

Monopoles and Instantons in String Theory [†]

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In recent work, several classes of solitonic solutions of string theory with higher-membrane structure have been obtained. These solutions can be classified according to the symmetry possessed by the solitons in the subspace of the spacetime transverse to the membrane. Solitons with four-dimensional spherical symmetry represent instanton solutions in string theory, while those with three-dimensional spherical symmetry represent magnetic monopole-type solutions. For both of these classes, we discuss bosonic as well as heterotic solutions.

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1. Introduction

In recent work classical solitonic solutions of string theory with higher-membrane structure have been investigated. These solutions can be classified according to the symmetry the solitons possess in the subspace of spacetime transverse to the membrane. In this paper, we discuss two classes of solutions, those with four-dimensional spherical symmetry, which possess instanton structure, and those with three-dimensional spherical symmetry, which represent magnetic monopole-type solutions in string theory.

For both instantons and monopoles, we review solutions in Yang-Mills field theory as well as axionic solitonic solutions for the massless fields of the bosonic string. In each case we combine the gauge theory solution with the corresponding bosonic solution to obtain an exact multi-soliton solution of heterotic string theory[1].

We begin section 2 with a review of the 't Hooft ansatz for the Yang-Mills instanton[2–6]. We then turn to the axionic instanton solution first mentioned in [7]. This tree-level solution is extended in [8] to an exact solution of bosonic string theory for the special case of a linear dilaton wormhole solution[9,10]. Exactness is shown by combining the metric and antisymmetric tensor in a generalized curvature, which is written covariantly in terms of the tree-level dilaton field, and rescaling the dilaton order by order in the parameter α' . The corresponding conformal field theory is written down.

An exact heterotic multi-soliton solution with YM instanton structure in the four dimensional transverse space can be obtained[11,12] by equating the curvature of the Yang-Mills gauge field with the generalized curvature derived in [8]. This solution represents an exact extension of the tree-level fivebrane solutions of [13,14,15] and combines the gauge and axionic instanton structures.

In section 3 we turn to the three-dimensional (monopole) solutions. We first discuss a multimonopole solution in YM field theory, which arises from a modification of the 't Hooft ansatz for the four-dimensional instanton[16,17]. We then mention the bosonic three-dimensional solution obtained in [18]. We complete this section with a review of the recently constructed exact multimonopole solution of heterotic string theory[16,17], which now combines the gauge and axionic monopole structures. Unlike the heterotic instanton solution, this solution does not lend itself easily to a CFT description. An interesting aspect of this string monopole solution, however, is that the divergences stemming from the YM sector are precisely cancelled by those coming from the gravity sector, thus resulting in a finite action solution.

We conclude in section 4 with a summary of these results and a brief discussion.

2. Four-Dimensional Instanton Solutions

In this section, we discuss four-dimensional, or instanton solutions in bosonic and heterotic string theory. We first summarize the 't Hooft ansatz for the Yang-Mills instanton, and then write down the tree-level bosonic axionic instanton solution of [7]. An exact extension of this solution can be obtained for the special case of a wormhole solution, and the corresponding conformal field theory is written down[8]. Finally, an exact multi-instanton solution of heterotic string theory is obtained, combining the Yang-Mills gauge solution with the bosonic axionic instanton[19,11,12].

Consider the four-dimensional Euclidean action

$$S = -\frac{1}{2g^2} \int d^4x \text{Tr} G_{\mu\nu} G^{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4. \quad (2.1)$$

For gauge group $SU(2)$, the fields may be written as $A_\mu = (g/2i)\sigma^a A_\mu^a$ and $G_{\mu\nu} = (g/2i)\sigma^a G_{\mu\nu}^a$ (where σ^a , $a = 1, 2, 3$ are the 2×2 Pauli matrices). The equation of motion derived from this action is solved by the 't Hooft ansatz[2-6]

$$A_\mu = i\bar{\Sigma}_{\mu\nu} \partial_\nu \ln f, \quad (2.2)$$

where $\bar{\Sigma}_{\mu\nu} = \bar{\eta}^{i\mu\nu} (\sigma^i/2)$ for $i = 1, 2, 3$, where

$$\begin{aligned} \bar{\eta}^{i\mu\nu} &= -\bar{\eta}^{i\nu\mu} = \epsilon^{i\mu\nu}, & \mu, \nu &= 1, 2, 3, \\ &= -\delta^{i\mu}, & \nu &= 4 \end{aligned} \quad (2.3)$$

and where $f^{-1} \square f = 0$. The ansatz for the anti-self-dual solution is similar, with the δ -term in (2.3) changing sign. To obtain a multi-instanton solution, one solves for f in the four-dimensional space to obtain

$$f = 1 + \sum_{i=1}^N \frac{\rho_i^2}{|\vec{x} - \vec{a}_i|^2}, \quad (2.4)$$

where ρ_i^2 is the instanton scale size and \vec{a}_i the location in four-space of the i th instanton. We will return to the 't Hooft ansatz when we consider a superstring model with YM coupling (the heterotic string[1]).

We now turn to the bosonic axionic instanton solution considered in [8]. We first derive the tree-level solution of [7] and then extend the single instanton wormhole solution to $O(\alpha')$ in the massless fields. For this purpose we use the theorem of equivalence of

the massless string field equations to the sigma-model Weyl invariance conditions (demonstrated to two-loop order by Metsaev and Tseytlin[20,21]), which require the Weyl anomaly coefficients $\overline{\beta}_{\mu\nu}^G$, $\overline{\beta}_{\mu\nu}^B$ and $\overline{\beta}^\Phi$ to vanish identically to the appropriate order in the parameter α' . The two-loop solution obtained by this method suggests a representation of the sigma model as the product of a WZW[22] model and a one-dimensional CFT (a Feigin-Fuchs Coulomb gas)[7]. This representation allows us to obtain an exact solution.

The bosonic sigma model action can be written as[23]

$$I = \frac{1}{4\pi\alpha'} \int d^2x \left(\sqrt{\gamma}\gamma^{ab}\partial_a x^\mu\partial_b x^\nu g_{\mu\nu} + i\epsilon^{ab}\partial_a x^\mu\partial_b x^\nu B_{\mu\nu} + \alpha'\sqrt{\gamma}R^{(2)}\phi \right), \quad (2.5)$$

where $g_{\mu\nu}$ is the sigma model metric, ϕ the dilaton and $B_{\mu\nu}$ the antisymmetric tensor, and where γ_{ab} is the worldsheet metric and $R^{(2)}$ the two-dimensional curvature. The Weyl anomaly coefficients are given by[20,21]

$$\begin{aligned} \overline{\beta}_{\mu\nu}^G &= \beta_{\mu\nu}^G + 2\alpha'\nabla_\mu\nabla_\nu\phi + \nabla_{(\mu}W_{\nu)}, \\ \overline{\beta}_{\mu\nu}^B &= \beta_{\mu\nu}^B + \alpha'H_{\mu\nu}{}^\lambda\partial_\lambda\phi + \frac{1}{2}H_{\mu\nu}{}^\lambda W_\lambda, \\ \overline{\beta}^\Phi &= \beta^\Phi + \alpha'(\partial\phi)^2 + \frac{1}{2}W^\lambda\partial_\lambda\phi, \end{aligned} \quad (2.6)$$

where $\beta_{\mu\nu}^G$, $\beta_{\mu\nu}^B$ and β^Φ are the RG β functions and where $H_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]}$ and $W_\mu = -(\alpha'^2/24)\nabla_\mu H^2$.

We first show that for any dilaton function satisfying $e^{-2\phi}\square e^{2\phi} = 0$ with

$$\begin{aligned} g_{\mu\nu} &= e^{2\phi}\delta_{\mu\nu} & \mu, \nu &= 1, 2, 3, 4, \\ g_{ab} &= \delta_{ab} & a, b &= 5, \dots, 26, \\ H_{\mu\nu\lambda} &= \pm\epsilon_{\mu\nu\lambda\sigma}\partial^\sigma\phi & \mu, \nu, \lambda, \sigma &= 1, 2, 3, 4 \end{aligned} \quad (2.7)$$

the $O(\alpha')$ Weyl anomaly coefficients vanish identically.

We define a generalized curvature $\hat{R}^i{}_{jkl}$ in terms of the standard curvature $R^i{}_{jkl}$ and $H_{\mu\alpha\beta}$ [24]:

$$\hat{R}^i{}_{jkl} = R^i{}_{jkl} + \frac{1}{2}(\nabla_l H^i{}_{jk} - \nabla_k H^i{}_{jl}) + \frac{1}{4}(H^m{}_{jk}H^i{}_{lm} - H^m{}_{jl}H^i{}_{km}). \quad (2.8)$$

One can also define $\hat{R}^i{}_{jkl}$ as the Riemann tensor generated by the generalized Christoffel symbols $\hat{\Gamma}^\mu_{\alpha\beta}$, where $\hat{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - (1/2)H^\mu{}_{\alpha\beta}$.

We follow Metsaev and Tseytlin's computation of the renormalization group beta functions for the general sigma-model and combine dimensional regularization and the

minimal subtraction scheme with the following generalized prescription for contraction of ϵ^{ab} tensors[20]:

$$\epsilon^{ab}\epsilon^{cd} = f(d) (\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}), \quad (2.9)$$

where $f(d) = 1 - f_1\epsilon + O(\epsilon^2)$ and $\epsilon = d - 2$. We note that the precise form of the renormalization group beta functions at two-loop order is not scheme-independent but depends on the choice of f_1 . Here we set $f_1 = -1$, for which Metsaev and Tseytlin obtain the following two-loop expressions for the Weyl anomaly coefficients[20,21]:

$$\begin{aligned} \overline{\beta}_{\mu\nu}^G &= \alpha'(\hat{R}_{(\mu\nu)} + 2\nabla_\mu\nabla_\nu\phi) \\ &\quad + \frac{\alpha'^2}{2} \left(\hat{R}^{\alpha\beta\gamma}{}_{(\mu}\hat{R}_{\nu)\alpha\beta\gamma} - \frac{1}{2}\hat{R}^{\beta\gamma\alpha}{}_{(\mu}\hat{R}_{\nu)\alpha\beta\gamma} + \frac{1}{2}\hat{R}_{\alpha(\mu\nu)\beta}(H^2)^{\alpha\beta} \right) + \nabla_{(\mu}W_{\nu)}, \\ \overline{\beta}_{\mu\nu}^B &= \alpha'(\hat{R}_{[\mu\nu]} + H_{\mu\nu}{}^\lambda\partial_\lambda\phi) \\ &\quad + \frac{\alpha'^2}{2} \left(\hat{R}^{\alpha\beta\gamma}{}_{[\mu}\hat{R}_{\nu]\alpha\beta\gamma} - \frac{1}{2}\hat{R}^{\beta\gamma\alpha}{}_{[\mu}\hat{R}_{\nu]\alpha\beta\gamma} + \frac{1}{2}\hat{R}_{\alpha[\mu\nu]\beta}(H^2)^{\alpha\beta} \right) + \frac{1}{2}H_{\mu\nu}{}^\lambda W_\lambda, \\ \overline{\beta}^\Phi &= \frac{D}{6} - \frac{\alpha'}{2} \left(\nabla^2\phi - 2(\partial\phi)^2 + \frac{1}{12}H^2 \right) \\ &\quad + \frac{\alpha'^2}{16} \left(2(H^2)^{\mu\nu}\nabla_\mu\nabla_\nu\phi + R_{\lambda\mu\nu\rho}^2 - \frac{11}{2}RHH + \frac{5}{24}H^4 + \frac{11}{8}(H_{\mu\nu}^2)^2 + \frac{4}{3}\nabla H \cdot \nabla H \right) \\ &\quad + \frac{1}{2}W^\lambda\partial_\lambda\phi, \end{aligned} \quad (2.10)$$

where $\nabla H \cdot \nabla H \equiv \nabla_\alpha H_{\beta\gamma\delta}\nabla^\alpha H^{\beta\gamma\delta}$. Unless otherwise indicated, all expressions are written to two loop order in the beta-functions, which corresponds to $O(\alpha')$ in the action. Also, all indices are in the curved four-space, as it is clear that the flat dimensions do not contribute.

The crucial observation for obtaining higher-loop and even exact solutions is the following. For any solution of the form (2.7), we can express the generalized curvature in covariant form in terms of the dilaton field ϕ :

$$\hat{R}^i{}_{jkl} = \delta_{il}\nabla_k\nabla_j\phi - \delta_{ik}\nabla_l\nabla_j\phi + \delta_{jk}\nabla_l\nabla_i\phi - \delta_{jl}\nabla_k\nabla_i\phi \pm \epsilon_{ijkm}\nabla_l\nabla_m\phi \mp \epsilon_{ijlm}\nabla_k\nabla_m\phi, \quad (2.11)$$

It follows from (2.11) that

$$\begin{aligned} \hat{R}_{(\mu\nu)} &= -2\nabla_\mu\nabla_\nu\phi, \\ \hat{R}_{[\mu\nu]} &= 0. \end{aligned} \quad (2.12)$$

It also follows from (2.7) that

$$\begin{aligned}\nabla^2\phi &= 0, \\ H_{\mu\nu}{}^\lambda\partial_\lambda\phi &= 0, \\ H^2 &= 24(\partial\phi)^2.\end{aligned}\tag{2.13}$$

From (2.12) and (2.13) it follows that the $O(\alpha')$ terms in the Weyl anomaly coefficients in (2.10) vanish identically for the ansatz (2.7). A tree-level multi-instanton solution is therefore given by (2.7) with the dilaton given by

$$e^{2\phi} = C + \sum_{i=1}^N \frac{Q_i}{|\vec{x} - \vec{a}_i|^2},\tag{2.14}$$

where Q_i is the charge and \vec{a}_i the location in the four-space (1234) of the i th instanton. We call (1234) the transverse space, as the solitons have the structure of 21+1-dimensional objects embedded in a 26-dimensional spacetime.

We now specialize to the spherically symmetric case of $e^{2\phi} = Q/r^2$ in (2.7) and determine the $O(\alpha')$ corrections to the massless fields in (2.7) so that the Weyl anomaly coefficients vanish to $O(\alpha'^2)$. For this solution we notice

$$\nabla_\mu\nabla_\nu\phi = 0,\tag{2.15}$$

and therefore from (2.11)

$$\hat{R}^i{}_{jkl} = 0,\tag{2.16}$$

and we have what is called a ‘‘parallelizable’’ space[20,21]. To maintain a parallelizable space to $O(\alpha')$ we keep $g_{\mu\nu}$ and $H_{\alpha\beta\gamma}$ in their lowest order form and assume that any corrections to (2.7) appear in the dilaton:

$$\begin{aligned}\phi &= \phi_0 + \alpha'\phi_1 + \dots \\ e^{2\phi_0} &= \frac{Q}{r^2}, \\ g_{\mu\nu} &= e^{2\phi_0}\delta_{\mu\nu}, \\ H_{\mu\nu\lambda} &= \pm\epsilon_{\mu\nu\lambda\sigma}\partial^\sigma\phi_0.\end{aligned}\tag{2.17}$$

It follows from (2.17) that $H^2 = 24(\partial\phi_0)^2 = 24/Q$ and thus $W_\mu = 0$. It follows from (2.16) that $\bar{\beta}_{\mu\nu}^G$ and $\bar{\beta}_{\mu\nu}^B$ vanish identically to two loop order and that

$$\begin{aligned}\bar{\beta}^\Phi &= \frac{D}{6} + \alpha' \left((\partial\phi)^2 - \frac{1}{Q} \right) \\ &\quad + \frac{\alpha'^2}{16} \left(R_{\lambda\mu\nu\rho}^2 - \frac{11}{2}RHH + \frac{5}{24}H^4 + \frac{11}{8}(H_{\mu\nu}^2)^2 + \frac{4}{3}\nabla H \cdot \nabla H \right).\end{aligned}\tag{2.18}$$

We use the relations in equation (34) in [20] for parallelizable spaces and the observation that $(H_{\mu\nu}^2)^2 = 2H^4 = 192/Q^2$ for our solution to get the identities

$$\begin{aligned} R_{\lambda\mu\nu\rho}^2 &= \frac{1}{8}H^4, \\ RHH &= \frac{1}{2}H^4, \\ \nabla H \cdot \nabla H &= 0. \end{aligned} \tag{2.19}$$

(2.18) then simplifies further to

$$\bar{\beta}^\Phi = \frac{D}{6} + \alpha' \left((\partial\phi)^2 - \frac{1}{Q} \right) + 2\frac{\alpha'^2}{Q^2}. \tag{2.20}$$

The lowest order term in $\bar{\beta}^\Phi$ is proportional to the central charge and the $O(\alpha')$ terms vanish identically. With the choice $\vec{\nabla}\phi_1 = -(1/Q)\vec{\nabla}\phi_0$, the $O(\alpha'^2)$ terms also vanish identically. The two-loop solution is then given by

$$\begin{aligned} e^{2\phi} &= \frac{Q}{r^{2(1-\frac{\alpha'}{Q})}}, \\ g_{\mu\nu} &= \frac{Q}{r^2}\delta_{\mu\nu}, \\ H_{\mu\nu\lambda} &= \pm\epsilon_{\mu\nu\lambda\sigma}\partial^\sigma\phi_0, \end{aligned} \tag{2.21}$$

which corresponds to a simple rescaling of the dilaton. A quick check shows that this solution has finite action near the singularity.

We now rewrite $\bar{\beta}^\Phi$ in (2.20) in the following suggestive form:

$$\begin{aligned} 6\bar{\beta}^\Phi &= (1 + 6\alpha'(\partial\phi)^2) + \left(3 - 6\frac{\alpha'}{Q} + 12\left(\frac{\alpha'}{Q}\right)^2 \right) \\ &= 4. \end{aligned} \tag{2.22}$$

The above splitting of the central charge $c = 6\bar{\beta}^\Phi$ suggests the decomposition of the corresponding sigma model into the product of a one-dimensional CFT (a Feigin-Fuchs Coulomb gas) and a three-dimensional WZW model with an $SU(2)$ group manifold [7,20,21]. This can be seen as follows. Setting $u = \ln r$, we can rewrite (2.5) for our solution[7] in the form $I = I_1 + I_3$, where

$$I_1 = \frac{1}{4\pi\alpha'} \int d^2x \left(Q(\partial u)^2 + \alpha' R^{(2)}\phi \right) \tag{2.23}$$

is the action for a Feigin-Fuchs Coulomb gas, which is a one-dimensional CFT with central charge given by $c_1 = 1 + 6\alpha'(\partial\phi)^2$ [25]. The imaginary charge of the Feigin-Fuchs Coulomb

gas describes the dilaton background growing linearly in imaginary time and I_3 is the Wess–Zumino–Witten[22] action on an $SU(2)$ group manifold with central charge

$$c_3 = \frac{3k}{k+2} \simeq 3 - \frac{6}{k} + \frac{12}{k^2} + \dots \quad (2.24)$$

where $k = Q/\alpha'$, called the “level” of the WZW model, is an integer. This can be seen from the quantization condition on the Wess-Zumino term[22]

$$\begin{aligned} I_{WZ} &= \frac{i}{4\pi\alpha'} \int_{\partial S_3} d^2x \epsilon^{ab} \partial_a x^\mu \partial_b x^\nu B_{\mu\nu} \\ &= \frac{i}{12\pi\alpha'} \int_{S_3} d^3x \epsilon^{abc} \partial_a x^\mu \partial_b x^\nu \partial_c x^\lambda H_{\mu\nu\lambda} \\ &= 2\pi i \left(\frac{Q}{\alpha'} \right). \end{aligned} \quad (2.25)$$

Thus Q is not arbitrary, but is quantized in units of α' .

We use this splitting to obtain exact expressions for the fields by fixing the metric and antisymmetric tensor field in their lowest order form and rescaling the dilaton order by order in α' . The resulting expression for the dilaton is

$$e^{2\phi} = \frac{Q}{r \sqrt{\frac{4}{1 + \frac{2\alpha'}{Q}}}}. \quad (2.26)$$

We now turn to the heterotic multi-instanton solution of [11,12]. The tree-level supersymmetric vacuum equations for the heterotic string are given by

$$\delta\psi_M = \left(\nabla_M - \frac{1}{4} H_{MAB} \Gamma^{AB} \right) \epsilon = 0, \quad (2.27)$$

$$\delta\lambda = \left(\Gamma^A \partial_A \phi - \frac{1}{6} H_{AMC} \Gamma^{ABC} \right) \epsilon = 0, \quad (2.28)$$

$$\delta\chi = F_{AB} \Gamma^{AB} \epsilon = 0, \quad (2.29)$$

where ψ_M , λ and χ are the gravitino, dilatino and gaugino fields. The Bianchi identity is given by

$$dH = \alpha' \left(\text{tr} R \wedge R - \frac{1}{30} \text{Tr} F \wedge F \right). \quad (2.30)$$

The $(9+1)$ -dimensional Majorana-Weyl fermions decompose down to chiral spinors according to $SO(9,1) \supset SO(5,1) \otimes SO(4)$ for the $M^{9,1} \rightarrow M^{5,1} \times M^4$ decomposition. Let $\mu, \nu, \lambda, \sigma = 1, 2, 3, 4$ and $a, b = 0, 5, 6, 7, 8, 9$. Then the ansatz

$$\begin{aligned} g_{\mu\nu} &= e^{2\phi} \delta_{\mu\nu}, \\ g_{ab} &= \eta_{ab}, \\ H_{\mu\nu\lambda} &= \pm \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \phi \end{aligned} \quad (2.31)$$

with constant chiral spinors ϵ_{\pm} solves the supersymmetry equations with zero background fermi fields provided the YM gauge field satisfies the instanton (anti)self-duality condition

$$F_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu}{}^{\lambda\sigma} F_{\lambda\sigma}. \quad (2.32)$$

An exact solution is obtained as follows. Define a generalized connection by

$$\Omega_{\pm M}^{AB} = \omega_M^{AB} \pm H_M^{AB} \quad (2.33)$$

embedded in an $SU(2)$ subgroup of the gauge group, and equate it to the gauge connection A_{μ} [26] so that $dH = 0$ and the corresponding curvature $R(\Omega_{\pm})$ cancels against the Yang-Mills field strength F . As in the bosonic case, for $e^{-2\phi} \square e^{2\phi} = 0$ with the above ansatz, the curvature of the generalized connection can be written in the covariant form[8]

$$\begin{aligned} R(\Omega_{\pm})_{\mu\nu}^{mn} = & \delta_{n\nu} \nabla_m \nabla_{\mu} \phi - \delta_{n\mu} \nabla_m \nabla_{\nu} \phi + \delta_{m\mu} \nabla_n \nabla_{\nu} \phi - \delta_{m\nu} \nabla_n \nabla_{\mu} \phi \\ & \pm \epsilon_{\mu mn\alpha} \nabla_{\alpha} \nabla_{\nu} \phi \mp \epsilon_{\nu mn\alpha} \nabla_{\alpha} \nabla_{\mu} \phi, \end{aligned} \quad (2.34)$$

from which it easily follows that

$$R(\Omega_{\pm})_{\mu\nu}^{mn} = \mp \frac{1}{2} \epsilon_{\mu\nu}{}^{\lambda\sigma} R(\Omega_{\pm})_{\lambda\sigma}^{mn}. \quad (2.35)$$

Thus we have a solution with the ansatz (2.31) such that

$$F_{\mu\nu}^{mn} = R(\Omega_{\pm})_{\mu\nu}^{mn}, \quad (2.36)$$

where both F and R are (anti)self-dual. This solution becomes exact since $A_{\mu} = \Omega_{\pm\mu}$ implies that all the higher order corrections vanish[26,27,28,11,12,19]. The self-dual solution for the gauge connection is then given by the 't Hooft ansatz

$$A_{\mu} = i \bar{\Sigma}_{\mu\nu} \partial_{\nu} \ln f. \quad (2.37)$$

For a multi-instanton solution f is again given by

$$f = e^{-2\phi_0} e^{2\phi} = 1 + \sum_{i=1}^N \frac{\rho_i^2}{|\vec{x} - \vec{a}_i|^2}, \quad (2.38)$$

where ρ_i^2 is the instanton scale size and \vec{a}_i the location in four-space of the i th instanton. An interesting feature of the heterotic solution is that it combines a YM instanton structure

in the gauge sector with an axionic instanton structure in the gravity sector. In addition, the heterotic solution has finite action.

Note that the single instanton solution in the heterotic case carries through to higher order without correction to the dilaton. This seems to contradict the bosonic solution by suggesting that the expansion for the Weyl anomaly coefficient $\bar{\beta}^\Phi$ terminates at one loop. This contradiction is resolved by noting that for a supersymmetric ansatz the bosonic contribution to the central charge is given by[29]

$$c_3 = \frac{3k'}{k' + 2}, \quad (2.39)$$

where $k' = k - 2$. This reduces to

$$\begin{aligned} c_3 &= 3 - \frac{6}{k} \\ &= 3 - \frac{6\alpha'}{Q}, \end{aligned} \quad (2.40)$$

which indeed terminates at one loop order. The exactness of the splitting then requires that c_1 not get any corrections from $(\partial\Phi)^2$ so that $c_1 + c_3 = 4$ is exact for the tree-level value of the dilaton[11,12,19].

3. Three-Dimensional Monopole Solutions

In this section we review the solutions with three-dimensional spherical symmetry, and which have monopole-like structure. We begin with a simple modification of the 't Hooft ansatz[16,17] which leads to a multimonopole solution in field theory, not in the BPS limit[30,31] and in itself far less interesting than the BPS solution. We then note that a tree-level bosonic multi-soliton solution with monopole-like structure can be written down[18]. Finally, we combine the gauge solution with the bosonic solution to obtain an exact heterotic multimonopole solution[16,17].

We now return to the 't Hooft ansatz and the four-dimensional Euclidean action

$$S = -\frac{1}{2g^2} \int d^4x \text{Tr} G_{\mu\nu} G^{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4 \quad (3.1)$$

with gauge group $SU(2)$. We obtain a multimonopole solution by modifying the 't Hooft ansatz

$$A_\mu = i\bar{\Sigma}_{\mu\nu} \partial_\nu \ln f \quad (3.2)$$

as follows. We single out a direction in the transverse four-space (say x_4) and assume all fields are independent of this coordinate. Then the solution for f satisfying $f^{-1}\square f = 0$ can be written as

$$f = 1 + \sum_{i=1}^N \frac{m_i}{|\vec{x} - \vec{a}_i|}, \quad (3.3)$$

where m_i is the charge and \vec{a}_i the location in the three-space (123) of the i th monopole. If we make the identification $\Phi \equiv A_4$ (we loosely refer to this field as a Higgs field in this paper, even though there is no apparent symmetry breaking mechanism), then the Lagrangian density for the above ansatz can be rewritten as

$$\begin{aligned} G_{\mu\nu}^a G_{\mu\nu}^a &= G_{ij}^a G_{ij}^a + 2G_{k4}^a G_{k4}^a \\ &= G_{ij}^a G_{ij}^a + 2D_k \Phi^a D_k \Phi^a, \end{aligned} \quad (3.4)$$

which has the same form as the Lagrangian density for YM + massless scalar field in three dimensions.

We now go to 3 + 1 dimensions with the Lagrangian density (signature $(- + ++)$)

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} - \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a, \quad (3.5)$$

and show that the above multimonopole ansatz is a static solution with $A_0^a = 0$ and all time derivatives vanish. The equations of motion in this limit are given by

$$\begin{aligned} D_i G^{jia} &= g \epsilon^{abc} (D^j \Phi^b) \Phi^c, \\ D_i D^i \Phi^a &= 0. \end{aligned} \quad (3.6)$$

It is then straightforward to verify that the above equations are solved by the modified 't Hooft ansatz

$$\begin{aligned} \Phi^a &= \mp \frac{1}{g} \delta^{ai} \partial_i \omega, \\ A_k^a &= \epsilon^{akj} \partial_j \omega, \end{aligned} \quad (3.7)$$

where $\omega \equiv \ln f$. This solution represents a multimonopole configuration with sources at $\vec{a}_i = 1, 2, \dots, N$. A simple observation of far field and near field behaviour shows that this solution does not arise in the Prasad-Sommerfield[31] limit. In particular, the fields are singular near the sources and vanish as $r \rightarrow \infty$. This solution can be thought of as a multi-line source instanton solution, each monopole being interpreted as an ‘‘instanton string’’ [32].

The topological charge of each source is easily computed ($\hat{\Phi}^a \equiv \Phi^a/|\Phi|$) to be

$$Q = \int d^3x k_0 = \frac{1}{8\pi} \int d^3x \epsilon_{ijk} \epsilon^{abc} \partial_i \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c = 1. \quad (3.8)$$

The magnetic charge of each source is then given by $m_i = Q/g = 1/g$. It is also straightforward to show that the Bogomoln'yi[30] bound

$$G_{ij}^a = \epsilon_{ijk} D_k \Phi^a \quad (3.9)$$

is saturated by this solution. Finally, it is easy to show that the magnetic field $B_i = \frac{1}{2} \epsilon_{ijk} F^{jk}$ (where $F_{\mu\nu} \equiv \hat{\Phi}^a G_{\mu\nu}^a - (1/g) \epsilon^{abc} \hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c$ is the gauge-invariant electromagnetic field tensor defined by 't Hooft[33]) has the the far field limit behaviour of a multimonopole configuration:

$$B(\vec{x}) \rightarrow \sum_{i=1}^N \frac{m_i (\vec{x} - \vec{a}_i)}{|\vec{x} - \vec{a}_i|^3}, \quad \text{as } r \rightarrow \infty. \quad (3.10)$$

As usual, the existence of this static multimonopole solution owes to the cancellation of the gauge and Higgs forces of exchange—the “zero-force” condition.

We have presented all the monopole properties of this solution. Unfortunately, this solution as it stands has divergent action near each source, and this singularity cannot be simply removed by a unitary gauge transformation. This can be seen for a single source by noting that as $r \rightarrow 0$, $A_k \rightarrow \frac{1}{2} (U^{-1} \partial_k U)$, where U is a unitary 2×2 matrix. The expression in parentheses represents a pure gauge, and there is no way to get around the $1/2$ factor in attempting to “gauge away” the singularity[34]. The field theory solution is therefore not very interesting physically. As we shall later in this section, however, we can obtain an analogous finite action solution in heterotic string theory. As in the previous section, we first consider a monopole-like solution in bosonic string theory.

If we again single out a direction (say x_4) in the transverse space (1234) of the bosonic string and assume all fields are independent of x_4 , then the tree-level bosonic multi-soliton solution to the string equations of motion with the ansatz (2.7) is given by[18]

$$\begin{aligned} e^{2\phi} &= C + \sum_{i=1}^N \frac{m_i}{|\vec{x} - \vec{a}_i|}, \\ g_{\mu\nu} &= e^{2\phi} \delta_{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4, \\ g_{ab} &= \eta_{ab}, \quad a, b = 0, 5, 6 \dots 25, \\ H_{\alpha\beta\gamma} &= \pm \epsilon_{\alpha\beta\gamma}{}^\mu \partial_\mu \phi, \quad \alpha, \beta, \gamma, \mu = 1, 2, 3, 4, \end{aligned} \quad (3.11)$$

where $\vec{x} = (x_1, x_2, x_3)$ is a three-dimensional coordinate vector in the (123) subspace of the transverse space. m_i represents the charge and \vec{a}_i the location in the three-space of the i th source.

By singling out a direction x_4 and projecting out all the field dependence on it, we destroy the $SO(4)$ invariance in the transverse space possessed by the instanton solution[8]. However, (3.11) is an equally valid solution to the string equations as the multi-instanton solution, since in both cases the dilaton field satisfies the Poisson equation $e^{-2\phi}\square e^{2\phi} = 0$. The projection is necessary to obtain the three-dimensional symmetry of a magnetic monopole.

Although the above bosonic multi-soliton solution (3.11) lacks the gauge and Higgs fields normally attributed to a magnetic monopole in field theory, one can think of the dual field in the transverse four-space $H_\mu^* \equiv \frac{1}{6}\epsilon_{\alpha\beta\gamma\mu}H^{\alpha\beta\gamma}$ as the magnetic field strength of a multimonopole configuration in the space (123) (note that $H_4^* = 0$).

Unlike the four-dimensional solutions, the three-dimensional solutions do not easily lend themselves to a CFT description, and it is therefore difficult to go beyond $O(\alpha')$ in obtaining stringy corrections to the tree-level fields. In [8], the $O(\alpha')$ correction was worked out for the special case of a single source with $C = 0$. As in the four-dimensional case, the metric and antisymmetric tensor are unchanged to $O(\alpha')$, but the dilaton is corrected:

$$e^{2\phi} = \frac{m}{r} \left(1 - \frac{\alpha'}{8mr} \right). \quad (3.12)$$

Unlike the four-dimensional solution, however, the dilaton correction is not a simple rescaling of the power of r to order α' . This fact is intimately connected with the difficulty in formulating a CFT description of the three-dimensional solution.

We now combine the above solutions to construct an exact multimonopole solution of heterotic string theory. The derivation of this solution closely parallels that of the multi-instanton solution reviewed in section 2, but in this case, the solution possesses three-dimensional (rather than four-dimensional) spherical symmetry near each source. Again the reduction is effected by singling out a direction in the transverse space. An exact solution is now given by

$$\begin{aligned} g_{\mu\nu} &= e^{2\phi}\delta_{\mu\nu}, & g_{ab} &= \eta_{ab}, \\ H_{\mu\nu\lambda} &= \pm\epsilon_{\mu\nu\lambda\sigma}\partial^\sigma\phi, \\ e^{2\phi} &= e^{2\phi_0}f, \\ A_\mu &= i\bar{\Sigma}_{\mu\nu}\partial_\nu\ln f, \end{aligned} \quad (3.13)$$

where in this case

$$f = 1 + \sum_{i=1}^N \frac{m_i}{|\vec{x} - \vec{a}_i|}, \quad (3.14)$$

where m_i is the charge and \vec{a}_i the location in the three-space (123) of the i th monopole. If we again identify the Higgs field as $\Phi \equiv A_4$, then the gauge and Higgs fields may be simply written in terms of the dilaton as

$$\begin{aligned} \Phi^a &= -\frac{2}{g} \delta^{ia} \partial_i \phi, \\ A_k^a &= -\frac{2}{g} \epsilon^{akj} \partial_j \phi \end{aligned} \quad (3.15)$$

for the self-dual solution. For the anti-self-dual solution, the Higgs field simply changes sign. Here g is the YM coupling constant. Note that ϕ_0 drops out in (3.15).

The above solution (with the gravitational fields obtained directly from (3.13) and (3.14)) represents an exact multimonopole solution of heterotic string theory and has the same structure in the four-dimensional transverse space as the above multimonopole solution of the YM + scalar field action. If we identify the (123) subspace of the transverse space as the space part of the four-dimensional spacetime (with some toroidal compactification, similar to that used in [35]) and take the timelike direction as the usual X^0 , then the monopole properties of the field theory solution carry over directly into the string solution.

The string action contains a term $-\alpha' F^2$ which also diverges as in the field theory solution. However, this divergence is precisely cancelled by the term $\alpha' R^2(\Omega_{\pm})$ in the $O(\alpha')$ action. This result follows from the exactness condition $A_{\mu} = \Omega_{\pm\mu}$ which leads to $dH = 0$ and the vanishing of all higher order corrections in α' . Another way of seeing this is to consider the higher order corrections to the bosonic action shown in [27,28]. All such terms contain the tensor T_{MNPQ} , a generalized curvature incorporating both $R(\Omega_{\pm})$ and F . The ansatz is constructed precisely so that this tensor vanishes identically[8,19]. The action thus reduces to its finite lowest order form and can be calculated directly for a multi-source solution from the expressions for the massless fields in the gravity sector.

The divergences in the gravitational sector in heterotic string theory thus serve to cancel the divergences stemming from the field theory solution. This solution thus provides an interesting example of how this type of cancellation can occur in string theory, and supports the promise of string theory as a finite theory of quantum gravity. Another point of interest is that the string solution represents a supersymmetric multimonopole solution

coupled to gravity, whose zero-force condition in the gravity sector (cancellation of the attractive gravitational force and repulsive antisymmetric field force) arises as a direct result of the zero-force condition in the gauge sector (cancellation of gauge and Higgs forces of exchange) once the gauge connection and generalized connection are identified.

4. Conclusion

We classified some of the recently obtained higher-membrane solitonic solutions of string theory according to the symmetry the solitons possess in the space transverse to the membrane. We considered in this paper two such classes: those with four-dimensional spherical symmetry, and which possess instanton structure, and those with three-dimensional symmetry, which represent magnetic monopole-like solutions in string theory.

We outlined in section 2 the 't Hooft ansatz for the Yang-Mills instanton, and then turned to the bosonic tree-level axionic instanton solution of [7], and its exact extension for the case of a single instanton wormhole solution[8]. A combination of the gauge instanton and axionic instanton solutions led to an exact multi-instanton solution in heterotic string theory[11,12].

In section 3 we considered some of the monopole-like solutions. In this case, a combination of the modified 't Hooft ansatz[16,17] and a bosonic three-dimensional solution[18] led to an exact heterotic multimonopole solution[16,17]. Unlike the instanton solutions, the monopole solutions do not seem to be easily describable in terms of conformal field theories, an unfortunate state of affairs from the point of view of string theory. An interesting aspect of this solution, however, is that the YM divergences of the modified 't Hooft ansatz solution are precisely cancelled in the string theory solution by similar divergences in the gravity sector, resulting in a finite action solution. This finding is significant in that it represents an example of how string theory incorporates gravity in such a way as to cancel infinities inherent in gauge theories, thus supporting its promise as a finite theory of quantum gravity.

Another class of solutions, which we did not consider here, are the eight dimensional instanton[36–38] solutions of string theory[39–41]. In this case, however, the exact extension is most natural in the context of a dual theory of fundamental fivebranes, which has not yet been constructed.

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