TORIC VARIETIES AND NUMERICAL ALGORITHMS FOR SOLVING POLYNOMIAL SYSTEMS

A Dissertation

by

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ABSTRACT

This work utilizes toric varieties for solving systems of equations. In particular, it includes two numerical homotopy continuation algorithms for numerically solving systems of equations. The first algorithm, the Cox homotopy, solves a system of equations on a compact toric variety. The Cox homotopy tracks points in the total coordinate space of the toric variety and can be viewed as a homogeneous version of the polyhedral homotopy of Huber and Sturmfels. The second algorithm, the Khovanskii homotopy, solves a system of equations on a variety in the presence of a finite Khovanskii basis. This homotopy takes advantage of Anderson's flat degeneration to a toric variety. The Khovanskii homotopy utilizes the Newton-Okounkov body of the system, whose normalized volume gives a bound on the number of solutions to the system. Both homotopy algorithms provide the computational advantage of tracking paths in a compact space while also minimizing the total number of paths tracked. The Khovanskii homotopy is optimal with respect to the number of paths tracked, and the Cox homotopy is optimal when the system is Bernstein-general.

DEDICATION

To my family – past, present, and future.

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1. INTRODUCTION

Polynomial systems appear throughout mathematics and its applications. Consequently, given a collection of polynomials $F = \{f_1, \ldots, f_r\} \subset \mathbb{C}[x_1, \ldots, x_n]$, a longstanding topic of research in algebraic geometry is *finding the solutions of the system* F. Explicitly, this translates to finding all points $(p_1, \ldots, p_n) \in \mathbb{C}^n$ such that $f_i(p_1, \ldots, p_n) = 0$ for all $i = 1, \ldots, r$. The set of all solutions of the system $F = 0$ is sometimes also called the *roots*, *zeros*, or *variety* of the system F. These terms are used interchangeably, with “solutions" and “variety" among the most common here.

Due to their nonlinearity, solving polynomial systems is challenging and can be computationally expensive. To help with these computations, a new research area emerged in the late 20th century called *numerical algebraic geometry*. This area of research uses computational techniques from numerical analysis to estimate solutions of polynomial systems. The main tool in numerical algebraic geometry is *homotopy continuation*, which numerically tracks paths that interpolate between the solutions of an already-solved system and the solutions of the system of interest. While homotopy continuation is extremely advantageous for providing solutions of systems accurately and quickly, there can be significant computational waste when the algebraic structure of the polynomials is ignored.

Indeed, the algebraic structure of the polynomials influences their solution sets. Such algebraic structure can lead to classes of polynomial systems and classes of the varieties that they define. One well-studied class of varieties consists of toric varieties. Their relative simplicity makes them ideal for examples, applications, and algorithms. This work investigates the use of toric varieties in homotopy algorithms for solving polynomial systems.

In particular, this work provides two homotopy continuation algorithms in which toric varieties play a crucial role. By leveraging the structure and advantages of toric varieties, these homotopy continuation algorithms are able to significantly reduce, and in some cases eliminate, any computational waste found in more traditional homotopy algorithms. The new homotopy algorithms developed in this dissertation are the *Cox homotopy*, which is found in Chapter 10, and the *Kho-*

vanskii homotopy, which is found in Chapter 11.

Chapters 2 – 9 provide the necessary background for these two homotopy continuation algorithms. Specifically, Chapters 2 – 7 provide various background material in algebraic geometry, commutative algebra, and surrounding topics, including toric varieties in Chapter 5 and the algebraic structure of systems in Chapter 6. Chapters 8 – 9 provide background on numerical algebraic geometry, including homotopy continuation in Chapter 9.

2. VARIETIES

Varieties, the zero sets of polynomial systems, are fundamental to algebraic geometry. This chapter contains the notions of varieties necessary for this work. For a more detailed introductory treatment, see [1].

2.1 Affine varieties

The standard notation $\mathbb{C}[x_1, \ldots, x_n]$ denotes the space of all polynomials in the variables $x_1, \ldots, x_n$ with complex coefficients. For a set of polynomials $F \subset \mathbb{C}[x_1, \ldots, x_n]$, the affine variety of F, denoted $\mathcal{V}(F)$, is the set of all points $(p_1, \ldots, p_n) \in \mathbb{C}^n$ where all polynomials in F vanish. Specifically,

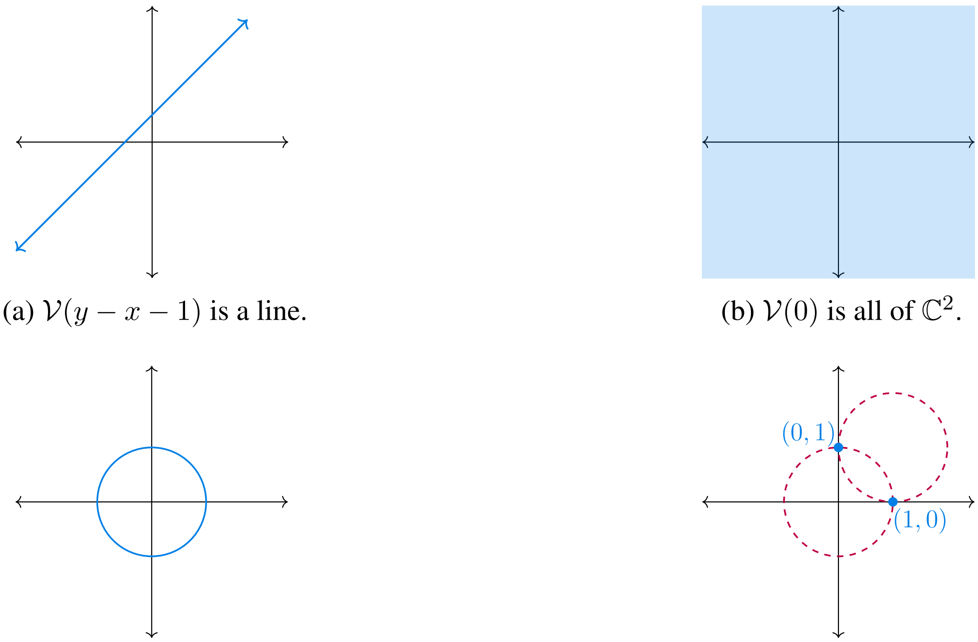
$$\mathcal{V}(F) := \{(p_1, \ldots, p_n) \in \mathbb{C}^n \mid f(p_1, \ldots, p_n) = 0 \text{ for all } f \in F\} \subset \mathbb{C}^n. \tag{2.1}$$

For example, the affine variety of the polynomial $0 \in \mathbb{C}[x_1, \ldots, x_n]$ is all of $\mathbb{C}^n$. A set of points $X \subset \mathbb{C}^n$ is an affine variety provided there exists a set of polynomials which vanish at exactly those points. Such polynomials are referred to as defining polynomials for X. Given that $\mathbb{C}^n$ is already an affine variety itself, X is a subvariety of $\mathbb{C}^n$. The real part of various examples of subvarieties of $\mathbb{C}^2$ are given in the following Figure 2.1. A variety X is said to be irreducible if there are no subvarieties $\emptyset \neq X_1, X_2 \subsetneq X$ such that $X = X_1 \cup X_2$.

One may note that defining polynomials for a variety X need not be unique. For instance, the point $(a, b) \in \mathbb{C}^2$ is defined by the system $\{x-a, y-b\}$, and also by the system $\{x+y-a-b, y-b\}$. Furthermore, one may consider all polynomials vanishing on a given set of points. Indeed, the collection of all polynomials vanishing on a set of points Z is called the ideal of Z, denoted $\mathcal{I}(Z)$:

$$\mathcal{I}(Z) := \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(p) = 0 \text{ for all } p \in Z \subset \mathbb{C}^n\}. \tag{2.2}$$

This is indeed an ideal as for any $f, g \in \mathcal{I}(Z)$, $h \in \mathbb{C}[x_1, \ldots, x_n]$, and $p \in Z$, $(f+g)(p) =$



(a) $\mathcal{V}(y - x - 1)$ is a line.

(b) $\mathcal{V}(0)$ is all of $\mathbb{C}^2$.

(c) $\mathcal{V}(x^2 + y^2 - 1)$ is the unit circle centered at the origin

(d) $\mathcal{V}(x^2 + y^2 - 1,\, (x-1)^2 + (y-1)^2 - 1)$ is the two intersection points of two unit circles.

Figure 2.1: Real part of subvarieties of $\mathbb{C}^2$.

$f(p) + g(p) = 0$ and $(fh)(p) = f(p)h(p) = 0$ since $f(p) = 0$. Similar to a polynomial system, the set of points where all polynomials in an ideal I vanish is the variety of I:

$$\mathcal{V}(I) := \{p \in \mathbb{C}^n \mid f(p) = 0 \text{ for all } f \in I\} \subset \mathbb{C}^n. \tag{2.3}$$

Algebraic geometry earns its name from the dictionary between algebra (polynomial ideals) and the geometry (varieties) they define. The following two lemmas establish part of this dictionary. Consider $\mathcal{I}$ as a function from varieties in $\mathbb{C}^n$ to ideals in $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathcal{V}$ as a function from ideals in $\mathbb{C}[x_1, \ldots, x_n]$ to varieties in $\mathbb{C}^n$.

Lemma 2.1.1. *The functions $\mathcal{I}$ and $\mathcal{V}$ are inclusion-reversing. That is,*

1. *for varieties $X \subset Y \subset \mathbb{C}^n$, $\mathcal{I}(Y) \subset \mathcal{I}(X)$, and*
2. *for ideals $I \subset J \subset \mathbb{C}[x_1, \ldots, x_n]$, $\mathcal{V}(J) \subset \mathcal{V}(I)$.*

Lemma 2.1.2. *Let $X \subset \mathbb{C}^n$ be a variety and $I \subset \mathbb{C}[x_1, \ldots, x_n]$ an ideal. Then*

1. *$\mathcal{V}(\mathcal{I}(X)) = X$, and*

2. *$I \subseteq \mathcal{I}(\mathcal{V}(I))$.*

This dictionary is necessary for defining a certain topology on X, called the Zariski topology. In this topology, all subvarieties of a variety $X \subset \mathbb{C}^n$ are designated as closed. To show that the Zariski topology is indeed a topology, one might use the Hilbert Basis Theorem (see Proposition 2.1.3), which establishes that any ideal in $\mathbb{C}[x_1, \ldots, x_n]$ is finitely generated. An ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is finitely generated if there is a finite set of polynomials $\{f_1, \ldots, f_r\} \subset \mathbb{C}[x_1, \ldots, x_n]$ such that the ideal generated by $\{f_1, \ldots, f_r\}$, denoted $\langle f_1, \ldots, f_r \rangle$, is equal to I.

Proposition 2.1.3 (Hilbert Basis Theorem [1])**.** *An ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is finitely generated. That is, for some $r \in \mathbb{N}$, there exist $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$ such that $I = \langle f_1, \ldots, f_r \rangle$.*

Lemma 2.1.4. *Given a variety X in $\mathbb{C}^n$,*

1. *X and $\emptyset$ are closed,*

2. *$\bigcap_{i \in \mathscr{I}} X_i$ is closed, where $X_i \subset X$ is a subvariety of X for all $i \in \mathscr{I}$, and*

3. *$X_1 \cup X_2$ is closed, where $X_1, X_2 \subset X$ are subvarieties of X.*

Proof. Let F be the set of defining polynomials for X. Then $\mathcal{V}(1) = \emptyset \subset X$ and $\mathcal{V}(F) = X \subset X$, which shows that both X and $\emptyset$ are closed. Next consider $\bigcap_{i \in \mathscr{I}} X_i$. First note that $\bigcap_{i \in \mathscr{I}} X_i \subset X$ as each $X_i \subset X$. Furthermore, $\mathcal{I}(\bigcap_{i \in \mathscr{I}} X_i) = \langle g_1, \ldots, g_s \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ for some $s \in \mathbb{N}$ by Proposition 2.1.3. Thus $\mathcal{V}(g_1, \ldots, g_s) = \bigcap_{i \in \mathscr{I}} X_i$, which shows that $\bigcap_{i \in \mathscr{I}} X_i$ is closed. Finally consider $X_1 \cup X_2$ where $X_1, X_2 \subset X$ are any subvarieties of X. As subvarieties, X_1 and X_2 are each defined by a finite set of polynomials, say $h_1, \ldots, h_r$ and $g_1, \ldots, g_s$, respectively. As $X_1, X_2 \subset X$, the set $X_1 \cup X_2$ is contained in X, and $\mathcal{V}(\{h_i g_j \mid i = 1, \ldots, r \text{ and } j = 1, \ldots, s\}) = X_1 \cup X_2$. $\square$

Fundamental to algebraic geometry is the notion that the functions defined on a variety capture much of the information of that variety. Given a variety $X \subset \mathbb{C}^n$, a polynomial in $\mathbb{C}[x_1, \dots, x_n]$ but restricted to X is a regular function on X. By definition, if $f \in \mathcal{I}(X)$, then $f(x) = 0$ for all $x \in X$. Consequently, two polynomials f, g on X are equal on X, i.e. $f(x) = g(x)$ for all $x \in X$, if and only if $f - g \in \mathcal{I}(X)$. Thus, the set of all regular functions on a variety X is the ring $\mathbb{C}[x_1, \dots, x_n]/\mathcal{I}(X)$. This ring is called the coordinate ring of X.

If X is an irreducible variety, then $\mathcal{I}(X)$ is prime. Consequently, its coordinate ring $\mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n]/\mathcal{I}(X)$ is an integral domain. The corresponding field of fractions of $\mathbb{C}[X]$ is the function field or field of rational functions of X, denoted by $\mathbb{C}(X)$.

2.2 Projective varieties

Complex projective space may be defined using the following equivalence relation. Two complex tuples $(p_0, \dots, p_n)$ and $(q_0, \dots, q_n)$ are equivalent, denoted $(p_0, \dots, p_n) \sim (q_0, \dots, q_n)$, if there exists some nonzero $c \in \mathbb{C}$ such that $(p_0, \dots, p_n) = c(q_0, \dots, q_n)$. Then n-dimensional complex projective space, denoted by $\mathbb{P}^n$, is given by:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim . \tag{2.4}$$

Equivalence classes in $\mathbb{C}^n$, i.e. points in $\mathbb{P}^n$, are denoted with square brackets, i.e . $[p_0 : \cdots : p_n]$.

Polynomials are not well-defined as functions on projective space. In particular, for any nonconstant polynomial f and point $p = [p_0 : \cdots : p_n] \in \mathbb{P}^n$, $f(p_0, \dots, p_n) \neq f(cp_0, \dots, cp_n)$. Zero sets of homogeneous polynomials are, however, well-defined. A polynomial is homogeneous of degree m if every nonzero term has the same degree m. That is, a polynomial $f = \sum_\alpha c_\alpha x^\alpha$ is m-homogeneous for some positive integer m if $\sum_{i=0}^n \alpha_i = m$ for all $c_\alpha \neq 0$. Supposing f is m-homogeneous, $f(p) = 0$, and $c \neq 0$, then $f(p) = c^m f(p) = f(cp)$, demonstrating that zero sets are well-defined. Homogeneous polynomials define projective varieties. A projective variety is always defined by a homogeneous ideals, which are ideals that can be generated by homogeneous polynomials. This concept of homogeneity generalizes beyond polynomials whose terms all have the same degree. Such generalizations can be seen in the definition of weighted projective space (see Example 2.2.1) and the Cox construction (see Section 10.2.2).

Example 2.2.1 (Weighted projective space)**.** One way to view $\mathbb{P}^n$ is as a copy of $\mathbb{C}^{n+1}$ where nonzero points on a line through the origin are identified. Weighted projective space generalizes this notion from lines to curves. That is, let $0 \neq w \in \mathbb{N}^{n+1}$ be a vector of positive integers where $\gcd(w_0, \dots, w_n) = 1$. Then $(p_0, \dots, p_n) \sim_w (q_0, \dots, q_n)$ if there exists some nonzero $c \in \mathbb{C}$ such that $(p_0, \dots, p_n) = (c^{w_0} q_0, \dots, c^{w_n} q_n)$. The complex weighted projective space with respect to weight w is

$$\mathbb{P}^n_w := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim_w . \tag{2.5}$$

Note that $\mathbb{P}^n_w = \mathbb{P}^n$ when $w = \mathbb{1} = (1, \dots, 1)$.

Weighted homogeneous polynomials define varieties in $\mathbb{P}^n_w$. A polynomial $f = \sum_\alpha c_\alpha x^\alpha$ is w-homogeneous for some $w \in \mathbb{N}^{n+1}$ if $\alpha \cdot w$ is constant for all $c_\alpha \neq 0$. This and other constructions of weighted projective space are found in Section 11.4. △

2.3 Properties of varieties

As demonstrated in Figure 2.1, varieties take on any number of shapes and sizes. Two essential characteristics of varieties are their dimension and degree.

2.3.1 Dimension

Let Z be an n-dimensional space, either $\mathbb{P}^n$ or $\mathbb{C}^n$, and suppose $X \subset Z$ is an irreducible variety. The dimension of X, $\dim X$, is the length d of the longest chain of strictly decreasing subvarieties of X, i.e. $X \supsetneq X_1 \supsetneq \cdots \supsetneq X_d \supsetneq \emptyset$ where each $X_i \subset X$ is an irreducible subvariety of X. If a variety is reducible, its dimension is the maximum of the dimensions of its irreducible components. Given X is a subset of Z, the codimension of X, $\operatorname{codim}_Z X$, is $n-d$. The subscript Z is dropped when the ambient space Z is clear. Varieties Y and X have complementary dimension if $\dim Y = \operatorname{codim} X$. A zero-dimensional variety is a finite set of points. The intersection of a variety X with a general linear subspace of complementary dimension is a linear section of X.

2.3.2 Degree

Let X be a variety and let $\mathcal{L}$ be the set of linear subspaces of complementary dimension to X such that $X \cap L$ is finite. The degree of a variety X, denoted $\deg X$, is the maximum number of points in a linear section of X. That is, $\deg X = \max_{L \in \mathcal{L}} |X \cap L|$.

3. COMPUTATIONAL ALGEBRAIC GEOMETRY

Computational algebraic geometry provides an algorithmic approach for solving polynomial systems, representing algebraic objects, and performing operations. The theory of Gröbner bases enables computation for ideals of polynomial rings. The theory of SAGBI and Khovanskii bases enables computation, though more limited, for subalgebras. Throughout this chapter, polynomial rings will be over the complex numbers, though as few complex numbers can be exactly represented on a computer, frequently in practice one will restrict to polynomial rings over the rational numbers for computer computations.

3.1 Monomial orders

The standard division algorithm for univariate polynomials relies on the natural ordering of the terms by degree. In order to have a division algorithm for multivariate polynomials, there likewise needs to be a way to order multivariate monomials.

Monomials may be represented by their exponent vectors. That is, a monomial $u \in \mathbb{C}[x_1, \ldots, x_n]$ has the form $u = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for nonnegative integers α_i. The exponent vector of u is $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and one may abbreviate $u = x^\alpha$. Some applications consider exponent vectors in $\mathbb{Z}^n$, in which case the corresponding monomial is a Laurent monomial.

Definition 3.1.1 (Monomial order)**.** A monomial order $\leq$ on $\mathbb{C}[x_1, \ldots, x_n]$ is a total order on the monomials of $\mathbb{C}[x_1, \ldots, x_n]$ such that:

1. $1 \leq u$ for all monomials $u \in \mathbb{C}[x_1, \ldots, x_n]$, and
2. if $u < v$ and z is any monomial in $\mathbb{C}[x_1, \ldots, x_n]$, then $uz < vz$.

There are infinitely many choices for monomial orders. Some of the more common orders are given in Example 3.1.1.

Example 3.1.1 (Monomial orders)**.**

1. *Lexicographic.* $x^\alpha < x^\beta$ if the left-most nonzero component of $\alpha - \beta$ is negative.

2. *Graded reverse lexicographic.* $x^\alpha < x^\beta$ if either $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, or $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and the right-most nonzero component of $\alpha - \beta$ is positive.

3. *Weighted.* Let $<$ be any monomial order, and $w \in \mathbb{R}^n$ has nonnegative entries. The weighted monomial order $<_w$ is given by: $x^\alpha < x^\beta$ if $\alpha \cdot w < \beta \cdot w$, or else $\alpha \cdot w = \beta \cdot w$ and $x^\alpha < x^\beta$, where $\cdot$ is the usual dot product and $\alpha \cdot w < \beta \cdot w$ is with respect to the usual ordering of $\mathbb{R}$.

$\triangle$

The vector w in the weighted monomial order is called a weight vector. A weight vector alone defines a partial monomial order. Given a finite set of polynomials F and monomial order $<$, there exists a weight vector w such that $\alpha \cdot w < \beta \cdot w$ for every pair of monomials $x^\alpha < x^\beta$ appearing in a polynomial of F [2]. Such a weight vector w is said to preserve $<$ on F.

For any polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ and fixed monomial order $<$, the leading monomial of f, $\mathrm{in}_<(f)$, is the largest monomial of f with respect to $<$. The coefficient c of $\mathrm{in}_<(f)$ is the leading coefficient, and $c\,\mathrm{in}_<(f)$ is the leading term. For an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, the lead ideal of I is the ideal generated by the leading monomials, i.e. $\mathrm{in}_<(I) := \langle \mathrm{in}_<(f) \mid f \in I, f \neq 0 \rangle$.

3.2 Gröbner bases

A Gröbner basis for an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ with respect to monomial order $<$ is a sequence $g_1, \ldots, g_m$ of elements in I such that $\mathrm{in}_<(I) = (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_m))$. Additionally, every Gröbner basis for an ideal I will also generate I.

Proposition 3.2.1 (Macaulay's Theorem, [3])**.** *Let $<$ be a monomial order on $\mathbb{C}[x_1, \ldots, x_n]$ and $I \subset \mathbb{C}[x_1, \ldots, x_n]$ an ideal. Then the monomials in $\mathbb{C}[x_1, \ldots, x_n]$ which do not belong to $\mathrm{in}_<(I)$ form a $\mathbb{C}$-basis of $\mathbb{C}[x_1, \ldots, x_n]/I$.*

The monomials in $\mathbb{C}[x_1, \ldots, x_n] \setminus \mathrm{in}_<(I)$ are called the standard monomials. While there are infinitely many monomial orders on $\mathbb{C}[x_1, \ldots, x_n]$, given an ideal I there are only finitely many distinct lead ideals $\mathrm{in}_<(I)$ [4, Proposition 2.7]. Monomial orders and Gröbner bases allow for multivariate division with a unique remainder.

Proposition 3.2.2 ([3])**.** *Fix a monomial order* $<$ *on* $\mathbb{C}[x_1,\ldots,x_n]$ *and let* $f, g_1, \ldots, g_m \in \mathbb{C}[x_1,\ldots,x_n]$ *be nonzero polynomials. There exist polynomials* $q_1,\ldots,q_m, r \in \mathbb{C}[x_1,\ldots,x_n]$ *such that*

1. $f = q_1 g_1 + \cdots + q_m g_m + r$*,*
2. *no nonzero monomial in* r *is contained in* $\langle \mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_m)\rangle$*, and*
3. $\mathrm{in}_<(f) \geq \mathrm{in}_<(q_i g_i)$ *for all* i*.*

Furthermore, if $\{g_1,\ldots,g_m\}$ *is a Gröbner basis for the ideal it generates, then the polynomial* r *is unique.*

The polynomial f in Proposition 3.2.2 is said to be divided by $g_1,\ldots,g_m$, and r is a remainder of f. A polynomial f reduces to 0 with respect to $g_1,\ldots,g_m$ if f has a remainder $r = 0$ when divided by $g_1,\ldots,g_m$. Given two polynomials f and g with respective leading coefficients c and d, their S-polynomial is:

$$S(f,g) := \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c\,\mathrm{in}_<(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{d\,\mathrm{in}_<(g)} g \tag{3.1}$$

Using Proposition 3.2.2 and S-polynomials, Buchberger's Criterion allows one to check whether a set of ideal generators is indeed a Gröbner basis for that ideal:

Proposition 3.2.3 (Buchberger's Criterion)**.** *Let* $<$ *be a monomial order on* $\mathbb{C}[x_1,\ldots,x_n]$ *and* $I = \langle g_1,\ldots,g_m\rangle$ *be an ideal in* $\mathbb{C}[x_1,\ldots,x_n]$ *with* $g_i \neq 0$ *for all* i*. The following are equivalent:*

1. $\mathcal{G} = \{g_1,\ldots,g_m\}$ *is a Gröbner basis for* I *with respect to* $<$*.*
2. $S(g_i, g_j)$ *reduces to 0 with respect to* $g_1,\ldots,g_m$ *for all* $i < j$*.*

One may extend Buchberger's Criterion to an algorithm for computing a Gröbner basis for an ideal I given a set of generators $\mathcal{G}$. Indeed, whenever an S-polynomial of the generators does not reduce to 0 with respect to $\mathcal{G}$, the resulting remainder is added to the list of generators and Buchberger's Criterion is again checked. Each time a remainder is added to $\mathcal{G}$, the corresponding monomial ideal $\langle \mathrm{in}_<(g) \mid g \in \mathcal{G}\rangle$ becomes strictly larger, thus creating a sequence of strictly increasing monomial ideals. Such a chain must eventually stabilize (see [4, Proposition 1.12]), which means that this process, known as Buchberger's algorithm, will eventually terminate. Thus, every ideal has a finite Gröbner basis, and one may compute a finite Gröbner basis from any set of ideal generators via Buchberger's algorithm. There are many mathematical software packages which will compute Gröbner bases, one of which is `Macaulay2` [5].

3.3 SAGBI bases

Gröbner bases enable computation with ideals, such as eliminating variables to solve systems, computing the intersection of ideals, ideal quotients, ideal membership, and so on. SAGBI bases seek to extend the utility of Gröbner bases to subalgebras. Let $A = \mathbb{C}[f_1, \ldots, f_r]$ be a subalgebra of $\mathbb{C}[x_1, \ldots, x_n]$ with a monomial order $<$. The lead algebra of A with respect to $<$, denoted $\mathrm{in}_< A$, is the $\mathbb{C}$-vector space spanned by $\{\mathrm{in}_<(h) \mid h \in A\}$. A set of polynomials $\mathcal{B} \subset A$ is a SAGBI basis for A if $\{\mathrm{in}_< b \mid b \in \mathcal{B}\}$ generates $\mathrm{in}_< A$ as a $\mathbb{C}$-algebra. Unlike Gröbner bases, finite SAGBI bases do not necessarily exist, as demonstrated by Example 3.3.1. This is because the lead algebra $\mathrm{in}_< A$ may not be finitely generated. When $\mathrm{in}_< A$ is not finitely generated, a finite SAGBI basis does not exist.

Example 3.3.1. Some subalgebras do not have a finite SAGBI basis for any term order, others have a finite SAGBI basis with respect to one term order but an infinite SAGBI basis with respect to a different term order, and some subalgebras have a universal SAGBI basis.

- *No finite SAGBI basis [6].* Consider the subalgebra A of polynomials in $\mathbb{C}[x_1, x_2, x_3]$ which are invariant under the cyclical action $x_1 \mapsto x_2, x_2 \mapsto x_3, x_3 \mapsto x_1$. This subalgebra has four generators, i.e. $A = \mathbb{C}[x_1+x_2+x_3, x_1x_2+x_1x_3+x_2x_3, x_1x_2x_3, (x_1-x_2)(x_1-x_3)(x_2-x_3)]$. As shown in [6], no matter the monomial order, the lead algebra $\mathrm{in}_< A$ will not be finitely generated.

- *Finite SAGBI basis conditional on $<$ [7].* Consider the subalgebra $A = \mathbb{C}[x, xy - y^2, xy^2]$. Whether or not A has a finite SAGBI basis depends upon the chosen monomial order, as shown in [7]. When the monomial order on $\mathbb{C}[x, y]$ is the lexicographic order with $x > y$, then $\mathrm{in}_< A$ is not finitely generated. When the monomial order on $\mathbb{C}[x, y]$ is the lexicographic order with $y > x$, then $\mathrm{in}_< A$ is finitely generated and has finite SAGBI basis $\{x, y^2 - yx, yx^2\}$.

- *Universal SAGBI bases [7].* The elementary symmetric polynomials form a SAGBI basis for the subalgebra they generate with respect to any term order, as shown in [7].

$\triangle$

When finite SAGBI bases exist, they exhibit similar properties of Gröbner bases, including that a SAGBI basis will necessarily generate A and there is a reduction property similar to multivariate division, known as subduction.

Algorithm 3.3.1 (Subduction Algorithm [6])**.**
Input: A SAGBI basis $\mathcal{B}$ with respect to $<$ for a subalgebra $A \subset \mathbb{C}[x_1, \ldots, x_n]$, and a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$.
Output: An expression of f as a polynomial in the elements of $\mathcal{B}$, provided $f \in A$.
Steps:
While f is not a constant in $\mathbb{C}$ do:

1. Find $b_1, \ldots, b_r$ in $\mathcal{B}$, exponents $i_1, \ldots, i_r$ in $\mathbb{N}$, and $c \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ such that

$$\mathrm{in}_< f = c(\mathrm{in}_< b_1)^{i_1} \cdots (\mathrm{in}_< b_r)^{i_r} \tag{3.2}$$

2. If no such representation (3.2) exists, then output "f does not belong to A" and STOP.

3. Otherwise output $p := c(b_1)^{i_1} \cdots (b_r)^{i_r}$, and replace f by $f - p$.

Output the constant f.

A presentation of a subalgebra is useful for determining whether a set of subalgebra generators is a SAGBI basis. Given subalgebra generators $\mathcal{B} = \{b_1, \dots, b_r\} \subset A \subset \mathbb{C}[x_1, \dots, x_n]$, a presentation of A is the kernel of the map

$$\begin{aligned} \varphi_{\mathcal{B}} : \mathbb{C}[z_1, \dots, z_r] &\to A \\ z_i &\mapsto b_i. \end{aligned}$$

Let $I_{\mathcal{B}}$ denote $\ker \varphi_{\mathcal{B}}$. Let $\mathcal{A}$ denote the lead terms of $\mathcal{B}$. One can similarly define a map

$$\begin{aligned} \varphi_{\mathcal{A}} : \mathbb{C}[z_1, \dots, z_r] &\to \text{in}_< A \\ z_i &\mapsto \text{in}_<(b_i). \end{aligned}$$

The kernel of $\varphi_{\mathcal{A}}$ is a toric ideal, which is denoted by $I_{\mathcal{A}}$.

The ideal $I_{\mathcal{A}}$ contains information about the lead ideal of $I_{\mathcal{B}}$. To make this relation between $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ concrete, let $\mathcal{A}$ also denote the $d \times r$ matrix where the ith column is the exponent vector of $\text{in}_<(b_i)$, and choose a weight vector w such that w preserves $<$ on $\mathcal{B}$. Then $\mathcal{A}^\top w$ is a weight vector in $\mathbb{R}^r$, and consequentially $\mathcal{A}^\top w$ defines a partial monomial order $<_{\mathcal{A}^\top w}$ on $\mathbb{C}[z_1, \dots, z_r]$. This partial monomial order can be used to define leading terms of polynomials in $I_{\mathcal{B}}$. As $\mathcal{A}^\top w$ defines a partial order, a polynomial in $I_{\mathcal{B}}$ may have more than one leading term with respect to $<_{\mathcal{A}^\top w}$. Specifically, if $f \in I_{\mathcal{B}}$, then $\text{in}_{\mathcal{A}^\top w}(\sum_{\alpha \in \text{supp}(f)} c_\alpha z^\alpha) = \sum_{\gamma \in \Gamma} c_\gamma z^\gamma$ where Γ is all exponent vectors in f which maximize the dot product with $\mathcal{A}^\top w$. That is, $\Gamma = \text{argmax}_{\alpha \in \text{supp}(f)}\{\alpha \cdot \mathcal{A}^\top w \mid c_\alpha \neq 0\}$. The following establishes a criterion for a set of subalgebra generators to be a SAGBI basis.

Proposition 3.3.2 ([6])**.** *Using the notation in the proceeding paragraph, a set $\mathcal{B} \subset \mathbb{C}[x_1, \dots, x_n]$ is a SAGBI basis for the subalgebra generated by $\mathcal{B}$ if and only if $\text{in}_{\mathcal{A}^\top w}(I_{\mathcal{B}}) = I_{\mathcal{A}}$.*

Remark 3.3.3 ([6])**.** For any $\mathcal{B} \subset \mathbb{C}[x_1, \dots, x_n]$, $\text{in}_{\mathcal{A}^\top w}(I_{\mathcal{B}}) \subset I_{\mathcal{A}}$.

The following corollary allows for an algorithmic approach for generating SAGBI bases.

Corollary 3.3.4 ([6])**.** *Let $p_1, \ldots, p_s$ be generators for $I_{\mathcal{B}}$. Then $\mathcal{B} = \{b_1, \ldots, b_r\}$ is a SAGBI basis for the subalgebra it generates if and only if Algorithm 3.3.1 reduces $p_i(b_1, \ldots, b_r)$ to a constant for all $i \in \{1, \ldots, s\}$.*

While finite SAGBI bases may not exist, and the existence of a finite SAGBI basis for a given subalgebra is generally not known, one may use Corollary 3.3.4 to create algorithms for computing finite SAGBI bases, if they exist. One implementation is the `SubalgebraBases` package for `Macaulay2` [8, 5]. This implementation has a built-in termination limit that the user can change in the case of large or infinite SAGBI bases.

3.4 Khovanskii bases

SAGBI bases were originally introduced independently by Kapur and Medlener in 1989 [9] and Robbiano and Sweedler in 1990 [7]. They have since been generalized by the notion of Khovanskii bases, which were introduced by Kaveh and Manon in 2016 [10]. The main difference with Khovanskii bases is that $\mathbb{Z}^d$-valuations replace the monomial orders of SAGBI bases. This adjustment gives the opportunity to consider subalgebras of algebras other than the polynomial ring. This section will consider subalgebras of $\mathbb{C}(X)$, where X is a variety of dimension d. The notation $A^\times$ denotes $A \setminus \{0\}$, and, in particular, $\mathbb{C}(X)^\times := \mathbb{C}(X) \setminus \{0\}$ is a multiplicative group.

Following [11], let $\succ$ be a total ordering on $\mathbb{Z}^d$ so that $\mathbb{Z}^d$ is an ordered abelian group under addition. A $\mathbb{Z}^d$-valuation on $\mathbb{C}(X)$ is a surjective group homomorphism $\nu \colon \mathbb{C}(X)^\times \to \mathbb{Z}^d$ such that for all $f, g \in \mathbb{C}(X)$ and $c \in \mathbb{C}^\times$,

$$\nu(f+g) \succeq \min\{\nu(f), \nu(g)\} \text{ and } \nu(c) = 0.$$

By convention, $\nu(0) = \infty$, $\infty \succeq \alpha$, and $\alpha + \infty = \infty$ for all $\alpha \in \mathbb{Z}^d$. Since $\dim X = d$, ν is a surjection, and $\mathbb{C}$ is algebraically closed, one can conclude that if $f, g \in \mathbb{C}(X)^\times$ with $\nu(f) = \nu(g)$, then there is a unique $c \in \mathbb{C}^\times$ with $\nu(f - cg) \succ \nu(f)$.

Example 3.4.1. Every monomial order $<$ defines a valuation. In this case, $\nu : \mathbb{C}[X]^\times \to \mathbb{Z}^d$ and for $f \in \mathbb{C}[X]$ with lead monomial $\text{in}_<(f) = c_\alpha x^\alpha$, the valuation of f is $\nu(f) = -\alpha$. △

The image of a subalgebra A under a $\mathbb{Z}^d$-valuation ν defines a monoid, often called the value semigroup of A and denoted $S(A, \nu)$. A subset $\mathcal{B} \subset A$ is a Khovanskii basis for A with respect to ν if the image of $\mathcal{B}$ under ν generates $S(A, \nu)$. As with SAGBI bases, some subalgebras may not have a finite Khovanskii basis, or the existence of a finite Khovanskii basis may depend upon the choice of valuation. Most notably, subalgebras may have a finite Khovanskii basis but no finite SAGBI basis.

Example 3.4.2 (Finite Khovanskii basis for alternating group [10, Example 7.7])**.** Recall the subalgebra of invariants of the alternating group in Example 3.3.1 did not have a finite SAGBI basis with respect to any term order. In contrast, there exist valuations such that the subalgebra generators, $x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3, (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, do form a finite Khovanskii basis for the subalgebra they generate. For details on such valuations, see Example 7.7 of [10]. $\triangle$

As finite Khovanskii bases may not exist, computing them is generally difficult. Kaveh and Manon give a method to compute a finite Khovanskii basis for V with respect to a valuation ν [10, Algorithm 2.18].

4. POLYTOPES

Convex geometry plays an important role in describing toric varieties and bounds on the number of solutions to polynomial systems. This chapter describes the necessary background for understanding bounds such as the mixed-volume bound of Bernstein's Theorem, as well as the basics on describing toric varieties with cones and fans. A more detailed treatment of convex geometry can be found in [12].

4.1 Polytopes

A subset $S \subset \mathbb{R}^n$ is convex if it contains every line segment between any two points in S. That is, S is convex if $\{\lambda p + (1-\lambda)q \mid \forall p, q \in S \text{ and } 0 \leq \lambda \leq 1\} \subset S$. The convex hull of S, denoted $\operatorname{conv}(S)$, is the smallest convex set containing S. Figure 4.1 gives an example of the convex hull of a finite set of points. A polytope is a convex set which is equal to the convex hull of a finite set of points.

The dimension of a polytope $\mathscr{P} \subset \mathbb{R}^n$ is the dimension of the smallest Euclidean space that can contain $\mathscr{P}$. More concretely, let

$$\operatorname{aff}(\mathscr{P}) := \{\lambda_1 p_1 + \cdots + \lambda_r p_r \mid p_i \in \mathscr{P}, \lambda_i \in \mathbb{R}, \lambda_1 + \cdots + \lambda_r = 1, \text{ and } r \in \mathbb{N}\}$$

be the affine hull or affine span of $\mathscr{P}$. Then $\dim(\mathscr{P}) = \dim(\operatorname{aff}(\mathscr{P}))$ [12].

A subset $\mathscr{Q} \subseteq \mathscr{P}$ is a face of $\mathscr{P}$ if there is a vector $v \in \mathbb{R}^n$ such that, for all $q \in \mathscr{Q}$,

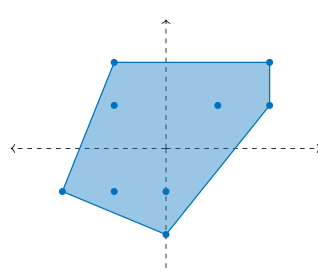


Figure 4.1: Convex hull of a finite set of points.

$q \cdot v = \min_{p \in \mathscr{P}}\{p \cdot v\}$, where $\cdot$ denotes the usual dot product. That is, $\mathscr{Q} = \operatorname{argmin}_{p \in \mathscr{P}}\{p \cdot v\}$. For a given v, the face exposed by v is $\mathscr{P}_v = \operatorname{argmin}_{p \in \mathscr{P}}\{p \cdot v\}$. Faces of dimensions $\dim \mathscr{P} - 1$, 1, and 0 are respectively facets, edges, and vertices of $\mathscr{P}$. A vector v exposing a facet $\mathscr{P}_v$ is called an inner normal of $\mathscr{P}_v$.

Lemma 4.1.1 ([13, Proposition 2.3]). *Given a polytope $\mathscr{P}$,*

1. *every face of $\mathscr{P}$ is a polytope, and*
2. *the nonempty intersection of two faces of $\mathscr{P}$ is also a face of $\mathscr{P}$.*

Given two polytopes $\mathscr{P}, \mathscr{Q} \subset \mathbb{R}^n$, their Minkowski sum is the polytope $\mathscr{P} + \mathscr{Q} = \{p + q \mid p \in \mathscr{P} \text{ and } q \in \mathscr{Q}\}$. Polytopes can also be scaled, that is for polytope $\mathscr{P}$ and $\lambda \geq 0$, $\lambda\mathscr{P} = \{\lambda p \mid p \in \mathscr{P}\}$ is also a polytope [13]. A polytope $\mathscr{P} \subset \mathbb{R}^n$ is normal if $(k\mathscr{P}) \cap \mathbb{Z}^n + (l\mathscr{P}) \cap \mathbb{Z}^n = (k + l)\mathscr{P} \cap \mathbb{Z}^n$ for all $k, l \in \mathbb{N}$. The n-dimensional Euclidean volume of a polytope $\mathscr{P} \subset \mathbb{R}^n$ is denoted $\operatorname{Vol}_n(\mathscr{P})$. Given a collection of polytopes $\mathscr{P}_1, \ldots, \mathscr{P}_n$ in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_n \geq 0$, the function $f(\lambda_1, \ldots, \lambda_n) = \operatorname{Vol}_n(\lambda_1\mathscr{P}_1 + \cdots + \lambda_n\mathscr{P}_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$ [12]. The coefficient of $\lambda_1 \cdots \lambda_n$ is called the mixed volume of $\mathscr{P}_1, \ldots, \mathscr{P}_n$, denoted $\operatorname{MV}(\mathscr{P}_1, \ldots, \mathscr{P}_n)$.

4.2 Subdivisions

Subdivisions of polytopes especially play an important role in computing roots and root counts for polynomial systems. The following definitions and notation are adapted from [14]. These definitions can be technical, but are demonstrated visually in Example 4.2.1.

Suppose $\mathcal{A}^\bullet = (\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(r)})$ is a collection of finite subsets of $\mathbb{R}^n$ whose union affinely spans $\mathbb{R}^n$. A cell of $\mathcal{A}^\bullet$ is an r-tuple $\mathcal{C}^\bullet = (\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(r)})$ such that each $\mathcal{C}^{(i)} \subset \mathcal{A}^{(i)}$ is nonempty. The concepts of convex hull, faces, and inner normals may be extended to cells. In particular, the convex hull of a cell is the convex hull of the sum of its components: $\operatorname{conv}(\mathcal{C}^\bullet) := \operatorname{conv}(\mathcal{C}^{(1)} + \cdots + \mathcal{C}^{(r)})$. A face of $\mathcal{C}^\bullet$ is a subcell $F = (F^{(1)}, \ldots, F^{(r)})$ such that some linear functional $\lambda \in (\mathbb{R}^n)^\vee$ attains its minimum over $\mathcal{C}^{(i)}$ at $F^{(i)}$ for $i = 1, \ldots, r$. Such a functional λ is an inner normal of

F. Let $|\mathcal{C}^\bullet| := |\mathcal{C}^{(1)}| + \cdots + |\mathcal{C}^{(r)}|$ denote the total number of points in $\mathcal{C}$. A subdivision of $\mathcal{A}^\bullet$ is defined as the following:

Definition 4.2.1. A subdivision of $\mathcal{A}^\bullet$ is a collection of cells $S^\bullet = \{\mathcal{C}_1^\bullet, \ldots, \mathcal{C}_m^\bullet\}$ satisfying the following:

1. $\dim(\operatorname{conv}\mathcal{C}_j^\bullet) = n$ for all $\mathcal{C}_j^\bullet$,
2. $\operatorname{conv}(\mathcal{C}_{j_1}^\bullet) \cap \operatorname{conv}(\mathcal{C}_{j_2}^\bullet)$ is a face of both $\operatorname{conv}(\mathcal{C}_{j_1}^\bullet)$ and $\operatorname{conv}(\mathcal{C}_{j_2}^\bullet)$ for all $\mathcal{C}_{j_1}^\bullet, \mathcal{C}_{j_2}^\bullet$ in $S^\bullet$,
3. $\bigcup_{j=1}^r \operatorname{conv}(\mathcal{C}_j^\bullet) = \operatorname{conv}(\mathcal{A}^\bullet)$.

A subdivision $S^\bullet$ is a mixed subdivision if it additionally satisfies:

4. $\sum_{i=1}^r \dim \mathcal{C}_j^{(i)} = n$ for all cells $\mathcal{C}_j^\bullet \in S^\bullet$.

A subdivision $S^\bullet$ is a fine mixed subdivision if $S^\bullet$ satisfies the stronger condition:

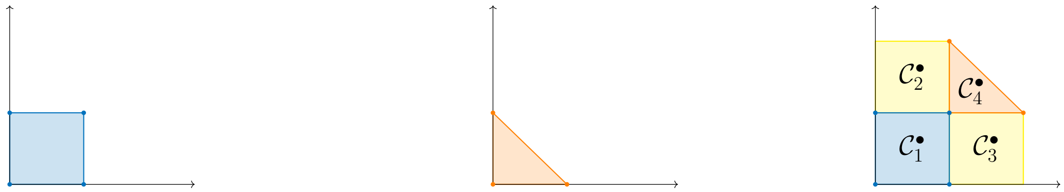
5. $\sum_{i=1}^r (|\mathcal{C}_j^{(i)}| - 1) = n$ for all cells $\mathcal{C}_j^\bullet \in S^\bullet$.

These definitions are best viewed through an example.

Example 4.2.1 (Mixed subdivision)**.** Let $\mathcal{A}^\bullet = (\mathcal{A}^{(1)}, \mathcal{A}^{(2)})$, where $\mathcal{A}^{(1)} = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\mathcal{A}^{(2)} = \{(0,0),(1,0),(0,1)\}$. Consider the polytopes $\mathscr{P}_1 = \operatorname{conv}(\mathcal{A}^{(1)})$, $\mathscr{P}_2 = \operatorname{conv}(\mathcal{A}^{(2)})$, and $\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2$. One mixed subdivision for $\mathcal{A}^\bullet$ consists of the four cells

$$\mathcal{C}_1^\bullet = (\{(0,0),(0,1),(1,0),(1,1)\}, \{(0,0)\}),$$
$$\mathcal{C}_2^\bullet = (\{(0,1),(1,1)\}, \{(0,0),(0,1)\}),$$
$$\mathcal{C}_3^\bullet = (\{(1,0),(1,1)\}, \{(0,0),(1,0)\}),$$
$$\mathcal{C}_4^\bullet = (\{(1,1)\}, \{(0,0),(0,1),(1,0)\}).$$

This gives a subdivision of $\mathscr{P}$, which is depicted in Figure 4.2c. $\triangle$



(a) Polytope $\mathscr{P}_1$. (b) Polytope $\mathscr{P}_2$. (c) Subdivision of $\mathscr{P}_1 + \mathscr{P}_2$.

Figure 4.2: Mixed subdivision from Example 4.2.1.

One way to construct subdivisions of a collection of finite subsets $\mathcal{A}^\bullet$ of $\mathbb{Z}^n$ is through functions. Let $\Gamma^{(i)} : \mathcal{A}^{(i)} \to \mathbb{R}$ be a real-valued function for each $i = 1, \ldots, r$. The r-tuple of functions $\Gamma^\bullet = (\Gamma^{(1)}, \ldots, \Gamma^{(r)})$ is called a lifting function on $\mathcal{A}^\bullet$. The graph of $\mathcal{A}^{(i)}$ under $\Gamma^{(i)}$, $\widehat{\mathcal{A}^{(i)}} = \{(a, \Gamma^{(i)}(a)) \in \mathbb{R}^{n+1} \mid a \in \mathcal{A}^{(i)}\}$, is the lift of $\mathcal{A}^{(i)}$. Extending the notation, let $\widehat{\mathcal{A}^\bullet} = (\widehat{\mathcal{A}^{(1)}}, \ldots, \widehat{\mathcal{A}^{(r)}})$.

Now let $S_{\Gamma^\bullet}$ be the set of cells $\mathcal{C}^\bullet$ in $\mathcal{A}^\bullet$ such that the lift $\widehat{\mathcal{C}^\bullet}$ of $\mathcal{C}^\bullet$ under $\Gamma^\bullet$ has the following properties:

1. $\dim(\operatorname{conv} \widehat{\mathcal{C}^\bullet}) = n$, and

2. $\widehat{\mathcal{C}^\bullet}$ is a face of $\widehat{\mathcal{A}^\bullet}$ whose inner normals $\alpha \in \mathbb{R}^{n+1}$ have positive last coordinate.

The convex hull of the lifted cells $\widehat{\mathcal{C}^\bullet}$ satisfying the second condition belong to the lower hull of $\operatorname{conv}(\widehat{\mathcal{A}^\bullet})$. That is, for any cell $\mathcal{C}^\bullet \in S_{\Gamma^\bullet}$, $\operatorname{conv} \widehat{\mathcal{C}^\bullet}$ is an n-dimensional face in the lower hull of $\operatorname{conv}(\widehat{\mathcal{A}^\bullet})$. The set $S_{\Gamma^\bullet}$ is a subdivision of $\mathcal{A}^\bullet$, called the subdivision induced by $\Gamma^\bullet$ [14, Lemma 2.6].

Example 4.2.2. The subdivision of Example 4.2.1 may be achieved through a lifting function. One function inducing this subdivision is $\Gamma^{\bullet} = (\Gamma^{(1)}, \Gamma^{(2)})$, where

$$\Gamma^{(1)}((0,0)) = 0 \qquad \Gamma^{(2)}((0,0)) = 0$$
$$\Gamma^{(1)}((0,1)) = 0 \qquad \Gamma^{(2)}((0,1)) = 1$$
$$\Gamma^{(1)}((1,0)) = 0 \qquad \Gamma^{(2)}((1,0)) = 1$$
$$\Gamma^{(1)}((1,1)) = 0.$$

$\triangle$

4.3 Cones

The following can be found in [15]. A convex cone is a set of points in $\mathbb{R}^n$ which is closed under addition and nonnegative scalar multiplication. A convex cone is a convex set. Cones other than $\{0\}$ are not compact. Given a finite set of points $S \subset \mathbb{R}^n$, the set S generates a convex polyhedral cone, $\operatorname{cone}(S) := \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\}$. Faces of cones may be defined similarly to faces of polytopes. That is, τ is a face of a cone σ, denoted $\tau \preceq \sigma$, if there is a vector $v \in \mathbb{R}^n$ such that $\tau = \operatorname{argmin}_{s \in \sigma}\{s \cdot v\}$, where $\cdot$ is again the usual dot product. In particular, for such a v and τ, $u \cdot v = 0$ for all $u \in \tau$ and v is said to expose the face τ.

Lemma 4.3.1 ([15]). *Given a cone σ,*

1. *Every face of σ is a cone.*
2. *The intersection of two faces of σ is also a face of σ.*

Like polytopes, the dimension of a cone $\sigma \subset \mathbb{R}^n$ is the dimension of the smallest linear subspace of $\mathbb{R}^n$ containing σ. Furthermore, faces of dimension $\dim \sigma - 1$ and those of dimension 1 are called facets and edges of σ, respectively. A vector v exposing a facet $\tau \prec \sigma$ is an inner normal of τ. Given a facet τ of a full dimensional cone σ, $\tau^{\perp}$ is the collection of all vectors normal to τ, i.e. $\tau^{\perp} := \{m \in \mathbb{R}^n \mid m \cdot u = 0 \text{ for all } u \in \tau\}$. The dual cone of a cone $\sigma \subset \mathbb{R}^n$ is $\sigma^{\vee} := \{m \in \mathbb{R}^n \mid m \cdot u \geq 0 \text{ for all } u \in \sigma\}$.

Proposition 4.3.2 ([15])**.** *The inner normals of a cone σ generate the dual cone $\sigma^\vee$.*

A cone σ is strongly convex if $\{0\}$ is a face of σ. Equivalently, $\sigma \cap (-\sigma) = \{0\}$, or $\dim \sigma^\vee = n$. A cone is rational if it is generated by a finite set $S \subset \mathbb{Z}^n$. The intersection of any edge ρ of σ with $\mathbb{Z}^n$ forms a monoid with a unique monoid generator. This unique monoid generator, denoted by u_ρ, generates the ray ρ and is called the minimal generator of ρ.

Lemma 4.3.3 ([15])**.** *A strongly convex rational cone is generated by the minimal ray generators of its edges.*

A cone is simplicial if its minimal generators are linearly independent over $\mathbb{R}$. For example, a full dimensional cone $\sigma \subset \mathbb{R}^n$ is simplicial if the number of minimal generators of σ is n.

4.4 Fans

As will be seen in Chapter 5, cones and collections of cones are foundational for describing normal toric varieties. A fan is a collection of cones with certain properties.

Definition 4.4.1 (Fan, [15])**.** A fan $\Sigma \subset \mathbb{R}^n$ is a finite collection of cones such that

1. every $\sigma \in \Sigma$ is a strongly convex rational cone,
2. for all $\sigma \in \Sigma$, each face of σ is also in Σ, and
3. for all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

Notationally, $\Sigma(k)$ denotes the set of all k-dimensional cones in Σ. A fan Σ is simplicial if every cone in Σ is simplicial. A fan is complete if the union of its cones is $\mathbb{R}^n$.

One may associate a complete fan to any full dimensional polytope $\mathscr{P} \subset \mathbb{R}^n$ [15]. Indeed, suppose that $\mathscr{P} \subset \mathbb{R}^n$ is a full dimensional lattice polytope. Recall that for each facet $F \subset \mathscr{P}$, $F = \operatorname{argmin}_{p \in \mathscr{P}}\{p \cdot u_F\}$, where u_F is an inner normal of F. Specifically, one can choose u_F such that it is the minimal generator of cone(u_F). As $\mathscr{P}$ is a lattice polytope, there is an integer a_F such that $-a_F = q \cdot u_F \leq p \cdot u_F$ for all $q \in F$ and $p \in \mathscr{P}$. Using this context, $\mathscr{P}$ has a representation

$$\mathscr{P} = \{p \in \mathbb{R}^n \mid \text{ for all facets } F \text{ of } \mathscr{P}, -a_F \leq p \cdot u_F\}.$$

The normal fan of $\mathscr{P}$, $\Sigma_{\mathscr{P}}$, consists of all cones σ_Q where Q is a face of $\mathscr{P}$ and $\sigma_Q = \text{cone}(u_F \mid Q \subset F)$. Note that each ray in $\Sigma_{\mathscr{P}}$ corresponds to a facet of $\mathscr{P}$.

Alternatively, one may equivalently define $\Sigma_{\mathscr{P}}$ by considering the vectors exposing a given face of $\mathscr{P}$. Specifically, each vector in $\mathbb{R}^n$ exposes exactly one face of $\mathscr{P}$. Let $\sim$ be an equivalence relation on the vectors in $\mathbb{R}^n$ such that $v_1 \sim v_2$ if v_1 and v_2 expose the same face of $\mathscr{P}$. The closure of each equivalence class of $\sim$ will form polyhedral cones. The collection of all of these polyhedral cones will also form the normal fan $\Sigma_{\mathscr{P}}$.

The following example gives a polytope with its corresponding normal fan $\Sigma_{\mathscr{P}}$.

(a) Polytope $\mathscr{P}$.

(b) Normal fan of $\mathscr{P}$ with black ray generators.

Figure 4.3: Polytope and its normal fan from Example 4.4.1.

Example 4.4.1 (Polytope with its normal fan)**.** Given the polytope $\mathscr{P} = \text{conv}((0,0),(1,0),(0,1))$, its normal fan is complete, with ray generators $(1,0)$, $(0,1)$, and $(-1,-1)$. $\triangle$

5. TORIC VARIETIES

Toric varieties appear naturally in algebraic geometry. Their combinatorial structure makes them ideal for computation and for examples. The d-dimensional algebraic torus is the group $(\mathbb{C}^{\times})^d := (\mathbb{C} \setminus \{0\})^d$ under the action of coordinate-wise multiplication. The d-dimensional algebraic torus is often referred to as simply a torus. Traditionally, a toric variety is an irreducible complex algebraic variety equipped with an action of a torus $\mathbb{T}$ such that X contains an isomorphic copy of $\mathbb{T}$ as an open dense orbit. There are several ways to construct toric varieties, including from monomial maps, cones, fans, and polytopes. This chapter reviews the basic constructions of toric varieties from maps and cones, and introduces the notation for toric varieties from fans and polytopes. Readers are referred [15, 16] for greater details.

5.1 Toric varieties from monomial maps

One may construct toric varieties, affine or projective, as the closure of a monomial map. To construct an affine toric variety from a monomial map, let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{Z}^d$ be such that $\mathbb{Z}\mathcal{A} = \mathbb{Z}^d$. This set is sometimes referred to as a configuration. The notation $\mathcal{A}$ is also used to denote the corresponding $d \times n$ matrix with columns $\alpha_1, \ldots, \alpha_n$. Using $\mathcal{A}$, one may embed $(\mathbb{C}^{\times})^d$ into $\mathbb{C}^n$ through the following map $\varphi_{\mathcal{A}}$:

$$\begin{aligned} \varphi_{\mathcal{A}} : (\mathbb{C}^{\times})^d &\to (\mathbb{C}^{\times})^n \subset \mathbb{C}^n \\ t &\mapsto (t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_n}) \end{aligned} \tag{5.1}$$

The Zariski closure of the image of $\varphi_{\mathcal{A}}$ defines a toric variety $X_{\mathcal{A}}$. That is, $X_{\mathcal{A}} := \overline{\varphi_{\mathcal{A}}((\mathbb{C}^{\times})^d)}$. The ideal of $X_{\mathcal{A}}$, denoted $I_{\mathcal{A}}$, is the ideal $\langle x^u - x^v \mid u, v \in \mathbb{N}^n, \mathcal{A}u = \mathcal{A}v \rangle$. This ideal $I_{\mathcal{A}}$ is also the kernel of the map

$$\begin{aligned} \mathbb{C}[x_1, \ldots, x_n] &\to \mathbb{C}[t_1^{\pm}, \ldots, t_d^{\pm}] \\ x_i &\mapsto t^{\alpha_i}. \end{aligned} \tag{5.2}$$

One can also use embeddings to define projective toric varieties. Given $\mathcal{A} = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{Z}^d$ with $\mathbb{Z}\mathcal{A} = \mathbb{Z}^d$, suppose that $\mathbb{1}$ is in the row span of $\mathcal{A}$. Then, the closure of the image of the following map defines a projective toric variety:

$$\begin{aligned} \varphi_{\mathcal{A}} : (\mathbb{C}^{\times})^d &\to \mathbb{P}^n \\ t &\mapsto [t^{\alpha_0} : t^{\alpha_1} : \cdots : t^{\alpha_n}]. \end{aligned} \tag{5.3}$$

In this projective case, $\varphi_{\mathcal{A}}$ is called a toric Kodaira map. Defining equations for this variety are computed similarly to the defining equations for the affine toric varieties.

Example 5.1.1 (Weighted projective space, Example 2.0.5 [15])**.** Weighted projective space can be embedded into projective space as a toric variety through a Kodaira map. For example, let $w = (1, 1, 2)$. Then $\mathbb{P}^2_w$ embeds into $\mathbb{P}^3$ via the following map:

$$\begin{aligned} \varphi_{\mathcal{A}} : \mathbb{P}^2_w &\to \mathbb{P}^3 \\ (t_0 : t_1 : t_2)_w &\mapsto [t_0^2 : t_0 t_1 : t_1^2 : t_2]. \end{aligned}$$

If $y_0, \ldots, y_3$ are the coordinates on $\mathbb{P}^3$, then the defining equation for $\mathbb{P}^2_w$ as a subvariety of $\mathbb{P}^3$ is $y_0 y_2 - y_1^2$. $\triangle$

One can also construct Kodaira maps for translated toric varieties. Let $\mathbb{T}$ be the dense torus of $\mathbb{P}^n$. The torus $\mathbb{T}$ acts on $\mathbb{P}^n$ by scaling each coordinate. That is, given a projective toric variety $X_{\mathcal{A}} \subset \mathbb{P}^n$ and $p = [p_0 : \cdots : p_n] \in \mathbb{T}$, the multiplication of p on $X_{\mathcal{A}}$ is the translated toric variety $p.X_{\mathcal{A}} := \{[p_0 x_0 : \cdots : p_n x_n] \mid [x_0 : \cdots : x_n] \in X\}$. The Kodaira map for the translated toric variety $p.X_{\mathcal{A}}$ is:

$$\begin{aligned} \varphi_{p,\mathcal{A}} : (\mathbb{C}^{\times})^d &\to \mathbb{P}^n \\ t &\mapsto [p_0 t^{\alpha_0} : p_1 t^{\alpha_1} : \cdots : p_n t^{\alpha_n}]. \end{aligned} \tag{5.4}$$

The ideal defining $p.X_{\mathcal{A}}$ is $\mathcal{I}_{p,\mathcal{A}} = \langle p^v x^u - p^u x^v \mid u, v \in \mathbb{N}^n, \mathcal{A}u = \mathcal{A}v \rangle$.

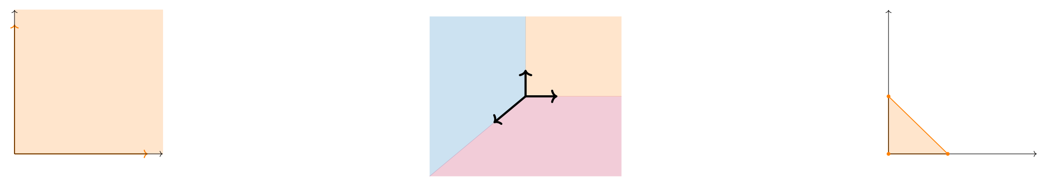
5.2 Toric varieties from combinatorial objects

Another common way to define toric varieties is through cones. Given the d-dimensional algebraic torus $\mathbb{T} = (\mathbb{C}^{\times})^d$, its character and cocharacter lattices are $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{C}^{\times}) \cong \mathbb{Z}^d$ and $N = \mathrm{Hom}(\mathbb{C}^{\times}, \mathbb{T}) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathbb{Z}^d$. Let $N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N$ and $M_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} M$. Given a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, the set $S_\sigma = \sigma^\vee \cap M$ forms a finitely generated semigroup and $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is an affine toric variety of dimension d. Furthermore, one can show that $\mathrm{Spec}(\mathbb{C}[S_\sigma]) \cong \mathrm{Hom}_{semi}(S_\sigma, \mathbb{C})$, where $\mathrm{Hom}_{semi}(S_\sigma, \mathbb{C})$ is the set of semigroup homomorphism from the additive semigroup S_σ to the multiplicative semigroup $\mathbb{C}$ [17].

Suppose that $\Sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^d$ is a fan. One may construct a toric variety X_Σ from Σ by considering all affine varieties U_σ from each cone $\sigma \in \Sigma$ and gluing along common faces. More specifically, if τ and σ are cones in Σ with τ a face of σ, then there is a natural inclusion $U_\tau \subset U_\sigma$. This inclusion is, in fact, functorial. One obtains a toric variety from Σ by gluing along these natural inclusions. As 0 is a cone in Σ, and contained in every cone in Σ, $U_0 \subset U_\sigma$ for all $\sigma \in \Sigma$. As $U_0 = \mathbb{T}$, X_Σ does indeed contain a dense torus [17]. Furthermore, using the equivalence to semigroup homomorphisms, $U_0 = \mathbb{T} \cong \mathrm{Hom}_{semi}(M, \mathbb{C}) \cong \mathrm{Hom}(M, \mathbb{C}^{\times})$. This can be used to show that $\mathbb{T}$ does indeed act on X_Σ, and, consequently, X_Σ is a toric variety [17]. Varieties constructed from a fan are normal [15, Theorem 3.1.5]. For details on this construction, see [15, 16].

A toric variety may also be constructed from a full dimensional normal polytope $\mathscr{P}$, see [15, 16]. When the vertices of the normal polytope $\mathscr{P}$ consist of integers, then the toric variety associated to $\mathscr{P}$ is isomorphic to the toric variety given by the normal fan of $\mathscr{P}$ (see Section 4.4). The work in Chapter 10 specifically constructs toric varieties from the normal fans of polytopes.

Example 5.2.1. Familiar spaces, such as $\mathbb{C}^d$ and $\mathbb{P}^d$, are toric varieties. For instance, the positive orthant in $\mathbb{R}^d$, along with all of its faces, is the cone associated to $\mathbb{C}^d$. The fan associated to $\mathbb{P}^d$ has ray generators given by the unit vectors $e_1, \ldots, e_d$ and the vector $-\mathbb{1}$. The standard d-simplex in $\mathbb{R}^d$ is a polytope which can also be used to construct $\mathbb{P}^d$. The 2-dimensional cases for these examples are depicted in Figure 5.1 $\triangle$



(a) Cone associated to $\mathbb{C}^2$. (b) Fan associated to $\mathbb{P}^2$. (c) Polytope associated to $\mathbb{P}^2$.

Figure 5.1: Combinatorial objects defining $\mathbb{C}^2$ and $\mathbb{P}^2$ as toric varieties.

5.3 Cox construction

This section describes normal toric varieties as almost geometric quotients. Basic definitions on (almost) geometric quotients can be found in Appendix A. Given a toric variety X_Σ, the goal is to find an integer k, base locus $Z \subset \mathbb{C}^k$, and reductive group $\mathbb{G}$ such that

$$X_\Sigma \cong (\mathbb{C}^k \setminus Z) /\!\!/ \mathbb{G}. \tag{5.5}$$

This construction provides a surjective parametrization of X_Σ by an open subset of $\mathbb{C}^k$. If Σ is simplicial, then the quotient is geometric and a point in X_Σ corresponds to a $\mathbb{G}$-orbit in $\mathbb{C}^k \setminus Z$. The familiar construction of $\mathbb{P}^d$, where $\mathbb{P}^d \cong \mathbb{C}^{d+1} \setminus \{0\} /\!\!/ \mathbb{C}^\times$, is an example of such a geometric quotient, see Example 5.3.1. Details on this construction are given in [15, 18], although earlier descriptions arose in the analytic category in [19]. The following summary can be found in [20].

Given the fan $\Sigma \subset \mathbb{R}^d \cong N_\mathbb{R}$, let $\Sigma(r)$ denote its r-dimensional cones and define $k = |\Sigma(1)|$. As discussed in Section 4.3, every $\rho_i \in \Sigma(1)$ has a unique minimal ray generator $u_i \in N$. Let u_i be the i^{th} column of the matrix $\mathbf{F} \in \mathbb{Z}^{d\times k}$, i.e. $\mathbf{F} = [u_1 \ \cdots \ u_k]$. This matrix $\mathbf{F}$ defines a lattice homomorphism $\mathbf{F} : N' \to N$, where $N' = \mathbb{Z}^k$. The next step is to construct a fan Σ' such that Σ' is the fan of $\mathbb{C}^k$ (i.e. the positive orthant in $\mathbb{R}^k$ and all its faces) with some cones missing such that $\mathbf{F}$ is compatible with $\Sigma' \subset N'_\mathbb{R}$ and $\Sigma \subset N_\mathbb{R}$. The lattice map $\mathbf{F} : N' \to N$ is compatible with Σ' and Σ if for every cone $\sigma' \in \Sigma'$ there exists a cone $\sigma \in \Sigma$ such that $\mathbf{F}_\mathbb{R}(\sigma') \subseteq \sigma$. Letting $\{e_\rho \mid \rho \in \Sigma(1)\}$ be the standard basis for $\mathbb{R}^{\Sigma(1)} \cong \mathbb{R}^k$, for each $\sigma \in \Sigma$ define $\tilde{\sigma} \subset \mathbb{R}^{\Sigma(1)}$ as $\operatorname{cone}(e_\rho \mid \rho \in \sigma(1))$. Let the fan Σ' be this collection of cones $\tilde{\sigma}$ with their faces, i.e. $\Sigma' = \{\tau \mid \tau \preceq \tilde{\sigma} \text{ for some } \sigma \in \Sigma\}$.

With $\mathbf{F}$ now compatible with Σ' and Σ, $\mathbf{F}$ defines a toric morphism $\pi : X_{\Sigma'} \to X_\Sigma$. Here, $X_{\Sigma'} = \mathbb{C}^k \setminus Z$, where Z is the union of coordinate subspaces corresponding to the missing cones. The affine space $\mathbb{C}^k$ is called the total coordinate space and Z is the base locus. The base locus is defined by the irrelevant ideal B of the total coordinate ring $S = \mathbb{C}[x_1, \ldots, x_k]$ of $\mathbb{C}^k$. This ideal B is a monomial ideal:

$$B = \left\langle \prod_{i \text{ s.t. } \rho_i \not\subset \sigma} x_i \mid \sigma \in \Sigma(d) \right\rangle \subset S \quad \text{and} \quad Z = \mathcal{V}_{\mathbb{C}^k}(B). \tag{5.6}$$

At this point, the remaining piece needed to define X_Σ as an almost geometric quotient is finding the appropriate reductive group $\mathbb{G}$. To find $\mathbb{G}$, first restrict the morphism π to the torus $(\mathbb{C}^\times)^k$. This will give the group homomorphism

$$\pi|_{(\mathbb{C}^\times)^k} = F \otimes_{\mathbb{Z}} \mathbb{C}^\times : (\mathbb{C}^\times)^k \to (\mathbb{C}^\times)^d \tag{5.7}$$

where $(z_1, \ldots, z_k) \overset{\pi}{\mapsto} (z^{\mathbf{F}_{1,:}}, \ldots, z^{\mathbf{F}_{n,:}})$, with $\mathbf{F}_{i,:}$ denoting the i^{th} row of $\mathbf{F}$. The kernel of $\pi|_{(\mathbb{C}^\times)^k}$ is a subgroup $\mathbb{G} \subset (\mathbb{C}^\times)^k$ which acts on $\mathbb{C}^k \setminus Z$ where the morphism π is constant on $\mathbb{G}$-orbits. The integer k, set Z, and group $\mathbb{G}$ discussed here are sufficient to define X_Σ as an almost geometric quotient, as described in the following theorem.

Theorem 5.3.1 ([18])**.** *The morphism $\pi : \mathbb{C}^k \setminus Z \to X_\Sigma$ coming from $\mathbf{F} = [u_1 \ \cdots \ u_k]$ is an almost geometric quotient for the action of $\mathbb{G}$ on $\mathbb{C}^k \setminus Z$. Moreover, the subset $U \subset X_\Sigma$ for which $\pi|_{\pi^{-1}(U)}$ is a geometric quotient is given by $X_{\Sigma_0} \subset X_\Sigma$, where $\Sigma_0 \subset \Sigma$ is the subfan of all simplicial cones of Σ. Furthermore, $(X_\Sigma \setminus U)$ has codimension at least 3 in X_Σ.*

Constructing toric varieties as almost geometric quotients can be seen as a generalization of the construction of projective space.

Example 5.3.1 (Projective space)**.** Projective space is readily constructed as a toric variety from a fan (see Example 5.2.1) and a geometric quotient. The quotient map for $\mathbb{P}^d$ is given by $\pi : \mathbb{C}^{d+1} \to \mathbb{P}^d$, where $(x_1, \ldots, x_{d+1}) \mapsto [x_1 : \cdots : x_{d+1}]$. The reductive group $\mathbb{G}$ in this case is $\mathbb{C}^\times$, where $\mathbb{C}^\times$

acts by scalar multiplication. Furthermore, because the fan for $\mathbb{P}^d$ is simplicial, π is a geometric quotient, i.e. $\mathbb{P}^d = \mathbb{C}^{d+1} \setminus \{0\} /\!\!/ \mathbb{C}^\times$. $\triangle$

5.4 Divisors on toric varieties

This subsection recalls basic definitions and results for divisors on normal toric varieties. For more information, see [15, 16].

Suppose X_Σ is a d-dimensional toric variety with fan Σ, dense torus $\mathbb{T}$, character lattice M, and cocharacter lattice N. A prime divisor of X_Σ is an irreducible subvariety of X_Σ of dimension $d-1$. The free abelian group generated by the prime divisors of X_Σ is denoted by $\mathrm{Div}(X_\Sigma)$. An element of $\mathrm{Div}(X_\Sigma)$ is called a Weil divisor. The set $\mathrm{Div}_{\mathbb{T}}(X_\Sigma) \subset \mathrm{Div}(X_\Sigma)$ is the subset of all prime divisors of X_Σ which are invariant under the action of $\mathbb{T}$. For each $\rho \in \Sigma(1)$, the subvariety $D_\rho \subset X_\Sigma$ corresponding to ρ is an element of $\mathrm{Div}_{\mathbb{T}}(X_\Sigma)$. Furthermore, $\mathrm{Div}_{\mathbb{T}}(X_\Sigma)$ is a free abelian group with basis $\{D_\rho \mid \rho \in \Sigma(1)\}$ [15, 16].

One may construct a Weil divisor from a nonzero rational function on X_Σ. For $f \in \mathbb{C}(X_\Sigma)^\times$ and D a prime divisor of X_Σ, let $\nu_D(f)$ denote the order of vanishing of f along D. Then the divisor of f is defined as

$$\mathrm{div}(f) := \sum_{D \in \mathrm{Div}(X_\Sigma)} \nu_D(f) D.$$

Such a divisor is called a principal divisor. The set of all principal divisors on X_Σ forms a group $\mathrm{Div}_0(X_\Sigma)$. Two Weil divisors D_1, D_2 are linearly equivalent if $D_1 - D_2 \in \mathrm{Div}_0(X_\Sigma)$.

Example 5.4.1 ([15]). Characters of $\mathbb{T}$ are rational functions on X_Σ. Given $m \in M$, one can compute the divisor of the character χ^m. If u_ρ is the minimal generator of a ray ρ, then the divisor of χ^m is

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho,$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing of M and N and corresponds with the dot product in $\mathbb{Z}^d$ [15, Proposition 4.1.2]. $\triangle$

A Weil divisor is a Cartier divisor if it is locally principal. That is, D is Cartier if there is an open cover $\{U_i\}_{i\in I}$ of X_Σ such that $D|_{U_i}$ is principal in U_i for all $i \in I$. The set of Cartier divisors on X_Σ forms a group $\mathrm{CDiv}(X_\Sigma)$. The subgroup of Cartier divisors which are $\mathbb{T}$-invariant is denoted by $\mathrm{CDiv}_{\mathbb{T}}(X_\Sigma)$.

Example 5.4.2 ([15, Proposition 4.2.2])**.** For a strongly convex cone $\sigma \subset N_{\mathbb{R}}$, every $\mathbb{T}$-invariant Cartier divisor on U_σ is the divisor of a character. △

Every principal divisor is Cartier, which means $\mathrm{Div}_0(X_\Sigma) \subset \mathrm{CDiv}(X_\Sigma) \subset \mathrm{Div}(X_\Sigma)$. The class group of X_Σ is $\mathrm{Cl}(X_\Sigma) := \mathrm{Div}(X_\Sigma)/\,\mathrm{Div}_0(X_\Sigma)$ and the Picard group of X_Σ is $\mathrm{Pic}(X_\Sigma) := \mathrm{CDiv}(X_\Sigma)/\,\mathrm{Div}_0(X_\Sigma)$.

Proposition 5.4.1 ([15, Theorem 4.1.3])**.** *Let Σ be a fan such that $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_{\mathbb{R}}$. The sequence*

$$0 \to M \xrightarrow{f} \mathrm{Div}_{\mathbb{T}}(X_\Sigma) \xrightarrow{g} \mathrm{Cl}(X_\Sigma) \to 0$$

is exact, where f is the map $m \mapsto \mathrm{div}(\chi^m)$ and the map g sends a $\mathbb{T}$-invariant divisor to its image in $\mathrm{Cl}(X_\Sigma) = \mathrm{Div}(X_\Sigma)/\,\mathrm{Div}_0(X_\Sigma)$.

Proposition 5.4.2 ([15, Theorem 4.2.1])**.** *Let Σ be a fan such that $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_{\mathbb{R}}$. The sequence*

$$0 \to M \xrightarrow{f} \mathrm{CDiv}_{\mathbb{T}}(X_\Sigma) \xrightarrow{g} \mathrm{Pic}(X_\Sigma) \to 0,$$

is exact, where f is the map $m \mapsto \mathrm{div}(\chi^m)$ and the map g sends a $\mathbb{T}$-invariant divisor to its image in $\mathrm{Pic}(X_\Sigma) = \mathrm{CDiv}(X_\Sigma)/\,\mathrm{Div}_0(X_\Sigma)$.

One can also construct divisors from normal polytopes. Suppose $\Sigma = \Sigma_{\mathscr{P}}$ is the normal fan of a full dimensional normal polytope $\mathscr{P} \subset M_{\mathbb{R}}$. For each facet F of $\mathscr{P}$, let u_F denote the inner normal of F such that u_F is also the minimal integer generator of the ray $\rho_F \in \Sigma_{\mathscr{P}}$. As u_F exposes F, there is an $a_F \in \mathbb{Z}$ such that $q \cdot u_F = -a_F$ for all $q \in F$. The collection of all rays in $\Sigma_{\mathscr{P}}$ is $\{\rho_F \mid F \text{ is a facet of } \mathscr{P}\}$. Denote the corresponding collection of $\mathbb{T}$-invariant prime divisors by $\{D_F \mid F \text{ is a facet of } \mathscr{P}\}$. Summing over the facets of $\mathscr{P}$, the divisor $D_{\mathscr{P}} := \sum_F a_F D_F$ is a $\mathbb{T}$-invariant Cartier divisor on $X_{\Sigma_{\mathscr{P}}}$ [15, Proposition 4.2.10].

6. BOUNDS ON THE NUMBER OF SOLUTIONS TO NONLINEAR SYSTEMS

Given a set of d homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_d]$, Bézout's Theorem states that if the number of solutions to the polynomial system in $\mathbb{P}^d$ is finite, then the number of solutions, when counted with multiplicity, is equal to the product of the degrees of the polynomials [21]. Consequently, the number of isolated solutions in $\mathbb{C}^d$ to any system of d polynomials from $\mathbb{C}[x_1, \ldots, x_d]$ is *at most* the product of the degrees of the polynomials. This bound on the number of isolated solutions is known as the Bézout bound [21]. Many polynomial systems of interest, however, have a sparse algebraic structure, which results in fewer solutions than the Bézout bound. Bounds which consider the sparse algebraic structure of the polynomial system frequently give better estimates for the number of isolated solutions, see [21, 22, 23].

This chapter discusses some of the well-known bounds on the number of isolated solutions to a system of equations. These bounds each hold generally with respect to the algebraic structure considered. That is, the algebraic structure of systems considered here defines families of systems, and a system in the family is generic or general with respect to a particular bound if all of its solutions are nondegenerate (see Section 9.1) and the number of solutions equals the considered bound. These families of systems depend on parameters, and there is a Zariski-open subset U of the parameter space such that all polynomial systems arising from U are general. Further discussions on these bounds with proofs and examples can be found in [21, 22, 23]. Part of the discussion on Newton-Okounkov bodies can be found in [11]*.

6.1 Sparse polynomials

The monomials appearing in a polynomial system play an important role in the number of solutions to that system. A sparse polynomial is a polynomial which is described by its monomial support. One may denote the monomial support of a sparse polynomial by its list $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset$

*Reprinted from "Numerical homotopies from Khovanskii bases" by M. Burr, F. Sottile, and E. Walker, to appear in *Mathematics of Computation* and published by the American Mathematical Society.

$\mathbb{Z}^d$ of exponent vectors of its nonzero terms. Given a polynomial $f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$, the Newton polytope of f is the convex hull of its exponent vectors. That is, $\mathrm{Newt}(f) = \mathrm{conv}\{\alpha \mid c_\alpha \neq 0\}$. Polyhedral bounds on the number of isolated solutions to a polynomial system are described in terms of the Newton polytopes of the polynomials [22, 23].

Fix a set $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\} \subset \mathbb{Z}^d$ of n vectors in $\mathbb{Z}^d$. The following theorem considers the number of solutions to a generic polynomial system $f_1 = \cdots f_d = 0$ where each polynomial $f_i \in \mathbb{C}[x_1, \dots, x_d]$ has the same Newton polytope, $\mathrm{Newt}(f_i) = \mathrm{conv}(\mathcal{A})$, and $\mathrm{supp}(f_i) \subset \mathcal{A}$. In this setting, $\mathcal{A}$ defines a family of polynomial systems where the coefficients of $x^{\alpha_1}, \dots, x^{\alpha_n}$ in $f_1, \dots, f_d$ are the family parameters. Let $\mathbb{C}^{\mathcal{A}_\bullet}$ denote this parameter space. There is a Zariski-open subset $U \subset \mathbb{C}^{\mathcal{A}_\bullet}$ such that every polynomial system with coefficients from U is generic and thus has the same number of solutions. That number of solutions to such a generic system is given in the following theorem.

Theorem 6.1.1 (Kushnirenko's Theorem [24])**.** *If $\{f_1, \dots, f_d\} \subset \mathbb{C}[x_1, \dots, x_d]$ is a generic set of sparse polynomials with support $\mathcal{A}$, then the number of isolated solutions to $f_1 = \cdots = f_d = 0$ in $(\mathbb{C}^\times)^d$ is $d!\,\mathrm{Vol}_d(\mathrm{conv}(\mathcal{A}))$. Alternatively, if $\{f_1, \dots, f_d\}$ is not generic, then the number of isolated solutions in $(\mathbb{C}^\times)^d$ is at most $d!\,\mathrm{Vol}_d(\mathrm{conv}(\mathcal{A}))$.*

Example 6.1.1. Consider polynomials f_1, f_2 with support $\mathcal{A} = \{1, x, y, x^2, y^2\}$. Both f_1 and f_2 have the same Newton polytope $\mathscr{P}$, which is displayed in Figure 6.1. The volume of $\mathscr{P}$ is 2. Thus by Theorem 6.1.1, the number of isolated solutions to $f_1 = f_2 = 0$ is at most $2!\,\mathrm{Vol}_2(\mathscr{P}) = 4$. △

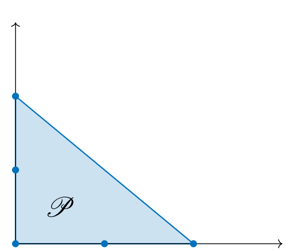


Figure 6.1: Newton polytope of f_1, f_2 from Example 6.1.1.

(a) $\mathrm{Newt}(f_1)$ from Example 6.1.2 (b) $\mathrm{Newt}(f_2)$ from Example 6.1.2

Figure 6.2: Newton polytopes of f_1, f_2, respectively, from Example 6.1.2.

In Kushnirenko's Theorem, every polynomial has the same Newton polytope. The mixed version of Kushnirenko's Theorem, where polynomials in a system can have different Newton polytopes, was proved by Bernstein in 1975 [25].

Theorem 6.1.2 (Bernstein's Theorem [25])**.** *If $\{f_1, \ldots, f_d\} \subset \mathbb{C}[x_1, \ldots, x_d]$ is a generic set of sparse polynomials with respective supports $\mathcal{A}_1, \ldots, \mathcal{A}_d$, then the number of isolated solutions to $f_1 = \cdots = f_d = 0$ in $(\mathbb{C}^\times)^d$ is $\mathrm{MV}(\mathrm{conv}(\mathcal{A}_1), \ldots, \mathrm{conv}(\mathcal{A}_d)) = \mathrm{MV}(\mathrm{Newt}(f_1), \ldots, \mathrm{Newt}(f_d))$. Alternatively, if $\{f_1, \ldots, f_d\}$ is not generic, then the number of isolated solutions in $(\mathbb{C}^\times)^d$ is at most $\mathrm{MV}(\mathrm{conv}(\mathcal{A}_1), \ldots, \mathrm{conv}(\mathcal{A}_d))$.*

The bound on the number of isolated solutions to a system $f_1 = \cdots = f_d = 0$ given in Theorem 6.1.2 is the mixed-volume bound. This bound is sharp. When the number of solutions to a system of polynomial equations is equal to its mixed-volume bound, then that system is said to be Bernstein-general.

Example 6.1.2. Consider polynomials f_1, f_2 which have supports $\mathcal{A}_1 = \{1, x, y, xy\}$ and $\mathcal{A}_2 = \{1, x^2y, xy^2\}$, respectively. The Newton polytopes of f_1 and f_2 are displayed in Figure 6.2. The mixed volume is $\mathrm{MV}(\mathrm{Newt}(f_1), \mathrm{Newt}(f_2)) = 4$. Thus by Theorem 6.1.2, the number of isolated solutions to $f_1 = f_2 = 0$ is at most 4. Note that this is strictly smaller than the bound given by Theorem 6.1.1. That is, if $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, then Theorem 6.1.1 would predict a bound of $2!\,\mathrm{Vol}_2(\mathrm{conv}(\mathcal{A})) = 5$. △

6.2 Birationally invariant intersection index

In [26, 27], Kaveh and Khovanskii generalize the polyhedral bounds found in Section 6.1. In particular, they describe the number of isolated solutions of systems of equations on a variety X, where systems are drawn from vector spaces of functions on X. Specifically, let X be an irreducible variety of dimension d and $V_1, \ldots, V_d \subset \mathbb{C}(X)$ be finite-dimensional vector spaces of rational functions on X. Let the set $U_V \subset X$ denote all the nonsingular points of X such that all functions in every V_i are defined on U_V. Then let $Z_V \subset U_V$ be the set of all points x in U_V where, for some i, all functions in V_i vanish at x. The birationally invariant intersection index, here referred to the intersection index, is the number of solutions in $U_V \setminus Z_V$ to a general system of equations $f_1 = \cdots = f_d = 0$ where $f_i \in V_i$. This intersection index is well-defined and is denoted by $[V_1, \ldots, V_d]$ [26, 27]. Generic systems with respect to the intersection index belong to a Zariski open subset of $V_1 \times \cdots \times V_d$, as described in the following proposition.

Proposition 6.2.1 ([26, Proposition 5.7])**.** *There is a proper subvariety $R \subset (V_1 \times \cdots \times V_d)$ such that for each d-tuple $(f_1, \ldots, f_d) \in (V_1 \times \cdots \times V_d) \setminus R$, the following hold:*

1. *The number of solutions of the system $f_1 = \cdots = f_d = 0$ in the set $U_V \setminus Z_V$ is independent of the choice of $f_1, \ldots, f_d$ and is equal to the intersection index $[V_1, \ldots, V_d]$.*

2. *Each solution $a \in U_V \setminus Z_V$ of the system $f_1 = \cdots = f_d = 0$ is nondegenerate.*

Example 6.2.1 ([27, Theorem 4.13, Theorem 4.14])**.** The intersection index generalizes both Bernstein's Theorem and Kushnirenko's Theorem. That is, let $X = (\mathbb{C}^\times)^d$ and consider when each V_i is the span of a finite set of monomials $\mathcal{A}_i$. Then $U_V = (\mathbb{C}^\times)^d$ and $Z_V = \emptyset$ as $Z_V \subset (\mathbb{C}^d) \setminus (\mathbb{C}^\times)^d$. Then $[V_1, \ldots, V_d] = \mathrm{MV}(\mathrm{conv}(\mathcal{A}_1), \ldots, \mathrm{conv}(\mathcal{A}_d))$. Furthermore if $\mathcal{A}_1 = \cdots = \mathcal{A}_d$, then $[V_1, \ldots, V_d] = d! \,\mathrm{Vol}(\mathrm{conv}(\mathcal{A}_1))$. This furthermore shows that the generic systems with respect to Bernstein's Theorem or Kushnirenko's Theorem belong to a Zariski open subset of $V_1 \times \cdots \times V_d$, as per Proposition 6.2.1. △

For vector spaces which are not the finite span of monomials, the intersection index can be difficult to compute. Kaveh and Khovanskii provide a formula for computing the self-intersection index, which is the intersection index $[V, \ldots, V]$ when $V = V_1 = \cdots = V_d$. Their formula requires the computation of a convex body associated to V called a Newton-Okounkov body. Newton-Okounkov bodies can be difficult to compute. The steps for defining a Newton-Okounkov body associated to a vector space V are given in Section 6.2.1.

6.2.1 Newton-Okounkov bodies

A Newton-Okounkov body is, in many ways, a generalization of the Newton polytope of a polynomial. In particular, a Newton-Okounkov body is a convex body associated to a vector space of rational functions on a variety X. The construction of a Newton-Okounkov body can be found in [27, 28], though the following summary of Newton-Okounkov bodies follows [27] and can be found in [11].

Let X be a variety of dimension d and let $V \subset \mathbb{C}(X)$ be a finite-dimensional vector space of rational functions on X. Let $\nu : \mathbb{C}(X)^{\times} \to \mathbb{Z}^d$ be a valuation. As in Section 3.4, the valuation is assumed to be surjective and $\mathbb{C}(X)^{\times} := \mathbb{C}(X) \setminus \{0\}$. Let $R(V)$ denote the graded ring $\bigoplus_{k \geq 0} V^k s^k$, where $V^k \subset \mathbb{C}(X)$ is the subspace spanned by all k-fold products of elements in V and s is a formal variable recording the grading. A nonzero element $f \in R(V)^{\times}$ is the sum of its homogeneous components,

$$f = f_k s^k + \cdots + f_1 s + f_0,$$

where $f_k \neq 0$ and $f_i \in V^i$ for all i. One may extend the valuation ν to $R(V)$ by defining $\nu(f) := (\nu(f_k), k) \in \mathbb{Z}^d \oplus \mathbb{N}$, and also extend $\succeq$ to $\mathbb{Z}^d \oplus \mathbb{N}$, where $(\alpha, k) \succ (\beta, l)$ if $k < l$ or else $k = l$ and $\alpha \succ \beta$ in the order on $\mathbb{Z}^d$. The direction of the inequality in $k < l$ is chosen to be consistent with $\nu(f) = (\nu(f_k), k)$ defining a valuation.

Let $S(V, \nu)$ denote the image $\{\nu(f) : f \in R(V)^{\times}\}$ of $R(V)^{\times}$ under ν. While this is a submonoid of $\mathbb{Z}^d \oplus \mathbb{N}$, $S(V, \nu)$ is commonly referred to as a value semigroup in the literature. The closure of the convex hull of $S(V, \nu)$ in $\mathbb{R}^d \times \mathbb{R}$ is the cone, $\text{cone}(V)$. Its base $\text{NO}_V := \text{cone}(V) \cap (\mathbb{R}^d \times \{1\})$ is the Newton-Okounkov body of V.

Example 6.2.2. This example was developed with Michael Burr in the preparation of [11]. Consider the vector space $V = \mathrm{span}_{\mathbb{C}}\{x^2, x-1, x^3+x^2+x+1\}$. Then $R(V) = \bigoplus_{k\geq 0} V^k s^k \cong \mathbb{C}[x^2\, s, (x-1)\, s, (x^3+x^2+x+1)\, s]$, where s is a formal grading variable. Consider the valuation on $R(V)$ where for $f = f_k s^k + \cdots + f_1 s + f_0$, $\nu(f) = (-\deg(f_k), k)$. The image of $R(V)$ under ν is the semigroup depicted in Figure 6.3. Taking the closure of the convex hull gives cone(V), which is the cone generated by $(0,1)$ and $(-3,1)$. Intersecting the cone at its base gives the Newton-Okounkov body NO_V as the line segment between $(-3,1)$ and $(0,1)$. $\triangle$

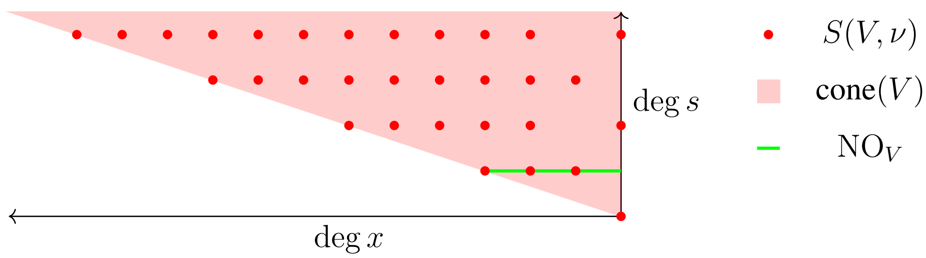


Figure 6.3: Value semigroup $S(V,\nu)$ (red points), cone(V) (shaded in pink), and Newton-Okounkov body NO_V (green line segment) from Example 6.2.2.

The Newton-Okounkov body carries a considerable amount of information about $R(V)$, see [27, 28]. In particular, when a system $f_1,\ldots,f_d$ is general in V, the number of solutions to $f_1 = \cdots = f_d = 0$ is equal to the normalized volume of the Newton-Okounkov body NO_V.

Theorem 6.2.2 ([27]). *If* $X = \mathrm{Proj}(R(V))$, *then* $[V,\ldots,V] = d!\,\mathrm{Vol}_d(\mathrm{NO}_V)$. *Alternatively, if* $X \neq \mathrm{Proj}(R(V))$, *then* $[V,\ldots,V] = d!\,\mathrm{Vol}_d(\mathrm{NO}_{\overline{V}})$, *where* $\mathrm{NO}_{\overline{V}}$ *denotes the Newton-Okounkov body of the integral closure of* $R(V)$.

The number given for $[V,\ldots,V]$ in Theorem 6.2.2 is referred to as the Newton-Okounkov body bound. One should note that the Newton-Okounkov body of $R(V)$ is not always equal to the Newton-Okounkov body of its integral closure, as demonstrated in the following example from [27].

Example 6.2.3 ([27, Example 4.12])**.** Let $X = \mathbb{C}$ with coordinate function z, $V = \text{span}\{1, z^2\}$, and choose $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}$ to be the order of vanishing at the point $z = 1$. First consider the order of vanishing at $z = 1$ of all the polynomials in V^1. Note that $1, z^2 \in V^1$ both do not vanish at $z = 1$, but $z^2 - 1 = (z-1)(z+1) \in V^1$ vanishes with order 1 at $z = 1$. No polynomial in V^1 will vanish with order greater than 1, so $NO_V = [0, 1]$. The integral closure of $R(V)$, however, has $z \in V^1$. Thus $(z-1)^2 \in V^1$, and so $\mathrm{NO}_{\overline{V}} = [0, 2]$. $\triangle$

The bound given in Theorem 6.2.2 can be tighter than Kushnirenko's Theorem, as demonstrated in Example 6.2.4.

Example 6.2.4. Consider a system F of two polynomials in $\mathbb{C}[x, y]$ where the support of every polynomial is given by $\mathcal{A} = \{1, x, y, x^2, y^2\}$. Further assume that the coefficients of x^2 and y^2 are equal for each polynomial in F. Example 6.1.1 gives the Newton polytope associated to $\mathcal{A}$ and the corresponding bound given by Kushnirenko's Theorem, which is 4. Note that F belongs to the vector space $V = \text{span}_{\mathbb{C}}\{1, x, y, x^2 + y^2\}$. Using the graded reverse lexicographic monomial term order on V and extending to a valuation on $R(V)$, the Newton-Okounkov body is the convex hull of the exponent vectors of the leading terms of $\{1, x, y, x^2 + y^2\}$. That is, $\mathrm{NO}_V = \mathrm{conv}((0,0), (1,0), (0,1), (2,0))$. One can check that $R(V)$ is integrally closed. Applying Theorem 6.2.2, the bound on the number of solutions to F when F belongs to V would be $2!\,\mathrm{Vol}(\mathrm{NO}_V) = 2 \cdot 1 = 2$. As both the mixed-volume bound and the bound given in Kushnirenko's Theorem are equal to 4, F is an example where the Newton-Okounkov body bound is tighter than the mixed-volume bound. Figure 6.4 displays the convex bodies associated to each bound. $\triangle$

(a) Convex hull of $\mathcal{A}$ from Example 6.2.4 (b) Newton-Okounkov body from Example 6.2.4

Figure 6.4: Convex bodies from Example 6.2.4.

7. FLAT FAMILIES AND DEGENERATIONS

A flat family of varieties over the complex numbers is a collection of varieties depending on a parameter $\tau \in \mathbb{C}$ such that the varieties vary continuously as the parameter τ moves continuously in $\mathbb{C}$. The convention $\mathbb{C}_\tau$ is used to denote the complex space with parameter τ. Flat families have many properties which are useful in homotopy continuation, as discussed in Chapter 11. The rigorous definition of flat families uses the concept of flat modules. This chapter reviews the definitions of flat families from the commutative algebra perspective, and relates these concepts back to the applied algebraic geometry context. One particular instance of flat families, namely toric degenerations, is discussed. For details on flat families, see [2]. Some of the examples and remarks in this chapter can be found in [11]*.

7.1 Flat families

Let R be a ring.

Definition 7.1.1 (Flat module)**.** An R-module F is flat if for every injective map of R-modules $M' \to M$, the induced map $F \otimes_R M' \to F \otimes_R M$ is also injective.

Proposition 7.1.2. *Suppose F is an R-module. If for every short exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$ the induced sequence $0 \to F \otimes_R M' \to F \otimes_R M \to F \otimes_R M'' \to 0$ is exact, then F is flat.*

Proof. Due to right exactness of $F \otimes_R -$, the sequence $F \otimes_R M' \to F \otimes_R M \to F \otimes_R M'' \to 0$ is an exact sequence. Thus if $0 \to F \otimes_R M' \to F \otimes_R M \to F \otimes_R M'' \to 0$ is exact, then in particular $F \otimes_R M' \to F \otimes_R M$ is injective, which implies F is flat by definition. □

Proposition 7.1.3 ([2])**.** *Every free module is flat.*

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Flat $\mathbb{C}[\tau]$-modules describe flat families of varieties over $\mathbb{C}_\tau$. That is, the points $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{C}_\tau$ (or $\mathcal{X} \subset \mathbb{C}^n \times \mathbb{C}_\tau$) with map $\pi : \mathcal{X} \to \mathbb{C}_\tau$ form a flat family of varieties over $\mathbb{C}_\tau$ if the parametrized coordinate ring representing $\mathcal{X}$, e.g. $\mathbb{C}[\tau][x_1, \ldots, x_n]/I_\tau$ where $\mathcal{V}(I_\tau) = \mathcal{X}$, is a flat $\mathbb{C}[\tau]$-module. For any $p \in \mathbb{C}_\tau$, the preimage $\pi^{-1}(p)$ is a fiber of $\mathcal{X}$. To emphasize that a fiber is one of the varieties in the family $\mathcal{X}$, fibers are frequently denoted as $\mathcal{X}_p$.

Example 7.1.1 (Flat family [2, pg. 157])**.** Consider the set of points $\mathcal{X} = \mathcal{V}(x^2 - \tau) \subset \mathbb{C}_x \times \mathbb{C}_\tau$. Let $\pi : \mathcal{X} \to \mathbb{C}_\tau$ be the projection onto the second coordinate. The parametrized coordinate ring representing $\mathcal{X}$ is $\mathbb{C}[\tau][x]/\langle x^2 - \tau\rangle$. Note that $\{1, x\}$ is a linearly independent set over $\mathbb{C}[\tau]$ which also spans $\mathbb{C}[\tau][x]/\langle x^2 - \tau\rangle$ over $\mathbb{C}[\tau]$. Thus $\mathbb{C}[\tau][x]/\langle x^2 - \tau\rangle$ is a free $\mathbb{C}[\tau]$-module with basis $\{1, x\}$, which implies $\mathbb{C}[\tau][x]/\langle x^2 - \tau\rangle$ is a flat $\mathbb{C}[\tau]$-module and $\mathcal{X}$ is a flat family over $\mathbb{C}_\tau$. The fiber over a point $p \in \mathbb{C}_\tau$ consists of the solutions in $\mathbb{C}_x$ to $x^2 - p$. $\triangle$

One reason why flat families are useful in numerical algebraic geometry is because in projective flat families, every fiber in the family has the same degree and dimension [29, Proposition III-56]. This fact leads to useful homotopy algorithms, as seen in Section 11.2. This utility of flat families motivates the question: given a variety X, is there a flat family of varieties $\pi : \mathcal{X} \to \mathbb{C}_\tau$ such that X belongs to $\mathcal{X}$ as a fiber of π? One way to construct such a flat family containing X is through a weight degeneration, see Example 7.2.1. In particular, a weight degeneration may be used to construct a toric degeneration, which is a particular type of flat family. The following section, Section 7.2, describes weight degenerations and when they lead to toric degenerations. For more information, see [2, Chapter 15.8].

7.2 Toric degenerations

One specific kind of flat family is a toric degeneration.

Definition 7.2.1 (Toric degeneration)**.** Let X be a projective variety. A surjective map $\pi : \mathcal{X} \to \mathbb{C}_\tau$ is a toric degeneration of X if

1. $\pi : \mathcal{X} \to \mathbb{C}_\tau$ defines a flat family of projective varieties $\mathcal{X}$,

2. the fiber $\mathcal{X}_1$ is equal to X, and

3. the fiber $\mathcal{X}_0$ is a toric variety.

The following examples and remarks are excerpts from Section 1.4 of [11].

Example 7.2.1 ([11, Example 6]). Weight degenerations induced by a $\mathbb{C}^\times$-action on $\mathbb{P}^n$ are a source of toric degenerations. Anderson [30] constructs a toric weight degeneration given a Khovanskii basis. The SAGBI homotopy [31] is also based on a toric weight degeneration.

As per [2, Section 15.8], one may construct flat families from $\mathbb{C}^\times$-actions in the following manner. Let $w \in \mathbb{Z}^{n+1}$ be a weight and define an action of the torus $\mathbb{C}^\times$ on $\mathbb{P}^n$ by

$$(x, \tau) \in \mathbb{P}^n \times \mathbb{C}^\times \mapsto \tau.x := [x_0\tau^{-w_0}, \ldots, x_n\tau^{-w_n}] \in \mathbb{P}^n.$$

The dual action on functions is $\tau.f(x) := f(\tau^{-1}.x)$, and it induces an action on polynomials. For a polynomial $f = \sum c_\alpha x^\alpha$,

$$\tau.\left(\sum c_\alpha x^\alpha\right) = \sum c_\alpha x^\alpha \tau^{w\cdot\alpha}, \tag{7.1}$$

where $w \cdot \alpha$ is the usual dot product. (To compare this to [2, Section 15.8], let $w = -\lambda$.) Let $w(f)$ be the minimum value of $w \cdot \alpha$ for $c_\alpha \neq 0$. Then define

$$f_\tau := (\tau.f)\tau^{-w(f)} = f_w + \tau\, g, \tag{7.2}$$

where the initial form f_w of f is the sum of its terms $c_\alpha x^\alpha$ where $w \cdot \alpha = w(f)$, and g is a polynomial in the variables $\tau, x_0, \ldots, x_n$.

Let $X \subset \mathbb{P}^n$ be a projective variety with ideal I. Define $\mathcal{X}^w \subset \mathbb{P}^n \times \mathbb{C}$ to be the Zariski closure of the family of translates of X, thus

$$\mathcal{X}^w := \overline{\{(x, \tau) \in \mathbb{P}^n \times \mathbb{C}^\times : x \in \tau.X\}} \subset \mathbb{P}^n \times \mathbb{C}_\tau.$$

For $\tau \neq 0$, observe that $\mathcal{X}^w_\tau = \tau.X$ and has ideal $\langle f_\tau : f \in I\rangle$. The following result establishes the flatness of this family:

Proposition 7.2.2 ([2, Theorem 15.17])**.** *The family $\mathcal{X}^w \to \mathbb{C}_\tau$ is flat. The fiber at $\tau = 0$ is the scheme with ideal*

$$I_w \;=\; \langle f_w : f \in I \rangle.$$

The proof uses a Gröbner basis $\mathcal{G}$ for I with respect to a weighted term order $\leq$ with weight $-w$ so that f_w consists of the $\leq$-leading terms for f. The weight $-w$ is used instead of w because the leading terms in the weighted term order $\leq_\omega$ for $\omega \in \mathbb{Z}^{n+1}$ is the sum of terms with highest ω-weight, which is opposite the convention from valuations. This distinction will be important in Chapter 11.

The family $\mathcal{X}^w$ is defined by the polynomials $\mathcal{G}_\tau := \{g_\tau : g \in \mathcal{G}\}$. A Gröbner basis for I_w is obtained by setting $\tau = 0$ in $\mathcal{G}_\tau$. The scheme at $\tau = 0$ may be neither reduced nor irreducible. If this scheme is a toric variety, then the weight degeneration $\mathcal{X}^w \to \mathbb{C}_\tau$ is a toric degeneration. $\triangle$

Remark 7.2.3 ([11, Remark 8])**.** Homotopy algorithms using weight degenerations appearing in the literature include the homotopy for solving the Kuramoto equations [32] and the Gröbner homotopy [31]. In these examples, I_w is a square-free monomial ideal so that the special fiber is a union of linear spaces. Such degenerations can be handled by Algorithm 11.2.3, see Remark 11.2.6.

Example 7.2.2 (Weight degeneration)**.** Consider the system $F = \{y^2 - x\,z - w,\; x^3 - z\,w - 2\,x\,w + 4\,y\,w\} \subset \mathbb{C}[w,x,y,z]$. This system is also a Gröbner basis for the ideal it generates with respect to the graded reverse lexicographic term order. Let $w = [0,-1,-2,2]$. The weight degeneration of F is calculated via the following steps:

$$\begin{aligned}
\tau.(y^2 - x\,z - w) &= (y^2\,\tau^{-2} - x\,z\,\tau^{-2} - w\,\tau^2)\tau^{-\min\{-2,2\}}\\
&= y^2 - x\,z - w\,\tau^4\\
\tau.(x^3 - z\,w - 2\,x\,w + 4\,y\,w) &= (x^3 - z\,w - 2\,x\,w\,\tau^2 + 4\,y\,w\,\tau)\tau^{-\min\{0,1,2\}}\\
&= x^3 - z\,w - 2\,x\,w\,\tau^2 + 4\,y\,w\,\tau
\end{aligned}$$

Thus $F_\tau = \{y^2 - x\,z - w\,\tau^4, x^3 - z\,w - 2\,x\,w\,\tau^2 + 4\,y\,w\,\tau\}$ defines a flat family of varieties $\mathcal{V}(F_\tau) = \mathcal{X}$ such that $\mathcal{V}(F) = \mathcal{X}_1$. Furthermore, F_0 consists of binomials which generate a prime binomial ideal. Thus $\mathcal{X}_0$ is a toric variety and $\mathcal{X}$ is a toric degeneration of $\mathcal{V}(F)$. $\triangle$

Example 7.2.3 ([11, Example 9])**.** Algebraic statistics gives examples of toric degenerations [33] which do not come from weight degenerations. Let G be a graph with vertex set $[m] := \{1, \ldots, m\}$ and edge set $E \subset \binom{[m]}{2}$. For each $i \in [m]$, let a_i be a parameter. For each $\{i, j\} \in E$, let $x_{ij} = x_{ji}$ and define p_{ij} and p_{ji} via the formula

$$p_{ij} := x_{ij}(1 + a_i - \tau a_j).$$

These polynomials give a map $p\colon \mathbb{C}^{|E|} \times \mathbb{C}^m \times \mathbb{C}_\tau \to \mathbb{P}^{2|E|-1} \times \mathbb{C}_\tau$ whose image is the family $\mathcal{QS}$ of quasi-symmetry models. This family contains two known quasi-symmetry models, the Pearsonian quasi-symmetry model at $\tau = 1$ and the toric quasi-symmetry model at $\tau = 0$. Polynomials associated to cycles in G generate the ideal of the family $\mathcal{QS}$. In the proof of this fact, one step is to show that this family is flat.

The family of quasi-symmetry models when G is a 3-cycle is the family of hypersurfaces defined by the cubic

$$\begin{aligned} P := {} & (1 + \tau + \tau^2)(p_{12}p_{23}p_{31} - p_{21}p_{32}p_{13}) + \\ & \tau(p_{12}p_{23}p_{13} + p_{12}p_{32}p_{31} + p_{21}p_{23}p_{31} - p_{12}p_{32}p_{13} - p_{21}p_{23}p_{13} - p_{21}p_{32}p_{31}). \end{aligned} \tag{7.3}$$

The fiber $\mathcal{QS}_0$ at $\tau = 0$ is the toric variety defined by the binomial $p_{12}p_{23}p_{31} - p_{21}p_{32}p_{13}$.

The family of quasi-symmetry models $\mathcal{QS}$ for a graph is typically not a weight degeneration. In particular, the family defined in Equation (7.3) is not a weight degeneration. Indeed, consider P as a polynomial in the p_{ij} with coefficients in $\mathbb{C}[\tau]$. Then in each of the eight terms of P, exactly one of p_{ij} or p_{ji} occurs. This implies that these terms correspond to the vertices of a cube. For example, if p_{ij} is identified with 0 when $i < j$ and 1 otherwise, then the monomial $p_{12}p_{23}p_{31}$ is identified with the point $(0, 0, 1)$. Note that for any weight w, P_w consists of the sum of terms identified with some face of the cube. Since the polynomial defining $\mathcal{QS}_0$ corresponds to a diagonal of the cube, it is not of the form P_w, for any w. $\triangle$

8. TWO NUMERICAL ALGORITHMS

When exact solutions to differential equations or nonlinear systems are computationally expensive or otherwise unattainable, numerical analysis provides a means to compute numerical approximations of the solutions. This chapter discusses Newton's Method for estimating a solution to a nonlinear system, as well as the Forward Euler Method for approximating the solution to an initial value ODE problem. Throughout this chapter, considered systems are assumed to be square, meaning the number of equations is equal to the number of variables. Further details and algorithms can be found in [34].

8.1 Newton's Method

Let F be a square system of equations and x^* a root of F. The goal of Newton's Method is to produce a numerical approximation of x^* given an initial root estimate x_0, where x_0 is in some regard close to x^*. Newton's Method is based on a sequence of Newton-Raphson iterations, where each iteration produces a new estimate for x^*. Specifically, if DF is the Jacobian of F, then a Newton-Raphson iteration calculates the zero of the linear approximation of F at x_0 in the following manner:

Algorithm 8.1.1 (Newton-Raphson iteration)**.**
Input: F, a square system of equations, and x_i, an estimate for a zero of F.
Iteration: Compute $x_{i+1} := x_i - [DF(x_i)]^{-1}F(x_i)$.
Output: x_{i+1}, the zero of the linear approximation of F at x_i.

Given an initial estimate x_0 for the zero x^* of a system F, repeatedly applying Newton-Raphson iterations generates a sequence of estimates $(x_i)_{i\geq 0}$ for x^*, where the desired outcome is for $(x_i)_{i\geq 0}$ to converge to x^*. A sequence $(x_i)_{i\geq 0}$ converges quadratically to x^* if there is a constant $C > 0$ such that $||x_{i+1} - x^*|| \leq C||x_i - x^*||^2$ for all $i \geq 0$. If a given initial root estimate of F is within the neighborhood of quadratic convergence for x^*, then the sequence of Newton-Raphson iterates will converge quadratically to x^*, as described in the Proposition 8.1.2. Let $C^2(U)$ denote the space of all functions whose second derivative is continuous on $U \subset \mathbb{C}^n$ and $B_\delta(p) \subset \mathbb{C}^n$ denote the n-dimensional ball of radius $\delta > 0$ centered at $p \in \mathbb{C}^n$.

Proposition 8.1.2 (Theorem 1, Chapter 3.2 [34])**.** *Suppose $F \in C^2(B_\delta(x^*))$ for $x^* \in \mathbb{C}^n$ a root of F of multiplicity one and $\delta > 0$. Define a sequence starting with x_0 such that $x_{i+1} := x_i - [DF(x_i)]^{-1}F(x_i)$. Then there exists an $\epsilon > 0$ and a constant $C > 0$ such that if $x_0 \in B_\epsilon(x^*)$, then $||x_{i+1} - x^*|| \leq C||x_i - x^*||^2$ for all $i \geq 0$.*

Proof. Define $g(x) = x - [DF(x)]^{-1}F(x)$. Because $F(x^*) = 0$, one can check that $g(x^*) = x^*$ and $Dg(x^*) = 0$. The Taylor expansion of g centered at x^* is

$$g(x) = x^* + \frac{1}{2}D^{(2)}g(a)(x - x^*)^2 \tag{8.1}$$

for some a such that $||a - x^*|| < ||x - x^*||$. Evaluating Equation (8.1) at x_i gives:

$$\begin{aligned} g(x_i) &= x^* + \frac{1}{2}D^{(2)}g(a)(x_i - x^*)^2 \\ x_{i+1} - x^* &= \frac{1}{2}D^{(2)}g(a)(x_i - x^*)^2 \\ ||x_{i+1} - x^*|| &\leq C||x_i - x^*||^2 \text{ for some } C > 0. \end{aligned}$$

□

To summarize, provided that x^* is a root of multiplicity one and initial approximation x_0 is within the neighborhood of quadratic convergence for x^*, then the sequence of Newton-Raphson iterations beginning with x_0 will converge quadratically to x^*. This iterative method for estimating x^* is known as Newton's Method.

8.2 Euler's Method

This section discusses the Forward Euler Method for computing a numerical solution to an initial value ODE problem. That is, suppose that $y' = f(x, y)$ is smooth in a neighborhood of (x_0, y_0), where $y(x_0) = y_0$. The goal is to provide a numerical approximation of y in a neighborhood of x_0.

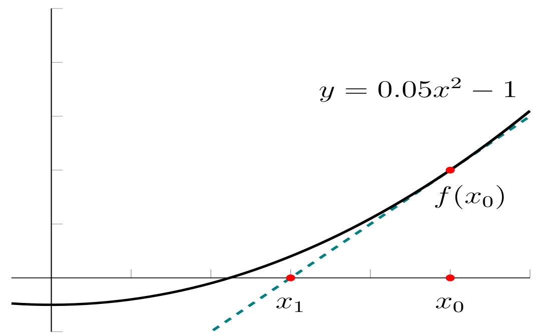


Figure 8.1: A geometric depiction of Newton's Method for the first iteration.

Specifically, given a step size $h > 0$ and positive integer N, the Forward Euler Method provides an approximation y_i of $y(x_0 + ih)$ for each $i = 1, \dots, N$.

Algorithm 8.2.1 (Forward Euler Method)**.**

Input: $y' = f(x, y)$ smooth in a neighborhood of (x_0, y_0), $y(x_0) = y_0$, step size $h > 0$, and positive integer N for number of iterations.

Output: $(y_i)_{0 \leq i \leq N}$, a sequence of approximations for respectively $(y(x_0 + ih))_{0 \leq i \leq N}$.

Steps: For $1 \leq i \leq N$,

- Set $x_{i+1} := x_i + h$.
- Set $y_{i+1} := y_i + hf(x_i, y_i)$.

Return $(y_i)_{0 \leq i \leq N}$.

The Forward Euler Method is globally a *first order* method, meaning the error in the Forward Euler Method is bounded by a multiple of h.

Proposition 8.2.2. *If $y \in C^2$, then the Forward Euler Method is first order.*

Proof. First one must compute the error for one step. The error for the first step suffices. This is found by computing the difference between a Taylor expansion for $y(x)$ centered at $x_1 = x_0 + h$ and the corresponding approximation y_1. The Taylor expansion for $y(x)$ centered at $x_1 = x_0 + h$

is:

$$y(x_0 + h) = y_0 + hy'(x_0) + \frac{h^2}{2}y''(x_0) + \mathcal{O}(h^3).$$

The difference between $y(x_0 + h) = y(x_1)$ and the approximation y_1 is:

$$\begin{aligned} |y(x_1) - y_1| &= \left|[y_0 + hy'(x_0) + \frac{h^2}{2}y''(x_0) + \mathcal{O}(h^3)] - [y_0 + hf(x_0, y_0)]\right| \\ &= \left|\frac{h^2}{2}y''(x_0) + \mathcal{O}(h^3)\right| \\ &\leq Ch^2 \text{ for some } C > 0. \end{aligned}$$

Thus, the error in one iteration is on the order of h^2. Now to calculate the error for computing $y(x_i)$, note that $\frac{x_i - x_0}{h}$ iterations would be performed, each with an error bounded by Ch^2, where $C > 0$ is some constant. Thus the global error will be bounded by $Ch^2(\frac{x_i - x_0}{h}) = C'h$, where $C' > 0$ is some other constant. $\square$

The Forward Euler Method belongs to a class of Runge-Kutta algorithms. Some Runge-Kutta algorithms have a higher order than the Forward Euler Method, and many implemented ODE solvers (e.g. `ode45` in MATLAB and `Bertini`) use 4th or 5th order Runge-Kutta algorithms.

9. NUMERICAL ALGEBRAIC GEOMETRY

Numerical algebraic geometry encompasses various numerical algorithms for understanding and computing varieties. The fundamental tool in numerical algebraic geometry is numerical homotopy continuation, which is a type of algorithm that computes the isolated solutions to a polynomial system F by numerically tracking paths that interpolate between the solutions of F and the already-known solutions of a similar system G. This chapter develops the general setup for homotopy continuation, and introduces some common homotopy algorithms. This chapter assumes square systems that define 0-dimensional affine varieties. Worth noting, however, is that homotopy continuation may also be used to understand or solve both over- and under-determined systems, as well as to compute projective varieties. For more information on homotopy continuation and its uses in numerical algebraic geometry, see [35, 36, 37].

9.1 Homotopy methods

Homotopy continuation algorithms seek to solve a target system $F \subset \mathbb{C}[x_1, \ldots, x_n]$ using a similar start system $G \subset \mathbb{C}[x_1, \ldots, x_n]$ where the points in $\mathcal{V}(G)$ are already known. Here both F and G are assumed to be square with $\mathcal{V}(F)$ and $\mathcal{V}(G)$ both 0-dimensional. Let $H(x;\tau) \subset \mathbb{C}[x_1, \ldots, x_n][\tau]$ be a one-parameter family of equations with parameter $\tau \in \mathbb{C}_\tau$ such that both F and G belong to the family. Typically, one might let $H(x;0) = G$ and $H(x;1) = F$. Then $\mathcal{V}(H) \subset \mathbb{C}^n \times \mathbb{C}_\tau$ is a complex family of varieties realized by the projection $\pi : \mathcal{V}(H) \to \mathbb{C}_\tau$ onto the last coordinate. Furthermore, $\mathcal{V}(H)$ contains $\mathcal{V}(F)$ in the fiber over $\tau = 1$, $\mathcal{V}(G)$ in the fiber over $\tau = 0$, and the dimension of each component of $\mathcal{V}(H)$ is at least one. Let C denote the union of all the 1-dimensional components of $\mathcal{V}(H)$ with non-constant projection to $\mathbb{C}_\tau$. Consequently, the projection onto $\mathbb{C}_\tau$ of each curve in C is dense. A point $(x;\tau) \in \mathcal{V}(H)$ is nondegenerate provided that the Jacobian of H with respect to the x-variables, D_xH, is invertible at $(x;\tau)$. Assuming C is nonempty, there a nonempty Zariski-open subset $U \subset \mathbb{C}_\tau$ such that $H(x;u) = 0$ has $\delta \in \mathbb{Z}_{>0}$ nondegenerate solutions for every $u \in U$ [35, 37]. The system $H(x;\tau)$

defines a homotopy for computing the isolated, nondegenerate points of $\mathcal{V}(F)$ if C is nonempty and $0 \in U$.

The points in $\mathcal{V}(F)$ are computed by tracking real 1-dimensional paths. More specifically, given an arc $\gamma \subset \mathbb{C}_\tau$ between 0 and 1, the restriction of the curve C to γ, denoted $C|_\gamma$, is a set of arcs. Every point in $\mathcal{V}(F)$ at $\tau = 1$ is connected to a point in $\mathcal{V}(G)$ at $\tau = 0$ by an arc in $C|_\gamma$. Provided every point in $C|_\gamma$ is nondegenerate, the arc γ is general and a numerical homotopy continuation algorithm will compute all points in $\mathcal{V}(F)$ by numerically tracking along each arc from $\tau = 0$ to $\tau = 1$. Numerical path tracking in this context amounts to numerically solving an initial value problem.

To see the initial value problem, first note that the assumption that every point in $C|_\gamma$ is nondegenerate means that every arc in $C|_\gamma$ is smooth. Thus, by the Implicit Function Theorem, there is a parametrization $x(\tau)$ for each arc. Recall that $x(0) = x_0$ is known for each arc as it is one of the points in $\mathcal{V}(G)$. Furthermore, $H(x(\tau);\tau) = 0$ for each arc $x(\tau)$. Differentiating $H(x(\tau);\tau) = 0$ with respect to τ gives a Davidenko differential equation:

$$D_x H \frac{dx}{d\tau} + \frac{\partial H}{\partial \tau} = 0. \tag{9.1}$$

By rearranging the differential equation and including the known $x(0) = x_0$, the parametrized path $x(\tau)$ is the unique solution to the following initial value problem:

$$\begin{cases} \dfrac{dx}{d\tau} & = -\left(D_x H\right)^{-1} \dfrac{\partial H}{\partial \tau} \\ x(0) & = x_0. \end{cases} \tag{9.2}$$

Given this initial value problem, one could solve for $x(\tau)$ simply by applying any numerical differential equation solver. For example, one could use Euler's Method (Algorithm 8.2.1). Every numerical differential equation solver introduces error, however, which may lead to poor approximations for $x(1)$. Better approximations may be achieved by utilizing predictor-corrector methods. Recall that for any τ^* along γ, $H(x,\tau^*)$ is a polynomial system and $x(\tau^*)$ is a solution to that poly-

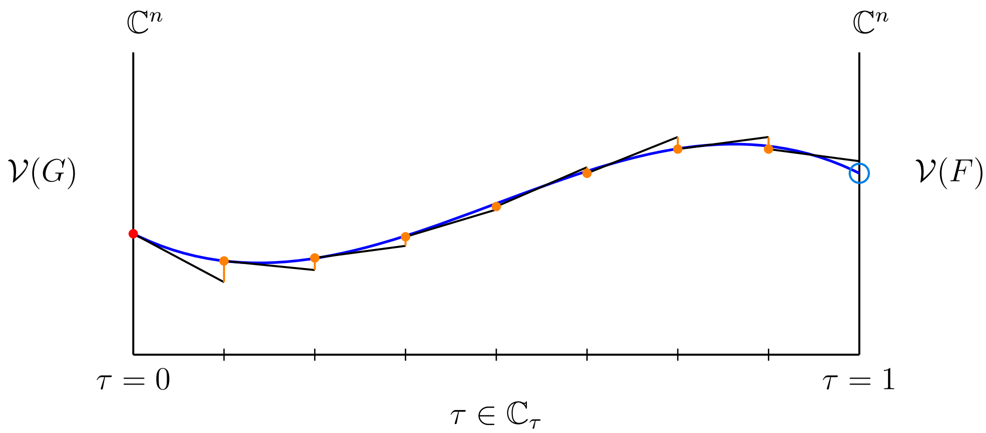


Figure 9.1: Cartoon of a prediction-correction path tracking algorithm.

nomial system. Thus each iteration from a numerical differential equation solver can be refined closer to $x(\tau^*)$ using Newton's Method. This pattern of predicting an iterate with a numerical differential equation solver and correcting with Newton's Method is used to estimate $x(\tau)$ up to $\tau = 1$.

The above description relied on every point of $C|_\gamma$ being nondegenerate. At $t = 1$, however, there is a possibility for $x(1)$ to be degenerate if $\mathcal{V}(F)$ contains singular solutions. Consequently, there is no general arc $\gamma \subset \mathbb{C}_\tau$ such that $C|_\gamma$ is nondegenerate at a singular $x(1)$. Furthermore, path tracking along an arc ending at $x(1)$ will fail as the degeneracy implies that the Jacobian $D_x H$ matrix will be singular at $x(1)$. As both Newton's Method and Euler's Method rely on inverting $D_x H$, these methods will fail close to $\tau = 1$.

There is still a way to approximate the singular solutions of F using path tracking, however. After choosing a $\gamma \subset \mathbb{C}_\tau$ such that $C|_\gamma$ is nondegenerate everywhere, except perhaps over $\tau = 1$, the first step is to implement path tracking along the arcs of $C|_\gamma$ starting at $\tau = 0$, but stopping just before $\tau = 1$. For instance, one might choose to stop path tracking when τ is 0.1 away from 1. The next step is to estimate $x(1)$ using an endgame, which is a type of algorithm that uses results in complex analysis to approximate $x(1)$ without relying on the Jacobian. One common endgame algorithm is the Cauchy endgame, which estimates $x(1)$ using Cauchy's integral formula and numerically estimating the integral [36]. By leveraging endgames, numerical path tracking can be used to compute $x(1)$, even when it may be singular. In this manner, homotopy continuation algorithms can estimate every isolated point in $\mathcal{V}(F)$ [35, 36, 37].

The general steps of numerical homotopy continuation are summarized Algorithm 9.1.1.

Algorithm 9.1.1.

Input: a square, 0-dimensional system F.

Output: approximations to the isolated points in $\mathcal{V}(F)$.

Steps:

1. Create a homotopy $H(x;\tau)$ such that $H(x;1) = F$ and the solutions to $G := H(x;0)$ are known.

2. Choose an arc $\gamma \subset \mathbb{C}_\tau$ connecting $\tau = 0$ and $\tau = 1$ such that the corresponding paths defined by $H(x;\tau) = 0$ are smooth for all $\tau \in \gamma \setminus \{1\}$.

3. For each point $x(0)$ in $\mathcal{V}(G)$:

   (a) Use prediction-correction path tracking, starting at $x(0)$, to estimate $x(\tau)$ for $\tau \in \gamma$.

   (b) Stop path tracking just before $\tau = 1$ and use an endgame to estimate $x(1)$.

4. Return all points $x(1)$.

In summary, given a homotopy $H(x;\tau)$ where $H(x;0) = G$, $H(x;1) = F$, and all paths are nondegenerate everywhere except perhaps over $\tau = 1$, then numerical path tracking techniques starting with the points in $\mathcal{V}(G)$ at $\tau = 0$ will find all isolated points in $\mathcal{V}(F)$. Assuming a start system G is already chosen, one example of a homotopy is the straight-line homotopy, which is defined by $H(x;\tau) = G(1-\tau) + F\tau$. Further details on choosing the start system G and the homotopy $H(x;\tau)$ are in the following Section 9.2.

9.2 Homotopy algorithms

The choice of start system G and homotopy H plays an important role in the efficacy and efficiency of homotopy algorithms. In particular, the number of paths tracked in a homotopy algorithm is equal to the number of solutions of the start system G. The definition of homotopy requires that G has at least the same number of solutions as F. If G has more solutions than F, then excess paths diverge to infinity. A homotopy $H(x;\tau)$ is optimal if F and G have the same number of solutions when counted with multiplicity, and if the paths connecting $\mathcal{V}(F)$ and $\mathcal{V}(G)$ are smooth everywhere except possibly at $\tau = 1$. Creating optimal homotopies can be challenging as the number of solutions to F is typically *a priori* unknown. Frequently homotopy methods are designed around bounds on the number of solutions to F. This is the case with both the Bézout homotopy and the polyhedral homotopy. Other homotopy methods, such as the coefficient-parameter homotopy, rely on a general start system.

9.2.1 Coefficient parameter homotopy

The coefficient-parameter homotopy creates a homotopy by first replacing the coefficients of the target system F with parameters. Restricting to an arc in the coefficient-parameter space where one endpoint of the arc is the point corresponding to the coefficients of F gives a homotopy for computing all solutions to F. One example of such a homotopy is the straight-line homotopy. The basis of coefficient-parameter homotopies is made more precise by the following theorem adapted from [35].

Theorem 9.2.1 ([35, Theorem 7.1.1])**.** *Let $F(x;q)$ be a system of polynomials in n variables $x_1,\ldots,x_n$, and m parameters $q_1,\ldots,q_m$,*

$$F(x;q) : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n.$$

That is, $F(x;q) = \{f_1(x;q),\ldots,f_n(x;q)\}$ and each $f_i(x;q)$ is a polynomial in both x and q. Furthermore, let $N(q)$ denote the number of nonsingular solutions of F as a function of q:

$$N(q) := \left|\{x \in \mathbb{C}^n \mid F(x;q) = 0, \det(D_xF(x;q)) \neq 0\}\right|.$$

Then,

1. *$N(q)$ is finite and the same, say $\mathcal{N}$, for almost all $q \in \mathbb{C}^m$;*

2. *For all $q \in \mathbb{C}^m$, $N(q) \leq \mathcal{N}$;*

3. *The subset of $\mathbb{C}^m$ for which $N(q) = \mathcal{N}$ is a Zariski-open set. That is, the exceptional set $Q^* := \{q \in \mathbb{C}^m \mid N(q) < \mathcal{N}\}$ is an affine subvariety of a variety of dimension $n-1$;*

4. *The homotopy $F(x; \phi(\tau)) = 0$ with continuous $\phi : [0,1] \to \mathbb{C}^m \setminus Q^*$ has $\mathcal{N}$ continuous, nonsingular solution paths $x(\tau) \subset \mathbb{C}^n$;*

5. *As $\tau \to 1$, the limits of the solution paths of the homotopy $F(x; \phi(\tau)) = 0$ with $\gamma(\tau) : [0,1] \to Q$ and $\gamma(\tau) \notin Q^*$ for $\tau \in [0,1)$ include all the nonsingular solutions in U of $F(x; \gamma(1)) = 0$.*

Therefore, a choice of $\phi : [0,1] \to \mathbb{C}^m \setminus Q^*$ such that the solutions of $F(x; \phi(0)) = 0$ are known will give a homotopy algorithm for computing all points in $\mathcal{V}(F) = \mathcal{V}(F(x; \phi(1)))$. Note that an analogous theorem, and consequentially homotopy, also holds for projective space, see [35, Theorem 7.1.4].

9.2.2 Bézout homotopy

The Bézout homotopy, sometimes also called the total degree homotopy, is designed around the Bézout bound. Recall from Chapter 6 that the Bézout bound is calculated as the product of the degrees of the polynomials in F. That is, if $F = \{f_1, \ldots, f_n\}$ where $\deg f_i = \delta_i$, then the Bézout bound for F is $\delta := \delta_1 \cdots \delta_n$. The Bézout homotopy works by constructing a start system G with δ solutions and then using the straight-line homotopy for $H(x;t)$. One way to construct a start system G with δ solutions is to take $G = \{x_i^{\delta_i} - 1\}$ for $i = 1, \ldots, d$ [35, 36].

9.2.3 Polyhedral homotopy

As discussed in Chapter 6, the Bézout homotopy will be optimal for general systems. Sparse systems, however, are often not general in the sense of the Bézout bound. For these systems, the mixed-volume bound of Bernstein's Theorem (Theorem 6.1.2) is typically a tighter bound. The polyhedral homotopy, introduced in [14], is designed precisely around the mixed-volume bound of Bernstein's Theorem.

Supposing δ is the mixed-volume bound, the polyhedral homotopy works by using combinatorial methods to create a start system with δ solutions while simultaneously defining a homotopy. More precisely, suppose $F = \{f_1, \ldots, f_n\}$ and let $\mathcal{A}^\bullet$ collect their respective supports, i.e. $\mathcal{A}^\bullet = (\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(n)})$ where $\mathcal{A}^{(i)} = \operatorname{supp}(f_i)$. The first step in the polyhedral homotopy is to find a lifting function $\Gamma^\bullet = (\Gamma^{(1)}, \ldots, \Gamma^{(n)})$, $\Gamma^{(i)} : \mathcal{A}^{(i)} \to \mathbb{N}$, for $\mathcal{A}^\bullet$ such that the induced subdivision is a fine mixed subdivision, see Section 4.2.

The next step is to construct a homotopy using a complex parameter $\tau \in \mathbb{C}_\tau$, $\Gamma^\bullet$, and the system F. Recall one may write $f_i = \sum_{\alpha \in \mathcal{A}^{(i)}} c_{i,\alpha} x^\alpha$ for each $i = 1, \ldots, n$. Then, for each $i = 1, \ldots, n$, define

$$f_{i,\Gamma}(x;\tau) := \sum_{\alpha \in \mathcal{A}^{(i)}} c_{i,\alpha} x^\alpha \tau^{\Gamma^{(i)}(\alpha)}.$$

Letting $F_{\Gamma^\bullet}(x;\tau) := \{f_{i,\Gamma} \mid i = 1, \ldots, n\}$, note that $F_{\Gamma^\bullet}(x;1) = F$. Huber and Sturmfels in [14] show that if F is Bernstein-general, then $F_{\Gamma^\bullet}(x;0)$ has δ solutions and the homotopy defined by $F_{\Gamma^\bullet}(x;\tau)$ is optimal. Since *a priori* whether or not F is Bernstein-general is unknown, the polyhedral homotopy algorithm first constructs a general system G with support $\mathcal{A}^\bullet$. Next the algorithm uses $G_{\Gamma^\bullet}(x;\tau)$ to solve G, and then uses G and its solutions as the start system for a straight-line homotopy for computing the solutions of F. For the detailed approach, see [14].

9.2.4 (Multi)homogenous homotopies

To avoid diverging paths, one common strategy in numerical homotopy continuation is to homogenize the system F to a compact space and then use a total-degree homotopy to track paths in that compact space. The most commonly used compact spaces are projective space $\mathbb{P}^n$ or a product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ where $n = n_1 + \cdots + n_s$. Different choices of compactification can result in a different number of solutions. The following example, which can be found in [20, Section 2.1], describes when this can happen.

Example 9.2.1 ([20, Example 2.1])**.** Consider $\hat{F}(t) = (\hat{f}_1(t), \hat{f}_2(t)) = 0$, where

$$\hat{f}_1 = 1 + t_1 + t_2 + t_1 t_2 + t_1^2 t_2 + t_1^3 t_2, \qquad \hat{f}_2 = 2 + t_2 + t_1 t_2 + t_1^2 t_2.$$

The system $\hat{F}(t) = 0$ has a mixed-volume bound of three, and indeed has three solutions in the algebraic torus $(\mathbb{C}^\times)^2 \subset \mathbb{C}^2$, given by

$$(t_1, t_2) = (-1, -2), (e^{-\sqrt{-1}\frac{\pi}{3}}, -e^{\sqrt{-1}\frac{\pi}{3}}), (e^{\sqrt{-1}\frac{\pi}{3}}, -e^{-\sqrt{-1}\frac{\pi}{3}}).$$

Two choices for compactifying the solution space are the usual embeddings $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$ and $\mathbb{C}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

First consider the compactification to $\mathbb{P}^2$. A total-degree homotopy may be specified in homogeneous coordinates on $\mathbb{P}^2$ by

$$H(x_0, x_1, x_2; \tau) = \gamma\,(1 - \tau)\,F(x_0, x_1, x_2) + \tau\,G(x_0, x_1, x_2),$$

where $\gamma \in \mathbb{C}^*$ is a generic constant, and the homogeneous start and target equations are given by $G(x) = (x_1^4 - x_0^4, x_2^3 - x_0^3)$ and $f_1(x_0, x_1, x_2) = x_0^4\,\hat{f}_1(x_1/x_0, x_2/x_0) \in \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(4))$, $f_2(x_0, x_1, x_2) = x_0^3\,\hat{f}_3(x_1/x_0, x_2/x_0) \in \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3))$ respectively. To get a unique representative for each point in $\mathbb{P}^2$, one may augment H with a generic equation of the form $*\,x_0 + *\,x_1 + *\,x_2 = 1$, representing an affine patch on $\mathbb{P}^2$. There are twelve start solutions. Genericity of γ and the patch imply that solution paths are smooth, and thus H recovers homogeneous representatives of the toric solutions. The nine remaining endpoints are $[x_0 : x_1 : x_2] = [0 : 0 : 1]$ and $[0 : 1 : 0]$ with multiplicities of 6 and 3, respectively.

For solutions in $\mathbb{P}^1 \times \mathbb{P}^1$, the system $\hat{F}$ is homogenized in each variable separately. Thus, the homogeneous target system becomes $F = (x_0^3\,y_0\,\hat{f}_1(x_1/x_0, y_1/y_0), x_0^2\,y_0\,\hat{f}_2(x_1/x_0, y_1/y_0)) = (f_1, f_2)$ with $f_1 \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 1))$, $f_2 \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1))$ and we may work on a patch defined by $*\,x_0 + *\,x_1 = 1$ and $*\,y_0 + *\,y_1 = 1$. The number of start solutions is now the corresponding multihomogeneous Bézout bound [35, Ch. 8, pp. 126–130], which in this case equals five. Besides the toric solutions, the multihomogeneous homotopy recovers an additional solution at infinity: $[1 : 0] \times [0 : 1]$ with multiplicity 2. $\triangle$

From Example 9.2.1, one can see that while compactification solves the problem of diverging paths, it can potentially introduce the problem of extraneous paths ending at singular solutions at infinity. The homotopy algorithms introduced in Chapters 10 and 11 are thoughtfully constructed to track all paths in a compact space without introducing new solutions at infinity.

9.3 Implementations

Several software implementations of homotopy continuation algorithms are available, including `Bertini` [38], `NumericalAlgebraicGeometry` [39], `HOM4PS` [40], `PHCpack` [41], and `HomotopyContinuation.jl` [42]. These packages differ with regards to their strengths and default homotopy algorithms. For instance, `Bertini` defaults to the total-degree homotopy but also has options for user-defined parameter-coefficient homotopies and (multi-)projective path tracking. In contrast, `HOM4PS` and `PHCpack` specifically implement the polyhedral homotopy. The software `HomotopyContinuation.jl` defaults to the polyhedral homotopy but has options for coefficient-parameter homotopies as well. The work in Chapter 11 uses `Bertini` and `PHCpack`, whereas the work in Chapter 10 uses primarily `HomotopyContinuation.jl`.

10. THE COX HOMOTOPY

Recall from Chapter 9 that the polyhedral homotopy is optimal for any Bernstein-general polynomial system. For polynomial systems which are not Bernstein-general or whose solutions lie outside of the torus, the polyhedral homotopy may track diverging paths. Diverging paths are expensive to track and require heuristics for determining termination. These diverging paths are possible in the polyhedral homotopy because it tracks paths in the dense torus, which is an affine variety. With Tim Duff, Simon Telen, and Thomas Yahl, one goal of [20] was to create a homogeneous homotopy where the number of paths tracked equals the mixed-volume bound. Such a homotopy would have the advantage that all paths converge, as well as being optimal for any Bernstein-general system. The remainder of this chapter is an edited excerpt of [20].

The Cox homotopy algorithm, introduced in [20], recognizes any sparse polynomial system as a system of polynomial equations on a compact toric variety X_Σ. The algorithm lends its name from a construction, described by Cox, of X_Σ as a(n almost) geometric quotient $X_\Sigma = (\mathbb{C}^k \backslash Z) /\!\!/ \mathbb{G}$, see Section 5.3. The Cox homotopy tracks paths in the total coordinate space $\mathbb{C}^k$ of X_Σ and can be seen as a homogeneous version of the standard polyhedral homotopy, which works on the dense torus of X_Σ. It furthermore generalizes the commonly used path tracking algorithms in (multi)projective spaces in that it tracks a set of homogeneous coordinates contained in the $\mathbb{G}$-orbit corresponding to each solution. The Cox homotopy combines the advantages of polyhedral homotopies and (multi)homogeneous homotopies, tracking only mixed volume-many solutions and providing an elegant way to deal with solutions on or near the torus-invariant divisors of X_Σ. In addition, the strategy may help to understand the deficiency of the root count for certain families of systems which are not Bernstein-general.

10.1 Introduction

The aim of the Cox homotopy is to compute the isolated solutions of a polynomial system $\hat{F}(x) = 0$, where $\hat{F} = (\hat{f}_1(x), \ldots, \hat{f}_n(x))$ and $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Of particular interest are the cases where some isolated solutions of $\hat{F}(x) = 0$ lie at or near infinity, as such systems may present challenges for standard homotopy algorithms. To make this precise, one must consider solutions in a suitable compactification of $\mathbb{C}^n$. Common choices for the compactification are the complex projective space $\mathbb{P}^n$ or, more generally, a product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ where $n_1 + \cdots + n_s = n$. In each of these compactifications, the number of solutions to a generic system is a fixed number given by Bézout's theorem or its extension to a product of projective spaces. The generality of Bézout's Theorem and its extension to products of projective spaces provides the basis for numerical homotopy continuation methods—specifically, the (homogeneous) total degree homotopy and more general multihomogeneous homotopies, see Section 9.2.

In practice, infinite or nearly-infinite solutions to the target system $\hat{F}$ present challenges for homotopy continuation. When tracking paths affinely, solutions of large magnitude are hard to estimate accurately. Premature path truncation may also occur, leading to a loss of solutions near infinity, as in Experiment 3 in Section 10.5. Moreover, a large discrepancy between the number of start and target solutions may result in wasted computational resources. Homotopy methods have varying strengths when addressing these challenges. (Multi)homogeneous homotopies have the advantage of working in a compact space so that all paths converge. The largest issue with these homotopies, however, is that compactification to projective spaces may introduce singular, or nearly singular, solutions near or at infinity (see Example 9.2.1).

In contrast to the traditional (multi)homogeneous start systems, which usually depend only on the degrees of $\hat{f}_1, \ldots, \hat{f}_n$, the polyhedral homotopy introduced in [14, 43] takes the Newton polytopes of the (Laurent) polynomials $\hat{f}_1, \ldots, \hat{f}_n$ into account, see Section 9.2. By taking the polynomial structure into account, the number of start solutions for the polyhedral homotopy is given by the mixed-volume bound from Bernstein's Theorem [25]. This bound is often tighter than the Bézout bound, so the polyhedral homotopy may track fewer extraneous paths towards infinity than (multi)projective homotopies. As the polyhedral homotopy tracks points in the torus $(\mathbb{C}^\times)^n$, the polyhedral endgame [44] may be used to detect solutions at infinity by numerically extrapolating coefficients of power series solutions, and scaling issues can be addressed by the

solution space	$\mathbb{P}^n$	$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$	X_Σ
root count	Bézout	multihomogeneous Bézout	Mixed volume
graded ring	$\mathbb{C}[x_0, \ldots, x_n]$	$\bigotimes_{i=1}^{s} \mathbb{C}[x_{i0}, \ldots, x_{in_i}]$	$\mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$
homotopy	homogeneous homotopy	multihomogeneous homotopy	Cox homotopy

Table 10.1: Schematic comparing (multi)homogeneous and Cox homotopies.

toric Newton method introduced in [45].

The proposed Cox homotopy combines the salient features of the multihomogeneous and polyhedral homotopies. A schematic situating the Cox homotopy approach is given in Figure 10.1. The chosen compactification for the Cox homotopy is an n-dimensional, compact, normal toric variety X_Σ, where the polyhedral fan Σ refines the normal fan of each Newton polytope of $(\hat{f}_1, \ldots, \hat{f}_n)$. A polynomial system then may be regarded as a section of a rank-n vector bundle on X_Σ. This section has a well-defined vanishing locus, which for a generic system has mixed volume-many isolated points (see Theorem 6.1.2).

This vanishing locus can be described globally by homogenizing the $\hat{f}_i$ to the total coordinate ring or Cox ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ of X_Σ (see Section 10.2.3). The polynomial ring S, together with its grading by the divisor class group of X_Σ and the irrelevant ideal, corresponds to the geometric construction of X_Σ as a(n almost) geometric quotient $X_\Sigma = (\mathbb{C}^k \setminus Z) /\!\!/ \mathbb{G}$ of a quasi-affine variety by the action of an algebraic reductive group $\mathbb{G}$ (much like the Proj-construction for $\mathbb{P}^n$). Here k denotes the number of rays in the fan Σ. This construction was described by Cox in [18], is summarized in Section 5.3, and is recalled briefly in Section 10.2.2. To represent points in X_Σ, the global coordinates x_ρ from the Cox construction are used, which are called the Cox coordinates. In analogy with choosing an affine patch in (multi)homogeneous homotopies, the $\mathbb{G}$-orbit corresponding to a point $p \in X_\Sigma$ is intersected with a linear space of complementary dimension in $\mathbb{C}^k$ to pick out finitely many sets of Cox coordinates representing p. These points in $\mathbb{C}^k \setminus Z$ are referred to as representatives for p.

The advantages of the Cox homotopy are listed below. Note that strict subsets of this list are shared with other homotopy algorithms, but the Cox homotopy is exceptional by exhibiting all these advantages.

1) By working in a compact space, the Cox homotopy can compute solutions at infinity and reduce the risk of prematurely truncating paths (see Experiments 1 and 3).

2) The Cox homotopy is flexible with its choice of linear space. Poor scaling or ill-conditioning may be mitigated by choosing a random linear space or the normal space to a $\mathbb{G}$-orbit (see Experiment 2).

3) The mixed-volume bound never exceeds any multihomogeneous Bézout bound, and may be substantially smaller (see Experiments 1-4), minimizing the total number of paths tracked.

4) Generalizing the multihomogeneous case, a solution at infinity lies on a divisor where some Cox coordinate equals 0. This can be used to heuristically establish when a system is not Bernstein-general by investigating certain face systems (see Experiment 4).

The main theoretical results behind the Cox homotopy are summarized in Theorem 10.1.1. See Theorems 10.4.1 and 10.4.6 for more precise statements concerning the generic and degenerate cases, respectively.

Theorem 10.1.1. *For a target system with prescribed supports, Algorithm 2 computes all isolated solutions in the simplicial locus $U \subset X_\Sigma$ (cf. Theorem 5.3.1) by tracking one representative in $\mathbb{C}^k \setminus Z$ for each path in X_Σ. The total number of tracked paths in X_Σ is given by the mixed-volume bound and hence is optimal with respect to the supports. Moreover, for a solution in the dense torus of X_Σ, any representative path converges to a set of homogeneous coordinates in $\mathbb{C}^k \setminus Z$. For the case of a solution at infinity in $U \subset X_\Sigma$, the number of representative paths which converge is at least the degree of the corresponding $\mathbb{G}$-orbit. This number can be computed explicitly by Proposition 10.3.6 as the suitably normalized volume of an orbit polytope.*

Theorem 10.4.6 is based on an analysis of simplicial $\mathbb{G}$-orbits given in Section 10.3, which may be of independent interest. Note that $U = X_\Sigma$ in the commonly encountered case where Σ is simplicial, such as a (multi)projective space, a weighted projective space, or any toric surface. In general, the codimension of the subvariety $(X_\Sigma \setminus U)$ in X_Σ is at least 3. Theorem 10.1.1 states that for solutions outside of the torus, it may happen that not all representative paths converge to a set of homogeneous coordinates. To remedy this, a specialized endgame (Algorithm 1) is proposed in Section 10.4, which detects non-converging representatives and switches to a converging representative at the end of the path.

Besides the theoretical results, the advantages of the Cox homotopy enumerated above are demonstrated through a proof-of-concept implementation applied to a variety of examples. These examples also illustrate the flexibility of the Cox homotopy approach. In one of these examples, the Cox homotopy is compared to the existing polyhedral homotopy methods to demonstrate the comparative robustness of the Cox homotopy algorithm when faced with infinite or nearly-infinite solutions (see Experiment 3).

The Cox homotopy belongs to a larger body of literature studying the uses of toric varieties in homotopy continuation. Indeed related works include the important role played by toric compactifications, albeit without Cox coordinates, in the complexity-theoretic study of sparse polynomial system solving, such as in the papers [46, 47, 48]. Another related work is a type of homogenization based on the Newton polytopes of the system used by Verschelde in the development of a toric Newton method in [45]. In essence, Verschelde's toric Newton algorithm is a generalized Newton iteration based on dynamic affine patch selection (i.e., dynamic scaling), and is proposed as a postprocessing step for paths tracked with a traditional polyhedral homotopy. While related to Cox coordinates, Verschelde's procedure uses both homogeneous and toric variables and is therefore not a homogenization into the total coordinate ring. In contrast, the Cox homotopy tracks solution paths in X_Σ via an explicit homotopy in the total space $\mathbb{C}^k$ of the Cox construction—see Remark 10.2.1 for further discussion of differences. Finally, the Cox homotopy also complements the recent use of Cox coordinates for dealing with non-toric solutions in a robust manner in numerical algebraic normal form methods [49, 50, 51]. The experiments in Section 10.5 utilizing the Cox homotopy paint a picture complementary to these works, showing that toric compactifications can be suitable in practical computing scenarios.

This chapter is organized as follows. Section 10.2 recalls the necessary background on homotopy continuation, toric varieties, and the Cox construction. Section 10.3 describes the closure of $\mathbb{G}$-orbits in $\mathbb{P}^k$. Section 10.4 presents the Cox homotopy algorithm (Algorithm 2), as well as an algorithm for orthogonal slicing (Algorithm 4), and includes the main theorems (Theorems 10.4.1 and Theorem 10.4.6) supporting the Cox homotopy. Finally Section 10.5 gives examples which demonstrate the advantages of the algorithms in Section 10.4 and compares the Cox homotopy to the polyhedral homotopy.

10.2 Preliminaries

10.2.1 Background on homotopy continuation

This section details a few choices on setup and notation regarding homotopy continuation. Further details on homotopy continuation are available in Chapter 9, as well as in sources such as [35, 52, 36]. For the purposes of this chapter, a homotopy $H(x;\tau)$ in variables $x = (x_1, \dots, x_n)$ consists of polynomials $H_1(x;\tau), \dots, H_n(x;\tau)$ depending on an additional tracking parameter τ. This chapter uses the convention that the start and target system are given by $G(x) = H(x, 1)$ and $F(x) = H(x, 0)$, respectively. Note that this differs from Chapters 9 and 11, where the start system is at $\tau = 0$ and the target system is at $\tau = 1$.

Often in homotopy continuation, polynomial systems are homogenized so path-tracking takes place in a compact space, such as (multi)projective spaces. This chapter uses the following convention to distinguish between homogeneous and non-homogeneous notation.

Convention 1. The variable $t = (t_1, \dots, t_n)$ denotes a set of affine/toric variables and $x = (x_1, \dots, x_k)$ denotes a set of homogeneous variables. Equations or systems of equations that are not homogeneous are indicated by a circumflex, which is dropped after homogenization. For instance, homogenizing the system $\hat{F}(t) = 0$ given by $\hat{f}_1(t) = \cdots = \hat{f}_n(t) = 0$ results in the system

$F(x) = 0$ given by $f_1(x) = \cdots = f_n(x) = 0$. $\triangle$

The choice of compactification/ homogenization influences the number of solutions to the system. For instance, Example 9.2.1 gives an instance of the family $\hat{F}(t_1, t_2; c) = (\hat{f}_1(t_1, t_2; c), \hat{f}_2(t_1, t_2; c))$ of systems with fixed monomial supports:

$$\hat{f}_1 = c_{11} + c_{12}t_1 + c_{13}t_2 + c_{14}t_1t_2 + c_{15}t_1^2t_2 + c_{16}t_1^3t_2, \quad \hat{f}_2 = c_{21} + c_{22}t_2 + c_{23}t_1t_2 + c_{24}t_1^2t_2.$$

A member of this family with generic coefficients c also has three toric solutions. Moreover, in either the $\mathbb{P}^2$-compactification or the $\mathbb{P}^1 \times \mathbb{P}^1$-compactification, the solutions at infinity are the same for any generic choice of c. Thus, one may naturally view these solutions at infinity as an artifact of the chosen compactification. Next to the introduction of (possibly infinitely many [49, Example 3.2.4]) spurious solutions at infinity, another issue with the standard compactifications is the artificial clustering effect of well-separated toric solutions, see for instance [45, Example 4.2].

If a toric compactification X_Σ is used instead, as proposed in the Cox homotopy and described in Section 10.2.3, then for generic choices of the c_{ij}, these homogenized equations only define three solutions on X_Σ, all contained in its dense torus. For this example, the toric variety X_Σ turns out to be a Hirzebruch surface, as later seen in Example 10.2.1.

10.2.2 Toric varieties and the Cox construction

This section details a few notational choices for toric varieties and the Cox construction. Further details on toric varieties and the Cox construction are available in Chapter 5, as well as in sources such as [15, 18].

Let $\mathbb{T} = (\mathbb{C}^\times)^n$ be the algebraic torus of dimension n. Its character and cocharacter lattices are denoted by $M = \mathrm{Hom}_\mathbb{Z}(T, \mathbb{C}^\times) \simeq \mathbb{Z}^n$ and $N = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$ respectively. Given a full-dimensional lattice polytope $\mathscr{P} \subset M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$, let Σ be its normal fan. This fan is complete, which means that $\bigcup_{\sigma \in \Sigma} \sigma = N_\mathbb{R}$, and the corresponding toric variety X is compact. The notation $X = X_\Sigma$ is sometimes used to emphasize the correspondence between X and its fan. The matrix $\mathbf{F} = [u_1 \ \cdots \ u_k] \in \mathbb{Z}^{n \times k}$, whose columns are the unique minimal ray generators u_i for each $\rho \in \Sigma(1)$, is called the facet matrix. In this context, the u_i represent the inner normals of the polytope $\mathscr{P}$.

Recall from the Cox construction (see Section 5.3) that $\mathbf{F}$ represents a lattice morphism. The idea is to view $\mathbf{F}$ as a map from the fan of $\mathbb{C}^k$ to the fan of X_Σ. In order for $\mathbf{F}$ to be compatible with the fans of $\mathbb{C}^k$ and X_Σ, some cones of the fan of $\mathbb{C}^k$ may need to be removed. The cones which are removed designate the base locus of $\mathbf{F}$, which is defined by the irrelevant ideal B (see Equation (5.6)) in the coordinate ring $S = \mathbb{C}[x_1, \dots, x_k]$ of $\mathbb{C}^k$. Viewing $\mathbf{F}$ as a map of fans, $\mathbf{F}$ furthermore describes a Laurent monomial map (5.7) whose kernel is a subgroup $\mathbb{G} \subset (\mathbb{C}^\times)^k$. Additionally, $\pi : \mathbb{C}^k \setminus Z \to X_\Sigma$ coming from $\mathbf{F}$ will be an almost geometric quotient for the action of $\mathbb{G}$ on $\mathbb{C}^k \setminus Z$, and is a geometric quotient for an open subset of $U \subset X_\Sigma$, see Theorem 5.3.1.

Consequently, there is a Zariski open subset $U \subset X_\Sigma$ such that $\mathbb{G}$-orbits in $\pi^{-1}(U)$ are in one-to-one correspondence with points of U. The fan of U is the subfan of Σ consisting of all its simplicial cones, and $X_\Sigma \setminus U$ is codimension at least three in X_Σ.

In order to interpret S (with its irrelevant ideal B) as the homogeneous coordinate ring or Cox ring of the toric variety X_Σ, S is equipped with a grading such that the vanishing locus in $\mathbb{C}^k \setminus Z$ of homogeneous elements is stable under the action of $\mathbb{G}$. The grading is by the divisor class group $\mathrm{Cl}(X_\Sigma)$ of X_Σ, which is the group of Weil divisors modulo linear equivalence. This group may be described explicitly. Let $D_1, \dots, D_k$ be the torus invariant prime divisors on X_Σ corresponding to $\rho_1, \dots, \rho_k$ respectively. By Proposition 5.4.1, the following is an exact sequence

$$0 \to M \xrightarrow{\mathbf{F}^\top} \bigoplus_{i=1}^{k} \mathbb{Z} \cdot D_i \xrightarrow{\mathcal{P}} \mathrm{Cl}(X_\Sigma) \to 0, \tag{10.1}$$

where the map $\mathbf{F}^\top$ sends a character to its divisor and $\mathcal{P}$ takes a torus invariant divisor to its class in $\mathrm{Cl}(X_\Sigma)$ (see [15, Theorem 4.1.3]). This is the same exact sequence as in Proposition 5.4.1, but with coordinates. The map $\mathbf{F}^\top$ is a lattice map $\mathbb{Z}^n \to \mathbb{Z}^k$ given by the transpose of the matrix $\mathbf{F}$. This exact sequence shows that $\mathrm{Cl}(X_\Sigma) \simeq \mathbb{Z}^k / \operatorname{im} \mathbf{F}^\top$ and every element of $\mathrm{Cl}(X_\Sigma)$ can be written as the class $[D]$ of some torus invariant divisor $D = \sum_{i=1}^k a_i D_i$. For an element $\alpha =$

$[\sum_{i=1}^k a_i D_i] \in \mathrm{Cl}(X_\Sigma)$, one may define the vector subspace

$$S_\alpha = \bigoplus_{\mathbf{F}^\top m + a \geq 0} \mathbb{C} \cdot x^{\mathbf{F}^\top m + a},$$

where the sum ranges over all $m \in M$ satisfying $\langle u_i, m \rangle + a_i \geq 0$. Here $\langle \cdot, \cdot \rangle$ denotes the usual pairing between $N \simeq \mathbb{Z}^n$ and its dual $M \simeq \mathbb{Z}^n$, and corresponds to the usual dot product in $\mathbb{Z}^n$. One can check that this definition is independent of the chosen representative for α and if $f = \sum_{\mathbf{F}^\top m + a \geq 0} c_m x^{\mathbf{F}^\top m + a} \in S_\alpha$, then for $g \in \mathbb{G} \subset (\mathbb{C}^\times)^k$,

$$f(g \cdot x) = \sum_{\mathbf{F}^\top m + a \geq 0} c_m (g \cdot x)^{\mathbf{F}^\top m + a} = g^a f(x).$$

It follows that $f \in S_\alpha$ has a well defined vanishing locus

$$\mathcal{V}_{X_\Sigma}(f) = \{p \in X_\Sigma \mid f(x) = 0 \text{ for some } x \in \pi^{-1}(p)\}$$

and this definition extends to $\mathcal{V}_{X_\Sigma}(f_1, \ldots, f_s) = \bigcap_{i=1}^s \mathcal{V}_{X_\Sigma}(f_i)$ for elements $f_i \in S_{\alpha_i}$. An element $f \in S_\alpha$ is called homogeneous of degree α. The ring S, with its grading by $\mathrm{Cl}(X_\Sigma)$ and its irrelevant ideal B, is called the Cox ring of X_Σ. The terminology related to the Cox construction of X_Σ is summarized in the table below.

	Algebra		Geometry	
Cox ring	$S = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} S_\alpha$	$\xrightarrow{\mathrm{MaxSpec}(\cdot)}$	$\mathbb{C}^k$	total coordinate space
irrelevant ideal	B	$\xrightarrow{V_{\mathbb{C}^k}(\cdot)}$	Z	base locus
class group	$\mathrm{Cl}(X_\Sigma)$	$\xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^\times)}$	$\mathbb{G}$	reductive group

Table 10.2: Terminology and notation related to the Cox construction.

10.2.3 Homogenization of sparse polynomial systems

Given a system of equations, this section briefly describes how to interpret the system as functions on a compact toric variety X. The homogenization procedure is described in detail in Section 3 of [50]. This homogenization process is a generalization of the standard homogenization used for sending polynomials to the multihomogeneous coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. Line bundles on X are in one-to-one correspondence with elements of the Picard group $\mathrm{Pic}(X) \subset \mathrm{Cl}(X)$, consisting of Cartier divisors modulo linear equivalence [15, Chapter 4]. Sections of these line bundles are homogeneous polynomials in the Cox ring S of X.

The homogenization process is as follows. Let $\hat{f}_1, \ldots, \hat{f}_n \in \mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be a given set of Laurent polynomials and let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be their supports. That is, $\hat{f}_i$ can be written as $\hat{f}_i = \sum_{m \in \mathcal{A}_i} c_{i,m} t^m$ where $c_{i,m} \neq 0$. For $i = 1, \ldots, n$, let $\mathscr{P}_i = \mathrm{conv}(\mathcal{A}_i)$ be the Newton polytope of $\hat{f}_i$ (see Section 6.1). The Minkowski sum of these polytopes is denoted by $\mathscr{P} = \mathscr{P}_1 + \cdots + \mathscr{P}_n$ and this sum is assumed to have dimension n. The normal fan $\Sigma_{\mathscr{P}}$ of $\mathscr{P}$ gives the toric variety $X = X_{\Sigma_{\mathscr{P}}}$. For each i, there is a canonical way of associating a torus-invariant Cartier divisor $D_{\mathscr{P}_i} = \sum_{j=1}^{k} a_{i,j} D_i$ to $\mathscr{P}_i$, see Section 5.4. The class of this divisor in $\mathrm{Pic}(X)$ is denoted by $\alpha_i = [D_{\mathscr{P}_i}] \in \mathrm{Pic}(X)$. The vector space of Laurent polynomials with Newton polytope $\mathscr{P}_i$ is the vector space of sections of the vector bundle $\mathscr{O}_X(\alpha_i)$ on X [15, Proposition 4.3.3]. Moreover, this vector space can be identified with the degree α_i part of the Cox ring S of X [15, Proposition 5.3.7] and $\mathscr{O}_X(\alpha_i)$ is generated by global sections. In summary,

$$\bigoplus_{m \in \mathscr{P}_i \cap M} \mathbb{C} \cdot t^m \simeq \Gamma(X, \mathscr{O}_X(\alpha_i)) \simeq S_{\alpha_i}.$$

The homogenization to the Cox ring of $\hat{f}_i$ is given by the coefficients $a_{i,j}$ defining $D_{\mathscr{P}_i}$. Explicitly, the homogeneous polynomials f_i are computed from $\hat{f}_i$ by the following:

$$\hat{f}_i = \sum_{m \in \mathcal{A}_i} c_{i,m} t^m \mapsto f_i = \sum_{m \in \mathcal{A}_i} c_{i,m} x^{\mathbf{F}^\top m + a_i}, \tag{10.2}$$

where $\mathbf{F}$ is the facet matrix. Note that $\mathcal{A}_i \subset \mathscr{P}_i \cap M$ implies that $\mathbf{F}^\top m + a_i \geq 0$, so that the f_i are indeed polynomials.

Remark 10.2.1. Note that the homogenization used in [45] differs from (10.2) in that the homogenized polynomials in [45] depend on the toric variables, as well as on the new, homogeneous variables. Moreover in [45], each of the $\hat{f}_i$ is 'homogenized' (in the sense of [45, Definition 2.1]) with respect to its own Newton polytope. The idea of [45] is to use the homogenization variables as auxiliary coordinates that help to scale the solution paths in a clever way. Another difference is that the normal fans of the $\mathscr{P}_i$ undergo a simplicial refinement in the symbolic step of the algorithm in [45]. This step is mentioned to be memory exhaustive in [45, Section 7.1], and the examples considered there are correspondingly small. In the homogenization described here, all of the $\hat{f}_i$ are homogenized with respect to the same polytope $\mathscr{P}$ and its normal fan need not be simplicial.

The n-tuple $(f_1, \ldots, f_n)$ is a section of the rank n vector bundle $\mathscr{O}_X(\alpha_1) \oplus \cdots \oplus \mathscr{O}_X(\alpha_n)$ on X. The zero locus of this section is the vanishing locus $\mathcal{V}_X(f_1, \ldots, f_n)$. It contains the points defined by $\hat{f}_1 = \cdots = \hat{f}_n = 0$ in $\mathbb{T}$, denoted by $\mathcal{V}_{\mathbb{T}}(\hat{f}_1, \ldots, \hat{f}_n)$. Since $\mathscr{O}_X(\alpha_1) \oplus \cdots \oplus \mathscr{O}_X(\alpha_n)$ is a rank n vector bundle on a variety of dimension n, the expected dimension of $V_X(f_1, \ldots, f_n)$ is 0. The number of points to expect is equal to the mixed-volume bound, see Theorem 6.1.2.

The examples of particular interest for the Cox homotopy are those where $(f_1, \ldots, f_n)$ defines points *outside of* $\mathbb{T}$, that is, on the boundary $X \setminus \mathbb{T}$ of the torus in X. The following example from [50] illustrates such points of interest.

Example 10.2.1. Consider the Laurent polynomials $\hat{f}_1, \hat{f}_2 \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ given by

$$\hat{f}_1 = 1 + t_1 + t_2 + t_1 t_2 + t_1^2 t_2 + t_1^3 t_2, \qquad \hat{f}_2 = 1 + t_2 + t_1 t_2 + t_1^2 t_2,$$

which is equal to the system in Example 9.2.1 up to the constant coefficient of $\hat{f}_2$. Although the mixed-volume bound for the system $\hat{\mathcal{F}} = (\hat{f}_1, \hat{f}_2) = 0$ equals $\mathrm{MV}(\mathscr{P}_1, \mathscr{P}_2) = 3$, the point $(-1, -1)$ is the unique solution (with multiplicity 1) in $\mathbb{T} = (\mathbb{C}^\times)^2$. To explain this discrepancy with respect to the mixed-volume bound, first extend the relations $\hat{f}_1 = \hat{f}_2 = 0$ to an appropriate

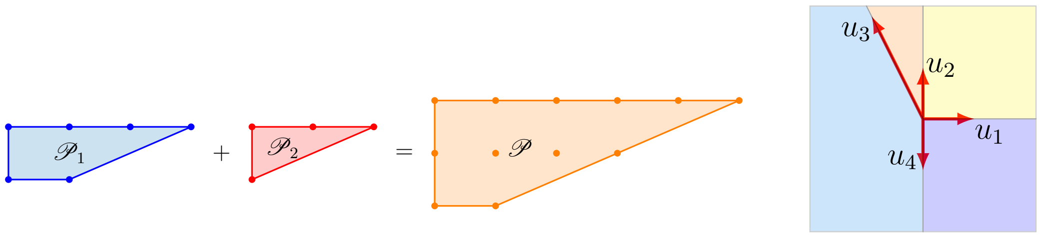


Figure 10.1: Polytopes and fan of the Hirzebruch surface from Example 10.2.1.

toric variety. The polytopes and the fan are illustrated in Figure 10.1. The facet matrix $\mathbf{F}$ is

$$\mathbf{F} = [u_1\ u_2\ u_3\ u_4] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}.$$

The toric variety $X = X_\Sigma$ is the Hirzebruch surface $\mathscr{H}_2$. The base locus in $\mathbb{C}^4$ is given by $Z = \mathcal{V}_{\mathbb{C}^4}(x_1, x_3) \cup \mathcal{V}_{\mathbb{C}^4}(x_2, x_4)$. The divisor $D_{\mathscr{P}_2}$ is $D_{\mathscr{P}_2} = D_4$ (i.e. $a_{2,1} = a_{2,2} = a_{2,3} = 0, a_{2,4} = 1$, or $a_2 = (0, 0, 0, 1)^\top$). The homogenization of the monomials t^m in $\hat{f}_2$ is given by $\mathbf{F}^\top m + a_2$:

$$\mathbf{F}^\top \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which gives $f_2 = x_4 + x_2x_3^2 + x_1x_2x_3 + x_1^2x_3 \in S_{[D_4]}$. Analogously, $f_1 = x_3x_4 + x_1x_4 + x_2x_3^3 + x_1x_2x_3^2 + x_1^2x_2x_3 + x_1^3x_2 \in S_{[D_3+D_4]}$. The vanishing locus $\mathcal{V}_X(f_1, f_2)$ on X consists of three points, with Cox coordinates

$$z_1 = (-1, -1, 1, 1),\ z_2 = (0, -1, 1, 1),\ z_3 = (1, -1, 0, 1).$$

Hence, the relations $\hat{f}_1 = \hat{f}_2 = 0$ define three isolated points on X, which is the expected number. Note that $\pi(z_1)$ is the toric solution $(-1, -1)$ (π denotes the quotient $\pi : \mathbb{C}^4 \setminus Z \to X$) and the other solutions are on the boundary of the torus: $\pi(z_2) \in D_1, \pi(z_3) \in D_3$. $\triangle$

10.3 $\mathbb{G}$-orbits in the Cox construction

Let $X = X_\Sigma$ be a compact toric variety corresponding to a complete fan Σ and let $U \subset X$ be as in Theorem 5.3.1. As mentioned in the introduction, the Cox homotopy algorithm will track a set of Cox coordinates for a point $p \in U \subset X$ by slicing the $\mathbb{G}$-orbit $\pi^{-1}(p)$ with a linear space of complementary dimension. In order to understand what this dimension is and how many representatives there are in such a linear space, this section is devoted to an explicit description of the dimension and degree of the projective closure of $\mathbb{G}$-orbits $\mathbb{G} \cdot z \subset \mathbb{C}^k \setminus Z$ in the Cox construction.

10.3.1 Orbit parametrization

Taking $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^\times)$ of the exact sequence (10.1) gives the explicit description for $\mathbb{G}$,

$$\mathbb{G} = \ker \mathrm{Hom}_\mathbb{Z}(\mathbf{F}^\top, \mathbb{C}^\times) = \{g \in (\mathbb{C}^\times)^k \mid g^{\mathbf{F}_{1,:}} = \cdots = g^{\mathbf{F}_{n,:}} = 1\},$$

as a subgroup of $(\mathbb{C}^\times)^k$. The first aim is to parametrize the orbit

$$\mathbb{G} \cdot z = \{g \cdot z = (g_1 z_1, \ldots, g_k z_k) \in (\mathbb{C}^\times)^k \mid g \in \mathbb{G}\}$$

for $z \in \mathbb{C}^k \setminus Z$. Note that if $\mathrm{Cl}(X) \simeq \mathbb{Z}^{k-n}$ is free, then $\mathbb{G} = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X), \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^{k-n}$ is a torus. In general, $\mathbb{G}$ is a quasitorus of dimension $k - n$, i.e. $\mathbb{G}$ is isomorphic to the direct sum of a torus and a finite abelian group.

To find the orbit parametrization of $\mathbb{G} \cdot z$, the first step is to compute the Smith normal form of the transpose of the facet matrix $\mathbf{F}^\top \in \mathbb{Z}^{k \times n}$:

$$\mathbf{P}\mathbf{F}^\top\mathbf{Q} = \mathrm{diag}(s_1, \ldots, s_n),$$

where $\mathrm{diag}(s_1, \ldots, s_n)$ is a diagonal matrix of size $k \times n$ with the invariant factors s_i of $\mathbf{F}^\top$ on its diagonal. Note that $s_i \neq 0$ since $\mathbf{F}$ comes from a complete fan. The submatrix $\mathbf{P}''$ of $\mathbf{P}$ containing the last $k-n$ rows of $\mathbf{P}$ is a $\mathbb{Z}$-basis for the kernel of $\mathbf{F}:\mathbb{Z}^k \to N$. The submatrix of $\mathbf{P}$ containing its first n rows is denoted by $\mathbf{P}'$. Note that

$$\mathrm{Cl}(X) \simeq \mathbb{Z}^k / \operatorname{im}\ \mathbf{F}^\top \simeq \mathbb{Z}/s_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/s_n\mathbb{Z} \oplus \mathbb{Z}^{k-n}.$$

Because the matrix $\mathbf{P}$ is a representation of the map $\mathcal{P}$ from (10.1), this isomorphism may be written explicitly as

$$[\sum_{i=1}^k a_i D_i] \mapsto (\underbrace{(\mathbf{P}'a)_1 + s_1\mathbb{Z}, \ldots, (\mathbf{P}'a)_n + s_n\mathbb{Z}}_{\mathbb{Z}/s_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/s_n\mathbb{Z}}, \underbrace{(\mathbf{P}''a)_1, \ldots, (\mathbf{P}''a)_{k-n}}_{\mathbb{Z}^{k-n}}) \tag{10.3}$$

where $(\mathbf{P}'a)_i$ is the i-th entry of the matrix-vector product $\mathbf{P}'a$, and likewise for $\mathbf{P}''a$. Furthermore,

$$\mathbb{G} = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{C}^\times) = W_1 \oplus \cdots \oplus W_n \oplus (\mathbb{C}^\times)^{k-n}$$

where $W_i \subset \mathbb{C}^\times$ is the multiplicative group of s_i-th roots of unity. The inclusion $\mathbb{G} \hookrightarrow (\mathbb{C}^\times)^k$ is the dual map $\mathcal{P}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}, \mathbb{C}^\times)$ of (10.3), given by

$$\mathbb{G} \simeq \left(\bigoplus_{i=1}^n W_i\right) \oplus (\mathbb{C}^\times)^{k-n} \to (\mathbb{C}^\times)^k, \quad (w,\lambda) \mapsto (w^{\mathbf{P}'_{:,1}}\lambda^{\mathbf{P}''_{:,1}}, \ldots, w^{\mathbf{P}'_{:,k}}\lambda^{\mathbf{P}''_{:,k}}), \tag{10.4}$$

where $\mathbf{P}'_{:,i}$ denotes the i-th column of $\mathbf{P}'$ and likewise for $\mathbf{P}''$. Thus, $\mathbb{G} \subset (\mathbb{C}^\times)^k$ is a union of tori isomorphic to

$$\mathbb{T}_{\mathbf{P}''} = \{(\lambda^{\mathbf{P}''_{:,1}}, \ldots, \lambda^{\mathbf{P}''_{:,k}}) \in (\mathbb{C}^\times)^k \mid \lambda \in (\mathbb{C}^\times)^{k-n}\} \simeq (\mathbb{C}^\times)^{k-n}.$$

Moreover, the following statement follows immediately from this discussion.

Lemma 10.3.1. *For $z \in \mathbb{C}^k \setminus Z$, the orbit $\mathbb{G}\cdot z$ of $z = (z_1, \ldots, z_k)$ is parametrized by the map*

$$\left(\bigoplus_{i=1}^n W_i\right) \oplus (\mathbb{C}^\times)^{k-n} \to \mathbb{C}^k \setminus Z \quad \textit{given by} \quad (w,\lambda) \mapsto (w^{\mathbf{P}'_{:,1}}\lambda^{\mathbf{P}''_{:,1}} z_1, \ldots, w^{\mathbf{P}'_{:,k}}\lambda^{\mathbf{P}''_{:,k}} z_k). \tag{10.5}$$

Remark 10.3.2. Note that the image of (10.4) is the orbit $\mathbb{G} \cdot z$ of $z = (1, \dots, 1)$.

For a subset $\mathcal{W} \subset \mathbb{C}^k \setminus Z$, its Zariski closure in $\mathbb{P}^k$ is denoted by

$$\overline{\mathcal{W}} = \overline{\{(1 : z_1 : \dots : z_k) \in \mathbb{P}^k \mid (z_1, \dots, z_k) \in \mathcal{W}\}} \subset \mathbb{P}^k.$$

Corollary 10.3.3 follows from applying Lemma 10.3.1 to $\overline{\mathbb{G} \cdot z}$.

Corollary 10.3.3. *For $z \in \mathbb{C}^k \setminus Z$, the dimension and degree of the projective variety $\overline{\mathbb{G} \cdot z}$ depend only on which $(\mathbb{C}^\times)^k$-orbit z belongs to. Equivalently, they only depend on the set of indices $\mathscr{I} = \{i \mid z_i \neq 0\}$.*

10.3.2 Orbit dimension

By Corollary 10.3.3, the dimension of $\overline{\mathbb{G} \cdot z}$ is constant on the dense torus $(\mathbb{C}^\times)^k$ of $\mathbb{C}^k \setminus Z$. In fact, it is constant on an even larger open subset of X by some results from geometric invariant theory.

Lemma 10.3.4. *If $z \in \pi^{-1}(U) \subset \mathbb{C}^k \setminus Z$, where U is as in Theorem 5.3.1, then* $\dim \overline{\mathbb{G} \cdot z} = k - n$.

Proof. The set $\pi^{-1}(U) \subset \mathbb{C}^k \setminus Z$ is the set of stable points for the action of $\mathbb{G}$ on $\mathbb{C}^k \setminus Z$ (these are the points z whose stabilizer $\mathbb{G}_z$ consists of finitely many points, see e.g. [53, Proposition 1.26] or [54, Chapter 1, §4]). Therefore $\dim \mathbb{G} \cdot z = \dim \overline{\mathbb{G} \cdot z} = \dim \mathbb{G} = k - n$. □

For points z outside of $\pi^{-1}(U)$, the orbit $\mathbb{G} \cdot z$ might not be closed in $\mathbb{C}^k \setminus Z$, and a general formula for the dimension is given by $\dim \mathbb{G} \cdot z = \dim \mathbb{G} - \dim \mathbb{G}_z$, where $\mathbb{G}_z$ is the stabilizer of $\mathbb{G}$ at z. The following example illustrates what may happen.

Example 10.3.1. Consider the toric threefold $X = X_\Sigma$ corresponding to a pyramid in $\mathbb{R}^3$ whose normal fan Σ has rays generated by the columns of

$$F = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{bmatrix}.$$

The polytope and fan are illustrated in Figure 10.2. The irrelevant ideal is $B = \langle x_1, x_2x_3, x_2x_5, x_3x_4, x_4x_5 \rangle$.

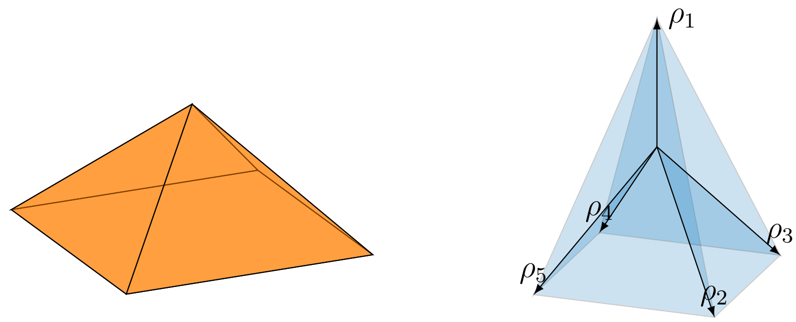


Figure 10.2: Polytope and normal fan from Example 10.3.1.

The corresponding two-dimensional base locus is $Z = V_{\mathbb{C}^k}(x_1, x_2, x_4) \cup V_{\mathbb{C}^k}(x_1, x_3, x_5)$. A Smith normal form computation yields

$$\begin{array}{c} \mathbf{P}' \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \mathbf{P}'' \left\{ \begin{array}{c} \\ \\ \end{array} \right. \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \end{bmatrix} \mathbf{F}^\top \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

such that by Lemma 10.3.1 the orbit of $(z_1, z_2, z_3, z_4, z_5)$ is parametrized by

$$(\lambda_1^2\lambda_2^2 z_1, \lambda_1 z_2, \lambda_2 z_3, \lambda_1 z_4, \lambda_2 z_5), \quad (\lambda_1, \lambda_2) \in (\mathbb{C}^\times)^2. \tag{10.6}$$

The subset $U \subset X$ for which $\pi^{-1}(U) \to U$ is geometric is the complement in X of the torus invariant point $p \in X$ corresponding to $\sigma = \text{cone}(\rho_2, \rho_3, \rho_4, \rho_5)$. It is clear that $z_{\{1\}} = (1, 0, 0, 0, 0) \in \pi^{-1}(p)$. From Equation (10.6), one can see that the orbit $\mathbb{G} \cdot z_{\{1\}}$ has dimension 1. This is the unique closed $\mathbb{G}$-orbit in $\pi^{-1}(p)$, and one can check that the stabilizer $\mathbb{G}_{z_{\{1\}}}$ is one-dimensional.

Since σ is the smallest cone of Σ containing ρ_3 and ρ_5, then the $(\mathbb{C}^\times)^5$-orbit

$$(\mathbb{C}^\times)^{\{1,2,4\}} = \{(z_1, z_2, 0, z_4, 0) \mid (z_1, z_2, z_4) \in (\mathbb{C}^\times)^3\} \subset \mathbb{C}^5 \setminus Z$$

is contained in $\pi^{-1}(p)$. The same holds for ρ_2 and ρ_4. The fiber $\pi^{-1}(p)$ has dimension 3. From Equation (10.6) it is clear that the orbit $\mathbb{G} \cdot z$ has dimension 2 for $z \in (\mathbb{C}^\times)^{\{1,2,4\}}$. Moreover, these orbits are not closed in $\mathbb{C}^k \setminus Z$, as they contain $\mathbb{G} \cdot z_{\{1\}}$ in their closure. $\triangle$

10.3.3 Orbit degree

For a finite set $\mathcal{A} = \{\alpha_0, \dots, \alpha_s\} \subset \mathbb{Z}^\ell$ let $X_\mathcal{A} \subset \mathbb{P}^s$ denote the projective toric variety obtained as the closure of the image of $(\mathbb{C}^\times)^\ell \to \mathbb{P}^s$, where $(t_1, \dots, t_\ell) \mapsto (t^{\alpha_0} : \cdots : t^{\alpha_s})$. The notation $\mathcal{A}$ is also used to denote the matrix of ℓ-tuples in $\mathcal{A}$, i.e. $\mathcal{A} = [\alpha_0 \ \cdots \ \alpha_s] : \mathbb{Z}^{s+1} \to \mathbb{Z}^\ell$. One may compute the degree of the projective variety $X_\mathcal{A}$ using the general version of Kushnirenko's Theorem.

Theorem 10.3.5 (Kushnirenko's Theorem [55]). *Let $X_\mathcal{A} \subset \mathbb{P}^s$ be the projective toric variety defined by*

$$\mathcal{A} = \begin{bmatrix} 1 & \cdots & 1 \\ m_0 & \cdots & m_s \end{bmatrix} : \mathbb{Z}^{s+1} \to \mathbb{Z}^\ell.$$

Let $\Delta = \operatorname{conv}(m_0, \dots, m_s) \subset \mathbb{R}^{\ell-1}$ be the polytope obtained by taking the convex hull of the lattice points $m_0, \dots, m_s \in \mathbb{Z}^{\ell-1}$ and suppose that Δ has dimension $\ell - 1$. Then,

$$\deg X_\mathcal{A} = \frac{(\ell-1)!}{q} \operatorname{Vol}(\Delta),$$

where $\operatorname{Vol}(\cdot)$ denotes the Euclidean volume and q is the lattice index of $\operatorname{im}([m_0 \ \cdots \ m_s] : \mathbb{Z}^{s+1} \to \mathbb{Z}^{\ell-1})$ in $\mathbb{Z}^{\ell-1}$. That is, for almost all choices of coefficients $c_{ij} \in \mathbb{C}$, the system of equations

$$\sum_{j=0}^{s} c_{ij} \lambda^{m_j} = 0, \qquad i = \ell - 1$$

has exactly $(\ell-1)!\,\mathrm{Vol}(\Delta)$ *solutions* $\lambda \in (\mathbb{C}^\times)^{\ell-1}$.

With the notation of Subsection 10.2.2, let $\mathscr{I} \subset \{1,\dots,k\}$ be a subset of indices such that there is a cone $\sigma \in \Sigma$ containing all $\rho_i \in \Sigma(1)$ for which $i \notin \mathscr{I}$, and none of the other rays. Equivalently, $\mathscr{I}$ is such that $\mathrm{cone}(e_i, i \notin \mathscr{I}) \in \Sigma'$, where e_i is the i-th standard basis vector of $\mathbb{R}^k$. The $(\mathbb{C}^\times)^k$-orbit $\{z \in \mathbb{C}^k \setminus Z \mid z_i \neq 0, \text{ for all } i \in \mathscr{I} \text{ and } z_i = 0 \text{ for all } i \notin \mathscr{I}\}$ is denoted by $(\mathbb{C}^\times)^{\mathscr{I}} \simeq (\mathbb{C}^\times)^{|\mathscr{I}|}$. Let $\mathcal{P}^{\mathscr{I}}$ be the restriction of $\mathcal{P}$ from (10.1) to $\bigoplus_{i\in\mathscr{I}} \mathbb{Z}\cdot D_i$ and consider the exact sequence

$$0 \longrightarrow \ker \mathcal{P}^{\mathscr{I}} \longrightarrow \bigoplus_{i\in\mathscr{I}} \mathbb{Z}\cdot D_i \xrightarrow{\mathcal{P}^{\mathscr{I}}} \mathrm{Cl}(X) \longrightarrow \mathrm{coker}\, \mathcal{P}^{\mathscr{I}} \longrightarrow 0. \tag{10.7}$$

Taking duals shows that $(\mathcal{P}^{\mathscr{I}})^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}^{\mathscr{I}}, \mathbb{C}^\times) : G \to (\mathbb{C}^\times)^{\mathscr{I}}$ is given in coordinates by

$$\mathbb{G} \simeq \left(\bigoplus_{i=1}^{n} W_i\right) \oplus (\mathbb{C}^\times)^{k-n} \to (\mathbb{C}^\times)^{\mathscr{I}} \simeq (\mathbb{C}^\times)^{|\mathscr{I}|}, \quad (w,\lambda) \mapsto (w^{\mathbf{P}'_{:,i}} \lambda^{\mathbf{P}''_{:,i}})_{i\in\mathscr{I}}. \tag{10.8}$$

By Lemma 10.3.1, the image of $(\mathcal{P}^{\mathscr{I}})^\vee$ is the orbit $\mathbb{G}\cdot z_{\mathscr{I}}$ of $z_{\mathscr{I}} = \sum_{i\in\mathscr{I}} e_i$, where e_i is the i-th standard basis vector of $\mathbb{C}^k$. The closure $\overline{\mathbb{G}\cdot z_{\mathscr{I}}}$ is a union of a number of copies of the projective toric variety $X_{\mathcal{A}_{\mathscr{I}}}$ where $\mathcal{A}_{\mathscr{I}}$ is given by the columns of

$$\mathcal{A}_{\mathscr{I}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & \mathbf{P}''_{:,\mathscr{I}} & \end{bmatrix},$$

and $\mathbf{P}''_{:,\mathscr{I}}$ is the submatrix of $\mathbf{P}''$ containing the columns indexed by $\mathscr{I}$.

Example 10.3.2. For $\mathscr{I} = \{1,\dots,k\}$, the associated toric variety is $X_{\mathcal{A}_{\mathscr{I}}} = \overline{\mathbb{T}_{\mathbf{P}''}}$. Indeed,

$$X_{\mathcal{A}} = X_{\mathcal{A}_{\mathscr{I}}} := \overline{\left\{(1 : \lambda^{\mathbf{P}''_{:,1}} : \dots : \lambda^{\mathbf{P}''_{:,k}}) \mid \lambda \in (\mathbb{C}^\times)^{k-n}\right\}} \subset \mathbb{P}^k,$$

where $\mathcal{A} = \mathcal{A}_{\mathscr{I}} = \{\alpha_0, \ldots, \alpha_k\} \subset \mathbb{Z}^{k-n}$ is given by the columns of

$$\mathcal{A} = \mathcal{A}_{\mathscr{I}} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & & \mathbf{P}'' & \end{bmatrix}.$$

The dimension of $\overline{\mathbb{T}_{\mathbf{P}''}} = X_{\mathcal{A}}$ is $k-n$, by the fact that $\mathbf{P}''$ has rank $k-n$ and [15, Proposition 2.1.2]. $\triangle$

By Theorem 10.3.5, under the assumption that $\mathbf{P}''_{:,\mathscr{I}}$ has rank $k-n$, the degree of $X_{\mathcal{A}_{\mathscr{I}}}$ is $(k-n)!/q_{\mathscr{I}} \operatorname{Vol}(\Delta^{\mathscr{I}})$, where $\Delta^{\mathscr{I}} = \operatorname{conv}(\{0\} \cup \{\mathbf{P}''_{:,i}, i \in \mathscr{I}\}) \subset \mathbb{R}^{k-n}$ and $q_{\mathscr{I}}$ is the lattice index of $\operatorname{im} \mathbf{P}''_{:,\mathscr{I}}$ in $\mathbb{Z}^{k-n}$. The polytope $\Delta^{\mathscr{I}}$ is the orbit polytope corresponding to $\mathscr{I}$.

Proposition 10.3.6. *Let $\mathscr{I} \subset \{1, \ldots, k\}$ be a subset of indices such that $z_{\mathscr{I}} = \sum_{i \in \mathscr{I}} e_i \in \pi^{-1}(U)$, with U as in Theorem 5.3.1. For all z in the $(\mathbb{C}^\times)^k$-orbit $(\mathbb{C}^\times)^{\mathscr{I}} \subset \mathbb{C}^k \setminus Z$,*

$$\deg \overline{\mathbb{G} \cdot z} = \frac{s_{\mathscr{I}}}{q_{\mathscr{I}}} (k-n)! \operatorname{Vol}(\Delta^{\mathscr{I}}),$$

where $q_{\mathscr{I}}, \Delta^{\mathscr{I}}$ are the lattice index and orbit polytope as above, $s_{\mathscr{I}}$ is the product of the invariant factors of $\ker \mathcal{P}^{\mathscr{I}}$, and $\operatorname{Vol}(\cdot)$ denotes the Euclidean volume. In particular, $\overline{\mathbb{G} \cdot z}$ is a union of $s_{\mathscr{I}}$ irreducible projective varieties of degree $\deg(\overline{\mathbb{G} \cdot z})/s_{\mathscr{I}}$, each of which is equal to the projective toric variety $X_{\mathcal{A}_{\mathscr{I}}} \subset \overline{(\mathbb{C}^\times)^{\mathscr{I}}} \subset \mathbb{P}^k$ up to an invertible diagonal scaling.

Proof. By Corollary 10.3.3, it suffices to show the proposition for $z = z_{\mathscr{I}}$. From (10.8), $\overline{\mathbb{G} \cdot z_{\mathscr{I}}}$ is a union of varieties obtained by scaling $X_{\mathcal{A}_{\mathscr{I}}}$. The dense torus of $X_{\mathcal{A}_{\mathscr{I}}}$, denoted by $\mathbb{T}_{\mathbf{P}''_{:,\mathscr{I}}}$, is the projection of $\mathbb{T}_{\mathbf{P}''}$ onto $(\mathbb{C}^\times)^{\mathscr{I}}$. The toric variety $X_{\mathcal{A}_{\mathscr{I}}}$ has dimension $k-n$ by Lemma 10.3.4 and degree $(k-n)!/q_{\mathscr{I}} \operatorname{Vol}(\Delta^{\mathscr{I}})$ by Theorem 10.3.5. By taking the dual of (10.7), one can see that $\mathbb{G} \cdot z_{\mathscr{I}}$ is isomorphic to the quasitorus

$$\operatorname{Hom}_{\mathbb{Z}} \left(\frac{\bigoplus_{i \in \mathscr{I}} \mathbb{Z} \cdot D_i}{\ker \mathcal{P}^{\mathscr{I}}}, \mathbb{C}^\times \right),$$

which is a direct sum of a finite group of order $s_{\mathscr{I}}$ and the torus $\mathbb{T}_{\mathbf{P}''_{:,\mathscr{I}}}$. $\square$

Remark 10.3.7. Note that when $\mathscr{I} = \{1, \ldots, k\}$, $\ker \mathcal{P}^{\mathscr{I}} = \ker \mathcal{P} = \operatorname{im} F^\top$, such that the invariant factors of $\ker \mathcal{P}$ are $s_1, \ldots, s_n$ and $s = s_{\mathscr{I}} = \prod_{i=1}^n s_i$. Moreover, the lattice map $\mathbf{P}'' = \mathbf{P}''_{:,\mathscr{I}}$ is surjective, such that $q_{\mathscr{I}} = 1$.

Corollary 10.3.8. *If $\mathrm{Cl}(X) \simeq \mathbb{Z}^{k-n}$ is free, then for $\mathscr{I}$ as in 10.3.6, the degree of $\overline{\mathbb{G} \cdot z} \subset \mathbb{P}^k$ is $\deg \overline{\mathbb{G} \cdot z} = \mathrm{Vol}(\Delta^{\mathscr{I}})$ for all z in the $(\mathbb{C}^\times)^k$-orbit $(\mathbb{C}^\times)^{\mathscr{I}}$. Moreover, $\overline{\mathbb{G} \cdot z} \subset \mathbb{P}^k$ is a(n irreducible) toric variety.*

Proposition 10.3.6 provides a direct way of computing $\deg(\overline{\mathbb{G} \cdot z})$ for any $z \in \pi^{-1}(U)$. This computation is implemented in Macaulay2 [5]. The code is available at `https://mathrepo.mis.mpg.de/CoxHomotopies/index.html`. The following two examples illustrate applications of Proposition 10.3.6.

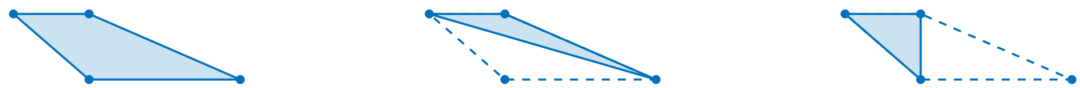
Example 10.3.3. The *double pillow* is the toric variety $X = X_{\mathcal{A}} \subset \mathbb{P}^4$ where $\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}$ [17, Subsection 3.3]. The fan corresponding to the double pillow has facet matrix $\mathbf{F} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ with Smith normal form

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \mathbf{F}^\top \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This shows that the class group has torsion: $\mathrm{Cl}(X) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^2$ via

$$[\sum_{i=1}^4 a_i D_i] \mapsto (a_2 - a_1 + 2\mathbb{Z}, a_2 + a_4, a_1 + a_3).$$

The reductive group $\mathbb{G}$ is isomorphic to $\{-1, 1\} \oplus (\mathbb{C}^\times)^2$ via $(w, \lambda_1, \lambda_2) \mapsto (w^{-1}\lambda_2, w\lambda_1, \lambda_2, \lambda_1)$. The closure is a union of two planes in $\mathbb{P}^4$, and every orbit closure $\overline{\mathbb{G} \cdot z}$ for $z \in (\mathbb{C}^\times)^k$ is equal to $\overline{\mathbb{G}}$



$\Delta^{\{1,2,3,4\}} = \Delta^{\{2,3,4\}} = \Delta^{\{1,2,4\}}$ $\quad \Delta^{\{1,2,3\}} = \Delta^{\{1,2\}} = \Delta^{\{2,3\}}$ $\quad \Delta^{\{1,3,4\}} = \Delta^{\{3,4\}} = \Delta^{\{1,4\}}$

Figure 10.3: Orbit polytopes from Example 10.3.4.

up to a diagonal change of coordinates. Therefore, these orbits have degree 2. The orbit polytope $\Delta^{\mathscr{I}}$ is the standard simplex in $\mathbb{R}^2$. $\triangle$

Example 10.3.4. Consider again the Hirzebruch surface $\mathscr{H}_2$ from Example 10.2.1. The Smith normal form of the facet matrix $\mathbf{F}^\top$ is given by

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \mathbf{F}^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{hence} \quad \mathbf{P}'' = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}.$$

The class group $\mathrm{Cl}(\mathscr{H}_2)$ is free, $\mathbb{G} \simeq (\mathbb{C}^\times)^2$, and the orbit of $z = (z_1, \ldots, z_4) \in \mathbb{C}^4 \setminus Z$ is parametrized by

$$\mathbb{G} \cdot z = \{(z_1\lambda_1^{-1}, z_2\lambda_1^2\lambda_2^{-1}, z_3\lambda_1^{-1}, z_4\lambda_2^{-1}) \mid (\lambda_1, \lambda_2) \in (\mathbb{C}^\times)^2\}. \tag{10.9}$$

The orbit polytopes are shown in Figure 10.3. By Corollary 10.3.8, if $z \in (\mathbb{C}^\times)^3$, $\overline{\mathbb{G} \cdot z}$ has degree $\mathrm{Vol}(\Delta^{\{1,2,3,4\}}) = 3$. If $z \in (\mathbb{C}^\times)^{\{1,3,4\}} \cup (\mathbb{C}^\times)^{\{1,2,3\}}$, the degree drops to $\mathrm{Vol}(\Delta^{\{1,3,4\}}) = \mathrm{Vol}(\Delta^{\{1,2,3\}}) = 1$. $\triangle$

10.4 Cox homotopies: coefficient-parameter theory and algorithms

As in the previous section, $X = X_\Sigma$ is an n-dimensional toric variety such that $\pi : \mathbb{C}^k \setminus Z \to X$ is an almost geometric quotient which is constant over $\mathbb{G}$-orbits on $\mathbb{C}^k \setminus Z$. Moreover, on the dense open subset $U \subset X$, $\pi_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is geometric.

This section describes the Cox homotopy algorithm for solving polynomial systems on X by tracking points in the total coordinate space $\mathbb{C}^k$. For any $p \in U$, the $\mathbb{G}$-orbit $\pi^{-1}(p)$ is sliced with a general linear space of dimension n to get degree-many representatives for $\pi^{-1}(p)$. Theorem 10.4.1 establishes that the orbits in the homotopy remain disjoint. Thus, the algorithm may track only one representative per orbit when the target system is Bernstein-general. Section 10.4.2 considers the case of a non-generic target system. This presents a subtlety not encountered in the multihomogeneous case—namely, some paths in the total space $\mathbb{C}^k$ may either diverge or converge to a point in the base locus (see Example 10.4.1). To address this subtlety, an endgame is proposed in Section 10.4.2 for finding a path whose endpoint represents a point in X. Pseudocode for the main algorithm and subroutines are in Section 10.4.3. Section 10.4.4 outlines the orthogonal patching strategy considered in Experiment 2.

10.4.1 Coefficient-parameter homotopy in the dense torus

As the Cox homotopy is akin to a coefficient-parameter homotopy, first consider the *family* of all systems with fixed supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ in M (cf. Subsection 10.2.3). Thus, consider $\hat{h}_i(t; c) = \sum_{m \in \mathcal{A}_i} c_{i,m} t^m$, $i = 1, \ldots, n$, where the parametric coefficients $c_{i,m}$ may vary. The affine space of parameters is denoted by $\mathbb{C}^{\mathcal{A}}$. This section furthermore assumes hereafter that the Minkowski sum $\mathscr{P} = \mathscr{P}_1 + \cdots + \mathscr{P}_n$, where $\mathscr{P}_i = \operatorname{conv}(\mathcal{A}_i)$, has full dimension n, so that the mixed-volume bound $\delta = \operatorname{MV}(\mathscr{P}_1, \ldots, \mathscr{P}_n)$ from Theorem 6.1.2 is positive.

The homotopy considered in the main algorithm (Algorithm 2) is given by

$$(H(x; \tau), L(x)) = 0, \tag{10.10}$$

where $L(x) = Ax + b$ determines an (affine) linear space $\{x \in \mathbb{C}^k \mid L(x) = 0\}$ (which is also denoted by L) and H is obtained by homogenizing

$$\hat{H}(t; \phi(\tau)) = \left(\hat{h}_1(t; \phi(\tau)), \ldots, \hat{h}_n(t; \phi(\tau))\right) = 0 \tag{10.11}$$

for path $\phi : [0,1] \to \mathbb{C}^{\mathcal{A}}$ which is assumed to be smooth. Furthermore, ϕ, A, and b must satisfy certain genericity conditions as outlined in Theorem 10.4.1. Such homotopies are referred to as Cox homotopies.

Theorem 10.4.1 gives the basis for Algorithm 2. To prove this theorem, it is natural to consider the incidence variety defined for fixed L by

$$\widehat{\mathcal{V}}_{\mathcal{A},L} = \left\{(x,c) \in \left((\mathbb{C}^k \setminus Z) \cap L\right) \times \mathbb{C}^{\mathcal{A}} \mid H(x;c) = 0\right\}. \tag{10.12}$$

Solution paths for the Cox homotopy in Equation (10.10) correspond to *lifts* of the parameter path ϕ to the variety $\widehat{\mathcal{V}}_{\mathcal{A},L}$. For generic ϕ, A, and b, Theorem 10.4.1 implies that there are exactly $d\delta$ such paths, where d denotes the degree of the $\mathbb{G}$-orbit closure $\overline{\mathbb{G} \cdot (1,1,\ldots,1)}$ and $\delta = \mathrm{MV}(\mathscr{P}_1, \ldots, \mathscr{P}_n)$. Note that d may be computed using Remark 10.3.7.

Theorem 10.4.1. *There exist Zariski open subsets $\mathcal{U}_{\mathcal{A}} \subset \mathbb{C}^{\mathcal{A}}$ and $\mathcal{U}_{A,b} \subset \mathbb{C}^{(k-n)\times k} \times \mathbb{C}^{k-n}$ such that for all $c^* \in \mathcal{U}_{\mathcal{A}}$ and all $(A,b) \in \mathcal{U}_{A,b}$, with $L(x) = Ax + b$,*

1. *$V_T(\hat{H}(t;c^*)) = \{\zeta_1, \ldots, \zeta_\delta\}$ consists of δ points,*
2. *$V_{(\mathbb{C}^\times)^k}(H(x;c^*), \mathcal{L}(x)) = \{z_{ij} \mid i = 1, \ldots, \delta, j = 1, \ldots, d\}$ consists of $d\delta$ points,*
3. *for some labeling of these points, $\pi(z_{ij}) = \zeta_i$ for all i, j.*

For fixed $(A,b) \in \mathcal{U}_{A,b}$ and a smooth function $\phi : [0,1] \to \mathcal{U}_{\mathcal{A}}$,

4. *the homotopy $\hat{H}(t;\phi(\tau)) = 0$ has δ trackable paths $\zeta_i : [0,1] \to (\mathbb{C}^\times)^n, i = 1, \ldots, \delta$,*
5. *the homotopy $(H(x;\phi(\tau)), \mathcal{L}(x)) = 0$ has $d\delta$ trackable paths $z_{ij} : [0,1] \to (\mathbb{C}^\times)^k$, $i = 1, \ldots, \delta, j = 1, \ldots, d$,*
6. *after relabeling, $\pi \circ z_{ij} = \zeta_i$ for all i, j.*

Proof. The variety $\mathcal{V}_{(\mathbb{C}^\times)^k}(H(x;c^*))$ is a union of $\mathbb{G}$-orbits, which have dimension $k-n$ by Lemma 10.3.4. Take $\mathcal{U}_{A,b}$ such that $\mathbb{G} \cdot (e_1 + \cdots + e_k) \cap L$ always consists of d points for L. Theorem 6.1.2, together with Remark 10.3.7, implies that the coordinate projection $\pi_{\mathcal{A}} : \widehat{\mathcal{V}}_{\mathcal{A},L} \to \mathbb{C}^{\mathcal{A}}$ is a dominant map with generically finite fibers. More precisely, $\pi_{\mathcal{A}}$ is a branched cover of degree $d\delta$, meaning that the fiber $\pi_{\mathcal{A}}^{-1}(c)$ consists of $d\delta$ points for all $c \in \mathcal{U}_{\mathcal{A}}$, where $\mathbb{C}^{\mathcal{A}} \backslash \mathcal{U}_{\mathcal{A}}$ denotes the branch locus of $\pi_{\mathcal{A}}$. The set $\mathcal{U}_{\mathcal{A}}$ is Zariski open by [56, Theorem 2.29]. Observe that $\pi_{\mathcal{A}}$ has a factorization induced by π. That is, one may may write $\pi_{\mathcal{A}} = \pi_{\mathcal{A}}^1 \circ \pi_{\mathcal{A}}^2$ where $\pi_{\mathcal{A}}^1$ and $\pi_{\mathcal{A}}^2$ are branched covers of degree d, δ, respectively. So far items (1)–(3) have been shown. Now, whenever $(z, c) \in \pi_{\mathcal{A}}^{-1}(c)$ and $c \in \mathcal{U}_{\mathcal{A}}$, the derivative $D_{z,c}\,\pi_{\mathcal{A}}$ is an invertible linear map by [56, Theorems 2.30]. This gives (5). Similarly, (4) follows for $\zeta = \pi(z)$ by considering $D_{\zeta,c}\,\pi_{\mathcal{A}}^1$ and applying the chain rule. For (6), simply note that $(A, b) \in \mathcal{U}_{A,b}$ implies there are exactly d paths $z_{i1}, \ldots, z_{id}$ for each ζ_i satisfying $\zeta_i = \pi \circ z_{ij}$. □

Remark 10.4.2. It is possible to allow the linear space L to change as τ moves from 1 to 0. In other words, one might consider a Cox homotopy given by $(H(x;\tau), L_\tau(x)) = 0$ where $L_\tau(x)$ defines a sufficiently generic, continuous path in the Grassmannian $Gr(n, k)$. Subsection 10.4.4 exploits this observation and proposes the strategy of orthogonal slicing for Cox homotopies.

Remark 10.4.3. Since the complement of $\mathcal{U}_{\mathcal{A}}$ is of complex codimension at least 1 in $\mathbb{C}^{\mathcal{A}}$, the image of a general 1-real-dimensional parameter path $\phi : [0, 1] \to \mathbb{C}^{\mathcal{A}}$ with $c_0 = \phi(0) \in \mathcal{U}_{\mathcal{A}}$, $c_1 = \phi(1) \in \mathcal{U}_{\mathcal{A}}$ is contained in $\mathcal{U}_{\mathcal{A}}$. A standard trick for obtaining such a general path is by setting $\phi(\tau) = (1 - \tau)c_0 + \gamma\tau c_1$, where $\gamma \in \mathbb{C}$ is a random constant. This is known as the gamma trick, see for instance [35, Lemma 7.1.3]. In this case, the Cox homotopy (H, L) is given by $H(x;\tau) = (1 - \tau)G + \gamma\tau F$, where $F(x) = H(x; 0)$ and $G(x) = H(x; 1)$. If the start system (G, L) is suitably generic, one can set $\gamma = 1$.

An important consequence of Theorem 10.4.1 is that for parameter paths $\phi : [0, 1] \to \mathcal{U}_{\mathcal{A}}$ it is sufficient to track only $\delta = \mathrm{MV}(\mathscr{P}_1, \ldots, \mathscr{P}_n)$ paths in the Cox homotopy (H, L) to track the paths defined by $H(x;\tau)$ on X. Indeed, for $i = 1, \ldots, \delta$, it suffices to track only one of the paths $z_{ij}(\tau), j = 1, \ldots, d$ for τ going from 1 to 0, since all of these will land on a representative of the same $\mathbb{G}$-orbit when $\tau = 0$.

10.4.2 Degenerating orbits and a specialized endgame

For a set $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of supports, fix a smooth parameter path $\phi : [0,1] \to \mathbb{C}^{\mathcal{A}}$ and consider the corresponding homotopy $\hat{H}(t;\tau)$. Denote $\hat{F}(t) = \hat{H}(t;0)$, $\hat{G}(t) = \hat{H}(t;1)$ and F, G, H for the associated homogeneous counterparts. Furthermore assume that $\hat{G}$ defines $\delta = \mathrm{MV}(\mathscr{P}_1, \ldots, \mathscr{P}_n)$ isolated solutions in $\mathbb{T} \subset X$, but $\hat{F}$ does not. That is, $|\mathcal{V}_{\mathbb{T}}(\hat{F})| < \delta$.

In this setup, if ϕ is suitably generic, Theorem 10.4.1 implies that $\hat{H}(t;\tau),\ \tau \in (0,1]$ defines δ disjoint paths $\tau \mapsto \zeta_i(\tau), i = 1, \ldots, \delta$ in the dense torus $\mathbb{T}$. This section investigates what happens for $\tau \to 0$. Since $|\mathcal{V}_{\mathbb{T}}(\hat{F})| < \delta$ by assumption, at least one of these paths, say $\zeta_i(\tau)$, moves out of the torus $\mathbb{T}$ for $\tau \to 0$. By compactness of $X \supset \mathbb{T}$, the endpoint $\lim_{\tau \to 0^+} \zeta_i(\tau)$ exists in X and it satisfies

$$\zeta_i(0) = \lim_{\tau \to 0^+} \zeta_i(\tau) \in X \setminus \mathbb{T}.$$

Assume that $\zeta_i(0) \in U$ is an isolated point of $\mathcal{V}_X(F)$, where U is as in Theorem 5.3.1. The first step is to show that in an analytic neighborhood $\mathcal{U} \subset \mathbb{C}$ of $\tau = 0$, the $\mathbb{G}$-orbit $\pi^{-1}(\zeta_i(\tau))$ has a representative $r(\tau) = (r_1(\tau), \ldots, r_k(\tau))$ given by Puiseux series. The notation $\mathbb{C}[[\tau]]$ is used for the power series ring, $\mathbb{C}\{\{\tau\}\}$ for the field of Puiseux series, and $\mathrm{val} : \mathbb{C}\{\{\tau\}\} \to \mathbb{Q}$ for its standard valuation. As in Section 10.3, let $(\mathbb{C}^\times)^{\mathscr{I}}$ be the $(\mathbb{C}^\times)^k$-orbit $\{z \in \mathbb{C}^k \setminus Z \mid z_i \neq 0, \text{ for all } i \in \mathscr{I} \text{ and } z_i = 0 \text{ for all } i \notin \mathscr{I}\}$ and work with index sets $\mathscr{I}$ such that $(\mathbb{C}^\times)^{\mathscr{I}} \subset \pi^{-1}(U)$.

Lemma 10.4.4. *Let $\tau \mapsto \zeta_i(\tau)$ be a solution path of $\hat{H}(t;\tau)$ such that $\zeta_i(0) \in U$ is an isolated point in $\mathcal{V}_X(F)$, where U is as in Theorem 5.3.1. In a neighborhood $\mathcal{U} \subset \mathbb{C}$ of $\tau = 0$, there is an algebraic function $r : \mathcal{U} \to \mathbb{C}^k \setminus Z,\ r \in \mathbb{C}\{\{\tau\}\}^k$ such that $\pi^{-1}(\zeta_i(\tau)) = \mathbb{G} \cdot r(\tau)$ for all $\tau \in \mathcal{U}$. Moreover, if $\mathscr{I}$ is such that $\zeta_i(0) \in \pi((\mathbb{C}^\times)^{\mathscr{I}})$, the valuations of $r(\tau) = (r_1(\tau), \ldots, r_k(\tau))$ satisfy*

$$\mathrm{val}(r_i(\tau)) > 0,\ i \notin \mathscr{I} \qquad \textit{and} \qquad \mathrm{val}(r_i(\tau)) = 0,\ i \in \mathscr{I}.$$

Proof. Since $\zeta_i(0) \in U \subset X$, $\overline{\pi^{-1}(\zeta_i(0))}$ has dimension $k - n$ (Lemma 10.3.4) and for almost all choices of the linear space L, $L \cap \overline{\pi^{-1}(\zeta_i(0))}$ consists of isolated points. For such a fixed choice of L, pick a point $r_0 \in L \cap \overline{\pi^{-1}(\zeta_i(0))} \subset \mathcal{V}_{\mathbb{C}^k \setminus Z}(F, L)$. Since $\zeta_i(0)$ is an isolated point in $\mathcal{V}_X(F)$, r_0 is an isolated solution of $(F(x), L(x)) = (H(x; 0), L(x))$ in $\mathbb{C}^k \setminus Z$ and there is an open neighborhood $\mathcal{U}$ of $\tau = 0$ and an algebraic function $r : \mathcal{U} \to \mathbb{C}^k \setminus Z$ satisfying $r(0) = r_0$ and $(H(r(\tau); \tau), L(r(\tau))) = 0$. The statement about the valuations follows immediately from $r(0) \in \pi^{-1}(\zeta_i(0)) \subset (\mathbb{C}^\times)^{\mathscr{I}}$. $\square$

Remark 10.4.5. By reparametrizing, e.g. by setting $\tau \leftarrow \tau^m$ for a positive integer m, it can be assumed that $r(\tau)$ in Lemma 10.4.4 is given by a power series.

Take $\tau^* \in \mathcal{U}$ where $\mathcal{U}$ is as in Lemma 10.4.4 and suppose that $r(0) \in (\mathbb{C}^\times)^{\mathscr{I}}$. The aim is to formalize the following intuition. A $\mathbb{G}$-orbit $\mathbb{G} \cdot r(\tau^*) \subset \mathbb{C}^k \setminus Z \subset \mathbb{P}^k$ for $r(\tau^*) \in (\mathbb{C}^\times)^k$ hits the general linear space L in d points, where $d = s(k-n)! \operatorname{Vol}(\Delta^{\{1,\dots,k\}})$ (see Remark 10.3.7). When $r(\tau^*)$ moves towards $r(0) \in (\mathbb{C}^\times)^{\mathscr{I}}$, the points in $(\mathbb{G} \cdot r(\tau^*)) \cap L$ move in L. By Proposition 10.3.6, as the representative $r(\tau)$ enters $(\mathbb{C}^\times)^{\mathscr{I}}$ the orbit degree might drop, meaning that possibly $|(\mathbb{G} \cdot r(\tau^*)) \cap L| > |(\mathbb{G} \cdot r(0)) \cap L|$. The 'missing' points in $(\mathbb{G} \cdot r(0)) \cap L$ may be due to some points traveling into $(L \cap Z)$, some points traveling to infinity out of the total coordinate space, or two different points colliding at $\tau = 0$. This begs the following question: how many of the points in $(\mathbb{G} \cdot r(\tau^*)) \cap L$ eventually land in $(\mathbb{G} \cdot r(0)) \cap L$? Equivalently, if $r(\tau)$ is a representative for $\pi^{-1}(\zeta_i(\tau))$, how many of the representative paths $z_{ij}(\tau)$ in $\mathbb{C}^k \setminus Z$ travel to the orbit $\mathbb{G} \cdot r(0)$ as $\tau \to 0$?

Theorem 10.4.6. *Let $\mathcal{U} \subset \mathbb{C}$ be an open neighborhood of $0 \in \mathbb{C}$ and let $r(\tau) = (r_1(\tau), \dots, r_k(\tau))$ be a map $\mathcal{U} \to \mathbb{C}^k \setminus Z$ such that $r_i(\tau) \in \mathbb{C}[[\tau]] \subset \mathbb{C}\{\{\tau\}\}$. Let $\mathscr{I}$ be such that $r(0) \in (\mathbb{C}^\times)^{\mathscr{I}}$ and suppose that*

$$\operatorname{val}(r_i(\tau)) > 0, \; i \notin \mathscr{I} \qquad \text{and} \qquad \operatorname{val}(r_i(\tau)) = 0, \; i \in \mathscr{I}. \tag{10.13}$$

For almost all affine maps $L(x) = Ax + b$, there are $s(k-n)! \operatorname{Vol}(\Delta^{\mathscr{I}})$ k-tuples

$$z(t) = (z_1(\tau), \dots, z_k(\tau)) \in \mathbb{C}[[\tau]]^k$$

such that

$$\mathrm{val}(z_i(\tau)) > 0,\ i \notin \mathscr{I} \qquad \textit{and} \qquad \mathrm{val}(z_i(\tau)) = 0,\ i \in \mathscr{I}$$

and $z(\tau) \in G \cdot r(\tau),\ A\, z(\tau) + b = 0,$ *for* τ *in a neighborhood* $\mathcal{U}' \subset \mathcal{U}$ *of* $0 \in \mathbb{C}$*.*

Proof. The proof uses some notation from Section 10.3. In particular, recall that $\mathbf{P}$ is one of the unimodular matrices in the Smith normal form of the transposed facet matrix $\mathbf{F}^\top$ and $\mathbf{P}''$ contains its last $k-n$ rows. First consider the case where $\mathrm{Cl}(X)$ is torsion free. By Lemma 10.3.1 it suffices to show that for almost all choices of A, b, there are $(k-n)!\,\mathrm{Vol}(\Delta^{\mathscr{I}})$ $(k-n)$-tuples $\lambda(\tau) = (\lambda_1(\tau), \ldots, \lambda_{k-n}(\tau)) \in \mathbb{C}\{\{\tau\}\}^{k-n}$ such that $\mathrm{val}(\lambda_i(\tau)) = 0, i = 1, \ldots, k-n$ and

$$A(r_1(\tau)\lambda(\tau)^{P''_{:,1}}, \ldots, r_k(\tau)\lambda(\tau)^{\mathbf{P}''_{:,k}})^\top + b = 0,$$

or $\sum_{j=1}^k A_{ij} r_j(\tau)\lambda(\tau)^{\mathbf{P}''_{:,j}} + b_i = 0, i = 1, \ldots, k-n$. The solutions $\lambda(\tau)$ with $\mathrm{val}(\lambda_i(\tau)) = 0, i = 1, \ldots, k-n$ are given by

$$\lambda(\tau) = (\ell_1, \ldots, \ell_{k-n}) + \text{ higher order terms},$$

where $(\ell_1, \ldots, \ell_{k-n}) \in (\mathbb{C}^\times)^{k-n}$ is a solution of

$$\sum_{j \in \mathscr{I}} A_{ij} r_j(0) \lambda^{\mathbf{P}''_{:,j}} + b_i = 0, \quad i = 1, \ldots, k-n. \tag{10.14}$$

Indeed, by (10.13) the terms where $j \in \mathscr{I}$ correspond to the facet with facet normal $(0, \ldots, 0, 1)$ on the lower hull of the lifted point set $\{(\mathbf{P}''_{:,j}, \mathrm{val}(r_j(\tau)))\}_{j=1,\ldots,k} \subset \mathbb{R}^{k-n+1}$, see e.g. [14, Section 2]. By Theorem 10.3.5 and the fact that $r_j(0) \neq 0$ for all $j \in \mathscr{I}$, (10.14) has $(k-n)!\,\mathrm{Vol}(\Delta^{\mathscr{I}})$ solutions for almost all choices of A, b.

It remains to show that each of these solutions $\lambda(\tau)$ gives a different k-tuple

$$z(\tau) = (r_1(\tau)\lambda(\tau)^{\mathbf{P}''_{:,1}}, \ldots, r_k(\tau)\lambda(\tau)^{\mathbf{P}''_{:,k}}).$$

For $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = \{\tau \in \mathbb{C} \mid |\tau| < \varepsilon\}$ and note that there exists $\varepsilon > 0$ such that for any two solutions $\lambda(\tau)$ and $\mu(\tau)$, $\lambda(\tau^*) \neq \mu(\tau^*)$ for all $\tau^* \in \mathcal{U}_\varepsilon$ and $r_i(t^*) \neq 0$ for all $\tau^* \in \mathcal{U}_\varepsilon \setminus \{0\}$. Since $\mathbf{P}'' : \mathbb{Z}^k \to \mathbb{Z}^{k-n}$ is surjective, this implies

$$(r_1(\tau^*)\lambda(\tau^*)^{\mathbf{P}''_{:,1}}, \ldots, r_k(\tau^*)\lambda(\tau^*)^{\mathbf{P}''_{:,k}}) \neq (r_1(\tau^*)\mu(\tau^*)^{\mathbf{P}''_{:,1}}, \ldots, r_k(\tau^*)\mu(\tau^*)^{\mathbf{P}''_{:,k}}), \quad \text{for } \tau^* \in \mathcal{U}_\varepsilon \backslash \{0\}.$$

Using an analogous argument, one can see that in the case where $\mathrm{Cl}(X)$ has torsion, each irreducible component of $G \cdot r(\tau)$ contributes $(k-n)! \operatorname{Vol}(\Delta^{\mathscr{I}})$ k-tuples $z(\tau)$. $\square$

Example 10.4.1. Continuing with Example 10.3.4, this example will show how the orbit $\mathbb{G} \cdot z$ 'degenerates' from a degree 3 surface to a plane as z moves into D_4. First consider the equations defining the closure $\overline{\mathbb{G} \cdot z}$ in $\mathbb{P}^4$. It can be seen from (10.9) that for $z \in (\mathbb{C}^\times)^4$,

$$\overline{\mathbb{G} \cdot z} = V_{\mathbb{P}^4}(z_3x_1 - z_1x_3,\ z_4x_1^2x_2 - z_1^2z_2x_0^2x_4), \tag{10.15}$$

where $x_1, \ldots, x_4$ are the Cox variables and $x_0 = 0$ is the hyperplane 'at infinity' in the total coordinate space. If $r(\tau) = (r_1(\tau), r_2(\tau), r_3(\tau), r_4(\tau)) = (z_1, z_2, z_3, \tau)$, then the variety $\overline{\mathbb{G} \cdot r(\tau)}$ degenerates to $\mathcal{V}_{\mathbb{P}^4}(z_3x_1 - z_1x_3, x_0^2x_4)$ for $\tau \to 0$, which is the union of a double plane at infinity and the plane $\overline{\mathbb{G} \cdot (z_1, z_2, z_3, 0)} = \mathcal{V}_{\mathbb{P}^4}(z_3x_1 - z_1x_3, x_4)$. This means that when slicing (10.15) with a general plane L and letting $z_4 \to 0$, two out of three intersection points drift off to infinity and the other one lands on the orbit $\mathbb{G} \cdot (z_1, z_2, z_3, 0)$.

Applying the same reasoning for $z_2 \to 0$, one can see that (10.15) degenerates to $\mathcal{V}_{\mathbb{P}^4}(z_3x_1 - z_1x_3, x_1^2x_2)$, which is the union of two planes where one plane is $\mathcal{V}_{\mathbb{P}^4}(x_1, x_3)$ with multiplicity 2 and the other plane is $\overline{\mathbb{G} \cdot (z_1, 0, z_3, z_4)} = \mathcal{V}_{\mathbb{P}^4}(z_3x_1 - z_1x_3, x_2)$. Note that the intersection of the first component $\mathcal{V}_{\mathbb{P}^4}(x_1, x_3)$ with $\mathbb{C}^4$ is a component of the base locus $Z \subset \mathbb{C}^4$. This means that if one slices (10.15) with a general plane L and lets $z_2 \to 0$, two out of three intersection points move towards the base locus and only one of them lands on the orbit $\mathbb{G} \cdot (z_1, 0, z_3, z_4)$. $\triangle$

Combining Lemma 10.4.4, Remark 10.4.5 and Theorem 10.4.6, one can see that if $\zeta_i(0) \in$

$\pi((\mathbb{C}^\times)^{\mathscr{I}})$, then a fraction of $\mathrm{Vol}(\Delta^{\mathscr{I}})/\,\mathrm{Vol}(\Delta^{\{1,\ldots,k\}})$ of the representatives $z_{ij}(\tau)$ of $\zeta_i(\tau)$ will travel to $\pi^{-1}(\zeta_i(0))$ as $\tau \to 0$. This means that, although for the purpose of tracking paths in $\mathbb{T} \subset X$ it suffices to track only one representative, at the very end of the tracking process there may be a need to switch representatives in order to find homogeneous coordinates of $\zeta_i(0)$. The proposed solution is to track only one representative per path in X for $\tau \in [\tau^*, 1]$, where τ^* is 'close' to 0, e. g. $\tau^* = 0.1$. At $\tau = \tau^*$, one may initialize a specialized endgame which tries to finish the path by switching representatives at $\tau = \tau^*$ until a point in $\pi^{-1}(\zeta_i(0))$ is reached. This procedure is made explicit in Algorithm 1.

Algorithm 1 A specialized endgame for Cox homotopies

1: **procedure** ENDGAME($(H, L), 0 < \tau^* \leq 1, z \in \mathcal{V}_{(\mathbb{C}^\times)^k}(H(x;\tau^*)) \cap L$)
2:     found $\leftarrow$ false
3:     **while** found == false **do**
4:         obtain z_{target} by tracking (H, L) for $\tau \in [0, \tau^*]$ with starting solution z
5:         **if** $z_{\text{target}} < \infty$ and $z_{\text{target}} \notin Z$ **then**
6:             found $\leftarrow$ true
7:         **else**
8:             $z \leftarrow$ SWITCHREPRESENTATIVE(z)
9:         **end if**
10:     **end while**
11:     **return** z_{target} ▷ A set of Cox coordinates for $\lim_{\tau\to 0^+} \zeta_i(\tau) \in \mathcal{V}_X(F)$
12: **end procedure**

A few comments are in order to clarify Algorithm 1. First of all, note that Theorem 10.4.6 guarantees that the endgame terminates and that the output z_{target} is such that $\pi(z_{\text{target}}) = \zeta_i(0)$ if $z = z_{ij}(\tau^*)$ for some j. In line 4 the assumption is that the output of the tracking algorithm is ∞ in case the path diverges, and to check whether $z_{\text{target}} \in Z$ one can see if the residual with respect to the monomial generators of the irrelevant ideal B is adequately small (using some sensible heuristic). In line 8, the routine SWITCHREPRESENTATIVE finds another representative z' of $\mathbb{G} \cdot z$ satisfying $L(z') = 0$. In the implementation accompanying this work, the user may choose to either enumerate representatives all at once, or dynamically by tracking monodromy loops on

$$\mathcal{L}(z_1\lambda^{\mathbf{P}''_{:,1}},\ldots,z_k\lambda^{\mathbf{P}''_{:,k}}) = \left(\sum_{j=1}^{k} A_{ij} z_j \lambda^{\mathbf{P}''_{:,j}} + b_i\right)_{i=1,\ldots,k-n} = 0$$

with seed $\lambda_0 = (1,\ldots,1)$ and setting $z' = (z_1\lambda^{\mathbf{P}''_{:,1}},\ldots,z_k\lambda^{\mathbf{P}''_{:,k}})$ for any solution $\lambda \neq \lambda_0$. In practice one should check that z' is a representative that has not been used before. In principal, though less straightforward to implement, the polyhedral homotopy could also be used in such an incremental strategy (cf. [57]). Still, monodromy may be preferable if computing mixed cells is a bottleneck. For more details on monodromy and efficiency considerations, see [58].

Remark 10.4.7. In the case of (multi)projective homotopies, $\mathrm{Vol}(\Delta^{\mathscr{I}}) = 1/(k-n)!$ and $s_{\mathscr{I}} = 1$ are constant for all $\mathscr{I}$, meaning that in this case all representative paths will land on $\pi^{-1}(\zeta_i(0))$ for $\tau \to 0$.

10.4.3 Solving equations on X

Let $\hat{f}_1,\ldots,\hat{f}_n \in \mathbb{C}[M]$ and let $\mathcal{A} = (\mathcal{A}_1,\ldots,\mathcal{A}_n)$ be the corresponding supports ($\mathcal{A}_i \subset M$). This subsection describes an algorithm for computing the solutions defined by $\hat{F} = (\hat{f}_1,\ldots,\hat{f}_n) = 0$ on X, where $X = X_\Sigma$ is the n-dimensional toric variety coming from $\mathcal{A}$ as in Subsection 10.2.3. That is, the algorithm computes all isolated points of $V_X(F)$, where F is obtained from $\hat{F}$ by homogenizing. Here, 'computing' a point $\zeta \in X$ means computing numerical approximations of a set of Cox coordinates of ζ in $\mathbb{C}^k \setminus Z$. It is assumed that a system of affine start equations $\hat{\mathcal{G}} = (\hat{g}_1,\ldots,\hat{g}_n)$ is given such that $\hat{g}_i$ has support $\mathcal{A}_i$ and $|\mathcal{V}_{\mathbb{T}}(\hat{G})|$ consists of the mixed-volume number δ many starting solutions $\{\zeta_i(1)\}_{i=1,\ldots,\delta}$, which are also given. As per usual, G denotes the homogenized start system. For a generic linear space L of $\mathbb{C}^k \setminus Z$, consider the Cox homotopy (H, L) where $H(x;\tau) = (1-\tau)G + \gamma\tau F$ and $\gamma \in \mathbb{C}$ is either a random complex constant or $\gamma = 1$ when G is sufficiently generic. Using insights from the previous sections, Algorithm 2 is now immediate. In line 5 of this algorithm, the given starting solutions $\{\zeta_1(1),\ldots,\zeta_\delta(1)\}$ are lifted to the points $\{z_1(1),\ldots,z_\delta(1)\}$ in the total coordinate space in such a way that $\pi(z_i(1)) = \zeta_i(1)$ and $\mathcal{L}(z_i(1)) = 0, i = 1,\ldots,\delta$. This is done using Algorithm 3, which is discussed below. In line

Algorithm 2 solve $\hat{F} = 0$ on X

1: **procedure** SOLVEVIACOXHOMOTOPY($\hat{F}, \hat{G}, \{\zeta_1(1), \ldots, \zeta_\delta(1)\}, \tau_{\text{EG}}$)
2: $\quad F, G \leftarrow$ homogenize $\hat{F}, \hat{G}$
3: $\quad H \leftarrow (1 - \tau)G + \gamma\tau F$
4: $\quad L \leftarrow$ random affine map $Ax + b$
5: $\quad \{z_1(1), \ldots, z_\delta(1)\} \leftarrow$ HOMOGENIZESTARTINGSOLUTIONS($\{\zeta_1(1), \ldots, \zeta_\delta(1)\}, G, L$)
6: $\quad \{z_1(\tau_{\text{EG}}), \ldots, z_\delta(\tau_{\text{EG}})\} \leftarrow$ track $\{z_1(1), \ldots, z_\delta(1)\}$ along (H, L) for $\tau \in [\tau_{\text{EG}}, 1]$
7: $\quad$ **for** $i = 1, \ldots, \delta$ **do**
8: $\qquad z_{i,\text{target}} \leftarrow$ ENDGAME($(H, L), \tau_{\text{EG}}, z_i(\tau_{\text{EG}})$)
9: $\quad$ **end for**
10: $\quad$ **return** $\{z_{1,\text{target}}, \ldots, z_{\delta,\text{target}}\}$ $\qquad \triangleright$ A set of Cox coordinates for each point in $\mathcal{V}_X(F)$
11: **end procedure**

6 of Algorithm 2, the homogenized starting solutions are tracked for τ going from 1 to τ_{EG}, which is a parameter indicating where the endgame operating region $\tau \in [0, \tau_{\text{EG}})$ starts. The proposed default value is $\tau_{\text{EG}} = 0.1$. The tracking can happen in only n instead of k variables, by setting $x = \hat{x} + Ky$, where $\hat{x} \in \mathbb{C}^k$ is any point in L and K is a matrix whose columns span $\ker A$, where A is the matrix from line 4. Line 8 uses Algorithm 1.

The following discusses what happens in line 5 in more detail. First, a set of points $\tilde{z}_i \in (\mathbb{C}^\times)^k$ satisfying $\pi(\tilde{z}_i(1)) = \zeta_i(1)$ is computed. Since $\zeta_i(1) = (t_{i1}, \ldots, t_{in}) \in \mathbb{T} = (\mathbb{C}^\times)^n, i = 1, \ldots, \delta$, it follows that $\tilde{z}_i(1)^{\mathbf{F}_{j,:}} = t_{ij}, j = 1, \ldots, n$ by (5.7). Writing $(v_{i1}, \ldots, v_{ik})$ for the (unknown) coordinates of $\tilde{z}_i(1)$ on $(\mathbb{C}^\times)^k$, one obtains the system of binomial equations

$$v_{i1}^{\mathbf{F}_{j,1}} v_{i2}^{\mathbf{F}_{j,2}} \cdots v_{ik}^{\mathbf{F}_{j,k}} = t_{ij}, \quad i = 1, \ldots, \delta, \quad j = 1, \ldots, \delta. \tag{10.16}$$

Taking $\log(\cdot)$ on both sides (using any choice of branch) gives

$$\mathbf{F} v_{\log} = t_{\log}, \tag{10.17}$$

where $\mathbf{F}$ is the facet matrix, $(v_{\log})_{ij} = \log v_{ji}$ and $(t_{\log})_{ij} = \log t_{ji}$. It is clear that a solution $v_{\log}$ of the linear equations (10.17) gives a solution $v_{ij} = \exp((v_{\log})_{ji})$ to (10.16). Let $\tilde{\mathbf{F}} = \mathbf{F}_{:,\{i_1,\ldots,i_n\}} = [u_{i_1} \ \cdots \ u_{i_n}]$ be an invertible submatrix of $\mathbf{F}$, consisting of the ray generators indexed by $\{i_1,\ldots,i_n\}$. Such a matrix $\tilde{\mathbf{F}}$ exists, since Σ is complete. A solution to (10.17) is given by

$$(v_{\log})_{\{i_1,\ldots,i_n\},:} = \tilde{\mathbf{F}}^{-1} t_{\log}, \qquad (v_{\log})_{\{1,\ldots,k\}\setminus\{i_1,\ldots,i_n\},:} = 0.$$

In order to reduce rounding errors in the computation of $\tilde{\mathbf{F}}^{-1} t_{\log}$, it is favorable to pick a well-conditioned submatrix $\tilde{\mathbf{F}}$. This can be done, for instance, using a strong rank-revealing QR factorization [59]. The obtained solutions $\tilde{z}_i(1)$ satisfy $\pi(\tilde{z}_i(1)) = \zeta_i(1)$, and hence $G(\tilde{z}_i(1)) = 0$. It remains to track the $\tilde{z}_i(1)$ through the $\mathbb{G}$-orbit $\pi^{-1}(\zeta_i(1))$ to obtain $z_i(1)$ satisfying both $G(z_i(1)) = 0$ and $L(z_i(1)) = 0$. For that, note that the points $\tilde{z}_i(1)$ satisfy the (very non-generic) linear conditions $\tilde{z}_i(1) \in L_1^{-1}(0)$, where $L_1(x) = (x_i - 1)_{i \in \{1,\ldots,k\}\setminus\{i_1,\ldots,i_n\}}$. Therefore, one can track the homotopy $(G, \gamma\tau L_1 + (1-\tau)L)$ which intersects $\mathcal{V}_{\mathbb{C}^k\setminus Z}(G)$ with a moving linear space for τ going from 1 to 0, with starting solutions $\tilde{z}_i(1)$. Homotopies with a moving linear space are fundamental in numerical algebraic geometry, particularly in homotopy membership testing and monodromy [35, §15.4]. This discussion is summarized in Algorithm 3.

Algorithm 3 Lift a set of affine starting solutions in $\mathbb{T}$ to $(\mathbb{C}^\times)^k$

---

procedure HOMOGENIZESTARTINGSOLUTIONS($\{\zeta_1(1),\ldots,\zeta_\delta(1)\}, G, L$)
  $\{i_1,\ldots,i_n\} \leftarrow$ a subset of indices in $\{1,\ldots,k\}$ such that $\tilde{\mathbf{F}} = \mathbf{F}_{:,\{i_1,\ldots,i_n\}}$ is invertible
  $\{\tilde{z}_1(1),\ldots,\tilde{z}_\delta(1)\} \leftarrow$ a solution of (10.16) obtained via (10.17)
  $L_1(x) \leftarrow (x_i - 1)_{i \in \{1,\ldots,k\}\setminus\{i_1,\ldots,i_n\}}$
  $\mathscr{L}(x;\tau) \leftarrow \gamma\tau L_1(x) + (\tau - 1)L(x)$
  $\{z_1(1),\ldots,z_\delta(1)\} \leftarrow$ track $\{\tilde{z}_1(1),\ldots,\tilde{z}_\delta(1)\}$ along $(G, \mathscr{L}(x;\tau))$ for $\tau \in [0,1]$
  return $\{z_1(1),\ldots,z_\delta(1)\}$ ▷ A set of points in $\mathcal{V}_{\mathbb{C}^k\setminus Z}(G, L)$
end procedure

10.4.4 Orthogonal slicing in $\mathbb{C}^k \setminus Z$

In the case where $X = \mathbb{P}^n$, the linear part L of the Cox homotopy (H, L) is a single linear equation in the x-variables ($k = n+1$ and the orbits have dimension 1). As pointed out in Remark 10.4.2, one may let $L(x;\tau)$ depend on the continuation parameter τ. The points in $\mathbb{C}^{n+1} \setminus \{0\}$ satisfying $L(x;\tau) = 0$ for $\tau \in (0,1]$ lie on a moving hyperplane. This can be thought of as a continuously varying affine patch in which the homotopy is being tracked. In [60], the authors propose several adaptive strategies for choosing this patch. One natural choice they propose is that of an orthogonal patch (see [60, Subsection 3.2]). This subsection discusses how this can be done quite naturally in the total coordinate space of any compact toric variety X.

Let $z \in (\mathbb{C}^\times)^k$ and consider the corresponding orbit $\mathbb{G} \cdot z$. Locally (and if $\mathrm{Cl}(X)$ is free, even globally), this orbit is parametrized by

$$(\lambda^{\mathbf{P}''_{:,1}} z_1, \ldots, \lambda^{\mathbf{P}''_{:,k}} z_k), \quad \lambda \in \mathbb{C}^{k-n},$$

where P'' comes from the Smith normal form of $F^\top$, see Lemma 10.3.1. The tangent space to the orbit at $(\lambda^{\mathbf{P}''_{:,1}} z_1, \ldots, \lambda^{\mathbf{P}''_{:,k}} z_k)$ is parametrized by

$$z + \sum_{i=1}^{n-k} c_i \frac{\partial}{\partial \lambda_i} (\lambda^{\mathbf{P}''_{:,1}} z_1, \ldots, \lambda^{\mathbf{P}''_{:,k}} z_k), \quad c_i \in \mathbb{C},$$

where z is considered as a row vector of length k. For $\lambda = (1, \ldots, 1)$, the tangent space to the orbit at z is given by the simple expression

$$z + c^\top \mathbf{P}'' \mathrm{diag}(z_1, \ldots, z_k), \quad c \in \mathbb{C}^{k-n},$$

where $\mathrm{diag}(z_1, \ldots, z_k)$ is a diagonal $k \times k$ matrix with the coordinates of z on its diagonal. It follows that x is in the normal space to the orbit $\mathbb{G} \cdot z$ at z if and only if $x - z$ is orthogonal to the rows of $\mathbf{P}'' \mathrm{diag}(z_1, \ldots, z_k)$.

For a smooth path $(z(\tau), \tau) \in (\mathbb{C}^k \setminus Z) \times (0, 1]$ satisfying $\mathcal{H}(z(\tau), \tau) = 0$, one may define

$$L(x; \tau) = \text{conj}(\mathbf{P}''\text{diag}(z_1(\tau), \dots, z_k(\tau)))(x - z(\tau)),$$

where conj$(\cdot)$ takes the (entry-wise) complex conjugate. This suggests Algorithm 4 for tracking one path in the total coordinate space of X, using orthogonal slicing for collecting representatives on the orbits.

Algorithm 4 Track one path in the total coordinate space using orthogonal slicing

1: **procedure** TRACKORTH$(G, F, z(1), \mathbf{P}'')$
2: $\quad z \leftarrow z(1)$ ▷ Starting solution: $G(z(1)) = 0$
3: $\quad H(x; \tau) \leftarrow (1 - \tau)G(x) + \gamma\tau F(x)$
4: $\quad \tau^* \leftarrow 1$
5: $\quad$ **while** $\tau^* > 0$ **do** ▷ The target parameter value for τ is 0
6: $\quad\quad L(x; \tau^*) \leftarrow \text{conj}(\mathbf{P}''\text{diag}(z_1, \dots, z_k))(x - z)$
7: $\quad\quad (\tilde{z}, \Delta\tau) \leftarrow \text{PREDICT}((H, L(x; \tau^*)), z, \tau^*)$ ▷ Adaptive stepsize predictor routine
8: $\quad\quad z \leftarrow \text{CORRECT}((H, L), \tilde{z}, \tau^* - \Delta\tau)$ ▷ Corrector routine, e.g. Newton iteration
9: $\quad\quad \tau^* \leftarrow \tau^* - \Delta\tau$
10: $\quad$ **end while**
11: $\quad$ **return** z ▷ z is the target solution.
12: **end procedure**

The algorithm uses the blackbox functions PREDICT (line 7) and CORRECT (line 8) which are assumed to implement a predictor-corrector path tracking scheme, possibly (and preferably) using an adaptive step size control [61, 62, 63, 64, 65]. The function PREDICT returns a point $\tilde{z}$ and a step size $\Delta\tau$ such that $\tilde{z}$ is an approximation for a solution of $(H(x; \tau^* - \Delta\tau), L(x; \tau^*))$ and $\Delta\tau$ is a 'safe' step size. The function CORRECT then refines $\tilde{z}$ using, for instance, Newton iteration on $(H(x; \tau^* - \Delta\tau), L(x; \tau^*))$ with starting point $\tilde{z}$.

10.5 Numerical examples

The algorithms in Section 10.4 have been implemented in Julia, making use of the packages Polymake.jl (v0.5.3) [66, 67] and HomotopyContinuation.jl (v2.3.1) [42]. This section presents a selection of experiments highlighting the advantages of Cox homotopies, as discussed in the introduction. More material and the code is available on `https://mathrepo.mis.mpg.de/CoxHomotopies/index.html`.

Experiment 1 (A problem from computer vision)**.** The 8-point problem for cameras with radial distortion [68] consists of 9 equations in 9 unknowns. Eight equations are given by

$$\begin{pmatrix} p'_{i,1} & p'_{i,2} & 1+r'_i\lambda \end{pmatrix} \underbrace{\begin{pmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & 1 \end{pmatrix}}_{\mathcal{F}} \begin{pmatrix} p_{i,1} \\ p_{i,2} \\ 1+r_i\lambda \end{pmatrix} = 0, \tag{10.18}$$

where the parameters $p_{i,j}, p'_{i,j}$ and r_i, r'_i are known and represent distorted image coordinates and distortion radii, respectively. The true image coordinates are known only after the radial distortion parameter λ is recovered. The matrix $\mathcal{F}$ is called the fundamental matrix and satisfies the additional constraint

$$\det \mathcal{F} = 0. \tag{10.19}$$

For more details on the model and problem, see [69, 68].

For generic parameters $p_{i,j}, p'_{i,j}$ and $r_i, r'_i,$, the number of solutions to equations (10.18) and (10.19) is exactly the mixed-volume bound, $\delta = 16$. For comparison, the Bézout bound is 768. Homogenizing these equations as in Section 10.2.3, there are 26 Cox coordinates. For this experiment, consider the *nearly degenerate systems* satisfying $q'_{i,1} = -r'_{i,1} + \epsilon$ for ϵ small. For $\epsilon = 10^{-7}$, the `solve` command in HomotopyContinuation.jl reports 8 solutions and 8 paths going towards infinity. The implementation of Algorithm 2 finds a representative in $\mathbb{C}^{26}$ for each endpoint of these 16 paths in the compact toric variety X_Σ.

Since the generic orbit degree $d = 4583$ appearing in Theorem 10.4.1 is quite large, one may dynamically enumerate the representatives of each path considered in Algorithm 1 using random monodromy loops. This proves to be far more efficient than tracking a total of $d\delta = 73328$ paths. Ignoring the pre-computation of the polytope $\mathscr{P}$ and facet matrix $\mathbf{F}^\top$, the entire procedure takes around 15–45s between several runs on the same machine. Most time is spent on the endgame in Algorithm 1. The 8 nearly-infinite, nearly-singular solutions in the torus $\mathbb{T} \subset \mathbb{C}^9$ are reasonably accurate in the sense of backward error. More precisely, the residuals for the 8 regular solutions are all on the order of unit roundoff $\approx 10^{-16}$, and for the other 8 are in the range from 10^{-7} to 10^{-15}. Here the definition of the *residual* of an approximate solution t to $\hat{f}_1 = \cdots = \hat{f}_s = 0$ is defined, as motivated in [49, App. C], as

$$\frac{1}{s} \sum_{i=1}^{s} \frac{|\hat{f}_i(t)|}{\sum_{m \in \mathcal{A}_i} |c_{i,m} t^m| + 1}.$$

$\triangle$

Experiment 2 (Intersecting curves on a Hirzebruch surface)**.** Consider again the Hirzebruch surface $X = \mathscr{H}_2$ from Example 10.2.1. Its fan Σ and facet matrix $\mathbf{F}$ are shown in Figure 10.1. This experiment illustrates the use of orthogonal slicing (see Subsection 10.4.4) as an adaptive strategy for tracking paths in a Cox homotopy. The first step was to generate Laurent polynomials $\hat{f}_1, \hat{f}_2 \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ with support $\mathcal{A}_1 = \mathcal{A}_2 = \mathscr{P}_i \cap M$, where $\mathscr{P}_i$ has facet representation given by $\mathbf{F}^\top$ and $a_i = (0, 0, 10, 10)^\top$ for $i = 1, 2$. These Laurent polynomials are random in the sense that the coefficients of the $\hat{f}_i$ have been drawn from a standard normal distribution. There are 400 solutions to $\hat{f}_1 = \hat{f}_2 = 0$ in $\mathbb{T} \subset X$. For illustration purposes, an implementation of Algorithm 4 with the naive predictor was used

$$\textsc{Predict}((H, L(x; \tau^*)), z, \tau^*) = (z, 0.001).$$

Three of the 256 paths $z(\tau)$ in the Cox homotopy were chosen randomly and the condition number of the Jacobian $J_{\tau^*} = \partial_x(H(x; \tau^*), L(x; \tau^*))|_{x = z(\tau^*)}$ was recorded for $\tau^* = 1, 1 - \Delta\tau, 1 - 2\Delta\tau, \ldots, 0$, with $\Delta\tau = 0.001$. For comparison, the same paths in X were tracked using 10 random slices $L_{\text{rand}}(x)$ given by $L_{\text{rand}}(x) = A(x - z(1))$ where A is a matrix with entries drawn from a complex standard normal distribution. The results are illustrated in Figure 10.4. The dashed (orange) curves are obtained by taking the geometric mean of the dotted (grey) curves, which represent the condition number of J_τ for 10 random slices. The blue curves represent the condition number of J_τ using Algorithm 4. The figure shows that for different randomly generated $L_{\text{rand}}(x)$,

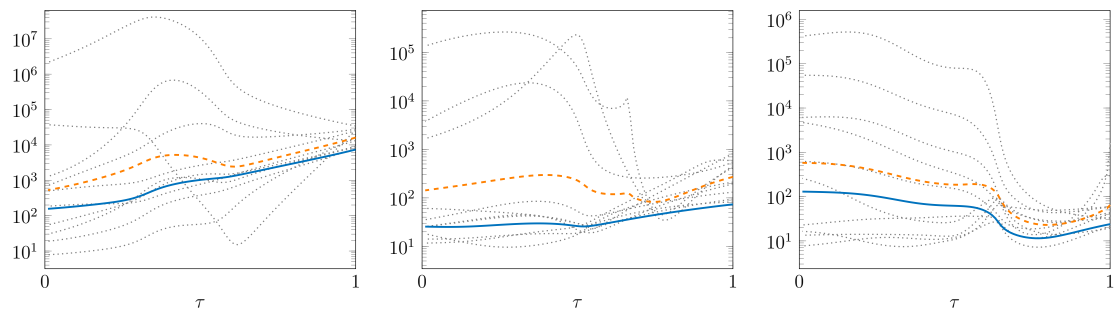


Figure 10.4: Condition number of the Jacobian along 3 paths of the homotopy in Experiment 2 using the orthogonal slicing strategy of Algorithm 4 (——) and the average condition number (- - -) for 10 random linear slices (⋯⋯).

the behavior of the condition number may vary significantly. Using orthogonal slicing, this experiment consistently obtains smaller condition numbers on average. Moreover, the computation of the orthogonal slice causes virtually no computational overhead. $\triangle$

Experiment 3 (Solving equations on a weighted projective space)**.** Consider the system of polynomial equations

$$\begin{aligned}
\hat{f}_1 &= (3+\varepsilon_1)t_1^2 + 7t_1t_2 + 7t_2^2 + 9t_1 + 3t_2 + 9t_3 + 2 = 0,\\
\hat{f}_2 &= (3+\varepsilon_2)t_1^2 + 7t_1t_2 + 7t_2^2 + 5t_1 + 2t_2 + 3t_3 + 4 = 0,\\
\hat{f}_3 &= (3+\varepsilon_3)t_1^2 + 7t_1t_2 + 7t_2^2 + 4t_1 + 8t_2 + 4t_3 + 9 = 0,
\end{aligned}$$

in the variables t_1, t_2, t_3. The parameters ε_i are assigned random complex values of modulus 10^{-12}. The normalized volume of the Newton polytope of these equations is 4, which is equal to the number of solutions in $\mathbb{T} = (\mathbb{C}^\times)^3$. However, the command `solve` in HomotopyContinuation.jl only finds 2 solutions. The polynomials $\hat{f}_1, \hat{f}_2, \hat{f}_3$ homogenize to degree 2 elements in the Cox ring $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ of the weighted projective threefold $X = \mathbb{P}_{1,1,2,1}$, where x_i corresponds to the facet normal e_i for $i = 1, 2, 3$ and $\{x_4 = 0\}$ is the divisor 'at infinity'. Using Algorithm 2, all 4 solutions in X are found. The Cox coordinates x_3 and x_4 have absolute value $\approx 10^{-12}$ for two of these solutions, which means they lie close to $D_3 \cap D_4$. The corresponding points in the torus have coordinates of modulus $\approx 10^{12}$. Tracking these solutions in $(\mathbb{C}^\times)^3$ causes premature truncation in the standard polyhedral homotopy.

The next step is to test the Cox homotopy on a different, larger system of equations with similar behavior. The chosen system is given by $\hat{f}_1 = \cdots = \hat{f}_5 = 0$, where the $\hat{f}_i$ have Newton polytope $\mathscr{P} = \{m \in \mathbb{R}^5 \mid F^\top m + a \geq 0\}$, with

$$\mathbf{F}^\top = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & -2 & -4 \end{bmatrix}, \qquad a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 12 \end{bmatrix}.$$

These correspond to degree 12 equations with 7776 solutions in the weighted projective space $X = \mathbb{P}_{1,2,2,2,4,1}$. By manipulating the coefficients, one can force 216 of these solutions to lie near $D_5 \cap D_6$, meaning that their Cox coordinates x_5 and x_6 have modulus $\approx \varepsilon$. More details on the setup can be found at `https://mathrepo.mis.mpg.de/CoxHomotopies/index.html`. For $\varepsilon = 10^{-7}$, in five different runs, the polyhedral homotopy used by `solve` terminates in only 59 up to 69 seconds, but misses between 201 and 207 solutions. The polyhedral homotopy implementation in the blackbox solver `phc -b` of PHCpack (v2.4.83) [41], for $\varepsilon = 10^{-7}$ and five different runs, reports between 211 and 218 solutions at infinity and between 30 and 66 path failures with a computation time ranging from 13 minutes and 11 seconds to 16 minutes and 57 seconds. Algorithm 2 finds *all* 7776 solutions in T within 7 minutes and 40 seconds. These computations were performed on a 16 GB MacBook Pro machine with an Intel Core i7 processor working at 2.6 GHz. $\triangle$

Experiment 4 (Equations on a Bott-Samelson variety)**.** Defining $B \subseteq GL_3$ as the subgroup of upper-triangular matrices, GL_3/B is birational to a Bott-Samelson variety. The considered system is a random complex square polynomial system on GL_3/B from the Khovanskii basis

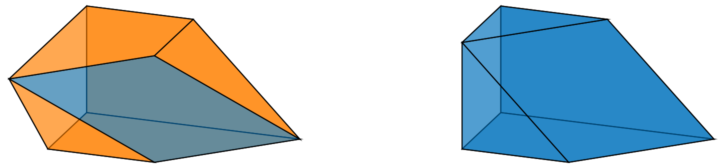


Figure 10.5: The Newton polytope $\mathscr{P}_i$ (left) of the equations in Experiment 4 and the Newton-Okounkov body associated to $\mathcal{B}$ (right). The face of $\mathscr{P}_i$ whose corresponding system has solutions at infinity is highlighted in blue.

$\mathcal{B} = \{1, x, y, z, xz, yz, x(xz + y), y(xz + y)\}$ [30]. That is, the considered system is

$$\hat{f}_i = c_{i,1}\ 1 + c_{i,2}\ x + c_{i,3}\ y + c_{i,4}\ z + c_{i,5}\ xz + c_{i,6}\ yz + c_{i,7}\ x^2z + c_{i,8}\ xyz + c_{i,7}\ xy + c_{i,8}\ y^2,$$

where $i \in \{1, 2, 3\}$ and $c_{i,j} \in \mathbb{C}$ for $i = 1, 2, 3$ and $j = 1, \ldots, 8$. The mixed-volume bound for $\hat{F} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ is 10, which is given the by the normalized volume of the Newton polytope $\mathscr{P}_i$ of $\hat{f}_i$. The number of solutions to $\hat{F} = 0$, however, is known to be equal to the normalized volume of the Newton-Okounkov body associated to $\mathcal{B}$, which is six. Both the Newton polytope $\mathscr{P}$ and the Newton-Okounkov body associated to $\mathcal{B}$ are depicted in Figure 10.5. This deficient root count (six) with respect to the mixed-volume bound (ten) suggests that there are solutions at infinity, which would correspond to solutions of face system(s) of $\hat{F}$. Solutions to face systems can be found via the Cox homotopy.

The first step is to homogenize $\hat{F}$ using the normal fan of $\mathscr{P}$, whose rays are recorded by the columns of

$$\mathbf{F} = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

Using the Cox homotopy, one finds that the homogenization of $\hat{F}$, F, has six nonsingular solutions in the torus and four singular solutions whose first Cox coordinate is zero. This indicates that there are solutions to the face system of $\hat{F}$ associated to the ray $(-1, -1, 0)$. This face system, which is given by $c_{i,7}\ x(xz + y) + c_{i,8}\ y(xz + y) = 0$ for $i = 1, 2, 3$, has infinitely many solutions along the curve C defined by $y = -xz$. Therefore, the root deficiency of $\hat{F}$ is explained by this curve at infinity. The orbit degree $\deg(\overline{\mathbb{G} \cdot z})$ in this example is five for $z \in (\mathbb{C}^\times)^k$. The degree $\deg(\overline{\mathbb{G} \cdot z})$ for $z \in \pi^{-1}(\zeta)$ drops to three for general points on $D_1 = \mathcal{V}_X(x_1)$. Using the specialized endgame in Algorithm 1, the Cox homotopy consistently finds the homogeneous coordinates of four points on C. Although Lemma 10.4.4 only accounts for isolated points in $\mathcal{V}_X(F)$, this example shows that the Cox homotopy can also detect positive dimensional components on $X \setminus \mathbb{T}$. This strategy provides an important first step towards generalizing [70] for computing numerical irreducible decompositions in X. △

10.6 Conclusion

Cox homotopies track solution paths of sparse polynomial systems in a compact toric variety X. The algorithm makes explicit use of the construction of X as a(n almost) geometric quotient of a quasi-affine space by the action of a reductive group $\mathbb{G}$. The degree of (the closure of) $\mathbb{G}$-orbits in this construction were described in terms of volumes of orbit polytopes, lattice indices, and invariant factors. As shown in the experiments, Cox homotopies, as a generalization of (multi)projective homotopies, provide a means to deal with solutions on or near the boundary of the dense torus in X in an elegant way, avoiding premature truncation of solution paths and providing insight in the solution structure 'at infinity'. It inherits the advantage of polyhedral homotopies that the number of paths tracked is equal to the mixed-volume bound. Experiments show that the Cox homotopy algorithms provide the first steps towards performing numerical irreducible decomposition in X.

11. THE KHOVANSKII HOMOTOPY

As discussed in Chapter 6, the Newton-Okounkov body bound for the number of solutions to a polynomial system is sometimes tighter than the mixed-volume bound. With Michael Burr and Frank Sottile, the aim in [11] was to create a homotopy whose number of tracked paths is equal to the Newton-Okounkov body bound. This would provide an optimal homotopy which also has the potential to be optimal in cases where the polyhedral homotopy is not optimal. The foundation of this homotopy is the toric degeneration given by Anderson in [30], which exists whenever the variety has an associated finite Khovanskii basis, as explained in Section 11.3. This toric degeneration is what guarantees the optimality of the Khovanskii homotopy. The rest of this chapter is an edited excerpt of [11]*.

11.1 Introduction

This chapter considers the problem of computing the isolated solutions to the system

$$f_1(z) = f_2(z) = \cdots = f_d(z) = 0, \tag{11.1}$$

where $f_1, \ldots, f_d$ are general members of a finite-dimensional vector space V of rational functions on a complex algebraic variety X of dimension d. Kaveh-Khovanskii [26, 27] and Lazarsfeld-Mustaţă [28] show that the number of solutions is the normalized volume of the Newton-Okounkov body associated to V. The accompanying theory extends many uses of Newton polytopes from toric varieties to general algebraic varieties. This theory lends itself to algorithms when V has a finite Khovanskii basis [10].

The evaluation of functions in V induces the rational Kodaira map $\varphi\colon X \dashrightarrow \mathbb{P}(V^*)$. The solutions to System (11.1) are the pull backs of the points of a linear section $\varphi(X) \cap L$ along φ. When V has a finite Khovanskii basis, Anderson [30] shows that (the closure of) $\varphi(X)$ has a

*Reprinted from "Numerical homotopies from Khovanskii bases" by M. Burr, F. Sottile, and E. Walker, to appear in *Mathematics of Computation* and published by the American Mathematical Society. Copyright 2021 by the American Mathematical Society.

flat degeneration to a toric variety associated to the Newton-Okounkov body of V. This chapter describes numerical algorithms for computing the linear section $\varphi(X) \cap L$ based on this toric degeneration and the polyhedral homotopy algorithm [14, 43]. Solving System (11.1) then requires computing the pull back of the linear section.

The numerical algorithms for computing a linear section given in this chapter are based on homotopy continuation [71]. Anderson's flat toric degeneration gives a homotopy where the start system is a linear section of a toric variety and the target system is a linear section of $\varphi(X)$. Flatness guarantees that the number of solutions to the start and target systems are equal. Thus the homotopy is optimal in the sense that no extraneous paths are tracked.

The start system in the Khovanskii homotopy is a linear section of a toric variety, which may be solved using the optimal polyhedral homotopy algorithm [14, 43]. Beyond those derived from a finite Khovanskii basis, there are many instances in which a projective variety has a flat degeneration into a toric variety. Section 11.2 contains a description of an optimal toric two-step homotopy algorithm for solving systems given a toric degeneration in an ambient projective space, and presents an example of such a flat degenerations into a toric variety. Further examples, including the construction of a toric degeneration from a weight degeneration, can be found in Chapter 7. Other examples of flat degenerations include the Gröbner homotopy of [31] and the homotopy for solving the Kuramoto equations of [32], both of which exploit a flat degeneration into a union of linear spaces.

When the Khovanskii basis is a subset of V, Anderson's degeneration may be embedded in the projective space $\mathbb{P}(V^*)$. Section 11.3 presents the Khovanskii homotopy algorithm, which uses this embedding and the toric two-step homotopy to solve System (11.1). For a general Khovanskii basis, Anderson's degeneration may only be embedded in a weighted projective space and System (11.1) is not a pull back of a general linear section. Section 11.4 describes how to adapt the toric algorithm to this general case of a Khovanskii basis.

The SAGBI homotopy of [31] for Schubert calculus is also a two-step homotopy in which a system of polynomials on $\mathbb{C}^n$ is deformed into a system of sparse polynomials, which is solved using the polyhedral homotopy. Despite the relationship between SAGBI bases and Khovanskii bases, the Khovanskii homotopy algorithms presented here do not generalize the SAGBI homotopy. The clearest distinction is that the SAGBI homotopy proceeds in local coordinates that parameterize the fibers in a toric degeneration, while the two-step homotopy proceeds in the coordinates of the ambient projective space of the toric degeneration.

Each section ends with a concrete example to illustrate these techniques and algorithms. These examples are computed with `Macaulay2` scripts [5], which are archived on `GitHub`:

`https://github.com/EliseAWalker/KhovanskiiHomotopy/`

These computations use the `NumericalAlgebraicGeometry` package [39] to call the packages `Bertini` [38] and `PHCpack` [41] for user-defined homotopies and the polyhedral homotopy, respectively. A discussion of practical issues that arise from using these software packages is in Section 11.5.

11.2 Preliminaries

This discussion describes homotopy algorithms that arise from flat families, including those arising from toric degenerations.

11.2.1 Homotopies from flat families

Suppose that $X \subset \mathbb{P}^n$ is a subvariety of dimension d. A linear section of X is a transverse intersection $X \cap L$ where $L \subset \mathbb{P}^n$ is a linear subspace of codimension d so that $X \cap L$ consists of $\deg X$ points.

Let $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{C}$ be a variety with a surjective map $\pi\colon \mathcal{X} \to \mathbb{C}$. Then π realizes $\mathcal{X}$ as a family of projective schemes over $\mathbb{C}$ where a point $\tau \in \mathbb{C}$ corresponds to the fiber $\mathcal{X}_\tau := \pi^{-1}(\tau) \subset \mathbb{P}^n$. There is an open subset $U \subset \mathbb{C}$ such that $\mathcal{X}$ is flat over U and all fibers are reduced. That is, $\pi^{-1}(U) \to U$ is a flat family of varieties.

Suppose that the fibers of a flat family of varieties $\mathcal{X}$ over $U \subset \mathbb{C}$ have dimension d and that $0, 1 \in U$. Let $L \subset \mathbb{P}^n$ be a general linear subspace of codimension d which meets both $\mathcal{X}_0$ and $\mathcal{X}_1$ transversally so that $\mathcal{X}_0 \cap L$ and $\mathcal{X}_1 \cap L$ are linear sections of varieties. As $\mathcal{X}_0$ and $\mathcal{X}_1$ have the same degree, $\mathcal{X}_0 \cap L$ and $\mathcal{X}_1 \cap L$ have the same number of points. Let $H(x;\tau)$ be finitely many polynomials defining $\mathcal{X}$ and d linear forms defining L. Then $H(x;\tau)$ is called a linear section homotopy.

Proposition 11.2.1 (Linear section homotopy)**.** *A linear section homotopy $H(x;\tau)$ is an optimal homotopy with start system $\mathcal{X}_0 \cap L$ and target system $\mathcal{X}_1 \cap L$.*

Proof. Let C be the union of components of $\mathcal{X} \cap L$ that contain both $\mathcal{X}_0 \cap L$ and $\mathcal{X}_1 \cap L$. Since these intersections are zero-dimensional, C is a curve. Furthermore, both $\tau = 0$ and $\tau = 1$ belong to the open subset U of $\mathbb{C}_\tau$ of regular values of the projection $C \to \mathbb{C}_\tau$. Thus, H is a homotopy. Since flatness and the generality of L imply that $\mathcal{X}_0 \cap L$ and $\mathcal{X}_1 \cap L$ have the same number of points, the homotopy H is optimal. □

A linear section is part of a witness set, which is a fundamental data structure in numerical algebraic geometry [36]. Specifically, a witness set for a d-dimensional variety $X \subset \mathbb{P}^n$ is a triple $(F, L, X \cap L)$ where F is a set of homogeneous polynomials (forms) on $\mathbb{P}^n$ defining X, L is a set of d linear forms defining a linear subspace (which is also denoted as L), and $X \cap L$ is a linear section.

In the linear section homotopy in Proposition 11.2.1, L is a fixed general linear space and the variety $\mathcal{X}_\tau$ moves. In contrast, the Khovanskii homotopy algorithms sometimes require linear spaces which are not general, so that $X \cap L$ need not be a witness set. For this situation, one would use a different homotopy in which the variety is fixed but the linear section moves. This homotopy is described in the following basic algorithm for moving a witness set:

Algorithm 11.2.2 (Witness Set Homotopy)**.**

Input: A witness set $(G, L', X \cap L')$ for X and a codimension d linear subspace L such that $X \cap L$ is finite.

Output: The points of $X \cap L$.

Do:

1. Let $H := (G, \tau L + (1-\tau)L')$, a homotopy with start system $X \cap L'$ and target system $X \cap L$.

2. Use path tracking starting from the points of $X \cap L'$ to compute the points of $X \cap L$.

When $X \cap L$ is transverse, $(G, L, X \cap L)$ is a witness set for X.

11.2.2 Toric degenerations

This section demonstrates how a toric degeneration gives rise to a homotopy algorithm. In particular, given a toric degeneration $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{C}_\tau$, the linear section homotopy leads to the toric two-step homotopy which is described in Algorithm 11.2.3. For details on toric varieties and toric degenerations, see Chapters 5 and 7, respectively.

Consider a d-dimensional toric variety $X_\mathcal{A} \subset \mathbb{P}^n$ with dense torus $\mathbb{T}$, a point $p \in \mathbb{T}$, and a linear subsection $L \subset \mathbb{P}^n$ of dimension complementary to $X_\mathcal{A}$. A linear section $p.X_\mathcal{A} \cap L$ of the translated toric variety $p.X_\mathcal{A}$ pulls back along $\varphi_{p,\mathcal{A}}$ to the following system of sparse polynomials on $(\mathbb{C}^\times)^d$ whose monomials have exponents in $\mathcal{A}$:

$$g_1(z) = g_2(z) = \cdots = g_d(z) = 0. \tag{11.2}$$

As System (11.2) is sparse, the polyhedral homotopy algorithm [14, 43] optimally solves System (11.2) as the number of solutions to the start system equals the number of solutions to a general system with support $\mathcal{A}$.

Let $\mathcal{X} \to \mathbb{C}_\tau$ be a toric degeneration with d-dimensional toric special fiber $p.X_\mathcal{A} = \mathcal{X}_0$. A general linear subspace L of codimension d gives linear sections $p.X_\mathcal{A} \cap L$ and $\mathcal{X}_1 \cap L$. Combining the linear section homotopy of Proposition 11.2.1 with the polyhedral homotopy gives the toric two-step homotopy algorithm for computing the points of the linear section $\mathcal{X}_1 \cap L$. Let $G_\mathcal{A}$ be System (11.2), which is given by the pull back of L along $\varphi_{p,\mathcal{A}}$.

Algorithm 11.2.3 (Toric two-step homotopy algorithm)**.**

Input: A toric degeneration $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{C}_\tau$ with $\mathcal{X}_0 = p.X_\mathcal{A}$ a toric variety and a general linear space $L \subset \mathbb{P}^n$ of codimension equal to the dimension of $\mathcal{X}_1$.

Output: All points of the linear section $\mathcal{X}_1 \cap L$.

Do:

1. Compute the system $G_{\mathcal{A}}$ on $(\mathbb{C}^{\times})^d$ by pulling L back along the Kodaira map $\varphi_{p,\mathcal{A}}$.

2. Use the polyhedral homotopy to solve $G_{\mathcal{A}}$.

3. Use $\varphi_{p,\mathcal{A}}$ to obtain the points of the linear section $p.X_{\mathcal{A}} \cap L$.

4. Use the linear section homotopy (Proposition 11.2.1) beginning with the points of $p.X_{\mathcal{A}} \cap L$ to obtain the points of the linear section $\mathcal{X}_1 \cap L$.

The discussion preceding Algorithm 11.2.3 justifies the following theorem:

Theorem 11.2.4. *Algorithm 11.2.3 is an optimal homotopy algorithm for computing $\mathcal{X}_1 \cap L$.*

Remark 11.2.5. If L is not general, then replace it by a general codimension d linear subspace L' in Algorithm 11.2.3 and add a fifth step that uses the Witness Set Homotopy 11.2.2 to move L' to L.

Remark 11.2.6. Algorithm 11.2.3 can also be used to compute $\mathcal{X}_1 \cap L$ when the definition of a toric degeneration is relaxed so that $\mathcal{X}_0$ is a union of toric varieties. The points in a general linear section $\mathcal{X}_0 \cap L$ in Algorithm 11.2.3 may be computed from systems of sparse polynomials for each toric component of $\mathcal{X}_0$.

The following Example 11.2.1 comes from [30, Section 6.4] and is an explicit application of Algorithm 11.2.3 .

Example 11.2.1. This example demonstrates a weight degeneration and illustrates Algorithm 11.2.3. For more information on weight degenerations, see Chapter 7. Let $X \subset \mathbb{P}^7$ be the closure of the image of the map $\varphi\colon \mathbb{C}^3 \to \mathbb{P}^7$ given by

$$\varphi(x,y,z) \;=\; [1, x, y, z, xz, yz, x(xz+y), y(xz+y)].$$

This subvariety has degree six and its ideal I has nine generators:

$$x_1x_3 - x_0x_4,\;\; x_2x_3 - x_0x_5,\;\; x_1x_2 - x_0x_6 + x_1x_4,\;\; x_2^2 - x_0x_7 + x_3x_6 - x_4^2,$$
$$x_2x_6 - x_1x_7,\;\; x_2x_5 - x_3x_7 + x_4x_5,\;\; x_1x_5 - x_3x_6 + x_4^2,\;\; x_2x_4 - x_1x_5,\;\; x_5x_6 - x_4x_7.$$

Let $w = (-2,-1,-1,-1,0,0,0,0)$. Use Equation 7.2 to compute the ideal of $\mathcal{X}^w$. The following thirteen polynomials form a Gröbner basis $\mathcal{G}_\tau$ for $\mathcal{X}^w$ with respect to the weighted term order $\leq_{-w}$:

$$\underline{x_1x_3 - x_0x_4}\,,\ \ \underline{x_2x_3 - x_0x_5}\,,\ \ \underline{x_1x_2 - x_0x_6} + \tau x_1x_4\,,\ \ \underline{x_2^2 - x_0x_7} + \tau x_3x_6 - \tau^2x_4^2\,,$$

$$\underline{x_2x_6 - x_1x_7}\,,\ \ \underline{x_2x_5 - x_3x_7} + \tau x_4x_5\,,\ \ \underline{x_1x_5 - x_3x_6} + \tau x_4^2\,,\ \ \underline{x_2x_4 - x_1x_5}\,,\ \ \underline{x_5x_6 - x_4x_7}\,,$$

$$\underline{x_0x_6^2 - x_1^2x_7} - \tau x_1x_4x_6\,,\ \ \underline{x_0x_5^2 - x_3^2x_7} + \tau x_3x_4x_5\,,$$

$$\underline{x_0x_4x_5 - x_3^2x_6} + \tau x_3x_4^2\,,\ \ \underline{x_3x_6^2 - x_1x_4x_7} - \tau x_4^2x_6.$$

The leading terms with respect to $\leq_{-w}$ are underlined, and these binomials generate the ideal I_w. This ideal is the toric ideal of the image of the map $\varphi_{\mathcal{A}}(x,y,z) = [1,x,y,z,xz,yz,xy,y^2]$ given by the lowest order monomials in φ. For the toric ideal statement, observe that if one sets $(x_0,x_1,x_2,x_3) = (1,x,y,z)$, then the first four underlined binomials in $\mathcal{G}_\tau$ express $x_4,\ldots,x_7$ as the monomials in x,y,z appearing in $\varphi_{\mathcal{A}}$. The exponent vectors of $\varphi_{\mathcal{A}}$ are the columns of the matrix $\mathcal{A}$ in Figure 11.1.

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

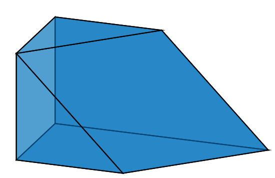


Figure 11.1: The weight vectors for the toric Kodaira map $\varphi_{\mathcal{A}}$ are the columns of matrix $\mathcal{A}$. The Newton polytope is the convex hull of these vectors.

Let $L \subset \mathbb{P}^7$ be the linear subspace of codimension three whose defining equations are $\ell_i = \sum c_{ij}x_j$ for $i = 1,2,3$, where $C = (c_{ij})$ is the 3×8 matrix

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & -4 & 5 & -6 & 7 & -8 \\ 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 \end{pmatrix}.$$

The subspace L meets both $\overline{\varphi(\mathbb{C}^3)} = \mathcal{X}_1$ and $\overline{\varphi_{\mathcal{A}}((\mathbb{C}^\times)^3)} = \mathcal{X}_0$ transversally in six points.

One may follow the steps of Algorithm 11.2.3 to compute $\mathcal{X}_1 \cap L$. First, compute the sparse system $G_{\mathcal{A}}$ in Step 1 of Algorithm 11.2.3 to arrive at the system

$$1 + x + y + z + xz + yz + xy + y^2 = 0$$
$$1 - 2x + 3y - 4z + 5xz - 6yz + 7xy - 8y^2 = 0$$
$$2 + 3x + 5y + 7z + 11xz + 13yz + 17xy + 19y^2 = 0.$$

In Step 2 of Algorithm 11.2.3, one computes the six solutions of the system $G_{\mathcal{A}}$, one of which is $\zeta = (-1.33613, 1.51406, -1.22871)$. The image $\varphi_{\mathcal{A}}(\zeta)$ in $\mathbb{P}^7$ is

$$[1,\ -1.33613,\ 1.51406,\ -1.22871,\ 1.64171,\ -1.86035,\ -2.02298,\ 2.29239].$$

In Step 3 of Algorithm 11.2.3, one computes the images of these six solutions under $\varphi_{\mathcal{A}}$, which forms the points of $\mathcal{X}_0 \cap L$. Therefore, the images of these points are the solutions to the start system for the linear section homotopy given by $H(x, \tau) = (\mathcal{G}_\tau, L)$. In Step 4, these solutions are followed from $\tau = 0$ to $\tau = 1$, computing the six points of the linear section $\mathcal{X}_1 \cap L$. One point of $\mathcal{X}_1 \cap L$ is

$$[1,\ -0.689522,\ 0.928435,\ -1.35986,\ 0.937652,\ -1.26254,\ -1.28671,\ 1.73254].$$

$\triangle$

11.3 Khovanskii homotopy

Let X be a complex variety of dimension d and $V \subset \mathbb{C}(X)$ be a finite-dimensional complex vector space of rational functions on X. The closure of the image of X under the Kodaira map $\varphi_V \colon X \dashrightarrow \mathbb{P}(V^*)$ has homogeneous coordinate ring $R(V)$ generated by V. When this ring has a finite Khovanskii basis contained in V, Anderson's toric degeneration embeds in $\mathbb{P}(V^*)$ as a weight degeneration. This degeneration is used in the Khovanskii homotopy algorithm (Algorithm 11.3.4) to compute a linear section $\varphi_V(X) \cap L$.

This section describes how to produce an embedding of Anderson's toric degeneration into $\mathbb{P}(V^*)$ when the Khovanskii basis is a subset of V. Steps detailing how to compute a Kodaira map of the toric special fiber are also given. With the embedding and toric Kodaira map, Algorithm 11.2.3 becomes an effective method to compute linear sections. Section 11.4 gives one way to modify this method for the general case when the Khovanskii basis is not a subset of V.

11.3.1 Valuations, Khovanskii bases, and Newton-Okounkov bodies

Suppose that X is a d-dimensional complex variety with function field $\mathbb{C}(X)$ and $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^d$ is a $\mathbb{Z}^d$-valuation on $\mathbb{C}(X)$ where $\succ$ is the total order on $\mathbb{Z}^d$. Following the definition of valuation in Section 3.4, ν is surjective. Let V be a finite-dimensional complex vector subspace of $\mathbb{C}(X)$ such that the image of $V^\times$ under ν generates $\mathbb{Z}^d$ (see Remark 11.3.1). Following Section 6.2.1, let $R(V)$ be the graded ring $\bigoplus_{k\geq 0} V^k s^k$. Furthermore, extend the valuation ν and the total order $\succ$ to $R(V)$ and let NO_V be the associated Newton-Okounkov body. Recall from Theorem 6.2.2 that the number of solutions to System (11.1) where $f_1, \ldots, f_d \in V$ are general (in this case, one can say that System (11.1) is drawn from V) is the normalized volume of NO_V.

Computations of NO_V are tractable when there is a finite Khovanskii basis for $R(V)$. Let $\mathcal{B} \subset R(V)$ be a finite, linearly linearly independent set which is a Khovanskii basis for $R(V)$ with respect to ν. Further assume that the elements of $\mathcal{B}$ are homogeneous so that for $b \in \mathcal{B}$ with $\nu(b) = (\alpha, k)$, $b \in V^k s^k$. Necessarily, $\mathcal{B}$ generates $R(V)$ and $\mathcal{B} \cap Vs$ is a basis for Vs. Note that given finite Khovanskii basis $\mathcal{B}$, Anderson [30] shows that NO_V is a rational polytope and that there exists a flat degeneration $\mathcal{X} \to \mathbb{C}_t$ of $\mathcal{X}_1 \simeq \mathrm{Proj}(R(V))$ to the toric variety $\mathcal{X}_0 \simeq \mathrm{Proj}(\mathbb{C}[S(V,\nu)])$. Anderson's degeneration [30] generalizes the toric degeneration of a spherical variety from [72].

The valuation ν on $R(V)$ induces a filtration on $R(V)$ by finite-dimensional subspaces indexed by elements $(\alpha, k) \in S(V, \nu)$. Let

$$\begin{aligned} R(V)_{(\alpha,k)} &:= \{f \in R(V) : \nu(f) \succeq (\alpha, k)\}, \text{ and} \\ R(V)^+_{(\alpha,k)} &:= \{f \in R(V) : \nu(f) \succ (\alpha, k)\}. \end{aligned}$$

Since $(\alpha, k) \in S(V, \nu)$, these subspaces satisfy $R(V)_{(\alpha,k)}/R(V)^+_{(\alpha,k)} \simeq \mathbb{C}$. Anderson's flat degeneration comes from the degeneration of the filtered algebra $R(V)$ to its associated graded algebra

$$\operatorname{gr} R(V) := \bigoplus_{(\alpha,k)\in S(V,\nu)} R(V)_{(\alpha,k)}/R(V)^+_{(\alpha,k)} \simeq \mathbb{C}[S(V,\nu)].$$

The toric fiber $\mathcal{X}_0$ of Anderson's degeneration is $\operatorname{Proj}(\operatorname{gr} R(V))$, and the isomorphism $\mathcal{X}_0 \simeq \operatorname{Proj}(\operatorname{gr} R(V))$ uses the isomorphism $\operatorname{gr} R(V) \simeq \mathbb{C}[S(V,\nu)]$.

As discussed in Section 3.4, computation of Khovanskii bases may be difficult. Finite Khovanskii bases are necessary for Anderson's degeneration, however. The Khovanskii homotopy consequently takes a finite Khovanskii basis as an input.

11.3.2 The Kodaira map and embedding the degeneration

To use Anderson's toric degeneration $\mathcal{X}$ in Algorithm 11.2.3, $\mathcal{X}$ must be embedded in a projective space. Suppose that a finite Khovanskii basis $\mathcal{B}$ for V is given such that $\mathcal{B} \subset Vs$. Therefore, $\mathcal{B}$ is a basis for Vs, by definition.

Let $X^\circ \subset X$ be the open subset of points of X where no function from V has a pole, and some function in V is nonzero. Evaluation of functions from V at a point $z \in X^\circ$ gives a nonzero linear map $\operatorname{ev}_z(f) := f(z)$ on V. Therefore, ev_z is a point in the projective space $\mathbb{P}(V^*)$, where V^* is the space of linear functions $V \to \mathbb{C}$. Thus the map $z \mapsto \operatorname{ev}_z$ induces a map $X^\circ \to \mathbb{P}(V^*)$, which is called the rational Kodaira map $\varphi_V : X \dashrightarrow \mathbb{P}(V^*)$. If one writes $\mathcal{B} = \{b_0 s, \ldots, b_n s\}$, then a Kodaira map can be explicitly written as $\varphi_{\mathcal{B}} \colon z \in X^\circ \mapsto [b_0(z), \ldots, b_n(z)] \in \mathbb{P}^n \simeq \mathbb{P}(V^*)$.

Remark 11.3.1. The Khovanskii homotopy algorithms compute the points of $\varphi_{\mathcal{B}}(X^\circ) \cap L$. Given these points, the solutions to System (11.1) on X° are their pull backs along $\varphi_{\mathcal{B}}$. Following Améndola and Rodriguez in [73], when the Kodaira map is not injective then the pull backs may be computed from the linear section and the points in a single general fiber of the Kodaira map.

Consequently, one may assume that the Kodaira map is an injection and replace X by its birational copy $\operatorname{Proj}(R(V))$, which is the closure of $\varphi_{\mathcal{B}}(X^\circ)$ in $\mathbb{P}^n$. In this case, $X = X^\circ$, V generates the function field $\mathbb{C}(X)$ of X, and the image of $V^\times$ under ν generates $\mathbb{Z}^d$. Thus the assumption that $X = \operatorname{Proj}(R(V))$ implies that the image of $V^\times$ under ν generates $\mathbb{Z}^d$.

The following summarizes the embedding of Anderson's toric degeneration $\mathcal{X}$ into $\mathbb{P}(V^*)$ [10, Section 2.2]. Let $\mathcal{A} := \nu(\mathcal{B})$ be the $(d+1) \times (n+1)$ matrix whose i^{th} column is $\nu(b_{i-1}s)$ for the Khovanskii basis $\mathcal{B} = \{b_0 s, \ldots, b_n s\} \subset Vs$. Note that the last row of $\mathcal{A}$ is $\mathbb{1} := (1, \ldots, 1)$. Define a partial order $>_{\mathcal{A}}$ on $\mathbb{Z}^{n+1}$ where $\beta >_{\mathcal{A}} \alpha$ if $\mathcal{A}\alpha \succ \mathcal{A}\beta$ in $\mathbb{Z}^{d+1}$. The initial form $\operatorname{in}_{\mathcal{A}}(f)$ of a polynomial f with respect to $>_{\mathcal{A}}$ is the sum of all terms $c_\alpha x^\alpha$ which minimize $\mathcal{A}\alpha$.

The ideal $I_{\mathcal{B}}$ of $X = \varphi_{\mathcal{B}}(X)$ is the kernel of the map $\mathbb{C}[x_0, \ldots, x_n] \to R(V)$ which takes x_i to $b_i s$. Define $\operatorname{in}_{\mathcal{A}}(I_{\mathcal{B}})$ to be the ideal generated by $\operatorname{in}_{\mathcal{A}}(f)$ for $f \in I_{\mathcal{B}}$. As shown in [74, Lemma 3.2] and [30, Lemma 8], there exists $w \in \mathbb{Z}^{d+1}$ such that if $\leq_{-w\mathcal{A}}$ is the weighted term order on $\mathbb{C}[x_0, \ldots, x_n]$ induced by $-w\mathcal{A}$, then the leading term ideal $\operatorname{lt}_{<_{-w\mathcal{A}}}(I_{\mathcal{B}})$ of $I_{\mathcal{B}}$ equals $\operatorname{in}_{\mathcal{A}}(I_{\mathcal{B}})$. Let w be such a weight vector and $\mathcal{G}$ denote a Gröbner basis for $I_{\mathcal{B}}$ with respect to a total order induced from the term order $\leq_{-w\mathcal{A}}$. The leading terms of elements of $\mathcal{G}$ with respect to $\leq_{-w\mathcal{A}}$ generate $\operatorname{in}_{\mathcal{A}}(I_{\mathcal{B}})$.

Let $g = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in $\mathcal{G}$, and define $w(g) := \min\{w\mathcal{A}\alpha : c_\alpha \neq 0\}$. Using Formula (7.2) (with $w\mathcal{A}$ in place of w), one can construct

$$g_\tau = \sum_\alpha c_\alpha x^\alpha \tau^{w\mathcal{A}\alpha - w(g)}. \tag{11.3}$$

Let $\mathcal{G}_\tau := \{g_\tau : g \in \mathcal{G}\}$. At $\tau = 0$, $\mathcal{G}_0$ generates $\operatorname{in}_{\mathcal{A}}(I_{\mathcal{B}})$ and at $\tau = 1$, $\mathcal{G}_1 = \mathcal{G}$ generates $I_{\mathcal{B}}$.

Finally, define $I_{\mathcal{A}}$ to be the kernel of the map $\mathbb{C}[x_0, \ldots, x_n] \to \operatorname{gr} R(V)$ which takes x_i to $\overline{b_i s} \in R(V)_{(\nu(b_i),1)}/R(V)^+_{(\nu(b_i),1)}$. Note that $I_{\mathcal{A}}$ is a toric ideal, and by [10, Theorem 2.17], $I_{\mathcal{A}} = \operatorname{in}_{\mathcal{A}}(I_{\mathcal{B}})$. Thus the toric weight degeneration can be embedded into $\mathbb{P}^n \simeq \mathbb{P}(V^*)$.

Proposition 11.3.2 ([30, Theorem 1]). *Let X be a variety and $V \subset \mathbb{C}(X)$ a finite-dimensional space of functions which has a finite Khovanskii basis $\mathcal{B} \subset Vs$. Then the family $\mathcal{X} \to \mathbb{C}_\tau$ defined by $\mathcal{G}_\tau$ is flat and embeds into $\mathbb{P}^n$ as the weight degeneration of $\mathcal{X}_1 = \operatorname{Proj}(R(V)) = \varphi_{\mathcal{B}}(X)$ induced by $w\mathcal{A}$. In particular, $\mathcal{X}_0 \simeq \operatorname{Proj}(\mathbb{C}[S(V,\nu)])$ and $\mathcal{X}$ is a toric degeneration.*

The following discusses the relationship between $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$. For $u \in \mathbb{N}^{n+1}$, write b^u for the product $\prod(b_i s)^{u_i}$ of elements in the Khovanskii basis. Since $\nu(b^u) = \mathcal{A}u$, when $\mathcal{A}u = \mathcal{A}v$ for some $u, v \in \mathbb{N}^{n+1}$, $\nu(b^u) = \nu(b^v)$ and there is a unique $c \in \mathbb{C}^\times$ such that

$$\mathcal{A}u = \mathcal{A}v \prec \nu(b^u - cb^v) \quad \text{and} \quad b^u - cb^v \in R(V)^+_{\mathcal{A}u}.$$

Since the last row of $\mathcal{A}$ is $(1, \ldots, 1)$, both b^u and $cb^v \in V^k s^k$ for some k and their difference is homogeneous.

The subduction algorithm [10, Algorithm 2.11] rewrites this difference as a homogeneous polynomial of degree k in the elements of the Khovanskii basis,

$$b^u - cb^v \ = \ h(b_0 s, b_1 s, \ldots, b_n s).$$

In particular, $g := x^u - cx^v - h(x_0, \ldots, x_n) \in I_{\mathcal{B}}$ with initial form $x^u - cx^v \in I_{\mathcal{A}}$. Applying Formula (11.3), we have that

$$g_\tau = x^u - cx^v - \tau^r h_\tau,$$

where $r = w(h) - w(g) > 0$.

Remark 11.3.3. Recall that the torus $\mathbb{T} = (\mathbb{C}^\times)^{n+1}/\Delta\mathbb{C}^\times \simeq (\mathbb{C}^\times)^n$ is the set of points in $\mathbb{P}^n$ with nonzero coordinates. A Kodaira map for the toric fiber $\mathcal{X}_0$ has the form $\varphi_{p,\mathcal{A}}$, as in Formula (5.4), for any $p \in \mathbb{T} \cap \mathcal{X}_0$. The following discussion provides a construction of such a point.

Let $x^u - cx^v \in I_{\mathcal{A}}$. Then $\mathcal{A}u = \mathcal{A}v$, so that $u - v \in \ker(\mathcal{A})$. Restricting this binomial to $\mathcal{X}_0 \cap \mathbb{T}$ results in the equation $c = x^{u-v}$. The constant c depends upon $u - v \in \ker(\mathcal{A})$, and one may write c_{u-v} for c. Thus a point $p \in \mathcal{X}_0 \cap \mathbb{T}$ satisfies equations of the form

$$c_u = p^u$$

for $u \in \ker(\mathcal{A})$. While every $u \in \ker(\mathcal{A})$ gives such an equation, an independent set of equations is given by a basis $u_1, \ldots, u_{n-d}$ for $\ker(\mathcal{A})$. The corresponding equations, $c_{u_i} = p^{u_i}$ for $i = 1, \ldots, n{-}d$, define $\mathcal{X}_0 \cap \mathbb{T}$ as a subvariety of $\mathbb{T}$.

To obtain a point of $\mathcal{X}_0 \cap \mathbb{T}$, one can construct d additional equations to these $n{-}d$ equations as follows: Since $\mathbb{1}$ is a row of $\mathcal{A}$, $\ker(\mathcal{A}) \subset \ker(\mathbb{1})$, which is a rank n sublattice of $\mathbb{Z}^{n+1}$. Let $v_1, \ldots, v_d \in \ker(\mathbb{1})$ be vectors such that $u_1, \ldots, u_{n-d}, v_1, \ldots, v_d$ are independent. Choose nonzero constants $c_{v_1}, \ldots, c_{v_d} \in \mathbb{C}^\times$ and consider the system of binomials

$$c_{u_i} - p^{u_i} = 0 = c_{v_j} - p^{v_j} \quad \text{for} \quad i = 1, \ldots, n{-}d \text{ and } j = 1, \ldots, d.$$

This system defines a finite set of points p in $\mathcal{X}_0 \cap \mathbb{T}$. An algorithm for solving such a system of binomials is given in [14, Lemma 3.2], which involves computing the Smith normal form of the matrix whose columns are $u_1, \ldots, u_{n-d}, v_1, \ldots, v_d$. Observe that only one solution is needed to obtain a Kodaira map.

11.3.3 Khovanskii homotopy

The procedure described in Section 11.3.2, combined with the toric two-step homotopy algorithm, Algorithm 11.2.3, forms the Khovanskii homotopy algorithm for computing the points of a linear section $\varphi_V(X) \cap L$.

Algorithm 11.3.4 (Khovanskii homotopy algorithm)**.**

Input: A finite-dimensional subspace $V \subset \mathbb{C}(X)$ for a variety $X = \mathrm{Proj}(R(V))$ of dimension d, a finite Khovanskii basis $\mathcal{B} \subset Vs$ for V, and a general linear subspace $L \subset \mathbb{P}^n$ of codimension d.

Output: All points in the linear section $\varphi_V(X) \cap L \subset \mathbb{P}(V^*)$.

Do:

1. Compute $I_{\mathcal{B}} = \ker(\mathbb{C}[x_0, \ldots, x_n] \to R(V))$ where $x_i \mapsto b_i s$.

2. Compute a weight vector w using [30, Lemma 2] so that $\mathrm{lt}_{<_{-w\mathcal{A}}}(I_{\mathcal{B}}) = \mathrm{in}_{\mathcal{A}}(I_{\mathcal{B}})$, where $\mathcal{A}$ is the matrix of values of $\mathcal{B}$.

3. Compute a Gröbner basis $\mathcal{G}$ for $I_{\mathcal{B}}$ using the weight $-w\mathcal{A}$.

4. Construct the homotopy $\mathcal{G}_\tau$ using Formula (11.3).

5. Construct the Kodaira map $\varphi_{p,\mathcal{A}}$ for $\mathcal{X}_0$ by following Remark 11.3.3.

6. Return the output $\varphi_V(X) \cap L$ of Algorithm 11.2.3 with inputs $\mathcal{G}_\tau$ and L.

Theorem 11.3.5. *Algorithm 11.3.4 is an optimal homotopy algorithm for computing all points of $\varphi_V(X) \cap L$.*

The correctness of Algorithm 11.3.4 follows from the discussion in Section 11.3.2.

Remark 11.3.6. In many cases, Algorithm 11.3.4 is applied to systems of functions where a finite Khovanskii basis is explicitly known from the theory (see Example 11.3.1). In this case, the theory not only includes the data for the finite Khovanskii basis $\mathcal{B}$, but also some or all of the data for Steps 1, 2, and 3 of Algorithm 11.3.4.

Example 11.3.1. Algorithm 11.3.4 and Remark 11.3.6 are illustrated here on a continuation of Example 11.2.1. In [30, Section 6.4], Anderson considers a particular three-dimensional Bott-Samelson variety X for $GL(3, \mathbb{C})$ In local coordinates (x, y, z) for X, the vector space V of considered functions on X has basis $\{1, x, y, z, xz, yz, x(xz + y), y(xz + y)\}$.

Anderson uses a valuation ν induced by the monomial valuation on $\mathbb{C}[x, y, z]$ defined by $\nu(f) = (a, b, c)$, where $x^a y^b z^c$ is the monomial of f that is minimal in the degree lexicographic order with $x > y > z$. The image $\mathcal{B} = \{1s, xs, ys, zs, xzs, yzs, x(xz + y)s, y(xz + y)s\}$ of this basis in Vs forms a Khovanskii basis for V. The corresponding matrix of valuations is

$$\mathcal{A} = \nu(\mathcal{B}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which is the matrix of Figure 11.1 after appending the row $\mathbb{1}$ for the exponents of s. The Newton-Okounkov body of V is also displayed in Figure 11.1.

Anderson provides the Khovanskii basis $\mathcal{B}$ for Algorithm 11.3.4, and Example 11.2.1 gives the general linear section L. For Step 1, generators of $I_{\mathcal{B}}$ are the generators of I in Example 11.2.1. The weight vector $w = (1, 1, 1, -2)$ suffices for Step 2. The vector $w\mathcal{A} = (-2, -1, -1, -1, 0, 0, 0, 0)$ appears as the weight in Example 11.2.1. The computations in Steps 3 and 4 are supplied by the elements in $\mathcal{G}_\tau$ in Example 11.2.1. Finally, for Step 5, the toric Kodaira map $\varphi_{\mathcal{A}}$ is also given in Example 11.2.1. $\triangle$

11.4 Khovanskii homotopy in weighted projective space

When a Khovanskii basis $\mathcal{B}$ for V contains elements of degree greater than 1, Anderson's toric degeneration naturally embeds into a weighted projective space [30]. This section explains how to lift the degeneration to a toric degeneration in ordinary projective space and use the toric two-step homotopy (Algorithm 11.2.3) to compute a linear section $\varphi_V(X) \cap L$ of the image of X under the Kodaira map $\varphi_V \colon X \dashrightarrow \mathbb{P}(V^*)$. Alternatively, one could use a projective embedding of the weighted projective space. This approach is not explored here, however, since it increases the ambient dimension and does not preserve the linear section.

11.4.1 Weighted projective spaces

This section first reviews the construction and some basic properties of weighted projective space, see [75]. Suppose that $a = (a_0, \ldots, a_{n+m})$ is a vector of mutually relatively prime positive integers. The weighted projective space $\mathbb{P}_a^{n+m}$ is $\mathrm{Proj}(\mathbb{C}[x_0, \ldots, x_{n+m}])$, where the grading on $\mathbb{C}[x_0, \ldots, x_{n+m}]$ is induced by setting the degree of x_j to a_j. Equivalently, $\mathbb{P}_a^{n+m}$ is the quotient of $\mathbb{C}^{n+m+1} \smallsetminus \{0\}$ by the $\mathbb{C}^\times$-action where $t.(x_0, \ldots, x_{n+m}) = (t^{a_0}x_0, \ldots, t^{a_{n+m}}x_{n+m})$, for $t \in \mathbb{C}^\times$. One may also construct $\mathbb{P}_a^{n+m}$ as a quotient of $\mathbb{P}^{n+m}$. To see this, let $\Delta\mathbb{C}^\times \subset (\mathbb{C}^\times)^{n+m+1}$ be the diagonal embedding of $\mathbb{C}^\times$ and let G_a be the image of the following product of groups of roots of unity in the dense torus $(\mathbb{C}^\times)^{n+m+1}/\Delta(\mathbb{C}^\times)$ of $\mathbb{P}^{n+m}$:

$$\mathrm{Hom}\left(\prod_j \mathbb{Z}/a_j\mathbb{Z}, \mathbb{C}^\times\right) = \prod_j \mathrm{Hom}\left(\mathbb{Z}/a_j\mathbb{Z}, \mathbb{C}^\times\right) \subset (\mathbb{C}^\times)^{n+m+1}.$$

Thus G_a acts faithfully on $\mathbb{P}^{n+m}$. As the a_j are mutually relatively prime, G_a is isomorphic to this product of groups of roots of unity. Let $\pi\colon \mathbb{P}^{n+m} \to \mathbb{P}_a^{n+m}$ be the quotient map by this G_a-action, which is a finite map of degree $|G_a| = \prod a_j$.

The weighted projective spaces that appear in the Khovanskii homotopy have the following special form: Let $W = \bigoplus_{k\geq 1} W_k$ be a finite-dimensional positively-graded vector space with $\dim W_1 = n{+}1 \geq 1$. Let $t \in \mathbb{C}^\times$ act on W_k as multiplication by t^{-k}, which gives a $\mathbb{C}^\times$-action on W. Identifying the dual space W^* with $\bigoplus_k W_k^*$, in the dual action, $t \in \mathbb{C}^\times$ acts on W_k^* as multiplication by t^k. Then the quotient of $W^* \smallsetminus \{0\}$ by $\mathbb{C}^\times$ is a weighted projective space.

This weighted projective space can be described explicitly. Suppose that $\dim W = n{+}m{+}1$, and let $a = (a_0, \ldots, a_{n+m})$ be a vector in which each $k \in \mathbb{N}$ occurs $\dim W_k$ times. Then $(W^* \smallsetminus \{0\})/\mathbb{C}^\times$ is isomorphic to $\mathbb{P}_a^{n+m}$, and $\mathbb{P}_a(W^*)$ is used to denote this quotient. The isomorphism depends upon the choice of an ordered basis for W^* which is a union of bases for each nontrivial summand W_k^* such that $a_j = k$ when the jth basis element lies in W_k^*. This choice of basis identifies W^* with $\mathbb{C}^{n+m+1}$, and allows one to define an action of G_a on the projective space $\mathbb{P}^{n+m}$ with quotient map $\pi\colon \mathbb{P}^{n+m} \to \mathbb{P}_a(W^*)$ as in the first paragraph above. Note that there is no natural identification of $\mathbb{P}(W^*)$ with $\mathbb{P}^{n+m}$ that is compatible with the map π, unless $\dim W_k \leq 1$ for all $k > 1$.

Let V be written for W_1. Under the $\mathbb{C}^\times$-action given by the weight a, the composition $V \hookrightarrow W \twoheadrightarrow V$ of the inclusion with the projection onto V is the identity and each map is $\mathbb{C}^\times$-equivariant. Taking linear duals gives the equivariant composition $V^* \hookrightarrow W^* \twoheadrightarrow V^*$, and this induces the composition $\mathbb{P}(V^*) \hookrightarrow \mathbb{P}_a(W^*) \dashrightarrow \mathbb{P}(V^*)$. One obtains ordinary projective space $\mathbb{P}(V^*)$ because $t \in \mathbb{C}^\times$ acts as multiplication by t on V^*. Write pr_a for the projection map $\mathbb{P}_a(W^*) \dashrightarrow \mathbb{P}(V^*)$, which is undefined on the image of the annihilator of V in $\mathbb{P}_a(W^*)$. In addition, let pr denote the composition $pr_a \circ \pi$. These maps are summarized in the following commutative diagram:

$$\begin{tikzcd}
G_a \curvearrowright \mathbb{P}^{n+m} \arrow[d, "\pi"'] \arrow[drr, dashed, "pr"] & & \\
\mathbb{P}_a^{n+m} \simeq \mathbb{P}_a(W^*) \arrow[rr, dashed, "pr_a"] & & \mathbb{P}(V^*) \simeq \mathbb{P}^n.
\end{tikzcd}$$

Let $X \subset \mathbb{P}(V^*)$ and $Z \subset \mathbb{P}_a(W^*)$ be varieties such that pr_a is an isomorphism between Z and X. In this case, a linear section $X \cap L$ pulls back along pr_a to $Z \cap pr_a^{-1}(L)$. Note that the subvariety $pr_a^{-1}(L)$, which is given by d forms that are linear in $x_0, \ldots, x_n$, is not general. For example, $pr_a^{-1}(L)$ includes $\mathcal{V}(x_0, \ldots, x_n)$, which contains the singular locus of $\mathbb{P}_a(W^*)$. Let $U \subset \mathbb{P}_a(W^*)$ be the open subset over which π is a covering space. For $u \in U$, G_a acts freely on the fiber $\pi^{-1}(z)$. The following lemma relates $Z \cap pr_a^{-1}(L)$ to $X \cap L$:

Lemma 11.4.1. *Let $Z \subset \mathbb{P}_a(W^*)$ be a subvariety of dimension d such that $Z \cap U$ is dense in Z and pr_a is an isomorphism between Z and $X := pr_a(Z)$. Let $Y := \pi^{-1}(Z) \subset \mathbb{P}^{n+m}$ be its inverse image. Suppose that $L \subset \mathbb{P}(V^*)$ is a general linear subspace of codimension d. Then,*

1. *$Z \cap pr_a^{-1}(L)$ is transverse and $pr_a \colon Z \cap pr_a^{-1}(L) \to X \cap L$ is a bijection.*

2. *$Y \cap pr^{-1}(L)$ is transverse and $\pi \colon Y \cap pr^{-1}(L) \to Z \cap pr_a^{-1}(L)$ is a $|G_a|$ to 1 surjection.*

3. *For any component Y' of Y, $\pi \colon Y' \cap pr^{-1}(L) \to Z \cap pr_a^{-1}(L)$ is a $|\mathrm{Stab}_{G_a}(Y')|$ to 1 surjection.*

Note that $Y = \pi^{-1}(Z)$ may not be irreducible. Each irreducible component, however, maps surjectively onto Z.

Proof. Transversality is addressed after establishing the set-theoretic assertions. For $x \in X \cap L$, let z be the unique point of Z with $pr_a(z) = x$. Since $z \in pr_a^{-1}(L)$, this completes the proof of the first statement.

Let $z \in Z \cap pr_a^{-1}(L)$. By the given assumptions, $Z \cap pr_a^{-1}(L) \subset U$, so $z \in U$. Then $\pi^{-1}(z) \subset Y \cap \pi^{-1} pr_a^{-1}(L) = Y \cap pr^{-1}(L)$. The second statement follows as $\pi\colon Y \to Z$ is $|G_a|$ to 1 over points of U.

For the third statement, observe that $pr^{-1}(L)$ is invariant under the G_a-action. Therefore, for all $g \in G_a$, $g.(Y' \cap pr^{-1}(L)) = (g.Y') \cap pr^{-1}(L)$. The claim follows from the second statement and a counting argument.

For transversality, let $x \in X \cap L$. As L is general, this intersection is transverse and the forms defining L generate the maximal ideal in the local ring of X at x. Transversality in the first statement follows since the map pr_a is an isomorphism between Z and X and $pr_a^{-1}(L)$ is defined by the same forms as L. Transversality in the second statement also follows, since the maximal ideal of Y at y is generated by the pull back of the maximal ideal of Z at $\pi(y)$ and $pr_a^{-1}(x) \in U$. $\square$

While $pr^{-1}(L)$ is a linear subspace, it is not general. A result similar to Lemma 11.4.1 is needed for a general linear subspace $\Lambda \subset \mathbb{P}^{n+m}$. Note that since Λ is general, $\pi^{-1}(\pi(\Lambda))$ consists of a union of $|G_a|$ linear subspaces.

Lemma 11.4.2. *Let $Z \subset \mathbb{P}_a(W^*)$ be a subvariety of dimension d such that $Z \cap U$ dense in Z. Let $Y := \pi^{-1}(Z) \subset \mathbb{P}^{n+m}$ be its inverse image, and suppose that $\Lambda \subset \mathbb{P}^{n+m}$ is a general linear subspace of codimension d. Then, $\pi\colon Y \cap \Lambda \to Z \cap \pi(\Lambda)$ is a bijection.*

Proof. Since Λ is general, $Z \cap \pi(\Lambda) \subset U$. Suppose that $q, q' \in Y \cap \Lambda$ are in the same fiber of π, and let $g \in G_a$ be defined by $q' = g.q$. Since Y is G_a-invariant, one can see that $q' \in Y \cap (g.\Lambda)$. Since Λ is general, $Y \cap \Lambda \cap (g.\Lambda)$ is empty unless g is the identity. Therefore, $q = q'$, and one may conclude that π is injective on $Y \cap \Lambda$.

This map is also surjective. If $p \in Z \cap \pi(\Lambda)$, then there is a point $q \in \pi^{-1}(p) \cap \Lambda$. As $Y = \pi^{-1}(Z)$, it contains $\pi^{-1}(p)$ and thus $q \in Y \cap \Lambda$ and $\pi(q) = p$. $\square$

11.4.2 Khovanskii bases and the degeneration

Let X be a d-dimensional complex variety and $V \subset \mathbb{C}(X)$ a finite-dimensional complex vector subspace. Suppose that the image of $V^\times$ under ν generates $\mathbb{Z}^d$ and V has a finite Khovanskii basis

$\mathcal{B}$ such that $\mathcal{B} \not\subset Vs$. For each $k \in \mathbb{N}$, let $W_k s^k := \mathrm{Span}(\mathcal{B} \cap V^k s^k) \subset V^k s^k$ be the span of the elements of $\mathcal{B}$ of homogeneous degree k. Define $W := \bigoplus_{k \geq 1} W_k$ where $V = W_1$ and construct the corresponding weighted projective space as in Section 11.4.1. Anderson's toric degeneration [30] naturally embeds into $\mathbb{P}_a(W^*)$. The weighted projective space $\mathbb{P}_a(W^*)$ is needed (rather than $\mathbb{P}(V^*)$) to accommodate the generators of $\mathrm{gr}\, R(V) \simeq \mathbb{C}[S(V,\nu)]$ which are not in V, as these are needed for embedding the toric fiber.

One can introduce coordinates by ordering the elements of $\mathcal{B} = \{b_0 s^{a_0}, \ldots, b_{n+m} s^{a_{n+m}}\}$ where $a_0 = \cdots = a_n = 1$, and for $n < j \leq n+m$, $a_j > 1$. Necessarily, $\{b_0, \ldots, b_n\} \subset V$, since Vs generates $R(V)$. Then, for each $n < j \leq n+m$, there is a homogeneous polynomial $h_j \in \mathbb{C}[z_0, \ldots, z_n]$ of degree a_j such that $b_j = h_j(b_0, \ldots, b_n)$.

Using the Khovanskii basis $\mathcal{B}$, the Kodaira map to $\mathbb{P}_a(W^*)$ from $X = \mathrm{Proj}(R(V))$ is $\varphi_{\mathcal{B}} \colon z \mapsto [b_0(z), \ldots, b_{n+m}(z)]$. Since, for $n < j \leq n+m$, $b_j = h_j(b_0, \ldots, b_n)$, the image of $\varphi_{\mathcal{B}}$ is a graph over the the image of φ_V in $\mathbb{P}(V^*) \subset \mathbb{P}_a(W^*)$.

The constructions of $I_{\mathcal{B}}$, $\mathcal{A}$, w, $\mathrm{in}_{\mathcal{A}}(I_{\mathcal{B}})$, and $\mathcal{G}_\tau$ from Section 11.3.2 all carry over to this general case since all of these ideals are a-homogeneous. Collectively, they embed Anderson's toric degeneration into the weighted projective space $\mathbb{P}_a(W^*)$. The special fiber $\mathcal{X}_0$ is a toric variety with ideal $\mathcal{G}_0$ and toric Kodaira map $\varphi_{p,\mathcal{A}}$, where $p \in \mathcal{X}_0 \cap \mathbb{T}_a$ (as before, the torus $\mathbb{T}_a \subset \mathbb{P}_a(W^*)$ consists of those points with nonzero coordinates).

One can pull back the embedded toric degeneration $\mathcal{X} \subset \mathbb{P}_a(W^*) \times \mathbb{C}_t$ along π to obtain a flat family $\mathcal{Y} \subset \mathbb{P}^{n+m} \times \mathbb{C}_\tau$ that is a toric degeneration in the sense of Remarks 11.2.6 and 7.2.3 as $\mathcal{Y}$ or $\mathcal{Y}_0$ may not be irreducible. The following explains how the equations defining the family $\mathcal{Y}$ may be obtained. Let $\mathbb{C}[y_0, \ldots, y_{n+m}]$ be the homogeneous coordinate ring of the projective space $\mathbb{P}^{n+m}$. The map $\pi \colon \mathbb{P}^{n+m} \to \mathbb{P}_a(W^*)$ corresponds to the map $\pi^* \colon \mathbb{C}[x_0, \ldots, x_{n+m}] \to \mathbb{C}[y_0, \ldots, y_{n+m}]$ induced by $x_i \mapsto y_i^{a_i}$. Let

$$\mathcal{F}_\tau := \{\pi^*(g_\tau) : g_\tau \in \mathcal{G}_\tau\} \tag{11.4}$$

be the pull back of the equations $\mathcal{G}_\tau$ for the embedded degeneration $\mathcal{X} \to \mathbb{C}_\tau$. Then $\mathcal{Y} = \mathcal{V}(\mathcal{F}_\tau) \subset \mathbb{P}^m \times \mathbb{C}_\tau$. This lifted family $\mathcal{Y} \to \mathbb{C}_\tau$ is the fiberwise pull back of Anderson's toric degeneration

$\mathcal{X} \to \mathbb{C}_\tau$ along the finite map π, where G_a acts on $\mathcal{Y}$ fiberwise.

11.4.3 Weighted Khovanskii homotopy

This section explains how to use the embedded degeneration $\mathcal{X}$ in $\mathbb{P}_a(W^*)$ to compute the linear section $\varphi_V(X) \cap L$. Since $\mathcal{X}_1 = \varphi_{\mathcal{B}}(X)$, it is natural to propose to compute $\mathcal{X}_1 \cap pr_a^{-1}(L)$ using an adaptation of the linear section homotopy to weighted projective space by following points of $\mathcal{X}_0 \cap pr_a^{-1}(L)$ along Anderson's degeneration. Unfortunately, $pr_a^{-1}(L)$ is not sufficiently general for the toric special fiber in Anderson's degeneration.

To avoid this problem, one may pull back the toric degeneration $\mathcal{X}$ along π to $\mathcal{Y}$ and use a linear section homotopy to compute the linear section $\mathcal{Y}_1 \cap pr^{-1}(L)$. Since $pr^{-1}(L)$ is not a general linear subspace, one can instead choose a general linear subspace $\Lambda \subset \mathbb{P}^{n+m}$ of codimension d. Next, one can use Algorithm 11.2.3 to compute $\mathcal{Y}_1 \cap \Lambda$, which is a witness set for $\mathcal{Y}_1$. Then, one can use the witness set homotopy (Algorithm 11.2.2) to compute $\mathcal{Y}_1 \cap pr^{-1}(L)$. Finally, $\varphi_V(X) \cap L$ is computed as $pr(\mathcal{Y}_1 \cap pr^{-1}(L))$.

Algorithm 11.4.3 (Weighted Khovanskii homotopy algorithm)**.**

Input: A finite-dimensional subspace $V \subset \mathbb{C}(X)$ for a variety $X = \mathrm{Proj}(R(V))$ of dimension d, finite Khovanskii basis $\mathcal{B} \not\subset Vs$ for V, and a general linear subspace $L \subset \mathbb{P}(V^*)$ of codimension d.

Output: Points in the linear section $\varphi_V(X) \cap L$ in the projective space $\mathbb{P}(V^*)$.

Do:

1. Follow Steps 1 through 5 of Algorithm 11.3.4, *mutatis mutandis*: The ideal $I_{\mathcal{B}}$ is the kernel of the map $\mathbb{C}[x_0, \ldots, x_{n+m}] \to R(V)$ where $x_i \mapsto b_i s^{a_i}$.

2. Pull back the family $\mathcal{X}$ along π to compute the family $\mathcal{Y}$ defined by $\mathcal{F}_\tau$, see Definition (11.4).

3. Compute Kodaira maps for each irreducible component of $\mathcal{Y}_0$.

4. Let $\Lambda \subset \mathbb{P}^{n+m}$ be a general linear subspace of codimension d and use Algorithm 11.2.3 to compute $\mathcal{Y}_1 \cap \Lambda$.

5. Use Algorithm 11.2.2 to compute $\mathcal{Y}_1 \cap pr^{-1}(L)$.

6. Return $\varphi_V(X) \cap L = pr(\mathcal{Y}_1 \cap pr^{-1}(L))$.

Remark 11.4.4. Consider Step 3 of Algorithm 11.4.3. As $\mathcal{Y}_0$ may consist of several components and $\mathcal{Y}_0 = \pi^{-1}(\mathcal{X}_0)$, the group G_a acts transitively on these components. Moreover, each component is a projective toric variety $X_{q,\mathcal{C}}$ for a point $q \in \mathcal{Y}_0 \cap \mathbb{T}$ and all have the same set of exponents, which are the columns of matrix $\mathcal{C}$. One can compute both q and $\mathcal{C}$ in the following manner.

From Step 5 of Algorithm 11.3.4, one obtains a toric Kodaira map $\varphi_{p,\mathcal{A}} \colon (\mathbb{C}^\times)^d \to \mathbb{P}^{n+m}_a$ such that $\mathcal{X}_0 = X_{p,\mathcal{A}}$. The image includes the point $p \in \mathcal{X}_0 \cap \mathbb{T}_a$. Points $q \in \pi^{-1}(p)$ are obtained by taking all a_j-th roots of the coordinate p_j of p, for all j,

$$\pi^{-1}(p) = \{q \in \mathbb{P}^{n+m} : q_j^{a_j} = p_j \text{ for } j = 0, \ldots, n+m+1\}.$$

It remains to determine the exponents $\mathcal{C}$ for $\pi^{-1}(X_{\mathcal{A}})$. As in Remark 11.3.3, one has a basis $u_1, \ldots, u_{n+m-d} \in \mathbb{Z}^{n+m+1}$ for $\ker(\mathcal{A})$. These vectors give equations $x^{u_i} = 1$ for $X_{\mathcal{A}} \cap \mathbb{T}_a$. Applying π^* substitutes $y_j^{a_j}$ for x_j and gives equations for $\pi^{-1}(X_{\mathcal{A}}) \cap \mathbb{T}$,

$$y^{v_i} = 1 \qquad i = 1, \ldots, n+m-d\,, \tag{11.5}$$

where v_i is obtained from u_i by multiplying its jth coordinate by a_j.

The System (11.5) for $\pi^{-1}(X_{\mathcal{A}}) \cap \mathbb{T}$ leads to equations for $Y := \pi^{-1}(X_{\mathcal{A}})$, which form a lattice ideal [76, Section 2] for the lattice K spanned by $\{v_1, \ldots, v_{n+m-d}\}$. That is, $Y = \mathcal{V}\langle y^\alpha - y^\beta \mid \alpha - \beta \in K\rangle$.

Let $(\gamma_1, \ldots, \gamma_d, \mathbb{1})$ be a basis for the annihilator of K in $\mathbb{Z}^{n+m+1}$. Suppose that $\mathcal{C}$ is the $d \times (n+m+1)$ matrix whose rows are $\gamma_1, \ldots, \gamma_d$. Then $X_{\mathcal{C}}$ is the component of Y containing the identity $\mathbb{1} \in \mathbb{T}$. Note that $\mathcal{C}$ may be computed from $v_1, \ldots, v_{n+m-d}$ using the Hermite normal form. All Kodaira maps needed in Step 3 of Algorithm 11.4.3 can then be computed by translations.

Remark 11.4.5. The number of components of $\mathcal{Y}$ or of $\mathcal{Y}_0$ impacts the number of Kodaira maps needed in Step 3 of Algorithm 11.4.3. Reductions in the number of Kodaira maps may significantly improve the efficiency of the algorithm.

When $\mathcal{Y}$ is known to be reducible, this structure may be exploited, as Statement 3 of Lemma 11.4.1 implies that it is enough to apply Algorithm 11.4.3 to a single component of $\mathcal{Y}$. In particular, the map π sends the curves in a linear section of one component onto $\mathcal{X} \cap pr^{-1}(L)$.

When $\mathcal{Y}_0$ has fewer than $|G_a|$ components, then there are redundant Kodaira maps constructed in Remark 11.4.4. More precisely, the number of redundant maps is the number of points of $\pi^{-1}(p)$ in a component of $\mathcal{Y}_0$. The following provides details on computing non-redundant Kodaira maps, assuming, as in Remark 11.4.4, that $\mathbb{1} \in \mathcal{Y}_0$. The general case is obtained by translation. Let

$$\operatorname{sat}(K) := \{w \in \mathbb{Z}^{n+m+1} : rw \in K \text{ for some } 0 \neq r \in \mathbb{Z}\}$$

be the saturation of K and $M = \ker(\mathbb{1}) \subset \mathbb{Z}^{n+m+1}$. Note that $\operatorname{sat}(K) \subset M$. Identify $\mathbb{T}$ with $\operatorname{Hom}(M, \mathbb{C}^\times)$ so that $\mathcal{Y}_0 \cap \mathbb{T} = \operatorname{Hom}(M/K, \mathbb{C}^\times)$, as these are the points satisfying System (11.5). The component of $\mathcal{Y}_0 \cap \mathbb{T}$ containing the identity $\mathbb{1} \in \mathbb{T}$ is $\operatorname{Hom}(M/\operatorname{sat}(K), \mathbb{C}^\times)$, and the group of components of $\mathcal{Y}_0 \cap \mathbb{T}$ is $\operatorname{Hom}(\operatorname{sat}(K)/K, \mathbb{C}^\times)$. Hence, the elements of $\operatorname{Hom}(\operatorname{sat}(K)/K, \mathbb{C}^\times)$ generate Kodaira maps to distinct components of $\mathcal{Y}_0$.

Proof of correctness of Algorithm 11.4.3. It suffices to show that the tracked paths provide enough points to compute $\varphi_V(X) \cap L$. By Statement 3 of Lemma 11.4.1, for each τ, the map $\pi : \mathcal{Y}_\tau \cap \Lambda \to \mathcal{X}_\tau \cap \pi(\Lambda)$ is a bijection.

The polyhedral homotopy correctly computes the points of $\mathcal{Y}_0 \cap \Lambda$. By Theorem 11.2.4, Algorithm 11.2.3 correctly computes the points of $\mathcal{Y}_1 \cap \Lambda$. Since the solution paths of the homotopy $\mathcal{X} \cap \pi(\Lambda)$ are disjoint, the solution paths of $\mathcal{Y} \cap \Lambda$ lie above paths of $\mathcal{X} \cap \pi(\Lambda)$. In fact, by Statement 3 of Lemma 11.4.1, π is a bijection between these sets of paths. Therefore, there is a bijection between the ends of the homotopy paths of $\mathcal{Y} \cap \Lambda$ and points in $\mathcal{X}_1 \cap \pi(\Lambda)$. The correctness of the final computation then follows from the correctness of Algorithm 11.2.2. □

Example 11.4.1. Let V be the space of cubic polynomials in $\mathbb{C}[x, y]$ which vanish at the points

$(4,4), (-3,-1), (-1,-1)$ and $(3,3)$. (This example is related to the example of [77, Section 5.1], which considers quartics vanishing at these points.) Then V is six-dimensional with a basis:

$$\begin{aligned}\{b_0,\ldots,b_5\} = \{&\underline{xy} - y^2 + x - y,\ \underline{x^2} - y^2 + 4x - 4y,\ \underline{y^3} - 6y^2 + 5y + 12,\\ &\underline{xy^2} - 6y^2 - x + 6y + 12,\ \underline{x^2y} - 6y^2 - 4x + 9y + 12,\ \underline{x^3} - 6y^2 - 13x + 18y + 12\}.\end{aligned}$$

A general linear section of $X = \mathrm{Proj}(R(V))$ in $\mathbb{P}(V^*) = \mathbb{P}^5$ is computed with Algorithm 11.4.3 in the following manner. Let $\succeq$ be the order on $\mathbb{Z}^2$ where $(a,b) \succeq (c,d)$ if $a+b < c+d$ or else $a+b = c+d$ and $a < c$. Define a valuation ν on $\mathbb{C}(X) = \mathbb{C}(x,y)$ as follows: for $f \in \mathbb{C}[x,y]$, $\nu(f) = (a,b)$ where (a,b) is the $\succeq$-minimal exponent of a term of f. This order and valuation ν are compatible with the `grevlex` order $\leq$ on $\mathbb{C}[x,y]$ with $x > y$ in that $(a,b) \succeq (c,d)$ if and only if $x^a y^b \leq x^c y^d$. Using the subduction algorithm, as implemented in the `Macaulay2` package `SubalgebraBases` [8] applied to $\{b_0 s, \ldots, b_5 s\}$, one obtains a Khovanskii basis $\mathcal{B} = \{b_0 s, \ldots, b_5 s,\ b_6 s^2, b_7 s^3\}$ with two additional generators, where

$$\begin{aligned} b_6 &:= \underline{xy^3} - y^4 + 10x^2y - 26xy^2 + 16y^3 + 10x^2 - 15xy + 5y^2 + 12x - 12y, \text{ and}\\ b_7 &:= \underline{10x^4y} - 49x^3y^2 + 89x^2y^3 - 71xy^4 + 21y^5 + 10x^4 - 18x^3y - 18x^2y^2\\ &\quad +50xy^3 - 24y^4 + 31x^3 - 83x^2y + 73xy^2 - 21y^3 + 24x^2 - 48xy + 24y^2.\end{aligned}$$

The corresponding matrix of valuations is

$$\mathcal{A} = \nu(\mathcal{B}) = \begin{pmatrix} 1 & 2 & 0 & 1 & 2 & 3 & 1 & 4\\ 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1\\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 \end{pmatrix}.$$

The Newton-Okounkov body, as displayed in Figure 11.2, is obtained by intersecting the cone generated by the columns of $\mathcal{A}$ with the hyperplane where the third coordinate is 1. The vertices $(1/2, 3/2)$ and $(4/3, 1/3)$ come from the initial (underlined) terms of b_6 and b_7. While they are not integers, the Newton-Okounkov body has normalized volume 5, which is the degree of X. One may interpret this volume as follows: Two cubics drawn from V meet in $5 = 3^2 - 4$ points outside the base locus $\mathcal{V}(V) = \{(4,4), (-3,-1), (-1,-1), (3,3)\}$.

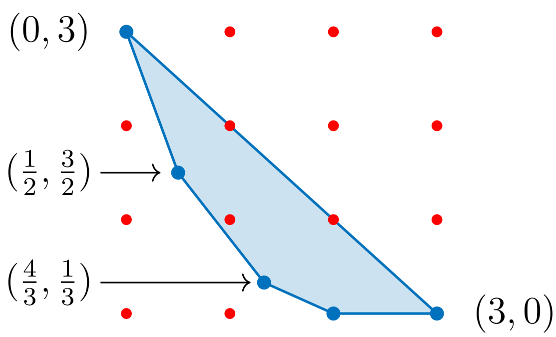


Figure 11.2: Newton Okounkov body for the space of cubic polynomials vanishing at $(4,4)$, $(-3,-1)$, $(-1,-1)$ and $(3,3)$.

The weight $w = (-6,-5,0)$ is compatible with the `grevlex` order $\leq$ on $\mathcal{B}$ in that for $b \in \mathcal{B}$, the $\leq$-leading term has lowest w-weight, so that $\mathrm{lt}_< b = b_w$. Choosing a term order on $\mathbb{C}[x_0, \dots, x_7]$ that is compatible with $w\mathcal{A}$, one may use `Macaulay2` to compute a Gröbner basis $\mathcal{G}$ for $I_{\mathcal{B}}$. This basis consists of 17 polynomials which are $a := (1,1,1,1,1,1,2,3)$-homogeneous. Let $\mathbb{C}^\times$ act on a-homogeneous polynomials using $w\mathcal{A}$ in place of w in Formula (7.1). Then one may compute $\mathcal{G}_\tau := \{g_\tau : g \in \mathcal{G}\}$ as in Formula (7.2), which defines a flat family $\mathcal{X} \subset \mathbb{P}^7_a \times \mathbb{C}_\tau$ with toric special fiber $\mathcal{X}_0$. This family pulls back along $\pi \colon \mathbb{P}^7 \to \mathbb{P}^7_a$ to a family $\mathcal{Y} \subset \mathbb{P}^7 \times \mathbb{C}_\tau$. The pull back $\mathcal{Y}_0$ of $\mathcal{X}_0$ under π is a toric variety as it is irreducible. From Remarks 11.3.3 and 11.4.4, a Kodaira map for $\mathcal{Y}_0$ is

$$\begin{aligned} \varphi_{p,\mathcal{A}} \colon (\mathbb{C}^*)^2 &\longrightarrow \mathbb{P}^7 \\ z &\longmapsto [z_1^6 z_2^3,\ z_1^4 z_2^3,\ z_1^6,\ z_1^4,\ z_1^2,\ 1,\ z_1^7 z_2^3,\ \sqrt[3]{10}\, z_1^6 z_2^4]. \end{aligned}$$

The polyhedral homotopy finds 30 points in $\mathcal{Y}_0 \cap \Lambda$. An application of the toric two-step algorithm (Algorithm 11.2.3) tracks these points to $\mathcal{Y}_1 \cap \Lambda$ with no paths diverging. Then, the witness set homotopy algorithm (Algorithm 11.2.2) moves Λ to $pr^{-1}(L)$ and finds the points of $\mathcal{Y}_1 \cap pr^{-1}(L)$. These 30 points project under $\pi\colon \mathbb{P}^7 \to \mathbb{P}^7_a$ to five points in $\mathcal{X}_1 \cap \ pr^{-1}(L)$. Finally, applying the map $pr\colon \mathbb{P}^7 \to \mathbb{P}^5 = \mathbb{P}(V^*)$ gives all five points in $\varphi_V(X) \cap L$. $\triangle$

11.5 Practical considerations

This section discusses how to compute a finite Khovanskii basis as well as options for tracking overdetermined homotopy systems.

11.5.1 Computing a Khovanskii basis.

Whether or not a given vector space V of functions has a finite Khovanskii basis is generally not known and depends on the choice of valuation. Given V and a valuation ν, the subduction algorithm [10, Algorithm 2.18] terminates and returns a finite Khovanskii basis, when one exists. When V is a space of polynomials and ν is induced by a term order, one can use the existing implementation in [5, 8] to compute finite Khovanskii bases. order [5, 8]. This implementation technically computes a SAGBI basis [9, 7].

11.5.2 Homotopy continuation for overdetermined systems.

Algorithms 11.3.4 and 11.4.3 generate a homotopy $(\mathcal{G}_\tau, L)$ from a Gröbner basis $\mathcal{G}$ defining $\mathcal{X}$. This is not typically square in that it has more equations than variables. As most implementations of homotopy continuation, including the user-defined homotopy in `Bertini`, require square systems, a method is needed to choose a square subsystem for tracking from $\tau = 0$ to $\tau = 1$.

Typically, a square subsystem is obtained by taking linear combinations of elements in a given system. There is an alternative for equations $\mathcal{G}_\tau$ from a toric degeneration. Let $\mathcal{A}$ be the matrix of exponents defining the Kodaira map for the toric special fiber, $\mathcal{X}_0$. The intersection $\mathcal{X}_0 \cap \mathbb{T}$ with the dense torus of $\mathbb{P}^n$ is the complete intersection defined by binomials $x^{u_i} - c_i x^{v_i}$ for $i = 1, \ldots, n{-}d$ such that $\{u_i - v_i : i = 1, \ldots, n{-}d\}$ form a basis for $\ker(\mathcal{A})$. The points of $\mathcal{X}_0 \cap L$ are smooth isolated solutions to the square system given by these binomials and the linear forms defining L.

If one chooses $\mathcal{F}_\tau \subset \mathcal{G}_\tau$ to consist of $n-d$ elements whose leading binomials are $x^{u_i} - c_i x^{v_i}$, then $(\mathcal{F}_\tau, L)$ is a square subsystem of $(\mathcal{G}_\tau, L)$ which defines curves containing $\mathcal{X}_0 \cap L$ and, therefore, is sufficient for homotopy continuation.

12. SUMMARY

Both the Khovanskii homotopy and the Cox homotopy provide numerical algorithms for estimating all solutions to a polynomial system. These homotopy methods have the advantage of tracking paths in compact spaces while also minimizing the total number of paths tracked. This advantage can lead to faster algorithms for solving polynomial systems.

The Cox homotopy has a `Julia` implementation, with examples, available at `https://mathrepo.mis.mpg.de/CoxHomotopies/index.html`. This implementation will compute the solutions to a given polynomial system using the algorithms detailed in Chapter 10. The examples available demonstrate the advantages of the Cox homotopy, as detailed in Section 10.5.

At this time, the Khovanskii homotopy does not have a general implementation, however examples using the Khovanskii homotopy are available at `https://github.com/EliseAWalker/KhovanskiiHomotopy/`. The primary reason that the Khovanskii homotopy does not have a general implementation is that it relies on the existence of a *finite* Khovanskii basis. Finite Khovanskii bases may not exist and there is no algorithm for determining whether or not a finite Khovanskii basis exists for a given set of subalgebra generators. As such, any implementation of the Khovanskii homotopy that includes the computation of a Khovanskii basis would be subject to the halting problem.

Even if a general implementation of the Khovanskii homotopy were written with a Khovanskii basis and corresponding valuation as input, then there is a need for the user to compute Khovanskii bases themselves. Thus providing robust software for these Khovanskii basis computations is a future direction for this work. With Michael Burr, Oliver Clarke, and Tim Duff, the software package `SubalgebraBases.m2`, which computes finite Khovanskii bases (if they exist) with respect to a chosen monomial term order, is under development. One of the future goals for expanding `SubalgebraBases.m2` includes implementing valuations to allow for more general computations of Khovanskii bases.

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APPENDIX A

GEOMETRIC QUOTIENTS

Let X and Y be varieties. Suppose a group $\mathbb{G}$ acts on X. Given $X = \operatorname{Spec}(R)$, the ring of invariants is denoted $R^{\mathbb{G}}$.

Definition A.0.1 (Categorical quotient, [15])**.** Given a group $\mathbb{G}$ which acts on variety X and a morphism of varieties $\pi : X \to Y$ which is constant on $\mathbb{G}$-orbits, the morphism π is a categorical quotient if:

1. If $U \subseteq Y$ is open, then the natural map $\mathscr{O}_Y(U) \to \mathscr{O}_X(\pi^{-1}(U))$ induces an isomorphism

$$\mathscr{O}_Y(U) \cong \mathscr{O}_X(\pi^{-1}(U))^{\mathbb{G}}.$$

2. If $W \subset X$ is closed and $\mathbb{G}$-invariant, then $\pi(W) \subset Y$ is closed.

3. If W_1, W_2 are closed, disjoint, and $\mathbb{G}$-invariant in X, then $\pi(W_1)$ and $\pi(W_2)$ are disjoint in Y.

A categorical quotient π is surjective [15, Theorem 5.0.6] and consequently is often written as $\pi : X \to X /\!\!/ \mathbb{G}$.

Proposition A.0.2 (Proposition 5.0.8 in [15])**.** *Let $\pi : X \to X /\!\!/ \mathbb{G}$ be a categorical quotient. Then the following are equivalent:*

1. *All $\mathbb{G}$-orbits are closed in X.*

2. *Given points $x, y \in X$, we have: $\pi(x) = \pi(y) \iff x$ and y lie in the same $\mathbb{G}$-orbit.*

3. *π induces a bijection, {$\mathbb{G}$-orbits in X} $\cong X /\!\!/ \mathbb{G}$.*

4. *The image of the morphism $\mathbb{G} \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$ is the fiber product $X \times_{X /\!\!/ \mathbb{G}} X$.*

Definition A.0.3 (Geometric quotient)**.** A categorical quotient $\pi : X \to X /\!\!/ \mathbb{G}$ is a geometric quotient if it satisfies the conditions in Proposition A.0.2.

Proposition A.0.4 (Proposition 5.0.11 in [15])**.** *Let $\pi : X \to X /\!\!/ \mathbb{G}$ be a categorical quotient. Then the following are equivalent:*

1. *X has a $\mathbb{G}$-invariant Zariski dense open subset U_0 such that $\mathbb{G} \cdot x$ is closed in X for all $x \in U_0$.*

2. *$X /\!\!/ \mathbb{G}$ has a Zariski dense open subset U such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a geometric quotient.*

Definition A.0.5 (Almost geometric quotient)**.** A good categorical quotient $\pi : X \to X /\!\!/ \mathbb{G}$ is an almost geometric quotient if it satisfies the conditions in Proposition A.0.4.