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# The Entropy for General Extremal Black Holes

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ABSTRACT

We use the Kerr/CFT correspondence to calculate the entropy for all known extremal stationary and axisymmetric black holes. This is done with the help of two ansatzs that are general enough to cover all such known solutions. Considering only the contribution from the Einstein-Hilbert action to the central charge(s), we find that the entropy obtained by using Cardy's formula exactly matches with the Bekenstein-Hawking entropy.

## Contents


**1 Introduction** **1**

**2 Two General Ansatzs for Stationary and Axisymmetric Black Holes** **3**

**3 The Near-Horizon Metric for Extremal Black Holes** **7**

**4 The Central Charge(s) of the Dual CFT(s)** **8**

**5 The Entropy** **9**

**6 Summary** **11**

**A Calculating the Central Term $K[\xi, \zeta]$** **12**

**B The Asymptotic Symmetry Group** **15**

**C Some Examples** **21**

C.1 Kerr-NUT-AdS Solutions in Diverse Dimensions . . . . . . . . . . . . . . . . 21

C.2 Extremal Static Black Holes in Supergravity Theories . . . . . . . . . . . . 24

C.3 Extremal Rotating Black Holes in Supergravity Theories . . . . . . . . . . . 27


# 1 Introduction

To successfully calculate the entropy for black holes is a challenge for all candidates of the quantum gravity theory. In reverse, helpful insight to quantum gravity may be obtained if one can find a general way to calculate the black hole entropy.

The Kerr/CFT correspondence [1, 2] has been quite successful with calculating the entropy for extremal black holes. The basic idea is to discuss dynamics on the near-horizon metric of the black holes. With appropriate boundary conditions, the corresponding phase space can be identified with that of a two dimensional conformal field theory. The entropy of the black hole can then be calculated from the corresponding central charge(s) by using Cardy's formula. After it was first proposed in [1], the method has been found to work for all the cases that have been checked (for refs, see [3]). It was suggested in [2] that the extremal condition may be at the heart of the correspondence. So the Kerr/CFT correspondence is also called the Extremal Black Hole/CFT correspondence.

In hindsight, several important points have also been raised in [2]. The first is related to the matter field contribution to the central charges of the dual CFTs. It was found in [2] that the gauge field does not contribute to the central charge for solutions in the Einstein-Maxwell system in four dimensions. This result was echoed in [4, 5], where it was shown by using examples in four and five dimensions that non-gravitational fields such as the scalar field, the Abelian gauge field and the antisymmetric tensor field do not contribute to the central charge(s). The second point is that the success of the Kerr/CFT calculation may partially due to the possibility that all near-horizon metrics share a particular common structure. The near-horizon metrics for some extremal black holes have been studied in [13, 14] in a different context. In four dimensions, the near-horizon metrics are found to be of the form

$$ds_4^2 = f(\theta)\Big[ - r^2dt^2 + \frac{dr^2}{r^2} + \alpha(\theta)d\theta^2\Big] + \gamma(\theta)(d\phi + krdt)^2 \,, \qquad (1.1)$$

while in higher dimensions they are found to be of the form

$$ds_d^2 = f(\theta^i)\Big[ - r^2dt^2 + \frac{dr^2}{r^2}\Big] + \alpha_i(\theta^j)d\theta^{i2} + \gamma_{ab}(\theta^i)(d\phi^a + k_a rdt)(d\phi^b + k_b rdt) \qquad (1.2)$$

for a certain class of solutions, where $k$ and $k_a$ are constants while all the functions depend on $\theta^i$'s only. It was then shown in [6] that (1.2) indeed plays a significant role when the Kerr/CFT correspondence is applied to various solutions in (gauged) supergravity theories. Further examples were also presented in [7]. Lastly, it was speculated in [2] that the Frolov-Thorne temperature may be of the general form $T_L = \frac{1}{2\pi k}$ in four dimensions. This was then generalized to higher dimensions in [6],

$$T_L^a = \frac{1}{2\pi k_a} \,, \qquad (1.3)$$

based on all the examples that have been studied. This result also plays a crucial role in applying the Kerr/CFT correspondence to various black hole solutions [6, 7].

In this paper, we present two ansatzs that are general enough to cover all known stationary and axisymmetric black holes. Extra constraints can be obtained by noticing that black hole horizons are intrinsically regular. We then show that (1.2) can be derived as soon as the near-horizon limit is taken for extremal black holes. As a result, (1.2) is valid for all known extremal stationary and axisymmetric black holes. The Frolov-Thorne temperature of the form (1.3) is also derived in a straight forward manor. Then we explicitly calculate the central charge(s) related to (1.2). When the microscopic entropy is calculated by using Cardy's formula, we find that the result exactly matches with the Bekenstein-Hawking entropy. In this way, we demonstrate in a general fashion that the Kerr/CFT correspondence is applicable to all known extremal stationary and axisymmetric black holes. What's more, empirical results such as (1.3) can also be derived without making extra assumptions.

Note earlier works have largely demonstrated the general applicability of the Kerr/CFT correspondence (see, e.g. [8, 6, 7]). So it is not our intention here to show this again. Rather, we are most interested to see to what extent can the calculation be carried out in a general fashion.

For practical reasons, we have only considered the contribution from the Einstein-Hilbert action to the central charge(s). The fact that the resulted microscopic entropy matches with the Bekenstein-Hawking entropy implies that the non-gravitational contributions to the central charge(s) are zero, which is consistent with the results found in [2, 4, 5]. One can certainly try to repeat the same process for more complicated theories. For example, it has been shown in [9] (See also [10] for an earlier work) that in a theory with higher-derivative corrections in the gravitational sector, the higher-derivative terms also contribute to the central charge(s) and the correct entropy is the one constructed by Iyer and Wald [11, 12]. However, it is obvious that a similar calculation will be extremely difficult.

The plan of the paper is as following. In section 2, we will present the two ansatzs for all known stationary and axisymmetric black holes. The near-horizon metric for extremal black holes will then be derived in section 3. The central charges will be calculated in section 4, but most of the extra detail will be contained in Appendix A. The microscopic entropy from the CFT side is then calculated in section 5. A summary will be given in section 6.

To make the whole calculation more accessible to most readers, we have included an introduction to the treatment of asymptotic symmetries by using the covariance phase space method in Appendix B. We will also revisit most of the examples studied in [8, 6, 7] in Appendix C, by using the new perspective that we gain from the present work.

## 2 Two General Ansatzs for Stationary and Axisymmetric Black Holes

The basics of the Kerr/CFT correspondence has been explained in [1] in much detail. Here we will go directly to the general case we want to study.

We will start with presenting two general ansatzs that cover all known stationary and axisymmetric black hole solutions. The construction will be partially based on our experience with all the solutions that are known.

Stationary and axisymmetric black hole solutions share some common features:

- By using the term "stationary and axisymmetric", one assumes that (i) a coordinate system exists where some of the coordinates can be identified with the asymptotic time direction $\hat{t}$ and the azimuthal directions $\hat{\phi}^a$, and (ii) the metric does not depend on $\hat{t}$ nor $\hat{\phi}^a$.

- Among the rest of the coordinates, one coordinate can be singled out as describing the radial direction $\hat{r}$. For all known solutions, the position of the black hole horizon ($\hat{r} = r_H$) is determined by a single function of $\hat{r}$ : $\Delta(r_H) = 0$.

- All other coordinates are then related to the latitudinal angles $\theta^i$. For a black hole in $d$-dimensional spacetime, there can be $[\frac{d-1}{2}]$ independent rotations. So $a = 1, \cdots, [\frac{d-1}{2}]$ and $i = 1, \cdots, [\frac{d}{2}] - 1$.

- For all known solutions, one can always chose the coordinate systems so that the metrics do not have any cross terms involving $d\hat{r}$ or $d\theta^i$.

- Near the black hole horizon, it can either be a term like $d\hat{t} + f_a(\hat{r}, \theta^i) d\hat{\phi}^a$ or a term like $f_a(\hat{r}, \theta^i) d\hat{\phi}^a$ playing the role of time.

Metrics reflecting such features can always be written as

$$ds_d^2 = -\frac{\Delta}{f_t}\left[d\hat{t} + f_a d\hat{\phi}^a\right]^2 + \frac{f_r}{\Delta} d\hat{r}^2 + g_{ij} d\theta^i d\theta^j + d\bar{s}_\phi^2 \,, \tag{2.1}$$

or

$$ds_d^2 = -\frac{\Delta}{f_t}\left[f_a d\hat{\phi}^a\right]^2 + \frac{f_r}{\Delta} d\hat{r}^2 + g_{ij} d\theta^i d\theta^j + d\bar{s}_\phi^2 \,, \tag{2.2}$$

with

$$d\bar{s}_\phi^2 = g_{ab}(d\hat{\phi}^a - \chi_a d\hat{t})(d\hat{\phi}^b - \chi_b d\hat{t}) + f_{tt} d\hat{t}^2 \,. \tag{2.3}$$

Note all the functions depend on $\hat{r}$ and $\theta^i$'s only, while $\Delta$ will be the function determining the location of the horizon and so it depends on $\hat{r}$ only. We have allowed $d\theta^i$'s to mix among themselves in (2.1) and (2.2), so both ansatzs can describe possibly slightly more general cases than listed above. We have also included the $f_{tt} d\hat{t}^2$ term in (2.3) to make (2.1) and (2.2) as general as possible. The assumption on $f_{tt}$ is that it should not play any significant role near the horizon. As we will see below, this means $f_{tt} \sim \Delta^2$ as $\hat{r} \to r_H$. As far as we can tell, all known stationary and axisymmetric black holes can either be written in the form of (2.1) or in the form of (2.2). We also notice that the two ansatz are actually general enough to go beyond black holes and cover objects such as the black ring [15].

Some extra constraints can be obtained for the functions in (2.1), (2.2) and (2.3) by noticing that black hole horizons are intrinsically regular. A regular horizon means that the metric (and the matter fields) should be manifestly regular on the horizon if the coordinate system is chosen appropriately.

To see how this can help us, note that the first two terms in (2.1) can be written as

$$\frac{\Delta}{f_t}\left(-\Big[d\hat{t}+f_a\,d\hat{\phi}^a\Big]^2+\frac{f_tf_r}{\Delta^2}d\hat{r}^2\right)=-\frac{\Delta}{f_t}\mathcal{A}^2+2\sqrt{f_r/f_t}\;d\hat{r}\mathcal{A}\,, \tag{2.4}$$

where

$$\mathcal{A}=d\hat{t}+f_a\,d\hat{\phi}^a+\frac{\sqrt{f_tf_r}}{\Delta}d\hat{r}\,. \tag{2.5}$$

The superficial singularity near the horizon comes solely from $\Delta(r_H)=0$. To make the metric regular on the horizon, one can try to make $\mathcal{A}$ regular first. This can be achieved if there exist functions $h_v=h_v(\hat{r})$, $h_a=h_a(\hat{r})$ and $h_{\mathcal{A}}=h_{\mathcal{A}}(\hat{r},\theta^i)$ being regular on the horizon and satisfying

$$\sqrt{f_tf_r}=h_v+f_ah_a+h_{\mathcal{A}}\Delta+\mathcal{O}(\Delta^2)\,. \tag{2.6}$$

In this case one can write $\mathcal{A}=dv+f_a\,d\psi^a+h_{\mathcal{A}}d\hat{r}+\mathcal{O}(\Delta)$ by using the coordinate transformation

$$dv=d\hat{t}+\frac{h_v(\hat{r})}{\Delta(\hat{r})}d\hat{r}\,,\quad d\psi^a=d\hat{\phi}^a+\frac{h_a(\hat{r})}{\Delta(\hat{r})}d\hat{r}\,. \tag{2.7}$$

We find that this process is possible for all know examples. For (2.3),

$$\begin{aligned} d\bar{s}^2_\phi \;&=\; g_{ab}\Big(d\psi^a-\chi_a dv-\frac{h_a-\chi_ah_v}{\Delta}d\hat{r}\Big)\Big(d\psi^b-\chi_b dv-\frac{h_b-\chi_bh_v}{\Delta}d\hat{r}\Big)\\ &\quad+f_{tt}\Big(dv-\frac{h_v}{\Delta}d\hat{r}\Big)^2\,. \end{aligned} \tag{2.8}$$

To make $d\bar{s}^2_\phi$ regular on the horizon, one must have

$$\chi_a=\frac{h_a+h^a_\chi\Delta}{h_v}+\mathcal{O}(\Delta^2)\,,\quad f_{tt}=h_{tt}\Delta^2+\mathcal{O}(\Delta^3)\,. \tag{2.9}$$

Again $h^a_\chi=h^a_\chi(\hat{r},\theta^i)$ and $h_{tt}=h_{tt}(\hat{r},\theta^i)$ must be regular on the horizon. Using these results and keeping only leading order corrections, one has for (2.1) at $\hat{r}\to r_H$,

$$\begin{aligned} ds^2_d \;&\approx\; f_r\Big\{-\Delta\frac{(d\hat{t}+f_ad\hat{\phi}^a)^2}{(h_v+f_ah_a+h_{\mathcal{A}}\Delta)^2}+\frac{d\hat{r}^2}{\Delta}\Big\}+g_{ij}d\theta^id\theta^j+h_{tt}\Delta^2d\hat{t}^2\\ &\quad+g_{ab}\Big(d\hat{\phi}^a-\frac{h_a+h^a_\chi\Delta}{h_v}d\hat{t}\Big)\Big(d\hat{\phi}^b-\frac{h_b+h^b_\chi\Delta}{h_v}d\hat{t}\Big)\,. \end{aligned} \tag{2.10}$$

If the same process is repeated for (2.2), one can find that when $\hat{r}\to r_H$,

$$ds^2_d \;\approx\; f_r\Big\{-\Delta\frac{(f_ad\hat{\phi}^a)^2}{(f_ah_a+h_{\mathcal{A}}\Delta)^2}+\frac{d\hat{r}^2}{\Delta}\Big\}+g_{ij}d\theta^id\theta^j+h_{tt}\Delta^2d\hat{t}^2$$

$$+g_{ab}\Big(d\hat{\phi}^a - \frac{h_a + h_\chi^a\Delta}{h_v}d\hat{t}\Big)\Big(d\hat{\phi}^b - \frac{h_b + h_\chi^b\Delta}{h_v}d\hat{t}\Big)\,. \tag{2.11}$$

As we will show in Appendix C, (2.10) with $h_{\mathcal{A}} = h_\chi^a = h_{tt} = 0$ is in fact exact (i.e., not an approximation) for a surprisingly large number of solutions.

Strictly speaking, our derivation of (2.10) and (2.11) is by no means the most general one. The whole process rests upon using the coordinate transformation (2.7) to render both $\mathcal{A}$ and $d\bar{s}_\phi^2$ finite on the horizon *separately*. One may as well try to think of other ways to make the whole metric (2.1) finite on the horizon all together. Since we have made no effort trying in such a direction, we will have nothing to say about this point. For the purpose of the paper, it is important to notice that (2.10) and (2.11) already appear to be general enough to cover all known stationary and axisymmetric black hole solutions.

For later convenience, lets calculate the black hole temperature for (2.10) and (2.11). For that purpose, we choose a static coordinate system with both $\hat{t}$ and $\hat{\phi}^a$ canonically normalized. The surface gravity is calculated with the particular Killing vector,

$$\xi = \partial_{\hat{t}} + \Omega_a \partial_{\hat{\phi}^a}\,. \tag{2.12}$$

Here the constants $\Omega_a$'s are chosen to make $\xi$ null on the (outer) horizon. They are interpreted as the angular velocities corresponding to the azimuthal angles $\hat{\phi}^a$. To see how $\Omega_a$'s can be calculated, note that for (2.10),

$$\xi^2 = \frac{-f_r\Delta\cdot(1+f_a\Omega_a)^2}{(h_v + f_a h_a + h_{\mathcal{A}}\Delta)^2} + g_{ab}\Big(\Omega_a - \frac{h_a + h_\chi^a\Delta}{h_v}\Big)\Big(\Omega_b - \frac{h_b + h_\chi^b\Delta}{h_v}\Big) + h_{tt}\Delta^2\,, \tag{2.13}$$

and for (2.11),

$$\xi^2 = \frac{-f_r\Delta\cdot(f_a\Omega_a)^2}{(f_a h_a + h_{\mathcal{A}}\Delta)^2} + g_{ab}\Big(\Omega_a - \frac{h_a + h_\chi^a\Delta}{h_v}\Big)\Big(\Omega_b - \frac{h_b + h_\chi^b\Delta}{h_v}\Big) + h_{tt}\Delta^2\,. \tag{2.14}$$

For both cases, to make $\xi$ vanish on the horizon one must have

$$\Omega_a = \frac{h_a^0}{h_v^0}\,, \quad h_a^0 = h_a(r_H)\,, \quad h_v^0 = h_v(r_H)\,. \tag{2.15}$$

Including corrections to the leading order, one has

$$\frac{h_a}{h_v} = \Omega_a + \Omega_a'\cdot(\hat{r} - r_H) + \mathcal{O}(\hat{r} - r_H)^2\,, \quad \Omega_a' \equiv \Big(\frac{h_a}{h_v}\Big)'\Big|_{\hat{r}=r_H}\,. \tag{2.16}$$

The surface gravity on the horizon can be calculated by using

$$\kappa^2 = \frac{(\partial\lambda)^2}{4\lambda}\Big|_{\hat{r}=r_H}\,, \quad \lambda = -\xi^2\,. \tag{2.17}$$

For non-extremal solutions, $\Delta(\hat{r}) = \Delta_0' \cdot (\hat{r} - r_H) + \mathcal{O}(\hat{r} - r_H)^2$ with $\Delta_0' = \Delta'(r_H)$. So to leading order,

$$\lambda = \frac{f_r^0}{h_v^{02}} \Delta_0' \cdot (\hat{r} - r_H) + \mathcal{O}(\hat{r} - r_H)^2 \,, \tag{2.18}$$

where $f_r^0 = f_r(r_H, \theta^i)$. The surface gravity (2.17) is then given by

$$\kappa^2 = \frac{g^{rr}\partial_{\hat{r}}\lambda \partial_{\hat{r}}\lambda}{4\lambda}\Big|_H = \frac{\Delta_0'^2}{4h_v^{02}} \,. \tag{2.19}$$

So the temperature of the black hole is given by

$$T_H = \frac{\kappa}{2\pi} = \frac{\Delta_0'}{4\pi h_v^0} \,. \tag{2.20}$$

For an extremal solution, $\Delta = \frac{1}{2}\Delta_0'' \cdot (\hat{r} - r_H)^2 + \mathcal{O}(\hat{r} - r_H)^3$ with $\Delta_0'' = \Delta''(r_H)$. One can find that $T_H = 0$. An easy way to see this is to start from (2.20) and then take the extremal limit

$$\Delta_0' \to 0 \quad \Longrightarrow \quad T_H \to 0 \,. \tag{2.21}$$

Note all the results starting from (2.15) are valid for both (2.10) and (2.11).

## 3 The Near-Horizon Metric for Extremal Black Holes

To get the near-horizon metric for an extremal black hole, one follows [16, 1, 6] and let

$$\hat{r} = r_H + y\lambda r_H \,, \quad \hat{t} = \frac{2h_v^0}{\lambda r_H \Delta_0''}\tilde{t} \,, \quad \hat{\phi}^a = \phi^a + \Omega_a \hat{t} \,. \tag{3.1}$$

Using $\Delta = \frac{1}{2}\Delta_0'' \cdot (\hat{r} - r_H)^2 + \mathcal{O}(\hat{r} - r_H)^3$ and after sending $\lambda \to 0$, one has for both (2.10) and (2.11),

$$\begin{aligned} ds^2 \;\; &= \;\; \frac{2f_r^0}{\Delta_0''}\Big( - y^2 d\tilde{t}^2 + \frac{dy^2}{y^2}\Big) + g_{ij}^0 d\theta^i d\theta^j \\ &\quad + g_{ab}^0 (d\phi^a + k^a y d\tilde{t})(d\phi^b + k^b y d\tilde{t}) \,, \end{aligned} \tag{3.2}$$

where $g_{ij}^0 = g_{ij}(r_H, \theta^i)$, and we have used (2.16) and have defined

$$k^a = -\frac{2h_v^0 \Omega_a'}{\Delta_0''} \,. \tag{3.3}$$

One can see that (3.2) is exactly of the form (1.2). Based on the argument made in the previous section, (3.2) is valid for all extremal stationary and axisymmetric black holes.

To get to the global coordinates, let

$$y = r + \sqrt{1 + r^2}\cos t \,, \quad \tilde{t} = \frac{\sqrt{1 + r^2}\sin t}{y} \,. \tag{3.4}$$

Then

$$
\begin{aligned}
-y^2 d\tilde{t}^2 + \frac{dy^2}{y^2} &= -(1+r^2)dt^2 + \frac{dr^2}{1+r^2}\,, \\
y d\tilde{t} &= r dt + d \ln\Big(\frac{1+\sqrt{1+r^2}\sin t}{\cos t + r\sin t}\Big)\,.
\end{aligned}
\tag{3.5}
$$

So by letting

$$
\phi^a \to \phi^a - k^a \ln\Big(\frac{1+\sqrt{1+r^2}\sin t}{\cos t + r\sin t}\Big)\,, \tag{3.6}
$$

one can rewrite the near-horizon metric (3.2) as

$$
\begin{aligned}
ds^2 &= \frac{2f_r^0}{\Delta_0''}\Big[-(1+r^2)dt^2 + \frac{dr^2}{1+r^2}\Big] + g_{ij}^0 d\theta^i d\theta^j \\
&\quad + g_{ab}^0 (d\phi^a + k^a r dt)(d\phi^b + k^b r dt)\,.
\end{aligned}
\tag{3.7}
$$

The significance of this form of the near-horizon metric in the context of the Kerr/CFT correspondence was first noticed in [2], then the importance was stressed upon again in [6] for black hole solutions in higher dimensions. More examples were then provided in [7].

# 4 The Central Charge(s) of the Dual CFT(s)

Following [1] one can try to calculate the black hole entropy by studying dynamics on the near-horizon metric (3.7), with the help of appropriate boundary conditions. The symmetries of the corresponding phase space are generated by $[\frac{d-1}{2}]$ commuting generators [6], namely

$$
\xi_m^a = -e^{-im\phi^a}\,\partial_{\phi^a} - imre^{-im\phi^a}\partial_r\,, \quad a = 1\,, \cdots, [\frac{d-1}{2}]\,. \tag{4.1}
$$

It is easy to check that

$$
i[\xi_m^a\,,\, \xi_n^a] = (m-n)\xi_{m+n}^a\,. \tag{4.2}
$$

These transformations generate $[\frac{d-1}{2}]$ commuting Virasoro algebras. For each Virasoro algebra, the phase space can be identified with that of a two-dimensional conformal field theory. The classical version of the charge $Q_{\xi_m^a}$ is defined in (B.31). To get the quantum version of the charge, we write

$$
Q_{\xi_m^a} = L_m^a - \alpha\delta_m\,, \tag{4.3}
$$

with $\alpha$ being some constant. From (B.31) and (B.47), it is easy to see that if $\xi_m^a$ is scaled by a factor, the right hand side of (4.3) also needs to be scaled by the same factor. Especially, one has

$$
Q_{[\xi_m^a, \xi_n^a]} = Q_{-i(m-n)\xi_{m+n}^a} = -i(m-n)\Big(L_{m+n}^a - \alpha\delta_{m+n}\Big)\,. \tag{4.4}
$$

So from (B.33),

$$
\begin{aligned}
[L^a_m\,,\,L^a_n] &= i\Big\{Q_{\xi^a_m}\,,\,Q_{\xi^a_n}\Big\}_{P.B.} = i\Big(Q_{[\xi^a_m,\xi^a_n]} + K[\xi^a_m,\xi^a_n]\Big) \\
&= (m-n)L_{m+n} - 2m\alpha\delta_{m+n} + iK[\xi^a_m,\xi^a_n]\,. 
\end{aligned} \tag{4.5}
$$

Comparing this with the usual relation,

$$
[L^a_m\,,\,L^a_n] = (m-n)L^a_{m+n} + \frac{c^a}{12}m(m^2-1)\delta_{m+n}\,, \tag{4.6}
$$

one gets

$$
K[\xi^a_m,\xi^a_n] = -i\frac{c^a}{12}m\Big(m^2-1+\frac{24\alpha}{c^a}\Big)\delta_{m+n}\,. \tag{4.7}
$$

So the central charge $c^a$ is determined by the coefficient of the $m^3$ term in $K[\xi^a_m,\xi^a_n]$. The term linear in $m$ is not so important because $\alpha$ is a free parameter.

The central term $K[\xi^a_m,\xi^a_n]$ corresponding to the near-horizon metric (3.7) is calculated in (A.12),

$$
K[\xi^a_m,\xi^a_n] = -\frac{i(m-n)n^2k^a}{16\pi}\delta_{m+n}\mathcal{A}_{rea}\,, \tag{4.8}
$$

with $\mathcal{A}_{rea}$ being the horizon area for either (2.10) or (2.11). Comparing this result with (4.7), one has

$$
c^a = \frac{3k^a}{2\pi}\mathcal{A}_{rea}\,. \tag{4.9}
$$

Note this result only contains the contribution from the Einstein-Hilbert action.

# 5 The Entropy

In the following, we shall try to relate the central charge to the entropy by using Cardy's formula. Again following [1], one can adopt the Frolov-Thorne vacuum [17] to provide a definition of the vacuum state for the extremal metric. One important task here is to derive the left-moving and right-moving temperatures. We will do it by starting with non-extremal metrics and then take the extremal limit.

Quantum fields for the general (non-extremal) metrics (2.1) and (2.2) can be expanded in eigenstates with asymptotic energy $\omega$ and angular momentum $m_a$, with $\hat{t}$ and $\hat{\phi}^a$ dependence $e^{-i\omega\hat{t}+im_a\hat{\phi}^a}$. In terms of the redefined $\tilde{t}$ and $\phi^a$ coordinates of the extremal near-horizon limit, given by (3.1), we have

$$
e^{-i\omega\hat{t}+im_a\hat{\phi}^a} = e^{-in_R\tilde{t}+in^a_L\phi^a}\,, \tag{5.1}
$$

with[1]

$$n_L^a = m_a\,, \quad n_R = \frac{2\tilde{h}_v^0}{\tilde{\Delta}_0'' r_H \lambda}(w - m_a\tilde{\Omega}_a)\,. \tag{5.2}$$

The left-moving and right-moving temperatures $T_L$ and $T_R$ are then defined by writing the Boltzmann factor as

$$e^{-(\omega - m_a\Omega_a)/T_H} = e^{-n_L^a/T_L^a - n_R/T_R}\,. \tag{5.3}$$

As a result,

$$T_L^a = \frac{T_H}{\tilde{\Omega}_a - \Omega_a}\,, \quad T_R = \frac{2\tilde{h}_v^0}{\tilde{\Delta}_0'' r_H \lambda} T_H\,. \tag{5.4}$$

In a black hole solution, there should always be a parameter corresponding to each global charge that the solution may have. For a rotation $\Omega_a$, the corresponding global charge is angular momentum, and let's suppose the corresponding parameter in the solution is given by $\ell_a$. To obtain the extremal limit for the temperatures, one can take $\ell_a$ to its extremal value $\tilde{\ell}_a$. On the horizon,

$$\Delta(r_H) = 0 \quad \Longrightarrow \quad 0 = \frac{d\Delta(r_H)}{d\ell_a} = \frac{\partial\Delta(r_H)}{\partial\ell_a} + \frac{\partial\Delta(r_H)}{\partial r_H}\frac{dr_H}{d\ell_a}\,. \tag{5.5}$$

Because $\partial\Delta(r_H)/\partial\ell_a$ is finite[2], one has in the extremal limit

$$\frac{\partial\Delta(r_H)}{\partial r_H} \quad \longrightarrow \quad 0 \quad \Longrightarrow \quad \frac{dr_H}{d\ell_a} = -\frac{\partial\Delta(r_H)}{\partial\ell_a}\Big/\frac{\partial\Delta(r_H)}{\partial r_H} \quad \longrightarrow \quad \infty\,. \tag{5.6}$$

So in the extremal limit, $T_R = 0$ and

$$\begin{aligned} T_L^a &= \frac{T_H}{\tilde{\Omega}_a - \Omega_a}\Big|_{\ell_a\to\tilde{\ell}_a} = -\Big(\frac{dT_H}{d\ell_a}\Big/\frac{d\Omega_a}{d\ell_a}\Big)\Big|_{\ell_a\to\tilde{\ell}_a} \\ &= -\Big(\frac{\partial T_H}{\partial\ell_a} + \frac{\partial T_H}{\partial r_H}\frac{dr_H}{d\ell_a}\Big)\Big/\Big(\frac{\partial\Omega_a}{\partial\ell_a} + \frac{\partial\Omega_a}{\partial r_H}\frac{dr_H}{d\ell_a}\Big)\Big|_{\ell_a\to\tilde{\ell}_a} \\ &= -\Big(\frac{\partial T_H}{\partial r_H}\Big/\frac{\partial\Omega_a}{\partial r_H}\Big)\Big|_{\ell_a\to\tilde{\ell}_a} = -\frac{\tilde{T}_H'(r_H)}{\tilde{\Omega}_a'} = -\frac{\tilde{\Delta}_0''}{4\pi\tilde{\Omega}_a'\tilde{h}_v^0} \\ &= \frac{1}{2\pi k^a}\,, \end{aligned} \tag{5.7}$$

where we have used (3.3). The result (5.7) was first speculated to be true for general extremal black holes in four dimensions in [2]. It was then generalized to solutions in arbitrary dimensions in [6] based on all the examples that are studied. Here we have shown that (5.7) is true for all known extremal stationary and axisymmetric black holes.

[1] From now on until (5.7), any quantity from the extremal solution will be distinguished with a tilde. For example, $\tilde{\Omega}_a$ is an angular velocity for the extremal solution, while $\Omega_a$ is its counterpart for the non-extremal solution.

[2] Note $\partial\Delta(r_H)/\partial\ell_a = 0$ corresponds to the case where $\Delta(r)$ does not contain the parameter $\ell_a$, which in turn means that $r_H$ is independent of $\ell_a$. This is unlikely to happen.

Now by using (4.9), (5.7) and Cardy's formula for the entropy of a unitary conformal field theory at temperature $T_L$, we find that the microscopic entropy is given by (no summation over $a$)

$$S = \frac{1}{3}\pi^2 \, c_L^a \, T_L^a = \frac{\mathcal{A}_{rea}}{4} \,, \tag{5.8}$$

where we have identified $c_L^a$ with $c^a$. We see that this result exactly matches with the Bekenstein-Hawking entropy.

Since the central charge $c^a$ in (4.9) only contains the contribution from the gravitational field, the fact that (5.8) matches with the Bekenstein-Hawking entropy implies that the non-gravitational contributions to the central charge(s) are zero. This is consistent with the results found in [2, 4, 5].

## 6 Summary

In this paper, we have calculated the microscopic entropy for all known extremal stationary and axisymmetric black holes by using the Kerr/CFT correspondence.

We started by presenting two ansatzs (2.1) and (2.2) that are general enough to cover all known stationary and axisymmetric black holes. Then more constraints on the metrics are introduced from the fact that the black hole horizons are regular. A common form of the near-horizon metric (3.7) can be derived when the near-horizon limit is taken for extremal black holes. By using this near-horizon metric, we explicitly show that the microscopic entropy calculated by using Cardy's formula exactly matches with the Bekenstein-Hawking entropy. In this way, we have shown that the Kerr/CFT correspondence is applicable to all known extremal stationary and axisymmetric black holes.

For practical reasons, we have only considered the contribution from the Einstein-Hilbert action to the central charges. And the match of the microscopic and the macroscopic entropies indicates that the non-gravitational fields do not contribute to the central charge(s). Although one can certainly try to repeat the same process for more complicated theories, such as what has been done in [9], the calculation will be much more complicated.

Finally, being able to calculate the entropy for a large class of black holes by using a general method is an encouraging progress. We hope that the result obtained in this work can help lead to some true understanding of the microscopic origin of the black hole entropy.

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## A Calculating the Central Term $K[\xi,\zeta]$

The central term $K[\xi^a_m,\xi^a_n]$ for (3.7) can be calculated by using (B.36) and (B.47), which are derived by using the Einstein-Hilbert action alone.

Lets first write down the non-vanishing metric elements in (3.7),[3]

$$\begin{aligned} G_{tt} &= -A(1+r^2)+k^2r^2\,, \\ G_{at} &= G_{ta}=k_a r\,, \\ G_{ab} &= g^0_{ab}\,, \\ G_{ij} &= g^0_{ij}\,, \\ G_{rr} &= \frac{A}{1+r^2}\,, \end{aligned} \tag{A.1}$$

where $k_a=g^0_{ab}k^b$, $k^2=g^0_{ab}k^ak^b$ and $A=2f^0_r/\Delta''_0$. Note $f^0_r=f_r(r_H,\theta^i)$, $g^0_{ij}=g_{ij}(r_H,\theta^i)$ and $g^0_{ab}=g_{ab}(r_H,\theta^i)$ are functions of $\theta^i$'s only, while $\Delta''_0=\Delta''(r_H)$ and $k^a$'s are constant. Let $(g^{0ab})$ be the inverse of $(g^0_{ab})$, and $(g^{0ij})$ be the inverse of $(g^0_{ij})$, one has

$$\begin{aligned} G^{tt} &= -\frac{1}{A(1+r^2)}\,, \\ G^{at} &= G^{ta}=\frac{k^a r}{A(1+r^2)}\,, \\ G^{ab} &= g^{0ab}-\frac{k^ak^br^2}{A(1+r^2)}\,, \\ G^{ij} &= g^{0ij}\,, \\ G^{rr} &= \frac{1+r^2}{A}\,. \end{aligned} \tag{A.2}$$

For later convenience, note that

$$\begin{aligned} \Gamma^t_{ra} &= -\frac{1}{2A(1+r^2)}k_a\,, \\ \Gamma^t_{rt} &= \frac{r}{1+r^2}-\frac{k^2 r}{2A(1+r^2)}\,, \end{aligned}$$

[3]In this section, we shall use the capital letter $G$ to denote the full metric (3.7), in order to distinguish it from the elements $g^0_{ij}$ and $g^0_{ab}$.

$$
\begin{aligned}
\Gamma^r_{rr} &= -\frac{r}{1+r^2}\,, \\
\Gamma^a_{rb} &= \frac{r}{2A(1+r^2)}k^a k_b\,, \\
\Gamma^i_{rj} &= 0\,, \\
\Gamma^t_{rr} &= 0\,, \\
\Gamma^a_{rt} &= \frac{1-r^2}{2(1+r^2)}k^a + \frac{k^2r^2}{2A(1+r^2)}k^a\,.
\end{aligned} \tag{A.3}
$$

Given a particular azimuthal angle $\phi^{\bar{a}}$, and the Killing vector

$$
\xi_n = -e^{-in\phi^{\bar{a}}}\,\partial_{\phi^{\bar{a}}} - inre^{-in\phi^{\bar{a}}}\partial_r\,, \tag{A.4}
$$

the nontrivial elements of

$$
h_{\mu\nu}(\xi_n) = \mathcal{L}_{\xi_n}G_{\mu\nu} = \xi^\rho_n\partial_\rho G_{\mu\nu} + G_{\mu\rho}\partial_\nu\xi^\rho_n + G_{\rho\nu}\partial_\mu\xi^\rho_n \tag{A.5}
$$

are given by

$$
\begin{aligned}
h_{rr} &= \xi^r_n\partial_r G_{rr} + 2G_{rr}\partial_r\xi^r_n = -\frac{2ine^{-in\phi^{\bar{a}}}A}{(1+r^2)^2}\,, \\
h_{ra} &= G_{rr}\partial_a\xi^r_n = -\frac{n^2re^{-in\phi^{\bar{a}}}A}{1+r^2}\delta_{a\bar{a}}\,, \\
h_{tt} &= \xi^r_n\partial_r G_{tt} = 2inr^2e^{-in\phi^{\bar{a}}}(A-k^2)\,, \\
h_{ta} &= \xi^r_n\partial_r G_{ta} + G_{tb}\partial_a\xi^b_n = -inre^{-in\phi^{\bar{a}}}(k_a - k_{\bar{a}}\delta_{\bar{a}a})\,, \\
h_{ab} &= G_{ac}\partial_b\xi^c_n + G_{cb}\partial_a\xi^c_n = ine^{-in\phi^{\bar{a}}}(g^0_{a\bar{a}}\delta_{\bar{a}b} + g^0_{b\bar{a}}\delta_{\bar{a}a})\,.
\end{aligned} \tag{A.6}
$$

As a result, $h = 0$ and

$$
\begin{aligned}
h^{rr} &= G^{rr}G^{rr}h_{rr} = -\frac{2ine^{-in\phi^{\bar{a}}}}{A}\,, \\
h^{ra} &= G^{rr}G^{ab}h_{rb} = -n^2re^{-in\phi^{\bar{a}}}\left(g^{0a\bar{a}} - \frac{r^2k^ak^{\bar{a}}}{A(1+r^2)}\right), \\
h^{rt} &= G^{rr}G^{ta}h_{ra} = -\frac{n^2r^2e^{-in\phi^{\bar{a}}}}{A(1+r^2)}k^{\bar{a}}\,, \\
h^{tt} &= G^{tt}G^{tt}h_{tt} + 2G^{tt}G^{ta}h_{ta} + G^{ta}G^{tb}h_{ab} = \frac{2inr^2e^{-in\phi^{\bar{a}}}}{A(1+r^2)^2}\,, \\
h^{ta} &= G^{tt}G^{at}h_{tt} + (G^{tt}G^{ab} + G^{tb}G^{at})h_{tb} + G^{tb}G^{ac}h_{bc} \\
&= \frac{inre^{-in\phi^{\bar{a}}}}{A(1+r^2)}\left(\frac{1-r^2}{1+r^2}k^a + k^{\bar{a}}\delta^{\bar{a}a}\right), \\
h^{ab} &= G^{at}G^{bt}h_{tt} + (G^{at}G^{bc} + G^{ac}G^{bt})h_{tc} + G^{ac}G^{bd}h_{cd} \\
&= ine^{-in\phi^{\bar{a}}}\Big[\delta^{a\bar{a}}g^{0b\bar{a}} + \delta^{b\bar{a}}g^{0a\bar{a}} - \frac{2r^2k^ak^b}{A(1+r^2)^2}
\end{aligned}
$$

$$-\frac{r^2 k^{\bar{a}}(\delta^{a\bar{a}}k^b + \delta^{b\bar{a}}k^a)}{A(1+r^2)}\Big] \,. \tag{A.7}$$

From (B.47), one has

$$\begin{aligned} k^{rt} &= \xi^t_m \nabla^r h - \xi^t_m \nabla_\rho h^{r\rho} + \frac{h}{2}\nabla^t \xi^r_m - h^{t\rho}\nabla_\rho \xi^r_m + \xi_{m\rho}\nabla^t h^{r\rho} \\ &\quad -\xi^r_m \nabla^t h + \xi^r_m \nabla_\rho h^{t\rho} - \frac{h}{2}\nabla^r \xi^t_m + h^{r\rho}\nabla_\rho \xi^t_m - \xi_{m\rho}\nabla^r h^{t\rho} \,. \end{aligned} \tag{A.8}$$

We are only interested in terms that will lead to $m^3$ when $m+n=0$ is applied,

$$\begin{aligned} \xi^r_m \nabla_\rho h^{t\rho} &= \xi^r_m(\partial_\rho h^{t\rho} + \Gamma^t_{\rho\sigma}h^{\sigma\rho} + \Gamma^\rho_{\rho\sigma}h^{t\sigma}) \\ &\approx \xi^r_m(\partial_{\bar{a}} h^{t\bar{a}} + \partial_r h^{tr} + 2\Gamma^t_{ra}h^{ra} + 2\Gamma^t_{rt}h^{rt} + \Gamma^\rho_{\rho r}h^{tr}) \,, \\ &= \frac{imn^2 r^2 e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)}\Big(\frac{2r^2-2}{1+r^2}\Big)k^{\bar{a}} \,, \\ -h^{t\rho}\nabla_\rho \xi^r_m &= -h^{t\rho}(\partial_\rho \xi^r_m + \Gamma^r_{\rho\sigma}\xi^\sigma_m) \\ &\approx -h^{t\bar{a}}\partial_{\bar{a}}\xi^r_m - h^{tr}(\partial_r \xi^r_m + \Gamma^r_{rr}\xi^r_m) \\ &= \frac{imn^2 r^2 e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)}\Big(\frac{4m/n-2}{1+r^2}\Big)k^{\bar{a}} \,, \\ h^{r\rho}\nabla_\rho \xi^t_m &= h^{r\rho}(\partial_\rho \xi^t_m + \Gamma^t_{\rho\sigma}\xi^\sigma_m) \\ &\approx (h^{ra}\Gamma^t_{ar} + h^{rt}\Gamma^t_{tr})\xi^r_m \\ &= \frac{imn^2 r^2 e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)}\Big(\frac{r^2-1}{1+r^2}\Big)k^{\bar{a}} \,, \\ \xi_{m\rho}\nabla^t h^{r\rho} &= \xi^\rho_m G^{rr}(G^{tt}\nabla_t h_{r\rho} + G^{ta}\nabla_a h_{r\rho}) \\ &= \xi^\rho_m G^{rr}G^{tt}(\partial_t h_{r\rho} - \Gamma^\sigma_{tr}h_{\sigma\rho} - \Gamma^\sigma_{t\rho}h_{r\sigma}) \\ &\quad +\xi^\rho_m G^{rr}G^{ta}(\partial_a h_{r\rho} - \Gamma^\sigma_{ar}h_{\sigma\rho} - \Gamma^\sigma_{a\rho}h_{r\sigma}) \\ &\approx \xi^r_m G^{rr}G^{tt}(-\Gamma^{\bar{a}}_{tr}h_{\bar{a}r} - \Gamma^{\bar{a}}_{tr}h_{r\bar{a}}) \\ &\quad +\xi^{\bar{a}}_m G^{rr}G^{t\bar{a}}\partial_{\bar{a}}h_{r\bar{a}} + \xi^r_m G^{rr}G^{t\bar{a}}\partial_{\bar{a}}h_{rr} \\ &\quad +\xi^r_m G^{rr}G^{ta}(-\Gamma^{\bar{a}}_{ar}h_{\bar{a}r} - \Gamma^{\bar{a}}_{ar}h_{r\bar{a}}) \\ &= \frac{imn^2 r^2 e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)}\Big(\frac{6-2r^2}{1+r^2} - \frac{2n}{m}\Big)k^{\bar{a}} \,, \\ -\xi_{m\rho}\nabla^r h^{t\rho} &= -\xi_{m\rho}G^{rr}(\partial_r h^{t\rho} + \Gamma^t_{r\sigma}h^{\sigma\rho} + \Gamma^\rho_{r\sigma}h^{t\sigma}) \\ &\approx -\xi_{mr}G^{rr}(\partial_r h^{tr} + \Gamma^t_{rt}h^{tr} + \Gamma^t_{r\bar{a}}h^{\bar{a}r} + \Gamma^r_{rr}h^{tr}) \\ &= \frac{imn^2 r^2 e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)}\Big(1 - \frac{4}{1+r^2}\Big)k^{\bar{a}} \,, \end{aligned} \tag{A.9}$$

where "≈" means only terms contributing to $m^3$ are preserved. The integral in (B.36) is done at $r \to +\infty$. In this limit, we have from (A.8) and (A.9),

$$k^{rt} = \frac{i(m-n)n^2 e^{-i(m+n)\phi^{\bar{a}}}}{A}k^{\bar{a}} \,. \tag{A.10}$$

Now using (B.36) and (B.47), and noticing that

$$\oint (d^{d-2}x)_{\mu\nu} k^{\mu\nu} = \oint 2(d^{d-2}x)_{rt} k^{rt} \,, \quad (d^{d-2}x)_{rt} = \frac{1}{2} A \sqrt{|g^0_{ij}|} \sqrt{|g^0_{ab}|} \prod_i d\theta^i \prod_a d\phi^a \,, \tag{A.11}$$

one has

$$\begin{aligned} K[\xi^{\bar{a}}_m, \xi^{\bar{a}}_n] &= -\frac{i(m-n)n^2 k^{\bar{a}}}{16\pi} \oint \sqrt{|g^0_{ij}|} \sqrt{|g^0_{ab}|} \prod_i d\theta^i \prod_a d\phi^a e^{-i(m+n)\phi^{\bar{a}}} \\ &= -\frac{i(m-n)n^2 k^{\bar{a}}}{16\pi} \delta_{m+n} \mathcal{A}_{rea} \,. \end{aligned} \tag{A.12}$$

Note $\mathcal{A}_{rea} = \oint \sqrt{|g^0_{ij}|} \sqrt{|g^0_{ab}|} \prod_i d\theta^i \prod_a d\phi^a$ is the horizon area for both (2.10) and (2.11).

# B The Asymptotic Symmetry Group

Asymptotic symmetries are transformations that leave the metric invariant up to what is allowed by given boundary conditions. One convenient way to treat asymptotic symmetries is the covariant phase space method as in [12, 18], which is also good for exact symmetries. The formalism was first used to calculate the central charge of conformal symmetries related to a black hole horizon in [19]. After that, there have been a lot of further developments. Some examples can be found in [20, 21, 22, 23].

To motivate for the covariant phase space method, one starts with the classical mechanics (see, e.g.[24]). The Lagrangian is given by $L = L(q, \dot{q})$, where $q = q(t)$ describes the classical trajectory of a particle. For a small variation of the path,

$$\delta L = \Big( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \Big) \delta q + \frac{d}{dt} \Big( \frac{\partial L}{\partial \dot{q}} \delta q \Big) \,. \tag{B.1}$$

The equation of motion is given by

$$E = \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \,. \tag{B.2}$$

When this is linearized, one has

$$\delta E = \frac{\partial^2 L}{\partial q^2} \delta q + \frac{\partial^2 L}{\partial \dot{q} \partial q} \delta \dot{q} - \delta \dot{p} = 0 \,, \quad p = \frac{\partial L}{\partial \dot{q}} \,. \tag{B.3}$$

From the boundary term in (B.1), one can define $\Theta(q, \delta) = p \delta q$ and

$$\begin{aligned} \Omega(q; \delta_1, \delta_2) &= \delta_1 \Theta(q, \delta_2) - \delta_2 \Theta(q, \delta_1) \\ &= \delta_1 p \delta_2 q - \delta_2 p \delta_1 q \,, \end{aligned} \tag{B.4}$$

where $\delta_1$ and $\delta_2$ stands for two independent variations. Notice that $\Omega(q;\delta_1,\delta_2)$ is time independent if both $\delta_1 q$ and $\delta_2 q$ satisfy (B.3),

$$\frac{d\Omega(q;\delta_1,\delta_2)}{dt}=\delta_1\dot{p}\delta_2 q+\delta_1 p\delta_2\dot{q}-\delta_2\dot{p}\delta_1 q-\delta_2 p\delta_1\dot{q}=0\,. \tag{B.5}$$

The Hamiltonian of the system can now be defined as

$$\delta H=\Omega\Big(q;\delta,\frac{d}{dt}\Big)=\delta\Theta\Big(q,\frac{d}{dt}\Big)-\frac{d}{dt}\Theta(q,\delta)=\delta p\dot{q}-\dot{p}\delta q\,. \tag{B.6}$$

Here we have taken the liberty to generalize $\delta$ to other possible operators, such as $d/dt$. In the case of a curved spacetime, one might also use the Lie derivative $\mathcal{L}_\xi$. It follows that

$$\dot{q}=\frac{\partial H}{\partial p}\,,\quad \dot{p}=-\frac{\partial H}{\partial q}\,. \tag{B.7}$$

Using generalized coordinates, $\phi^a=\{q,p\}\,,\ a=1,2$, one can write

$$\Omega(\phi^a;\delta_1,\delta_2)=\Omega_{ab}\delta_1\phi^a\delta_2\phi^b\,,\quad (\Omega_{ab})=\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\,. \tag{B.8}$$

Let $(\Omega^{ab})$ be the inverse of $(\Omega_{ab})$,

$$(\Omega^{ab})=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\,, \tag{B.9}$$

the Poisson bracket of any two functions is then given by

$$\Big\{f\,,\,g\Big\}_{P.B.}=\Omega^{ab}\partial_a f\partial_b g=\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\,. \tag{B.10}$$

A special example is that, for $f=f(q,p)$,

$$\frac{df}{dt}=\frac{\partial f}{\partial q}\dot{q}+\frac{\partial f}{\partial p}\dot{p}=\frac{\partial f}{\partial q}\frac{\partial H}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial H}{\partial q}=\Big\{f\,,\,H\Big\}_{P.B.}\,. \tag{B.11}$$

For a more general system, there can be more coordinates than just $\{q,p\}$ and $\Omega_{ab}$ can be more complicated than in (B.8). By analogy to (B.6), one can try to construct a charge $Q_\xi$ corresponding to any symmetric transformation $\delta_\xi$,

$$\delta Q_\xi=\Omega(\phi^a;\delta,\delta_\xi)=\Omega_{ab}\delta\phi^a\delta_\xi\phi^b\,. \tag{B.12}$$

To make $Q_\xi$ a physically meaningful charge, the variation (B.12) needs to be integrable and $\Omega(\phi^a;\delta,\delta_\xi)$ needs to be constant in time. This will put extra constraints on $\delta\phi^a$ and $\delta_\xi\phi^a$, just as in the case above. Given two charges as defined in (B.12), the Poisson bracket is

$$\Big\{Q_\xi\,,\,Q_\zeta\Big\}_{P.B.}=\Omega^{ab}\frac{\delta Q_\xi}{\delta\phi^a}\frac{\delta Q_\zeta}{\delta\phi^b}=\Omega(\phi^a;\delta_\zeta,\delta_\xi)\,. \tag{B.13}$$

This result will play a central role in the treatment that follows.

Now consider a system with the Lagrangian density $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a, \partial_\mu \partial_\nu \phi^a, \cdots)$. The actions is

$$S = \int_{\mathcal{M}} \mathbf{L}\,, \quad \mathbf{L} = \mathcal{L}\sqrt{|g|}\; d^n x = \mathcal{L} * \mathbf{1}\,. \tag{B.14}$$

A symmetric transformation should leave the integrand $\mathbf{L}$ invariant or up to a total derivative which integrates to zero,

$$\delta_\epsilon \mathbf{L} = d\mathbf{M}_\epsilon\,, \quad \delta S = \int_{\mathcal{M}} d\mathbf{M}_\epsilon = \oint_{\partial\mathcal{M}} \mathbf{M}_\epsilon = 0\,. \tag{B.15}$$

On the other hand,

$$\delta_\epsilon \mathbf{L} = E_a \delta_\epsilon \phi^a * \mathbf{1} + d\mathbf{\Theta}(\phi^a, \delta_\epsilon)\,, \tag{B.16}$$

where all the terms involving a derivative on $\delta_\epsilon \phi^a$ have been moved into the $d\mathbf{\Theta}$ term. It is easy to see that $E_a = 0$ is the usual Euler-Lagrange equation for $\phi^a$. From (B.15) and (B.16), one can define a Noether current,

$$\mathbf{J}_\epsilon = \mathbf{\Theta}(\phi^a, \delta_\epsilon) - \mathbf{M}_\epsilon\,, \tag{B.17}$$

which becomes a closed form when the equations of motion are satisfied, $d\mathbf{J}_\epsilon = -E_a \cdot \delta_\epsilon \phi^a * \mathbf{1}$. So when $E_a = 0$, one should locally have $\mathbf{J}_\epsilon = d\mathbf{Q}_\epsilon$, with $\mathbf{Q}_\epsilon$ being some $n-2$ form. Now with appropriate boundary conditions, a conserved charge can be defined as

$$Q_\epsilon = \int_V d\mathbf{Q}_\epsilon = \oint_{\partial V} \mathbf{Q}_\epsilon\,, \tag{B.18}$$

where $V$ is a space-like slice of the spacetime manifold $\mathcal{M}$. The charge $\mathbf{Q}_\epsilon$ is defined up to an arbitrary closed form, but this ambiguity drops out in (B.18).

For a transformation generated by the Lie derivative, $\delta_\xi \phi^a = \mathcal{L}_\xi \phi^a$, one has

$$\begin{aligned} \delta_\xi \mathbf{L} &= E_a \cdot \mathcal{L}_\xi \phi^a * \mathbf{1} + d\mathbf{\Theta}(\phi^a, \mathcal{L}_\xi) \\ &= \mathcal{L}_\xi \mathbf{L} = d(i_\xi \mathbf{L})\,. \end{aligned} \tag{B.19}$$

The Noether current (B.17) is

$$\mathbf{J}_\xi = \mathbf{\Theta}(\phi^a, \mathcal{L}_\xi) - i_\xi \mathbf{L}\,. \tag{B.20}$$

By analogy to (B.4), one can define

$$\Omega(\phi^a; \delta_1, \delta_2) = \int_V \mathbf{w}(\phi^a; \delta_1, \delta_2)\,, \tag{B.21}$$

$$\mathbf{w}(\phi^a; \delta_1, \delta_2) = \delta_1 \mathbf{\Theta}(\phi^a, \delta_2) - \delta_2 \mathbf{\Theta}(\phi^a, \delta_1)\,. \tag{B.22}$$

The quantity $\Omega(\phi^a;\delta_1,\delta_2)$ is conserved if

$$d\mathbf{w}(\phi^a;\delta_1,\delta_2)=0 \quad\Longrightarrow\quad \oint_{\partial\mathcal{M}}\mathbf{w}=\int_{\mathcal{M}}d\mathbf{w}=0\,. \tag{B.23}$$

Notice that,

$$0 \;=\; (\delta_1\delta_2-\delta_1\delta_2)(\mathcal{L}*\mathbf{1}) \quad\Longleftrightarrow\quad \delta_1\delta_2\phi^a=\delta_1\delta_2\phi^a\,, \tag{B.24}$$

$$\;=\; (\delta_1 E_a\delta_2\phi^a-\delta_2 E_a\delta_1\phi^a)*\mathbf{1}+d\mathbf{w}(\phi^a;\delta_1,\delta_2)\,. \tag{B.25}$$

As a result,

$$d\mathbf{w}(\phi^a;\delta_1,\delta_2)=0 \quad\Longrightarrow\quad \delta_1 E_a=\delta_2 E_a=0\,. \tag{B.26}$$

So $\delta_1\phi^a$ and $\delta_2\phi^a$ must both satisfy the linearized equations of motion for $\phi^a$, in order that $\Omega(\phi^a;\delta_1,\delta_2)$ can be constant in time. When this condition is satisfied, one can try to construct a charge corresponding to $\delta_\xi=\mathcal{L}_\xi$, by analogy to (B.6),

$$\delta Q_\xi=\Omega(\phi^a;\delta,\mathcal{L}_\xi)=\int_V\mathbf{w}(\phi^a;\delta,\mathcal{L}_\xi)\,. \tag{B.27}$$

The variation of the Noether current (B.20) is

$$\begin{aligned}\delta\mathbf{J}_\xi &= \delta\mathbf{\Theta}(\phi^a,\mathcal{L}_\xi)-i_\xi\delta\mathbf{L}\\ &= \delta\mathbf{\Theta}(\phi^a,\mathcal{L}_\xi)-\mathcal{L}_\xi\mathbf{\Theta}(\phi^a,\delta)+d\Big[i_\xi\mathbf{\Theta}(\phi^a,\delta)\Big]\,,\end{aligned} \tag{B.28}$$

where the second line is obtained for $E_a=0$. As a result,

$$\begin{aligned}\mathbf{w}(\phi^a;\delta,\mathcal{L}_\xi) &= \delta\mathbf{\Theta}(\phi^a,\mathcal{L}_\xi)-\mathcal{L}_\xi\mathbf{\Theta}(\phi^a,\delta)=d\mathbf{k}_\xi(\phi^a,\delta)\,,\\ \Longrightarrow\quad \delta Q_\xi &= \oint_{\partial V}\mathbf{k}_\xi(\phi^a,\delta)\,,\end{aligned} \tag{B.29}$$

with

$$\mathbf{k}_\xi(\phi^a,\delta)=\delta\mathbf{Q}_\xi-i_\xi\mathbf{\Theta}(\phi^a,\delta)\,. \tag{B.30}$$

Note that $\delta(\mathcal{L}_\xi\phi^a)=\mathcal{L}_\xi(\delta\phi^a)$, so both $\delta$ and $\mathcal{L}_\xi$ satisfy the assumption made about the operators $\delta_1$ and $\delta_2$ in (B.24). From (B.29),

$$Q_\xi(\phi)=\int_{\bar\phi}^{\phi}\delta Q_\xi+Q_\xi(\bar\phi)=\int_{\bar\phi}^{\phi}\oint_{\partial V}\mathbf{k}_\xi(\phi^a,\delta)+Q_\xi(\bar\phi)\,, \tag{B.31}$$

where $Q_\xi(\bar\phi)$ is the value of the charge on a given background. For the charge $Q_\xi(\phi)$ to be well defined, one expects the integral to be finite. Now given two such charges (say $Q_\xi$ and $Q_\zeta$), the Poisson bracket is found by analogy to (B.13),

$$\Big\{Q_\xi\,,\,Q_\zeta\Big\}_{P.B.} \;=\; \Omega(\phi^a;\mathcal{L}_\zeta,\mathcal{L}_\xi)=\oint_{\partial V}\mathbf{k}_\xi(\phi^a,\mathcal{L}_\zeta)\,. \tag{B.32}$$

It was shown in [25, 26] that with appropriate boundary conditions, the Poisson bracket $\{Q_\xi\,,\,Q_\zeta\}_{P.B.}$ of any differentiable generators $Q_\xi$ and $Q_\zeta$ takes the form

$$\Big\{Q_\xi\,,\,Q_\zeta\Big\}_{P.B.}=Q_{[\xi,\zeta]}+K[\xi,\zeta]\,, \tag{B.33}$$

where $K[\xi,\zeta]$ is a potential central extension to the algebra. It is demonstrated in [26] that a constant shift in the charges will not affect the nontrivial part of $K[\xi,\zeta]$. Using this, we can shift the charges by some constant and let $Q_{[\xi,\zeta]}(\bar\phi)=0$ in a chosen background. Then we get

$$K[\xi,\zeta]=\Big\{Q_\xi\,,\,Q_\zeta\Big\}_{P.B.}=\oint_{\partial V}\mathbf{k}_\xi(\bar\phi^a,\mathcal{L}_\zeta)\,. \tag{B.34}$$

Note that if instead of using (B.27), had we chosen to define

$$\delta Q_\xi=-\Omega(\phi^a;\delta,\mathcal{L}_\xi)=-\int_V\mathbf{w}(\phi^a;\delta,\mathcal{L}_\xi)\,, \tag{B.35}$$

we would have got

$$K[\xi,\zeta]=\Big\{Q_\xi\,,\,Q_\zeta\Big\}_{P.B.}=-\Omega(\phi^a;\mathcal{L}_\xi,\mathcal{L}_\zeta)=-\oint_{\partial V}\mathbf{k}_\xi(\phi^a,\mathcal{L}_\zeta)\,. \tag{B.36}$$

This result was used in the calculation of the Kerr/CFT correspondence [1].

In the case of pure gravity supplemented with a cosmological constant, the Lagrangian density is given by

$$\mathcal{L}=\frac{R-2\Lambda}{16\pi}\,. \tag{B.37}$$

For an infinitesimal variation of the metric,

$$\delta\mathbf{L}=\frac{1}{16\pi}\Big(-R^{\mu\nu}+\frac{R-2\Lambda}{2}g^{\mu\nu}+\nabla^\mu\nabla^\nu-g^{\mu\nu}\nabla_\rho\nabla^\rho\Big)\delta g_{\mu\nu}*\mathbf{1}\,. \tag{B.38}$$

Einstein's equations are

$$E^{\mu\nu} \;=\; R^{\mu\nu}-\frac{R-2\Lambda}{2}g^{\mu\nu}=0\,, \tag{B.39}$$

$$\Longrightarrow\quad R_{\mu\nu} \;=\; \frac{2\Lambda}{n-2}g_{\mu\nu}\,,\quad R=\frac{2n\Lambda}{n-2}\,. \tag{B.40}$$

When (B.39) is linearized, one has

$$\begin{aligned}0=\delta E_{\mu\nu} \;=\;& \frac{1}{2}\Big[\nabla^\rho(\nabla_\mu h_{\nu\rho}+\nabla_\nu h_{\mu\rho})-\partial^\rho\partial_\rho h_{\mu\nu}-\nabla_\mu\nabla_\nu h\Big]\\ &-\frac{1}{2}\Big[\nabla_\mu\nabla_\nu h^{\mu\nu}-\partial^\rho\partial_\rho h-R^{\rho\sigma}h_{\rho\sigma}\Big]g_{\mu\nu}-\frac{R-2\Lambda}{2}h_{\mu\nu}\,,\end{aligned} \tag{B.41}$$

where $h_{\mu\nu}=\delta g_{\mu\nu}$ and $h=g^{\mu\nu}h_{\mu\nu}$. Taking the trace of (B.41), one has

$$\nabla_\mu\nabla_\nu h^{\mu\nu}-\partial^\rho\partial_\rho h-R^{\mu\nu}h_{\mu\nu}=0\,. \tag{B.42}$$

From (B.19),

$$
\begin{aligned}
\mathbf{\Theta}(g_{\mu\nu},\delta) &= \frac{1}{16\pi}(d^{n-1}x)_\mu\Big[\nabla_\nu h^{\mu\nu}-\nabla^\mu h\Big]\,,\\
\Longrightarrow\quad i_\xi\mathbf{\Theta}(g_{\mu\nu},\delta) &= \frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}2\xi^\nu(\nabla_\nu h^{\mu\nu}-\nabla^\mu h)\\
&= \frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}(-I^{\mu\nu}_{\Theta_\xi})\,,
\end{aligned}
\tag{B.43}
$$

where

$$
I^{\mu\nu}_{\Theta_\xi}=\xi^\mu\nabla_\rho h^{\nu\rho}-\xi^\nu\nabla_\rho h^{\mu\rho}+\xi^\nu\nabla^\mu h-\xi^\mu\nabla^\nu h\,. \tag{B.44}
$$

The Noether current (B.20) is

$$
\begin{aligned}
\mathbf{J}_\xi &= \frac{1}{16\pi}(d^{n-1}x)_\mu\Big[\nabla^\nu\nabla^\mu\xi_\nu+\partial^\rho\partial_\rho\xi^\mu-2\nabla^\mu\nabla^\nu\xi_\nu-(R-2\Lambda)\xi^\mu\Big]\\
&= -\frac{1}{16\pi}(d^{n-1}x)_\mu\nabla_\nu\Big[\nabla^\mu\xi^\nu-\nabla^\nu\xi^\mu\Big]\,,\\
\Longrightarrow\quad \mathbf{Q}_\xi &= -\frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}(\nabla^\mu\xi^\nu-\nabla^\nu\xi^\mu)\,,
\end{aligned}
\tag{B.45}
$$

where we have used (B.39). Note that $\delta\mathbf{Q}_\xi=\frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}I^{\mu\nu}_{Q_\xi}$, with

$$
\begin{aligned}
I^{\mu\nu}_{Q_\xi}=-\frac{h}{2}(\nabla^\mu\xi^\nu-\nabla^\nu\xi^\mu)+h^{\mu\rho}\nabla_\rho\xi^\nu-h^{\nu\rho}\nabla_\rho\xi^\mu\\
-(\nabla^\mu h^{\nu\rho}-\nabla^\nu h^{\mu\rho})\xi_\rho\,.
\end{aligned}
\tag{B.46}
$$

From (B.30), one gets that

$$
\begin{aligned}
\mathbf{k}_\xi(g_{\mu\nu},\delta) &= \frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}k^{\mu\nu}\,,\\
k^{\mu\nu}=I^{\mu\nu}_{Q_\xi}+I^{\mu\nu}_{\Theta_\xi} &= \xi^\nu\nabla^\mu h-\xi^\nu\nabla_\rho h^{\mu\rho}+\frac{h}{2}\nabla^\nu\xi^\mu-h^{\nu\rho}\nabla_\rho\xi^\mu+\xi_\rho\nabla^\nu h^{\mu\rho}\\
&\quad-(\mu\leftrightarrow\nu)\,.
\end{aligned}
\tag{B.47}
$$

This result matches with that given in [5] up to a trivial term. Note [1] uses a formula for $\mathbf{k}_\xi(g_{\mu\nu},\delta)$ with the opposite sign, for which to make sense, we need to use (B.35) and (B.36).

To clarify the notations involved, note that we write a $p$-form as

$$
\mathbf{w}_p=\frac{1}{p!}w_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,. \tag{B.48}
$$

Its Hodge-$*$ dual is defined by (note $|\epsilon_{\dots}|=\sqrt{|g|}$)

$$
*\,\mathbf{w}_p=w^{\mu_1\cdots\mu_p}\frac{1}{p!(n-p)!}\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{n-p}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{n-p}}\,. \tag{B.49}
$$

One can also write it as

$$
*\,\mathbf{w}_p = (d^{n-p}x)_{\mu_1\cdots\mu_p}w^{\mu_1\cdots\mu_p}\,, \tag{B.50}
$$

$$(d^{n-p}x)_{\mu_1\cdots\mu_p} = \frac{1}{p!(n-p)!}\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{n-p}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{n-p}}\,. \tag{B.51}$$

With this, Stokes's theorem $\int_\Sigma d*\mathbf{w}_p = \oint_{\partial\Sigma}*\mathbf{w}_p$ can be written as

$$\int_\Sigma (d^{n-p+1}x)_{\mu_2\cdots\mu_p}\nabla_{\mu_1}w^{\mu_1\mu_2\cdots\mu_p} = \oint_{\partial\Sigma}(d^{n-p}x)_{\mu_2\cdots\mu_p\mu_1}w^{\mu_1\mu_2\cdots\mu_p}\,. \tag{B.52}$$

# C Some Examples

In this section, we use some examples to illustrate some major points made in the main context. The majority of the examples have been studied in [8, 6, 7]. Here we discuss them again by using the new perspective that we have gained from the present work. Since all the calculations after (2.10) and (2.11) evolve in a straight forward manor, our goal here is to show that all the examples can be put into the form of either (2.10) or (2.11) as $\hat{r}\to r_H$.

One intriguing result we find is that a surprisingly large number of solutions are exactly of the form (2.10) with $h_{\mathcal{A}} = h^a_\chi = h_{tt} = 0$. This feature could be helpful when one is trying to look for new solutions.

## C.1 Kerr-NUT-AdS Solutions in Diverse Dimensions

Lets start with examples studied in [8].

The first example is the Kerr-AdS solution in four dimensions [27],

$$\begin{aligned} ds^2 &= \rho^2\Big(\frac{d\hat{r}^2}{\Delta}+\frac{d\theta^2}{\Delta_\theta}\Big)+\frac{\Delta_\theta\sin^2\theta}{\rho^2}\Big(ad\hat{t}-\frac{\hat{r}^2+a^2}{\Xi}d\hat{\phi}\Big)^2-\frac{\Delta}{\rho^2}\Big(d\hat{t}-\frac{a\sin^2\theta}{\Xi}d\hat{\phi}\Big)^2\,,\\ \rho^2 &= \hat{r}^2+a^2\cos^2\theta\,,\qquad \Delta=(\hat{r}^2+a^2)(1+\hat{r}^2\ell^{-2})-2M\hat{r}\,,\\ &\Delta_\theta = 1-a^2\ell^{-2}\cos^2\theta\,,\qquad \Xi=1-a^2\ell^{-2}\,. \end{aligned} \tag{C.1}$$

It is is a solution to the equations of motion $R_{\mu\nu} = -3\ell^{-2}\,g_{\mu\nu}$. Comparing with (2.4) and (2.5), it is easy to see that

$$\begin{aligned} \mathcal{A} &= d\hat{t}-\frac{a\sin^2\theta}{\Xi}d\hat{\phi}+\frac{\rho^2}{\Delta}dr\\ &= d\hat{t}-\frac{a\sin^2\theta}{\Xi}d\hat{\phi}+\frac{r^2+a^2-a^2\sin^2\theta}{\Delta}dr\,,\\ \Longrightarrow\quad h_v &= r^2+a^2\,,\quad h_\phi = a\,\Xi\,,\quad h_{\mathcal{A}}=0\,. \end{aligned} \tag{C.2}$$

One sees that the metric is exactly of the form (2.10) with $h_{\mathcal{A}} = h^\phi_\chi = h_{tt} = 0$.

The second example is the five-dimensional rotating black hole with $S^3$ horizon topology. The solutions was obtained by Hawking, Hunter and Taylor-Robinson [28], satisfying the equations of motion $R_{\mu\nu} = -4\ell^{-2}\, g_{\mu\nu}$. The metric, which generalizes the Ricci-flat rotating black hole of Myers and Perry [29], is given by

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}(d\hat{t} - \frac{a\sin^2\theta}{\Xi_a}d\phi_1 - \frac{b\cos^2\theta}{\Xi_b}d\phi_2)^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}(ad\hat{t} - \frac{(\hat{r}^2+a^2)}{\Xi_a}d\phi_1)^2 \\ &\quad + \frac{\Delta_\theta\cos^2\theta}{\rho^2}(bd\hat{t} - \frac{(\hat{r}^2+b^2)}{\Xi_b}d\phi_2)^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &\quad + \frac{1+\hat{r}^2\ell^{-2}}{\hat{r}^2\rho^2}\Big(abd\hat{t} - \frac{b(\hat{r}^2+a^2)\sin^2\theta}{\Xi_a}d\phi_1 - \frac{a(\hat{r}^2+b^2)\cos^2\theta}{\Xi_b}d\phi_2\Big)^2\,, \end{aligned} \tag{C.3}$$

where

$$\begin{aligned} &\Delta = \frac{1}{\hat{r}^2}(\hat{r}^2+a^2)(\hat{r}^2+b^2)(1+\hat{r}^2\ell^{-2}) - 2M\,, \quad \Delta_\theta = 1 - a^2\ell^{-2}\cos^2\theta - b^2\ell^{-2}\sin^2\theta\,, \\ &\rho^2 = \hat{r}^2 + a^2\cos^2\theta + b^2\sin^2\theta\,, \qquad \Xi_a = 1 - a^2\ell^{-2}\,, \qquad \Xi_b = 1 - b^2\ell^{-2}\,. \end{aligned} \tag{C.4}$$

Note that in this coordinate system, the metric is asymptotic to $AdS_5$ in a rotating frame, with angular velocities $\Omega^\infty_{\phi_1} = -a\ell^{-2}$ and $\Omega^\infty_{\phi_2} = -b\ell^{-2}$. By letting

$$\phi_1 \to \phi_1 - a\ell^{-2}\hat{t}\,, \quad \phi_2 \to \phi_2 - b\ell^{-2}\hat{t}\,, \tag{C.5}$$

one can change to an asymptotically static coordinate system. The metric is now given by

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}\left[\Big(1 + \frac{a^2\ell^{-2}\sin\theta^2}{\Xi_a} + \frac{b^2\ell^{-2}\cos\theta^2}{\Xi_b}\Big)d\hat{t} - \frac{a\sin^2\theta}{\Xi_a}d\phi_1 - \frac{b\cos^2\theta}{\Xi_b}d\phi_2\right]^2 \\ &\quad + \frac{\rho^2}{\Delta}d\hat{r}^2 + \frac{\Delta_\theta\sin^2\theta(\hat{r}^2+a^2)^2}{\rho^2\Xi_a^2}\Big(d\phi_1 - \frac{a(1+\hat{r}^2\ell^{-2})}{\hat{r}^2+a^2}d\hat{t}\Big)^2 \\ &\quad + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta\cos^2\theta(\hat{r}^2+b^2)^2}{\rho^2\Xi_b^2}\Big(d\phi_2 - \frac{b(1+\hat{r}^2\ell^{-2})}{\hat{r}^2+b^2}d\hat{t}\Big)^2 \\ &\quad + \frac{a^2b^2(1+\hat{r}^2\ell^{-2})}{\hat{r}^2\rho^2}\Bigg\{\frac{(\hat{r}^2+a^2)\sin^2\theta}{a\Xi_a}\Big(d\phi_1 - \frac{a(1+\hat{r}^2\ell^{-2})}{\hat{r}^2+a^2}d\hat{t}\Big) \\ &\qquad + \frac{(\hat{r}^2+b^2)\cos^2\theta}{b\Xi_b}\Big(d\phi_2 - \frac{b(1+\hat{r}^2\ell^{-2})}{\hat{r}^2+b^2}d\hat{t}\Big)\Bigg\}^2\,. \end{aligned} \tag{C.6}$$

From (2.4) and (2.5),

$$\begin{aligned} \mathcal{A} &= \Big(1 + \frac{a^2\ell^{-2}\sin\theta^2}{\Xi_a} + \frac{b^2\ell^{-2}\cos\theta^2}{\Xi_b}\Big)d\hat{t} \\ &\quad - \frac{a\sin^2\theta}{\Xi_a}d\phi_1 - \frac{b\cos^2\theta}{\Xi_b}d\phi_2 + \frac{\rho^2}{\Delta}dr\,. \end{aligned} \tag{C.7}$$

Comparing (C.6) with (2.10), we find

$$\begin{aligned} h_v &= \frac{(\hat{r}^2+a^2)(\hat{r}^2+b^2)}{\hat{r}^2}\,, \quad h_1 = \frac{a(1+\hat{r}^2\ell^{-2})}{\hat{r}^2+a^2}h_v\,, \\ h_2 &= \frac{b(1+\hat{r}^2\ell^{-2})}{\hat{r}^2+b^2}h_v\,, \quad h_{\mathcal{A}} = 0\,. \end{aligned} \tag{C.8}$$

It is easy to see that (C.6) is of the form (2.10) with $h_{\mathcal{A}} = h^a_\chi = h_{tt} = 0$.

In the following, we shall consider the general Kerr-NUT-AdS solutions found in [30], which solve the Einstein equation $R_{\mu\nu} = -(d-1)\ell^{-2}\, g_{\mu\nu}$. The case of Kerr-AdS solutions have been studied in [8] and [6]. Since the NUT parameters will not affect anything in the process, here we will include them as well. Also, we will choose to write the metric by analogy to (40) and (48) in [30], which specialized to seven and six dimensions respectively. In even dimensions, $d = 2n$, the metric is given by

$$
\begin{aligned}
ds^2_{2n} &= \sum_{i=1}^{n} \Big(\frac{f_i dx_i^2}{X_i} + \frac{X_i}{f_i}\mathcal{A}_i^2\Big)\,, \quad f_i = \prod_{j\neq i}(x_i^2 - x_j^2)\,, && \text{(C.9)}\\
\mathcal{A}_i &= dt + \sum_{j\neq i} x_j^2 d\phi_1 + \sum_{j,k\neq i} x_j^2 x_k^2 d\phi_2 + \cdots + \prod_{j\neq i} x_j^2 d\phi_{n-1}\,,\\
X_i &= 2M_i x_i + \sum_{j=0}^{n-1} c_{2j} x_i^{2j} + g^2 x_i^{2n}\,. && \text{(C.10)}
\end{aligned}
$$

In odd dimensions, $d = 2n+1$, the metric is given by

$$
ds^2_{2n+1} = ds^2_{2n} + \frac{c_n}{\prod_{i=1}^n x_i^2}\mathcal{A}_n^2\,, \qquad \text{(C.11)}
$$

with

$$
\begin{aligned}
\mathcal{A}_n &= dt + \sum_{i=1}^{n} x_i^2 d\phi_1 + \sum_{i,j=1}^{n} x_i^2 x_j^2 d\phi_2 + \cdots + \prod_{i=1}^{n} x_i^2 d\phi_n\,,\\
X_i &= (-1)^{\frac{d-1}{2}}\frac{c_n}{x_i^2} + 2M_i + \sum_{j=1}^{n-1} c_{2j} x_i^{2j} + g^2 x_i^{2n}\,,\\
\mathcal{A}_{i\neq 1} &= dt + \sum_{j\neq 1,i} x_j^2 d\phi_1 + \sum_{j,k\neq 1,i} x_j^2 x_k^2 d\phi_2 + \cdots + \prod_{j\neq 1,i} x_j^2 d\phi_{n-2}\\
&\quad -r^2\Big(d\phi_1 + \sum_{j\neq 1,i} x_j^2 d\phi_2 + \cdots + \prod_{j\neq 1,i} x_j^2 d\phi_{n-1}\Big)\\
&= dt - r^2 d\phi_1 + \sum_{j\neq 1,i} x_j^2 (d\phi_1 - r^2 d\phi_2) + \cdots\\
&\quad + \prod_{j\neq 1,i} x_j^2 \Big(d\phi_{n-2} - r^2 d\phi_{n-1}\Big)\,. && \text{(C.12)}
\end{aligned}
$$

Note we have wick rotated the radial direction $r^2 \to -x_1^2$ so that the metrics (C.9) and (C.11) can be put into a compact form. To get back to the Lorentzian signature black hole metric, one needs to wick rotate back, $x_1^2 \to -r^2$. Especially, one has

$$
\begin{aligned}
f_1 &= (-1)^{n-1}\tilde{f}_1(r)\,, \quad X_1 = (-1)^n X(r)\,,\\
\tilde{f}_1(r) &= r^{2(n-1)} + r^{2(n-2)}\sum_{j>1} x_j^2 + r^{2(n-3)}\sum_{j,k>1} x_j^2 x_k^2 + \cdots + \prod_{j>1} x_j^2\,,
\end{aligned}
$$

$$X(r) = g^2 r^{2n} + \cdots . \tag{C.13}$$

Now from (2.4) and (2.5), one has for both (C.9) and (C.11),

$$\mathcal{A} = \mathcal{A}_1 + \frac{\tilde{f}_1 dr}{X} . \tag{C.14}$$

As a result, for both even and odd dimensions ($i \le n-1$),

$$h_v = r^{2(n-1)} , \quad h_i = r^{2(n-1-i)} , \quad h_{\mathcal{A}} = 0 . \tag{C.15}$$

From (C.12),

$$\begin{aligned} \mathcal{A}_{i\neq 1} &= dt - \frac{h_v}{h_1} d\phi_1 + \sum_{j\neq 1,i} x_j^2 \Big[ \Big( d\phi_1 - \frac{h_1}{h_v} dt \Big) - r^2 \Big( d\phi_2 - \frac{h_2}{h_v} dt \Big) \Big] \\ &\quad + \cdots + \prod_{j\neq 1,i} x_j^2 \Big[ \Big( d\phi_{n-2} - \frac{h_{n-2}}{h_v} dt \Big) - r^2 \Big( d\phi_{n-1} - \frac{h_{n-1}}{h_v} dt \Big) \Big] . \end{aligned} \tag{C.16}$$

In odd dimensions, we also have

$$\begin{aligned} \mathcal{A}_n &= dt + \sum_{i=1}^{n} x_i^2 d\phi_1 + \sum_{i,j=1}^{n} x_i^2 x_j^2 d\phi_2 + \cdots + \prod_{i=1}^{n} x_i^2 d\phi_n , \\ &= dt + \sum_{j\neq 1,i} x_j^2 d\phi_1 + \sum_{j,k\neq 1,i} x_j^2 x_k^2 d\phi_2 + \cdots + \prod_{j\neq 1,i} x_j^2 d\phi_{n-1} \\ &\quad - r^2 \Big( d\phi_1 + \sum_{j\neq 1,i} x_j^2 d\phi_2 + \cdots + \prod_{j\neq 1,i} x_j^2 d\phi_n \Big) \\ &= dt - r^2 d\phi_1 + \sum_{j\neq 1,i} x_j^2 (d\phi_1 - r^2 d\phi_2) + \cdots \\ &\quad + \prod_{j\neq 1,i} x_j^2 \Big( d\phi_{n-1} - r^2 d\phi_n \Big) \\ &= dt - \frac{h_v}{h_1} d\phi_1 + \sum_{j\neq 1,i} x_j^2 \Big[ \Big( d\phi_1 - \frac{h_1}{h_v} dt \Big) - r^2 \Big( d\phi_2 - \frac{h_2}{h_v} dt \Big) \Big] \\ &\quad + \cdots + \prod_{j\neq 1,i} x_j^2 \Big[ \Big( d\phi_{n-1} - \frac{h_{n-1}}{h_v} dt \Big) - r^2 \Big( d\phi_n - \frac{h_{n-1}}{r^2 h_v} dt \Big) \Big] . \end{aligned} \tag{C.17}$$

So it is obvious that both (C.9) and (C.11) can be put into the form of (2.10), with $h_{\mathcal{A}} = h_{\chi}^a = h_{tt} = 0$.

## C.2 Extremal Static Black Holes in Supergravity Theories

Here we turn to the examples studied in [7]. A key feature here is that all the solutions are charged but static. In order to use the Kerr/CFT correspondence, which only works with rotating black holes, the strategy used in [7] is to lift the charged static solutions into higher dimensions by using some consistent Kaluza-Klein reduction procedure. The electric charges of the static black holes then acquire the interpretation of rotations in the internal dimensions after the lifting.

Here we will discuss the same examples from the perspective of using (2.10), but we will still be using the same strategy as employed in [7]. For this purpose, we start with the various reduction ansatz given in [31]:

- For the $S^5$ reduction of type IIB supergravity, the ansatz for the ten-dimensional metric is

$$ds_{10}^2 = \sqrt{\widetilde{\Delta}}\, ds_5^2 + \frac{1}{g^2\sqrt{\widetilde{\Delta}}} \sum_{i=1}^{3} X_i^{-1} \Big[d\mu_i^2 + \mu_i^2\,(d\phi_i + g\,A^i)^2\Big]\,, \tag{C.18}$$

  where $X_1\,X_2\,X_3 = 1$.

- For the $S^7$ reduction of $D = 11$ supergravity, the ansatz for the eleven-dimensional metric is

$$ds_{11}^2 = \widetilde{\Delta}^{2/3}\, ds_4^2 + g^{-2}\,\widetilde{\Delta}^{-1/3} \sum_i X_i^{-1} \Big[d\mu_i^2 + \mu_i^2\,(d\phi_i + g\,A^i_{(1)})^2\Big]\,, \tag{C.19}$$

  where $\widetilde{\Delta} = \sum_{i=1}^4 X_i\,\mu_i^2$, and $\sum_i^4 \mu_i^2 = 1$ and $X_1X_2X_3X_4 = 1$.

- For the $S^4$ reduction of $D = 11$ supergravity, the ansatz for the eleven-dimensional metric is

$$\begin{aligned} ds_{11}^2 \;&=\; \widetilde{\Delta}^{1/3}\, ds_7^2 + g^{-2}\,\widetilde{\Delta}^{-2/3} \Big\{ X_0^{-1}\, d\mu_0^2 \\ &\quad + \sum_{i=1}^{2} X_i^{-1} \Big[d\mu_i^2 + \mu_i^2\,(d\phi_i + g\,A^i_{(1)})^2\Big] \Big\}\,, \end{aligned} \tag{C.20}$$

  where $\widetilde{\Delta} = \sum_{\alpha=0}^2 X_\alpha\,\mu_\alpha^2$ with $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$, and the auxiliary variable $X_0 \equiv (X_1X_2)^{-2}$.

- For the $S^4$ reduction of type IIA supergravity, the ansatz for the ten-dimensional metric is found in [32],

$$\begin{aligned} d\hat{s}_{10}^2 \;&=\; (\sin\xi)^{\frac{1}{12}}\, X^{\frac{1}{8}} \Big[\Delta^{\frac{3}{8}}\, ds_6^2 + 2g^{-2}\,\Delta^{\frac{3}{8}}\, X^2\, d\xi^2 \\ &\quad + \frac{1}{2} g^{-2}\,\Delta^{-\frac{5}{8}}\, X^{-1}\, \cos^2\xi \sum_{i=1}^{3} (\sigma^i + g\,A^i_{(1)})^2\Big]\,, \end{aligned} \tag{C.21}$$

  where $X = e^{-\frac{1}{2\sqrt{2}}\phi}$, and $\Delta = X\cos^2\xi + X^{-3}\sin^2\xi$. The quantities $\sigma^i$ are left-invariant 1-forms on $S^3$, which satisfy $d\sigma^i = -\frac{1}{2}\epsilon_{ijk}\,\sigma^j \wedge \sigma^k$. One can parameterize them as

$$\sigma_1 = d\theta\,, \quad \sigma_2 = \sin^2\theta d\phi\,, \quad \sigma_3 = d\psi + \cos\theta d\phi\,. \tag{C.22}$$

For all the examples that will be discussed in the following, the lower dimension metrics will be static. So the metric will not have any cross terms involving $d\hat{t}$ and the azimuthal angles. So for the terms involved in (2.4) and (2.5), one will have $f_a = 0$. What's more, all the gauge fields are of the particular form, $A^i = \Phi^i(r)d\hat{t}$; and for (C.21), only $A^3_{(1)} \neq 0$. So it is easy to see that $h_i/h_v = -g\Phi^i(r)$. It is then obvious that all the metrics (C.18), (C.19), (C.20), and (C.21) will be of the form (2.10). Now lets look at explicit examples.

The first example is with the maximal gauged supergravity in $D = 5$. It has $SO(6)$ gauge symmetry. The Cartan subgroup is $U(1)^3$. The five-dimensional three-charge static AdS black hole solution was constructed in [33]. We adopt the convention of [31], and the solution is given by

$$\begin{aligned}
ds_5^2 &= -\mathcal{H}^{-2/3} f\, d\hat{t}^2 + \mathcal{H}^{1/3}(f^{-1}d\hat{r}^2 + \hat{r}^2 d\Omega_{3,\epsilon}^2)\,,\\
X_i &= H_i^{-1}\mathcal{H}^{1/3}\,, \quad A^i_{(1)} = \Phi_i\, d\hat{t}\,, \quad \Phi_i = -(1 - H_i^{-1})\alpha_i\,,\\
f &= \epsilon - \frac{\mu}{\hat{r}^2} + g^2\hat{r}^2\mathcal{H}\,, \quad \mathcal{H} = H_1 H_2 H_3\,, \quad H_i = 1 + \frac{\ell_i^2}{\hat{r}^2}\,,\\
\alpha_i &= \frac{\sqrt{1 + \epsilon \sinh^2\beta_i}}{\sinh\beta_i}\,, \quad \ell_i^2 = \mu \sinh^2\beta_i\,,
\end{aligned} \tag{C.23}$$

where $d\Omega_{3,\epsilon}^2$ is the unit metric for $S^3$, $T^3$ or $H^3$ for $\epsilon = 1, 0$ or $-1$, respectively. If all the charge parameters $\beta_i$ are set equal, the solution becomes the five-dimensional Reissner-Nordström AdS black hole. We see that

$$\begin{aligned}
\frac{h_i}{h_v} &= -g\Phi_i\,, \quad h_\chi^\phi = h_{tt} = 0\,,\\
\mathcal{A} = d\hat{t} + \frac{\sqrt{\mathcal{H}}}{f}dr \quad &\Longrightarrow \quad h_v = \sqrt{\mathcal{H}}\,, \quad f_i = 0\,, \quad h_{\mathcal{A}} = 0\,.
\end{aligned} \tag{C.24}$$

The second example is with the maximum gauged supergravity in $D = 4$. It has $SO(8)$ gauge group, with the Cartan subgroup $U(1)^4$. The four-charge static AdS black hole was constructed in [34, 35]. Following the convention of [31], the four-dimensional 4-charge AdS black hole solution is given by

$$\begin{aligned}
ds_4^2 &= -\mathcal{H}^{-1/2} f\, d\hat{t}^2 + \mathcal{H}^{1/2}(f^{-1}d\hat{r}^2 + \hat{r}^2 d\Omega_{2,\epsilon}^2)\,,\\
X_i &= H_i^{-1}\mathcal{H}^{1/4}\,, \qquad A^i_{(1)} = \Phi_i\, d\hat{t}\,, \quad \Phi_i = -(1 - H_i^{-1})\alpha_i\,,\\
f &= \epsilon - \frac{\mu}{\hat{r}} + 4g^2\hat{r}^2\mathcal{H}\,, \qquad \mathcal{H} = H_1 H_2 H_3 H_4\,, \qquad H_i = 1 + \frac{\ell_i}{\hat{r}}\,,\\
\alpha_i &= \frac{\sqrt{1 + \epsilon \sinh^2\beta_i}}{\sinh\beta_i}\,, \qquad \ell_i = \mu \sinh^2\beta_i\,,
\end{aligned} \tag{C.25}$$

where $d\Omega_{2,\epsilon}^2$ is the unit metric for $S^2$, $T^2$ or $H^2$ for $\epsilon = 1, 0$ or $-1$, respectively. If the charge parameters $\beta_i$ are set equal, the solution becomes the standard Reissner-Nordström AdS black hole. We see that

$$
\begin{aligned}
\frac{h_i}{h_v} &= -g\Phi_i\,, \quad h^\phi_\chi = h_{tt} = 0\,, \\
\mathcal{A} = d\hat{t} + \frac{\sqrt{\mathcal{H}}}{f}dr \quad &\Longrightarrow \quad h_v = \sqrt{\mathcal{H}}\,, \quad f_i = 0\,, \quad h_{\mathcal{A}} = 0\,. \qquad \text{(C.26)}
\end{aligned}
$$

The third example is with the maximal gauged supergravity in $D = 7$. It has $SO(5)$ gauge symmetry, whose Cartan subgroup is $U(1)^2$. The seven-dimensional 2-charge AdS black hole solution is given by [31]

$$
\begin{aligned}
ds_7^2 &= -\mathcal{H}^{-4/5} f\, d\hat{t}^2 + \mathcal{H}^{1/5}(f^{-1}d\hat{r}^2 + \hat{r}^2 d\Omega^2_{5,\epsilon})\,, \\
X_i &= H_i^{-1}\mathcal{H}^{2/5}\,, \quad A^i_{(1)} = \Phi_i\, d\hat{t}\,, \quad \Phi_i = -(1 - H_i^{-1})\alpha_i\,, \\
f &= \epsilon - \frac{\mu}{\hat{r}^4} + \frac{1}{4}g^2\hat{r}^2\mathcal{H}\,, \quad \mathcal{H} = H_1 H_2\,, \quad H_i = 1 + \frac{\ell_i^4}{\hat{r}^4}\,, \\
\alpha_i &= \frac{\sqrt{1 + \epsilon \sinh^2\beta_i}}{\sinh\beta_i}\,, \quad \ell_i^4 = \mu \sinh^2\beta_i\,, \qquad \text{(C.27)}
\end{aligned}
$$

where $d\Omega^2_{5,\epsilon}$ is the unit metric for $S^5$, $T^5$ or $H^5$ for $\epsilon = 1, 0$ or $-1$, respectively. We see that

$$
\begin{aligned}
\frac{h_i}{h_v} &= -g\Phi_i\,, \quad h^\phi_\chi = h_{tt} = 0\,, \\
\mathcal{A} = d\hat{t} + \frac{\sqrt{\mathcal{H}}}{f}dr \quad &\Longrightarrow \quad h_v = \sqrt{\mathcal{H}}\,, \quad f_i = 0\,, \quad h_{\mathcal{A}} = 0\,. \qquad \text{(C.28)}
\end{aligned}
$$

The last example is with the gauged supergravity in $D = 6$ constructed in [36]. It has a $SU(2)$ gauge symmetry. The $U(1)$ charged AdS black hole was constructed in [32],

$$
\begin{aligned}
ds_6^2 &= -H^{-3/2} f\, d\hat{t}^2 + H^{1/2}(f^{-1}d\hat{r}^2 + \hat{r}^2 d\Omega^2_{4,\epsilon})\,, \\
X &= H^{-1/4}\,, \quad A_{(1)} = \Phi\, d\hat{t}\,, \quad \Phi = -\sqrt{2}(1 - H^{-1})\alpha\, d\hat{t}\,, \\
f &= \epsilon - \frac{\mu}{\hat{r}^3} + \frac{2}{9}g^2\hat{r}^2 H^2\,, \quad H = 1 + \frac{\ell^3}{\hat{r}^3}\,, \\
\alpha &= \frac{\sqrt{1 + \epsilon\, \sinh^2\beta}}{\sinh\beta}\,, \quad \ell^3 = \mu\, \sinh^2\beta\,. \qquad \text{(C.29)}
\end{aligned}
$$

We see that

$$
\begin{aligned}
\frac{h_{\sigma^3}}{h_v} &= -g\Phi\,, \quad h_{\sigma^1} = h_{\sigma^2} = h^\phi_\chi = h_{tt} = 0\,, \\
\mathcal{A} = d\hat{t} + \frac{H}{f}dr \quad &\Longrightarrow \quad h_v = H\,, \quad f_i = 0\,, \quad h_{\mathcal{A}} = 0\,. \qquad \text{(C.30)}
\end{aligned}
$$

## C.3 Extremal Rotating Black Holes in Supergravity Theories

The Kerr/CFT correspondence for rotating black hole solutions in supergravity theories were studied in [6]. Here we will revisit some of the examples by comparing them with (2.10) and (2.11).

In the five dimensional (un)gauged supergravities, there are three non-extremal solutions that cannot accommodate each other. They are the three-charge two-rotation Cvetič-Youm solution [37] in the ungauged supergravity, the three-charge equal-rotation solution [38] and the three-charge (two of which equal) two-rotation solution [39] in the gauged supergravity.

The Cvetič-Youm solution is given by

$$\begin{aligned}
ds^2 &= (H_1H_2H_3)^{1/3}\left[\frac{dx^2}{4X}+\frac{dy^2}{4Y}+\frac{U}{G}\left(d\chi-\frac{Z}{U}d\sigma\right)^2+\frac{XY}{U}d\sigma^2\right] \\
&\quad -\frac{G\left(dt+\widetilde{\mathcal{A}}\right)^2}{(H_1H_2H_3)^{2/3}}\,, \\
\widetilde{\mathcal{A}} &= \frac{2mc_1c_2c_3\left[(a^2+b^2-y)d\sigma-abd\chi\right]}{x+y-2m}-\frac{2ms_1s_2s_3(abd\sigma-yd\chi)}{x+y}\,, \\
X &= (x+a^2)(x+b^2)-2mx\,,\quad Y=-(a^2-y)(b^2-y)\,, \\
U &= yX-xY\,,\quad Z=ab(X+Y)\,,\quad G=(x+y)(x+y-2m)\,, \\
\mathcal{A}_i &= \frac{2m}{H_i}\Big\{c_is_idt+s_ic_jc_k\Big[abd\chi+(y-a^2-b^2)d\sigma\Big] \\
&\qquad +c_is_js_k(abd\sigma-yd\chi)\Big\}\,,\quad i\neq j\neq k\,, \\
X_i &= \frac{H_1^{1/3}H_2^{1/3}H_3^{1/3}}{H_i}\,,\quad H_i=x+y+2ms_i^2\,,
\end{aligned}\tag{C.31}$$

where $s_i=\sinh\delta_i$ , $c_i=\cosh\delta_i$ and $i,j,k=1,2,3$. The variables $\chi$ and $\sigma$ are related to the canonical azimuthal angles by

$$\sigma=\frac{a\hat\phi_1-b\hat\phi_2}{a^2-b^2}\,,\quad \chi=\frac{b\hat\phi_1-a\hat\phi_2}{a^2-b^2}\,.\tag{C.32}$$

Near the horizon, $\sigma$ is playing the role of the time direction as in the Schwarzschild solution. We have for (2.4) and (2.5),

$$\mathcal{A}=d\sigma+\frac{(a^2-b^2)\sqrt{x}\,dx}{2X}\sqrt{1-\frac{yX}{xY}}\,.\tag{C.33}$$

By comparing various terms, we find that

$$\begin{aligned}
h_v &= \frac{ab(c_1^2c_2^2c_3^2+s_1^2s_2^2s_3^2)-(a^2+b^2-2m)c_1c_2c_3s_1s_2s_3}{abc_1c_2c_3+xs_1s_2s_3}m\sqrt{x}\,, \\
h_1 &= \frac{a(b^2+x)s_1s_2s_3-b(b^2-2m+x)c_1c_2c_3}{2(abc_1c_2c_3+xs_1s_2s_3)}\sqrt{x}\,, \\
h_2 &= \frac{b(a^2+x)s_1s_2s_3-a(a^2-2m+x)c_1c_2c_3}{2(abc_1c_2c_3+xs_1s_2s_3)}\sqrt{x}\,,
\end{aligned}\tag{C.34}$$

and so

$$d\sigma = \frac{a}{a^2-b^2}d\hat\phi_1-\frac{b}{a^2-b^2}d\hat\phi_2\,,$$

$$
\begin{aligned}
-\frac{U}{4Y} &= \Big(\frac{a}{a^2-b^2}h_1 - \frac{b}{a^2-b^2}h_2\Big)^2 - \frac{yX}{4Y}\,,\\
d\chi - \frac{Z}{U}d\sigma &= -\frac{(\frac{x}{x+y}+\frac{a^2}{x+y-2m})(a^2-y)b}{(\frac{xy}{x+y}+\frac{a^2b^2}{x+y-2m})(a^2-b^2)}\Big(d\hat\phi_1 - \frac{h_1}{h_v}dt\Big)\\
&\quad +\frac{(\frac{x}{x+y}+\frac{b^2}{x+y-2m})(b^2-y)a}{(\frac{xy}{x+y}+\frac{a^2b^2}{x+y-2m})(a^2-b^2)}\Big(d\hat\phi_2 - \frac{h_2}{h_v}dt\Big)\\
&\quad -\frac{(\frac{abs_1s_2s_3}{x+y-2m}-\frac{c_1c_2c_3y}{x+y})X\sqrt{x}\,dt}{2h_v(\frac{xy}{x+y}+\frac{a^2b^2}{x+y-2m})(abc_1c_2c_3+s_1s_2s_3x)}\,,\\
dt+\widetilde{\mathcal{A}} &= \frac{2m(a^2-y)}{a^2-b^2}\Big(\frac{ac_1c_2c_3}{x+y-2m}-\frac{bs_1s_2s_3}{x+y}\Big)\Big(d\hat\phi_1-\frac{h_1}{h_v}dt\Big)\\
&\quad +\frac{2m(b^2-y)}{a^2-b^2}\Big(\frac{as_1s_2s_3}{x+y}-\frac{bc_1c_2c_3}{x+y-2m}\Big)\Big(d\hat\phi_2-\frac{h_2}{h_v}dt\Big)\\
&\quad +\frac{2m^2X\sqrt{x}\;c_1c_2c_3s_1s_2s_3dt}{h_v(abc_1c_2c_3+s_1s_2s_3x)(x+y-2m)(x+y)}\,. \qquad (C.35)
\end{aligned}
$$

It is obvious that (C.31) is of the form (2.11) with $h_{\mathcal{A}}, h_\chi^1, h_\chi^2 \neq 0$ but $h_{tt}=0$. As a side remark, note the gauge fields can be written as

$$
\begin{aligned}
\mathcal{A}_i &= \frac{2m}{(a^2-b^2)h_i}\bigg\{(bc_is_js_k - as_ic_jc_k)(a^2-y)\Big(d\hat\phi_1-\frac{h_1}{h_v}dt\Big)\\
&\quad +(bs_ic_jc_k - ac_is_js_k)(b^2-y)\Big(d\hat\phi_2-\frac{h_2}{h_v}dt\Big)\bigg\}\\
&\quad +\frac{abc_is_i(c_j^2c_k^2+s_j^2s_k^2)-c_jc_ks_js_k[x+c_i^2(a^2+b^2-2m)]}{(abc_ic_jc_k+s_is_js_kx)h_v/(m\sqrt{x})}dt\\
&\quad +\frac{c_jc_ks_js_kXm\sqrt{x}}{(abc_ic_jc_k+s_is_js_kx)h_ih_v}dt\,,\quad i\neq j\neq k\,. \qquad (C.36)
\end{aligned}
$$

When transforming to the coordinates on the horizon by (2.7), only the third line will lead to a divergence, but which can be absorbed as pure gauge.

For the three-charge equal-rotation solution in the gauged supergravity [38], the result is given by

$$
\begin{aligned}
ds^2 &= R\bigg\{-\frac{X}{f_1}dt^2+\frac{r^2}{X}dr^2+d\theta^2+\cos^2\theta\sin^2\theta(d\phi-d\psi)^2\\
&\quad +\frac{f_1}{R^3}\Big(\cos^2\theta d\phi+\sin^2\theta d\psi-\frac{f_2}{f_1}dt\Big)^2\bigg\}\,,\\
X &= r^4-2m(r^2-\ell^2)+g^2f_1\,,\quad f_1=2m\ell^2(r^2+2m\tilde{s})+R^3\,,\\
f_2 &= 2m\ell r^2(c_1c_2c_3-s_1s_2s_3)+4m^2\ell s_1s_2s_3\,,\\
R &= (H_1H_2H_3)^{1/3}\,,\quad H_i=r^2+2ms_i^2\,,\ i=1,2,3\,,\\
\tilde{s} &= 2s_1s_2s_3(c_1c_2c_3-s_1s_2s_3)-s_1^2s_2^2-s_1^2s_3^2-s_2^2s_3^2\,,
\end{aligned}
$$

$$\mathcal{A}_i = \frac{2m}{h_i}\Big[c_i s_i dt + \ell(c_i s_j s_k - s_i c_j c_k)(\cos^2\theta d\phi + \sin^2\theta d\psi)\Big]\,. \tag{C.37}$$

It is easy to tell that the metric is of the (2.10) with

$$h_v = r\sqrt{f_1}\,,\quad h_\phi = h_\psi = \frac{r f_2}{\sqrt{f_1}}\,,\quad h_{\mathcal{A}} = h_\chi^\phi = h_\chi^\psi = h_{tt} = 0\,. \tag{C.38}$$

After using (2.7), the gauge fields are also regular on the horizon up to some divergence which can be absorbed as pure gauge.

The three-charge (two of which equal) two-rotation solution in the gauged supergravity was found in [39], and the result is given by

$$\begin{aligned}
ds^2 &= H_1^{2/3}H_3^{1/3}\Bigg\{(x^2-y^2)\left(\frac{dx^2}{X}-\frac{dy^2}{Y}\right) - \frac{x^2X(dt+y^2d\sigma)^2}{(x^2-y^2)fH_1^2} \\
&\qquad + \frac{y^2Y\left[dt+(x^2+2ms_1^2)d\sigma\right]^2}{(x^2-y^2)(\gamma+y^2)H_1^2} \\
&\qquad - U\left(dt + y^2 d\sigma + \frac{(x^2-y^2)fH_1\left[abd\sigma+(\gamma+y^2)d\chi\right]}{ab(x^2-y^2)H_3 - 2ms_3c_3(\gamma+y^2)}\right)^2\Bigg\}\,, \\
\mathcal{A}^1 &= \mathcal{A}^2 = \frac{2ms_1c_1(dt+y^2d\sigma)}{(x^2-y^2)H_1}\,, \\
\mathcal{A}^3 &= \frac{2m\left\{s_3c_3(dt+y^2d\sigma) - (s_1^2-s_3^2)\left[abd\sigma+(\gamma+y^2)d\chi\right]\right\}}{(x^2-y^2)H_3}\,, \\
X_1 &= X_2 = \left(\frac{H_3}{H_1}\right)^{1/3}\,,\quad X_3 = \left(\frac{H_1}{H_3}\right)^{2/3}\,, \\
f &= x^2+\gamma+2ms_3^2\,,\quad \gamma = 2abs_3c_3 + (a^2+b^2)s_3^2\,, \\
U &= \frac{\left[ab(x^2-y^2)H_3 - 2ms_3c_3(\gamma+y^2)\right]^2}{(x^2-y^2)^2(\gamma+y^2)fH_1^2H_3}\,, \\
H_1 &= 1+\frac{2ms_1^2}{x^2-y^2}\,,\quad H_3 = 1+\frac{2ms_3^2}{x^2-y^2}\,, \\
X &= \frac{-2mx^2+(\tilde a^2+x^2)(\tilde b^2+x^2)}{x^2} \\
&\qquad + \frac{g^2(\tilde a^2+2ms_1^2+x^2)(\tilde b^2+2ms_1^2+x^2)(2ms_3^2+\gamma+x^2)}{x^2}\,, \\
Y &= \frac{(\tilde a^2+y^2)(\tilde b^2+y^2)\left[1+g^2(\gamma+y^2)\right]}{y^2}\,, \\
s_i &= \sinh\delta_i\,,\quad c_i = \cosh\delta_i\,,\quad \tilde a = ac_3+bs_3\,,\quad \tilde b = bc_3+as_3\,.
\end{aligned} \tag{C.39}$$

Comparing with (2.4) and (2.5), we see that

$$\begin{aligned}
\mathcal{A} &= dt + y^2 d\sigma + \frac{(x^2-y^2)\sqrt{f}H_1}{xX}dx \\
&= dt + y^2 d\sigma + \frac{(x^2-y^2+2ms_1^2)\sqrt{f}}{xX}dx\,,
\end{aligned}$$

$$\Longrightarrow \quad h_v = \frac{(x^2+2ms_1^2)\sqrt{f}}{x}\,, \quad h_\sigma = -\frac{\sqrt{f}}{x}\,. \tag{C.40}$$

As a result,

$$dt + (x^2+2ms_1^2)d\sigma \propto d\sigma - \frac{h_\sigma}{h_v}dt\,, \tag{C.41}$$

and with $h_\chi = \dfrac{ab+2mc_3s_3}{x\sqrt{f}}$ ,

$$\begin{aligned} & dt + y^2 d\sigma + \frac{(x^2-y^2)fH_1\big[ab d\sigma + (\gamma+y^2)d\chi\big]}{ab(x^2-y^2)H_3 - 2ms_3c_3(\gamma+y^2)} \\ = \; & \Big\{x + 2ms_1^2 + \frac{(ab+2mc_3s_3)(x^2-y^2)H_1(y^2+\gamma)}{ab(x^2-y^2)H_3 - 2mc_3s_3(y^2+\gamma)}\Big\}\Big(d\sigma - \frac{h_\sigma}{h_v}dt\Big) \\ & + \frac{(y^2+\gamma)(x^2-y^2)fH_1}{ab(x^2-y^2)H_3 - 2mc_3s_3(y^2+\gamma)}\Big(d\chi - \frac{h_\chi}{h_v}dt\Big)\,. \end{aligned} \tag{C.42}$$

Now it is obvious that the metric in (C.39) is of the form (2.10). For the gauge fields, one has

$$\begin{aligned} \mathcal{A}_1 &= \mathcal{A}_2 = \frac{2mc_1s_1y^2}{(x^2-y^2)H_1}\Big(d\sigma - \frac{h_\sigma}{h_v}dt\Big) + \frac{2mc_1s_1}{x^2+2ms_1^2}dt\,, \\ \mathcal{A}_3 &= -\frac{2m}{(x^2-y^2)H_3}\bigg\{\Big[ab(s_1^2-s_3^2) - c_3s_3y^2\Big]\Big(d\sigma - \frac{h_\sigma}{h_v}dt\Big) \\ &\qquad + (s_1^2-s_3^2)(y^2+\gamma)\Big(d\chi - \frac{h_\chi}{h_v}dt\Big)\bigg\} \\ &\qquad + \frac{2m\Big[c_3s_3f + (ab+2mc_3s_3)(s_1^2-s_3^2)\Big]}{f(x^2+2ms_1^2)}dt\,. \end{aligned} \tag{C.43}$$

Again, when (2.7) is used, the divergent pieces can be absorbed as pure gauge.

In the following, we consider a few more solutions in dimensions other than five. Again, all these have been studied in [6]. We include them here just to show the general applicability of the metric (2.10) and (2.11).

The first example is the four-charge black hole of the ungauged supergravity in four dimension [40, 41],

$$ds_4^2 = -\frac{\rho^2 - 2m\hat{r}}{W}(d\hat{t} + B\,d\hat{\phi})^2 + W\Big(\frac{d\hat{r}^2}{\Delta} + d\theta^2 + \frac{\Delta\,\sin^2\theta\,d\hat{\phi}^2}{\rho^2-2m\hat{r}}\Big)\,. \tag{C.44}$$

The detail of various functions can be found in [6]. Notably,

$$\begin{aligned} \Delta &= \hat{r}^2 - 2m\hat{r} + a^2\,, \quad \rho^2 = \hat{r}^2 + a^2\cos^2\theta\,, \quad W = W(r)\,, \\ B &= \frac{2ma^2\sin^2\theta[\hat{r}c_1c_2c_3c_4 - (\hat{r}-2m)s_1s_2s_3s_4]}{a(\rho^2-2m\hat{r})}\,. \end{aligned} \tag{C.45}$$

Note $\rho^2 - 2m\hat{r} = \Delta - a^2\sin^2\theta$. So when it comes close to the horizon, $d\hat{\phi}$ replaces $d\hat{t} + B\,d\hat{\phi}$ and become the time direction. What's more,

$$\begin{aligned} B &= -\frac{1}{B_0}\Big(1 + \frac{\Delta}{a^2\sin^2\theta}\Big) + \mathcal{O}(\Delta^2)\,, \\ B_0 &= \frac{a}{2m[\hat{r}c_1c_2c_3c_4 - (\hat{r}-2m)s_1s_2s_3s_4]}\,. \end{aligned} \tag{C.46}$$

Comparing (C.44) with (2.4), we have for (2.5),

$$\begin{aligned} \mathcal{A} &= d\hat{\phi} + \frac{\sqrt{a^2\sin^2\theta - \Delta}}{\Delta\sin\theta}d\hat{r} \\ &\approx d\hat{\phi} + \frac{a}{\Delta}d\hat{r} - \frac{d\hat{r}}{2a\sin^2\theta}\,, \\ \Longrightarrow \quad h_{\hat{\phi}} &= a\,, \quad h_{\mathcal{A}} = -\frac{1}{2a\sin^2\theta}\,. \end{aligned} \tag{C.47}$$

By letting $h_v = \dfrac{a}{B_0}$ and $h_\chi^{\hat{\phi}} = -\dfrac{1}{a\sin^2\theta}$ , we also have

$$d\hat{t} + B\,d\hat{\phi} \propto d\hat{\phi} - \frac{h_{\hat{\phi}} + h_\chi^{\hat{\phi}}\Delta}{h_v}d\hat{t} + \mathcal{O}(\Delta^2)\,. \tag{C.48}$$

So (C.44) is of the form (2.11) with $h_{tt} = 0$.

The next example is the rotating black hole solution in four-dimensional U(1)$^4$ gauged supergravity with the four U(1) charges pairwise equal [41]. The metric is

$$\begin{aligned} ds^2 &= H\Big[-\frac{R}{H^2(\hat{r}^2+y^2)}\left(d\hat{t} - \frac{a^2-y^2}{\Xi a}d\hat{\phi}\right)^2 + \frac{\hat{r}^2+y^2}{R}d\hat{r}^2 + \frac{\hat{r}^2+y^2}{Y}dy^2 \\ &\quad + \frac{Y}{H^2(\hat{r}^2+y^2)}\left(d\hat{t} - \frac{(\hat{r}+q_1)(\hat{r}+q_2)+a^2}{\Xi a}d\hat{\phi}\right)^2\Big]\,, \end{aligned} \tag{C.49}$$

where

$$\begin{aligned} R &= \hat{r}^2 + a^2 + g^2(\hat{r}+q_1)(\hat{r}+q_2)[(\hat{r}+q_1)(\hat{r}+q_2)+a^2] - 2m\hat{r}\,, \\ Y &= (1-g^2y^2)(a^2-y^2)\,, \quad \Xi = 1 - g^2a^2\,, \\ H &= \frac{(\hat{r}+q_1)(\hat{r}+q_2)+y^2}{\hat{r}^2+y^2}\,, \quad q_I = 2ms_I^2\,, \quad s_I = \sinh\delta_I\,. \end{aligned} \tag{C.50}$$

Comparing (C.49) with (2.4), we have for (2.5),

$$\begin{aligned} \mathcal{A} &= d\hat{t} - \frac{a^2-y^2}{\Xi a}d\hat{\phi} + \frac{(\hat{r}+q_1)(\hat{r}+q_2)+y^2}{R}d\hat{r}\,, \\ \Longrightarrow \quad h_v &= \frac{(\hat{r}+q_1)(\hat{r}+q_2)+a^2}{R}\,, \quad h_{\hat{\phi}} = \frac{\Xi a}{R}\,. \end{aligned} \tag{C.51}$$

It is easy to see that

$$d\hat{t} - \frac{(\hat{r}+q_1)(\hat{r}+q_2)+a^2}{\Xi a}d\hat{\phi} \quad \propto \quad d\hat{\phi} - \frac{h_{\hat{\phi}}}{h_v}d\hat{t}\,. \tag{C.52}$$

So (C.49) is of the form (2.10) with $h_{\mathcal{A}} = h_\chi^{\hat\phi} = h_{tt} = 0$.

A single-charge two-rotation solution to the six-dimensional SU(2) gauged supergravity was found in [42]. The metric is

$$\begin{aligned} ds^2 &= H^{1/2}\Big\{ -\frac{R}{H^2U}\widetilde{\mathcal{A}}^2 + \frac{(\hat r^2+y^2)(y^2-z^2)}{Y}dy^2 + \frac{Y\widetilde{\mathcal{A}}_Y^2}{(\hat r^2+y^2)(y^2-z^2)} \\ &\qquad + \frac{U}{R}d\hat r^2 + \frac{(\hat r^2+z^2)(z^2-y^2)}{Z}dz^2 + \frac{Z\widetilde{\mathcal{A}}_Z^2}{(\hat r^2+z^2)(z^2-y^2)}\Big\}, \end{aligned} \tag{C.53}$$

$$\begin{aligned} \widetilde{\mathcal{A}}_Y &= d\hat t - (\hat r^2+a^2)(a^2-z^2)\frac{d\hat\phi_1}{\epsilon_1} - (\hat r^2+b^2)(b^2-z^2)\frac{d\hat\phi_2}{\epsilon_2} - \frac{q\hat r\widetilde{\mathcal{A}}}{HU}, \\ \widetilde{\mathcal{A}}_Z &= d\hat t - (\hat r^2+a^2)(a^2-y^2)\frac{d\hat\phi_1}{\epsilon_1} - (\hat r^2+b^2)(b^2-y^2)\frac{d\hat\phi_2}{\epsilon_2} - \frac{q\hat r\widetilde{\mathcal{A}}}{HU}, \end{aligned} \tag{C.54}$$

where the various functions and constants can be found in [6]. The ones relevant for us are

$$\begin{aligned} U &= (\hat r^2+y^2)(\hat r^2+z^2), \quad H = 1 + \frac{q\hat r}{U}, \\ \widetilde{\mathcal{A}} &= d\hat t - (a^2-y^2)(a^2-z^2)\frac{d\hat\phi_1}{\epsilon_1} - (b^2-y^2)(b^2-z^2)\frac{d\hat\phi_2}{\epsilon_2}. \end{aligned} \tag{C.55}$$

Comparing (C.53) with (2.4), we have for (2.5),

$$\mathcal{A} = \widetilde{\mathcal{A}} + \frac{HU}{R}dr. \tag{C.56}$$

By comparing various terms, one can find

$$\begin{aligned} h_v &= (\hat r^2+a^2)(\hat r^2+b^2) + q\hat r, \\ h_1 &= \frac{\hat r^2+b^2}{a^2-b^2}\epsilon_1, \quad h_2 = \frac{\hat r^2+a^2}{b^2-a^2}\epsilon_2, \end{aligned} \tag{C.57}$$

and

$$\begin{aligned} \widetilde{\mathcal{A}}_Y &= \frac{(z^2-a^2)[q\hat r + (\hat r^2+a^2)(\hat r^2+z^2)](\hat r^2+y^2)}{HU\epsilon_1}\Big(d\hat\phi_1 - \frac{h_1}{h_v}d\hat t\Big) \\ &\quad + \frac{(z^2-b^2)[q\hat r + (\hat r^2+b^2)(\hat r^2+z^2)](\hat r^2+y^2)}{HU\epsilon_2}\Big(d\hat\phi_2 - \frac{h_2}{h_v}d\hat t\Big), \\ \widetilde{\mathcal{A}}_Z &= \frac{(y^2-a^2)[q\hat r + (\hat r^2+a^2)(\hat r^2+y^2)](\hat r^2+z^2)}{HU\epsilon_1}\Big(d\hat\phi_1 - \frac{h_1}{h_v}d\hat t\Big) \\ &\quad + \frac{(y^2-b^2)[q\hat r + (\hat r^2+b^2)(\hat r^2+y^2)](\hat r^2+z^2)}{HU\epsilon_2}\Big(d\hat\phi_2 - \frac{h_2}{h_v}d\hat t\Big). \end{aligned} \tag{C.58}$$

So (C.53) is of the form (2.10) with $h_{\mathcal{A}} = h_\chi^{\hat\phi} = h_{tt} = 0$.

The single-charge three-rotation black hole solution to the seven-dimensional SO(5) gauged supergravity was found in [43]. The metric is

$$ds^2 = H^{2/5}\Big\{ -\frac{R}{H^2U}\widetilde{\mathcal{A}}^2 + \frac{U}{R}d\hat r^2 + \frac{(\hat r^2+y^2)(y^2-z^2)}{Y}dy^2$$

$$
\begin{aligned}
&+\frac{(\hat{r}^2+z^2)(z^2-y^2)}{Z}dz^2+\frac{Y\widetilde{\mathcal{A}}_Y^2}{(\hat{r}^2+y^2)(y^2-z^2)} \\
&+\frac{Z\widetilde{\mathcal{A}}_Z^2}{(\hat{r}^2+z^2)(z^2-y^2)}+\frac{a_1^2a_2^2a_3^2}{\hat{r}^2y^2z^2}\widetilde{\mathcal{A}}_7^2\Bigg\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathcal{A}}_Y &= d\hat{t}-\sum_{i=1}^{3}\frac{(\hat{r}^2+a_i^2)\gamma_i}{a_i^2-y^2}\frac{d\hat{\phi}_i}{\epsilon_i}-\frac{q}{HU}\widetilde{\mathcal{A}}\,,\\
\widetilde{\mathcal{A}}_Z &= d\hat{t}-\sum_{i=1}^{3}\frac{(\hat{r}^2+a_i^2)\gamma_i}{a_i^2-z^2}\frac{d\hat{\phi}_i}{\epsilon_i}-\frac{q}{HU}\widetilde{\mathcal{A}}\,,\\
\widetilde{\mathcal{A}}_7 &= d\hat{t}-\sum_{i=1}^{3}\frac{(\hat{r}^2+a_i^2)\gamma_i}{a_i^2}\frac{d\hat{\phi}_i}{\epsilon_i}-\frac{q}{HU}\left(1+\frac{gy^2z^2}{a_1a_2a_3}\right)\widetilde{\mathcal{A}}\,,
\end{aligned}
\tag{C.59}
$$

where the various functions and constants can be found in [6]. The ones relevant for us are

$$
\begin{aligned}
U &= (\hat{r}^2+y^2)(\hat{r}^2+z^2)\,,\quad \gamma_i=a_i^2(a_i^2-y^2)(a_i^2-z^2)\,,\\
H &= 1+\frac{q}{(\hat{r}^2+y^2)(\hat{r}^2+z^2)}\,,\quad \widetilde{\mathcal{A}}=d\hat{t}-\sum_{i=1}^{3}\gamma_i\frac{d\hat{\phi}_i}{\epsilon_i}\,.
\end{aligned}
\tag{C.60}
$$

Comparing (C.59) with (2.4), we have for (2.5),

$$
\mathcal{A}=\widetilde{\mathcal{A}}+\frac{HU}{R}dr\,. \tag{C.61}
$$

By comparing various terms, one can find

$$
\begin{aligned}
h_v &= \frac{(r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2)+q(r^2-ga_1a_2a_3)}{r^2}\,,\\
h_i &= \frac{a_i(r^2+a_j^2)(r^2+a_k^2)-gqa_ja_k}{a_i(a_i^2-a_j^2)(a_i^2-a_k^2)r^2}\epsilon_i\,,\quad i\neq j\neq k\,,
\end{aligned}
\tag{C.62}
$$

and

$$
\begin{aligned}
\widetilde{\mathcal{A}}_Y &= \sum_{i=1}^{3}\frac{(z^2-a_i^2)[q+(\hat{r}^2+a_i^2)(\hat{r}^2+z^2)](\hat{r}^2+y^2)a_i^2}{HU\epsilon_i}\Big(d\hat{\phi}_i-\frac{h_i}{h_v}d\hat{t}\Big)\,,\\
\widetilde{\mathcal{A}}_Z &= \sum_{i=1}^{3}\frac{(y^2-a_i^2)[q+(\hat{r}^2+a_i^2)(\hat{r}^2+y^2)](\hat{r}^2+z^2)a_i^2}{HU\epsilon_i}\Big(d\hat{\phi}_i-\frac{h_i}{h_v}d\hat{t}\Big)\,,\\
\widetilde{\mathcal{A}}_7 &= \sum_{i=1}^{3}\frac{\gamma_i\Big[\frac{q(a_1a_2a_3+gy^2z^2)}{HU}-\frac{a_1a_2a_3}{a_i^2}(r^2+a_i^2)\Big]}{a_1a_2a_3\epsilon_i}\Big(d\hat{\phi}_i-\frac{h_i}{h_v}d\hat{t}\Big)\,.
\end{aligned}
\tag{C.63}
$$

So (C.59) is of the form (2.10) with $h_{\mathcal{A}}=h_\chi^{\hat{\phi}}=h_{tt}=0$.

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