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Multi-Spin Giants

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Abstract

We examine spherical p-branes in $AdS_m \times S^n$, that wrap an S^p in either AdS_m $(p = m-2)$ or S^n $(p = n-2)$. We first construct a two-spin giant solution expanding in S^n and has spins both in AdS_m and S^n. For $(m,n) = \{(5,5),(4,7),(7,4)\}$, it is 1/2 supersymmetric, and it reduces to the single-spin giant graviton when the AdS spin vanishes. We study some of its basic properties such as instantons, non-commutativity, zero-modes, and the perturbative spectrum. All vibration modes have real and positive frequencies determined uniquely by the spacetime curvature, and evenly spaced. We next consider the $(0+1)$-dimensional sigma-models obtained by keeping generally time-dependent transverse coordinates, describing warped product of a breathing-mode and a point-particle on S^n or $AdS_m \times S^1$. The BPS bounds show that the only spherical supersymmetric solutions are the single and the two-spin giants. Moreover, we integrate the sigma-model and separate the canonical variables. We quantize exactly the point-particle part of the motion, which in local coordinates gives Pöschl-Teller type potentials, and calculate its contribution to the anomalous dimension.

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1. INTRODUCTION

Giant gravitons were first proposed in [1] in order to explain the stringy exclusion principle [2] where the bound on the R-charge of CFT operators was related to the bound on the angular momentum in the supergravity picture. They are probe brane solutions in an $AdS_m \times S^n$ background with fluxes, obtained by wrapping an $(n-2)$-brane on an S^{n-2} sphere rotating inside S^n. Later it was shown that [3, 4] an $(m-2)$-brane wrapped on S^{m-2} at constant radius in AdS_m and rotating inside S^n carries the same quantum numbers. Together with the Kaluza-Klein point-like excitation, they constitute three different states representing the same graviton. However, the states are expected to be mixed with each other in quantum theory due to the existence of instantons that would allow semi-classical tunneling [3–5] which may resolve this puzzle.

The giant graviton is an example of how semi-classically stable string/brane solutions may be helpful in understanding different aspects of the AdS/CFT correspondence. Basically, this is due to the fact that large charges suppress quantum fluctuations and thus connect regimes where both the bulk theory and the CFT have meaningful perturbative expansions. From the bulk point of view, the main idea is to zoom in on a sub-sector of states carrying large charges scaling like the tension of a p-brane, and consider various semi-classical expansion schemes of the probe brane quantum field theory based on the identification of some small parameters. Typically, one would expect small parameters to measure the deviation from being BPS, though interestingly enough there also exist meaningful expansions in regimes far from being BPS.

This strategy has been successfully utilized in the BMN limit [6] where the relevant states are near BPS and represented as small closed strings in $AdS_5 \times S^5$ with center of mass rotating around a large circle of S^5 with large angular momentum J. One then considers $J \gg 1$ with λ/J^2 held fixed, where λ is the 't Hooft coupling. The limit $J \to \infty$, removes higher order corrections to the sigma-model leaving the pp-wave geometry, while the SYM side narrows down to the tower of "doped" operators built on top of the 1/2 BPS single-trace ground state.

Another interesting sector of states that have been studied along similar lines, have large spin S in AdS [7]. These arise as long rotating strings, corresponding to towers of single-trace operators doped by derivatives. The rotation induces a strongly coupled world-sheet sigma model. If S is the only semi-classically large parameter, the normal-coordinate expansion gives $1/\sqrt{\lambda}$-corrections to the AdS energy, that are difficult to match directly with the weakly coupled CFT, though other qualitative features do match. However, it was discovered that if one considers states which carry an additional large S^5 spin J, then the classical AdS energy has a regular expansion in λ/J^2. This prompted the proposal that the AdS/CFT duality can be tested in a non-BPS sector by comparing the λ/J^2 expansion of the AdS energy obtained from the classical string sigma-model, with the corresponding quantum anomalous dimensions in perturbative SYM theory. This has indeed been supported by recent results in a series of papers (see for example [8]-[18])

Quite generally, brane physics exhibits UV/IR mixing in the sense that energetic branes tend to grow large transverse directions probing more and more of the background curvature. In the context of AdS/CFT correspondence, this means that already the leading order of the probe sigma-model expansion share some qualitative features with the corresponding subset of CFT operators, most notably the leading linear relation between AdS energy E and other charges. The existence of semi-classically stable large strings, or other p-branes, therefore points to a sub-sector of (non-BPS) operators with parametrically large bare dimensions and suppressed anomalous dimensions.

Returning back to giants, they are stabilized by balancing the tension against electric or magnetic fluxes (the cosmological constants in $AdS_m \times S^n$), leaving a finite net tension independent of the AdS scale. Moreover, the string tension runs in AdS. Hence, in the IR limit of the AdS_m the usual flat space hierarchy is reversed, such that the excitations of the $(m-2)$-brane field theory become much lighter than the massive stringy, or M-theoretic, excitations of the brane.

In the case of the Type IIB theory on $AdS_5 \times S^5$ the following picture emerges (see figure 1): as the energy E of string states increases, the flat space Regge trajectories, where E scales like

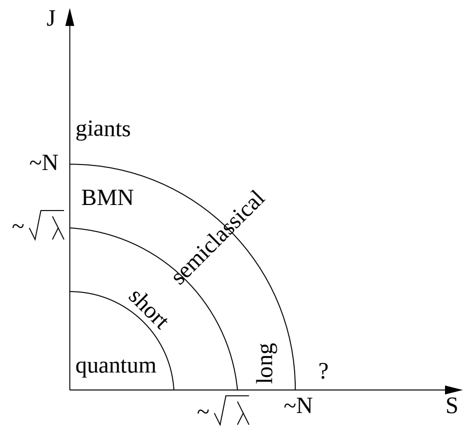

FIG. 1: Different descriptions of the states in $AdS_5 \times S^5$.

square-roots of charge, start bending into linear trajectories as $E \sim g_s N$. Roughly speaking, the energy and some charge of the string concentrate along nearly light-like portions of the world-sheet, while the transverse directions can extend resulting for example in the strings on the pp-wave or the long strings discussed above. At higher energies, which scale faster with N than the string tension, the semi-classical string description becomes strongly coupled. Awaiting some exact world-sheet formulation, the natural semi-classical probe is instead the $D3$-brane whose tension scales like N. In fact, the tension is of order 1 in units of the ten-dimensional Planck length, which means that the giant $D3$ is the last description prior to complete breakdown of the geometrical picture in the IIB theory.

The giant picture arises also in M theory on $AdS_{4/7} \times S^{7/4}$, where $M2$ and $M5$ branes can be dynamically stabilized against collapse. The semi-classical limits are essentially the same as for the $D3$-brane, since the tension is given by the eleven-dimensional Planck length (though here there is no clear analog on the CFT side of the stringy exclusion principle). Hence giants appear to capture universal features of the holography, valid both in string and M-theory and relying only on the notion of expansion in $1/N$, the bulk Planck's constant.

The above discussion suggests that the appropriate semi-classical treatment of p-brane giants is to expand in $\eta = (E - J)/J$ in the regime $E \sim J \sim N \gg 1$, $E - J \ll N$. The quantity $E - J$ is the total energy of the open plus closed string excitations above the giant ground state, therefore small η is the same as considering a few massless quanta on the giant, described by the p-brane field theory. One may also consider finite values of η, as long as one stays safely away from the Planck regime where $\eta \gg 1$. As in the case of the string sigma-model, the problem of analyzing the normal-coordinate expansion simplifies further in the double-scaling limit $N \to \infty$, with η fixed, where the p-brane field theory reduces to the ground state described by the classical solution plus the quadratic fluctuations on top of it. The spectrum of normal frequencies for these vibrations for 1/2 supersymmetric single-spin giant gravitons was calculated in [19], and found to be independent of the size of the brane, and thus the angular momentum. This implies that in the large N limit of the dual theory, the corresponding R-charged chiral operator, which is realized as a sub-determinant [20–24], has associated with it a sector of mainly non-BPS operators with level spacings independent of the ground state R-charge. In [24] this sector was constructed as impurities inserted into the ground state sub-determinant mixed with separate single-traces, shown to produce the structure of a Fock space of mixed open plus closed string excitations.

In this work, motivated by the fact that semi-classical string solutions with large AdS spin and its multi-spin generalizations successfully mimic the BMN strategy, we aim to show that similar ideas can be extended to giant p-branes. The plan of our paper is as follows.

In section 2, we construct a 1/2 supersymmetric[1], spherically symmetric giant $(n-2)$-brane solution in $AdS_m \times S^n$ that spins both in AdS_m and S^n. These two rotations are rigid and the field equations fix the angular velocities in terms of the curvature scales, while the spins are fixed by the radii of the circles of rotation. By adjusting the former radius, we can take the AdS spin to be small or large. We show the saturation of a BPS bound, where the energy is equal to the sum of the two angular momenta. There is also a point particle limit with non-vanishing conserved quantities, connected to the giant by an instanton (with finite action). In [25], it was suggested that imposing supersymmetry causes non-commutativity in the phase space. We indeed observe this to happen also in the two-spin case, with additional Dirac brackets between the radial AdS coordinate and the two cyclic coordinates in AdS_m and S^n used for the rotations.

In Section 3, we examine various aspects of the vibration spectrum of the single and the two-spin BPS giants. The bosonic fluctuations of the two-spin BPS giant (we are only considering the scalar fluctuations and leave vector and tensor fluctuations on $D3$ and $M5$ branes for future work) has two interesting features. Firstly, the frequencies depend only on the curvature scales of AdS_m and S^n, despite the fact that there are three more length scales, namely the radius of the rotation in AdS_m, the tension and the size of the brane. As a consequence, the vibration spectrum is evenly spaced, which is in agreement with the large-N Fock space picture of [24]. Secondly, for generic m and n, the vibration spectrum of the corresponding single-spin giant graviton is contained as a subset in the spectrum of two-spin giant. In fact, when $(m,n) = \{(5,5),(4,7),(7,4)\}$, the two spectra become identical but with different degeneracy. We also work out the fermionic vibration modes for single and two-spin BPS giant $M2$ in $AdS_7 \times S^4$ background.

In Section 4 we construct more general spherical giants where all available coordinates are assumed to depend on time after identifying the brane directions in space-time. They can support maximum possible number of independent spins both for the branes expanding in AdS and on the sphere. We find that the truncated p-brane equations can be integrated, and the canonical variables become separated, leading to an interesting set of potentials both for the "breathing mode" and the remaining "point-particle" motion. In fact, the latter we find to be governed by trigonometric and hyperbolic generalizations of the Calogero model, known as Pöschl-Teller Type I and II potentials, which are exactly solvable quantum mechanics models (see [26] and [27] for review). Alternatively, the point-particle sector can be quantized using global coordinates on embedding space leading to ordinary spherical harmonics. We also derive BPS bounds on the energy, and show that they can be saturated only by the single-spin and the two-spin solutions found in this paper, which are hence the only BPS spherical giants. Finally, we quantize the $(0+1)$-dimensional sigma-model, treating the point-particle motion exactly while borrowing the results from [28] for the breathing

[1] The supersymmetry of the solution can be established only when there exist a suitable κ-symmetric brane action coupled to a supergravity background. The main examples are $(m,n) = \{(5,5),(4,7),(7,4)\}$.

mode obtained using the Bohr-Sommerfeld recipe [29]. Interestingly enough, the exact energy spectrum from the (in general non-supersymmetric) point-particle motion is evenly spaced, while the breathing gives complicated corrections to the energy. This leads to a prediction for anomalous dimensions, however it is difficult to pinpoint the operators precisely.

In Section 5 we conclude and discuss several open problems. The Appendices contain details of the methods used to quantize the point-particle sector of the 0+1 dimensional sigma model, namely the global coordinates formulation, the eigenvalue spectra of Pöschl-Teller Type I and II potentials, and a comparison of the exact results with the Bohr-Sommerfeld method.

Note Added: In an earlier version of this paper, solutions of (2.24) which enhance 1/4 supersymmetry to 1/2 were overlooked and the two-spin giant solution presented in section 2 was erroneously claimed to be 1/4 supersymmetric. This mistake was corrected after the appearance of [30] where it was shown that the two-spin solution is related to the single-spin solution by an AdS isometry and thus has 1/2 supersymmetry.

2. 1/2 SUPERSYMMETRIC TWO-SPIN GIANTS

In this section we construct a 1/2 supersymmetric two-spin giant graviton wrapped on S^{n-2} inside S^n of $AdS_m \times S^n$ and rotating simultaneously on S^n and AdS_m. The bosonic p-brane action can be written as

$$S = -T_p \int d^{p+1}\sigma \sqrt{-\gamma}\ \left[1 + \frac{1}{(p+1)!}\ \epsilon^{\alpha_0 \ldots \alpha_p} \partial_{\alpha_0} X^{M_0} .. \partial_{\alpha_p} X^{M_p} A_{M_0..M_p}\right] \tag{2.1}$$

where $\gamma_{\alpha\beta}$ is the pull-back of the space-time metric to the world-volume. In some cases like $M5$ or $D3$-branes, there are additional world-volume fields which can be consistently set to zero. The field equations of the above action are

$$\begin{aligned} \frac{1}{\sqrt{-\gamma}} \partial_\alpha \left[\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\beta X^N g_{MN}\right] - \frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha X^N \partial_\beta X^P \partial_M g_{NP} = & \\ \frac{1}{(p+1)!}\ \epsilon^{\alpha_0 \ldots \alpha_p}\, \partial_{\alpha_0} X^{M_0} \ldots \partial_{\alpha_p} X^{M_p} H_{M M_0 \ldots M_p}, & \end{aligned} \tag{2.2}$$

where H is the field strength of the $(p+1)$-form potential, i.e. $H = dA$. The metric of $AdS_m \times S^n$ is:

$$ds^2 = -f dt^2 + f^{-1}\, dr^2 + r^2 d\Omega_{m-2}^2 + L^2 d\Omega_n^2 \tag{2.3}$$

where $f = 1 + r^2/\tilde{L}^2$ and

$$d\Omega_n^2 \;=\; d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta \left[d\chi_1^2 + \sin^2\chi_1(... + \sin^2\chi_{n-3} d\chi_{n-2}^2)\right]\ , \tag{2.4}$$

$$d\Omega_{m-2}^2 \;=\; d\alpha_1^2 + \sin^2\alpha_1 \left[d\alpha_2^2 + \sin^2\alpha_2(... + \sin^2\alpha_{m-3} d\alpha_{m-2}^2)\right]. \tag{2.5}$$

Here L and $\tilde{L}$ are the radius of curvatures which are related as $(m-1)L = (n-1)\tilde{L}$. When appropriate form fields are turned on which have non-zero fluxes on AdS or on the sphere, these geometries compromise maximally supersymmetric backgrounds of the corresponding supergravities for certain values of m and n. In this section we are interested in the magnetic backgrounds and thus should turn on the flux on the sphere. In the above coordinates the appropriate $(n-1)$-form potential supporting this flux becomes $A_{\phi\chi_1..\chi_{n-2}} = L^{n-1}(\sin\theta)^{n-1}\sqrt{g^\chi}$, where $(g^\chi)_{ij}$ is the metric on the unit S^{n-2} in (2.4) parametrized by χ coordinates. [2]

It is easy to verify that the following configuration solves the field equations (2.2):

$$\begin{aligned} & t=\tau, \quad \chi^i=\sigma^i, \\ & \phi=\tau/L, \quad \alpha\equiv\alpha_{m-2}=\tau/\tilde{L}, \\ & \theta=\theta_0,\ \ r=r_0,\ \ \alpha_1=..=\alpha_{m-3}=\pi/2 \end{aligned} \tag{2.6}$$

The brane wraps an $(n-2)$-sphere in S^n and rotates both on S^n and in AdS_m with constant angular velocities which are fixed by the corresponding curvature scales.

2.1. BPS Bound

From the action (2.1), the Lagrangian with $\alpha=\alpha(\tau)$ and $\phi=\phi(\tau)$ can be obtained as

$$\mathcal{L}=\frac{N}{L}\left[-(\sin\theta)^{n-2}\Delta+L(\sin\theta)^{n-1}\dot{\phi}\right], \tag{2.7}$$

where $\Delta^2=f-r^2\dot{\alpha}^2-L^2(\cos\theta)^2\dot{\phi}^2$. Here we have used the flux quantization $T_pA_p=N/L^{p+1}$ where A_p is the area of the unit p-sphere. The conserved angular momenta become

$$P_\phi = \frac{NL(\sin\theta)^{n-2}(\cos\theta)^2\dot{\phi}}{\Delta}+N(\sin\theta)^{n-1}, \tag{2.8}$$

$$P_\alpha = \frac{Nr^2(\sin\theta)^{n-2}\dot{\alpha}}{L\Delta}, \tag{2.9}$$

and the Hamiltonian can be written as

$$H=P_\phi\dot{\phi}+P_\alpha\dot{\alpha}-\mathcal{L}=\frac{N(\sin\theta)^{n-2}f}{L\Delta}. \tag{2.10}$$

The gauge condition $t=\tau$ implies $H=-P_t=E$, where E is the conserved AdS energy. From (2.8) and (2.9) it is possible to express Δ in terms of P_ϕ and P_α which gives the Hamiltonian as

$$H=\sqrt{f}\left[\frac{P_\alpha^2}{r^2}+\frac{P_\phi^2}{L^2}+\frac{N^2}{L^2}\tan^2\theta(\frac{P_\phi}{N}-\sin^{n-3}\theta)^2\right]^{1/2} \tag{2.11}$$

We see from (2.11) that the Hamiltonian obeys

$$H\geq\sqrt{f}\sqrt{\frac{P_\alpha^2}{r^2}+\frac{P_\phi^2}{L^2}}=\sqrt{\left[\frac{P_\phi}{L}+\frac{P_\alpha}{\tilde{L}}\right]^2+\left[\frac{P_\phi r}{L\tilde{L}}-\frac{P_\alpha}{r}\right]^2}, \tag{2.12}$$

which implies the BPS bound

$$H\geq\frac{P_\phi}{L}+\frac{P_\alpha}{\tilde{L}}. \tag{2.13}$$

[2] In some cases, such as $D3$, the n-form flux is self-dual and the potential has also an electric part. However, this does not affect the field equations for our ansatz. This remark is valid for all solutions found in this paper.

For fixed angular momenta, the extremum of the Hamiltonian can be found from $\partial_r H = 0$ and $\partial_\theta H = 0$. In the equilibrium r is uniquely determined by $r_0^2 = P_\alpha L\tilde{L}/P_\phi$. On the other hand, the roots of θ turn out to be equal to the ones found in [3], which are at $\theta = 0$ and $(\sin\theta)^{n-3} = P_\phi/N$. For the solution (2.6), the conserved quantities are

$$\begin{aligned} P_\phi &= N(\sin\theta_0)^{n-3}, && (2.14)\\ P_\alpha &= \frac{Nr_0^2}{L\tilde{L}}(\sin\theta_0)^{n-3}, && (2.15)\\ H &= \frac{P_\phi}{L} + \frac{P_\alpha}{\tilde{L}}. && (2.16) \end{aligned}$$

As for ordinary giants, P_ϕ depends only on the size of the brane. On the other hand, P_α is fixed both by the size of the brane and the radius of the rotation in AdS space. Note that $P_\phi \leq N$ but P_α is unbounded. Also, as $r_0 \to 0$, we have $P_\alpha \to 0$ and the single spin solution is recovered.

2.2. Supersymmetry

Eq. (2.16) shows that the energy saturates the BPS bound (2.13) and thus one expects this configuration to be supersymmetric. We will demonstrate this explicitly for a giant $M2$ brane in $AdS_7 \times S^4$. For other cases when $(m,n) = \{(5,5),(4,7)\}$ the calculation is similar. As discussed in [31], this solution will have residual supersymmetry if the following constraint is satisfied

$$\Gamma\epsilon = \epsilon, \tag{2.17}$$

where

$$\Gamma = -\frac{1}{3!}\,\epsilon^{\alpha_0\alpha_1\alpha_2}\partial_{\alpha_0}X^M\partial_{\alpha_1}X^N\partial_{\alpha_2}X^P\Gamma_{MNP}, \tag{2.18}$$

$\epsilon = \epsilon(X)|_{M2}$ and $\epsilon(X)$ is the Killing spinor in $AdS_7 \times S^4$ which can be found explicitly as (see, e.g., [3])[3]

$$\begin{aligned} \epsilon(X) \;=\; & e^{\frac{1}{2}\theta\gamma\Gamma^\theta}e^{\frac{1}{2}\phi\gamma\Gamma^\phi}e^{-\frac{1}{2}\chi_1\Gamma^{\chi_1\theta}}e^{-\frac{1}{2}\chi_2\Gamma^{\chi_2\chi_1}} \\ & e^{\frac{1}{2}u\Gamma^r\gamma}e^{-\frac{t}{2\tilde{L}}\Gamma^t\gamma}e^{-\frac{1}{2}\alpha_1\Gamma^{\alpha_1 r}}e^{-\frac{1}{2}\alpha_2\Gamma^{\alpha_2\alpha_1}}\ldots\, e^{-\frac{1}{2}\alpha_5\Gamma^{\alpha_5\alpha_4}}\epsilon_0, \end{aligned} \tag{2.19}$$

where

$$\sinh u = r/\tilde{L}, \qquad \gamma = \Gamma^{\theta\phi\chi_1\chi_2}, \tag{2.20}$$

ϵ_0 is a constant spinor and the indices on the gamma matrices refer to the tangent space. For this background we have $\tilde{L} = 2L$. For the solution (2.6), eq. (2.17) can be written as (after multiplying by $\Gamma^{\phi\chi_1\chi_2}$ from the left)

$$\left[\Gamma^{t\phi}\cosh u - \Gamma^{\alpha_5\phi}\sinh u - \gamma\Gamma^\theta\sin\theta + \cos\theta\right]\epsilon = 0. \tag{2.21}$$

[3] Note the sign differences with [3] in some exponentials.

The last two terms can be grouped as $\exp[-\theta\gamma\Gamma^\theta]$. Using this and commuting other factors with the first line of (2.19) we find that (2.21) is equivalent to

$$\Big[\Gamma^{t\phi}\cosh u - \Gamma^{\alpha_5\phi}\sinh u + I\Big]\, e^{\frac{1}{2}u\Gamma^r\gamma} e^{-\frac{\tau}{2L}\Gamma^t\gamma} e^{-\frac{1}{2}\alpha_1\Gamma^{\alpha_1 r}} e^{-\frac{1}{2}\alpha_2\Gamma^{\alpha_2\alpha_1}} ...\; e^{-\frac{\tau}{2L}\Gamma^{\alpha_5\alpha_4}}\epsilon_0 = 0. \qquad (2.22)$$

Multiplying from the left by $\exp[\frac{1}{2}u\Gamma^r\gamma]$ one obtains

$$\Big[\Gamma^{t\phi}\cosh u - \Gamma^{\alpha_5\phi}\sinh u + \cosh u + \Gamma^r\gamma\sinh u\Big]\, e^{-\frac{1}{2}\alpha_1\Gamma^{\alpha_1 r}} e^{-\frac{1}{2}\alpha_2\Gamma^{\alpha_2\alpha_1}} ...\; e^{-\frac{\tau}{2L}(\Gamma^{\alpha_5\alpha_4}+\Gamma^t\gamma)}\epsilon_0 = 0,$$

where we have also carried $\exp[-\frac{\tau}{2L}\Gamma^t\gamma]$ to the right. To take care of the angular dependencies in the middle, we first multiply the above expression from the left by $\exp[\frac{1}{2}\alpha_1\Gamma^{\alpha_1 r}]$ which commutes with the first three terms and anti-commutes with the fourth one. This last term gives the combination

$$(\sinh u)\Gamma^r\,\gamma\, e^{-\alpha_1\Gamma^{\alpha_1 r}} = -(\sinh u)\Gamma^r\,\gamma\,\Gamma^{\alpha_1 r} = (\sinh u)\,\gamma\,\Gamma^{\alpha_1} \qquad (2.23)$$

where we have used the fact that $\alpha_1 = \pi/2$. Carrying out the same calculation for α_2, α_3 and α_4 we finally get

$$\Big[\Gamma^{t\phi}\cosh u - \Gamma^{\alpha_5\phi}\sinh u + \cosh u + \Gamma^{\alpha_4}\gamma\sinh u\Big]\;\; e^{-\frac{\tau}{2L}(\Gamma^{\alpha_5\alpha_4}+\Gamma^t\gamma)}\epsilon_0 = 0. \qquad (2.24)$$

For this equation to hold, ϵ_0 should obey

$$\Big[I + \Gamma^{t\phi} - \tanh u\,\Gamma^{\alpha_5\phi}(I + \Gamma^{\alpha_5\alpha_4}\gamma\Gamma^\phi)\Big]\,\epsilon_0 = 0, \qquad (2.25)$$

$$\Big[I + \Gamma^{t\phi} + \tanh u\,\Gamma^{\alpha_5\phi}(I + \Gamma^{\alpha_5\alpha_4}\gamma\Gamma^\phi)\Big]\,\Big[I - \Gamma^{\alpha_5\alpha_4 t}\gamma\Big]\,\epsilon_0 = 0, \qquad (2.26)$$

where the first and the second conditions are implied by the even and the odd powers of τ in (2.24), respectively. Decomposing ϵ_0 as

$$\epsilon_0 = \epsilon^{++} + \epsilon^{+-} + \epsilon^{-+} + \epsilon^{--}, \qquad (2.27)$$

where

$$\Gamma^{t\phi}\,\epsilon^{s_1 s_2} \;=\; s_1\,\epsilon^{s_1 s_2}, \qquad \Gamma^{\alpha_5\alpha_4}\gamma\Gamma^\phi\,\epsilon^{s_1 s_2} \;=\; s_2\,\epsilon^{s_1 s_2}, \qquad s_{1,2} = \pm, \qquad (2.28)$$

one finds that (2.25) gives

$$\epsilon^{++} = 0, \qquad \epsilon^{+-} = \tanh u\,\Gamma^{\alpha_5\phi}\epsilon^{-+}, \qquad (2.29)$$

and (2.26) is satisfied identically. Thus the two-spin giant configuration preserves 1/2 of the supersymmetries of eleven dimensional supergravity. The single-spin solution can be recovered by letting $u \to 0$ (i.e. $r \to 0$) so that the motion in α_5 disappears. In that case, the Killing spinor condition (2.25) becomes

$$(I + \Gamma^{t\phi})\epsilon_0 = 0, \qquad (2.30)$$

which is the projection found in [3, 4]. Indeed, even for non-zero u it is possible to obtain (2.30) from (2.25) by a similarity transformation

$$\epsilon_0 \;\to\; \mathbb{S}^{-1}\,\epsilon_0 \qquad\qquad \Gamma^A \;\to\; \mathbb{S}^{-1}\,\Gamma^A\,\mathbb{S} \qquad (2.31)$$

where

$$\mathbb{S} = \exp(-\frac{1}{2}u\,\Gamma^{t\alpha_5})\exp(\pm\frac{1}{2}\delta\,\Gamma^{\phi\alpha_1\alpha_2\alpha_3\alpha_5 r})\;, \qquad \cos\delta = 1/\cosh u\;, \qquad (2.32)$$

and the sign in $\mathbb{S}$ is correlated with $\Gamma^{tr\alpha_1..\alpha_5}\gamma = \pm I$.

2.3. Instantons

From the experience with the single spin giants, it is plausible to expect a point particle limit. Indeed, taking $\sin\theta = \epsilon$, $L^2\dot{\phi}^2 = (1-\epsilon^{2n-4})/(1-\epsilon^2)$ and letting $\epsilon \to 0$, we get finite P_ϕ and P_α. Quantum mechanically, there may exist tunneling between the expanded and the point-like configurations. Following [3], to construct the relevant instanton solution we first assume $\theta = \theta(\tau)$ and let $\tau \to iz$. Using the corresponding conserved Euclidean energy we get a first order equation

$$P_\phi\, L\,\frac{d\theta}{dz} = \tan\theta[P_\phi - N(\sin\theta)^{n-3}]\,, \tag{2.33}$$

which is exactly the same one obtained in [3]. The solution is

$$(\sin\theta)^{n-3} = \frac{P_\phi\; e^{(n-3)z/L}}{1+\,N\; e^{(n-3)z/L}} \tag{2.34}$$

which interpolates between the extrema of θ as $z \to \pm\infty$. The total Euclidean action for this instanton is finite and thus one would expect mixing between the expanded and zero-size branes by tunneling. Note that unlike the ordinary giants, here there is no dual 1/2 BPS spherical configuration corresponding to a brane expanding in AdS. Because of this, no puzzle arises in solving the stringy exclusion principle.

2.4. Non-commutativity

In [25], a relation between supersymmetry and non-commutativity in the phase space was proposed, i.e. the BPS condition gives constraints and Dirac type canonical quantization leads to non-commutativity. One may wonder the consequences of having spin in AdS, as in our solution, for this analysis. For that purpose we relax the conditions on r and θ coordinates in (2.6) and let them to be dynamical. This modifies the Hamiltonian (2.11) so that there are additional fP_r^2 and P_θ^2/L^2 terms inside the square-root. For our 1/2 supersymmetric two-spin configuration we now have two primary constraints

$$\psi_1 \equiv P_\theta = 0, \quad \psi_2 \equiv P_r = 0. \tag{2.35}$$

The Poisson brackets of these with the Hamiltonian give two secondary constraints which are

$$\psi_3 = \frac{dV}{d\theta}, \quad \psi_4 = \frac{r^4}{\tilde{L}^2}\left[V(\theta) + \frac{P_\phi^2}{L^2}\right] - P_\alpha^2, \tag{2.36}$$

where $V(\theta) = N^2\tan^2\theta(P_\phi/N - \sin^{n-3}\theta)^2/L^2$. There are no further constraints since the Poisson brackets become $\{H,\psi_3\} \sim P_\theta$ and $\{H,\psi_4\} \sim P_r$. The constraints are second class with the algebra

$$\{\psi_1,\psi_3\} = -\frac{d^2V}{d\theta^2}, \quad \{\psi_2,\psi_4\} = -\frac{4P_\alpha^2}{r}. \tag{2.37}$$

We also have $\{\psi_1,\psi_4\} \sim \psi_3$ which vanishes on the restricted surface. The canonical structure on this constrained phase space can now be described by defining a Dirac bracket as follows:

$\{f,g\}_{DB} \equiv \{f,g\} - \{f,\psi_i\}(C^{-1})^{ij}\{g,\psi_j\}$ where $C_{ij} \equiv \{\psi_i,\psi_j\}$. This gives

$$\begin{aligned}
\{\theta,\phi\}_{DB} &= \frac{1}{N(n-3)(\sin\theta)^{n-4}\cos\theta}\,, \\
\{r,\alpha\}_{DB} &= \frac{L\tilde{L}}{2Nr(\sin\theta)^{n-3}}\,, \\
\{r,\phi\}_{DB} &= \frac{rL^2\tilde{L}^2}{2N(\sin\theta)^{n-3}}\,.
\end{aligned} \tag{2.38}$$

As in [25], non-commutativity is proportional to $1/N$. Note that it occurs not only between the sphere coordinates θ and ϕ but also between AdS coordinates r and α and between r and ϕ. For the single spin case, $P_\alpha = 0$ is an extra primary constraint due to which $\{\psi_2,\psi_4\} = 0$ and non-commutativity exists only between θ and ϕ [25].

2.5. Validity of the Solution

One can trust the solution only when the corrections to the space-time geometry and the Born-Infeld action can be ignored. In string theory the former requires $g_s \ll 1$ and $g_s N \gg 1$. On the other hand, the corrections to the Born-Infeld action are suppressed if the induced curvature scale on the world-volume is much larger than the string scale which gives $L\sin\theta_0 \gg \sqrt{\alpha'}$. For $AdS_5\times S^5$, $L^4 = 4\pi g_s N\alpha'^2$ and thus we have $(Ng_s)^{1/4}\sin\theta_0 \gg 1$. This implies from (2.14) $P_\phi \gg \sqrt{N/g_s}$. In M-theory, we still need $N \gg 1$ to rely on background geometry. To remove higher derivative corrections to $M2$ and $M5$ brane actions it is necessary to have $L\sin\theta_0 \gg l_p$. For $M2$ $L \sim l_p N^{1/6}$ and for $M5$ $L \sim l_p N^{1/3}$ which yields $P_\phi \gg N^{2/3}$ and $P_\phi \gg N^{1/3}$ respectively. Therefore, the giant graviton picture is reliable only when P_ϕ is large.

2.6. The Algebra of Unbroken Symmetries

The bosonic field theory defined by the action (2.1) in $AdS_m \times S^n$ background carries a representation of the isometry group $SO(2,m-1)\times SO(n+1)$, generated by the Killing vectors, $\delta X^M = K^M(X^N)$. The Noether charges are denoted by

$$(\widehat{\mathcal{M}}_{ab},\widehat{\mathcal{P}}_a)\ , \qquad a,b = (t,r,\alpha,\alpha_{m-3},\alpha_s)\ , \quad s = 1,..,m-4\ , \tag{2.39}$$

$$(\widehat{\mathcal{M}}_{IJ},\widehat{\mathcal{P}}_I)\ , \qquad I,J = (\theta,\phi,\chi^i)\ , \quad i = 1,\ldots,n-2\ , \tag{2.40}$$

where the indices are flat. The generators are anti-hermitian which are normalized such that

$$\begin{aligned}
&[\widehat{\mathcal{M}}_{ab},\widehat{\mathcal{M}}^{cd}] = 4\delta^{[c}_{[a}\widehat{\mathcal{M}}_{b]}{}^{d]}\ , && [\widehat{\mathcal{M}}_{IJ},\widehat{\mathcal{M}}^{KL}] = 4\delta^{[K}_{[I}\widehat{\mathcal{M}}_{J]}{}^{L]}\ , \\
&[\widehat{\mathcal{M}}_{ab},\widehat{\mathcal{P}}^{c}] = 2\delta^{c}_{[a}\widehat{\mathcal{P}}_{b]}\ , && [\widehat{\mathcal{M}}_{IJ},\widehat{\mathcal{P}}^{K}] = 2\delta^{K}_{[I}\widehat{\mathcal{P}}_{J]}\ , \\
&[\widehat{\mathcal{P}}_a,\widehat{\mathcal{P}}_b] = -\widehat{\mathcal{M}}_{ab}/\tilde{L}^2\ , && [\widehat{\mathcal{P}}_I,\widehat{\mathcal{P}}_J] = \widehat{\mathcal{M}}_{IJ}.
\end{aligned} \tag{2.41}$$

In determining the supersymmetry algebra in the background specified by a solution, the anti-commutator of two unbroken supersymmetries closes on unbroken bosonic symmetries. For both single and two-spin solutions these symmetries are

$$U(1)_{H'} \times SO(m-1) \times SO(n-1)_\chi \ , \tag{2.42}$$

where H' is the generator of τ-translations and $SO(n-1)_\chi$ are rotations of the S^p $(p = n-2)$ in the p-brane worldvolume, i.e.

$$H' \;=\; \begin{cases} E - J/L - S/\tilde{L} & \text{two-spin} \\ E - J/L & \text{single-spin} \end{cases} \tag{2.43}$$

$$SO(n-1)_\chi \quad : \qquad (\widehat{\mathcal{M}}_{\theta i}, \widehat{\mathcal{M}}_{ij}) \ , \qquad i = 1, \ldots, n-1 \ , \tag{2.44}$$

where

$$E \equiv -i\widehat{\mathcal{P}}_t \ , \qquad S \equiv i\widehat{\mathcal{M}}_{\alpha\alpha_{m-3}} \ , \qquad J \equiv i\widehat{\mathcal{P}}_\phi \ . \tag{2.45}$$

The form of H' follows from that $t = \tau = \phi/L = \alpha/\tilde{L}$ in the case of the two-spin solution, and $t = \tau = \phi/L$ in the case of the single-spin solution. In the linearized theory, H' becomes the fluctuation Hamiltonian. The unbroken $SO(m-1)$ is an internal symmetry from the point of view of the p-brane worldvolume theory. In the case of single-spin all $SO(m-1)$ rotations are manifest, and its generators are

$$SO(m-1) : \quad \mathcal{M}_{I'J'} = (\mathcal{M}_{r\alpha_r}, \mathcal{M}_{\alpha_r\alpha_s}) \ , \qquad I' = 1, ..., m-1 \ , \quad r = 1, \ldots, m-2 \ , \tag{2.46}$$

In the case of two-spin an $SO(m-3) \subset SO(m-1)$ subgroup is manifest, and generated by the rotations preserving the equator of $S^{m-2} \subset S^m$, i.e.

$$SO(m-3) : \quad (\widehat{\mathcal{M}}_{r\alpha_r}, \widehat{\mathcal{M}}_{\alpha_r\alpha_s}) \ , \qquad r = 1, \ldots, m-4 \ . \tag{2.47}$$

In addition to the unbroken symmetries, the linearized p-brane action is invariant under separate shifts in the cyclic t, α and ϕ coordinates generated by E, S and J defined in (2.45). Of these isometries only the combined shift generated by H' given in (2.43) is an unbroken isometry. However, the cyclic coordinates are axionic fields on the p-brane and hence E, S and J remain conserved in the linearized theory. These charges generate outer automorphisms of the the supersymmetry algebra in the background of the solution. We determine the $U(1)$-charges of the unbroken supersymmetry charges and fluctuation fields in Section 3.6 in the case of the single-spin solution.

As an example, which will be used in Section 3.6, let us determine the unbroken superalgebra in presence of a single-spin giant $M2$-brane in $AdS_7 \times S^4$. The supersymmetry algebra of M-theory expanded around a solution with non-trivial four-form fluxes is given in [32] and on $AdS_7 \times S^4$ it becomes $(\tilde{L} = 2L)$

$$\begin{aligned}
&\{\widehat{\mathcal{Q}}, \widehat{\bar{\mathcal{Q}}}\} = -2\Gamma^a\,\widehat{\mathcal{P}}_a + \frac{1}{\tilde{L}}\,\Gamma^{ab}\,\gamma\,\widehat{\mathcal{M}}_{ab} - \frac{2}{L}\,\Gamma^I\,\widehat{\mathcal{P}}_I - \frac{1}{L}\,\Gamma^{IJ}\,\gamma\,\widehat{\mathcal{M}}_{IJ}, \\
&[\widehat{\mathcal{P}}_a, \widehat{\bar{\mathcal{Q}}}] = \frac{1}{2\tilde{L}}\widehat{\bar{\mathcal{Q}}}\Gamma_a\gamma, \qquad [\widehat{\mathcal{M}}_{ab}, \widehat{\bar{\mathcal{Q}}}] = \frac{1}{2}\widehat{\bar{\mathcal{Q}}}\Gamma_{ab}, \\
&[\widehat{\mathcal{P}}_I, \widehat{\bar{\mathcal{Q}}}] = -\frac{1}{2}\widehat{\bar{\mathcal{Q}}}\Gamma_I\gamma, \qquad [\widehat{\mathcal{M}}_{IJ}, \widehat{\bar{\mathcal{Q}}}] = \frac{1}{2}\widehat{\bar{\mathcal{Q}}}\Gamma_{IJ}.
\end{aligned} \tag{2.48}$$

The full $M2$ brane supersymmetry generators are given by

$$\widehat{\bar{\mathcal{Q}}}\epsilon_0 = \int d^2\sigma\sqrt{-\gamma}\,\gamma^{0i}\,\bar{\epsilon}(1-\Gamma)\Gamma_i\Theta\ , \tag{2.49}$$

where $\epsilon = g(X)\epsilon_0$ is the Killing spinor (2.19).

From (2.30), the unbroken supersymmetry charges for the single-spin solution are given by

$$Q = P_+\widehat{\mathcal{Q}}\ , \qquad P_+ = \frac{1}{2}(1+\Gamma^{t\phi}). \tag{2.50}$$

These obey

$$\{Q,\bar{Q}\} = 2P_+\left(i\Gamma^t H' + \frac{1}{2\tilde{L}}\Gamma^{I'J'}\gamma\,\mathcal{M}_{I'J'} - \frac{1}{2L}\Gamma^{i'j'}\gamma\,\mathcal{M}_{i'j'}\right)\ , \tag{2.51}$$

and $[\tilde{L}E,\bar{Q}] = [J,\bar{Q}] = -\frac{i}{2}\bar{Q}\Gamma^t\gamma$. Next, we define

$$Q^\pm = \Pi_\pm Q\ , \qquad \Pi_\pm = \frac{1}{2}(1\pm i\Gamma^t\gamma)\ , \tag{2.52}$$

which commute with P_+ and the unbroken bosonic symmetries. Hence

$$[H',Q^\pm] = \mp\frac{1}{4L}Q^\pm\ , \qquad [J,Q^\pm] = \pm\frac{1}{2}\ , \tag{2.53}$$

and Q^+ and Q^- transform as $(4,2)$ and $(\bar{4},2)$ under $SO(6)\times SO(3)_\chi$ which we denote them by Q^A_α and $\bar{Q}^\alpha_A$, respectively, with[4] $\alpha = 1,...,4$ and $A = 1,2$. We use the $SO(6)\sim SU(4)$ chiral notation, in which chiral and anti-chiral spinors always have upper and lower indices, respectively. No raising and lowering of chiral indices can be done, while the doublet indices can of course be raised and lowered as usual. Under hermitian conjugation, $(Q^A_\alpha)^\dagger = \bar{Q}^{\alpha A}$. Multiplying (2.51) by $i\Gamma^t$ from the right and projecting by Π_+ we find that the non-vanishing commutators are

$$\begin{aligned}
\{Q^A_\alpha,\bar{Q}^\beta_B\} &= 2\delta^\beta_\alpha\delta^A_B H' + \frac{2}{L}\,\delta^A_B J_\alpha{}^\beta - \frac{i}{L}\,\delta^\beta_\alpha L^A{}_B\ ,\\
[H',Q_{\alpha A}] &= -\frac{1}{4L}\,Q_{\alpha A}\ , \qquad [H',\bar{Q}^\alpha_A] = \frac{1}{4L}\,\bar{Q}^\alpha_A\ ,\\
[J_\alpha{}^\beta,Q_{\gamma A}] &= \delta^\beta_\gamma\,Q_{\alpha A} - \frac{1}{4}\delta^\beta_\alpha\,Q_{\gamma A}\ , \qquad [L_{AB},Q_{\gamma C}] = -2i\epsilon_{C(A}Q_{\gamma B)}\ ,\\
[J_\alpha{}^\beta,Q^\gamma_A] &= \delta^\gamma_\alpha\,Q^\beta_A - \frac{1}{4}\delta^\beta_\alpha\,Q^\gamma_A\ , \qquad [L_{AB},Q^\gamma_C] = -2i\epsilon_{C(A}Q^\gamma{}_{B)}\ ,\\
[J_a{}^\beta,J_\gamma{}^\delta] &= \delta^\delta_\alpha\,J_\gamma{}^\beta - \delta_\gamma{}^\beta\,J^\delta_\alpha\ ,\\
[L_{AB},L_{CD}] &= -2i\epsilon_{C(A}L_{B)D} + (C\leftrightarrow D)\ ,
\end{aligned} \tag{2.54}$$

where L_{AB} are the $SO(3)_\chi$ generators and we have defined the $SO(6)$ generators $J_\alpha{}^\beta \equiv -\frac{1}{4}M^{I'J'}(\Gamma_{I'J'})_\alpha{}^\beta$. The non-trivial Jacobi identity $\{Q_{\alpha A},\{Q_{\beta B},Q^\gamma_C\}\} + \text{cyclic} = 0$ is indeed satisfied, and the fact that H' does not commute with the supercharges is essential for this to happen. Despite this non-commutativity the ground state energy may still vanish [33].

For the two-spin solution, the superalgebra can be obtained using the similarity transformation (2.31), noting that $\mathbb{S}^\dagger\Gamma^t = \Gamma^t\mathbb{S}^{-1}$.

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[4] We use the following conventions: hermitian generators L_i of $SO(3)$ and Pauli matrices obey $[L_i,L_j] = i\epsilon_{ijk}L_k$ and $\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k + \delta^{ij}$. Doublet indices are contracted using $\epsilon_{AB} = (\epsilon^{AB})^\dagger$, $\epsilon_{AB}\epsilon^{CD} = 2\delta^{CD}_{AB}$. The symmetric Pauli matrices $(\sigma^i)_{AB} \equiv (\sigma^i\epsilon)_{AB}$ obey $(\sigma^i)_{AB}(\sigma^j)_{CD} = -i\epsilon^{ijk}\epsilon_{AC}(\sigma^k)_{BD} - \frac{2}{3}\delta^{ij}\epsilon_{AC}\epsilon_{BD}$, where symmetrizations on AB and CD are suppressed. The corresponding anti-hermitian generators $\widehat{\mathcal{M}}_{ij}$ and Dirac-matrices Γ^{ij} are related by $\Gamma^{ij}\widehat{\mathcal{M}}_{ij} = -iL_{AB}$, where $L_{AB} \equiv 2i(\sigma^i)_{AB}L_i = (L^{AB})^\dagger$.

3. VIBRATION SPECTRUM

In this section we examine the spectrum of small fluctuations around the 1/2 BPS single and two-spin solutions, including fermions for the $M2$ in $AdS_7 \times S^4$. For single-spin giants the scalar fluctuations were analyzed in [25] (the vibration spectrum for giants in the PP-wave background was studied in [34–36]). There are several motivations for this. Firstly, the solution defines the ground state for a subsector of the p-brane field theory, based on normal coordinate expansion, which is stable only if all fluctuation-modes have real and positive frequencies. There is also the issue of singling out zero-modes describing continuous shifts in the semi-classical parameters.

Secondly, the full normal-coordinate expansion, including loops, becomes a $1/N$-expansion after absorbing a power of $\sqrt{N}$ into the fluctuation fields (the ℓ-loop contributions to the n-point diagrams scale like $N^{1-n/2-\ell}$), which reduces to the free quadratic Lagrangian in the limit $N \to \infty$. Hence, from the point of view of the dual CFT, the frequencies have a direct interpretation in terms of the large N limit of the scaling dimensions of a subset of operators forming a "tower" on top of a specific operator corresponding to the ground state of the expansion on the p-brane [6, 23, 24]. Another basic idea is that the tower has the structure of a Fock space in the large N limit. Indeed, as in the case of the original giant gravitons [25], the vibration modes of the two-spin giants have evenly spaced frequencies fixed by the radii of curvature.

For the calculations in this section, we find it convenient to use the following parametrization of the sphere S^{m-2} in (2.3):

$$d\Omega_{m-2}^2 = (1-y^2)d\alpha^2 + (\delta_{mn} + \frac{y_m y_n}{1-y^2})dy^m dy^n. \tag{3.1}$$

In these coordinates the S^{m-2} part of the solution (2.6) is given by

$$\alpha = \tau/\tilde{L}, \quad y^m = 0. \tag{3.2}$$

3.1. Bosonic Oscillations

In the physical gauge that we employed, the coordinates can be perturbed as

$$\begin{aligned} &\phi = \tau/L + \delta\phi, \quad \alpha = \tau/\tilde{L} + \delta\alpha, \\ &\theta = \theta_0 + \delta\theta, \quad r = r_0 + \delta r, \quad y^m = \delta y^m, \end{aligned} \tag{3.3}$$

where the fluctuations depend on all of the world-volume coordinates. Expanding the action (2.1) to the linear order in perturbations we find that the variation vanishes since the background obeys the field equations. On the other hand, to the second order, the action becomes

$$S_2 = \int d^{n-1}\sigma \sqrt{-\gamma^{(0)}}\, \mathcal{L}_2 \tag{3.4}$$

where

$$2\,\mathcal{L}_2 = \delta y^m \left[\Box - \frac{1}{\tilde{L}^2 \sin^2\theta_0}\right] \delta y^m + \delta r \left[\frac{\Box}{f_0}\right] \delta r + \delta\alpha \left[(1 + \frac{r_0^2}{\tilde{L}^2 \sin^2\theta_0}) r_0^2 \Box\right] \delta\alpha$$

$$+\delta\phi\left[L^2\cot^2\theta_0\Box\right]\delta\phi+\delta\theta\left[L^2\Box\right]\delta\theta+\frac{4r_0}{\tilde{L}\sin^2\theta_0}\delta r\,\partial_\tau\,\delta\alpha \tag{3.5}$$
$$+\frac{2(n-3)L\cos\theta_0}{\sin^3\theta_0}\delta\theta\,\partial_\tau\,\delta\phi+\frac{2(n-3)r_0^2\cos\theta_0}{\tilde{L}\sin^3\theta_0}\delta\theta\,\partial_\tau\,\delta\alpha+\delta\alpha\left[\frac{2r_0^2L}{\tilde{L}}\cot^2\theta_0\Box\right]\delta\phi,$$

$\Box$ is the D'Alembertian for the background world-volume metric $\gamma^{(0)}_{\alpha\beta}$ which is given by

$$\gamma^{(0)}_{\alpha\beta}=\begin{pmatrix}-\sin^2\theta_0 & 0\\ 0 & L^2\,\sin^2\theta_0\,(g^\chi)_{ij}\end{pmatrix} \tag{3.6}$$

and $f_0=1+r_0^2/\tilde{L}^2$. In writing the above Lagrangian some terms are integrated by parts; there is no surface contribution coming from the spatial part of the world-volume since it is a closed surface and the variations are assumed to vanish at $\tau=\pm\infty$. We expand a generic perturbation as

$$\delta X=\sum_l \delta X_0\,e^{i\omega_l\tau}\,Y_l \tag{3.7}$$

where Y_l are spherical harmonics on the unit $(n-2)$-sphere obeying

$$(g^\chi)^{ij}\partial_i\partial_j\,Y_l=-Q_l\,Y_l, \tag{3.8}$$

with $Q_l=l(l+n-3)$. Then, $\Box$ acting on the l'th mode becomes

$$\Box\to\frac{1}{\sin^2\theta_0}(\omega_l^2-\frac{Q_l}{L^2})\equiv D_l. \tag{3.9}$$

From the above quadratic Lagrangian, we see that δy^m perturbations decouple and have the normal frequencies

$$\omega_l^2=\frac{1}{\tilde{L}^2}+\frac{Q_l}{L^2}. \tag{3.10}$$

On the other hand, δr, $\delta\alpha$, $\delta\phi$ and $\delta\theta$ modes are coupled. The resulting frequencies are determined from the following equation

$$\begin{bmatrix}\frac{D_l}{f_0} & \frac{2i\omega_l r_0}{\tilde{L}\sin^2\theta_0} & 0 & 0\\ -\frac{2i\omega_l r_0}{\tilde{L}\sin^2\theta_0} & (1+\frac{r_0^2}{\tilde{L}^2\sin^2\theta_0})r_0^2D_l & \frac{r_0^2L}{\tilde{L}}\cot^2\theta_0D_l & -\frac{i\omega_l(n-3)r_0^2\cos\theta_0}{\tilde{L}\sin^3\theta_0}\\ 0 & \frac{r_0^2L}{\tilde{L}}\cot^2\theta_0D_l & L^2\cot^2\theta_0D_l & -\frac{i\omega_l(n-3)L\cos\theta_0}{\sin^3\theta_0}\\ 0 & \frac{i\omega_l(n-3)r_0^2\cos\theta_0}{\tilde{L}\sin^3\theta_0} & \frac{i\omega_l(n-3)L\cos\theta_0}{\sin^3\theta_0} & L^2D_l\end{bmatrix}\begin{bmatrix}\delta r\\ \delta\alpha\\ \delta\phi\\ \delta\theta\end{bmatrix}=0. \tag{3.11}$$

Calculating the determinant, we see that (for non-zero r_0, $\cos\theta_0$ and $\sin\theta_0$) it factorizes into two quadratic equations for ω_l^2 from which the following normal frequencies can be obtained

$$\omega_{l\pm}^2 = \frac{1}{L^2}\left[Q_l+\frac{(n-3)^2}{2}\pm(n-3)\sqrt{Q_l+\frac{(n-3)^2}{4}}\right] \tag{3.12}$$

$$\omega_{l\pm}^2 = \frac{1}{\tilde{L}^2}\left[\frac{\tilde{L}^2}{L^2}Q_l+2\pm2\sqrt{\frac{\tilde{L}^2}{L^2}Q_l+1}\right]. \tag{3.13}$$

There is no unstable mode in the system since all ω_l^2 are real and non-negative. Using the fact $Q_l = l(l+n-3)$, (3.12) simplifies to $\omega_{l+} = (l+n-3)/L$ and $\omega_{l-} = l/L$.

Frequencies (3.10) and (3.12) constitute the vibration spectrum of the single-spin giant wrapped in S^n [19]. Therefore, the modes (3.13) can be thought to arise due to the spin in AdS. The rotation disappears when $r_0 \to 0$ and the single spin solution is recovered. In this limit $(r,\, \alpha,\, y^m)$ coordinate system is not well defined. To read the eigenfrequencies one should either introduce flat coordinates or just treat $(r_0\delta\alpha)$ as the true perturbation. In the later case, it is easy to see that δr and $r_0\delta\alpha$ perturbations decouple from $\delta\phi$ and $\delta\theta$ modes and (3.13) should be replaced with (3.10) [5].

It is remarkable that for $(m,n) = \{(5,5),(4,7),(7,4)\}$, $L = (n-3)\tilde{L}/2$ and (3.12) becomes equal to (3.13). In this case, the eigenfrequencies are given by

$$\omega_l = \frac{l+(n-3)/2}{L}, \qquad \omega_{l+} = \frac{(l+n-3)}{L}, \qquad \omega_{l-} = \frac{l}{L}, \tag{3.14}$$

where the first ω_l is for δy^m with degeneracy $(m-3)$ and the others are for the mixing of δr, $\delta\alpha$, $\delta\phi$, $\delta\theta$, each frequency occurring with degeneracy 2. The eigenfrequencies for the single spin giant is given by (3.14) but the degeneracies are $(m-1)$, 1, 1, respectively.

3.2. Bosonic Zero Modes and Spectrum

As discussed in [19], some of the above excitations (zero modes) correspond to the collective motion of the brane since there are continuous families of equilibrium configurations. These modes will change the quantum numbers (i.e. the conserved quantities like P_ϕ and P_α) and should be removed from the spectrum since they can no longer be viewed to belong to the giant we started with. To linear order, there can be no change in the conserved quantities due to $l \neq 0$ modes, since these are calculated at a fixed world-volume time as integrations over the sphere and we have $\int Y_l = 0$ when $l \neq 0$. Note that Y_0 is the constant harmonic and we have $Q_0 = 0$.

For $l = 0$, we see from (3.10) that the frequencies for δy^m perturbations become $\omega_0 = 1/\tilde{L}$. These modes correspond to the shifts in the great circle in S^{m-2}. To see this, following [19], one can embed S^{m-2} in $(m-1)$-dimensional flat space with coordinates x_1, x_2, y^m. The unit sphere can be defined as $x_1^2 + x_2^2 + y^m y^m = 1$. The coordinate system (3.1) corresponds to the parametrization

$$x_1 = \cos\beta\cos\alpha, \quad x_2 = \cos\beta\sin\alpha, \quad y^m y^m = \sin^2\beta \tag{3.15}$$

The brane is circling in (x_1-x_2) plane at $\beta = 0$ (and thus $x_1 = \cos\alpha$, $x_2 = \sin\alpha$ and $y^m = 0$). One can now rotate this plane by an angle δ:

$$x_1' = x_1\cos\delta - y_1\sin\delta, \qquad y_1' = x_1\sin\delta + y_1\cos\delta. \tag{3.16}$$

Recalling that $x_1 = \cos\alpha$ and $y^m = 0$, for small δ one has

$$x_1' = x_1, \qquad y_1' = \delta\cos(\tau/\tilde{L}) \tag{3.17}$$

[5] Another limit that one may wish to consider is when the rotation in ϕ disappears, i.e. $\theta_0 \to \pi/2$. In this case, P_ϕ does not vanish, on the contrary it reaches its maximum (2.14). This is the maximal giant and the angular momentum arises due to the coupling of the background flux to the brane. The spectrum does not change. On the other hand in $\theta_0 \to 0$ limit the brane collapses to a point and fluctuation analysis becomes ill defined. Indeed, before reaching this value, the world-volume becomes highly curved and the probe brane approximation fails.

This change in y_1 is precisely equal to the perturbation with $l = 0$. Thus these modes represent shifts in the direction of P_α without changing its magnitude, and $l = 0$ frequencies in (3.10) should be removed from the spectrum.

From (3.12) and (3.13), there are four more frequencies with $l = 0$. These are $\omega_0^{(1)} = \omega_0^{(2)} = 0$, $\omega_0^{(3)} = (n-3)/L$ and $\omega_0^{(4)} = 2/\tilde{L}$. The modes corresponding to $\omega_0^{(1)}$ and $\omega_0^{(2)}$ represent shifts in the values of r_0 and θ_0. From (2.14) and (2.15), a change in r_0 modifies P_α and a change in θ_0 alters both P_ϕ and P_α. Therefore, these modes should not be in the spectrum.

On the other hand, the perturbations corresponding to $\omega_0^{(3)}$ and $\omega_0^{(4)}$ respectively obey

$$\delta r = \delta\alpha = 0, \qquad \delta\phi = i(\tan\theta_0)\,\delta\theta, \tag{3.18}$$

$$\delta r = \frac{iL(\tilde{L}^2 + r_0^2)}{\tilde{L} r_0}\,\delta\phi, \qquad \delta\alpha = -\frac{L\tilde{L}}{r_0^2}\,\delta\phi, \qquad \delta\theta = 0. \tag{3.19}$$

It is now straightforward to verify that the change in P_ϕ and P_α is zero under these perturbations. For example, from (2.8) the variation of P_ϕ under (3.18) becomes

$$\delta P_\phi = \frac{\partial P_\phi}{\partial \theta}\,\delta\theta + \frac{\partial P_\phi}{\partial \dot{\phi}}\,\delta\dot{\phi}. \tag{3.20}$$

Using $\delta\dot{\phi} = i\omega_0^{(3)}\delta\phi$ and (3.18) one finds $\delta P_\phi = 0$. Therefore, the zero modes corresponding to $\omega_0^{(3)}$ and $\omega_0^{(4)}$ should be kept in the spectrum.

After the elimination of these zero modes, we end up with the following spectrum of small scalar field fluctuations for $(m, n) = \{(5, 5), (4, 7), (7, 4)\}$ given in units of $1/L$ (note that l is shifted compared to (3.14)):

Bosonic ω_l $(l \geq 0)$	$l + (n-1)/2$	$l + n - 3$	$l + 1$
Multiplicity (two-spin)	$(m-3)_{l+1}$	$(2)_l$	$(2)_{l+1}$
Multiplicity (single-spin)	$(m-1)_{l+1}$	$(1)_l$	$(1)_{l+1}$

where the suffix indicates the leading $SO(n-1)_\chi$ highest weight label.

3.3. Fermionic Oscillations of $M2$-Brane (Two-spin)

We calculate the spectrum of the fermionic oscillations for an $M2$ brane in $AdS_7 \times S^4$. For the membrane, the quadratic action for the fermion fluctuations in a general bosonic background of $D = 11$ supergravity was derived in [37] which can be written as

$$\mathcal{L}_\Theta = i\sqrt{-\gamma}\,\gamma^{\alpha\beta}\,\partial_\alpha X^M\; E_M^A\,\bar{\Theta}\,(I - \Gamma)\,\Gamma_A\,\tilde{\nabla}_\beta\Theta \tag{3.21}$$

where

$$\tilde{\nabla}_\alpha = \partial_\alpha \,+\, \partial_\alpha X^M\left[\frac{1}{4}\omega^{AB}{}_M\Gamma_{AB} + \frac{1}{288}(8\delta_M^P\Gamma^{QRS} - \Gamma_M{}^{PQRS})H_{PQRS}\right], \tag{3.22}$$

$\gamma_{\alpha\beta}$ is the induced metric and E^A_M is an orthonormal basis in the space-time. The non-vanishing components of the connection one-forms[6] that contribute to (3.22) are

$$\omega^{\hat\phi}{}_{\hat\theta\hat\phi} = -\frac{\tan\theta_0}{L}, \quad \omega^{\hat i}{}_{\hat\theta\hat j} = -\frac{\cot\theta_0}{L}\delta^i{}_j, \quad \omega^{\hat i}{}_{\hat j\hat k} = \frac{\omega^{(1)\hat i}{}_{\hat j\hat k}}{L\sin\theta_0}, \qquad (3.23)$$
$$\omega^{\hat t}{}_{\hat r\hat t} = \frac{r_0}{\tilde L^2\sqrt{f_0}}, \quad \omega^{\hat\alpha}{}_{\hat r\hat\alpha} = \frac{\sqrt{f_0}}{r_0},$$

where a subscript '(1)' on a quantity indicates that it is defined on a unit 2-sphere. Using the solution (2.6) and $H_{\hat\theta\hat\phi\hat1\hat2} = 3/L$ we find

$$\begin{aligned}\tilde\nabla_\tau &= \partial_\tau + \frac{1}{2\tilde L}\left[\frac{r_0}{\tilde L}\Gamma_t + \sqrt{f_0}\Gamma_\alpha\right]\Gamma^r - \frac{1}{4L}\left[\frac{r_0}{\tilde L}\Gamma_\alpha + \sqrt{f_0}\Gamma_t\right]\gamma \\ &- \frac{1}{2L}\left[\sin\theta_0\,\Gamma^\phi + \cos\theta_0\Gamma^{12}\right]\Gamma^\theta \qquad (3.24)\\ \tilde\nabla_i &= \nabla_i^{(1)} - \frac{1}{2}\left[\cos\theta_0\,\Gamma^\theta + \sin\theta_0\,\gamma\right]\Gamma_i^{(1)} \qquad (3.25)\\ \Gamma &= \frac{1}{\sin\theta_0}\left[\sqrt{f_0}\,\Gamma_{t12} + \frac{r_0}{\tilde L}\Gamma_{\alpha12} + \cos\theta_0\,\Gamma_{\phi12}\right] \qquad (3.26)\end{aligned}$$

where $\gamma = \Gamma_{\theta\phi12}$, indices (1,2) refer to χ_1 and χ_2 directions and recall that the indices on the gamma matrices are flat. To fix κ symmetry, we impose $\Gamma\Theta = -\Theta$ which also gives $\bar\Theta\Gamma = -\bar\Theta$. Using this gauge condition in (3.21) we obtain

$$\mathcal{L}_\Theta = -iL^2\sin^2\theta_0\sqrt{-\gamma^{(1)}}\left[\bar\Theta\Gamma_{12}\tilde\nabla_\tau\Theta - \frac{1}{L}\bar\Theta\Gamma^i_{(1)}\tilde\nabla_i\Theta\right]. \qquad (3.27)$$

To proceed we use the following representation of the 11-dimensional gamma matrices:

$$\begin{aligned}&\Gamma_t = i\sigma_2\otimes\gamma_5\otimes\sigma_3\otimes\sigma_3, &&\Gamma_r = \sigma_3\otimes\gamma_5\otimes\sigma_3\otimes\sigma_3,\\ &\Gamma_\alpha = \sigma_1\otimes\gamma_5\otimes\sigma_3\otimes\sigma_3, &&\Gamma_m = I_2\otimes\gamma_m\otimes\sigma_3\otimes\sigma_3, \qquad (3.28)\\ &\Gamma_\theta = I_2\otimes I_4\otimes\sigma_2\otimes\sigma_3, &&\Gamma_\phi = I_2\otimes I_4\otimes\sigma_1\otimes\sigma_3,\\ &\Gamma_i = I_2\otimes I_4\otimes I_2\otimes\sigma_i, &&C = i\sigma^2\otimes c\otimes\sigma^1\otimes i\sigma^2,\end{aligned}$$

where $i = 1,2,\ m = 1,..,4$ and $\gamma^m c$ are symmetric. Then (3.26) become

$$\Gamma = \frac{i}{\sin\theta_0}\left[\left(\frac{r_0}{\tilde L}\sigma_1 + i\sqrt{f_0}\,\sigma_2\right)\otimes\gamma_5\otimes\sigma_3\otimes I_2 + \cos\theta_0\ I_2\otimes I_4\otimes\sigma_1\otimes I_2\right] \qquad (3.29)$$

and writing $\mathcal{L} = 2\bar\Theta M\Theta$ we find

$$\begin{aligned}M &= I_2\otimes I_4\otimes I_2\otimes\left(\sigma_3\partial_\tau + \frac{i}{L}\sigma^i\nabla_i^{(1)}\right) - \frac{1}{4L}\left(\frac{r_0}{\tilde L}\sigma_1 + i\sqrt{f_0}\,\sigma_2\right)\otimes(I_4+\gamma_5)\otimes I_2\otimes\sigma_3\\ &+ \frac{i\sin\theta_0}{2L}I_2\otimes I_4\otimes\sigma_3\otimes\sigma_3 + \frac{i\cos\theta_0}{2L}I_2\otimes I_4\otimes\sigma_2\otimes\sigma_3. \qquad (3.30)\end{aligned}$$

[6] Our convention is $de^A + \omega^A{}_B\wedge e^B = 0$ where we expand $\omega^A{}_B = \omega^A{}_{BC}\,e^C$.

We solve the Γ projection condition on Θ as

$$\Theta \;=\; \sum_{\epsilon,\epsilon'=\pm1} (P^{(1)}_{\epsilon} \otimes P^{(2)}_{\epsilon'} \otimes P^{(3)}_{\epsilon\epsilon'} \otimes I_2)\Theta_{\epsilon,\epsilon'}\ , \tag{3.31}$$

$$P^{(1)}_{\epsilon} \;=\; \frac{1}{2}(I_2 + \epsilon(i\frac{r_0}{\tilde{L}}\sigma_1 - \sqrt{f_0}\sigma_2)) \tag{3.32}$$

$$P^{(2)}_{\epsilon'} \;=\; \frac{1}{2}(I_4 + \epsilon'\gamma^5)\ , \tag{3.33}$$

$$P^{(3)}_{\epsilon\epsilon'} \;=\; \frac{1}{2}(I_2 - \epsilon\epsilon'(\sigma_3 + i\cos\theta_0\sigma^1)/\sin\theta_0) \tag{3.34}$$

where $\Theta_{\epsilon,\epsilon'}$ are unrestricted. Each term in the sum is a product of three projectors of half maximal rank, which implies that 28 of the 32 components in each $\Theta_{\epsilon,\epsilon'}$ are set to zero, such that Θ has $4\times4=16$ components. From (3.32) and (3.34) one verifies

$$(\frac{r_0}{\tilde{L}}\sigma_1 + i\sqrt{f_0}\sigma_2)P^{(1)}_{\epsilon} \;=\; -i\epsilon P^{(1)}_{\epsilon}\ , \tag{3.35}$$

$$P^{(3)}_{\epsilon\epsilon'}(\cos\theta_0\sigma_2 + \sin\theta_0\sigma_3)P^{(3)}_{\epsilon\epsilon'} \;=\; -\epsilon\epsilon' P^{(3)}_{\epsilon\epsilon'}, \tag{3.36}$$

which shows that the masses are independent of both r_0 and θ_0, while of course the direction in spinor space of the projected Θ depends on r_0 and θ_0 via (3.32) and (3.34). To extract the normal frequencies we expand $\Theta_{\epsilon,\epsilon'}$ in terms of spherical spinor harmonics on S^2 and substitute

$$\partial_\tau \to i\omega_l \qquad i\sigma^i\nabla^{(1)}_i \to (l+1/2)\sigma_1, \quad l = 1/2, 3/2, 5/2, ... \tag{3.37}$$

The resulting characteristic equation has the matrix

$$M \;=\; iP^{(1)}_{\epsilon} \otimes P^{(2)}_{\epsilon'} \otimes P^{(3)}_{\epsilon\epsilon'} \otimes \left((\omega_l + \frac{1}{4L}\epsilon(1-\epsilon'))\sigma_3 - \frac{i}{L}(l+1/2)\sigma_1\right) \tag{3.38}$$

which yields the (positive[7]) eigenfrequencies ($l = 1/2, 3/2, 5/2, \ldots$)

$$\omega_l = \frac{(l+1/2)}{L}\ \{\epsilon'=1\},\quad \omega_l = \frac{(l+1)}{L}\ \{\epsilon'=\epsilon=-1\},\quad \omega_l = \frac{l}{L}\ \{\epsilon'=-1,\epsilon=1\}. \tag{3.39}$$

The equations of motion (note that the Dirac operator projects by half in the last slot) eliminate half of the fermionic degrees of freedom, which means that the first ω_l occurs with degeneracy 4 and the others with degeneracy 2, giving 8 frequencies for each l, forming four doublets $(2_{\epsilon\epsilon'})_l$ under the $SO(3)_{\epsilon'} \subset SO(4)$, where $SO(4)$ is the manifest part of the unbroken $SO(6)$.

3.4. Fermionic Oscillations of $M2$-Brane (Single-spin)

For the 1/2 supersymmetric single-spin giant gravitons the spectrum of the fermionic oscillations has not been examined before. In this subsection we fill this gap for a giant $M2$ in $AdS_7 \times S^4$. The calculation is very similar to the two-spin membrane studied above. Mainly one should take $r_0 \to 0$ limit and remember the fact that α is now equal to a constant which makes a difference

[7] There are also conjugate modes which have negative frequencies with the same degeneracy

when pulling back the objects to the world-volume. We find that only the first line of (3.23) contributes to (3.22). Also, (3.24) becomes

$$\tilde{\nabla}_\tau = \partial_\tau \; - \frac{1}{4L}\,\Gamma_t\gamma \; - \; \frac{1}{2L}\left[\sin\theta_0\,\Gamma^\phi \; + \; \cos\theta_0\Gamma^{12}\right]\Gamma^\theta, \tag{3.40}$$

and (3.25) is not modified. Moreover, in (3.26) one should set $r_0 = 0$ which shows that Γ projection condition on Θ can be solved as above. However, the mass matrix (3.30) changes to

$$\begin{aligned} M \;&=\; I_2 \otimes I_4 \otimes I_2 \otimes \left(\sigma_3\partial_\tau + \frac{i}{L}\sigma^i\nabla_i^{(1)}\right) \; - \; \frac{i}{4L}\,\sigma_2 \,\otimes\, \gamma_5 \,\otimes I_2 \otimes \sigma_3 \\ &+ \; \frac{i\sin\theta_0}{2L}\, I_2 \otimes I_4 \otimes \sigma_3 \otimes \sigma_3 + \frac{i\cos\theta_0}{2L}\, I_2 \otimes I_4 \otimes \sigma_2 \otimes \sigma_3. \end{aligned} \tag{3.41}$$

After expanding Θ in terms of spherical spinor harmonics on S^2 we obtain the eigenfrequencies ($l = 1/2, 3/2, \ldots$)

$$\omega_l = \frac{(l + 3/4)}{L}\;\{\epsilon' \times \epsilon = 1\}, \quad \omega_l = \frac{(l + 1/4)}{L}\;\{\epsilon' \times \epsilon = -1\}, \tag{3.42}$$

where each ω_l occurs on-shell with degeneracy 4, giving rise to two $SO(6) \simeq SU(4)$ spinors, that we shall denote by $(\bar{4})_l$ ($\epsilon\epsilon' = 1$) and $(4)_l$ ($\epsilon\epsilon' = -1$).

3.5. Fermionic Zero Modes and Spectrum

As for bosons, some of these fermionic oscillations correspond to the collective motion of the giant graviton (in spinor space) and should be removed from the spectrum. These are precisely the modes generated by the broken supersymmetries, i.e. $\Theta = (1 - \Gamma)\epsilon$ where ϵ is the space-time Killing spinor $\epsilon(X)$ in (2.19) evaluated on the membrane. Let us emphasize that such a mode does not necessarily obey the equations of motion (the spinor ϵ satisfies $\tilde{\nabla}_\beta\epsilon = 0$, however one may have $[\Gamma^\beta\tilde{\nabla}_\beta, \Gamma] \neq 0$), so one should directly examine the field equations to extract zero modes.

In our case, there is a short way to proceed; for a maximal giant ($\theta_0 = \pi/2$) we have $[M, \Gamma] = 0$ where M is the operator given in (3.41) and (3.30). Therefore $M(1 - \Gamma)\epsilon = (1 - \Gamma)M\epsilon = 0$, so the zero modes are given by the spinors ϵ obeying $(1 + \Gamma)\epsilon = 0$.

For the single-spin maximal giant, it is easy to show that $(1 \pm \Gamma)\epsilon = 0$ for $(1 \mp \Gamma^{t\phi})\epsilon_0 = 0$ where ϵ_0 is the constant spinor in (2.19). So, all 16 modes generated by the broken supersymmetries $\Gamma^{t\phi}\epsilon_0 = \epsilon_0$ are zero modes. Half of them obeying $\gamma\Gamma^\phi\epsilon_0 = \epsilon_0$ have the frequency $3/(4L)$ and other half have $-3/(4L)$. Comparing with (3.42), one finds that the $l = 1/2$ modes with $\epsilon'\epsilon = -1$ should be eliminated.

For the two-spin maximal giant, the projection $(1 + \Gamma)\epsilon = 0$ can be solved as in subsection 2.2. With the notation used in that subsection, one finds that the zero modes are generated by the spinors ϵ^{+-} and ϵ^{++} (here $\pm$ assignments on ϵ_0 are different than ϵ and ϵ' values). Decomposing further, the fermions generated by $\gamma\Gamma^\phi\,\epsilon^{+-} \;=\; \pm\,\epsilon^{+-}$ give the zero modes with the frequencies $\omega = \pm 1/(2L)$. On the other hand, the fermions generated by $\Gamma^{\alpha_4\alpha_5}\,\epsilon^{++} \;=\; \pm\,\epsilon^{++}$ give the zero modes with $\omega = \pm 1/L$. From (3.39), we see that the $l = 1/2$ modes with $\epsilon = 1, \epsilon' = -1$ and the modes with $\epsilon = -1$,$\epsilon' = 1$ should be removed from the spectrum.[8] Summarizing, we have the following eigenfrequencies after the elimination of the zero modes (in units of $1/L$ and where the suffices indicate $SO(3)_\chi$ spin and (ϵ, ϵ') assignments):

Fermionic ω_l $(l \geq 1/2)$	$l+1+\eta/4$	$l+1/2+\eta/4$	$l+3/2$
Multiplicity (two-spin) $\eta = 0$	$(2_{+-})_{l+1} + (2_{--})_l$	$(2_{++})_l$	$(2_{-+})_{l+1}$
Multiplicity (single-spin) $\eta = 1$	$(4)_{l+1}$	$(\bar{4})_l$	

3.6. Comments on the Supermultiplet Structure

The vibration spectra of single and two-spin 1/2 BPS membranes in $AdS_7 \times S^4$ can be arranged into multiplets of the unbroken supersymmetry algebras found in subsection 2.6. Here, we would like to comment on some salient features of this computation and give the multiplet for the single-spin solution.

Let us start with the bosonic fluctuations. For the two-spin solution, $(\delta r, \delta\alpha, \delta\theta, \delta\phi)$-sector of the bosonic quadratic Lagrangian can be diagonalized by introducing two complex fields $\delta z(\tau, \sigma^i)$ and $\delta w(\tau, \sigma^i)$. These fields must be complex due to the first order time-derivatives, and can be chosen as

$$\delta z = L(\delta\theta + i\delta\phi/\cos\theta_0) + \mathcal{O}(\cos\theta_0) \ , \qquad \delta w = \delta r + i\tilde{L}\delta\alpha + \mathcal{O}(\cos\theta_0) \ . \tag{3.43}$$

Since the frequencies are independent of θ_0, it suffices to consider the limit $\sin\theta_0 \to 1$ (keeping δz fixed). Performing the expansion using the harmonics Y_l on S^2 gives the frequencies as:

$$z_l(\tau) \ = \ e^{i\omega_l^+\tau} a_l^\dagger + e^{-i\omega_l^-\tau} b_l \ , \tag{3.44}$$
$$w_l(\tau) \ = \ e^{i\omega_l^+\tau} c_l^\dagger + e^{-i\omega_l^-\tau} d_l \ , \tag{3.45}$$

where $\omega_l^- = l/L$ and $\omega_l^+ = (l+1)/L$. Note the shift in the negative frequency relative to the positive frequency. The conjugate oscillators are contained in $(\delta z(\tau, \sigma^i))^\dagger$ and $(\delta w(\tau, \sigma))^\dagger$, which have the frequency parts

$$\bar{z}_l(\tau) \ = \ e^{i\omega_l^-\tau} b_l^\dagger + e^{-i\omega_l^+\tau} a_l \ , \tag{3.46}$$
$$\bar{w}_l(\tau) \ = \ e^{i\omega_l^-\tau} d_l^\dagger + e^{-i\omega_l^+\tau} c_l \ . \tag{3.47}$$

There is no corresponding shift in the real δy^m-fields, which transform as a 4-plet under $SO(4)$. For the single-spin solution, δr and $\delta\alpha$ perturbations combine with δy^m-fields to form a 6-plet under $SO(6)$. Here, there is no need to introduce the complex scalar $\delta\omega$ and the expansion of δz is identical to (3.44).

In the fermionic sector, firstly eq. (2.49) implies that $U(1)_S \times SO(4) \times U(1)_J \times SO(3)_\chi$ (two-spin) or $SO(6) \times U(1)_J \times SO(3)_\chi$ (single-spin) rotations of the unbroken supercharges Q are related by

[8] As shown above, for the single-spin solution the zero modes have the chiralities $\epsilon\epsilon' = -1$. As $r_0 \to 0$ the two-spin solution reduces to the single-spin one and thus its zero modes should have the same chiralities $\epsilon\epsilon' = -1$.

conjugation by $g(X_0)$ to rotations of Θ. The former representation is generated by the set of Dirac matrices found in the anti-commutator (see e.g. (2.51)), while the latter representation is diagonal with respect to the decomposition defined by $P^{(1)}_{\epsilon} \otimes P^{(2)}_{\epsilon'} \otimes P^{(3)}_{\epsilon\epsilon'} \otimes I_2$ introduced in Section 3.3 (such that $U(1)_S$ acts in the first slot or $SO(6)$ acts in the first two slots and so on in the indicated order). Hence, for the two-spin solution using $SO(4) = SO(3)_+ \times SO(3)_-$ and the notation where $(\alpha, \dot{a})$ are the doublet indices of $SO(3)_+ \times SO(3)_\chi$, the fermionic $(2_{\pm+})_l$ states are contained in the mode-expansion of a complex fermion $\Theta_\alpha(\tau, \sigma^i)$ and its hermitian conjugate $\bar{\Theta}_\alpha(\tau, \sigma^i)$, while the $(2_{\pm-})_l$ states are contained in $\Theta_{\dot{\alpha}}(\tau, \sigma^i)$ and $\bar{\Theta}_{\dot{\alpha}}(\tau, \sigma^i)$. Similarly, for the single-spin solution $(4)_{l+1}$ states are contained in the mode expansion of a complex $SU(4)$ 4-plet Θ_α and $(\bar{4})_l$ states are contained in $\bar{4}$-plet $\bar{\Theta}^\alpha$.

Next, we turn to the identification of the S and J charges. From the geometric sigma-model picture, it follows that a full $U(1)_J$-transformation generated by $J = J_{(0)} + J_{(1)} + J_{(2)} + \cdots$, decomposes into a constant shift of ϕ followed by a rotation (by the same angle) in the tangent space spanned by $(\delta\theta, \delta\phi)$. In the linearized theory the shift is generated by the zero-mode in $J_{(0)}$ while the tangent space rotation is generated by $J_{(2)}$. Similarly, $S_{(1)}$ shifts α, and $S_{(2)}$ rotates $(\delta r, \delta\alpha)$. Hence $J_{(2)}$ and $S_{(2)}$ generate symmetries of the quadratic action. Clearly, these charges can be computed from the normal coordinate expansion, though this is an unnecessarily tedious procedure, given the fact that the charges of all fluctuations are fixed (up to an overall sign) by supersymmetry.

Let us illustrate this for the single-spin solution. Recall that the spectrum now consists of a real $SO(6)$-vector $\delta y^{I'}$ $(I' = 1, \ldots, 6)$, a complex scalar δz $(J = 1)$, a 4-plet Θ_α $(J = -1/2)$ and a $\bar{4}$-plet $\bar{\Theta}^\alpha$ $(J = 1/2)$. As we will show, the supersymmetry is consistent with the charge assignments. Note that the positive frequencies in δz and $\delta\bar{z}$ are given by $\omega^+_l = (l+1)/L$ and $\omega^-_l = l/L$, respectively, and that the negative ones are shifted. In (2.53) we found unbroken supercharges Q^A_α $((E, J) = (-\frac{1}{4L}, \frac{1}{2}))$ and $\bar{Q}^\alpha_A$ $((E, J) = (\frac{1}{4L}, -\frac{1}{2}))$ transforming as $(4, 2)$ and $(\bar{4}, 2)$ under $SO(6) \times SO(3)_\chi$. The spectrum of non-zero modes now fits into a single tower $(l = 0, 1, 2, \ldots)$:

$$
\begin{aligned}
&(1_1; (l+1)_l) \xrightarrow{Q^\dagger} (\bar{4}_{1/2}; (l+5/4)_{l+1/2}) \xrightarrow{Q^\dagger} (6_0; (1+3/2)_{l+1}) \\
&\xrightarrow{Q^\dagger} (4_{-1/2}; (l+7/4)_{l+3/2}) \xrightarrow{Q^\dagger} (1_{-1}; (1+2)_{l+2}),
\end{aligned}
\tag{3.48}
$$

where the quantum numbers are listed as $(R_J; (L\omega)_l)$, where R is $SO(6)$ irreps.

There remains the following zero-mode oscillators:

$$
b^\dagger(\omega = 0)\ , \qquad y^\dagger_{I'}(\omega = 1/2)\ , \qquad \Theta^{\dagger\alpha}_A(\omega = 3/4)\ , \qquad b^\dagger_{AB}(\omega = 1)\ . \tag{3.49}
$$

Since $b^\dagger$ has vanishing frequency and $SO(6) \times SO(3)_\chi$ charges, it is a supersymmetry singlet. The remaining states form a multiplet (with 9 bosons and 8 fermions) with supercharge and bosonic generators given by

$$
\begin{aligned}
Q_{\alpha A} &= \frac{1}{\sqrt{L}} \left(2b^{\dagger}{}_A{}^B \Theta_{\alpha C} + (\gamma^{I'})_{\alpha\beta} y_{I'} \bar{\Theta}^{\dagger\beta}_A \right)\ , \\
H' &= \frac{1}{L} \left(b^{\dagger AB} b_{AB} + \frac{3}{4} \bar{\Theta}^{\dagger\alpha}_A \Theta^A_\alpha + \frac{1}{2} y^\dagger_{I'} y_{I'} \right)\ , \\
\widehat{\mathcal{M}}_{I'J'} &= -\left(2y^\dagger_{[I'} y_{J']} + \frac{1}{2} \bar{\Theta}^{\dagger\alpha}_A (\gamma_{I'J'})_\alpha{}^\beta \Theta^A_\beta \right)\ , \\
L_{AB} &= 2i \left(2b^{\dagger}_{(A}{}^C b_{B)C} + \bar{\Theta}^{\dagger\alpha}_{(A} \Theta_{B)\alpha} \right).
\end{aligned}
\tag{3.50}
$$

For the two-spin solution a similar analysis can be repeated, and it would be interesting to examine how the similarity transformation (2.31) connects the two multiplet structures.

4. GENERAL SPHERICAL GIANTS

In this section we examine the most general ansatz for a spherical giant configuration. After identifying the brane directions in space-time, the spherical symmetry implies that all coordinate fields depend only on time. Hence, while their shape is fixed, these branes can "breathe" and perform point-like motion in the remaining directions. As we shall see, the whole system consists of a warped product of a breathing mode and a relativistic point particle on S^n (for the branes expanding in AdS) or in $AdS_m \times S^1$ (for the branes expanding in the sphere) with an additional velocity dependent potential on S^1. The warping means that the point particle motion decouples from the breathing in a suitable world-volume time. We show that the equations of motion are integrable. This can easily be verified using flat embedding coordinates where the solution takes a simple form. In terms of the usual spherical or AdS coordinates, one obtains first order equations that are nested such that they can be integrated further one by one. Moreover, switching to canonical fields leads to complete separation of variables, with the emergence of relatively simple potentials, known as Pöschl-Teller Type I and II, for all point-particle coordinates, while the breathing mode is governed by a seemingly more complicated potential.

We also derive BPS bounds for the energy as a function of the constants of motion, and show that these are saturated only by the 1/2 supersymmetric single-spin and the two-spin giants found in section 2. The quantization of these objects and some comparisons with the CFT side are discussed in subsections 4.3 and 4.4.

To facilitate the analysis we use the AdS metric

$$ds^2 = \tilde{L}^2 \left(-\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, d\Omega_{m-2}^2\right) + L^2 d\Omega_n^2\,, \tag{4.1}$$

where the line-elements on the unit spheres are specified in more detail below. The t coordinate is now dimensionless, which brings an extra $\tilde{L}$ factor in time-derivatives when compared to some of our previous results.

4.1. Branes Expanding in AdS (Electric)

In this case the background supports an m-form field strength which is given in the tangent space basis by $H_{\hat{r}\hat{t}\hat{\alpha}_1..\hat{\alpha}_{m-2}} = (m-1)/\tilde{L}$ (see also footnote 2). We choose the static gauge and identify the $(m-2)$-brane world-volume coordinates σ_i with the coordinates of S^{m-2} in (4.1):

$$t = \sigma_0 = \tau, \qquad \alpha_1 = \sigma_1,\ \dots\ , \alpha_{m-2} = \sigma_{m-2}\,. \tag{4.2}$$

We write the metric on the unit n sphere in (4.1) as

$$d\Omega_n^2 = G_{ab}\, d\phi^a\, d\phi^b, \qquad a, b = 1, .., n, \tag{4.3}$$

and assume a solution of the form:

$$r = r(\tau), \qquad \phi^a = \phi^a(\tau). \tag{4.4}$$

This is a generalization of the dual giant configurations found in [3, 4]. The pull-back of the spacetime metric to the world-volume is given by:

$$\gamma_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N g_{MN} = \begin{pmatrix} -\Delta^2 & 0 \\ 0 & \tilde{L}^2 \sinh^2 r \ (g^\alpha)_{mn} \end{pmatrix} \tag{4.5}$$

where $(g^\alpha)_{mn}$ is the metric on the unit $(m-2)$-sphere and

$$\Delta^2 = \tilde{L}^2 (\cosh^2 r - \dot{r}^2) - L^2 G_{ab} \dot{\phi}^a \dot{\phi}^b. \tag{4.6}$$

In this section, dot always denotes derivative with respect to τ. In (2.2) for $M = \alpha_m$ the equations are obeyed trivially and for $M = t$ one gets

$$\Delta = \frac{\tilde{L} (\sinh r)^{m-2} \ \cosh^2 r}{(\sinh r)^{m-1} + k}, \tag{4.7}$$

where k is an integration constant.

Using (2.1), the remaining field equations can be consistently derived from a truncated action

$$S = -\frac{\tilde{N}}{\tilde{L}} \int d\tau \left[(\sinh r)^{m-2} \Delta - \tilde{L} (\sinh r)^{m-1} \right] , \tag{4.8}$$

which gives

$$\partial_\tau \left[\frac{\tilde{L} (\sinh r)^{m-2} \dot{r}}{\Delta} \right] = \cosh r (\sinh r)^{m-3} \left[(m-1) \sinh r - \frac{(m-2)\Delta}{\tilde{L}} - \frac{\tilde{L} (\sinh r)^2}{\Delta} \right] \tag{4.9}$$

$$\partial_\tau \left[\frac{(\sinh r)^{m-2} \ G_{ab} \ \dot{\phi}^b}{\Delta} \right] = \frac{(\sinh r)^{m-2}}{2\Delta} \partial_a G_{bc} \, \dot{\phi}^b \, \dot{\phi}^c \, . \tag{4.10}$$

In (4.8), we introduced $\tilde{N}$ so that $T_p A_p = \tilde{N}/\tilde{L}^{m-1}$. In particular, for $(m,n) = \{(5,5),(4,7),(7,4)\}$ we have $\tilde{N} = N$, $\tilde{N} = \sqrt{N/2}$ and $\tilde{N} = 2N^2$, respectively. Using (4.7) in (4.28), one finds

$$E = \tilde{N} k. \tag{4.11}$$

As expected, k is related to AdS energy since it arises from fixing the time-reparametrization by the static gauge choice $t = \tau$.

To integrate the equations, we introduce flat $(n+1)$-dimensional coordinates x^A with the constraint $x^A x^A = 1$. Then the truncated action (4.8) can be rewritten using a Lagrange multiplier

$$S = -\frac{\tilde{N}}{\tilde{L}} \int d\tau \left[(\sinh r)^{m-2} \Delta - \tilde{L} (\sinh r)^{m-1} + L^2 \Lambda (x^A x^A - 1) \right] , \tag{4.12}$$

where now $\Delta^2 = \tilde{L}^2 (\cosh^2 r - \dot{r}^2) - L^2 \dot{x}^A \dot{x}^A$. The field equations for x^A are

$$\partial_\tau \left[\frac{(\sinh r)^{m-2} \dot{x}^A}{\Delta} \right] = -2\Lambda \, x^A, \qquad x^A \, x^A = 1 \, . \tag{4.13}$$

By contracting (4.13) with x^A, it is easy to see that $(\sinh r)^{m-2}\Lambda/\Delta = \Lambda_0$, where $\Lambda_0 > 0$ is a constant. The sphere part can be decoupled from the *breathing mode* r by introducing a new time coordinate

$$d\tilde{\tau} = \frac{\Delta}{(\sinh r)^{m-2}}\, d\tau. \tag{4.14}$$

In terms of $\tilde{\tau}$, (4.10) is equivalent to a point particle moving on S^n. Then, eq. (4.13) becomes

$$\frac{d^2x^A}{d\tilde{\tau}^2} = -2\,\Lambda_0\, x^A, \tag{4.15}$$

which can be solved as

$$x^A = x_1^A \cos(\sqrt{2\Lambda_0}\tilde{\tau}) + x_2^A \sin(\sqrt{2\Lambda_0}\tilde{\tau}), \tag{4.16}$$

where x_1^A and x_2^A are constants. Imposing $x^Ax^A = 1$ yields $x_1^Ax_1^A = x_2^Ax_2^A = 1$ and $x_1^Ax_2^A = 0$. With these constraints the total number of integration constants $(x_1^A, x_2^A, \Lambda_0)$ is $2n$. To determine the moduli space of the solutions, note that x_1^A defines an n-sphere. Being perpendicular to x_1^A and having unit length, x_2^A defines an $(n-1)$-sphere for each x_1^A. So, the moduli space of the solutions is R^+ (corresponding to Λ_0) times an S^{n-1} bundle over S^n.

The conserved Noether charges for the $SO(n+1)$ invariance of the action can easily be determined from (4.12) to be

$$J^{AB} = \frac{2\Lambda_0\tilde{N}L^2}{\tilde{L}}\left[x_1^Ax_2^B - x_1^Bx_2^A\right]. \tag{4.17}$$

Therefore, together with all $[(n+1)/2]$ Cartan generators, the non-commuting components of J_{AB} can also be activated.

Now let us study the integrability of equations using an explicit metric on S^n. One preferable choice is

$$d\Omega_n^2 = d\phi_1^2 + \cos^2\phi_1\, d\phi_2^2 + \sin^2\phi_1\left[d\phi_3^2 + \cos^2\phi_3^2 d\phi_4^2 + \sin^2\phi_3\left(... + \sin^2\phi_{n-1}d\phi_n^2\right)\right], \tag{4.18}$$

so that all Cartan generators of $SO(n+1)$ are manifestly realized as translations along the cyclic coordinates ϕ_2, ϕ_4,..,ϕ_n. Here, the non-cyclic coordinates are defined in the interval $[0, \pi/2]$. One may also consider

$$d\Omega_n^2 = d\phi_1^2 + \sin^2\phi_1\left[d\phi_2^2 + \sin^2\phi_2(... + \sin^2\phi_{n-1}d\phi_n^2)\right], \tag{4.19}$$

where S^n is parametrized as nested lower dimensional spheres. Here, only ϕ_n is cyclic and others are defined in $[0, \pi]$. It is also possible to take combinations of (4.18) and (4.19). Now, (4.10) can be integrated one by one in the order $(\phi_n, \phi_{n-1}, .., \phi_1)$ which yields

$$\dot{\phi}_a^2 = \frac{\Delta^2}{L^2(\sinh r)^{2m-4}G_{aa}^2} \times \begin{cases} q_a^2 & \text{if } \phi_a \text{ is cyclic} \\ q_a^2 - \frac{q_{a+1}^2}{\cos^2\phi_a} - \frac{q_{a+2}^2}{\sin^2\phi_a} & \text{if } \phi_a \in [0, \pi/2], \\ q_a^2 - \frac{q_{a+1}^2}{\sin^2\phi_a} & \text{if } \phi_a \in [0, \pi], \end{cases} \tag{4.20}$$

(no summation is implied in G_{aa}) where q_a's are dimensionless integration constants with $q_{n+1} = 0$. The motion along the cyclic coordinates is monotonic. A non-cyclic coordinate has two turning points where the time derivatives vanish which are fixed by the constants q_a. The positivity of the velocity-squares imply

$$q_a \geq q_{a+1} + q_{a+2}, \quad (\phi_a \in [0, \pi/2]) \quad \text{ or } \quad q_a \geq q_{a+1} \quad (\phi_a \in [0, \pi]). \tag{4.21}$$

Above we assumed that all ϕ_a depend non-trivially on τ. Otherwise, one has to analyze the field equations to find out the implications (see below for details). The metrics (4.18) or (4.19) have coordinate singularities and they are not globally well-defined on S^n. Thus (4.20) does not cover the whole solution space and q_a's are not globally well-defined moduli coordinates unlike the constants x_1^A and x_2^A.

In order to exhibit the integrable structure more clearly, one can express the field equations using the canonical momenta derived from the action (4.8), which gives

$$P_a^2 = \left(\frac{\tilde{N}L}{\tilde{L}}\right)^2 \times \begin{cases} q_a^2 & \text{if } \phi_a \text{ is cyclic,} \\ q_a^2 - \frac{q_{a+1}^2}{\cos^2\phi_a} - \frac{q_{a+2}^2}{\sin^2\phi_a} & \text{if } \phi_a \in [0, \pi/2], \\ q_a^2 - \frac{q_{a+1}^2}{\sin^2\phi_a} & \text{if } \phi_a \in [0, \pi]. \end{cases} \tag{4.22}$$

These equations define a canonical transformation $(\phi^a, P_a) \to (\phi^a, q_a)$, such that in the new variables the equations of motion are $\dot{q}_a = 0$ and $\dot{\phi}^a$ is given by (4.20). Clearly, q_a's corresponding to cyclic coordinates are related to Cartan symmetry generators of $SO(n+1)$. Other q_a's are "hidden" charges from the point of view of the sigma model written with the metric (4.18) or (4.19) (for example by evaluating Δ, one can see that $q_1^2 = 2L^2\Lambda_0$).

The potential that appears in (4.22) is of Pöschl-Teller Type I (for a review see, e.g., [27]). These belong to a large class, known as shape invariant potentials [26], which arises naturally in supersymmetric quantum mechanics and can be solved exactly, i.e. the energy eigenvalues, eigenfunctions as well as the scattering matrix can be given explicitly. We shall come back to this later in subsection 4.3.

Let us now return to the radial equation which can be fixed using (4.6), (4.7) and (4.20) as

$$\frac{1}{2}\dot{r}^2 + V(r) = 0, \tag{4.23}$$

where

$$V(r) = \frac{\cosh^2 r}{2[(\sinh r)^{m-1} + k]^2}\left[(q_1^2 - k^2)\cosh^2 r \, + \, (k - \sinh^{m-3} r)^2 \sinh^2 r\right]. \tag{4.24}$$

Eq. (4.23) is equivalent to one dimensional motion of a particle in the effective potential $V(r)$ with zero total energy. In terms of canonical momenta one has

$$\frac{1}{2}P_r^2 \, + \, \frac{\tilde{N}^2}{2\cosh^2 r}\left[(q_1^2 - k^2)\cosh^2 r \, + \, (k - \sinh^{m-3} r)^2 \sinh^2 r\right] = 0. \tag{4.25}$$

Unlike the angular part, the potential in (4.25) is rather complicated.

The motion is allowed in the region $V \leq 0$ which implies

$$k \geq q_1. \tag{4.26}$$

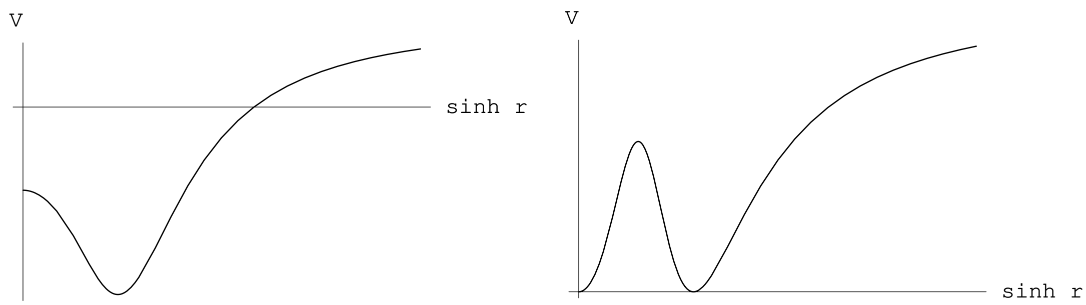

FIG. 2: The potential $V(r)$ in (4.24) for $k > q_1$ (on the left) and for $k = q_1$ (on the right).

For $k > q_1$, $V = 0$ has a single root at some $r = r_0$ (fixed by k, q_1 and m) and $V > 0$ for $r > r_0$ and $V < 0$ for $r < r_0$ (see figure 2). When the brane is given a small kick at $r < r_0$ (a kick is necessary since the total energy in this motion is zero), it either climbs the hill and reaches to $r = r_0$ and rolls back to hit $r = 0$ or directly moves through $r = 0$. In either case the brane totally collapses in finite amount of time and re-expands again. When $k = q_1$, the brane should be placed at a constant radial distance either at $r = 0$ or at $(\sinh r)^{m-3} = k$ where $V = 0$ (see figure 2).

As mentioned above, when a coordinate is set to a constant in the solution one should check the field equations for its consequences. These can be summarized as

(1) $r = r_0$: From (4.23) one finds $r_0 = 0$ or $(\sinh r_0)^{m-3} = k$. There is also a third root in (4.9) with $k = (m-2)(\sinh r_0)^{m-3} + (m-3)(\sinh r_0)^{m-1}$ which is not seen in figure 2. One should also set $q_1 = k$.

(2) $\phi_a = \phi_0$ (non-cyclic): This is only possible when $\phi_a = \{0, \pi/2, \pi\}$; or when $\dot{\phi}_b = 0$, $b > a$ for $\phi_a \in [0, \pi]$; or when all non-cyclic $\dot{\phi}_b = 0$, $b > a$ and the cyclic motion in $b > a$ is confined in a single plane defined with $\phi_a = \phi_0$ for $\phi_a \in [0, \pi/2]$. One can still use (4.20) by choosing the integration constants appropriately (for example if $\phi_a \in [0, \pi]$, then $q_a \sin^2 \phi_a = q_{a+1}$.)

(3) $\phi_a = \phi_0$ (cyclic): One should set the corresponding integration constant to zero.

With these taken into account, (4.20) and (4.23) constitute our most general configuration. Now, we would like to indicate some special solutions. One can for example set $r = r_0$, $\theta = 0$, $\phi_a = \pi/2$ (i.e. all $q_a = 0$). Here, one has to take $r_0 \to \infty$ and this is the "p-brane at the end of the universe" [31]. Now if we let $\phi_n = \phi_n(\tau)$ then we get the 1/2 supersymmetric dual giant studied in [3, 4]. in this case, we have $q_1^2 = k^2 = q_n^2 = (\sinh r_0)^{2m-6}$ which implies $\dot{\phi}_n = \tilde{L}/L$. (This solution was previously obtained for $AdS_4 \times S^7$ in [38] where S^7 is parametrized as a U(1) bundle over CP^3.) A further modification of this configuration is to let $r = r(\tau)$ which was considered in [28].

Finally, we would like to check whether any of the above solutions preserve some supersymmetry. For that purpose we derive a BPS bound for energy whose saturation is a necessary condition. Actually, using the inequalities between the integration constants (4.21), (4.26), and the conserved charges (4.11), (4.22), one already finds

$$H \geq \frac{\tilde{L}}{L} \sum_a P_a, \tag{4.27}$$

where the sum is over cyclic coordinates. Another way of deriving the same result is to analyze the Hamiltonian obtained from (4.8) as

$$\begin{aligned} H = P_{m'} \dot{X}^{m'} - \mathcal{L} &= \frac{\tilde{N}\tilde{L}(\sinh r)^{m-2}\cosh^2 r}{\Delta} - \tilde{N}(\sinh r)^{m-1}, \\ &= \cosh r \left[\tilde{L}^2 P^2 + \tilde{N}^2 (\sinh r)^{2m-4}\right]^{1/2} - \tilde{N}(\sinh r)^{m-1}, \end{aligned} \tag{4.28}$$

where $P^2 \equiv P_r^2/\tilde{L}^2 + G^{ab} P_a P_b / L^2$. Eq. (4.28) can be rewritten as

$$H = \left[(\tilde{L}P + \tilde{N}(\sinh r)^{m-1})^2 + (\tilde{L}P \sinh r - \tilde{N}(\sinh r)^{m-2})^2\right]^{1/2} - \tilde{N}(\sinh r)^{m-1}. \tag{4.29}$$

This gives $H \geq \tilde{L}P$ and thus

$$H \geq \frac{\tilde{L}}{L} \sum_a \left[G^{aa} P_a P_a\right]^{1/2}, \tag{4.30}$$

where the sum is over cyclic directions. Repeatedly using the inequality

$$\left[\frac{A^2}{\cos^2\theta} + \frac{B^2}{\sin^2\theta}\right]^{1/2} \geq A + B \ , \tag{4.31}$$

one finds (4.27).

To saturate the bound (4.27), all other momenta have to vanish. In this case, a detailed investigation of the field equations shows that *all* rotations have the same angular velocity and the circular motion is actually confined in a single plane. This leads us back to the single spin solution after a global $SO(n+1)$ rotation. Therefore, the only supersymmetric spherical brane expanding inside AdS is the 1/2 BPS single-spin giant graviton.

4.2. Branes Expanding in Sphere (Magnetic)

In this case the background supports an n-form field strength given by $H_{\hat{\theta}\hat{\phi}\hat{\chi}_1 \cdots \hat{\chi}_{n-2}} = (n-1)/L$ in the orthonormal basis (see footnote 2). We rewrite (4.1) as

$$ds^2 = \tilde{L}^2 \, G_{\mu\nu} dy^\mu dy^\nu + L^2 d\Omega_n^2, \qquad \mu, \nu = 0, .., m-1, \tag{4.32}$$

where $G_{\mu\nu}$ is the metric on the unit AdS_m, $y^0 = t$ and the sphere parametrization is given in (2.4)

$$d\Omega_n^2 = d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta \left[d\chi_1^2 + \sin^2\chi_1(... + \sin^2\chi_{n-3} d\chi_{n-2}^2)\right] \ . \tag{4.33}$$

We identify the world-volume coordinates with

$$t = \sigma_0 = \tau, \qquad \chi_1 = \sigma_1, \ ..., \ \chi_{n-2} = \sigma_{n-2}, \tag{4.34}$$

and assume a solution of the form:

$$\phi = \phi(\tau), \qquad \theta = \theta(\tau), \qquad y^\mu = y^\mu(\tau). \tag{4.35}$$

The induced metric becomes

$$\gamma_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N g_{MN} = \begin{pmatrix} -\Delta^2 & 0 \\ 0 & L^2 \sin^2\theta \; (g^\chi)_{mn} \end{pmatrix}, \tag{4.36}$$

where $(g^\chi)_{mn}$ is the metric on the unit $(n-2)$-sphere and

$$\Delta^2 = -\tilde{L}^2 \, (G_{\mu\nu} \dot{y}^\mu \dot{y}^\nu) - L^2 \, (\dot{\theta}^2 + \dot{\phi}^2 \cos^2\theta). \tag{4.37}$$

Eq. (2.2) is satisfied trivially for $M = \chi_i$, and $M = t$ component fixes the on-shell value of Δ as

$$\Delta = k \, \tilde{L} \, (\sin\theta)^{n-2} \cosh^2 r, \tag{4.38}$$

where k is an integration constant.

Using (2.1), the remaining equations can be obtained from the following one dimensional truncated action

$$S = -\frac{N}{L} \int d\tau \left[(\sin\theta)^{n-2} \Delta - L \, \dot{\phi} \, (\sin\theta)^{n-1} \right] \,, \tag{4.39}$$

which gives

$$\partial_\tau \left[\frac{L \, (\sin\theta)^{n-2} \cos^2\theta \, \dot{\phi}}{\Delta} \right] = -\partial_\tau (\sin\theta)^{n-1}, \tag{4.40}$$

$$\partial_\tau \left[\frac{L \, (\sin\theta)^{n-2} \, \dot{\theta}}{\Delta} \right] = \cos\theta (\sin\theta)^{n-3} \left[(n-1) \sin\theta \, \dot{\phi} - \frac{(n-2)\Delta}{L} - \frac{L \, \sin^2\theta}{\Delta} \dot{\phi}^2 \right], \tag{4.41}$$

$$\partial_\tau \left[\frac{(\sin\theta)^{n-2} \, G_{\mu\nu} \, \dot{y}^\nu}{\Delta} \right] = \frac{(\sin\theta)^{n-2}}{2\Delta} \, \partial_\mu \, G_{\nu\rho} \, \dot{y}^\nu \dot{y}^\rho. \tag{4.42}$$

Using the on-shell value of Δ (4.38) in (4.62) one gets

$$E = \frac{N\tilde{L}}{kL}. \tag{4.43}$$

As before the integration constant k is related to AdS energy.

Similar to the previous subsection, one can introduce a new world-volume time coordinate as follows:

$$d\tilde{\tau} = \frac{\Delta}{(\sin\theta)^{n-2}} \, d\tau. \tag{4.44}$$

In terms of $\tilde{\tau}$, the *breathing mode* θ and ϕ coordinate are decoupled from the rest and (4.42) corresponds to a point particle moving in AdS_m. On the other hand, (4.40) and (4.41) give motion on S^2 with an additional velocity dependent potential.

Let us first analyze the AdS part. If Y_P are $(m+1)$-dimensional embedding coordinates, then the AdS space is defined by $\eta^{PQ} Y_P Y_Q + 1 = 0$, where $\eta_{PQ} = (-1, -1, +1, .., +1)$. We take all Y_P

to be dynamical (namely, we are not imposing the static gauge). Using a Lagrange multiplier, the action (4.39) can be rewritten as

$$S=-\frac{N}{L}\int d\tau\left[(\sin\theta)^{n-2}\Delta-L\,\dot{\phi}\,(\sin\theta)^{n-1}+\tilde{L}^2\Lambda(\eta^{PQP}Y_PY_Q+1)\right]. \tag{4.45}$$

where now $\Delta^2=-\tilde{L}^2\,\eta^{PQ}\dot{Y}_P\dot{Y}_Q-L^2\,(\dot{\theta}^2+\dot{\phi}^2\cos^2\theta)$ and dot again denotes differentiation with respect to τ. Varying for Y_P and Λ one obtains

$$\partial_\tau\left[\frac{(\sin\theta)^{n-2}\,\dot{Y}^P}{\Delta}\right]=-2\Lambda\,Y^P,\qquad \eta^{PQ}Y_PY_Q=-1. \tag{4.46}$$

Contracting (4.46) with Y_P, one finds $(\sin\theta)^{n-2}\Lambda/\Delta=\Lambda_0$, where Λ_0 is an arbitrary real number. Then (4.46) becomes

$$\frac{d^2Y^P}{d\tilde{\tau}^2}=-2\,\Lambda_0\,Y^P, \tag{4.47}$$

which can be integrated in terms of elementary functions. The solution space consists of three disjoint parts parametrized by $\Lambda_0<0$, $\Lambda_0=0$ and $\Lambda_0>0$. The global $SO(2,m)$ charges can be calculated as

$$\begin{aligned}S_{PQ}&=\frac{\tilde{L}^2N(\sin\theta)^{n-2}}{L\Delta}\left(Y_P\dot{Y}_Q-Y_Q\dot{Y}_P\right)\\&=\frac{\tilde{L}^2N}{L}\left(Y_P\partial_{\tilde{\tau}}Y_Q-Y_Q\partial_{\tilde{\tau}}Y_P\right),\end{aligned} \tag{4.48}$$

which is clearly conserved by (4.47). Note that the AdS energy obtained in this way will in general differ from (4.43), since the later was calculated in the static gauge.

It is interesting to analyze the integrability of the equations using local coordinates on AdS

$$G_{\mu\nu}dy^\mu dy^\nu=-\cosh^2r\,dt^2+dr^2+\sinh^2r\,G_{ij}\,d\alpha^id\alpha^j, \tag{4.49}$$

where G_{ij} is the metric on unit S^{m-2}. One can solve the equations (4.42) starting from the spherical part which is essentially the same problem worked in the previous subsection. Using the parametrizations given in (4.19) or (4.18), the result is

$$\dot{\alpha}_i^2=\frac{k^2\cosh^4r}{G_{ii}^2\sinh^4r}\times\begin{cases}q_i^2 & \text{if } \alpha_i \text{ is cyclic}\\ q_i^2-\frac{q_{i+1}^2}{\cos^2\alpha_i}-\frac{q_{i+2}^2}{\sin^2\alpha_i} & \text{if } \alpha_i\in[0,\pi/2]\\ q_i^2-\frac{q_{i+1}^2}{\sin^2\alpha_i} & \text{if } \alpha_i\in[0,\pi],\end{cases} \tag{4.50}$$

(no summation in G_{ii}) and

$$\dot{r}^2=\cosh^4r\left[-q^2-\frac{q_1^2\,k^2}{\sinh^2r}+\frac{1}{\cosh^2r}\right] \tag{4.51}$$

where q and q_i $(q_{m-1}=0)$ are dimensionless constants of motion which should obey

$$\begin{aligned}&1\geq q_1\,k+q,\\&q_i\geq q_{i+1}+q_{i+2},\quad(\alpha_i\in[0,\pi/2])\quad\text{ or }\quad q_i\geq q_{i+1}\quad(\alpha_i\in[0,\pi]).\end{aligned} \tag{4.52}$$

The non-cyclic α_i and the coordinate r pulsate between two turning points fixed by q and q_i. The canonical momenta for the angular coordinates α_i are given by

$$P_i^2 = \left(\frac{N\tilde{L}}{L}\right)^2 \times \begin{cases} q_i^2 & \text{if } \alpha_i \text{ is cyclic,} \\ q_i^2 - \frac{q_{i+1}^2}{\cos^2\alpha_i} - \frac{q_{i+2}^2}{\sin^2\alpha_i} & \text{if } \alpha_i \in [0, \pi/2], \\ q_i^2 - \frac{q_{i+1}^2}{\sin^2\alpha_i} & \text{if } \alpha_i \in [0, \pi], \end{cases} \tag{4.53}$$

On the other hand (4.51) become

$$P_r^2 = \left(\frac{N\tilde{L}}{kL}\right)^2 \left[-q^2 - \frac{q_1^2 k^2}{\sinh^2 r} + \frac{1}{\cosh^2 r}\right] . \tag{4.54}$$

The potential for P_r in (4.54) is of Pöschl-Teller Type II [26].

Returning to the θ and ϕ coordinates, we see that (4.40) readily determines ϕ

$$\dot{\phi} = \frac{k\,\tilde{L}\cosh^2 r}{L\cos^2\theta}\,[p - (\sin\theta)^{n-1}], \tag{4.55}$$

where p is an integration constant. On the other hand, θ can be fixed from (4.37) and (4.38) which yields

$$\frac{1}{2}\,\dot{\theta}^2 + V(\theta) = 0, \tag{4.56}$$

where

$$V(\theta) = \frac{\tilde{L}^2\cosh^4 r}{2L^2}\left[k^2p^2 - q^2 + \frac{k^2\sin^2\theta}{\cos^2\theta}(p - \sin^{n-3}\theta)^2\right]. \tag{4.57}$$

This is a one-dimensional motion in the potential $V(\theta)$ with zero total energy. The requirement $V(\theta) \leq 0$ implies

$$q \geq kp. \tag{4.58}$$

When expressed in terms of momenta, (4.55) and (4.56) become

$$P_\phi = Np, \tag{4.59}$$

$$P_\theta^2 = \frac{N^2}{k^2}\left[k^2p^2 - q^2 + \frac{k^2\sin^2\theta}{\cos^2\theta}(p - \sin^{n-3}\theta)^2\right]. \tag{4.60}$$

The separation in canonical variables is manifest in (4.53), (4.54), (4.59) and (4.60). Unlike the exactly solvable potentials we encountered above, the potential on the right hand side of (4.60) is more complicated.

Let us now discuss the motion for the breathing coordinate θ in more detail. When $q < kp$, $V(\theta)$ is nowhere negative or zero. For $q = kp$, $V(\theta) = 0$ only at $\theta = 0$ and at $(\sin\theta)^{n-3} = p$ and $V(\theta) > 0$ otherwise (see figure 3). This enforces the brane to locate at either root and note that the second zero exists only for $p \leq 1$. When $q > kp$ there are two possibilities. Firstly, if $p = 1$ $V(\theta)$ is always negative and in this case θ reaches 0 or $\pi/2$ in a finite amount of time. Secondly, if $p \neq 1$ there is a root θ_0 so that $V(\theta) \leq 0$ for $\theta \leq \theta_0$ and $V(\theta) > 0$ for $\theta > \theta_0$ (such a potential is

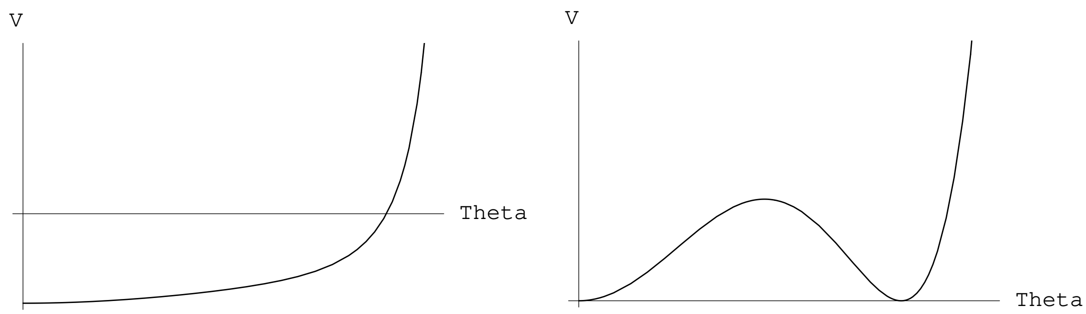

FIG. 3: The potential $V(\theta)$ in (4.57) for $q < kp$ (on the left) and for $q = kp$ (on the right).

drawn in figure 3). In this case, the brane either contracts directly to $\theta = 0$ or it expands till $\theta = \theta_0$ and then collapses to $\theta = 0$ in a finite amount of time. Note that $V(\theta)$ is scaled by $\tilde{L}^2 \cosh^4 r/L^2$ which does not alter the zeroes but affect the shape of the potential when r is time dependent.

Equations (4.50) and (4.51) hold under the assumption that the coordinates have non-trivial time dependences. Otherwise the original field equations may impose further restrictions. These can be summarized as:

(1) $\phi = \phi_0$: From (4.40) this happens only when $\theta = \theta_0$ and from (4.41) one has $\theta_0 = 0, \pi/2$. The constant p should be fixed as $p = 1$.

(2) $\theta = \theta_0$: This requires $q = kp$ and from (4.42) one finds $(\sin\theta_0)^{n-3} = p,\ \theta = 0,$ or $\theta = \pi/2$. There is also a fourth root with $p = (n-2)(\sin\theta_0)^{n-3} - (n-3)(\sin\theta_0)^{n-1}$.

(3) $r = r_0$: From (4.41) and (4.51) we see that $q^2 = 1/\cosh^4 r_0$ and $q_1 k = \sinh^2 r_0/\cosh^2 r_0$.

(4) $\alpha_i = \alpha_0$: This is exactly the same with the special cases (2) and (3) discussed in the previous subsection.

Upto these four cases, (4.50), (4.51), (4.55) and (4.56) give the most general configuration. The 1/2 supersymmetric two-spin solution (2.6) is obtained when we have $\alpha_1 = ... = \alpha_{m-3} = \pi/2$ and $\theta = \theta_0,\ r = r_0$. Using the above conditions one finds $q_1 = ... = q_{m-2},\ p = (\sin\theta)^{n-3}$, $q = kp = 1/\cosh^2 r$ and $q_1 k = \sinh^2/\cosh^2 r$ which imply from (4.55) and (4.50) that $\dot{\phi} = \tilde{L}/L$ and $\dot{\alpha}_{m-2} = 1$ (recall that here t is dimensionless). If we further set $\alpha_{m-2} = \pi/2$, then $q_1 = .. = q_{m-2} = 0$ which gives $r = 0$. This is the 1/2 supersymmetric giant graviton solution of [1]. For this case, $\theta = \theta(\tau)$ is studied in [28]. There is also the trivial solution with $r = 0$, $\theta = \pi/2$ and all other coordinates are set to constants.

Let us conclude this subsection by deriving a BPS bound for the energy given the conserved charges. This can readily be obtained using the inequalities between the integration constants (4.52), (4.58), and the expressions for conserved quantities (4.43), (4.53), (4.59) which gives

$$H \geq \frac{\tilde{L}}{L} P_\phi + \sum_i P_i, \tag{4.61}$$

where the sum is over cyclic coordinates. To obtain the same result in a different way, the Hamiltonian can be found from (4.39) as

$$
\begin{aligned}
H = P_{m'}\dot{X}^{m'} - \mathcal{L} \; &= \; \frac{\tilde{L}^2 N(\sin\theta)^{n-2}\cosh^2 r}{L\Delta}, \\
&= \; \tilde{L}\cosh r\left[P^2 + \frac{N^2}{L^2}(\sin\theta)^{2n-4}\right]^{1/2}, \qquad (4.62)
\end{aligned}
$$

where $P^2 \equiv P_\phi^2/(L^2\cos^2\theta) + G^{\mu\nu}P_\mu P_\nu/L^2$. The last term in (4.62) can be combined with $P_\phi^2/(L^2\cos^2\theta)$ to give P_ϕ^2/L^2 plus an exact square as in (2.11). Then, by a calculation similar to the one done in (2.12), and using the inequality (4.31) repeatedly, the Hamiltonian (4.62) can be shown to subject to (4.61).

This BPS bound can only be realized when all other momenta except P_ϕ and cyclic P_i vanish which enforce the corresponding coordinates to be constants. However, as in the previous subsection, one can see that the circular motion in S^{m-2} is along the equator, which is equivalent to the 1/2 supersymmetric two-spin solution. This reduces to the single-spin case when the radius of the embedded S^{m-2} goes to zero.

4.3. Quantization of Spherical Giants

We have examined spherically symmetric p-branes in $AdS_m \times S^n$, which are defined by (4.4) in the electric case ($p = m-2$) and (4.35) in the magnetic case ($p = n-2$). At the quantum level, the spherical truncation is performed by first fixing a physical gauge and expanding in normal coordinates $\xi^{m'}$ defined by $X^M(\sigma^\mu) = \left(\sigma^\mu, X^{m'}(\tau) + \xi^{m'}(\sigma^\mu)\right)$, where $X^{m'}(\tau)$ describe the 0+1 dimensional sigma-model. The classical consistency of the truncation implies that the action has no linear terms in $\xi^{m'}$. For large brane tension, the normal coordinates $\xi^{m'}$ become free, and thus yield a one-loop determinant contribution to the 0+1 dimensional sigma model for $X^{m'}$. We shall omit this contribution, based on the fact that the free spectrum for $\xi^{m'}$ is evenly spaced both for the single-spin and the two-spin solutions and only depends on the AdS and sphere radii. In this approximation, the spherical giant is described by a wave-function $\psi(X^{m'})$, where $X^{m'}$ are the transverse coordinates and the canonical momenta are realized by $P_{m'} = -i\partial/\partial X^{m'}$.

In the classical theory, the spherical brane consists of a warped product of a breathing-mode and a relativistic, massive point-particle. The latter lives on $R\times S^n$ in the electric case, where R is time, and on $AdS_m \times S^1$ in the magnetic case, where S^1 is the cyclic direction in S^n transverse both to the brane and the breathing direction. We have found that the resulting 0+1 dimensional sigma model is an integrable system[9], and that the breathing mode is governed by a potential that depends on the total angular momentum of the point-particle.

The quantization can be performed in terms of global embedding coordinates. This is discussed in Appendix A. In this subsection, we shall instead quantize using local spherical or AdS coordinates, leading to Pöschl-Teller potentials. In doing so, we parametrize the spheres using *maximal*

[9] See [29] in the case of the string in $AdS_5 \times S^5$. Also, non-relativistic point particles on spheres with potential appear in a similar context, namely as truncation of the string sigma-model, corresponding to picking a particular solution to conformal gauge, leading to the soluble N-R model, see e.g. [11], [39] and [40].

number of cyclic coordinates, i.e. (4.18), and impose (4.22) in the electric case, and (4.53), (4.54), and (4.59) in the magnetic case. We then proceed with the breathing modes, using the results of [28].

The classical solutions are parameterized by constants of motion, namely (k, q_a) defined in (4.22) and (4.25) in the electric case, and (k, q, q_i) defined in (4.53), (4.54) and (4.60) in the magnetic case, and one should ask whether they are actually limits of states in the quantum theory. To begin with, it follows from their definition that their mutual Poisson brackets vanish. The constant k determines the AdS energy E. The AdS energy, which is identified as the p-brane Hamiltonian, is quantized since the spatial world-volume is compact, though in the absence of additional symmetries there is no mechanism preventing non-integer energies. To be more precise, k sets the energy levels of the potential for the breathing modes given in (4.25) and (4.60), which become discrete in the quantum theory.

The point-particle motion may therefore be thought of as an internal sector, analogous to the orbital angular momentum in a central force problem. In this sector the Cartan subalgebra generators of $SO(m-1) \times SO(n+1)$ that are not set to zero in the spherical reduction are determined by some of the q_a and (q, q_i), namely the momenta of the cyclic transverse spatial coordinates. These symmetries are realized in the world-volume quantum theory, in the limit where this theory becomes reliable. Hence, the non-vanishing Cartan subalgebra spins are integers, which we shall denote by $S_i = P_i = n_i \subset SO(m-1)$ and $J_i = P_i = n_i \subset SO(n+1)$

The remaining (q_a) and (q, q_a) are activated by the oscillatory point-particle motion in $AdS_m \times S^n$, i.e. they set energy-like levels for the Pöschl-Teller potentials, which become quantized with discrete spectra (there is also a continuum in the Type II potential). This quantization can also be understood as quantization of the Casimirs of the chains $SO(m-1) \supset SO(m-3) \supset \cdots \supset SO(m-1-2[(m-4)/2])$ and $SO(n+1) \supset SO(n-1) \supset \cdots \supset SO(n+1-2[(m-2)/2])$, arising in the parametrization of the sphere line elements using maximal number of cyclic coordinates.[10]

The Pöschl-Teller potentials belong to a large class of exactly solvable quantum mechanical models, defined by superpotentials with a special property known as shape invariance [26]. There is a group-theoretic approach to solving the Pöschl-Teller potentials, based on coset representations of S^2 and AdS_2 (see, e.g., [27]). In Appendix B, we summarize its energy spectrum.

From the parametrization of the Type I potentials given in (4.22) and (4.53) and using (B4) one finds (ϕ_i oscillatory):

$$T\, q_i = 1 + \sqrt{\frac{1}{4} + (T\, q_{i+1})^2} + \sqrt{\frac{1}{4} + (T\, q_{i+2})^2} + 2n_i\ , \tag{4.63}$$

where $T = L\tilde{N}/\tilde{L}$ (electric) and $T = \tilde{L}N/L$ (magnetic). (To compare with WKB approach see Appendix C). Note that $T\, q_{i+1}$ is a spin, and also $T\, q_{i+2}$ is a spin in case i is the last oscillatory coordinate.

In the electric case we define $Q = T\, q_1$, and the above formula gives

$$Q = J_1 + \cdots J_v + 2(n_1 + \cdots + n_{n-v}) + \frac{(n-1)}{2} + \mathcal{O}(1/T), \tag{4.64}$$

[10] We thank K. Murakami for this observation.

where J_i denote the spins in the Cartan subalgebra, and $v = [(n+1)/2]$.

In the magnetic case the remaining point-particle motion in r is governed by the Type II potential given in (4.54). From (B5), the discrete spectrum is given by

$$-Q = 1 + Tq_1 - E + 2n_r + \mathcal{O}(1/T) \ , \qquad 0 \leq n_r \leq (E - Tq_1)/2, \tag{4.65}$$

where $Q = Tq/k$, $E = N\tilde{L}/(kL)$ (and we have chosen the sign leading to a positive contribution to E), and

$$Tq_1 = S_1 + \cdots + S_v + 2(n_1 + \cdots + n_{m-1-v}) + \frac{(m-3)}{2} + \mathcal{O}(1/T) \ , \tag{4.66}$$

where $v = [(m-1)/2]$. As one approaches the continuum of the Type II potential, Q decreases, which leads to that the breathing mode becomes strongly coupled. This is an interesting region, as the p-brane starts probing large r, but for simplicity we shall continue under the assumption $n_r \ll (E - Tq_1)$.

The quantization of the breathing modes, i.e. r and θ are governed by (4.25) and (4.60) in the electric and magnetic cases, respectively, was studied separately in [28], using semi-classical techniques, and we shall review these results in the present context below

Here the wave-function is approximated by the exponential of the action integral, and the boundary conditions are approximated by the Bohr-Sommerfeld quantization conditions

$$\begin{aligned} I_r &= \oint dr\, P_r = 2\pi(n_r + \frac{1}{2}) \\ &= \tilde{N} \oint \frac{dr}{\cosh r} \left[(k^2 - q_1^2)\cosh^2 r \ - \ (k - \sinh^{m-3} r)^2 \sinh^2 r\right]^{1/2} \ , \end{aligned} \tag{4.67}$$

$$\begin{aligned} I_\theta &= \oint d\theta\, P_\theta = 2\pi(n_\theta + \frac{1}{2}) \\ &= \frac{N}{k} \oint d\theta \left[q^2 - k^2 p^2 - \frac{k^2 \sin^2\theta}{\cos^2\theta}(p - \sin^{n-3}\theta)^2\right]^{1/2} . \end{aligned} \tag{4.68}$$

This approximation method is good if $1 \ll n \ll N$, where the upper bound is set such that the total AdS energy will not become too large, while the lower bound is set so that the WKB approximation can be trusted. Clearly, only the upper bound is necessary from the spacetime point of view, while the lower bound is simply an artifact of the particular approximation method used to solve the Schrödinger problem. The upper bound implies that it makes sense to expand in n/T, where T is the tension defined under (4.63). Adapting the results of [28] to our cases, the quantization of r in the electric case with $(E - \tilde{L}Q/L) \ll \tilde{N}$, Q gives (up to n_r^3 terms)

$$E - \frac{\tilde{L}}{L}Q = \frac{2\tilde{L}}{L}(n_r + \frac{1}{2}) - \begin{cases} \frac{6Nn_r^2}{Q^2} \ , & (m,n) = (5,5), \quad N = \tilde{N} \\ \frac{12Nn_r^2}{Q^3}, & (m,n) = (4,7), \quad N = 2\tilde{N}^2 \\ \frac{15Nn_r^2}{2Q^{3/2}}, & (m,n) = (7,4). \quad N = \sqrt{\frac{\tilde{N}}{2}} \end{cases} \tag{4.69}$$

The θ quantization in the magnetic case with $(Q - \tilde{L}P_\phi/L) \ll N$ gives

$$Q - \frac{\tilde{L}}{L}P_\phi = 2(n_\theta + \frac{1}{2}) - \begin{cases} \frac{6Nn_\theta^2}{P_\phi^2} \ , & (m,n) = (5,5), \\ \frac{15N^{1/2}n_\theta^2}{2P_\phi^{3/2}}, & (m,n) = (4,7), \\ \frac{6N^2 n_\theta^2}{P_\phi^3}, & (m,n) = (7,4), \end{cases} \tag{4.70}$$

Compared to (4.63), here we have a series expansion in n which makes one wonder whether the Bohr-Sommerfeld method used in [28] would reproduce the exact result (4.63) for Pöschl-Teller type potential. As we show in Appendix C, this indeed happens up to fourth order in n but $1/T$ corrections that are present in (4.64), (4.65) and (4.66) are not observed.

4.4. Holography

The above results can be summarized by the following expressions for the AdS energy in the case of electric and magnetic spherical p-branes for $(m,n) = \{(5,5),(4,7),(7,4)\}$:

$$\begin{aligned} E_{\rm el} &- \frac{\tilde{L}}{L}\left[J_1 + \cdots + J_v\right] = \frac{\tilde{L}}{L}\left[2(n_r + n_1 + \cdots + n_{n-v}) + \frac{(n+1)}{2}\right] \\ &- \frac{(n-1)(n+1)N}{2^{[(n+3)/4]}Q^{(n-1)/2}} n_r^2 + \mathcal{O}(n_r^3) + \mathcal{O}(1/\tilde{N}) \ , \end{aligned} \tag{4.71}$$

$$\begin{aligned} E_{\rm mag} &- \left[S_1 + \cdots + S_v\right] - \frac{\tilde{L}}{L}P_\phi = 2(n_\theta + n_r + n_1 + .. + n_{m-1-v}) \\ &+ \frac{(m+1)}{2} - \frac{(m-1)(m+1)N^{(m-3)/2}}{2^{[(m-1)/2]}P_\phi^{(m-1)/2}} n_\theta^2 + \mathcal{O}(n_\theta^3) + \mathcal{O}(1/N), \end{aligned} \tag{4.72}$$

where $v = [(n+1)/2]$ (electric) and $v = [(m-1)/2]$ (magnetic). The finite n and m dependent shifts should cancel in the supersymmetric completions (in order not to violate the unitarity bounds imposed by the superalgebra). The above expressions for the AdS energy are valid for bosonic p-branes. They may be altered in the case of supersymmetric p-branes due to contributions from the fermions, since in the quantum mechanical Hamiltonian the fermions cannot be truncated.

Following the proposal of [20, 21], we assume that a bulk $D3$-brane wave-function(al) with energy E, compact $SO(4) \subset SO(4,2)$ spins (S_1, S_2), and $SO(6)$ Cartan spins (J_1, J_2, J_3), corresponds to a component of an operator $\mathcal{O}_{[R](\Delta;S_L S_R;m_1 m_2 m_3)}$ in the dual SYM theory with scaling dimension $\Delta = E$, non-compact $SL(2,C)$ (Lorentz) spin $(S_L S_R)$, and $SO(6)$ highest weights $(m_1 m_2 m_3)$. Note that under $SO(6) \to U(1)^3$ the operator decomposes into components with Cartan spins (J_1, J_2, J_3) obeying $|J_1|+|J_2|+|J_3| = m_1$. The index $[R]$ refers to a representation of S_N, i.e. a Young-tableau of size k, used to construct the $SU(N)$ invariant [21]. The two extreme cases are single-column Young-tableaux, i.e. subdeterminants, which correspond to magnetic $D3$-branes, and single-row Young-tableaux which correspond to electric $D3$-branes. In these cases, an operator of size k has the form

$$\mathcal{O}_{[\pm]} = \mathcal{O}_{(\lambda_1)\dots(\lambda_k)}\delta^{i_1\dots i_k}_{\{j_1\dots j_k\}_\pm} W^{(\lambda_1)j_1}_{\ \ i_1} \cdots W^{(\lambda_k)j_k}_{\ \ i_k}, \tag{4.73}$$

where $\{\cdots\}_+$ and $\{\cdots\}_-$ denote symmetrization and anti-symmetrization, respectively, and $W^{(\lambda)}$ denote derivatives of the $SU(N)$ valued SYM superfield X^A. The tensor $\mathcal{O}_{\lambda_1\dots\lambda_k}$ picks out some irrep and contains an N-dependent normalization of two-point function. Two basic properties of these type of operators are that 1) they form a diagonal set for large $k \sim N$, which is possible because they contain multi-trace contributions which are not suppressed for large N; and 2) the constituents are automatically symmetrized (for both $\pm$). Consider a fixed $\mathcal{O}_0$ of size $k \sim N$, $N - k \gg 1$, and the space of excitations built on $\mathcal{O}_0$ by inserting impurities while keeping R in single column or row form. The above properties imply a Fock space structure [23, 24], which is isomorphic to the space of multi-particle states (i.e. composites) of the super Maxwell field theory arising from normal-coordinate expansion around the semi-classical $D3$-brane giant solution. In particular, the super Maxwell ground state carries the same charges as $\mathcal{O}_0$.

Let us consider the wave function of a spherically symmetric electric $D3$-brane giant with quantum numbers $(J_1, J_2, J_3; n_1, n_2; n_r)$. The corresponding operator $\mathcal{O}_{(J_1,J_2,J_3;n_1,n_2;n_r)}$ is a Lorentz scalar with $SO(6)$ Cartan spins (J_1, J_2, J_3). Eq. (4.71) suggests that the bare scaling dimension is given by

$$\Delta_{\text{bare}} = J_1 + J_2 + J_3 + 2(n_1 + n_2 + n_r) = Q + 2n_r, \tag{4.74}$$

which implies that for n_r the operator $\mathcal{O}_{(J_1,J_2,J_3;n_1,n_2;0)}$ is protected. Hence, the set of operators

$$\mathcal{O}_{(J_1,J_2,J_3;n_1,n_2;0)} \ , \qquad J_1 + J_2 + J_3 = J, \qquad J + 2(n_1 + n_2) = Q, \tag{4.75}$$

may be identified with the components of the protected scalar chiral primary operator $\mathcal{O}_{(J;00;J00)}$:

$$\mathcal{O}_{(J;00;J00)} = \mathcal{O}_{A_1 \ldots A_J} \delta^J X^{A_1} \cdots X^{A_J} \tag{4.76}$$

where A is the $SO(6)$ vector index, X^A the $SU(N)$ valued singletons and $\mathcal{O}_{A_1 \ldots A_J}$ is a constant traceless $SO(6)$ tensor, and δ^J denotes the symmetric $SU(N)$ invariant of size J in (4.73). Under $SO(6) \to U(1)^3$, the singleton superfields decompose as $X^A \to (Z_1, Z_2, Z_3)$, where $Z_i = X^{2i-1} + iX^{2i}$. If one lets

$$K_{\alpha_1,\alpha_2,\alpha_3} = \alpha_1 |Z_1|^2 + \alpha_2 |Z_2|^2 + \alpha_3 |Z_3|^2 \ , \tag{4.77}$$

then $\mathcal{O}_{(\Delta;00;J00)}$ decomposes into the set of components

$$\delta^J Z_1^{J_1} Z_2^{J_2} Z_3^{J_3} K_{2,-1,-1}^{l_1} K_{-1,2,-1}^{l_2} \ , \qquad J_1 + J_2 + J_3 = J \ , \qquad J + 2(l_2 + l_2) = Q, \tag{4.78}$$

where $K_{2,-1,-1}$ and $K_{-1,2,-1}$ is a choice of basis for traceless, $U(1)^3$-invariant bilinears. Elementary counting shows that the two sets of operators (4.75) and (4.78) are isomorphic. The identification becomes manifest in global coordinates, where the giant wave-function is given by (A6) for $p = 0$.

From (4.71), we see that starting from the protected operators making up the components of $\mathcal{O}_{(J;00;J00)}$, and switching on finite breathing number, $n_r = 1, 2...$, adds a bare dimension $2n_r$ and a negative anomalous dimension, $-6Nn_r^2/J$. Finite breathing number implies that the giant wave-function depends on the AdS radius r. In Poincaré coordinates, $ds^2 = L^2(u^2(-dt^2 + dx^2) + du^2/u^2 + d\Omega_5^2)$, the energy scale combines with S^5 into $du^2/u^2 + d\Omega_5^2 = dX^A dX^A/(L^2 u^2)$, suggesting that radial breathing translates to insertions of $SO(6)$-traces $K_{1,1,1}$ into the dual operator. We propose that the operators $\mathcal{O}_{(J_1,J_2,J_3;n_1,n_2;n_r)}$ corresponding to the giant wave-functions with fixed breathing number n_r are the components of the operator $\mathcal{O}_{(\Delta;00;J00)(n_r)}$ obtained by inserting the $SO(6)$ trace-part $K_{1,1,1}^{n_r}$ into (4.76), i.e.

$$\mathcal{O}_{(\Delta;00;J00)(p)} = \mathcal{O}_{A_1 \ldots A_J} \delta^{J+2p} X^{A_1} \cdots X^{A_J} (X^A X^A)^p \ , \quad \Delta = J + 2p - 6Np^2/J^2, \ p = 0, 1, \ldots \tag{4.79}$$

when $J \sim N$.

The anomalous dimension is negative[11] and independent of the 't Hooft coupling $\lambda = g^2_{\rm YM}N$. However, the results are valid for $\lambda \gg 1$ and there may be additional contributions from fermions as pointed out above. From the bulk point of view, the negative contributions can be interpreted as the binding energy of the closed strings on the giant. This energy should depend on the bulk coupling constant, i.e. the bulk Planck's constant $1/N^2$, but not the masses of the individual strings, i.e. the bulk string tension λ/R^2. This is in sharp contrast to the anomalous dimensions of single-trace operators, which are positive and depend on λ, as expected from the bulk picture where they represent individual closed strings.

Analogous operator constructions are relevant also for CFT duals of electric $M2$ giants. Here one starts by building operators in the UV from 8 free $SU(N)$-valued $OSp(8|4)$ supersingletons, X^A, $A = 1, \ldots, 8$, and let these flow to the IR under deformations of the free singleton theory, where they should correspond to the giants. Hence, in the IR one should find

$$\mathcal{O}_{(\Delta;0;J000)(p)} = \mathcal{O}_{A_1 \ldots A_J} \delta^{J+2p} X^{A_1} \cdots X^{A_J} (X^A X^A)^p \ , \quad \Delta = J + 2p - 12Np^2/J^3, \ p = 0, 1, \ldots \tag{4.80}$$

when $J \sim N^{1/2}$.

In the case of magnetic $D3$ and $M2$ branes, the energy formula (4.72) suggests that dual operators are built from subdeterminants involving P_ϕ scalar fields and $S_1 + \cdots + S_v + 2(n_r + n_\theta + n_1 + \cdots + n_{m-1-v})$ derivatives, such that the operators are protected when $n_\theta = 0$.

5. CONCLUSIONS

In this paper we have shown that the $(p+1)$-dimensional field theory of a p-brane in $AdS_m \times S^n$ admits consistent KK sphere reductions on either $S^p \subset AdS_m$, $m = p + 2$, or $S^p \subset S^n$, $n = p + 2$. The resulting $(0 + 1)$-dimensional models are integrable, the canonical variables separate and the quantum mechanics consists of a breathing mode with non-trivial potential times a set of oscillators which describe the overall transverse motion. These models contain the previously known 1/2 supersymmetric single-spin giant gravitons, that have one spin in S^n and expand spherically in S^n [1] or AdS_m [3, 4]. The magnetic model also includes a new 1/2 supersymmetric two-spin giant, that has one extra spin in AdS_m and expands in S^n. The BPS bounds show that these are the only supersymmetric solutions of these particular spherically symmetric truncation.[12]

There are several directions in which the $(0 + 1)$-dimensional sigma models should be explored. In the cases where the original p-brane is supersymmetric, one should consider supersymmetric completions by including fermions and possibly extra bosons, and in particular examine their contributions to the AdS energies. In the case of $D3$ and $M5$ branes, in analogy with sphere reductions of supergravity, we expect the extra bosons to be embedded into the $(p+1)$-dimensional vector and tensor fields together with certain low-lying spherical vector and tensor harmonics on S^3

[11] Negative anomalous dimensions are not unusual. Indeed there are several examples of multi-trace operators with anomalous dimensions that are negative both perturbatively and in the supergravity limit, where the anomalous dimension tends to zero from below as $N \to \infty$. We thank M. Bianchi and Y. Stanev for discussions on this point.

[12] There are also other supersymmetric giants in the literature, based on wrapping p-branes on supersymmetric cycles in $AdS_m \times S^n$ or $AdS_5 \times T^{1,1}$ [41, 42].

and S^5, respectively. It is worth investigating this in detail. Having obtained the supersymmetric $(0+1)$-dimensional sigma models, it would be interesting to examine to what extent the salient features of the bosonic quantum mechanics prevails.

An obvious generalization is to consider the effect on the $(0+1)$-dimensional model from k-fold wrapping of the p-brane on S^p. This gives rise to k copies of the quantum mechanical system moded out by cyclic symmetry taking the i'th copy to the $(i+1)$'th copy mod k (this is a global reparametrization). In the point-particle sector, the resulting giant wave-functions in global coordinates are built from a set of copied oscillators $X_i^A(\xi)$, $\xi = 1, ..., k$, obeying commutation rules $[X_i^A(\xi), X_j^B(\eta)] = i\delta_{\xi\eta}\eta^{AB}\epsilon_{ij}$, where η^{AB} has appropriate signature. The wave-functions now involve Young-tableaux of up to k rows, in rough agreement with the proposals on the field theory side [21].

Another line of generalization is to include higher modes of the KK spectrum on S^p and obtain sigma-models in dimensions between $(0+1)$ and $(p+1)$. For example, for the $M2$ brane, we can consistently set to zero all harmonics on S^2 with non-vanishing L_z eigenvalue, i.e. drop the dependence on the cyclic coordinate, χ_2 say, while keeping the full dependence on the remaining polar coordinate, χ_1 say $(0 \leq \chi_1 \leq \pi)$. This should lead to a non-trivial $(1+1)$-dimensional sigma-model with generally (τ, χ_1)-dependent fields, and it would be interesting to study whether the integrability of the $(0+1)$-dimensional model extends to $(1+1)$ dimensions. Similarly, the $D3$-brane on S^3 and the $M5$ on S^5 with trivial dependence on the cyclic coordinates should give interesting $(1+1)$ and $(2+1)$-dimensional sigma models (on S^3, χ_i, $i = 2, 3$, are cyclic and we keep $X^{m'}(\sigma^\mu)$ and $A_i(\sigma^\mu)$, $\sigma^\mu = (\tau, \chi_1)$; and on S^5, χ_i, $i = 2, 4, 5$, are cyclic, and we keep $X^{m'}(\sigma^\mu)$ and $b_{ij}(\sigma^\mu)$, $\sigma^\mu = (\tau, \chi_1, \chi_3)$).

As discussed in the Introduction, electric p-branes in AdS_{p+2} have semi-classical scaling behavior which make them suitable probes for examining holography at high energies. Could they also be used for actually defining the bulk dynamics in some certain limit? Consider, for example, the open/closed string quantum theory on a 1/2 BPS electric $D3$ giant graviton of radius r_0. The running string tension, which sets the scale for massive string excitations on top of the giant ground state, is given by $L^2 T_s(r_0) \sim L^2 \cosh^2 r_0/\alpha' \sim E_0\sqrt{g_s/N} \gg 1$. Hence, between the ground state and the first massive string states, there is a large number of massless open string excitations with energy $E \sim E_0 \gg 1$, and $E - E_0 \ll L^2 T_s(r_0)$. These are composite operators in the vector multiplet living on the $D3$-brane (i.e. they are multi-particle states from the world-volume point of view), which in the physical gauge describe one-particle states in the bulk. Consider a process with "in-state" prepared by first letting the breathing mode inhale until $r \gg r_0$ and then placing out operators, carrying distinct energy and spins. We may assume the operators to be separated initially, so that an observer in spacetime would see localized concentrations of energy and spin densities on the brane. During the subsequent time-evolution, the brane first exhales. For a spacetime observer this looks like particles falling inward to a scattering region. The brane then breathes in again and finally reaches large size, at which point the result of the scattering can be obtained by computing the correlator with an "out-state".

The question is, how good an approximation it is to describe the whole scattering process using only the massless field theory on the p-brane. Clearly, the initial excitation energies should not be too high. However, as discussed in Section 4, the breathing may cause the brane to implode, or at least pass through some region of large world-volume curvature. For example, from the $D3$-brane field theory point of view, the formation of strings may be thought of as the AdS analog of the BIon formation on a $D3$ brane in flat space [43]. It would be interesting to examine to what extent these stringy excitations of the $D3$-brane may behave differently in an AdS background as opposed to flat space. A related question is whether a rotating long string with energy-momentum propagating along the directions of a giant $D3$-brane could be realized as a weakly coupled state on the giant. Similar considerations could be undertaken for membrane-like excitations of electric $M5$ branes described by self-dual string solutions [44] and rotating membranes [45–47]

Finally, let us point out the relation with the old ideas of p-branes "at the end of the universe" [31, 38]. For example, the transition from a strongly coupled string world-sheet to a weakly coupled $D3$-brane world-volume at high energies and fixed R^2T_s, suggests a similar transition at fixed energy and small R^2T_s, i.e. the tensionless string limit. Indeed, for any finite R^2T_s, the running string tension $R^2T_s(r_0)$ diverges in the IR. This limit, which is most easily examined by replacing the spherical physical gauge by another physical gauge given in Poincaré coordinates [48], yields a superconformal $D3$-brane "at the end of the universe". This world-volume theory (which should not be confused with the dual CFT) is completely decoupled from string excitations, and therefore remains weakly coupled in the tensionless string limit. Similar limits exist also for the $M5$ and $M2$ branes. The perturbations of the conformal $M2/D3/M5$ branes have loop expansion in inverse powers of N. This suggests that a natural starting point for describing holography is to start close to the boundary with operator insertions on conformal p-branes, and then study the deformation of this system into the bulk by switching on perturbations corresponding to the breathing mode.

The conformal p-branes provide a link between the original supergravity theories in the UV region of the bulk, and higher spin gauge theories in the IR region of the bulk. Indeed the (unperturbed) conformal M2 brane world-volume is a free $OSp(8|4)$ supersingleton field theory, with conserved higher spin currents in the world-volume [49]. The conformal $D3$ and $M5$ brane world-volume theories are $PSU(2,2|4)$ and $OSp(8^*|4)$ supersingleton field theories with interactions stemming from the magnetic background fluxes, and it is desirable to study their applications for the higher spin symmetries. It would also be interesting to examine the conformal limit of the $Sp(2)$-covariant quantization of the point-particle sector (see Appendix A), since the $Sp(2)$-gauged version of the $SO(m-1,2)$-covariant oscillators play a central role in formulating the massless higher spin dynamics as formulated in [50].

In summary, giant p-branes in AdS backgrounds deserve further study as they have many intriguing properties both from the point of view of holography and for the understanding of the nature of fundamental interactions of M-theory and string theory at high energies.

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APPENDIX A: QUANTIZATION IN GLOBAL COORDINATES

In this appendix, we briefly discuss the canonical quantization of the global x^A coordinates governed by the action (4.12). It is clear that $P_\Lambda = 0$ is a primary constraint. The Poisson bracket of P_Λ with the Hamiltonian leads to a chain of secondary constraints, which can be summarized in terms of the $Sp(2)$ generators as

$$L_{ij} \equiv X_i^A X_{Aj} = M_{ij} \ , \qquad P_\Lambda = 0 \ , \tag{A1}$$

where $X_i^A = (x^A, P^A)$, $M_{11} = 1$, $M_{12} = 0$, $M_{22} = 2\tilde{N}^2 L^4 \Lambda_0 / \tilde{L}^2$. Here, Λ_0 is the constant that appears in (4.15). The four constraints are second class.

In quantum theory, the canonical commutation relations read

$$[\hat{X}_i^A, \hat{X}_j^B] = i\epsilon_{ij}\delta^{AB} \ . \tag{A2}$$

The ordering ambiguity in (A1) can be cured by demanding that the operators $\hat{L}_{ij} = \hat{X}^A_{(i}\hat{X}_{j)A}$ generate the $Sp(2)$ algebra. One can now impose the following Casimir constraint:

$$\left(\hat{L}^{ij}\hat{L}_{ij} - M^{ij}M_{ij}\right)|\Psi\rangle = 0 \ , \tag{A3}$$

together with $L_{11}|\Psi\rangle = P_\Lambda|\Psi\rangle = 0$. Using the oscillator algebra we find

$$\frac{1}{2}\hat{L}^{ij}\hat{L}_{ij} = \frac{1}{2}\hat{J}^{AB}\hat{J}_{AB} + \frac{1}{4}(n+1)(n-3) \ , \tag{A4}$$

where $\hat{J}_{AB} = \epsilon^{ij}\hat{X}_{[Ai}\hat{X}_{B]j}$ are the generators of $SO(n+1)$. Using the expression for M_{ij} given below (A1) and the fact that Ψ has $SO(n+1)$ highest weight $(J0\ldots0)$, we find (note that $q_1^2 = 2L^2\Lambda_0$, see below (4.22))

$$1 + \frac{\tilde{N}^2 L^2}{\tilde{L}^2}\, q_1^2 = (J + \frac{n-1}{2})^2 \ , \tag{A5}$$

which is in agreement with (4.63) for large J. The wave-functions are given by the spherical harmonics

$$\Psi_{(J00)p}(X^A, \Lambda) = \Psi_{A_1 \ldots A_J} X^{A_1} \cdots X^{A_J} \ , \tag{A6}$$

where $\Psi_{A_1\cdots A_J}$ is traceless and symmetric. It would be interesting to repeat the above analysis also for the magnetic case leading to $Sp(2) \times SO(m-1,2)$ covariant oscillators Y_i^P.

APPENDIX B: THE ENERGY SPECTRUM OF PÖSCHL-TELLER POTENTIALS

In this appendix, following [26], we summarize the spectrum of Pöschl-Teller potentials. A superpotential, $W(X)$ say, determines two "partner Hamiltonians", $H_+ = A^\dagger A$ and $H_- = AA^\dagger$, where $A = \frac{d}{dX} + W(X)$ and $A^\dagger = -\frac{d}{dX} + W(X)$, and the partner potentials are given by the Riccati equations $V_\pm = W^2 \mp W'$. A family of superpotentials $W(a;X)$, where a denotes a set of parameters, is said to be shape invariant if $A(a)A^\dagger(a) = A^\dagger(f(a))A(f(a)) + R(a)$, i.e. $V_-(a;X) =$

$V_+(f(a);X)+R(a)$, where f is a fixed function and $R(a)$ is a constant (independent of X). The eigenvalue problem for H_+ then has a generalized oscillator solution ($n=0,1,2,\ldots$):

$$\begin{aligned}
\Psi^+_{(n)}(X) &= A^\dagger(a)A^\dagger(f(a))\cdots A^\dagger(f^{n-1}(a))\exp\left[-\int^X dYW(f^n(a);Y)\right] , \\
E^+_{(n)} &= \sum_{k=0}^{n-1} R(f^k(a)) \ .
\end{aligned} \tag{B1}$$

The Pöschl-Teller Type I and II superpotentials are given by ($0\le\theta\le\pi/(2\alpha)$, $r>0$)

$$W_I(A,B,\alpha;\theta) = A\tan\alpha\theta - B\cot\alpha\theta \ , A,B>0 \ , \tag{B2}$$

$$W_{II}(A,B,\alpha;r) = A\tanh\alpha r - B\coth\alpha r \ , A>B>0, \tag{B3}$$

and the associated potentials, shape transformations and bound state eigenvalues read

$$\begin{aligned}
V_{I\pm} &= -(A+B)^2+\frac{A(A\mp\alpha)}{\cos^2\theta}+\frac{B(B\mp\alpha)}{\sin^2\theta} \ , \\
f_I(A,B,\alpha) &= (A+\alpha,B+\alpha,\alpha) \ , \qquad R_I=(A+B+2\alpha)^2-(A+B)^2 \ , \\
E^{I+}_{(n)} &= (A+B+2n\alpha)^2-(A+B)^2 \ , \quad n=0,1,\ldots
\end{aligned} \tag{B4}$$

$$\begin{aligned}
V_{II\pm} &= (A-B)^2-\frac{A(A\pm\alpha)}{\cosh^2\alpha r}+\frac{B(B\mp\alpha)}{\sinh^2\alpha r} \ , \\
f_{II}(A,B,\alpha) &= (A-\alpha,B+\alpha,\alpha) \ , \qquad R_{II}=(B-A)^2-(B-A+2\alpha)^2 \ , \\
E^{II+}_{(n)} &= (A-B)^2-(A-B-2n\alpha)^2, \quad n=0,\ldots,(A-B)/2.
\end{aligned} \tag{B5}$$

The wave-functions obey Dirichlet conditions at $\theta=0,\pi/(2\alpha)$ and $r=0$. In the Type II case there is a finite number of bound states and then a continuum, $E^{II+}\ge(A-B)^2$. One can extend Type I and II to $B=0$, provided the Dirichlet condition is dropped in the case of Type II and imposed at $\theta=\pm\pi/2$ in the case of Type I.

In applying to the motion on a sphere, one has to be careful with the Dirichlet conditions. If one switches on a spin, generated by a vector field V, by imposing $V\Psi=in\Psi$, then Ψ has to vanish at points where V has zero norm. In the present parametrization, there is a one-to-one correspondence between the spins in the Cartan subalgebra and the cyclic coordinates. Hence, if ϕ_i is an oscillatory coordinate, then both $\phi_i=0$ and $\phi_i=\pi/2$ are vanishing points for spins in the Cartan subalgebra, and hence all the Dirichlet conditions are globally well-defined.

APPENDIX C: COMPARISON WITH BOHR-SOMMERFELD

As we have seen above the quantum mechanical problem involving the Pöschl-Teller potentials is exactly solvable. Now, we will use Bohr-Sommerfeld approximation for Type I Pöschl-Teller potential for comparison. The action integral is

$$\begin{aligned}
I_\theta &= 2\pi(n_\theta+\frac{1}{2}) \ , \qquad n_\theta=0,1,2,\ldots \\
&= \frac{L\tilde{N}}{\tilde{L}}\oint d\theta\left[q^2-\frac{p^2}{\cos^2\theta}-\frac{u^2}{\sin^2\theta}\right]^{1/2} .
\end{aligned} \tag{C1}$$

To evaluate the integral perturbatively, we first set $x = \sin^2\theta$ which yields

$$I_\theta = \frac{L\tilde{N}}{\tilde{L}} q \int_{b_1}^{b_2} dx \frac{\sqrt{f(x)}}{x(1-x)} = 2\,\pi\,(n_\theta + \frac{1}{2}), \tag{C2}$$

where

$$f(x) = -x^2 + [\frac{q^2 - p^2 + u^2}{q^2}]x - \frac{u^2}{q^2} \equiv (x - b_1)(b_2 - x), \tag{C3}$$

and $0 < b_1, b_2 < 1$. For small oscillations we need $(b_2 - b_1) \sim 0$ and $(b_1 + b_2) \sim 1$, implying $p \sim u$ and $q \sim 2u$, and we can expand in

$$\eta \equiv (q - p - u). \tag{C4}$$

Defining $2x = (b_2 - b_1)y + (b_2 + b_1)$ and using the approximations

$$[1 - (b_2 + b_1)]^2 \simeq \frac{(p-u)^2}{(p+u)^2}\left[1 - \frac{4}{p+u}\eta\right] \tag{C5}$$

$$\frac{(b_2 - b_1)^2}{4} \simeq \frac{2pu}{(p+u)^3}\left[\eta + \frac{u^2 - 5pu + p^2}{2pu(p+u)}\eta^2 - \frac{3u^2 - 10pu + 3p^2}{2pu(p+u)^2}\eta^3\right]$$

$$\left[(\frac{b_1 + b_2}{2})(1 - \frac{b_1 + b_2}{2})\right]^{-1} \simeq \frac{(p+u)^2}{pu}\left[1 - \frac{(p-u)^2}{pu(p+u)}\eta + \frac{(p-u)^2(2p^2 + pu + 2u^2)}{2p^2u^2(p+u)^2}\eta^2\right]$$

the integral (C2) up to fourth order in η reads (the odd powers of y do not contribute)

$$\left[2\eta - \frac{p^2 - pu + u^2}{pu(p+u)}\eta^2 + \frac{p^4 - p^2u^2 + u^4}{p^2u^2(p+u)^2}\eta^3 + ...\right] \int_{-1}^{1} dy \sqrt{1 - y^2}\,[1 + h_1 y^2 + h_2 y^4 + ...] \tag{C6}$$

where

$$h_1 = \frac{2(p^2 - pu + u^2)}{pu(p+u)}\eta - \frac{3(p^4 - p^2u^2 + u^4)}{p^2u^2(p+u)^2}\eta^2\,, \tag{C7}$$

$$h_2 = \frac{4[p^4 - p^3u + P^2u^2 - pu^3 + u^4]}{p^2u^2(p+u)^2}\eta^2\,. \tag{C8}$$

Up to this order, the integration gives

$$2n_\theta + 1 = \frac{L\tilde{N}}{\tilde{L}}\eta. \tag{C9}$$

As in the exact result (4.63) there is no expansion in n. However, $1/T$ corrections that are present in (4.64) and (4.66) are not observed.

The above analysis also applies to Type II Pöschl-Teller potential which appeared in (4.54). One needs to introduce a new variable $x = -\sinh^2 r$ after which the action integral becomes (C3).

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