Error Threshold for Color Codes and Random 3-Body Ising Models

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We study the error threshold of color codes, a class of topological quantum codes that allow a direct implementation of quantum Clifford gates suitable for entanglement distillation, teleportation and fault-tolerant quantum computation. We map the error-correction process onto a statistical mechanical random 3-body Ising model and study its phase diagram via Monte Carlo simulations. The obtained error threshold of $p_c = 0.109(2)$ is very close to that of Kitaev's toric code, showing that enhanced computational capabilities do not necessarily imply lower resistance to noise.

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Protecting quantum states from external noise and errors is central for the future of quantum information technology. Because interaction with the environment is unavoidable, active quantum error-correction techniques based on quantum codes have been devised to restore the damaged quantum states from errors caused by decoherence [1, 2]. These approaches are, in general, cumbersome and require many additional quantum bits, thus making the system more error prone. An imaginative and fruitful approach to quantum protection is to exploit topological properties of a system, e.g., by using the nontrivial topology of a surface to encode quantum states at the logical level [3]. Topology is thus considered as a resource, much like entanglement is a resource for quantum information tasks. Topological quantum computation is the combination of these two resources with the aim of winning the battle against decoherence. These topological quantum error-correcting codes are instances of stabilizer quantum codes [4], in which errors are diagnosed by measuring certain check operators or stabilizers. In topological codes these check operators are local, which, in practice, is an important advantage. Moreover, error correction has a deep connection to random spin models in statistical mechanics and lattice gauge theories [5].

One of the original motivations for introducing surface codes was to achieve error protection at the physical level through energy barriers that would remove the need for external recovery actions. Only the application of strong magnetic fields (compared to the topological coupling) destabilizes the topological phase [6]. However, several studies [5, 7, 8, 9, 10] and a rigorous proof [8] have shown that the toric code (TC) is not stable against thermal excitations, except in four dimensions [5, 9].

Therefore, the study of active error correction in topological codes [5] is fully justified. Ultimately, the goal is not only to achieve good quantum memories, but also to perform quantum computations with them. In this regard, the TC [3] is somehow limited since it only allows for a convenient (transversal) implementation of a limited set of quantum gates: Pauli gates of X and Z type, and the CNOT gate. To overcome this limitation, topological color codes (TCC) have been introduced [11, 12]. Using TCC, it is possible to implement the *whole* Clifford group of quantum gates and thus realize quantum distillation, teleportation, etc. Notice that, although we use the mapping of Ref. [5], there is a difference regarding the issue of types of homology involved: our model has a colored homology, while the Kitaev model has a simple homology. As a result, unlike the standard Ising model, the resulting statistical mechanical model has 3-body interactions with a value of p_c a priori unknown, thus motivating the present study.

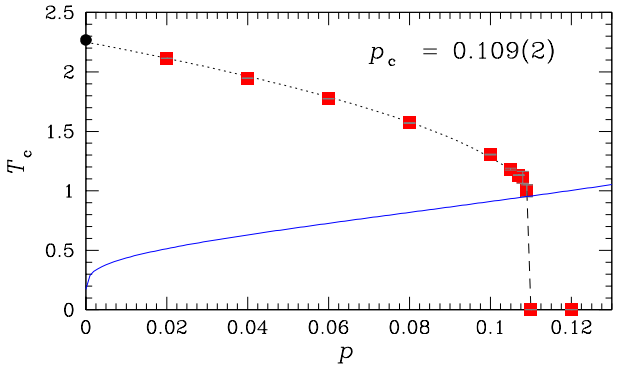

FIG. 1: (Color online) p–T_c phase diagram for the random 3-body Ising model. For $p > p_c \approx 0.109$ the ferromagnetic order is lost. The dotted line is a guide to the eye, the black circle represents the analytically known transition temperature of the 2D Ising model. The blue (solid) line represents the N-line. In the regime marked by a dashed line the exact determination of $T_c(p)$ is difficult.

The question arises as to whether the wider computational capabilities of TCCs imply a lower resistance to noise. We address this problem and show that the (error) threshold value is $p_c = 0.109(2)$, which is comparable to values for the TC [13, 14, 15]. To compute p_c, we derive a statistical mechanical model describing the error-correction process; a random 3-body Ising model, with (classical) spins located at the vertices of a triangular lattice. In addition to thermal fluctuations, the mapping requires the introduction of quenched randomness to the sign of the interactions that correspond to faulty bits.

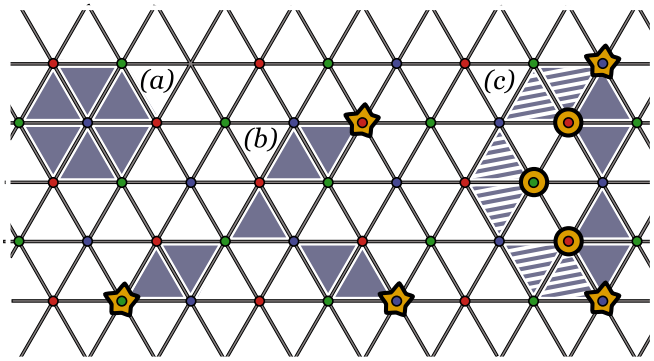

FIG. 2: (Color online) Lattice for the TCCs with 3-colored vertices. Physical qubits of the error-correcting code correspond to triangles (stars mark the 'boundaries' of the sets of triangles displayed). (a) Boundary of a vertex v. The stabilizer operators X_v , Z_v have support on the corresponding qubits. (b) Error pattern in the form of a string net. The three vertices that form its boundary are all the information we have to correct the error. (c) Two error patterns with the same boundary. Because together they form the boundary of the three vertices marked with a circle, they are equivalent.

One can then study the p–T_c phase diagram of the model, see Fig. 1, where p is the probability for wrong-sign couplings to appear. For low T, p the model orders, which corresponds to feasible error correction. The critical p_c for error correction is recovered from the critical p along the Nishimori (N) line [16] in the p–T plane.

The disordered 3-body Ising model on a triangular lattice has not been studied before. However, in the absence of randomness it is known to have a different universality class than the standard Ising model, but with the same critical temperature [17]. Furthermore, the critical exponents can be computed exactly ($\nu = \alpha = 2/3$), which allows us to test the numerical results in the $p = 0$ limit.

Topological color codes.— To construct a TCC $\mathcal{C}$ we start from any two-dimensional (2D) lattice in which all plaquettes are triangles and vertices are 3-colorable, such that no link connects vertices of the same color. The lattice is embedded in a compact surface of arbitrary topology. Since information is encoded in topological degrees of freedom, the code is nontrivial only when the topology of the surface is nontrivial, e.g., a torus of genus $g \geq 1$. So far, color codes have been introduced in the dual lattice (2-colex [11]). Here we prefer to work in the triangular lattice to have a more direct mapping, see Fig. 2.

We consider a physical system with a qubit at each lattice triangle, and introduce the following vertex operators that generate the stabilizer group of $\mathcal{C}$. For each vertex v we have two types of operators which correspond to Pauli operators of X or Z type, i.e., $X_v := \bigotimes_{\triangle : v \in \triangle} X_\triangle$ and $Z_v := \bigotimes_{\triangle : v \in \triangle} Z_\triangle$. Thus, a vertex operator acts on all nearby triangles, see Fig. 2. Vertex operators pairwise commute and square to identity. The code $\mathcal{C}$ is defined as the subspace with $X_v = Z_v = 1\ \forall v$. To perform error correction one measures vertex operators. The resulting collection of ± 1 eigenvalues is the error syndrome.

Error correction.— Color codes have a structure with stabilizer generators which are either products of X or Z Pauli operators, but not both. This allows us to treat bit-flip and phase errors separately, making the procedure classical: X-type (Z-type) errors produce violations of Z-type (X-type) vertex operators. Without loss of generality, let us consider the bit-flip case, that is, errors of the form $X_E := \bigotimes_{\triangle \in E} X_\triangle$, where E is the subset of triangles that suffered a bit-flip. Let ∂E be the collection of vertices that are part of an odd number of triangles in E, i.e., the boundary of a set of triangles E is chosen so that the error X_E gives rise to a syndrome with $Z_v = -1$ at those vertices $v \in \partial E$, see Fig. 2. In trying to correct the error, we apply to the system bit-flips $X_{E'}$ with the same boundary, $\partial E' = \partial E$. This is only successful as long as $X_{E'} X_E =: X_{E+E'}$ is an element of the stabilizer group. Geometrically, $D = E + E'$ is a cycle: its boundary ∂D is empty. Given a vertex v, let ∂v be the subset of triangles meeting at v. We say that D is a boundary if $D = \sum_V \partial v$ for some subset of triangles V. In that case, X_D is an element of the stabilizer group. Thus, error correction is successful whenever D is a boundary, i.e., if D has trivial homology. In that case the real error E and the guessed error E' belong to the same homology class.

Mapping to a random 3-body Ising model.— We consider a standard error model based on stochastic errors in which phase errors Z and qubit bit-flip errors X are uncorrelated and occur with probability p at each qubit. We focus on the correction of bit-flip errors.

Let $P(E)$ be the probability for a given set of bit-flip errors E. Up to a p-dependent factor, $P(E) \propto [p/(1-p)]^{|E|}$. We may also consider the total probability for the corresponding homology class $\bar{E}$ of errors, $P(\bar{E}) := \sum_D P(E + D)$, where D runs over all boundaries. If we measure a syndrome ∂E, then the probability that it was caused by an error in the homology class $\bar{E}$ is

$$P(\bar{E}|\partial E) = \frac{P(\bar{E})}{\sum_i P(\bar{E} + \bar{D}_i)}, \tag{1}$$

where the D_i are representatives of the homology classes of cycles [5]. Then, error correction is achievable if in the limit of infinite system size we have $\sum_E P(E)\, P(\bar{E}|\partial E) \to 1$. That is, $p < p_c$ if for those syndromes which have a nonnegligible probability to appear the error can be guessed with total confidence.

Following Ref. [5], we set $\exp(-2K) := p/(1-p)$ (with $K = J/T$, T the temperature) for the N-line so that $P(E) \propto \exp(K \sum_\triangle \tau_\triangle)$, where the sum is over all the triangular plaquettes (qubits) and $\tau_\triangle = \pm 1 < 0$ when $\triangle \in E$. By inserting classical spin variables $\sigma_i = \pm 1$ at the vertices and labeling the triangles $\triangle$ with triplets of vertices $\langle ijk \rangle$ we write $P(\bar{E})$ as a partition function

$$P(\bar{E}) \propto Z[K, \boldsymbol{\tau}] := \sum_{\boldsymbol{\sigma}} e^{K \sum_{\langle ijk \rangle} \tau_{ijk} \sigma_i \sigma_j \sigma_k}. \tag{2}$$

TABLE I: Simulation parameters: L is the system size, $N_{\rm sa}$ is the number of disorder samples, $t_{\rm eq} = 2^b$ is the number of equilibration sweeps, $T_{\rm min}$ [$T_{\rm max}$] is the lowest [highest] temperature, and $N_{\rm T}$ the number of temperatures used.

p	L	$N_{\rm sa}$	b	$T_{\rm min}$	$T_{\rm max}$	$N_{\rm T}$
0.00	12, 18	20	18	2.200	2.350	31
0.00	24, 30	20	19	2.200	2.350	31
0.00	36	20	20	2.200	2.350	31
0.02	12, 18	5 000	18	1.900	2.400	51
0.02	24, 30	5 000	19	1.900	2.400	51
0.02	36	5 000	20	1.900	2.400	51
0.04	12, 18	5 000	18	1.700	2.200	51
0.04	24, 30	5 000	19	1.700	2.200	51
0.04	36	5 000	20	1.700	2.200	51
0.06	12, 18	5 000	18	1.600	2.100	51
0.06	24, 30	5 000	19	1.600	2.100	51
0.06	36	5 000	20	1.600	2.100	51
0.08	12, 18	5 000	18	1.400	2.000	61
0.08	24, 30	5 000	19	1.400	2.000	61
0.08	36	5 000	20	1.400	2.000	61
0.10 — 0.12	12, 18	5 000	18	0.750	2.600	38
0.10 — 0.12	24, 30	5 000	19	0.750	2.600	38
0.10 — 0.12	36	5 000	20	0.750	2.600	38

Equation (2) is a 3-body classical Ising model with the couplings' sign given by $\boldsymbol{\tau}$. When all $\tau_\triangle = 1$ the model is ferromagnetically ordered at low T. Negative $\tau_\triangle$ introduce frustration in the form of *nets* of domain walls. These can branch, a new feature not present in the random bond Ising model associated with the TC.

The relative importance of the different error homology classes $P(\bar{E} + \bar{D}_i) = Z[K, \boldsymbol{\tau}_i]$ in (1) is given by the free energy cost of introducing a domain wall D_i, because

$$\Delta_i(\boldsymbol{\tau}) = \beta F(K, \boldsymbol{\tau}_i) - \beta F(K, \boldsymbol{\tau}) = \ln\left(\frac{Z[K, \boldsymbol{\tau}]}{Z[K, \boldsymbol{\tau}_i]}\right). \quad (3)$$

The cost Δ_i must be averaged over all coupling configurations, with p the probability for any triangle to have $\tau_\triangle = -1$. Thus we are led to the study of a random 3-body Ising model. For low p and T (high $K = J/T$) the system is ordered and domain-wall fluctuations are suppressed: Δ_i diverges with the system size for nontrivial domain walls. The critical error threshold p_c for error correction is recovered from the p–T phase diagram as the critical p along the N-line $e^{-2J/T} = p/(1-p)$ [18].

Numerical details.— To determine the existence of a ferromagnetic phase we compute the finite-size correlation length [19]. We start by determining the wave-vector-dependent susceptibility given by $\chi(k) = (1/L^2)\sum_{ij}\langle S_i S_j\rangle_T \exp[i\mathbf{k}\cdot(\mathbf{R}_i - \mathbf{R}_j)]$. Here $\langle\cdots\rangle_T$ denotes a thermal average and $\mathbf{R}_i$ the spatial location of the spins. The correlation length is given by

$$\xi_{\rm m} = (1/2)\sin^{-1}(q/2)\sqrt{[\chi(0)]_{\rm av}/[\chi(q)]_{\rm av} - 1}, \quad (4)$$

where $q = (2\pi/L, 0)$ is the smallest nonzero wave-vector and $[\cdots]_{\rm av}$ represents an average over $N_{\rm sa}$ disorder (error) samples. $\xi_{\rm m}/L \sim \widetilde{X}(L^{1/\nu}[T - T_{\rm c}])$, i.e., if there is a transition at $T = T_c$, data for $\xi_{\rm m}/L$ for different system sizes L cross at T_c [see, for example Fig. 3(a)]. The critical exponent ν for the correlation length can be determined by a full scaling of the data, as shown in Fig. 3(b). We also probe the existence of a spin-glass phase by computing the spin-glass finite-size correlation length.

The disorder in Eq. (2) increases the numerical complexity of the problem drastically with a behavior reminiscent of spin glasses [20]. To speed up the simulations, we use the exchange Monte Carlo method [21]. Equilibration is tested by a logarithmic binning of the data. Once the last three bins agree within errors, we define the system to be equilibrated. Simulation parameters are shown in Table I.

Error threshold.— Figure 3 shows the temperature-dependent finite-size correlation length for different values of p. (a) Data for $p = 0$, the ferromagnetic case. The dashed line represents the transition temperature of the 2D Ising model $T_c \simeq 2.2692$ [22]. The agreement with the numerical data is excellent, suggesting that corrections to scaling are negligible. (b) Finite-size scaling analysis of the data in (a) using the exact exponent $\nu = 2/3$. (c) — (h) Finite-size correlation length for different p values. For $p = 0.108$ marginal behavior appears and the determination of the transition is difficult. Because $p = 0.107$ shows a transition, and $p = 0.109$ shows marginal behavior, whereas $p = 0.110$ shows no sign of a transition, we conservatively estimate $p_c = 0.109(2)$ [23]. This is close to estimates for the TC where $p_c^{\rm TC}$ has been continuously improved from $0.1094(2)$ [13] to $0.1093(2)$ [14] and 0.109187 [15]. The p–T_c phase diagram is shown in Fig. 1; the solid (blue) line being the N-line. We have also verified that there is no spin-glass order in the model (not shown). Finally, we ensure that our results do not violate the quantum Gilbert-Varshamov bound [13, 14, 15, 24] where the encoding rate $R(p)$ must satisfy $R(p) \leq 1 - 2H(p)$, $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$ the Shannon entropy, [25, 26, 27]. For our estimate the bound is satisfied, since it lies under the zero-rate probability $p \simeq 0.110027$.

Conclusions.— In summary, we have computed the error threshold for TCCs on a triangular lattice by mapping the problem onto a 3-body random Ising model on a triangular lattice. Using Monte Carlo simulations we find for the error threshold $p_c = 0.109(2)$ [28]. Therefore, TCCs are as robust as the Kitaev toric code with the added benefit of being able to represent the whole Clifford group of quantum gates. The studied 3-body random Ising model highlights the relationship between spin-glass physics and information theory [29], e.g., fully connected systems, and presents a new class of system exhibiting glassy behavior via 3-body interactions, without spin-reversal symmetry. Future work will focus on the impact of faulty measurements and the corresponding mapping to a $(2+1)$-dimensional random gauge model.

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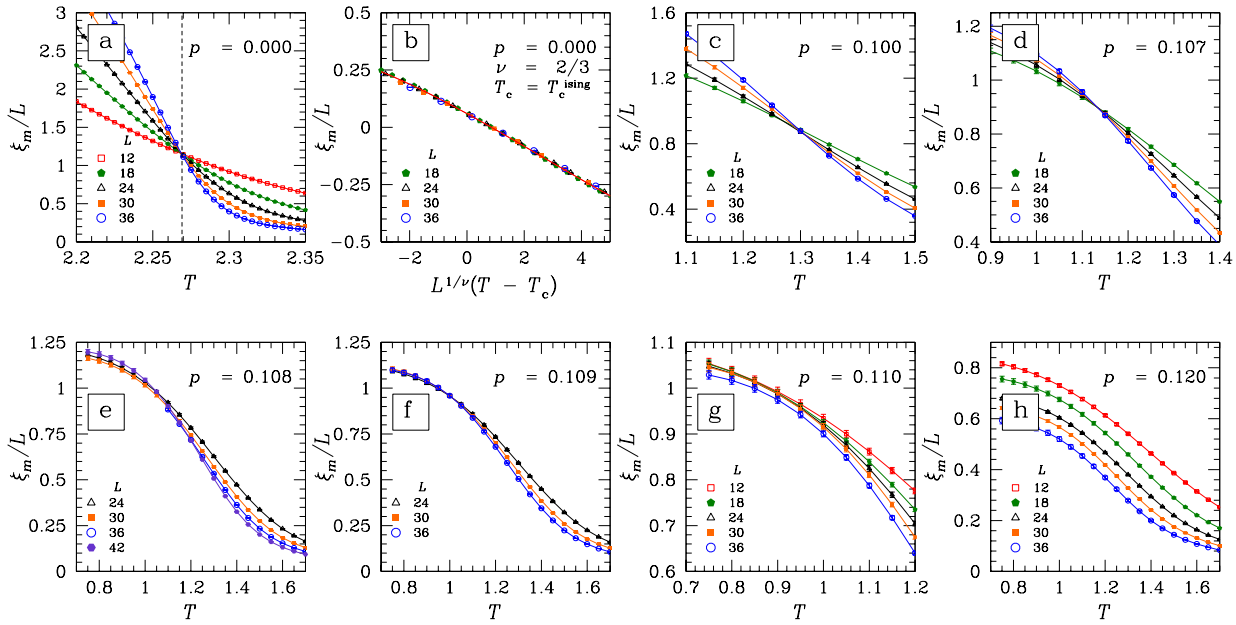

FIG. 3: (Color online) Finite-size correlation length $\xi_{\rm m}/L$ as a function of temperature T for different values of p. (a) $p = 0$. The data cross at the critical temperature of the 2D Ising model (dashed line). (b) Finite-size scaling analysis of the data for $p = 0$ using $\nu = 2/3$. The scaling is very good showing that corrections to scaling are negligible. (c)—(f) For $p \lesssim p_c = 0.109$ there is signature of a transition (data for different L cross) whereas for $p > p_c$ the transition vanishes [panels (g)—(h)].

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