FOLD & CUT

A Thesis

by

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ABSTRACT

This paper describes a method of constructing 3d fractal objects by stretching/folding about simple geometric objects. The history of fold/cut construction as related to Mathematics is explored. The major new contribution of this paper is to generalize the concept of folding and cutting paper to folding and cutting space.

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Contributors

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1. INTRODUCTION

Here I will describe a method of constructing 3d fractal objects by stretching/folding about simple geometric objects.

1.1 Fold-And-Cut Theorem

Any shape with straight sides can be cut from a sheet of paper by folding it and making a single cut.

This is a problem that has puzzled many for centuries. With some ingenuity, and a bit of magic, many cleaver foldings have been invented that produce quite complex shapes— the most noteworthy being the classic folding of a five pointed star that requires only 4 folds! Shown below is an image from Houdini's 1922 book Paper Magic demonstrating how this trick is performed.

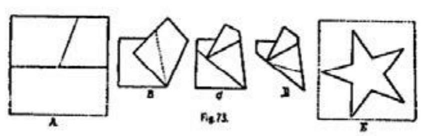

Figure 1. *Houdini's The Five-Pointed Star trick. [3]*

This trick can be traced back even further. According to American folk lore, the star on the U.S. flag was changed from six-pointed to five-point after Betsy Ross demonstrated to George Washington that a five-point star was more easily cut by the 4 fold trick.

In general, most shapes cannot be folded as elegantly as the five pointed star, and there is a limit to how much one can fold an actual piece of paper. In 2007, even with a football field-sized sheet of paper, the so called 'Myth Busters' were only able to make 11 folds—resulting in a thickness of $2^{11} = 2048$ sheets! [5] So practically speaking, you cannot fold and cut just any shape. However we may ignore physical constraints, and define our own types of Mathematical folding. Imagining a piece of paper with zero thickness, we may fold as many times as we like making the Fold and Cut theorem seem plausible.

The first official proof was given in 1999 and many algorithms have since been developed to determine a method of folding and cutting any general shape. [4] However, these algorithms rely mainly on brute force tactics. The resulting folding then necessary to cut out some general object by such a method may then be quite complicated and inefficient, and offer no real benefit over just cutting out the shape directly. Instead of brute force tactics, we will seek here elegant folding that capitalizes on the underlying properties and symmetries of the object we cut. Special objects that omit such symmetry, such as the five pointed star, will be our focus.

Let's begin with an explicit construction of a triangle. Our folds will all be reflections about lines going through the origin, hence we may define each fold in terms of the normal of the folding line. Here we will consider the dot product of all points in the entire space *Ω* with this normal—if negative then we reflect.

$$\text{Reflect}(v): \forall p \in \Omega, \text{ such that } p.v < 0,\ p \leftarrow p - 2|v.p|v/|v|$$

Multiple reflections with be given as a list, i.e. *Reflect({a, b, ...}) := Reflect(a), Reflect(b), …* A triangle fold may be defined as follows:

Triangle Fold:

$$\text{Reflect}(\{(\sqrt{3},-1), (\sqrt{3}, 1)\})$$

Next the cut. We will first consider a cut as defining a *trapping region*.

1.2 Definition of Trapping Region

We define a cutting line that partitions the plane into two regions, those points that are included in our object and those points that are excluded. The area that is included is called the *Trapping Region*.

This definition of a trapping region becomes particularly important when we cut shapes that result in more than two pieces as it gives natural assignment of the pieces as being included in one group or the other. A classic example of this is the 1960 checker-board puzzle by Martin Gardner for Scientific American that asks the puzzler to separate the red squares from the black squares of a checkerboard with only one cut. Spoiler: fold first diagonally. When folded correctly, all red squares will lie on top of one another and all black squares will lie on top of one another in an arrangement such that one may hold the folded board by the red region and cut away all black squares with one cut. [6]

Back to the cutting the triangle. Since we are cutting symmetric objects, we may define a cut by a normal vector v of the cutting line and a cut–off distance *d*:

$$\mathrm{Cut}(v, d) : \{p \in \Omega \mid p.v < d\}$$

The Triangle Cut is then defined as *Cut((√3, 1), 1/4).*

The resulting fold/cut triangle is shown below:

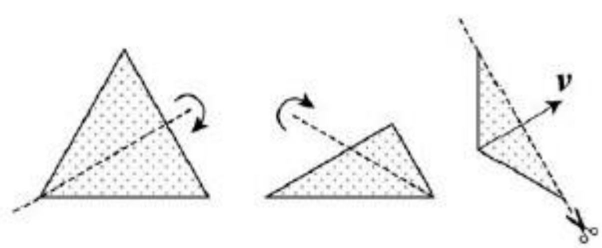

Figure 2. Fold/Cut triangle.

What is more, the dot product in the cut gives an approximate distance from the object's surface. With this, distance estimation renderings may be produced of the object. This technique is how the 3d renderings in this paper were produced. For more information see John C. Hart's paper on *Sphere Tracing*. [2]

The idea of folding and cutting paper may be extended to three dimensional space. This has many different variations depending on what one considers a 'fold' and a 'cut.' Also, recall that we are now considering mathematical folds, without the constraints imposed by the physical properties of paper. We already discarded the restriction on the number of folds, hence we may, if we so please, discard the constraint of rigidity and consider our paper to be elastic. In 3d then the folding/cutting can be roughly imagined as kneading, then cutting dough. A natural generalization to the fold cut theorem in 3d might then be like what is summarized in the following section.

1.3 Fold-And-Cut Theorem in 3d

Any shape with flat faces can be 'cut' from a single 'sheet' of space by folding/stretching (about planes) and making a single cut (along a cutting plane).

Here we will take a fold about a plane to be equivalent to a reflection about a plane. Since we allow for elastic folding, the theorem is trivial to prove—any general shape can be divided into simplexes that are homomorphic to a standard simplex. We can therefore fold all the shapes we want to cut out to a standard simplex that we then fold/cut. As an explicit example, the Sierpinski Tetrahedron construction discussed later folds all the tetrahedrons to a standard reference tetrahedron that is then cut with a single cut.

Note that while we may allow for any number of folds, we cannot by our construction fold to infinity. Our algorithm is fold, then cut. If we fold to infinity, then we will always be folding, and never get to cut! Since we cannot fold infinitely, we need instead to define some limiting object.

Consider a sequence of folds f_n and a cut *c* of the space *Ω* such that $c\ (f_{n+1}(\Omega)) \subseteq c\ (f_n\ (\Omega))$ for all *n*, we define the *limit set* to be:

$$\Lambda := \cap_{n\geq 0}\ c\ (f_n\ (\Omega))$$

We may now use Λ in place of what we would have considered the infinitely folded object.

2. CHAOS GAME

Before moving on, it will be worthwhile here to describe a well-known fractal construction that is in essence a folding construction. The classical Chaos Game is played by choosing a point at random inside a given polygon and then moving some given fractional distance from that point to a randomly chosen vertex of the polygon. This process is repeated, each time with a new randomly chosen vertex. When several such ‘walks’ are performed, some limiting object is obtained. If a regular triangle and scaling factor 2 is used, the result is a Sierpinski Triangle. If a tetrahedron and scaling factor 2 is used, the result is a Sierpinski Tetrahedron. [1]

We will investigate here the Sierpinski Triangle and show its similarities to folding construction. This will give incite on the motivation for choosing folding for our purposes. When rendering a Sierpinski Triangle, we draw a binary graph where each pixel of the graph is colored black or white depending on whether it is included or not included in the object. By the Chaos Game construction we color this graph by random walks. Each random walk will terminate at some pixel that is then colored, however we cannot anticipate which pixel. The result initially is a grainy under covering of the object. If we run the program long enough, it should eventually cover the entire object but, mathematically speaking, we cannot be 100% sure of this. Another alternative might be to loop through every possible walk beginning at any point in the starting triangle, but this still leads to problems and is in effect like shooting at a target in the dark.

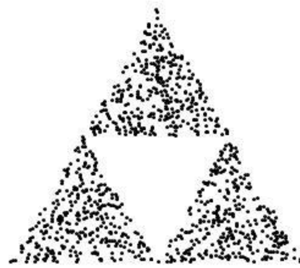

Figure 3. First iteration of the Sierpinski Triangle Chaos Game produced by 1000 random walks.

Instead of walking forward, what if we walk *backwards*? Instead of terminating at the grid pixel, lets start at a pixel and iterate through the transformations in the opposite direction. Once a backwards walk is complete, we may then consider its inclusion or exclusion depending on whether or not it is in our trapping region—here the starting triangle.

To begin, let's first define a scaling of the entire space Ω by r about a point c as:

$$\text{Stretch}(r, c): \forall p \in \Omega$$

$$p \leftarrow r(p - c) + c$$

We seek to cut out the first iteration of the Sierpinski Triangle pictured above. Let's color our triangle by our desired trapping region and consider the following effect of folding along two lines of symmetry and scaling about the bottom right vertex by a factor of r = 2:

$$\text{Reflect}((\sqrt{3},-1))$$

$$\text{Reflect}((1, 0))$$

$$\text{Stretch}(2, (\sqrt{3},-1)/2)$$

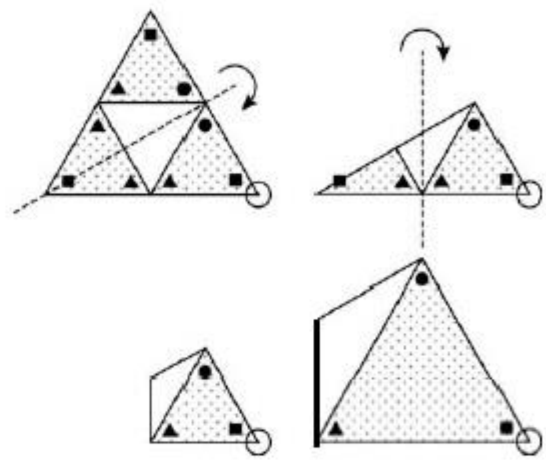

Figure 4. A folding/stretching of a triangle.

With this second construction, we see that the colored region returns to our original triangle with the surface remaining on the surface and interior regions remaining overlapping other interior regions—this overlapping is similar to the checker squares in the Martin Gardner puzzle mentioned earlier. We may now perform our original fold/cut construction to obtain our desired first iteration Sierpinski.

It is easy to see that this construction is similar to a backwards version of the Chaos Game. Choose three points with the same relative position to the three triangle vertices as illustrated by the black shapes in the previous image. Followed backwards, this set of points will end up collectively covering the same positions that they would if iterated by the Chaos Game. One can deduce from here that the trace of each construction is the same.

A general fold/cut Sierpinski Triangle for scale *r* and iterations *N* may be obtained as follows:

Sierpinski Triangle Fold/Stretch/Cut:

Do N times {

Reflect(($\sqrt{3}$,−1))

Reflect((1, 0))

Streatch(r, ($\sqrt{3}$,−1)/2)

}

TriangleFold

Cut(($\sqrt{3}$, 1), 1/4)

3. TETRAHEDRON SIERPINSKI

A technique of 3d ‘fractal folding’ construction was invented by ‘knighty’ at Fractal Forums. [7] I will explore here variations based on his idea—first a folding of a Sierpinski Tetrahedron. A Sierpinski Polyhedron is constructed from a polyhedron with *m* vertices by placing *m* copies of itself strictly inside itself so that a vertex of each copy is located at a vertex of the original polyhedron. Each copy is reduced in size by a scaling factor, and for certain polyhedron there exists a scaling factor so that each copy touches (but does not overlap) all its neighboring copies. We will call a Sierpinski Polyhedron with this scaling factor a flake. Only certain select polyhedron omit this construction. Here we will consider the platonic solids which can all be make into flakes.

To begin, we will first construct a Tetrahedron. Define the Tetrahedron folding/cutting as follows:

Tetrahedron Fold/Cut:

Reflect({(1, 1, 0), (1, 0, 1), (0, 1, 1)})

Cut((1, 1, 1), 1)

To fold a Sierpinski Tetrahedron, we may use a simple iterative version of the folding Tetrahedron above. For the Tetrahedron fold, we fold all the faces of the Tetrahedron together before cutting. Here, since we seek to place sub-copies of the Tetrahedron at its vertices, we will fold the vertices together instead. This is accomplished by folding with the dual of the

Tetrahedron (which is a reflected Tetrahedron). Once folded, we stretch about the location of the folded vertices. The algorithm for scale *r* and iterations *N* is as follows:

Sierpinski Tetrahedron Fold/Stretch/Cut:

Do N times {

TetrahedronFold

Stretch(r, (1, 1, 1))

}

$\Omega \leftarrow -\Omega$

TetrahedronFold

Cut((1, 1, 1), 1)

Shown below are Sierpinski Tetrahedrons for N = 1, 2, 3. Note that this is still a 'fold and cut' object, in that we fold (and stretch) and then create the object with a single cut.

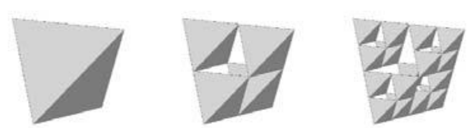

Figure 5. Sierpinski Tetrahedron flakes for N = 1, 2, 3.

Below shows the effect of changing the scale which can result in a connected or disconnected object:

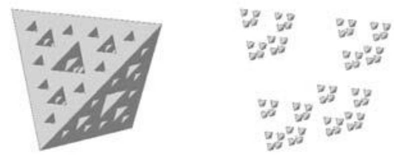

Figure 6. Sierpinski Tetrahedrons for different scales.

For disconnected or flake scale, we see that the limit set Λ will result in a 'dust' consisting of the limit of the vertices of the nested Tetrahedrons. In this regard, we call these vertex limits the *attractor* of the sequence. Moreover, since each iteration is a subset of the previous, we may consider Λ to be *invariant*. Shown below are half slices for connected scale. Note in particular how the interior of the object at right is filled with infinity many enclosed holes—however its outside is smooth. We have thus constructed a fractal object with a flat smooth outside surface!

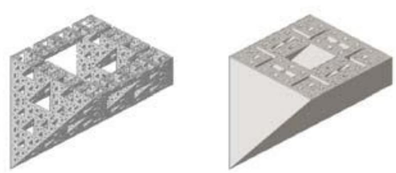

Figure 7. Half-sliced Sierpinski Tetrahedrons for connected scales.

Let's calculate the volume. First for a disconnected or 'flake' scale ($r \geq 2$). A Tetrahedron with side length l has volume:

$$V_T(l) := l^3/(6\sqrt{2})$$

Hence the *N*th iteration of our construction will result in the volume:

$$V_N = V_T(l/r^N)\ m^N$$

Where m is the number of replacement Tetrahedrons. As N get larger, this will better approximate the volume of the limit set. Taking the limit as $N \to \infty$ we see that $V_N = 0$ and thus the limit set will have no volume.

For connected, non-flake scale, calculating the volume becomes a bit more complicated due to 'overlap.' Instead of considering the solid volume, let's consider instead the negative space that surrounds it. Each iteration can be seen to cut-out some (or none) shapes. The volume is thus the volume of the starting Tetrahedron minus the cut–outs.

Figure 8. Inverse Sierpinski Tetrahedrons for N = 1, 2, 3.

Notice how the cut–out shapes are a union of truncated Tetrahedrons. These cut–out shapes will form only for a scaling factor that places the sub-Tetrahedrons where they do not contain the center of the super- Tetrahedron. A cut–out forms only for $r > 4/3$ as proven next.

3.1 Proof that a cut–out forms only for *r* > 4/3

First note that the insphere of a polyhedron is the sphere that touches each face of the polyhedron. Similarly the circumsphere of a polyhedron is the sphere that touches each vertex. The ratio of the radius l of the insphere to radius L of the circumsphere of a regular Tetrahedron is 1/3—with this one may deduce that the scaling factor that places the first 4 sub-Tetrahedrons in our construction just touching at the center of the starting Tetrahedron is $r = (l + L)/L = 4/3$. Hence for scaling factor $r > 4/3$ the first iteration of the Sierpinski Tetrahedron construction will not contain the center of the starting Tetrahedron. *q.e.d.*

Clearly for $1 \leq r \leq 4/3$, the limit set is the original solid starting Tetrahedron. It remains only to consider the case $4/3 < r < 2$. Due to 'overlap,' finding an exact formula for the volume would be a complicated undertaking. Instead, we will consider the following theorem:

3.2 Theorem regarding the volume of a folded Sierpinski Tetrahedron

The limit set of a folded Sierpinski Tetrahedron will have no volume if it does not contain the center of the starting Tetrahedron upon the first iteration.

3.3 Proof of Theorem 3.2

Let V0 be the starting volume and call the center volumes removed per iteration a 'bite.' Suppose that the percent not bitten after bite *i* is δ_I so that $V_1 = \delta_1 V_0$ and in general $V_N = \delta_N V_N - 1 = V_0 \prod_{i=1}^{N} \delta_i$. With this, we see that the limit set will have no volume if the δ_i do not tend towards 1 as $i \to \infty$. This is indeed the case for 'overlapping' scale—the percent not bitten is the same percent per sub-Tetrahedron, however because of how the sub-Tetrahedron 'overlap,' the overall percent not bitten will tend to decrease per iteration. For non-overlapping scale, either a flake forms for which the δ_1 remain constant, otherwise the limit is a disconnected 'dust' of no volume. *q.e.d.*

With this theorem we may say for certain that a Sierpinski Tetrahedron will have no volume for $4/3 < r < 2$.

We may also consider whether or not our object has an interior as summarized in the theorem that follows. It is important to consider this separately. Volume does not imply interior—an object may have no interior but yet have volume.

3.4 Theorem regarding the interior of a folded Sierpinski Tetrahedron

The limit set of a folded Sierpinski Tetrahedron will have no interior if it does not contain the center of the starting Tetrahedron upon the first iteration.

3.5 Proof of Theorem 3.2

Consider a Tetrahedron that fits in a ball of diameter l. Suppose that the first iteration of our folded Sierpinski Tetrahedron for scaling factor *r* carves out a void in the center of this starting Tetrahedron. Then upon the *n*th iteration a point is either in a void or within l/r^n distance from the surface of Λ. This shows that any point in Λ is arbitrarily near the surface of Λ since, for any arbitrarily small distance δ we choose, there exists an *N* such that for all $n > N$ we know that upon the *n*th iteration of our construction our point is in a void or is less than δ distance from the surface of Λ. *q.e.d.*

With this theorem we may say for certain that a Sierpinski Tetrahedron will have no interior for $r > 4/3$.

Despite having no volume, an object may appear to take up some type of bulk in space. As another method of measure, we may consider the amount to which the object 'fills' up space. For this we use the fractal dimension. The fractal dimension of the limit set of a Sierpinski polyhedron is $\log m/\log r$ where *m* is the number of recursive replacement polyhedrons and *r* is the scale factor. A Sierpinski Tetrahedron 'flake' forms for $r = 2$ and thus its limit set has fractal dimension $\log m/\log r = 2$, making it comparable to a flat piece of paper which also has dimension 2.

4. OCTAHEDRON SIERPINSKI

Next we will fold a Sierpinski Octahedron. As before we must fold first with the dual of the object, namely a box. Octahedron and Box folds are given below.

Octahedron Fold:

$$\text{Reflect}(\{\ (1, 0, 0), (0, 1, 0), (0, 0, 1)\ \})$$

Box Fold:

$$\text{Reflect}(\{\ (1, 1, 0), (1, 0, 1), (1, -1, 0), (1, 0, -1)\ \})$$

With these we may construct an Octahedron Sierpinski as follows:

Sierpinski Octahedron Fold/Stretch/Cut:

```
Do N times {
BoxFold
Stretch(r, (1, 0, 0))
}
OctahedronFold
Cut((1, 1, 1), 1)
```

Plotted below is the resulting Sierpinski Octahedron flake for $N = 1, 2, 3$ and $r = 2$. The limit set has fractal dimension $\log 6/\log 2 \approx 2.58$.

Figure 9. Sierpinski Octahedron flakes for N = 1, 2, 3.

5. HEXAHEDRON SIERPINSKI

A Sierpenski Hexahedron (Box) is constructed opposite of a Sierpinski Octahedron, namely Octahedron fold/stretch, then Box fold/cut. The case for $r > 2$ is shown below and is commonly called Cantor Cubes.

Figure 10. Sierpinski Box for N = 1, 2, 3, and r > 2.

Cantor Cubes have no volume in limit and form Cantor dust—a three dimensional version of the Cantor set. For $1 \leq r \leq 2$ we obtain a solid box, with fractal dimension log 8/log 2 = 3 as we would expect.

6. DODECAHEDRON SIERPINSKI

To make a Sierpinski Dodecahedron we will first need a Dodecahedron fold, as well as a fold for an Icosahedron—it's dual. Both are constructed using the vertices of a standard cube and thus both can make use of the Octahedron fold.

Dodecahedron Fold:

OctahedronFold

Reflect({ $(-\varphi(1+\varphi), \varphi^2 - 1, 1+\varphi)$

$(1+\varphi, -\varphi(1+\varphi), \varphi^2 - 1)$ })

Here $\varphi = (1+\sqrt{5})/2$ is the golden ratio. This cuts with $(1, 0, \varphi)$.

Icosahedron Fold:

OctahedronFold

Reflect({ $(\varphi^2, 1, -\varphi)$

$(-\varphi, \varphi^2, 1)$

$(1, -\varphi, \varphi^2)$ })

This cuts with $v = (1, 1, 1)$.

With these we may construct an Octahedron Sierpinski as follows:

Sierpinski Dodecahedron Fold/Stretch/Cut:

Do N times {

IcosahedronFold

Stretch(r, (1, 1, 1))

}

DodecahedronFold

Cut((1, 0, φ),1 + φ)

Plotted below is the flake for $N = 1, 2, 3$ and $r = 2+\varphi$. The limit set has fractal dimension log 20/log $(2+\varphi) \approx 2.33$.

Figure 11. Sierpinski Octahedron flakes for N = 1, 2, 3 and r = 2+φ.

The same idea but reversed results in a Sierpinski Icosahedron:

Sierpinski Icosahedron Fold/Stretch/Cut:

Do N times {

DodecahedronFold

Stretch(r, (1+φ)(1, 0, φ)/|(1, 0, φ)|)

}

IcosahedronFold

Cut((1, 1, 1),2 + φ)

Plotted below is the flake for $N = 1, 2, 3$ and $r = 1+\varphi$. The limit set has fractal dimension log 12/log $(1+\varphi) \approx 2.58$.

Figure 12. Sierpinski Dodecahedron flakes for N = 1, 2, 3 and r = 1+φ.

Plotting the negative space that surrounds a Sierpinski Dodecahedron for overlapping scale results in a 'star dust' of pseudo Great Dodecahedrons—in a magic trick that rivals Houdini, we may for a sufficiently large *N* cut an arbitrarily large number of five pointed stars all with a single cut! 'Fractal hunting' such as this can be a fun pastime.

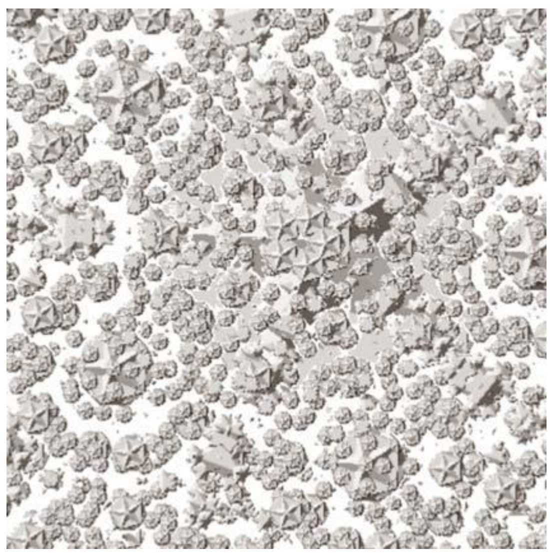

Figure 13. Great Dodecahedron star dust.

7. BIOFORMS & CONCLUSIONS

We have now made all the platonic Sierpenskis. With this base construction we may make alterations to produce an infinite variety of new shapes. Shown next are Sierpinski Dodecahedron and Icosahedron variations created by slightly altering the folding and cutting vectors.

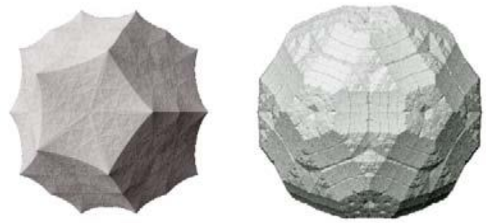

Figure 14. Sierpinski Dodecahedron and Icosahedron variations.

As another variation, rotations can be added to the iterated folding. Since the rotations along with the scaling chain recursively, this will naturally result in logarithmic spirals. As such, these new forms will mimic natural biological forms—they often form Pythagoras trees.

First we need to define a rotation. There are many possible ways to rotate—here we will rotate about a fixed unit axis k through the origin by an angle θ. This is most efficiently preformed using Rodrigues' rotation formula:

$$\text{Rotate}(k, \theta): \forall p \in \Omega,$$

$$p \leftarrow p \cos\theta + (k \times p) \sin\theta + k(k.v)(1 - \cos\theta)$$

The rotations may then be added to the iteration loop. An example of a Sierpinski Tetrahedron with rotations is given below.

Sierpinski Tetrahedron Fold/Stretch/Cut with rotations:

```
Do N times {
TetrahedronFold
Stretch(r, (1, 1, 1))
Rotate((1, 0, 0), θ1)
Rotate((0, 1, 0), θ2)
Rotate((0, 0, 1), θ3)
}
Ω ← −Ω
TetrahedronFold
Cut((1, 1, 1), 1)
```

Shown below are examples for different θ values.

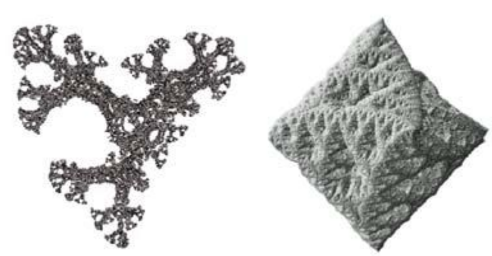

Figure 15. Sierpinski Tetrahedron with rotations

Recall, these are still made with a single cut! The possibilities are endless, Playing with different folds/cuts, one can create (discover?) a plethora of interesting forms.

What interesting objects will you discover? Happy hunting!

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