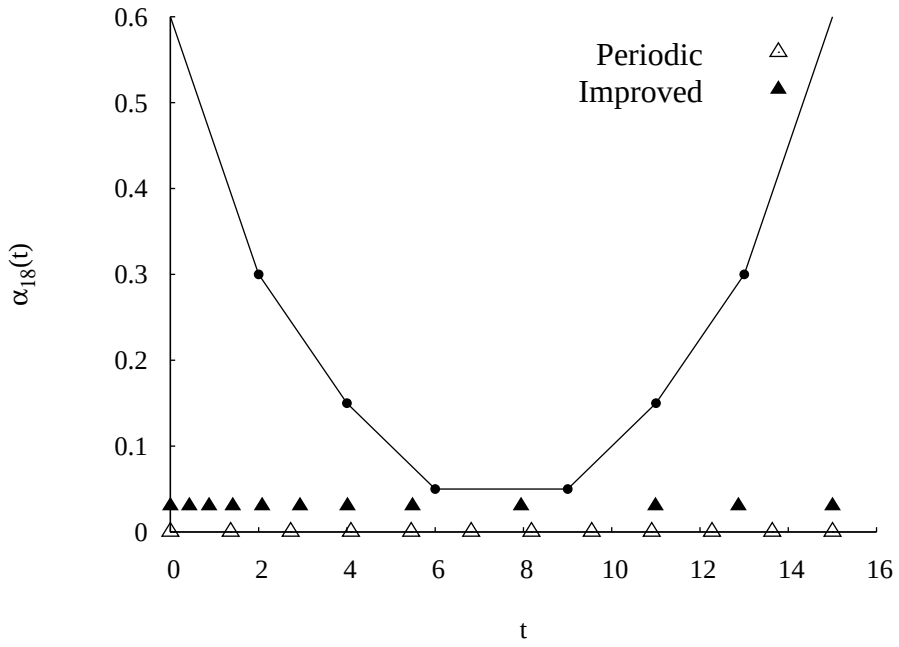

Fig. 47. Improved and periodic inspections for $\alpha_{18}(\cdot)$ when $k^* = 10$

Table XXII. Improved and periodic inspection schedules for $\alpha_{18}(\cdot)$ when $k^* = 10$

i	$PI(\tau)$	$\tilde{\tau}_i$	$\Delta\tilde{\tau}_i$
1	1.364	0.4271	0.4271
2	2.727	0.8775	0.4504
3	4.091	1.416	0.5388
4	5.455	2.081	0.6642
5	6.818	2.937	0.8567
6	8.182	4.014	1.077
7	9.545	5.485	1.471
8	10.91	7.945	2.46
9	12.27	10.99	3.045
10	13.64	12.87	1.877
11	15	15	2.132

APPENDIX F

NUMERICAL RESULTS FOR CONVERGENCE OF IMPROVEMENT ALGORITHM FOR NON-REPLACEABLE SYSTEMS

This appendix provides more numerical results for the convergence of $\tau_i^{(n)}, i = 1, \ldots, k^*$ for improved inspection schedules in Chapter IV. Throughout this part, we use $N_{max} = 30$ and $\epsilon = 0.0001$, $a = 0$, $\gamma = 0$ and $k^* = 6$.

$$\alpha_{20}(t) = \begin{cases} 0.05, & [0,1) \\ 0.05 + 0.025 \cdot (t-1), & [1,2) \\ 0.075 + 0.045 \cdot (t-2), & [2,3) \\ 0.12 + 0.06 \cdot (t-3), & [3,4) \\ 0.18 + 0.07 \cdot (t-4), & [4,5) \\ 0.25 + 0.1 \cdot (t-5), & [5,6] \end{cases}$$

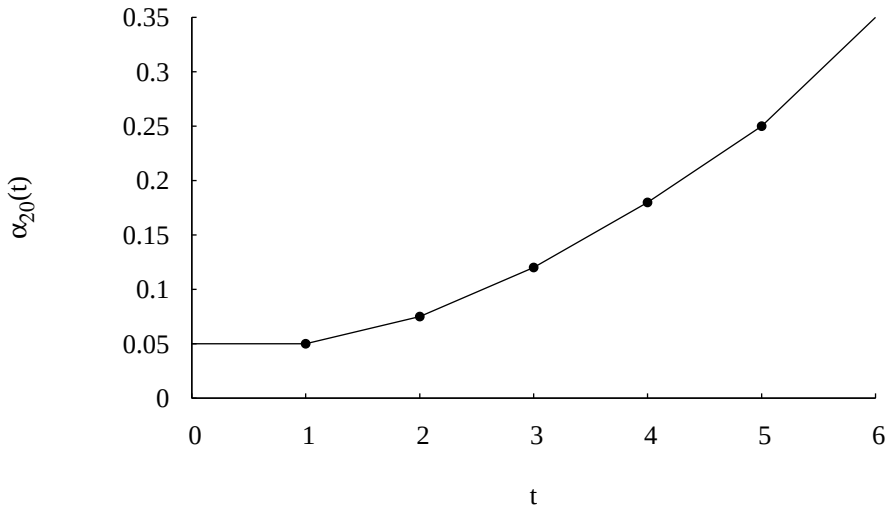

Fig. 48. The hazard rate function $\alpha_{20}(t)$ versus t

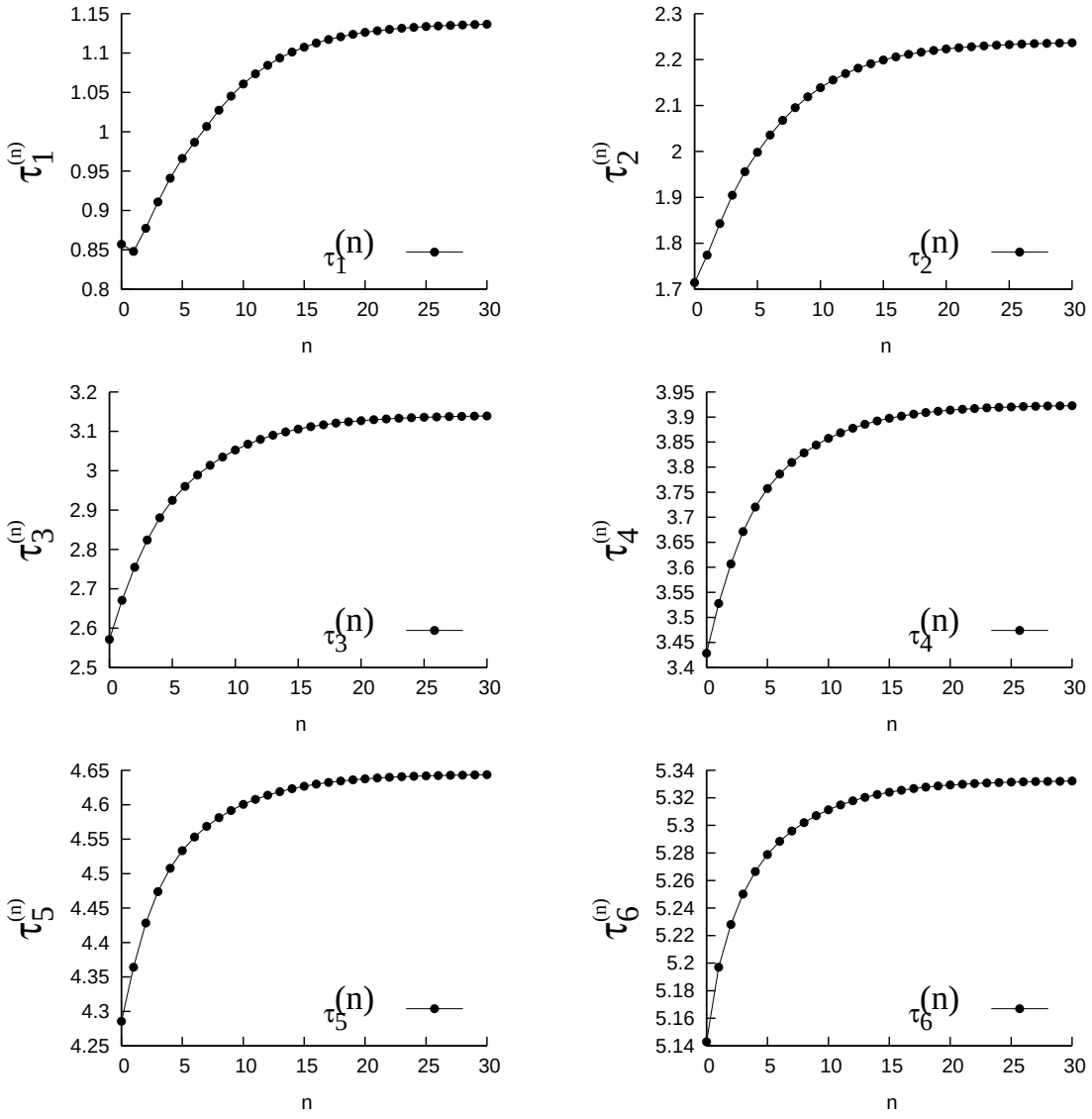

Fig. 49. $\tau_i^{(n)}$, $i = 1, \ldots, 6$ versus n for $\alpha_{20}(\cdot)$

$$\alpha_{21}(t) = \begin{cases} 0.35 - 0.1 \cdot t, & [0,1) \\ 0.25 - 0.07 \cdot (t-1), & [1,2) \\ 0.18 - 0.06 \cdot (t-2), & [2,3) \\ 0.12 - 0.045 \cdot (t-3), & [3,4) \\ 0.075 - 0.025 \cdot (t-4), & [4,5) \\ 0.05 - 0.025 \cdot (t-5), & [5,6] \end{cases}$$

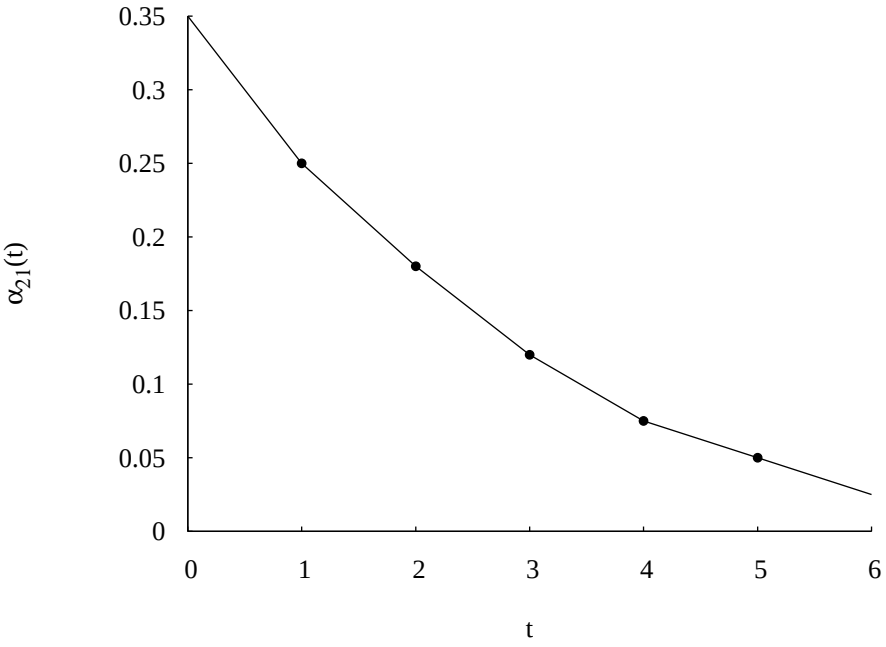

Fig. 50. The hazard rate function $\alpha_{21}(t)$ versus t

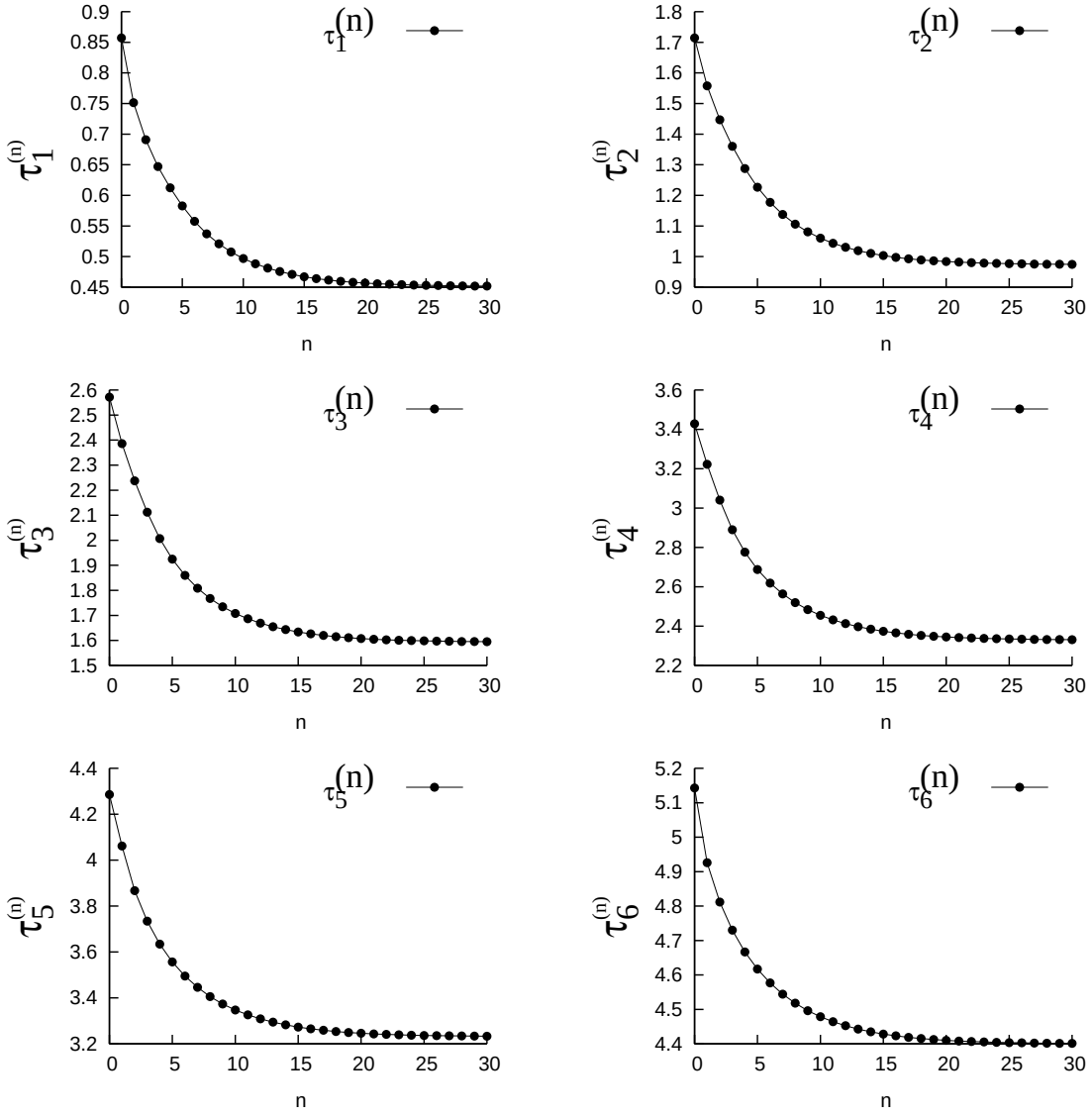

Fig. 51. $\tau_i^{(n)}$, $i = 1, \ldots, 6$ versus n for $\alpha_{21}(\cdot)$

$$\alpha_{22}(t) = \begin{cases} 0.25, & [0,1) \\ 0.05 + 0.05 \cdot (t-1), & [1,2) \\ 0.1 + 0.025 \cdot (t-2), & [2,3) \\ 0.125, & [3,5.5) \\ 0.125 - 0.025 \cdot (t-5.5), & [5.5,6.5) \\ 0.1 - 0.05 \cdot (t-6.5), & [6.5,7.5) \\ 0.025, & [7.5,8.5] \end{cases}$$

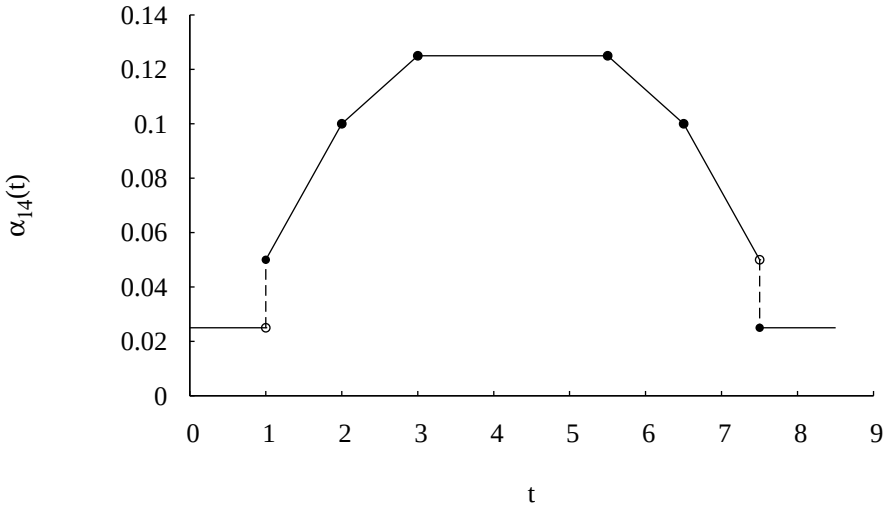

Fig. 52. The hazard rate function $\alpha_{22}(t)$ versus t

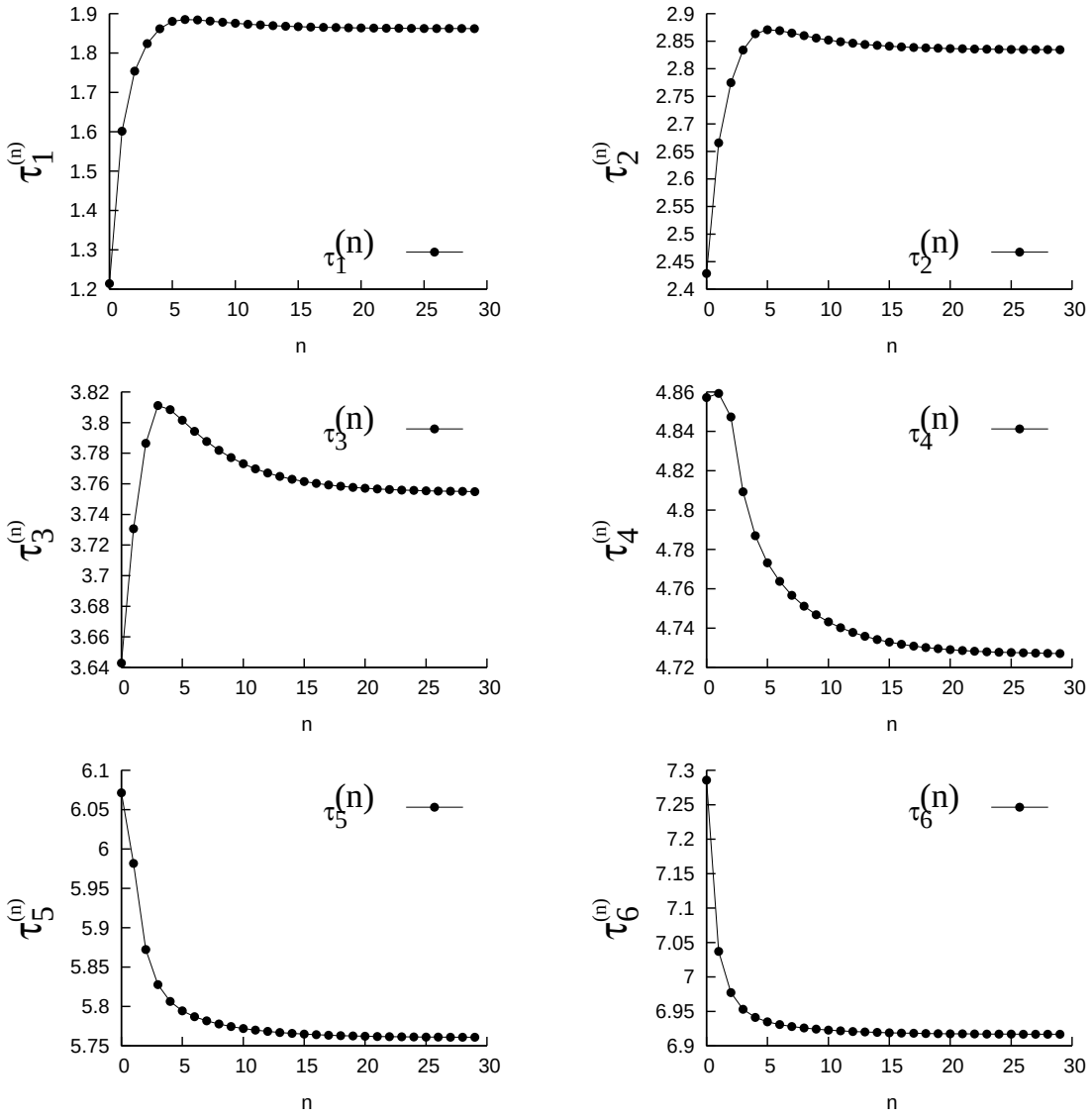

Fig. 53. $\tau_i^{(n)}$, $i = 1, \ldots, 6$ versus n for $\alpha_{22}(\cdot)$

$$\alpha_{23}(t) = \begin{cases} 0.2 - 0.05, & [0,1) \\ 0.1 - 0.035 \cdot (t-1), & [1,2) \\ 0.065 - 0.025 \cdot (t-2), & [2,3) \\ 0.05, & [3,4.5) \\ 0.05 - 0.015 \cdot (t-4.5), & [4.5,5.5) \\ 0.05 + 0.015 \cdot (t-5.5), & [5.5,6.5) \\ 0.015 + 0.05(t-6.5), & [6.5,7.5] \end{cases}$$

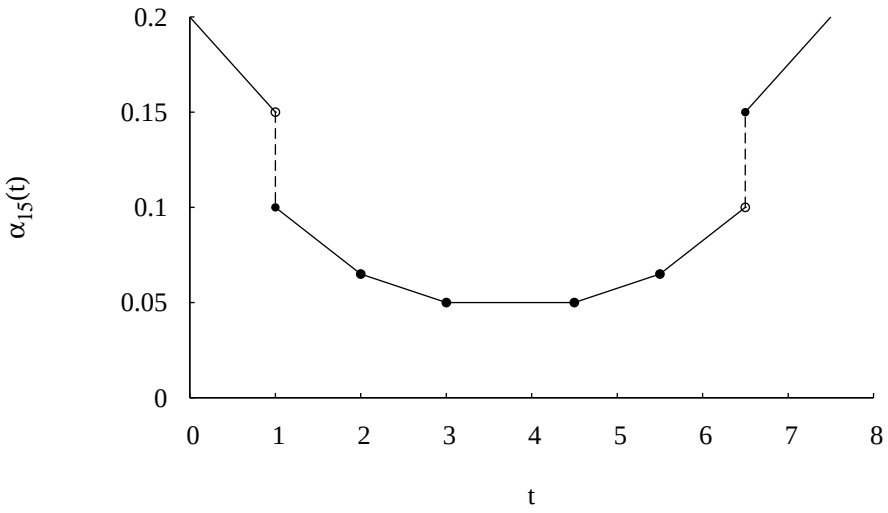

Fig. 54. The hazard rate function $\alpha_{23}(t)$ versus t

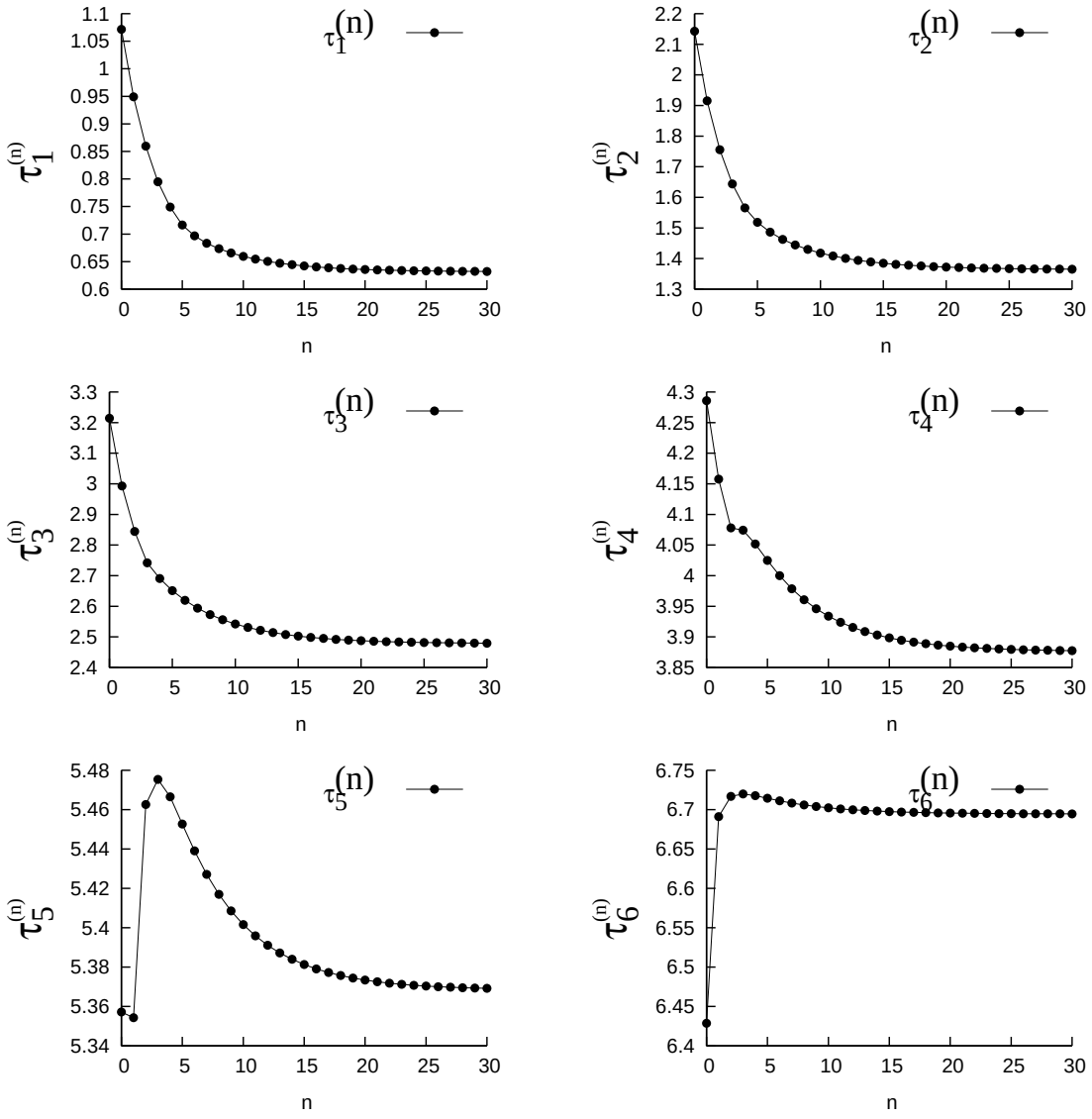

Fig. 55. $\tau_i^{(n)}$, $i = 1, \ldots, 6$ versus n for $\alpha_{23}(\cdot)$

APPENDIX G

NUMERICAL RESULTS FOR CONVERGENCE TO OPTIMAL SCHEDULE OF IMPROVEMENT ALGORITHM FOR NON-REPLACEABLE WHEN $K^* = 2$

In this appendix, the expected delay versus (τ_1, τ_2) are presented for the case $k^* = 2$. The same hazard rate functions in Appendix F are used ($\alpha_{20}(t)$, $\alpha_{21}(t)$, $\alpha_{22}(t)$ and $\alpha_{23}(t)$). Additionally, $N_{max} = 30$, $a = 0$, $\gamma = 0.1$ and $\epsilon = 0.0001$ are used.

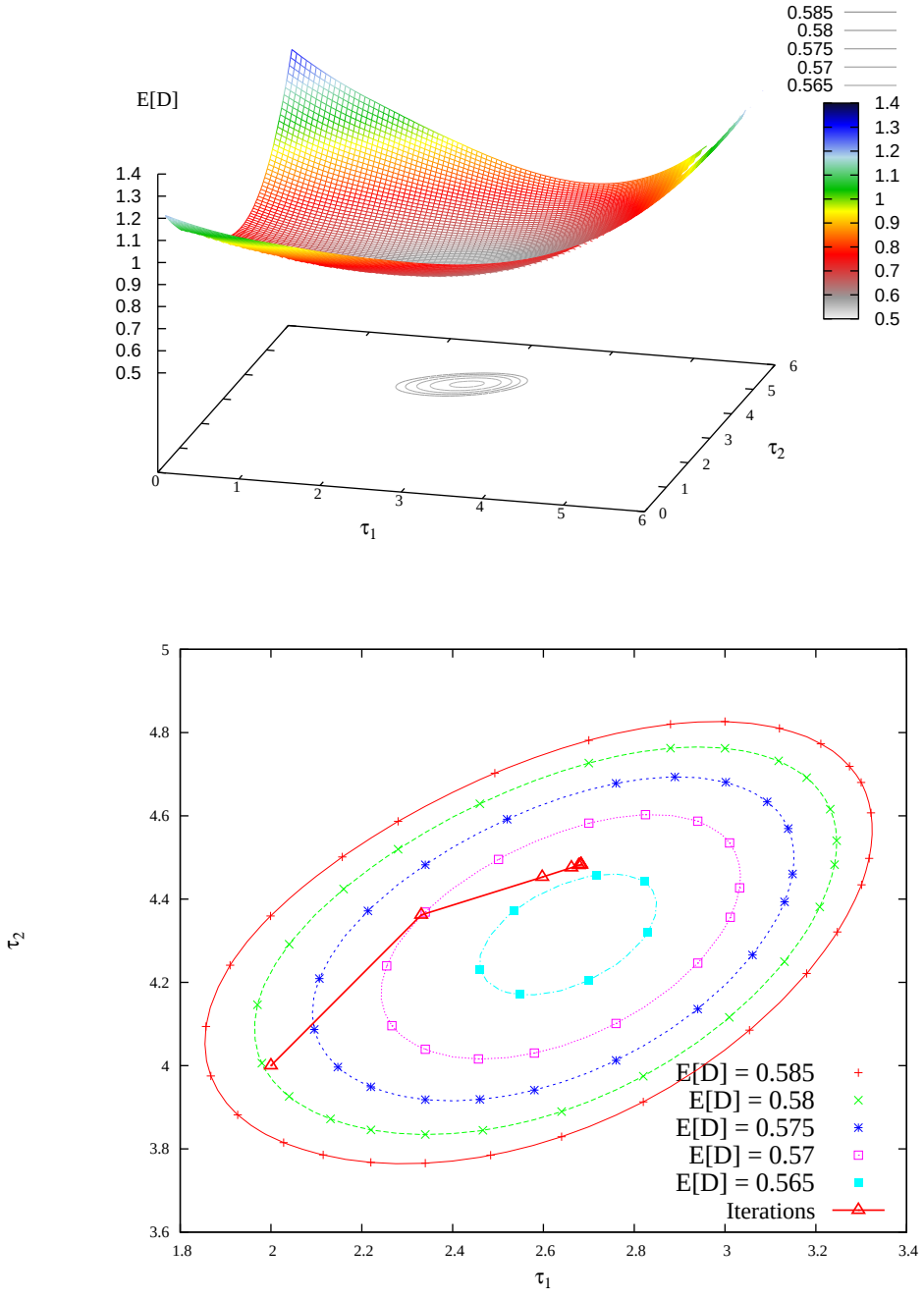

Fig. 56. $\mathbb{E}[D]$ versus (τ_1, τ_2) and iterations of the improvement algorithm for $\alpha_{20}(\cdot)$

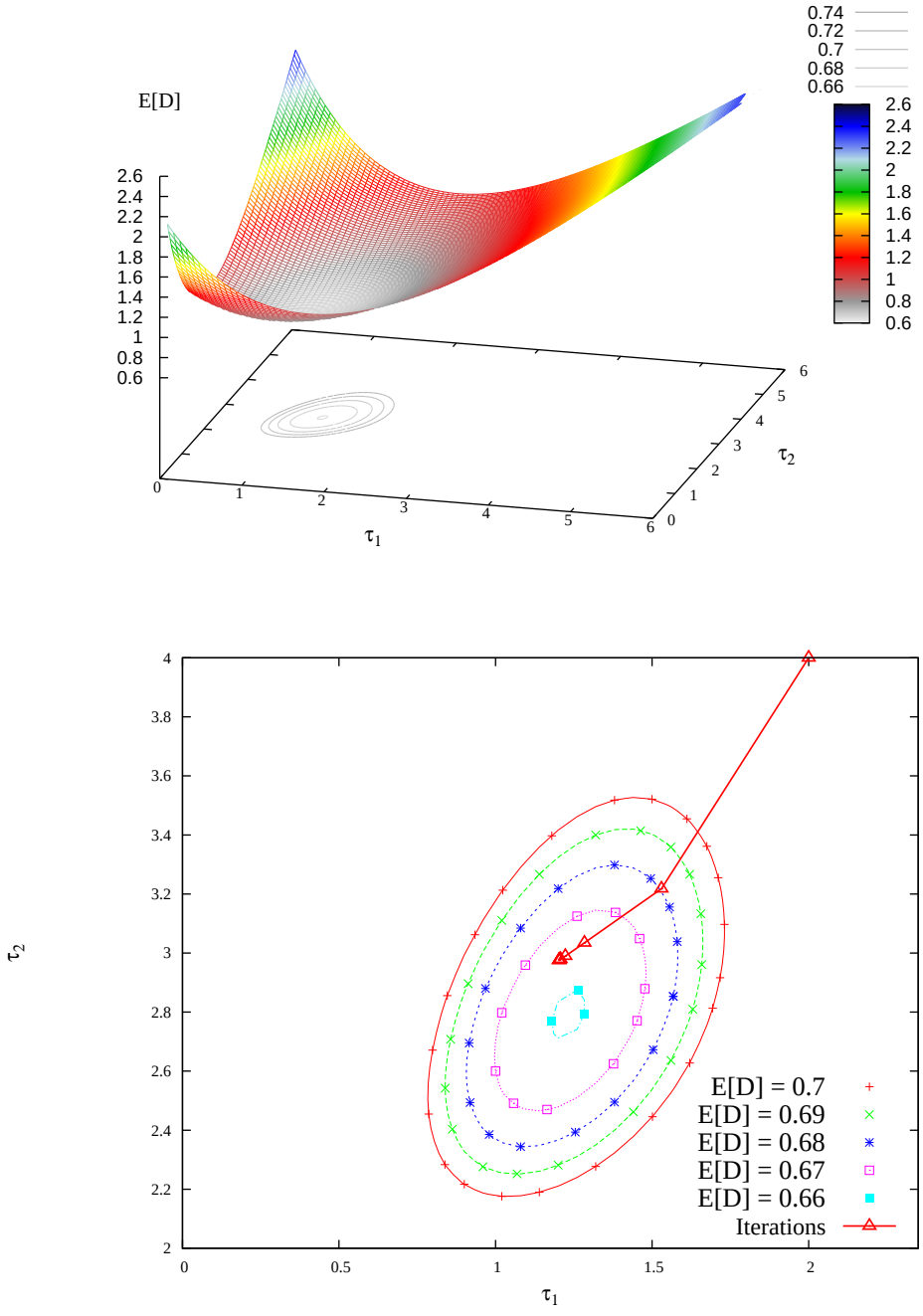

Fig. 57. $\mathbb{E}[D]$ versus (τ_1, τ_2) and iterations of the improvement algorithm for $\alpha_{21}(\cdot)$

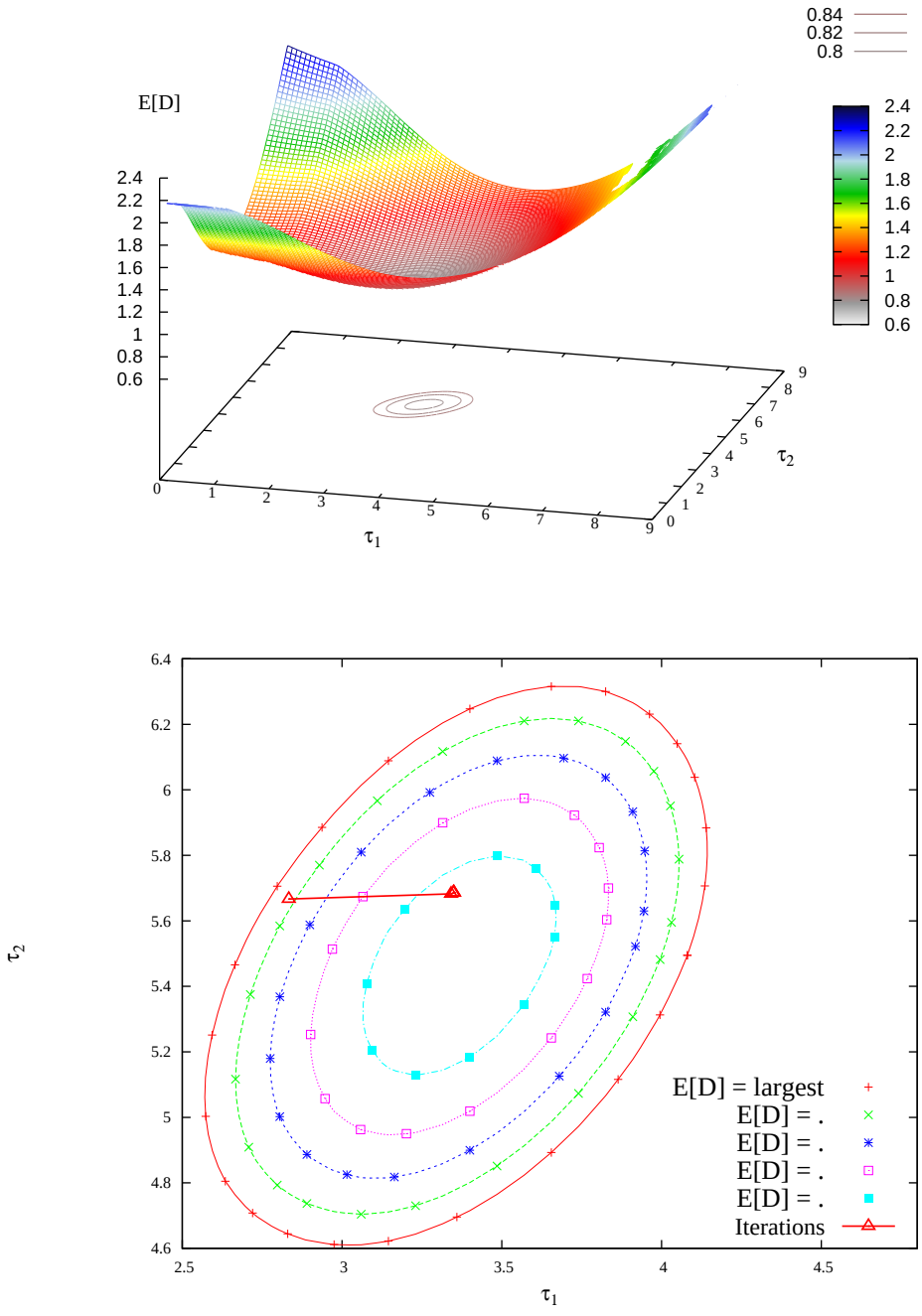

Fig. 58. $\mathbb{E}[D]$ versus (τ_1, τ_2) and iterations of the improvement algorithm for $\alpha_{22}(\cdot)$

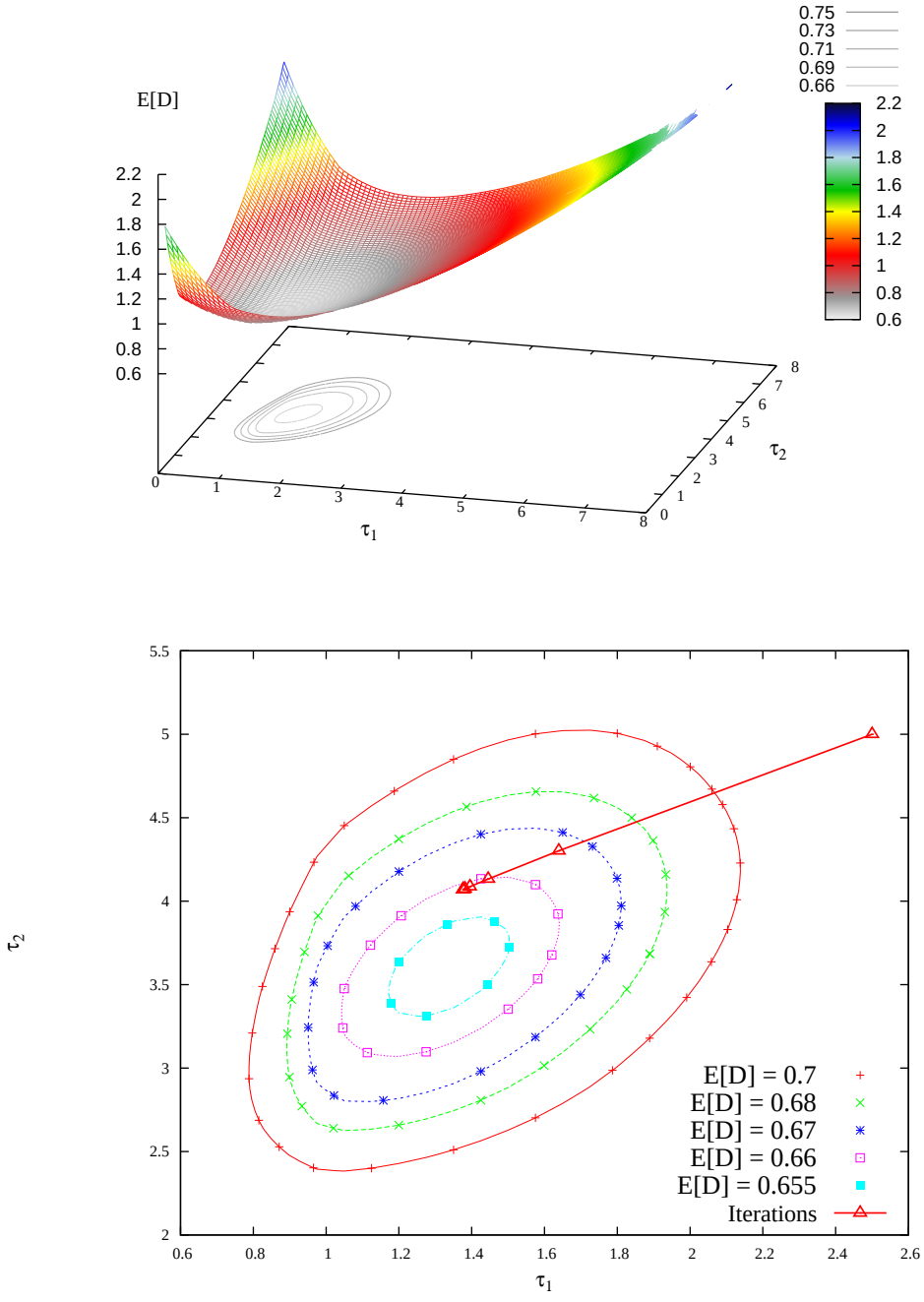

Fig. 59. $\mathbb{E}[D]$ versus (τ_1, τ_2) and iterations of the improvement algorithm for $\alpha_{23}(\cdot)$

APPENDIX H

NUMERICAL RESULTS FOR PERFORMANCE OF DIFFERENT INSPECTIONS STRATEGIES FOR NON-REPLACEABLE SYSTEMS

The expected delay for periodic inspectins and the improved inspectins are compared in this appendix. Comparison are made for the same hazard rate functions in Appendix F ($\alpha_{20}(t)$, $\alpha_{21}(t)$, $\alpha_{22}(t)$ and $\alpha_{23}(t)$). We generate Tables for $\gamma = 0.1$, $\gamma = 0.3$ using $N_{max} = 30$, $a = 0$, $\epsilon = 0.0001$.

Table XXIII. Comparions of periodic and improved inspections for $\alpha_{20}(\cdot)$ and $\gamma = 0.1$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	0.793	0.772	-2.68%
2	0.579	0.563	-2.76%
3	0.457	0.443	-2.99%
4	0.377	0.365	-3.35%
5	0.321	0.31	-3.35%
6	0.279	0.27	-3.36%
7	0.247	0.238	-3.61%
8	0.222	0.213	-3.76%
10	0.184	0.177	-3.81%
15	0.129	0.124	-3.72%

Table XXIV. Comparions of periodic and improved inspections for $\alpha_{20}(\cdot)$ and $\gamma = 0.3$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	0.731	0.73	-0.112%
2	0.603	0.593	-1.76%
3	0.515	0.499	-3.12%
4	0.45	0.431	-4.17%
5	0.398	0.379	-4.63%
6	0.356	0.338	-4.91%
7	0.322	0.305	-5.19%
8	0.294	0.278	-5.35%
10	0.25	0.236	-5.5%
15	0.181	0.171	-5.43%

Table XXV. Comparions of periodic and improved Inspections for $\alpha_{21}(\cdot)$ and $\gamma = 0.1$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	1.21	1.04	-13.6%
2	0.791	0.659	-16.6%
3	0.582	0.48	-17.6%
4	0.458	0.377	-17.8%
5	0.378	0.31	-17.9%
6	0.321	0.264	-17.8%
7	0.279	0.229	-17.9%
8	0.247	0.202	-18%
10	0.2	0.164	-18%
15	0.136	0.112	-17.3%

Table XXVI. Comparions of periodic and improved inspections for $\alpha_{21}(\cdot)$ and $\gamma = 0.3$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	1.27	1.09	-14%
2	0.971	0.781	-19.5%
3	0.772	0.605	-21.7%
4	0.635	0.494	-22.2%
5	0.536	0.416	-22.3%
6	0.462	0.36	-22.1%
7	0.405	0.317	-21.9%
8	0.361	0.283	-21.6%
10	0.296	0.233	-21.1%
15	0.203	0.162	-20.2%

Table XXVII. Comparions of periodic and improved inspections for $\alpha_{22}(\cdot)$ and $\gamma = 0.1$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	1.17	1.17	-0.0935%
2	0.832	0.793	-4.78%
3	0.648	0.601	-7.18%
4	0.528	0.484	-8.32%
5	0.444	0.404	-9%
6	0.383	0.348	-9.17%
7	0.337	0.306	-9.24%
8	0.3	0.272	-9.16%
10	0.245	0.224	-8.69%
15	0.169	0.155	-8.53%

Table XXVIII. Comparions of periodic and improved inspections for $\alpha_{22}(\cdot)$ and $\gamma = 0.3$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	1.17	1.16	-0.433%
2	0.957	0.889	-7.05%
3	0.813	0.721	-11.2%
4	0.7	0.606	-13.4%
5	0.611	0.522	-14.5%
6	0.539	0.46	-14.8%
7	0.482	0.41	-14.9%
8	0.434	0.37	-14.8%
10	0.36	0.31	-14.1%
15	0.253	0.22	-12.8%

Table XXIX. Comparions of periodic and improved inspections for $\alpha_{23}(\cdot)$ and $\gamma = 0.1$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	1.11	0.958	-13.7%
2	0.754	0.653	-13.4%
3	0.566	0.496	-12.4%
4	0.452	0.406	-10.3%
5	0.376	0.344	-8.64%
6	0.322	0.298	-7.31%
7	0.282	0.261	-7.46%
8	0.251	0.231	-8.02%
10	0.207	0.191	-7.69%
15	0.143	0.132	-7.21%

Table XXX. Comparions of periodic and improved inspections for $\alpha_{23}(\cdot)$ and $\gamma = 0.3$

k^*	$\mathbb{E}[D_{periodic}]$	$\mathbb{E}[D_{improved}]$	$100 \cdot \frac{\mathbb{E}[D_{periodic}]-\mathbb{E}[D_{improved}]}{\mathbb{E}[D_{periodic}]}$
1	1.12	0.959	-14.7%
2	0.871	0.714	-18%
3	0.7	0.575	-17.9%
4	0.581	0.487	-16.1%
5	0.495	0.425	-14%
6	0.43	0.376	-12.6%
7	0.381	0.337	-11.7%
8	0.343	0.306	-11%
10	0.287	0.259	-9.61%
15	0.202	0.186	-8.03%

VITA

Bahadır Aral received his B.S. degree in industrial engineering from Boğaziçi University in 2001 M.S and Ph.D. in industrial engineering from Texas A&M University in 2003 and 2007.

His research interest include stochastic processes, probabilistic modeling and reliability. His mailing address is Ismetpasa Mah., Bulvar Apt, No: 52/7, 48200, Milas, Mugla, Turkey.