ESSAYS ON HOUSING MARKET PROBLEM

A Dissertation

by

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ABSTRACT

In my dissertation, I focus on resource reallocation problem. Specifically, I consider the housing market problem. In this problem, there is a group of agents and a group of objects. Each agent owns at most one object and each object is owned by at most one agent. Agents have preferences over objects. The goal is to reallocate these objects among agents while satisfying desirable properties; *Pareto efficiency* (not possible to make someone better-off without making someone worse-off), *individual rationality* (each agent is assigned an object at least as good as her endowment), *strategy proofness* (no agent has an incentive to lie) and *weak-core selection* (no group of agents can trade among themselves such that each of them becomes better-off). In addition, I consider this problem while allowing agents to be indifferent between objects.

Recently, favorable results have been established for such problems. It has been proved that *Pareto efficient*, *weak-core selecting* (hence, *individually rational*) and *strategy proof* rules exist for such problems. I consider additional properties for the housing market problem with indifferences. I show that there are rules which, in addition to the aforementioned properties, satisfy *no justified-envy for agents with identical endowments* and *weak group strategy proofness* even though *Pareto efficiency* and *group strategy proofness* are incompatible under the assumption of indifferences. I achieve this by providing sufficient conditions for *weak group strategy proofness.* Then, I propose a procedural enhancement which prioritizes the outcome achieved without violating *strategy proofness.* I show that some of the existing rules do not satisfy this criterion. So, I propose a new mechanism which satisfies this property in addition to other desirable results. Additionally, I present an amended version of sufficient condition for *strategy proofness* for housing market problem with weak preferences.

I also consider random assignment solutions to housing market problem which is referred to as fractional housing market problem in literature. For general and strict preferences, several impossibility results have been established for such problems. I show that for a restricted class of preferences, trichotomous preferences, these impossibility results do not hold.

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CHAPTER I

INTRODUCTION

In this chapter, I provide a brief introduction of the research presented in the following chapters. This dissertation is a study of the housing market problem. This problem was first modeled by Shapley & Scarf [28]. In this problem, there is a set of agents and a set of objects. Each agent owns at most one object and each object is owned by at most one agent. Agents have preferences over the objects which are to be reassigned. Preferences can be broadly categorized as; general (weak preferences) and restricted (includes, but not limited to, no indifferences with endowment, strict, trichotomous, etc). Solution concept for the problem is of the following types: deterministic (an object is assigned to an agent or not) or random/fractional (an agent can be assigned fractions of objects).

The second chapter considers housing market problem with weak preferences for deterministic solutions. The main objective of that chapter is to extend results already established in this setting. In certain real life applications of housing market problem, social ranking of agents[1] could be of importance i.e. it might be of interest to treat agents with higher social ranking systematically better in assignment of objects. Social ranking of agents can arise in real-life applications of the housing market problem: donor lists in organ donor markets; first-come, first-serve criterion for campus housing and seminar slots; seniority of employees for office assignment, etc. Social ranking of agents allows us to consider fairness notions for the housing market problem. However, for deterministic solutions, fairness cannot be examined in a very meaningful manner since an agent either receives an object or not. A fairness notion that can be considered is *no justified-envy for agents with identical endowments.* This property states that for agents with identical endowments[2], the agent with a higher social rank should receive an object she likes at least as much as the other agent. The existing rules for the housing market problem with weak preferences use priority orderings over agents and/or objects [3, 14, 27]. An intuitive and simple solution would be to use social ranking of agents as the priority orderings required by these rules; if the rule requires priority ordering of agents, use social ranking as the priority ordering whereas if the rule requires priority ordering of objects, endowments of agents are ranked according to the social rank. As it turns out, when priority orderings reflect social ranking of agents, rules proposed in [3, 14, 27] satisfy *no*

[1] I assume that no two agents are ranked identically under the social ranking.
[2] Every agent is indifferent between endowments of these agents.

justified-envy for agents with identical endowments.

Then, I present sufficient conditions for *weak group strategy proofness* which states that no group of agents can misreport their preferences such that each of agent in the group becomes better-off. Additionally, I show that the rule proposed in [3] satisfies *weak group strategy proofness.* Hence, for housing market problem with weak preferences, there are rules which are *Pareto efficient*, *weak core selecting*, *core selecting* (whenever *core* is non-empty), *weakly group strategy proof* and satisfy *no justified-envy for agents with identical endowments.*

Under weak preferences, housing market problem can have several solutions satisfying the same desirable properties. Additionally, *no justified-envy for agents with identical endowments* states only how agents with identical endowments are treated under the rule. It might be of interest to direct how the rule selects the solution to these problems. So, I propose a procedural enhancement; *prioritized treatment of market-equal unsatisfied agents.* This criterion prioritizes the treatment of unsatisfied agents during the course of the algorithm while satisfying conditions for strategy proofness as proposed by Saban & Sethuraman [27]. I show that the existing rules do not satisfy this property when priority orderings reflect social ranking of agents. Arguably, priority orderings could be found, in certain situations, such that this property is satisfied by existing rules. However, this would require finding such priority orderings for each housing market problem which might not be a trivial task. So, I propose a new rule, *Modified Top Cycles* rule. I study this rule for preferences for which there are no indifferences with endowment. The results for this rule show that for housing market problem, there are rules which are *Pareto efficient*, *weak core selecting*, *strategy proof* and satisfy *no justified-envy for agents with identical endowments* and *prioritized treatment of market-equal unsatisfied agents.*

Third chapter is a note on Saban & Sethuraman [27]. Using *Modified Top Cycles* rule, I was able to identify an oversight in their sufficient conditions for *strategy proofness* i.e. their result does not hold in general. Then, I provide three conditions under which results of Saban & Sethuraman [27] are valid; no indifferences with endowment, ineffective status change of agents and ineffective status change of objects.

In the final chapter, I consider a random assignment solution to the housing market problem i.e. in this setting, agents can be assigned fractions of objects. The fractions of objects can be interpreted as partial ownership of the object or the probability of receiving an object. Random assignment solution for housing market problem are of interest because fairness can be considered in a more meaningful manner in this setting. Since several impossibility results have been reported for random assignment solutions to housing market problem under weak and strict preferences, I consider the problem under restricted setting of trichotomous preferences.

For random assignment solutions, preferences of agents need to be extended to random assignments. To this end, I employ stochastic dominance relation and use that to define desirable properties. Using rules in class of mechanisms presented by Saban & Sethuraman [27], I generate a random assignment solution to the housing market problem for trichotomous preferences. I show that this rule is *efficient*, *core-stable*, *strategy proof* and satisfies *no-envy for agents with identical endowments*, *no justified-envy* and *equal treatment of equals*. Therefore, several impossibility results do not hold under the setting examined in this chapter.

For each of the following chapters, I introduce the problem along with related literature, model the problem[3], report the results with some discussion and finally provide concluding remarks.

[3]Except for Chapter III for which the model is same as that for Chapter II.

CHAPTER II

HOUSING MARKET WITH WEAK PREFERENCES: ADDITIONAL SELECTION CRITERION

Introduction

I consider problem of reallocating objects among a set of agents. Specifically, I consider problems where each agent has to be assigned (at most) one object and is endowed with (at most) one object. Agents have preferences over objects and these objects are to be reassigned without any monetary transfers. Such reallocation problems are referred to as housing market problems in literature and were first modeled by Shapley & Scarf [28]. This simple economy has several real-life applications such as allocation of housing [2], offices, seminar slots, and organs for transplant [25].

Top Trading Cycles (TTC) rule, attributed to David Gale, was proposed by Shapley & Scarf [28] for housing market problems. Under strict preferences, TTC proceeds by repeating the following until no agent is left in the problem: Each agent points at an agent holding her most preferred object. Since each agent is pointing at someone and there are finite number of agents, there is at least one cycle. Each agent in a cycle is assigned object of the agent she is pointing at and are removed from the problem with this object. The outcome TTC rule satisfies several desirable properties. Roth & Postlewaite [24] show that TTC outcome is the unique allocation in the *core* and it is also the unique *competitive allocation*. Moreover, TTC mechanism is *strategy proof* [23], no agent has an incentive to misreport her preferences, and *group strategy proof* [7], no group of agents has an incentive to misreport their preferences such that no agent in the group is made worse-off and at least one agent is made better-off. Additionally, TTC is the only mechanism which satisfies *Pareto efficiency*, *individual rationality* and *strategy proofness* [17, 29]. Also, as shown by Miyagawa [19], TTC is *anonymous*, independent of how agents are named, and *non-bossy*, no agent can influence welfare of other agents without affecting her own welfare.

Considering weak preferences is a natural extension to the housing market problem. Under weak preferences, agents are allowed to be indifferent between objects. Indifferences among objects can arise when there is not enough information to break ties. Similarly, an agent might be indifferent between some objects if those objects are of similar importance to her e.g. organs for transplant can be identical for an agent when considering properties like blood and tissue type of the donor.

In accounting for indifferences, some of the desirable properties cannot be achieved. In presence of indifferences, a *core* allocation may not exist [28], *competitive allocation* does not coincide with *core* [31] and *Pareto efficiency* is incompatible with *group strategy proofness* [12]. Additionally, *Pareto efficiency*, *individual rationality* and *strategy proofness* are not compatible in general [29]. Moreover, following impossibility results hold under weak preferences: (1) no rule is *Pareto efficient*, *strategy proof* and *anonymous*, and (2) no rule is *Pareto efficient*, *strategy proof*, *individually rational* and *non-bossy* [8, 14].

Moreover, TTC algorithm is limited to strict preferences for agents. An intuitive work around would be to arbitrarily break ties between objects and then applying TTC algorithm to the resulting housing market problem with strict preferences. This straightforward rule is *weakly Pareto efficient*, *individually rational*, *strategy proof*, *non-bossy* and *consistent* [13]. Additionally, this rule can be generalized to agents owning multiple objects [21, 22]. Unfortunately, *weak Pareto efficiency* is quite weak in this setting since any assignment in which one agent gets one of her most preferred objects is *weakly Pareto efficient*. In fact, examples can be found where no matter how ties are broken, outcome of TTC with fixed tie-breaking is not *Pareto efficient* [14].

Even though several desirable results, in case of strict preferences, do not hold for the housing market problem with weak preferences, some appropriate results can still be achieved since *weak core* is non-empty [28] and incompatibility of *Pareto efficiency*, *individual rationality* and *strategy proofness* holds only under certain assumptions on preference domains [30]. Utilizing this, much progress has been made recently for the housing market problem with weak preferences. Alcalde-Unzu & Molis [3] and Jaramillo & Manjunath [14] independently proposed generalizations of TTC algorithm to account for indifferences; *Top Trading Absorbing Sets* ($TTAS$) rule and *Top Cycles* (TC) rule , respectively. Both these rules are *Pareto efficient*, *individually rational*, *strategy proof*, *weak core selecting* and *core selecting* (whenever *core* is non-empty) [3, 6, 14]. Moreover, TC has a polynomial running time whereas $TTAS$ has an exponential running time in the worst possible case [6, 14]. Saban & Sethuraman [27] establish sufficient conditions for *strategy proofness* and employ that condition to come up with fast algorithms. They provide a class of rules for which each member satisfies *Pareto efficiency*, *individual rationality*, *weak core selection* and *strategy proofness*, namely; *common ordering on agents*, *individual ordering on objects* ($CAIO$). Moreover, they propose a member from this class of rules, *Highest Priority Object* (HPO) rule, which has a polynomial running time.

Algorithms for the three rules are quite similar. Each rule is iterative where each step consists of three phases; *departure*, *pointing* and *trading*. In the *departure* phase, a group of agents are chosen to depart in a manner so that there are no beneficial trades possible involving any agent from the departing group. When agents are allowed to be indifferent between objects, it possible to have multiple most preferred objects. Each of these rules employ priority ordering over agents and/or objects to determine a unique pointee for each agent in the *pointing* phase. In the *trading* phase, objects are exchanged according to the cycles formed in *pointing* phase.

In real-life applications of the housing market problem, social rankings of agents might arise e.g. seniority of people in office assignment problem, donor lists in organ donor market, first-come, first-served criterion in on-campus housing problem, etc. It might be of interest to treat agents with higher social ranking better than agents with lower social ranking. A simple solution would be to use social ranking of agents as priority orderings for the existing rules. Specifically, for rules which use priority ordering of agents, social ranking of agents can be used as the priority orderings and rules which use priority ordering of objects, social ranking of agents could be used to rank endowments of agents. Then, it might be of interest to determine if these rules satisfy some fairness criterion associated with social ranking of agents because using certain priority orderings could potentially lead to systematic favoring of some agents. *No justified-envy for agents with identical endowments* is an appropriate fairness criterion for the housing market problem. This property states that if agents have identical endowments, then agent with higher social ranking should be favored by the mechanism.

Next, I show that even though *group strategy proofness* is incompatible with *Pareto efficiency*, *weak group strategy proofness* can still be achieved for the housing market problem with weak preferences. I present two sufficient conditions for *weak group strategy proofness* and show that $TTAS$ is *weakly group strategy proof*.

Finally, I propose a procedural enhancement which I refer to as *prioritized treatment of market-equals unsatisfied agents*. This property prioritizes trading cycles which occur at each step based on priority ordering of agents and/or objects without violating *strategy proofness*. I show that TC, $TTAS$ and HPO rules do not satisfy this property. So, I propose a rule which satisfies this property along with other desirable properties.

To the best of my knowledge, additional properties for housing market problem with weak preferences have not been explored as of yet. In the next section, I present the model and provide some relevant notation.

Model

Let N be set of agents and O be set of objects. Without loss of generality, it can be assumed that $|N| = |O|$ [14]. Each agent is endowed with an object and that object is denoted by the bijection $\omega : N \to O$. For each $i \in N$, agent i's endowment is denoted as $\omega(i)$ and for any $M \subseteq N$, let $\omega(M) \equiv \cup_{i \in M} \{\omega(i)\}$.

Let $\mathcal{R}$ be set of all possible preference relations over O. For a given $R \in \mathcal{R}^N$, preference relation for $i \in N$ is denoted as R_i and for each $a, b \in O$; (1) a being at least as good as b for agent i is represented as aR_ib, (2) a being preferred to b by agent i is represented as aP_ib and (3) agent i being indifferent between a and b is denoted as aI_ib. R_{-i} is used to denote preferences of everyone other than agent i. For any $M \subseteq N$, R_M denotes preferences of everyone in M and R_{-M} denotes preferences of everyone other than M. For any $R \in \mathcal{R}^N$, $i \in N$ and $O' \subseteq O$, let $\tau(R_i, O')$ represent agent i's most preferred objects in O' under R_i. Formally, $\tau(R_i, O') \equiv \{a \in O' : aR_ib \ \forall b \in O'\}$.

Let A be set of all possible allocations i.e. it contains all bijections from N to O. For any allocation $\alpha \in A$, let object allocated to person i under α be denoted as $\alpha(i)$. Moreover, for any $M \subseteq N$, let $\alpha(M) \equiv \cup_{i \in M}\{\alpha(i)\}$.

Priority orderings over objects and agents are required for the housing market problem with weak preferences. Let $\prec$ denote some complete, transitive and antisymmetric priority ordering over agents in N or over objects in O. With slight abuse of notation, I use same notation for priority ordering of agents or objects. For ordering over agents, $\prec$, agent i has higher priority ordering than agent j if $i \prec j$. For ordering over objects, $\prec$, object a has higher priority ordering than object b if $a \prec b$.

The quadruple (O, N, ω, R) denotes a *housing market* problem with set of agents N, objects O, endowment ω and preference profile R. An allocation rule, $\varphi : \mathcal{R}^N \times A \to A$, gives an allocation for a given housing market problem.

Let $\varphi : \mathcal{R}^N \times A \to A$ be a rule which is iterative and each of its steps has three phases: departure, pointing and trading. In the departure phase, some agents and objects are chosen to be removed from the problem. In the pointing phase, each (or some) agent points at a unique agent and in trading phase, agents trade objects in accordance with the cycles formed in the pointing phase. For any step t, $p_t^{\varphi}(i)$ will denote the agent pointed at by agent i in pointing phase of φ at step t. Let $h_t^{\varphi}(i)$ be the object held by agent i at beginning of step t under φ. Moreover, for any $M \subseteq N$, let $h_t^{\varphi}(M) \equiv \cup_{i\in M}\{h_t^{\varphi}(i)\}$. Let N_t^{φ} and O_t^{φ} be set of agents and objects, respectively, remaining after departure phase of φ in step t. Any $i \in N_t^{\varphi}$ is said to be satisfied if $h_t^{\varphi}(i) \in \tau(R_i, O_t^{\varphi})$. Let S_t^{φ} be set of all satisfied agents in N_t^{φ}. Any agent $i \in N_t^{\varphi}$ who is not satisfied is referred to as an unsatisfied agent and set of all unsatisfied agents in N_t^{φ} are denoted as U_t^{φ}. At step t, let G_t^{φ} denote the graph where there is a directed arc from each $i \in N_t^{\varphi}$ to $j \in N_t^{\varphi}$ such that $h_t^{\varphi}(j) \in \tau(R_i, O_t^{\varphi})$.

In the next section, I formally describe some desirable properties for allocations in the housing market problem.

Some Properties

Consider any $(R, \omega) \in \mathcal{R}^N \times A$. An allocation $\alpha \in A$ Pareto dominates $\beta \in A$ if $\alpha(i) R_i \beta(i)$ for all $i \in N$ and $\alpha(j) P_j \beta(j)$ for some $j \in N$. An allocation rule, $\varphi : \mathcal{R}^N \times A \to A$, is *Pareto efficient* if for all $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega)$ is not Pareto dominated by any allocation in A.

An allocation rule, φ, is *individually rational* if for all $(R, \omega) \in \mathcal{R}^N \times A$ and $i \in N$, $\varphi(R, \omega)(i)\, R_i \omega(i)$ i.e. each agent receives an object at least as good as her endowment.

An allocation rule is *strategy proof* if no agent has an incentive to misreport her preferences i.e. for each $i \in N$, $R \in \mathcal{R}^N$ and $R_i' \in \mathcal{R}$, $\varphi(R, \omega)(i)\, R_i \varphi(R', \omega)(i)$ where $R' = (R_{-i}, R_i')$.

An allocation rule is *weakly group strategy proof* if no group of agents can misreport preferences such that every agent in the group is made better-off i.e. for any $M \subseteq N$, there are no $R \in \mathcal{R}^N$ and $R_M' \in \mathcal{R}^M$ such that $\varphi(R', \omega)(i)\, P_i \varphi(R, \omega)(i)$ for each $i \in M$ where $R' = (R_{-M}, R_M')$. It should be obvious that *weak group strategy proofness* implies *strategy proofness* but converse is not true in general.

For each allocation $\alpha \in A$ and $M \subseteq N$, α is said to be *blocked* by M if $\exists \beta \in A$ such that $\beta(M) = \omega(M)$ and for each $i \in M$, $\beta(i) P_i \alpha(i)$. An allocation $\alpha \in A$ is said to be *weakly blocked* by $M \subseteq N$ if $\exists \beta \in A$ such that $\beta(M) = \omega(M)$, $\beta(i)\, R_i \alpha(i)$ for all $i \in M$ and $\beta(j)\, P_j \alpha(j)$ for some $j \in M$. An allocation is in the *weak core* if it is not *blocked* by any subset of N. An allocation is in the *core* if it is not *weakly blocked* by any subset of N. An allocation rule is said to be *weak core selecting* if it always finds allocations in the *weak core* and *core selecting* if it finds allocations in the *core* whenever the *core* is non-empty.

Existing Rules

In this section, I briefly describe three of the existing mechanisms proposed for housing market problem with weak preferences. First, I present the common *departure* condition for the three rules. Then, I describe the properties required for sufficient conditions of *strategy proofness* as given by Saban & Sethuraman [27]. Finally, I describe the *pointing* phase of each of the three rules.

Departure Condition

At each step of the algorithm, agents and objects are chosen to depart. Unlike TTC rule, in presence of indifferences among objects, agents cannot be allowed to depart after they have been part of a trading cycle since some beneficial trades might still be possible. So, in order to achieve *Pareto efficiency*, following *departure* condition is used for the existing rules and I state it for a general rule φ: A group of agents is selected to depart if every person in the group is satisfied and the group, as a whole, owns all their most preferred objects among the remaining objects. Formally, a set of agents, M, is chosen to depart at step t if $h_t^{\varphi}(i) \in \tau(R_i, O')$ for all $i \in M$ and $h_t^{\varphi}(M) = \cup_{i \in M} \tau(R_i, O')$ where $O' \subseteq O_{t-1}^{\varphi}$ and $O_{t-1}^{\varphi} \backslash O'$ are the objects removed at step t before group M is chosen for departure. The process is repeated until no other group of agents satisfies the *departure* condition.

This *departure* condition is equivalent to condition of *paired-symmetric absorbing sets* used for $TTAS$ by Alcalde-Unzu & Molis [3] and *terminal sinks* used for HPO by Saban & Sethuraman [27]. This condition ensures *Pareto efficiency* for these rules because all possible beneficial trades are exhausted.

Independence of Unsatisfied Agents and Persistence

Saban & Sethuraman [27] establish sufficient conditions for *strategy proofness*. They show that *independence of unsatisfied agents* and *persistence* play an important role for *strategy proofness* of rules for housing market problem under weak preferences. I present these properties for a general rule φ.

Independence of unsatisfied agents states that *pointing* phase should be independent of most preferred objects (among the remaining ones) of unsatisfied agents. Consider any step t and $i \in U_t^{\varphi}$, then by *independence of unsatisfied agents*, changing outgoing edges of agent i in G_t^{φ} should not change the unique pointee selected for any $j \in N_t^{\varphi} \setminus \{i\}$.

Persistence states that if an unsatisfied agent was pointed at (directly or indirectly) by some agents, then those agents should keep on pointing (directly or indirectly) at that agent until the unsatisfied agent becomes part of a trading cycle or leaves the problem. Formally, if at step t, there is $\{i_1, \cdots, i_m\} \subseteq N_t^{\varphi}$ such that $p_t^{\varphi}(i_r) = i_{r+1}$ for all $r \in \{1, \cdots, m-1\}$, $i_m \in U_t^{\varphi}$ and $t' > t$ be the first step agent i_m becomes part of a trading cycle or leaves the problem, then $p_{\ddot{t}}^{\varphi}(i_r) = i_{r+1}$ for all $r \in \{1, \cdots, m-1\}$, $\ddot{t} \in \{t, \cdots, t'-1\}$, if agent i_m departs at step t', and $\ddot{t} \in \{t, \cdots, t'\}$, if agent i_m becomes part of a trading cycle at step t'. This condition ensures that any object made available to an unsatisfied agent once, should remain available to that agent until she becomes part of a trading cycle or leaves the problem.

Top Cycles Rule

In this section, I briefly describe *Top Cycles* (TC) rule proposed by Jaramillo & Manjunath [14] and provide some relevant notation. Let $\prec$ be some priority ordering of agents. Step t of the algorithm is as follows:

1. Group of agents satisfying *departure* condition are chosen to depart until no more group of agents satisfy the *departure* condition. Each departing agent is assigned the object she is holding i.e. if agent i was chosen to depart, then agent i is assigned $h_t^{TC}(i)$.

2. Each agent points at an agent holding one of her most preferred objects among the remaining objects. If there are more than one such agents, the unique agent pointed at is determined in the following manner:

   (a) *(TC-persistence)* For any agent j who holds the same object as in the previous step, agents pointing at agent j in the previous step, point at agent j in the current step i.e. if $h_t^{TC}(j) = h_{t-1}^{TC}(j)$, for each $i \in N_t^{TC}$ such that $p_{t-1}^{TC}(i) = j$, $p_t^{TC}(i) = j$[4].

   (b) If at least one of the most preferred objects of an agent is held by an unsatisfied agent, that agent points at the unsatisfied agent with the highest priority under $\prec$.

   (c) Any agent who is not pointing must have all her most preferred objects held by satisfied agents. If some of those satisfied agents point at an unsatisfied agent, the agent points

[4] TC rule explicitly enforces *persistence*. However, TC-*persistence* is more restrictive than *persistence* and might result in some cycles not having any unsatisfied agents [27].

at whoever points at the higher priority unsatisfied agent. If two or more satisfied agents point at the unsatisfied agent with highest priority, the agent points at the satisfied agent having a higher priority under $\prec$. If none of the satisfied agents point at an unsatisfied agents, the agent points at whoever points at someone who points at an unsatisfied agent with the highest priority. If two or more satisfied agents point at someone who points at the unsatisfied agent with highest priority, among these agents, the agent points at the satisfied agent with a higher priority and so on.

(d) Any agent unable to reach an unsatisfied agent points at the highest priority agent, other than herself, holding one of her most preferred objects.

3. Since at each step, every agent is pointing at someone, there is at least one cycle among the remaining agents. In the next step of TC rule, every agent in a cycle holds object of the agent she was pointing at i.e. if agent i is part of a trading cycle at step t, $h_{t+1}^{TC}(i) = h_t^{TC}\left(p_t^{TC}(i)\right)$.

The second phase of TC ensures that each cycle has at least one unsatisfied person in absence of *TC-persistent* pointing. For any $(R, \omega) \in \mathcal{R}^N \times A$ and priority ordering $\prec$, outcome of TC rule is denoted as $TC^{\prec}(R, \omega)$. TC rule is *Pareto efficient*, *individually rational*, *strategy proof*, *weak core selecting* and *core selecting* (whenever *core* is non-empty) [6, 14]. Moreover, it has been shown to have a polynomial running time[14].

Top Trading Absorbing Sets Rule

I briefly describe $TTAS$ rule given by Alcalde-Unzu & Molis [3]. Let $\prec$ be some ordering over objects.

There is a path from node v to node v' if there are nodes $\{v_1, \cdots, v_m\}$ such that there is an arc from v_{l-1} to v_l for all $l \in \{2, \cdots, m\}$, $v_1 = v$ and $v_m = v'$. A set of nodes, V, is said to be an absorbing set if for all $v, v' \in V$, there is a path from v to v' and for each $v \in V$, there is no path from v to some $v' \notin V$. Absorbing set V is paired-symmetric if for all $v \in V$, $\exists v' \in V$ such that there is an arc from v to v' and an arc from v' to v. Step t of the algorithm proceeds in the following manner[5]:

[5]Even though for $TTAS$ rule, each agent points at an object and each object points at an agent, I describe the rule such that agents are pointing at agents for notational congruity.

1. Each remaining agent points at each agent holding one of her most preferred objects (among the remaining ones) i.e. each $i \in N_t^{TTAS}$ points at all agents who own an object in $\tau\left(R_i, O_t^{TTAS}\right)$,.

2. Group of agents are chosen to depart according to the *departure* condition until no more group of agents satisfy *departure* condition. Each departing agent is assigned the object she is holding i.e. if agent i was chosen to depart, then agent i is assigned $h_t^{TTAS}(i)$.

3. Now consider remaining absorbing sets, if any. For each agent in the absorbing set with multiple most preferred objects (among the remaining ones), a unique agent is chosen to point at in the following manner: among the most preferred objects that have not been assigned to the agent yet, the agent points at the agent who owns the object with the highest priority ordering under $\prec$. If all most preferred objects have been assigned to the agent at least m times, the agent points at the agent who owns the highest priority most preferred object which has not been assigned to her $m+1$ times. For this criterion, endowment of an agent is considered as a previously assigned object.

4. Since each agent, in an absorbing set, is pointing at an object there is at least one cycle. Each agent and object in a cycle are kept in the algorithm. However, in step $t+1$, each agent in a cycle is assigned the object she was pointing at in the cycle i.e. if agent i is in a cycle at step t, $h_{t+1}^{TTAS}(i) = h_t^{TTAS}\left(p_t^{TTAS}(i)\right)$.

(3) and (4) can be considered as *pointing* and *trading* phase of $TTAS$, respectively. The algorithm ends when every agent and object has departed. For any $(R, \omega) \in \mathcal{R}^N \times A$ and priority ordering of objects $\prec$, outcome of $TTAS$ rule will be denoted as $TTAS^{\prec}(R, \omega)$. $TTAS$ rule is *Pareto efficient*, *individually rational*, *strategy proof*, *weak core selecting* and *core selecting* (whenever *core* is non-empty) [3]. However, $TTAS$ rule can have an exponential running time in the worst possible case [6].

Highest Priority Object Rule

I briefly describe HPO rule proposed by Saban and Sethuraman [27]. HPO rule requires priority ordering over objects which is then used to induce an ordering over agents at each step. However, the induced ordering of agents is not used for HPO rule and was used only to show that HPO belongs to the class of rules; *common ordering on agents*, *individual ordering on objects* [27]. So, I ignore induced ordering of agents in the following description. Let $\prec$ be some priority ordering over objects. Step t of HPO proceeds as follows:

1. Group of agents are chosen to depart according to the *departure* condition until no more agents satisfy *departure* condition. Each departing agent is assigned the object she is holding i.e. if agent i was chosen to depart, then agent i is assigned $h_t^{HPO}(i)$.

2. Unique pointee for each agent in N_t^{HPO} is determined in the following manner:

   (a) (*Persistence*) For any $i \in U_t^{HPO}$, if there are $\{i_1, \cdots, i_m\} \subseteq N_{t-1}^{HPO}$ such that $p_{t-1}^{HPO}(i_r) = i_{r+1}$ for all $r \in \{1, \cdots, m-1\}$ and $p_{t-1}^{HPO}(i_m) = i$, then $p_t^{HPO}(i_r) = i_{r+1}$ for all $r \in \{1, \cdots, m-1\}$ and $p_t^{HPO}(i_m) = i$.

   (b) For each $i \in U_t^{HPO}$ who is not pointing yet, let $M \subseteq N_t^{HPO}$ be such that for each $j \in M$, $h_t^{HPO}(j) \in \tau\left(R_i, O_t^{HPO}\right)$. Then, agent i points at the agent in M who owns the highest priority object under $\prec$.

   (c) Repeat the following until everyone is pointing: agents who are already pointing at someone are referred to as labeled agents and denoted as L. Moreover, agents adjacent to labeled agents are denoted as AL i.e. for each $i \in AL$, there is $j \in L$ such that $h_t^{HPO}(j) \in \tau\left(R_i, O_t^{HPO}\right)$. Select the agent in AL who owns the highest priority object, say agent i. Let $M \subseteq L$ be such that for each $j \in M$, $h_t^{HPO}(j) \in \tau\left(R_i, O_t^{HPO}\right)$. Then, agent i points at an agent in M who owns the highest priority object i.e. $p_t^{HPO}(i) = j$ if for each $j, j' \in M$, $j \neq j'$ and $h_t^{HPO}(j) \prec h_t^{HPO}(j')$. Add agent i to L and each $j \in N_t^{HPO} \backslash L$ such that $h_t^{HPO}(i) \in \tau\left(R_j, O_t^{HPO}\right)$ to AL.

3. Since each agent is pointing, there is at least one cycle of remaining agents and by (2), each cycle has at least one unsatisfied agent. In the next step of HPO rule, each agent in a cycle holds object of the agent she was pointing at i.e. if agent i is part of a trading cycle at step t, $h_{t+1}^{HPO}(i) = h_t^{HPO}\left(p_t^{HPO}(i)\right)$.

For any $(R, \omega) \in \mathcal{R}^N \times A$ and priority ordering of objects $\prec$, outcome of HPO rule is denoted as $HPO^{\prec}(R, \omega)$. HPO rule is *Pareto efficient*, *individually rational*, *strategy proof* and *weak core selecting* [27]. Moreover, HPO rule can be implemented in $O\left(n^2 \log n + n^2 \gamma\right)$ where γ is the maximum number of objects agents are indifferent between for a given preference list.

Results for Existing Rules

In this section, I present results for the existing rules. First, I present a fairness criterion, *no justified-envy for agents with identical endowments*, and show that it is satisfied by TC, $TTAS$ and HPO rules. Next, I show that $TTAS$ satisfies *weak group strategy proofness* and also present selection criteria which induces *weak group strategy proofness.* I show that $TTAS$ satisfies one of these sufficient conditions and hence, is *weakly group strategy proof.* Then, I present a procedural enhancement which prioritizes the trades occurring during the algorithm while satisfying *strategy proofness.* I show that this criterion is not satisfied by TC, $TTAS$ or HPO rule.

No Justified-Envy for Agents with Identical Endowments

In real-life applications of housing market problem, social ranking of agents might arise and it might be of interest to treat agents in accordance with this ranking. Examples of social rankings arising in housing market problems are as follows: seniority of employees in office assignment, donor lists in organ donor market, first-come, first-serve criterion for on-campus housing, etc. The aforementioned rules, TC, $TTAS$ and HPO, make use of priority ordering of agents or objects. So, which priority ordering should be used for a given housing market problem for these rules? An intuitive and simple solution to this question would be to use the social ranking of agents as the priority ordering. If priority ordering is over agents, it could be same as social ranking of agents whereas if priority ordering is over objects, social ranking of agents can be used to induce an ordering of objects in the following manner: if agent i has a higher social ranking than agent j, then $\omega(i)$ has a higher priority ordering than $\omega(j)$.

Then, a fairness notion of interest could be that a higher priority agent should receive an object she likes at least as much as that of a lower priority agent when endowments of these agents are considered to be identical by everyone. Formally, a rule φ with priority ordering $\prec$ over agents or objects, $\varphi : \mathcal{R}^N \times A \to A$ satisfies *no justified-envy for agents with identical endowments* if for any $i, j \in N$ such that $i \prec j$ or $\omega(i) \prec \omega(j)$ and $\omega(i) I_k \omega(j)$ for all $k \in N$, then $\varphi^{\prec}(R, \omega)(i) R_i \varphi^{\prec}(R, \omega)(j)$. As it turns out, *no justified-envy for agents with identical endowments* is satisfied by TC, $TTAS$ and HPO rules when priority ordering reflects social ranking of agents.

Proposition 2.1. For ordering $\prec$ and any $(R, \omega) \in \mathcal{R}^N \times A$, TC, $TTAS$ and HPO rules satisfy *no justified-envy for agents with identical endowments.*

Proof. Let φ be some *individually rational* rule. If there are $i, j \in N$ such that $i \prec j$ or $\omega(i) \prec \omega(j)$, $\omega(i) I_k \omega(j)$ for each $k \in N$ and $\varphi(j) P_i \varphi(i)$, then it cannot be that $\varphi(j) = \omega(j)$ since then $\omega(i) P_i \varphi(i)$. So, $\varphi(j) \neq \omega(j)$. For TC, $TTAS$ and HPO rules, this implies that agent j is part of a trading cycle at least once.

(TC rule) On contrary, there is $(R, \omega) \in \mathcal{R}^N \times A$, ordering over agents $\prec$, $i, j \in N$ such that $i \prec j$ and $\omega(i) I_k \omega(j)$ for all $k \in N$ but $\alpha(j) P_i \alpha(i)$ where $\alpha \equiv TC^{\prec}(R, \omega)$. Let t be the first step agent j becomes part of a trading cycle. So, $h_t^{TC}(j) = \omega(j)$ and $\alpha(j) \in \tau\left(R_j, O_t^{TC}\right)$. It must be the case that i has neither departed nor been part a trading cycle for any $t' \leq t$ i.e. $h_t^{TC}(i) = \omega(i)$. If i departed in some $t' \leq t$, then $\alpha(i) R_i b$ for all $b \in O_{t'}^{TC}$ and since $O_t^{TC} \subseteq O_{t'-1}^{TC}$, $\alpha(i) R_i \alpha(j)$. If i has been part of a trading cycle for some $t' \leq t$, then $\alpha(i) \in \tau\left(R_i, O_{t'}^{TC}\right)$ and since $O_t^{TC} \subseteq O_{t'}^{TC}$, we again have a contradiction. Moreover, it cannot be the case that agent i is satisfied at step t because if $\omega(i) \in \tau\left(R_i, O_t^{TC}\right)$, $\omega(i) R_i \alpha(j)$. Now, let $k \in N_t^{TC}$ be the agent pointing at agent j in the trading cycle including agent j. Then, $\omega(j) \in \tau\left(R_k, O_t^{TC}\right)$ and since $\omega(i) I_k \omega(j)$, $\omega(i) \in \tau\left(R_k, O_t^{TC}\right)$. Since agent k is not pointing at agent i, it must be the case that agent k is pointing at agent j by *TC-persistence* of pointing phase. Let $t_k < t$ be the first step agent k points at agent j. Since $\omega(i), \omega(j) \in \tau(R_k, O_{t_k})$, $i, j \in U_{t_k}^{TC}$ and $i \prec j$, it cannot be the case that agent k points at agent j in step t_k.

($TTAS$ rule) On contrary, there exists $(R, \omega) \in \mathcal{R}^N \times A$, ordering over objects $\prec$, $i, j \in N$ such that $\omega(i) \prec \omega(j)$ and $\omega(i) I_k \omega(j)$ for all $k \in N$ but $\alpha(j) P_i \alpha(i)$ where $\alpha \equiv TTAS^{\prec}(R, \omega)$. Let t be the first step agent j is part of a cycle. So, we have $h_t^{TTAS}(j) = \omega(j)$ and $\alpha(j) \in \tau\left(R_j, O_t^{TTAS}\right)$. It cannot be the case that agent i has departed at some $t' \leq t$ since then $\alpha(i) R_i b$ for all $b \in O_{t'}^{TTAS}$ and so, $\alpha(i) R_i \alpha(j)$ because $O_t^{TTAS} \subseteq O_{t'}^{TTAS}$. Also, agent i could not have been part of a cycle at some $t' \leq t$ since then $\alpha(i) \in \tau\left(R_i, O_{t'}^{TTAS}\right)$ and $\alpha(i) R_i \alpha(j)$ because $O_t^{TTAS} \subseteq O_{t'}^{TTAS}$. So, $h_t^{TTAS}(i) = \omega(i)$. Now, let $k \in N_t^{TTAS}$ be the agent pointing at agent j in the cycle containing agent j. Let S be the absorbing set containing agents j and k. It must be the case that the agent i is in this absorbing set as well because there is a path from agent k to agent i. Neither $\omega(i)$ nor $\omega(j)$ have been assigned to agent k previously since agents i and j have not been part of a cycle before t. Then, it cannot be the case that agent k points at agent j in step t because $\omega(i) \prec \omega(j)$.

(HPO rule) On contrary, there exist $(R, \omega) \in \mathcal{R}^N \times A$, ordering over objects $\prec$, $i, j \in N$ such that $\omega(i) \prec \omega(j)$ and $\omega(i) I_k \omega(j)$ for all $k \in N$ but $\alpha(j) P_i \alpha(i)$ where $\alpha \equiv HPO^{\prec}(R, \omega)$. Let t be the first step agent j is part of a trading cycle. Then, we have $h_t^{HPO}(j) = \omega(j)$. Since $\alpha(j) P_i \alpha(i)$, it cannot be the case that agent i departed at or became part of a trading cycle at some step $t' \leq t$. So, $h_t^{HPO}(i) = \omega(i)$. Moreover, $i \in U_t^{HPO}$. Let k be the agent pointing at agent j in the trading cycle consisting of agent j. Note that it must be the case that agent k is *persistently* pointing at agent j because (1) if $k \in U_t^{HPO}$, agent k points at whoever owns a higher priority object among agents holding one of agent k's most preferred objects and since $\omega(i) \prec \omega(j)$, $p_t^{HPO}(k) \neq j$, and (2) if $k \in S_t^{HPO}$, whenever unique pointee of agent k is to be determined, agent k cannot point at agent j because $\omega(i) \prec \omega(j)$ and agent i is always labeled because $i \in U_t^{HPO}$.

Let $t_k < t$ be the first step agent k points at agent j. If $k \in U_{t_k}^{HPO}$, agent k points at whoever owns a higher priority object among agents holding one of agent k's most preferred objects and since $\omega(i)\, I_k \omega(j)$ and $\omega(i) \prec \omega(j)$, it cannot be the case that $p_{t_k}^{HPO}(k) = j$. If $k \in S_{t_k}^{HPO}$, whenever unique pointee of agent k is to be determined, agent k cannot point at agent j because $\omega(i) \prec \omega(j)$ and agent i is already labeled because $i \in U_{t_k}^{HPO}$. This completes the proof. ■

Weak Group Strategy Proofness

Even though *group strategy proofness* is incompatible with *Pareto efficiency* for the housing market problem under weak preferences [12], *weak group strategy proofness* is still compatible with *Pareto efficiency* as I show in this subsection.

For notational simplicity, I drop φ in the superscript. Let $R, R' \in \mathcal{R}^N$. At step t of rule φ, let N_t (resp. N'_t) and O_t (resp. O'_t) be set of remaining agents and objects, respectively, after departure phase of step t under R (resp. R'). Let $p_t(i)$ (resp. $p'_t(i)$) denote the agent pointed at by agent i at step t under R (resp. R'). Let $h_t(i)$ (resp. $h'_t(i)$) denote the object held by agent i at step t under R (resp. R'). For any $R \in \mathcal{R}^N$, let $C_i(R, t)$ denote set of all agents having a path to agent i at step t, under preferences R, via pointing (including agent i) i.e.

$$C_i(R,t) \equiv \left\{ j \in N_t : \begin{array}{c} j = i, \\ p_t(j) = i, \\ p_t(p_t(j)) = i, \\ \vdots \end{array} \right.$$

Independence of unsatisfied agents is defined for one unsatisfied agent. However, if a rule satisfies this property, it holds for multiple unsatisfied agents as well. Consider the following: Let G_0 be some

$TTC-$graph and $\{i_1, \cdots, i_m\}$ be some set of unsatisfied agents. Let $G_1, \cdots, G_m$ be $TTC-$graphs such that G_k and G_0 differ only in outgoing edges from agents $i_1, \cdots, i_k$ for each $k \in \{1, \cdots, m\}$. Let F be the unique pointee selection criterion under rule φ which satisfies *independence of unsatisfied agents*. Since G_{k-1} and G_k differ only in outgoing edges from agent i_k, $F(G_{k-1})$ and $F(G_k)$ differ only in outgoing edge from agent i_k, by *independence of unsatisfied agents*, for all $k \in \{1, \cdots, m\}$. Then, graphs $F(G_0)$ and $F(G_m)$ differ only in outgoing edges from agents $i_1, \cdots, i_m$. However, *independence of unsatisfied agents* may not be enough to ensure *weak group strategy proofness*. So, I introduce a stronger notion of independence.

Comprehensive Independence of Agents: Let G_1 and G_2 be two $TTC-$graphs which differ only in outgoing edges of an agent i. Let F be the unique pointee selection criterion under rule φ where $F(G)$ represents graph obtained after applying F criterion to graph G. Then, rule φ satisfies *comprehensive independence of agents* if $F(G_1)$ and $F(G_2)$ differ only in outgoing edge from agent i.

In contrast to *independence of unsatisfied agents*, *comprehensive independence of agents* holds for all agents rather than just for unsatisfied agents. Hence, *comprehensive independence of agents* implies *independence of unsatisfied agents*. Similar to *independence of unsatisfied agents*, *comprehensive independence of agents* also holds for multiple agents. Similar to *independence of unsatisfied agents*, consider the following: Let G_0 be some $TTC-$graph and $\{i_1, \cdots, i_m\}$ be any set of agents. Let $G_1, \cdots, G_m$ be $TTC-$graphs such that G_k and G_0 differ only in outgoing edges from agents $i_1, \cdots, i_k$ for each $k \in \{1, \cdots, m\}$. Let F be the unique pointee selection criterion under rule φ which satisfies *comprehensive independence of agents*. Since G_{k-1} and G_k differ only in outgoing edges from agent i_k, $F(G_{k-1})$ and $F(G_k)$ differ only in outgoing edge from agent i_k for all $k \in \{1, \cdots, m\}$, by *comprehensive independence of agents*. Then, graphs $F(G_0)$ and $F(G_m)$ differ only in outgoing edges from agents $i_1, \cdots, i_m$.

Now, I introduce a notion which restricts how unique pointee selection criterion, F, operates when a group of agents are misreporting their preferences. I refer to this notion as *consistent pointing* and it is defined below.

Consistent Pointing: Consider any $R \in \mathcal{R}^N$ and let $R' = (R_{-M}, R'_M)$ for any $M \subseteq N$. Let t (resp. t') be the first step an agent in M departs, becomes satisfied or becomes part of a trading cycle under R (resp. R'). Suppose that $t' < t$. Let $C_M(R, \cdot) = \cup_{i \in M} C_i(R, \cdot)$. Suppose there is step $\tilde{t} < t$ such that for all $\ddot{t} < \tilde{t}$:

1. $\begin{array}{c} N'_{\ddot{t}} \subseteq N_{\ddot{t}} \\ N_{\ddot{t}} \backslash N'_{\ddot{t}} \subseteq C_M\left(R, \ddot{t}-1\right) \end{array}$ and $\begin{array}{c} O'_{\ddot{t}} \subseteq O_{\ddot{t}} \\ O_{\ddot{t}} \backslash O'_{\ddot{t}} \subseteq h_{\ddot{t}}\left(C_M\left(R, \ddot{t}-1\right)\right) \end{array}$,

2. for all $j \in N_{\ddot{t}} \backslash C_M\left(R, \ddot{t}\right)$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, and

3. for all $j \in N_{\ddot{t}} \backslash C_M\left(R, \ddot{t}\right)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$, then

if $N'_{\tilde{t}} \subseteq N_{\tilde{t}}$, $N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right)$, $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$, and $O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{\tilde{t}}\left(C_M\left(R, \tilde{t}-1\right)\right)$, then the unique pointee selection criterion for rule φ is said to be *consistent* if $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for each $j \in N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$.

If unique pointee selection criterion of rule φ is *consistent*, I say that the rule φ satisfies *consistent pointing*.

Proposition 2.2. An *individually rational* rule satisfying *comprehensive independence of agents*, *persistence* and *consistent pointing* properties satisfies *weak group strategy proofness.*

Proof. On contrary, suppose there is $M \subseteq N$ and $R'_M \in \mathcal{R}^M$ such that $\varphi(R', \omega)(i) P_i \varphi(R, \omega)(i)$ for all $i \in M$ where $R' = (R_{-M}, R'_M)$ where φ is an *individually rational* rule. Let $\alpha \equiv \varphi(R, \omega)$ and $\alpha' \equiv \varphi(R', \omega)$. Let t (resp. t') be the first step in φ under R (resp. R') where either of the following is true for some agent $i \in M$:

1. i departs at step t (resp. t') under R (resp. R'),

2. i becomes satisfied at step t (resp. t') under R (resp. R'), or

3. i becomes part of a trading cycle at step t (resp. t') under R (resp. R').

Claim 1. $\alpha'(i) P_i \omega(i)$ for each $i \in M$.

Proof. By *individual rationality* of φ, $\alpha(i) R_i \omega(i)$ for each $i \in N$. By initial assumption, $\alpha'(i) P_i \alpha(i)$ for all $i \in M$. Then, it must be the case that $\alpha'(i) P_i \omega(i)$ for all $i \in M$.

So, it must be the case that each agent in M becomes part of a trading cycle at least once under R'. So, t' represents the first step an agent in M becomes satisfied or becomes part of a trading cycle under R'. □

Claim 2. No agent in M departs at step 1 of φ under either R or R'.

Proof. This follows for R' directly from Claim 1. On contrary, suppose some agent in M departs at step 1 under R. Let $i \in M$ be the first such agent. Let $N_{i,1}$ be set of agents who departed before agent i at step 1. Since these agents depart without ever trading, we have $\alpha(j) = \omega(j)$ for all $j \in N_{i,1}$. Since $\alpha'(i) P_i \alpha(i)$, owing to the departure condition, some agent in $N_{i,1}$ would have to be made strictly worse off under R'. This violates *individual rationality* of rule φ because $\alpha(j) = \omega(j)$ for all $j \in N_{i,1}$. □

Claim 3. No agent in M is satisfied at step 1 under R.

Proof. Note that if $\omega(i) \in \tau(R_i, O_1)$ for some $i \in M$, then, by departure condition, i can be made strictly better off under R' only if some person in $N \backslash N_1$ is made strictly worse off. This violates *individual rationality* because $\alpha(j) = \omega(j)$ for all $j \in N \backslash N_1$. □

Claim 4. $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$, $h_{\tilde{t}} = h'_{\tilde{t}}$ and for $j \in N_{\tilde{t}} \backslash M$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for all $\tilde{t} < \underline{t} = \min\{t, t'\}$. Moreover, $h_{\underline{t}} = h'_{\underline{t}}$. Also, if no agent in M departs at step $\underline{t}$ under R, $N_{\underline{t}} = N'_{\underline{t}}$ and $O_{\underline{t}} = O'_{\underline{t}}$.[6]

Proof. If $\underline{t} = 1$, the claim holds vacuously. Now suppose that $\underline{t} > 1$. Consider $\tilde{t} = 1$. No one in M departs at step 1 under either R or R' (Claims 1-2). Then, set of departing agents and objects should be the same because $R'_j = R_j$ for all $j \in N \backslash M$. So, $N_1 = N'_1$ and $O_1 = O'_1$. Moreover, each agent holds her endowment at step 1 so that $h_1 = h'_1$. By *comprehensive independence of agents*, we have $p_1(j) = p'_1(j)$ for each $j \in N_1 \backslash M$.

Now, suppose claim holds for some $\tilde{t} < \underline{t} - 1$. We want to show that the claim is true for step $\tilde{t} + 1$. By assumption, we have $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$, $h_{\tilde{t}} = h'_{\tilde{t}}$ and $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for each $j \in N_{\tilde{t}} \backslash M$. Since $\tilde{t} < \underline{t}$, no agent in M is part of a trading cycle under either R or R' at step $\tilde{t}$. So, same cycles occur under both R and R' because $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for each $j \in N_{\tilde{t}} \backslash M$. Hence, $h_{\tilde{t}+1} = h'_{\tilde{t}+1}$. Since $\tilde{t} + 1 < \underline{t}$, no agent in M departs at step $\tilde{t} + 1$ under either R or R'. So, $N_{\tilde{t}+1} = N'_{\tilde{t}+1}$ and $O_{\tilde{t}+1} = O'_{\tilde{t}+1}$ because $h_{\tilde{t}+1} = h'_{\tilde{t}+1}$ and $R'_j = R_j$ for all $j \in N \backslash M$. By *persistence* and *comprehensive independence of agents*, $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ for all $j \in N_{\tilde{t}+1} \backslash M$.

To show that $h_{\underline{t}} = h'_{\underline{t}}$, we simply need to show that $h_{\tilde{t}+2} = h'_{\tilde{t}+2}$. Since $\tilde{t} + 1 < \underline{t}$, no agent in M is part of a trading cycle at step $\tilde{t} + 1$ under R and R'. Then, same cycles occur under R and R' at step $\tilde{t} + 1$ because $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ for all $j \in N_{\tilde{t}+1} \backslash M$ and so, we have $h_{\tilde{t}+2} = h'_{\tilde{t}+2}$.

Now, we show that last part of the claim holds true. Since $h_{\underline{t}} = h'_{\underline{t}}$ and no agent in M departs at step $\underline{t}$ under either R (by assumption) or R' (Claim 1), same agents depart at step $\underline{t}$ under R and R' because $R'_j = R_j$ for all $j \in N \backslash M$. Hence, $N_{\underline{t}} = N'_{\underline{t}}$ and $O_{\underline{t}} = O'_{\underline{t}}$. □

Claim 5. Suppose $t' < t$. For any step $\tilde{t}$ and $R \in \mathcal{R}^N$, let $C_M\left(R, \tilde{t}\right) = \cup_{i \in M} C_i\left(R, \tilde{t}\right)$. Then, for all $\tilde{t} \in \{t', \cdots, t-1\}$:

1. $$\begin{array}{c} N'_{\tilde{t}} \subseteq N_{\tilde{t}} \\ N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array} \quad \text{and} \quad \begin{array}{c} O'_{\tilde{t}} \subseteq O_{\tilde{t}} \\ O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{\tilde{t}}\left(C_M\left(R, \tilde{t}-1\right)\right) \end{array},$$

[6] It should be noted that the proof for Claim 4 does not require the rule to satisfy *consistent pointing*.

2. $\begin{array}{c} S_{\tilde{t}} \subseteq S'_{\tilde{t}} \\ S'_{\tilde{t}} \backslash S_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array}$ or equivalently, $\begin{array}{c} U'_{\tilde{t}} \subseteq U_{\tilde{t}} \\ U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array}$,

3. for each $j \in N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$, and

4. for each $j \in N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$, $h_{\tilde{t}+1}(j) = h'_{\tilde{t}+1}(j)$.

Proof. Consider $\tilde{t} = t'$. By Claim 4, $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$ and $h_{\tilde{t}} = h'_{\tilde{t}}$. So, (1) holds for step $\tilde{t}$.

An agent in $N_{\tilde{t}} \backslash M$ is satisfied at step $\tilde{t}$ under R if and only if she is satisfied at step $\tilde{t}$ under R' because $h_{\tilde{t}} = h'_{\tilde{t}}$ and $R'_j = R_j$ for all $j \in N \backslash M$. Moreover, each agent in M is unsatisfied at step $\tilde{t}$ under R because $\tilde{t} < t$. So, $S_{\tilde{t}} \subseteq S'_{\tilde{t}}$. If each agent in M is unsatisfied at step $\tilde{t}$ under R', we have $S_{\tilde{t}} = S'_{\tilde{t}}$. Now suppose some agent $j \in M$ is satisfied at step $\tilde{t}$ under R'. Then, $j \in S'_{\tilde{t}} \backslash S_{\tilde{t}}$. By construction, $j \in C_M\left(R, \tilde{t}-1\right)$. So, (2) holds at step $\tilde{t}$.

Since $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$ and $h_{\tilde{t}} = h'_{\tilde{t}}$, the graphs at step $\tilde{t}$ under R and R' differ only in outgoing edges of agents in M. Then, by *comprehensive independence of agents*, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for all $j \in N_{\tilde{t}} \backslash M$. So, (3) holds at step $\tilde{t}$.

By (3), we know that for each $j \in N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$. So, any trading cycles consisting only of agents in $N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$ at step $\tilde{t}$, occur under both R and R'. So, $h_{\tilde{t}+1}(j) = h'_{\tilde{t}+1}(j)$ and hence, (4) holds at step $\tilde{t}$.

Now, suppose that the claim holds for some $\tilde{t}$ such that $t' \leq \tilde{t} < t-1$. We want to show that the claim is true for step $\tilde{t}+1$. Note that any cycle that occurs at step $\tilde{t}$ under R must consist entirely of agents in $N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$ and these cycles occur at step $\tilde{t}$ under R' as well. Moreover, any cycle that occurs under R' but not under R, at step $\tilde{t}$, must consist entirely of agents in $C_M\left(R, \tilde{t}\right)$. This implies (1) and (2) at step $\tilde{t}+1$.

Next, we want to show that (3) is true at step $\tilde{t}+1$. By Claim 4 and induction hypothesis, conditions of *consistent pointing* are satisfied for all $\ddot{t} < \tilde{t}+1$. Since $\tilde{t}+1 < t$, we have $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ for all $j \in N_{\tilde{t}+1} \backslash C_M\left(R, \tilde{t}+1\right)$ by *consistent pointing*.

By (4) at step $\tilde{t}$ and (3) at step $\tilde{t}+1$, we have $h_{\tilde{t}+2}(j) = h'_{\tilde{t}+2}(j)$ for all $j \in N_{\tilde{t}+1} \backslash C_M\left(R, \tilde{t}+1\right)$ so that (4) holds at step $\tilde{t}+1$. □

Remark 2.1. Claim 5 holds at step t till an agent in M departs or becomes satisfied at step t under R.

Claim 6. It cannot be the case that $t' < t$.

Proof. On contrary, suppose that $t' < t$. Note that if any agent $i \in M$ departs at step $\tilde{t}$, under R', such that $\tilde{t} < t$[7], then $\alpha'(i) \in h_{t-1}(C_M(R, t-1))$. This is the case because for any step $\ddot{t} < t$, if $j \in C_M(R, \ddot{t})$, then $h'_{\ddot{t}+1}(j) \in h_{\ddot{t}}(C_M(R, \ddot{t}))$. So, $\alpha'(i) \in h_{\tilde{t}-1}(C_M(R, \tilde{t}-1))$ and, by *persistence*, $h_{\tilde{t}-1}(C_M(R, \tilde{t}-1)) \subseteq h_{t-1}(C_M(R, t-1))$.

By definition, at least one agent in M departs, becomes satisfied or becomes part of a trading cycle at step t under R. Let $i \in M$ be the first agent to depart or become satisfied at step t under R. Let $N_{i,t} \subseteq N_{t-1}$ and $O_{i,t} \subseteq O_{t-1}$ be set of agents and objects, respectively, departing before agent i departs or becomes satisfied at step t under R. Then, by *persistence*, $N_{i,t} \cap C_M(R, t-1) = \phi$ and $O_{i,t} \cap h_{t-1}(C_M(R, t-1)) = \phi$. If agent i departed at some step $\tilde{t} < t$ under R', then $\alpha'(i) \in h_{t-1}(C_M(R, t-1))$. Since $h_{t-1}(C_M(R, t-1)) \subseteq O_{t-1} \backslash O_{i,t}$, $\alpha(i) R_i \alpha'(i)$. Now, suppose agent i departs at some step $\tilde{t} \geq t$ under R'. Note that, by Remark 2.1, $N_{i,t}$ and $O_{i,t}$ depart at step t under R' as well. Moreover, if any agents and objects depart with (or before) $N_{i,t}$ and $O_{i,t}$ under R', then those agents and objects must be in $C_M(R, t-1)$ and $h_t(C_M(R, t-1))$, respectively. However, by *persistence*, $h_t(C_M(R, t-1)) \subseteq O_{t-1} \backslash O_{i,t}$ and so, $\alpha(i) R_i \alpha'(i)$[8].

Now, suppose that no agent in M departs or becomes satisfied at step t under R. Let $i \in M$ be an agent who becomes part of a trading cycle at step t under R. So, we have $\alpha(i) \in \tau(R_i, O_t)$. If agent i departs at some step $\tilde{t} < t$ under R', then $\alpha'(i) \in h_{t-1}(C_M(R, t-1))$. By *persistence*, $h_{t-1}(C_M(R, t-1)) \subseteq O_t$ and so, $\alpha(i) R_i \alpha'(i)$. Now, suppose agent i departs at step t under R'. Then, $\alpha'(i) \in h_{t-1}(C_M(R, t-1))$ and so, $\alpha(i) R_i \alpha'(i)$. Now, suppose that agent i departs at some step $\tilde{t} > t$ under R'. We have $O'_{\tilde{t}-1} \subseteq O_t$ because $O'_t \subseteq O_t$ by Remark 2.1. Then, $\alpha(i) R_i \alpha'(i)$. □

Claim 7. It cannot be the case that $t \leq t'$.

Proof. On contrary, suppose that $t \leq t'$. Consider the following cases:

Case 1. No agent in M departs at step t under R.

Then, we have $N_t = N'_t$ and $O_t = O'_t$ because $h_t = h'_t$ and $R'_i = R_i$ for each $i \in N \backslash M$. By definition of t and assumption, there is at least one agent in M who either becomes satisfied or becomes part of a trading cycle at step t under R.

Let $i \in M$ be the first agent to become satisfied at step t under R i.e. after departure of some agents, agent i is holding one of her most preferred objects and so, $\alpha(i) I_i \omega(i)$. However, by Claim 4, $N_t = N'_t$ and $O_t = O'_t$. Then, for each $a \in O_t$, we have $\alpha(i) R_i a$ so that it must be the case that $\alpha(i) R_i \alpha'(i)$.

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[7]By Claim 1, $t' < \tilde{t}$.

[8]If agent i departs with (or before) $N_{i,t}$ at step t under R', then $\alpha'(i) \in h_t(C_M(R, t-1))$. If agent i departs at some step $\tilde{t} > t$ under R', then $O'_{\tilde{t}} \subseteq O_{t-1} \backslash O_{i,t}$.

Now, let $i \in M$ be an agent who becomes part of a trading cycle at step t under R. Then, $\alpha(i) \in \tau(R_i, O_t)$. Suppose agent i first becomes part of a trading cycle at step t'_i under R'. Then, $t \leq t'_i$ and so, $O'_{t'_i} \subseteq O_t$[9]. Since $\alpha'(i) \in \tau\left(R'_i, O'_{t'_i}\right)$ and $O'_{t'_i} \subseteq O_t$, we have $\alpha(i) R_i \alpha'(i)$.

Case 2. Some agents in M depart at step t under R.

Let $i \in M$ be the first agent in M to depart at step t under R. Since no agent in M was part of a trading cycle at step $t-1$ under R, we have $\alpha(i) = \omega(i)$. Let $N_{i,t}$ and $O_{i,t}$ be the set of agents and objects departing before agent i at step t under R, respectively. Then, $N_{i,t}$ and $O_{i,t}$ depart at step t under R' as well. Then, $\alpha'(i) \in O_t \backslash O_{i,t}$ because by Claim 4, $h_t(N_{i,t}) = h'_t(N_{i,t})$. Then, $\alpha(i) R_i \alpha'(i)$ by departure condition. □

Claims (6) and (7) give a contradiction. This completes the proof. ■

So, Proposition 2.2 gives a sufficient condition for a rule to satisfy *weak group strategy proofness*. Next, I show that $TTAS$ rule is *weakly group strategy proof.* I prove this by showing that $TTAS$ satisfies *comprehensive independence of agents* and *consistent pointing.*

Proposition 2.3. *Top Trading Absorbing Sets Rule* ($TTAS$) satisfies *weak group strategy proofness.*

Proof. We know that $TTAS$ satisfies *persistence* and *individual rationality.* At any step of $TTAS$, unique pointee of an agent is determined by priority ordering of objects and number of times an object was previously held by that agent. So, $TTAS$ satisfies *comprehensive independence of agents* as well.

Now, we need to show that $TTAS$ satisfies *consistent pointing.* Consider any $R \in \mathcal{R}^N$ and let $R' = (R_{-M}, R'_M)$ for some $M \subseteq N$. Suppose $t' < t$. By Claim 4, we have $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$, $h_{\tilde{t}} = h'_{\tilde{t}}$ and for $j \in N_{\tilde{t}} \backslash M$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for all $\tilde{t} \leq t'$[10]. So, $TTAS$ satisfies *consistent pointing* for $\tilde{t} \leq t'$.

Now, suppose that $TTAS$ satisfies *consistent pointing* property for some $\tilde{t}$ such that $t' \leq \tilde{t} < t-1$. We want to show that $TTAS$ satisfies the property for step $\tilde{t}+1$. So, for all $\ddot{t} \leq \tilde{t}$, we have:

1. $$\begin{array}{c} N'_{\ddot{t}} \subseteq N_{\ddot{t}} \\ N_{\ddot{t}} \backslash N'_{\ddot{t}} \subseteq C_M\left(R, \ddot{t}-1\right) \end{array} \quad \text{and} \quad \begin{array}{c} O'_{\ddot{t}} \subseteq O_{\ddot{t}} \\ O_{\ddot{t}} \backslash O'_{\ddot{t}} \subseteq h_{\ddot{t}}\left(C_M\left(R, \ddot{t}-1\right)\right) \end{array},$$
2. for all $j \in N_{\ddot{t}} \backslash C_M\left(R, \ddot{t}\right)$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, and
3. for all $j \in N_{\ddot{t}} \backslash C_M\left(R, \ddot{t}\right)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

[9]This is the case because, by Claim 4, $O_t = O'_t$ and $O'_{t'_i} \subseteq O'_t$.

[10]This is the case because Claim 4 does not require *consistent pointing* to be true.

Suppose that $N'_{\tilde{t}+1} \subseteq N_{\tilde{t}+1}$, $N_{\tilde{t}+1} \backslash N'_{\tilde{t}+1} \subseteq C_M\left(R, \tilde{t}\right)$, $O'_{\tilde{t}+1} \subseteq O_{\tilde{t}+1}$, and $O_{\tilde{t}+1} \backslash O'_{\tilde{t}+1} \subseteq h_{\tilde{t}}\left(C_M\left(R, \tilde{t}\right)\right)$. Consider any $j \in N_{\tilde{t}+1} \backslash C_M\left(R, \tilde{t}+1\right)$. We need to show that $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$. In order to do that, we first show that for each $a \in \tau\left(R_j, O_{\tilde{t}+1}\right)$ and $b \in \tau\left(R_j, O'_{\tilde{t}+1}\right)$, aI_jb. Then, we show that $\tau\left(R_j, O'_{\tilde{t}+1}\right) \subseteq \tau\left(R_j, O_{\tilde{t}+1}\right)$.

First, note that aR_jb because $O'_{\tilde{t}+1} \subseteq O_{\tilde{t}+1}$. Now, if aP_jb, then it must be the case that $\tau\left(R_j, O_{\tilde{t}+1}\right) \cap O'_{\tilde{t}+1} = \phi$. But then, $\tau\left(R_j, O_{\tilde{t}+1}\right) \subseteq h_{\tilde{t}+1}\left(C_M\left(R, \tilde{t}\right)\right)$ because $O_{\tilde{t}+1} \backslash O'_{\tilde{t}+1} \subseteq h_{\tilde{t}+1}\left(C_M\left(R, \tilde{t}\right)\right)$. So, at step $\tilde{t}+1$ under R, agent j points at some agent holding an object in $\tau\left(R_j, O_{\tilde{t}+1}\right)$ but then, $j \in C_M\left(R, \tilde{t}+1\right)$ because each agent who owns an object in $\tau\left(R_j, O_{\tilde{t}+1}\right)$ is in $C_M\left(R, \tilde{t}+1\right)$. So, aI_jb for each $a \in \tau\left(R_j, O_{\tilde{t}+1}\right)$ and $b \in \tau\left(R_j, O'_{\tilde{t}+1}\right)$. Hence, it must be the case that $\tau\left(R_j, O'_{\tilde{t}+1}\right) \subseteq \tau\left(R_j, O_{\tilde{t}+1}\right)$ because $O'_{\tilde{t}+1} \subseteq O_{\tilde{t}+1}$.

By *persistence*, $j \in N_{\ddot{t}} \backslash C_M\left(R, \ddot{t}\right)$ for all $\ddot{t} \leq \tilde{t}+1$ and so, same cycles occur for agent j for all steps $\ddot{t} \leq \tilde{t}$ under R and R'. Since same cycles occurred for agent j for all $\ddot{t} \leq \tilde{t}$, objects in $\tau\left(R_j, O'_{\tilde{t}+1}\right)$ would have been assigned same number of times under R and R'. Let $a \in \tau\left(R_j, O'_{\tilde{t}+1}\right)$ be the object that is assigned least number of times[11] to agent j. Then, under R', agent j points at the agent who owns object a at step $\tilde{t}+1$. Under R, it cannot be the case that agent j points at an agent holding an object in $O_{\tilde{t}+1} \backslash O'_{\tilde{t}+1}$ because $O_{\tilde{t}+1} \backslash O'_{\tilde{t}+1} \subseteq h_{\tilde{t}}\left(C_M\left(R, \tilde{t}\right)\right)$. So, agent j points at an agent holding object a at step $\tilde{t}+1$ under R. Let $p_{\tilde{t}+1}(j) = k$ and $p'_{\tilde{t}+1}(j) = k'$ under R and R'. Since $j \in N_{\tilde{t}+1} \backslash C_M\left(R, \tilde{t}+1\right)$, it must be the case that $k, k' \in N_{\tilde{t}+1} \backslash C_M\left(R, \tilde{t}+1\right)$. Then, by *persistence*, $k, k' \in N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$ and so, $h_{\tilde{t}+1}(k) = h'_{\tilde{t}+1}(k)$ and $h_{\tilde{t}+1}(k') = h'_{\tilde{t}+1}(k')$. Then, we have $k = k'$ and so, $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ which completes the proof. ■

Based on Proposition 2.1, Proposition 2.3 and results already proved for $TTAS$ rule, following theorem can be stated.

Theorem 2.1. For a housing market problem with weak preferences, there are rules which are *Pareto efficient*, *weak core selecting* (hence, *individually rational*), *weakly group strategy proof* (hence, *strategy proof*), *core selecting* (whenever, *core* is non-empty) and satisfy *no justified-envy for agents with identical endowments.*

The sufficient condition given in Proposition 2.2 might be too restrictive especially because rules satisfying *comprehensive independence of agents* might be computationally complex which is true for $TTAS$ rule. This complexity in running time might arise because a rule satisfying *comprehensive independence of agents* cannot ensure that each trading cycle consists of at least one unsatisfied

[11] If there are multiple such objects, ties are broken according to priority ordering of objects.

agent. So, I provide an alternative sufficient condition for *weak group strategy proofness*. Consider the following variation of *consistent pointing*:

Consistent* Pointing: For any $M \subseteq N$ and $R, R' \in \mathcal{R}^N$ such that $R' = (R_{-M}, R'_M)$. Suppose there is step $\tilde{t}$ for rule φ under preferences R and R' such that:

1. $\forall j \in N_{\tilde{t}-1} \backslash C_M\left(R, \tilde{t}-1\right)$, $h_{\tilde{t}}(j) = h'_{\tilde{t}}(j)$,

2. $\begin{array}{c} N'_{\tilde{t}} \subseteq N_{\tilde{t}} \\ N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array}$ and $\begin{array}{c} O'_{\tilde{t}} \subseteq O_{\tilde{t}} \\ O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{\tilde{t}}\left(C_M\left(R, \tilde{t}-1\right)\right) \end{array}$, and

3. $\begin{array}{c} S_{\tilde{t}} \subseteq S'_{\tilde{t}} \\ S'_{\tilde{t}} \backslash S_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array}$ or equivalently, $\begin{array}{c} U'_{\tilde{t}} \subseteq U_{\tilde{t}} \\ U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array}$, then

rule φ is said to satisfy *consistent* pointing* if for all $j \in N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$.

The next result shows that an *individually rational* rule satisfying *independence of unsatisfied agents*, *persistence* and *consistent* pointing* satisfies *weak group strategy proofness*.

Proposition 2.4. An *individually rational* rule satisfying *independence of unsatisfied agents*, *persistence* and *consistent* pointing* satisfies *weak group strategy proofness*.

Proof. On contrary, suppose there is $M \subseteq N$ and $R'_M \in \mathcal{R}^M$ such that $\varphi(R', \omega)(i)\, P_i \varphi(R, \omega)(i)$ for all $i \in M$ where $R' = (R_{-M}, R'_M)$. Let $\alpha \equiv \varphi(R, \omega)$ and $\alpha' \equiv \varphi(R', \omega)$. Let t (resp. t') be the first step in φ under R (resp. R') where either of the following is true for some agent $i \in M$:

1. i departs at step t (resp. t') under R (resp. R'),

2. i becomes satisfied at step t (resp. t') under R (resp. R'), or

3. i becomes part of a trading cycle at step t (resp. t') under R (resp. R').

I prove this result by showing that claims made in proof of Proposition 2.2 are true under these properties as well. Note that Claims 1-3 hold for rule φ. Now I show that Claim 4 is true when *comprehensive independence of agents* is replaced with *independence of unsatisfied agents*.

Claim 4. $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$, $h_{\tilde{t}} = h'_{\tilde{t}}$ and for $j \in N_{\tilde{t}} \backslash M$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for all $\tilde{t} < \underline{t} = \min\{t, t'\}$. Moreover, $h_{\underline{t}} = h'_{\underline{t}}$. Also, if no agent in M departs at step $\underline{t}$ under R, $N_{\underline{t}} = N'_{\underline{t}}$ and $O_{\underline{t}} = O'_{\underline{t}}$.

Proof. If $\underline{t} = 1$, the claim holds vacuously. Now suppose that $\underline{t} > 1$. Consider $\tilde{t} = 1$. No agent in M departs at step 1 under either R or R' (Claims 1-2). Then, set of departing agents and objects should be the same because $R'_j = R_j$ for all $j \in N \backslash M$. So, $N_1 = N'_1$ and $O_1 = O'_1$. Moreover, each agent holds her endowment at step 1 so that $h_1 = h'_1$. Since each agent in M is unsatisfied at step 1 under R and R', by *independence of unsatisfied agents*, we have $p_1(j) = p'_1(j)$ for each $j \in N_1 \backslash M$.

Now, suppose claim is true for some $\tilde{t} < \underline{t} - 1$. We want to show that the claim is true for $\tilde{t} + 1$. By induction hypothesis, we have $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$, $h_{\tilde{t}} = h'_{\tilde{t}}$ and $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for each $j \in N_{\tilde{t}} \backslash M$. Since $\tilde{t} < \underline{t}$, no agent in M is part of a trading cycle under either R or R' at step $\tilde{t}$. So, same cycles occur under both R and R' because $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for each $j \in N_{\tilde{t}} \backslash M$. Hence, $h_{\tilde{t}+1} = h'_{\tilde{t}+1}$. Since $\tilde{t} + 1 < \underline{t}$, no agent in M departs at step $\tilde{t} + 1$ under either R or R'. So, $N_{\tilde{t}+1} = N'_{\tilde{t}+1}$ and $O_{\tilde{t}+1} = O'_{\tilde{t}+1}$ because $h_{\tilde{t}+1} = h'_{\tilde{t}+1}$ and $R'_j = R_j$ for all $j \in N \backslash M$. Moreover, since no agent in M is satisfied at step $\tilde{t}+1$ under either R or R', by *independence of unsatisfied agents*, $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ for all $j \in N_{\tilde{t}+1} \backslash M$.

To show that $h_{\underline{t}} = h'_{\underline{t}}$, we simply need to show that $h_{\tilde{t}+2} = h'_{\tilde{t}+2}$. Since $\tilde{t} + 1 < \underline{t}$, no agent in M is part of a trading cycle at step $\tilde{t} + 1$ under R and R'. Then, same cycles occur under R and R' at step $\tilde{t} + 1$ because $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ for all $j \in N_{\tilde{t}+1} \backslash M$ and so, we have $h_{\tilde{t}+2} = h'_{\tilde{t}+2}$.

Now we show that last part of the claim holds true. Since $h_{\underline{t}} = h'_{\underline{t}}$ and no agent in M departs at step $\underline{t}$ under either R (by assumption) or R' (Claim 1), same agents depart at step $\underline{t}$ under R and R' because $R'_j = R_j$ for all $j \in N \backslash M$. Hence, $N_{\underline{t}} = N'_{\underline{t}}$ and $O_{\underline{t}} = O'_{\underline{t}}$. □

Claim 5. Suppose $t' < t$. For any step $\tilde{t}$ and $R \in \mathcal{R}^N$, let $C_M\left(R, \tilde{t}\right) = \cup_{i \in M} C_i\left(R, \tilde{t}\right)$. Then, for all $\tilde{t} \in \{t', \cdots, t-1\}$:

1. $\begin{array}{c} N'_{\tilde{t}} \subseteq N_{\tilde{t}} \\ N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array}$ and $\begin{array}{c} O'_{\tilde{t}} \subseteq O_{\tilde{t}} \\ O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{\tilde{t}}\left(C_M\left(R, \tilde{t}-1\right)\right) \end{array}$,

2. $\begin{array}{c} S_{\tilde{t}} \subseteq S'_{\tilde{t}} \\ S'_{\tilde{t}} \backslash S_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array}$ or equivalently, $\begin{array}{c} U'_{\tilde{t}} \subseteq U_{\tilde{t}} \\ U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq C_M\left(R, \tilde{t}-1\right) \end{array}$,

3. for each $j \in N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$, and

4. for each $j \in N_{\tilde{t}} \backslash C_M\left(R, \tilde{t}\right)$, $h_{\tilde{t}+1}(j) = h'_{\tilde{t}+1}(j)$.

Proof. Consider $\tilde{t} = t'$. By Claim 4, $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$ and $h_{\tilde{t}} = h'_{\tilde{t}}$. So, (1) holds for step $\tilde{t}$.

An agent in $N_{\tilde{t}} \backslash M$ is satisfied at step $\tilde{t}$ under R if and only if she is satisfied at step $\tilde{t}$ under R' because $h_{\tilde{t}} = h'_{\tilde{t}}$, $R'_j = R_j$ for all $j \in N \backslash M$ and $O_{\tilde{t}} = O'_{\tilde{t}}$. Moreover, each agent in M is unsatisfied at step $\tilde{t}$ under R because $\tilde{t} < t$. So, $S_{\tilde{t}} \subseteq S'_{\tilde{t}}$. If each agent in M is unsatisfied at step $\tilde{t}$ under R', we have $S_{\tilde{t}} = S'_{\tilde{t}}$. Now, suppose some agent $j \in M$ is satisfied at step $\tilde{t}$ under R'. Then, $j \in S'_{\tilde{t}} \backslash S_{\tilde{t}}$. By construction, $j \in C_M\left(R, \tilde{t}-1\right)$. So, (2) holds at step $\tilde{t}$.

Since $h_{\tilde{t}} = h'_{\tilde{t}}$ (by Claim 4), (1) and (2) hold at step $\tilde{t}$ and the rule satisfies *consistent* pointing*, we have $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for all $j \in N_{\tilde{t}} \backslash C_M(R, \tilde{t})$ i.e. (3) holds at step $\tilde{t}$.

By (3), we know that for each $j \in N_{\tilde{t}} \backslash C_M(R, \tilde{t})$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$. So, any trading cycles consisting only of agents in $N_{\tilde{t}} \backslash C_M(R, \tilde{t})$ at step $\tilde{t}$, occur under both R and R'. So, $h_{\tilde{t}+1}(j) = h'_{\tilde{t}+1}(j)$ and hence, (4) holds at step $\tilde{t}$.

Now, suppose that the claim holds for some $\tilde{t}$ such that $t' \leq \tilde{t} < t - 1$. We want to show that it is true for step $\tilde{t} + 1$. Note that any cycle that occurs at step $\tilde{t}$ under R must consist entirely of agents in $N_{\tilde{t}} \backslash C_M(R, \tilde{t})$ and these cycles occur at step $\tilde{t}$ under R' as well. Moreover, any cycle that occurs under R' but not under R, at step $\tilde{t}$, must consist entirely of agents in $C_M(R, \tilde{t})$. This implies (1) and (2) at step $\tilde{t} + 1$.

Next, we want to show that (3) is true at step $\tilde{t} + 1$. Since (4) holds at step $\tilde{t}$, (1) and (2) are satisfied at step $\tilde{t} + 1$ and the rule satisfies *consistent* pointing*, we have $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ for all $j \in N_{\tilde{t}+1} \backslash C_M(R, \tilde{t} + 1)$.

By (4) at step $\tilde{t}$ and (3) at step $\tilde{t}+1$, we have $h_{\tilde{t}+2}(j) = h'_{\tilde{t}+2}(j)$ for all $j \in N_{\tilde{t}+1} \backslash C_M(R, \tilde{t} + 1)$ so that (4) holds at step $\tilde{t} + 1$. □

Claim 6. It cannot be the case that $t' < t$.

Proof. On contrary, suppose that $t' < t$. Note that if any agent $i \in M$ departs at step $\tilde{t}$, under R', such that $\tilde{t} < t$[12], then $\alpha'(i) \in h_{t-1}(C_M(R, t-1))$. This is the case because for any step $\ddot{t} < t$, if $j \in C_M(R, \ddot{t})$, then $h'_{\ddot{t}+1}(j) \in h_{\ddot{t}}(C_M(R, \ddot{t}))$. So, $\alpha'(i) \in h_{\tilde{t}-1}(C_M(R, \tilde{t} - 1))$ and, by *persistence*, $h_{\tilde{t}-1}(C_M(R, \tilde{t} - 1)) \subseteq h_{t-1}(C_M(R, t-1))$.

By definition, at least one agent in M departs, becomes satisfied or becomes part of a trading cycle at step t under R. Let $i \in M$ be the first agent to depart or become satisfied at step t under R. Let $N_{i,t} \subseteq N_{t-1}$ and $O_{i,t} \subseteq O_{t-1}$ be set of agents and objects, respectively, departing before agent i at step t under R. Then, by *persistence*, $N_{i,t} \cap C_M(R, t-1) = \phi$ and $O_{i,t} \cap h_{t-1}(C_M(R, t-1)) = \phi$. If agent i departed at some step $\tilde{t} < t$ under R', then $\alpha'(i) \in h_{t-1}(C_M(R, t-1))$. Since $h_{t-1}(C_M(R, t-1)) \subseteq O_{t-1} \backslash O_{i,t}$, $\alpha(i) R_i \alpha'(i)$. Now, suppose agent i departs at some step $\tilde{t} \geq t$ under R'. Note that, by Remark 2.1, $N_{i,t}$ and $O_{i,t}$ depart at step t under R' as well. Moreover, if any agents and objects depart with (or before) $N_{i,t}$ and $O_{i,t}$ under R', then those agents and objects must be in $C_M(R, t-1)$ and $h_t(C_M(R, t-1))$, respectively. However, by *persistence*,

[12]By Claim 1, $t' < \tilde{t}$.

$h_t(C_M(R,t-1)) \subseteq O_{t-1} \backslash O_{i,t}$ and so, $\alpha(i) R_i \alpha'(i)$[13].

Now, suppose that no agent in M departs or becomes satisfied at step t under R. Let $i \in M$ be an agent who becomes part of a trading cycle at step t under R. So, we have $\alpha(i) \in \tau(R_i, O_t)$. If agent i departs at some step $\tilde{t} < t$ under R', then $\alpha'(i) \in h_{t-1}(C_M(R,t-1))$. By *persistence*, $h_{t-1}(C_M(R,t-1)) \subseteq O_t$ and so, $\alpha(i) R_i \alpha'(i)$. Now, suppose agent i departs at step t under R'. Then, $\alpha'(i) \in h_{t-1}(C_M(R,t-1))$ and so, $\alpha(i) R_i \alpha'(i)$. Now suppose that agent i departs at some step $\tilde{t} > t$ under R'. We have $O'_{\tilde{t}-1} \subseteq O_t$ because $O'_t \subseteq O_t$ by Remark 2.1. Then, $\alpha(i) R_i \alpha'(i)$. □

Claim 7. It cannot be the case that $t \leq t'$.

Proof. On contrary, suppose that $t \leq t'$. Consider the following cases:

Case 1. No agent in M departs at step t under R.

Then, we have $N_t = N'_t$ and $O_t = O'_t$ because $h_t = h'_t$ and $R'_i = R_i$ for each $i \in N \backslash M$. By definition of t and assumption, there is at least one agent in M who either becomes satisfied or becomes part of a trading cycle at step t under R.

Let $i \in M$ be the first agent to become satisfied at step t under R i.e. after departure of some agents, agent i is holding one of her most preferred objects and so, $\alpha(i) I_i \omega(i)$. However, by Claim 4, $N_t = N'_t$ and $O_t = O'_t$. Then, for each $a \in O_t$, we have $\alpha(i) R_i a$ so that it must be the case that $\alpha(i) R_i \alpha'(i)$.

Now, let $i \in M$ be an agent who becomes part of a trading cycle at step t under R. Then, $\alpha(i) \in \tau(R_i, O_t)$. Suppose agent i first becomes part of a trading cycle at step t'_i under R'. Then, $t \leq t'_i$ and so, $O'_{t'_i} \subseteq O_t$[14]. Since $\alpha'(i) \in \tau\left(R'_i, O'_{t'_i}\right)$ and $O'_{t'_i} \subseteq O_t$, we have $\alpha(i) R_i \alpha'(i)$.

Case 2. Some agents in M depart at step t under R.

Let $i \in M$ be the first agent in M to depart at step t under R. Since no agent in M was part of a trading cycle at step $t-1$ under R, we have $\alpha(i) = \omega(i)$. Let $N_{i,t}$ and $O_{i,t}$ be the set of agents and objects departing before agent i at step t under R, respectively. Then, $N_{i,t}$ and $O_{i,t}$ depart at step t under R' as well. Then, $\alpha'(i) \in O_t \backslash O_{i,t}$ because by Claim 4, $h_t(N_{i,t}) = h'_t(N_{i,t})$ and agent i does not depart at step t under R'. Then, $\alpha(i) R_i \alpha'(i)$ by departure condition. □

Claims (6) and (7) give us a contradiction. This completes the proof. ■

[13]If agent i departs with (or before) $N_{i,t}$ at step t under R', then $\alpha'(i) \in h_t(C_M(R,t-1))$. If agent i departs at some step $\tilde{t} > t$, then $O'_{\tilde{t}} \subseteq O_{t-1} \backslash O_{i,t}$.

[14]This is the case because, by Claim 4, $O_t = O'_t$ and $O'_{t'_i} \subseteq O'_t$.

Prioritized Treatment of Market-Equal Unsatisfied Agents

I resume the use of notation with φ in super-script. In this section, I propose a new criterion; namely, *prioritized treatment of market-equal unsatisfied agents.* Before defining this property formally, I provide some intuition for this property using an example.

Example 2.1: Consider the following housing market problem: $N = \{1, 2, 3, 4, 5\}$, $O = \{a, b, c, d, e\}$, $\omega = (a, b, c, d, e)$ and preference profile:

R_1	R_2	R_3	R_4	R_5
bcd	ab	a	a	a
a	$\vdots$	ce	d	e
e		$\vdots$	$\vdots$	$\vdots$

In this problem, *Pareto efficient* and *individually rational* assignments are: (c, b, a, d, e), (d, b, c, a, e) and (c, b, e, d, a). In each of these assignments, exactly one agent in $\{3, 4, 5\}$ gets her unique most preferred object a. Which of these assignments should be outcome of the housing market problem though? As mentioned earlier, social ranking of agents can arise in real-life applications of the housing market problem. So, a potential solution could be to select whoever has highest priority among agents 3, 4 and 5 to receive object a. However, note that no rule employing trading cycles can ever select the outcome (c, b, e, d, a) because agents 1 and 2 find object e to be worse than their endowments and agents 3 and 4 cannot trade with agent 5 until object a has been removed from the problem. On the other hand, agents 1, 2, 3 and 4 would want to trade with each other because there is a potential for beneficial trades i.e. for each agent $i \in \{1, 2, 3, 4\}$, there is an agent $j \in \{1, 2, 3, 4\}$ such that agent j holds one of agent i's most preferred object (among the remaining ones) and all most preferred objects (among the remaining ones) of agent i are held by agents in $\{1, 2, 3, 4\}$. I refer to such agents as *market-equals.* Then, by definition, agents in an absorbing set of a graph are *market-equals.* So, at any step of rule φ, I refer to agents in the same absorbing set as *market-equals.* Satisfied (resp. unsatisfied) agents in this absorbing set are referred to as *market-equal* satisfied (resp. unsatisfied) agents.

Note that object a is one of the most preferred objects for agent 2 as well. However, agent 2 already holds one of her most preferred objects in the problem. Existing rules ensure that an agent's welfare does not decrease throughout the algorithm, so it would make sense to prioritize treatment of *market-equal* agent who does not hold one of her most preferred objects (among the remaining ones). Hence, for this criterion, I consider only *market-equal* unsatisfied agents. So, it might be of interest to require that among *market-equals*, highest priority unsatisfied agent receives one of her most preferred objects (among the remaining ones). However, if *strategy proofness* is required, it might not always be possible to achieve this. So, in defining *prioritized treatment of market-equal unsatisfied agents*, *independence of unsatisfied agents* and *persistence* need to be considered.

Consequences of *persistence* are straightforward. By enforcing *persistence*, certain trading cycles might not be achievable because unique pointees for some agents are determined from the previous step. So, only those paths in an absorbing set can be considered which do not violate *persistence*. Consequences of *independence of unsatisfied agents* are slightly more complicated. This condition requires that unique pointee of each agent are determined independent of preferences of other unsatisfied agents. Consider an absorbing set. Suppose agent i_1 is the highest priority unsatisfied agent in this absorbing set. Additionally, suppose that for each cycle containing agent i_1 in this absorbing set, there are at least three unsatisfied agents (including agent i_1). However, by *independence of unsatisfied agents*, it cannot be ensured that a cycle consisting of agent i_1 occurs at that step. Suppose the cycle has unsatisfied agents i_1, i_2 and i_3 such that there is a path from agent i_1 to agent i_2, path from agent i_2 to agent i_3 and path from agent i_3 to agent i_1. Then, a rule satisfying *independence of unsatisfied agents* cannot ensure that path from agent i_2 to i_3 occurs via pointing.

So, I define the following condition which ensures that *independence of unsatisfied agents* and *persistence* do not interfere with assigning the highest priority *market-equal* unsatisfied agent her most preferred object (among the remaining ones):

Strategy proofness compliance: At any step of a rule φ, an absorbing set satisfies *strategy proofness compliance* if there is at least one cycle such that:

1. each agent in the cycle is either not pointing based on *persistence* or is pointing at the same agent/object as in the cycle, and

2. there are at most two unsatisfied agents in the cycle who have a higher priority than every other unsatisfied agent in the absorbing set.

Now, *prioritized treatment of market-equal unsatisfied agents* can be defined as follows:

Prioritized Treatment of Market-Equal Unsatisfied Agents: A rule φ satisfies *prioritized* *treatment of market-equal unsatisfied agents* if, at any step of the rule, whenever an absorbing set satisfies *strategy proofness compliance*, the highest priority unsatisfied agent in the absorbing set receives one of her most preferred objects (among the remaining ones).

Now that this new criterion has been defined, it would be of interest to determine if some existing rules satisfy this property. The next result shows that TC, $TTAS$ and HPO rules do not satisfy *prioritized treatment of market-equal unsatisfied agents* when priority orderings required for these rules is based on social ranking of agents.

Proposition 2.5. *TC, $TTAS$ and HPO rules do not satisfy prioritized treatment of market-equal unsatisfied agents.*

Proof. Consider the following housing market problem: $N = \{1, 2, \cdots, 8\}$, $O = \{a, b, \cdots, h\}$, $\omega = (a, b, c, d, e, f, g, h)$, priority ordering $1 \prec \cdots \prec 8$ ($a \prec \cdots \prec h$ for $TTAS$ and HPO) and preference profile:

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
d	d	ac	$defg$	eh	cf	bg	d
a	b	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	h
$\vdots$	$\vdots$						$\vdots$

Note that in this housing market problem, all agents are *market-equals* and it satisfies *strategy proofness compliance* because in the first step of $\varphi \in \{TC, TTAS, HPO\}$ *persistence* does not play any role and agent 1 is the highest priority unsatisfied agent according to the priority orderings. Now, consider outcomes of TC, $TTAS$ and HPO: $TC^{\prec}(R, \omega) = (a, d, c, g, e, f, b, h)$ and $TTAS^{\prec}(R, \omega) = HPO^{\prec}(R, \omega) = (a, b, c, e, h, f, g, d)$. So, agent 1 does not receive her most preferred object. ■

Note that for TC rule, no priority orderings can be found where agent 1 gets object d in counter-example of Proposition 2.5 because for any priority ordering, agent 4 points at either agent 5 or agent 7. Arguably, for $TTAS$ and HPO rules, for a given housing market problem, it might be possible to find priority orderings such that agent 1 receives object d. However, this would require determining priority orderings for each housing market problem so that these rules satisfy *prioritized treatment of market-equal unsatisfied agents*. In the next section, I propose a rule which satisfies *prioritized treatment of market-equal unsatisfied agents* when priority ordering reflects social ranking of agents i.e. there is no need to determine priority orderings in order to satisfy *prioritized treatment of market-equal unsatisfied agents*. Additionally, I show that this rule satisfies several desirable properties.

Modified Top Cycles Rule

Here, I present a new rule; namely *Modified Top Cycles* (MTC) rule. However, I drop MTC in the super-script since I discuss only MTC from here on. The goal of proposing this rule is to achieve *prioritized treatment of market-equal unsatisfied agents* along with other desirable results. In this and the following section, I assume that there are no indifferences with endowments. Let this class of preferences be represented as $\bar{\mathcal{R}}$. Then, for any $(R, \omega) \in \bar{\mathcal{R}}^N \times A$, $i \in N$ and $a \in O$, $aI_i\omega_i$ implies that $a = \omega_i$[15]. Also, priority ordering over agents $\prec$ is used for MTC.

Like all the rules discussed in this chapter, each step of MTC proceeds in three phases; *departure*, *pointing* and *trading*. Step t of the algorithm proceeds as follows:

1. Agents satisfying *departure* condition are chosen to depart until no more agents satisfy the *departure* condition. Each departing agent is assigned the object she is holding at that step i.e. if agent i was chosen to depart, then agent i is assigned $h_t(i)$.

2. Each agent points at an agent holding one of her most preferred objects (among the remaining ones). If there are more than one such people, the unique pointee is determined in the following manner:

   (a) (*Persistence*) For any $i \in U_t$, if there are $\{i_1, \cdots, i_m\} \subseteq N_{t-1}$ such that $p_{t-1}(i_r) = i_{r+1}$ for all $r \in \{1, \cdots, m-1\}$ and $p_{t-1}(i_m) = i$, then $p_t(i_r) = i_{r+1}$ for all $r \in \{1, \cdots, m-1\}$ and $p_t(i_m) = i$.

   (b) Here we determine a unique pointee for each agent in S_t who is not pointing as yet. Repeat the following until all agents in U_t have been considered[16]:

      i. Let $j \in U_t$ be the highest priority unsatisfied agent who has not been considered yet. For any $i \in S_t$ who is not pointing yet and $h_t(j) \in \tau(R_i, O_t)$, agent i points at agent j i.e. $p_t(i) = j$. If there is no such agent, move on to the next highest priority unsatisfied agent who has not been considered yet, otherwise move to (ii).

[15]The reasoning behind this assumption will become apparent in the next chapter.
[16]Note that after (b), each satisfied agent must be pointing due to the *departure* condition.

ii. For each $i \in S_t$, let $X(i)$ denote the first unsatisfied agent that can be reached by following pointing of agent i i.e. i points at $X(i)$, i points at a satisfied agent who points at $X(i)$, i points at a satisfied agent who points at a satisfied agent who points at $X(i)$ and so on. For any $k \in S_t$ who is not pointing yet, if there is an $i \in S_t$ such that $X(i) = j$ and $h_t(i) \in \tau(R_k, O_t)$, then agent k points at agent i. If there are more than one such agents, agent k points at agent i who has higher priority under $\prec$ and $X(i) = j$. Repeat until each agent $k \in S_t$, who is not pointing yet, any satisfied agent i who holds one of her most preferred objects (among the remaining ones), $X(i) \neq j$ or $X(i)$ is not defined. Return to (i).

(c) Now we determine a unique agent to point at for any agent in U_t who is not pointing as yet. For each $j \in U_t$, let $X(j) = j$. Consider any $i \in U_t$ who is not pointing yet. Agent i points at whoever holds one of her most preferred objects and has a higher priority unsatisfied agent reachable under X. If there are multiple such agents, agent i points at whoever among these agents has a higher priority. Formally, let $K \subseteq N_t$ be such that for each $k \in K$, $h_t(k) \in \tau(R_i, O_t)$. Let $J \subseteq K$ be such that for each $j \in J$, $X(j) \prec X(k)$ where $k \in K \backslash J$ and for each $j, j' \in J$, $X(j) = X(j')$. Agent i points at whoever has highest priority in J under $\prec$ i.e. $p_t(i) = j$ if $j \prec j'$ for each $j' \in J \backslash \{j\}$.

3. Since every agent is pointing at another agent, there is at least one cycle of agents in the problem. In the step next step, each agent in a cycle holds object of the agent she was pointing at.

For any $(R, \omega) \in \bar{\mathcal{R}}^N \times A$ and priority ordering of agents $\prec$, *Modified Top Cycles* rule outcome is denoted as $MTC^{\prec}(R, \omega)$. As I show in the next section, MTC satisfies *prioritized treatment of market-equal unsatisfied agents* along with several desirable properties.

Results for Modified Top Cycles Rule

In this section, I present results for MTC rule. Proposition 2.6 states that MTC rule is *Pareto efficient, individually rational* and *weak core selecting.* These properties follow directly from Propositions 3 and 4 of Jaramillo & Manjunath [14] and so, I do not provide a formal proof.

Proposition 2.6. For each priority ordering $\prec$ and $(R, \omega) \in \bar{\mathcal{R}}^N \times A$, $MTC^{\prec}(R, \omega)$ is *Pareto* *efficient*, *individually rational* and *weak-core selecting.*

Next, I establish that MTC is *strategy proof* i.e. no agent has an incentive to misreport her preferences. Notice that *persistence* has been explicitly enforced in algorithm of MTC. Moreover, MTC satisfies *independence of unsatisfied agents*. This is the case because every agent, who is not *persistently* pointing, points at an agent leading to a path to the highest priority unsatisfied agent that she can reach and that decision is independent of preferences of unsatisfied agents. Then, by Theorem 3 of Saban & Sethuraman [27] and Theorems 3.1 and 3.2, presented in Chapter III, *strategy proofness* of MTC is equivalent to *local invariance* which is defined as follows:

Local Invariance: Consider any $(R,\omega) \in \bar{\mathcal{R}}^N \times A$. Suppose some agent $i \in N$ receives $a \in O$ such that $aP_i\omega(i)$ under rule φ. Let $R_i' \in \bar{\mathcal{R}}$ be such that $R_i'|_{O\setminus\{a\}} = R_i|_{O\setminus\{a\}}$ i.e. ordering of objects in $O\setminus\{a\}$ is same under both R_i and R_i'. Moreover, $bP_ia \iff bP_i'a$ and $aR_ib \iff aP_i'b$ for all $b \in O\setminus\{a\}$. The rule φ is said to satisfy *local invariance* if $\varphi(R',\omega)(i) = a$ where $R' = (R_{-i}, R_i')$.

In order to establish *local invariance* of MTC, I use some additional notation. Consider $R, R' \in \bar{\mathcal{R}}^N$ such that $R' = (R_{-i}, R_i')$, $\omega \in A$ and some priority ordering $\prec$ over agents. Let N_t (resp. N_t') and O_t (resp. O_t') be set of agents and objects remaining after departure phase of step t for $MTC^{\prec}(R,\omega)$ (resp. $MTC^{\prec}(R',\omega)$), respectively. Let $S_t \subseteq N_t$ (resp. $S_t' \subseteq N_t'$) and $U_t \subseteq N_t$ (resp. $U_t' \subseteq N_t'$) be set of satisfied and unsatisfied agents at step t for $MTC^{\prec}(R,\omega)$ (resp. $MTC^{\prec}(R,\omega)$), respectively. Let $h_t(j)$ (resp. $h_t'(j)$) denote object held by agent j at beginning of step t of $MTC^{\prec}(R,\omega)$ (resp. $MTC^{\prec}(R',\omega)$). For any step t and $j \in N_t$ (resp. $j \in N_t'$), let $p_t(j)$ (resp. $p_t'(j)$) represent the agent pointed at by agent j in pointing phase of step t for $MTC^{\prec}(R,\omega)$ (resp. $MTC^{\prec}(R',\omega)$). Let $C_i(R,t)$ (resp. $C_i(R',t)$) be set of all $j \in N_t$ (resp. $j \in N_t'$) such that $j = i$, $p_t(j) = i$ (resp. $p_t'(j) = i$), $p_t(p_t(j)) = i$ (resp. $p_t'(p_t'(j)) = i$) and so on under R (resp. R').

Now, fix agent i and assume that only agent i is misreporting her preferences. Let t (resp. t') be the first step agent i becomes part of a trading cycle or becomes satisfied under $MTC^{\prec}(R,\omega)$ (resp. $MTC^{\prec}(R',\omega)$). Let $\underline{t} = \min\{t,t'\}$. Following result says that state of algorithm is same before step $\underline{t}$ regardless of whether agent i reports R_i or R_i'.

Lemma 2.1. For all $0 \le \tilde{t} < \underline{t}$, $N_{\tilde{t}} = N_{\tilde{t}}'$, $O_{\tilde{t}} = O_{\tilde{t}}'$, $h_{\tilde{t}} = h_{\tilde{t}}'$ and for each $j \in N_{\tilde{t}}\setminus\{i\}$, $p_{\tilde{t}}(j) = p_{\tilde{t}}'(j)$. Moreover, $h_{\underline{t}} = h_{\underline{t}}'$ and if i is not satisfied at step $\underline{t}$ under R and R', then $N_{\underline{t}} = N_{\underline{t}}'$, $O_{\underline{t}} = O_{\underline{t}}'$, and for each $j \in N_{\underline{t}}\setminus\{i\}$, $p_{\underline{t}}(j) = p_{\underline{t}}'(j)$.

Proof. If $\underline{t} = 1$, the claim holds vacuously for $\tilde{t} = 0$. Now suppose $\underline{t} > 1$. Then, it must be the case that $i \in U_1$ and $i \in U_1'$. The set of departing agents in step 1 should be the same under R

and R' because i does not depart and for each $j \in N\backslash\{i\}$, $R'_j = R_j$. So, it must be the case that $N_1 = N'_1$ and $O_1 = O'_1$. Before first step, no trading has taken place so each agent is holding her endowments i.e. $h_1(j) = h'_1(j) = \omega(j)$ for all $j \in N$. Moreover, each $j \neq i$ points at the same person at step 1 because i is unsatisfied under both R and R' and MTC satisfies *independence of unsatisfied agents*. So, the claim holds for step 1.

Now, suppose the claim holds for some $\tilde{t} < \underline{t} - 1$. We need to show that it holds for step $\tilde{t} + 1$. By assumption, we have $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$, $h_{\tilde{t}} = h'_{\tilde{t}}$ and for each $j \in N_{\tilde{t}}\backslash\{i\}$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$. Since $\tilde{t} < \underline{t}$, agent i is not part of a trading cycle under R or R'. So, same trading cycles are formed for both R and R' at step $\tilde{t}$. Hence, $h_{\tilde{t}+1} = h'_{\tilde{t}+1}$. Moreover, since $\tilde{t} + 1 < \underline{t}$, agent i does not depart in departure phase of step $\tilde{t} + 1$ under either R or R'. Then, it must be that $N_{\tilde{t}+1} = N'_{\tilde{t}+1}$ and $O_{\tilde{t}+1} = O'_{\tilde{t}+1}$. Moreover, it must be that for each $j \in N_{\tilde{t}+1}\backslash\{i\}$, $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ because $i \in U_{\tilde{t}+1}$, $i \in U'_{\tilde{t}+1}$ and MTC satisfies *independence of unsatisfied agents*.

Now, we prove second part of the lemma. We know that $N_{\underline{t}-1} = N'_{\underline{t}-1}$, $O_{\underline{t}-1} = O'_{\underline{t}-1}$, $h_{\underline{t}-1} = h'_{\underline{t}-1}$ and for each $j \in N_{\underline{t}-1}\backslash\{i\}$, $p_{\underline{t}-1}(j) = p'_{\underline{t}-1}(j)$. Agent i is not part of a trading cycle under R or R' at step $\underline{t} - 1$. So, we have same trading cycles under both R and R'. Hence, $h_{\underline{t}} = h'_{\underline{t}}$. If agent i is not satisfied under either R or R' at step $\underline{t}$, we have $N_{\underline{t}} = N'_{\underline{t}}$ and $O_{\underline{t}} = O'_{\underline{t}}$. Also, for each $j \in N_{\underline{t}}\backslash\{i\}$, $p_{\underline{t}}(j) = p'_{\underline{t}}(j)$ because MTC satisfies *independence of unsatisfied agents*. This completes the proof. ■

The following result proves that MTC satisfies *local invariance* for general preferences.

Proposition 2.7. For each priority ordering $\prec$ and $(R,\omega) \in \mathcal{R}^N \times A$, $MTC^{\prec}(R,\omega)$ satisfies *local invariance*.

Proof. Consider any $(R,\omega) \in \mathcal{R}^N \times A$ and $i \in N$ such that $MTC^{\prec}(R,\omega)(i) = a$ such that $aP_i\omega(i)$. Let $R'_i \in \mathcal{R}$ be such that $R'_i|_{O\backslash\{a\}} = R_i|_{O\backslash\{a\}}$, $bP_ia \iff bP'_ia$ and $aR_ib \iff aP'_ib$ for all $b \in O\backslash\{a\}$. Let $\alpha \equiv MTC^{\prec}(R,\omega)$ and $\alpha' \equiv MTC^{\prec}(R',\omega)$. We want to show that $\alpha'(i) = a$. Notice that since $\alpha(i)P_i\omega(i)$, it cannot be the case that agent i becomes satisfied before trading phase of step t because $h_t(i) = \omega(i)$.

Note that it must be the case that $\alpha(i) \in O'_{\underline{t}}$. If not, $\alpha(i)$ would have departed at (or before) step $\underline{t}$ under R as well because $h_{\underline{t}} = h'_{\underline{t}}$ by Lemma 2.1. This would imply that agent i cannot receive $\alpha(i)$ under R. Since $h_{\underline{t}}(i) = h'_{\underline{t}}(i)$, it cannot be the case that agent i is satisfied at step $\underline{t}$ under either R or R' because we have $\alpha(i)P'_i\omega(i)$. So, it cannot be the case that agent i departs in departure phase of step $\underline{t} = \min\{t,t'\}$ under R or R'. Moreover, it cannot be the case that

$\alpha'(i)\, I'_i \omega(i)$ since then it cannot be possible for agent i to receive $\alpha(i)$ under R. This is the case because if $t' \leq t$, then it means that each $b \in O$ such that $bP'_i\omega(i)$ has departed by departure phase of step t' under R' but by construction of R'_i, $bP_i\omega(i) \iff bP'_i\omega(i)$, and so agent i cannot receive $\alpha(i)$ under R. If $t < t'$, then the trading cycle agent i is part of at step t under R, occurs as a chain in pointing phase at step t under R' and keeps on occurring until agent i becomes satisfied by *persistence.* In other words, under R', agent i can always receive $h_{t+1}(i)$ and since $h_{t+1}(i)\, P_i\omega(i)$, we have $h_{t+1}(i)\, P'_i\omega(i)$. So, agent i cannot become satisfied before trading phase of step t' under R'.

Step 1: $\alpha(i)\, R_i\alpha'(i)$. Consider the following cases:

Case 1: $\underline{t} = t \leq t'$.

Since agent i does not depart at step $\underline{t}$, by Lemma 2.1, $O_{\underline{t}} = O'_{\underline{t}}$. Moreover, under R, agent i is part of a trading cycle at step $\underline{t}$. So, $\alpha(i)\, I_i h_{\underline{t}+1}(i)$ and $h_{\underline{t}+1}(i) \in \tau\left(R_i, O_{\underline{t}}\right)$. Since $O'_{t'} \subseteq O_{\underline{t}}$ and $\alpha'(i) \in O_{\underline{t}}$, we have $\alpha(i)\, R_i\alpha'(i)$.

Case 2: $\underline{t} = t' < t$.

On contrary, suppose that $\alpha'(i)\, P_i\alpha(i)$. Since agent i does not depart at step $\underline{t}$, by Lemma 2.1, $N_{\underline{t}} = N'_{\underline{t}}$, $O_{\underline{t}} = O'_{\underline{t}}$ and $h_{\underline{t}} = h'_{\underline{t}}$. Since $\underline{t} = t'$, agent i is part of a trading cycle at step $\underline{t}$ under R'. Let the cycle be $C = \{i_1, \cdots, i_m\}$ such that $p'_{\underline{t}}(i_1) = i_2$, $p'_{\underline{t}}(i_2) = i_3, \cdots$, $p'_{\underline{t}}(i_m) = i_1$. Without loss of generality, let $i_m = i$ and so, $\alpha'(i)\, I'_i h_{\underline{t}}(i_1)$. Since $\alpha'(i)\, P_i\alpha(i)$, we have $\alpha'(i) \neq \alpha(i)$. Then, by construction of R'_i, $h_{\underline{t}}(i_1) \neq \alpha(i)$. So, $\alpha'(i)\, I_i h_{\underline{t}}(i_1)$.

Note that, by Lemma 2.1, each $j \in N_{\underline{t}} \backslash \{i\}$ points at same agent at step $\underline{t}$ under R and R'. So, the trading cycle C occurs as a chain at step $\underline{t}$ under R i.e. $p_{\underline{t}}(i_1) = i_2$, $p_{\underline{t}}(i_2) = i_3, \cdots$, $p_{\underline{t}}(i_{m-1}) = i_m$ and $i_m = i$[17]. Then, by *persistence*, this chain keeps occurring as long as agent i is unsatisfied. Agent i remains unsatisfied until trading phase of step t under R. So, this chain occurs up to step t and we have $h_t(j) = h_{\underline{t}}(j)$ for all $j \in \{i_1, \cdots, i_m\}$. Specifically, $h_{\underline{t}}(i_1) \in O_t$. Since agent i is part of a trading cycle at step t, $\alpha(i)\, I_i h_{t+1}(i)$ and $h_{t+1}(i) \in \tau(R_i, O_t)$. So, $\alpha(i)\, R_i\alpha'(i)$ which is a contradiction.

Step 2: $\alpha'(i)\, R'_i\alpha(i)$ when $\underline{t} = t' \leq t$. Since agent i does not depart at step $\underline{t}$, by Lemma 2.1, $O_{\underline{t}} = O'_{\underline{t}}$. Moreover, under R', agent i is part of a trading cycle at step $\underline{t}$. So, $\alpha'(i)\, I'_i h'_{\underline{t}+1}(i)$ and $h'_{\underline{t}+1}(i) \in \tau\left(R'_i, O_{\underline{t}}\right)$. Since, $O_t \subseteq O_{\underline{t}}$, $\alpha(i) \in O'_{\underline{t}}$. This implies that $\alpha'(i)\, R'_i\alpha(i)$.

Step 3: $\alpha'(i) = \alpha(i)$ or $\alpha'(i)\, R'_i\alpha(i)$ when $\underline{t} = t < t'$. In order to prove this, we follow the

[17] Under R, this is not a cycle because $t' < t$.

approach of Jaramillo & Manjunath [14]. Consider the following claim:

Post-trade Inclusion Claim: Let $t < t'$ i.e. $\underline{t} = t$. Then, for all $\ddot{t} \in \{t, \cdots, t'\}$:

1. $\begin{array}{c} O_{\ddot{t}} \subseteq O'_{\ddot{t}} \\ O'_{\ddot{t}} \backslash O_{\ddot{t}} \subseteq h_{\ddot{t}} \left(C_i \left(R', \ddot{t} - 1\right)\right) \end{array}$, and $\begin{array}{c} N_{\ddot{t}} \subseteq N'_{\ddot{t}} \\ N'_{\ddot{t}} \backslash N_{\ddot{t}} \subseteq C_i \left(R', \ddot{t} - 1\right) \end{array}$,

2. $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ and $S_{\ddot{t}} \backslash S'_{\ddot{t}} \subseteq C_i \left(R', \ddot{t} - 1\right)$,

3. for each $j \in N'_{\ddot{t}} \backslash C_i \left(R', \ddot{t}\right)$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, and

4. for each $j \in N'_{\ddot{t}} \backslash C_i \left(R', \ddot{t}\right)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

Proof. Consider $\ddot{t} = t$. Then, by Lemma 2.1; $N_t = N'_t$, $O_t = O'_t$, $h_t = h'_t$ and for each $j \in N_t \backslash \{i\}$, $p_t(j) = p'_t(j)$. So, $(1) - (3)$ hold for $\ddot{t} = t$.

Since for each $j \in N_t \backslash \{i\}$, we have $p_t(j) = p'_t(j)$, it must be that $C_i(R, t) = C_i(R', t)$. So, if any $j \in N_t \backslash C_i(R, t)$ is part of a trading cycle at step t under R, that trading cycle must consist only of agents in $N_t \backslash C_i(R, t)$ otherwise we would have $j \in C_i(R, t)$. Then, for any $j \in N_t \backslash C_i(R', t)$, same trading cycles occur under both R and R'. Hence, for each $j \in N'_t \backslash C_i(R', t)$, we would have $h_{t+1}(j) = h'_{t+1}(j)$ because $h_t = h'_t$. So, (4) holds for $\ddot{t} = t$.

Now, suppose $(1) - (4)$ hold for all $\ddot{t} < t' - 1$. We want to show that the claim is true for step $\ddot{t} + 1$. Note that at step $\ddot{t} + 1$, no agent in $C_i\left(R', \ddot{t}\right)$ departs under R' because agent i does not become satisfied before trading phase of step t'. So, any agents (along with their objects) departing at step $\ddot{t} + 1$ under R' must be in $N'_{\ddot{t}} \backslash C_i\left(R', \ddot{t}\right)$. By induction hypothesis, we know that for any $j \in N'_{\ddot{t}} \backslash C_i\left(R', \ddot{t}\right)$, $h_{\ddot{t}+1}(j) = h_{\ddot{t}+1}(j)$ and $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$. So, if a group of agents, $G \subseteq N'_{\ddot{t}}$ departs at step $\ddot{t} + 1$ under R', $G \subseteq N'_{\ddot{t}} \backslash C_i\left(R', \ddot{t}\right)$. Notice that $G \subseteq N_{\ddot{t}}$ otherwise there is $j \in G$ such that $j \in C_i\left(R', \ddot{t} - 1\right)$ and by *persistence*, $C_i\left(R', \ddot{t} - 1\right) \subseteq C_i\left(R', \ddot{t}\right)$ so that $j \in C_i\left(R', \ddot{t}\right)$. Let $G = G_1 \cup \cdots \cup G_T$ where G_1 is the first group to depart, G_2 is second group to depart,$\cdots$, G_T is the T-th (and last) group to depart at step $\ddot{t} + 1$ under R'. Moreover, let $A_k = h'_{\ddot{t}+1}(G_k)$ for $k \in \{1, \cdots, T\}$. Then, by departure condition, for each $j \in G_k$, $h'_{\ddot{t}+1}(j) \in \tau\left(R_j, O'_{\ddot{t}} \backslash \left(\cup_{l=1}^{k-1} A_l\right)\right)$ and $h'_{\ddot{t}+1}(G_k) = \cup_{j \in G_k} \tau\left(R_j, O'_{\ddot{t}} \backslash \left(\cup_{l=1}^{k-1} A_l\right)\right)$ for $k \in \{1, \cdots, T\}$. Then, we have for each $j \in G_k$, $h_{\ddot{t}+1}(j) \in \tau\left(R_j, O_{\ddot{t}} \backslash \left(\cup_{l=1}^{k-1} A_l\right)\right)$, and $h_{\ddot{t}+1}(G_k) = \cup_{j \in G_k} \tau\left(R_j, O_{\ddot{t}} \backslash \left(\cup_{l=1}^{k-1} A_l\right)\right)$ for $k \in \{1, \cdots, T\}$ because $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$ from (1) and $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$ for all $j \in G$ from (4). So, any agents, along with their objects, departing at step $\ddot{t} + 1$ under R', depart at step $\ddot{t} + 1$, along with their objects, under R as well. So, we have $N_{\ddot{t}+1} \subseteq N'_{\ddot{t}+1}$ and $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$ because $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$ and $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$.

Note that for any trading cycle, say C, that occurs at step $\ddot{t}$ under R but not under R', it must be the case that $C \subseteq C_i\left(R', \ddot{t}\right)$ by (3) at step $\ddot{t}$. Moreover, any trading cycle that occurs at step $\ddot{t}$ under R', also occurs under R because no agent in $C_i\left(R', \ddot{t}\right)$ is part of a trading cycle at step $\ddot{t}$ under R'. Then, it must be the case that $S'_{\ddot{t}+1} \subseteq S_{\ddot{t}+1}$ and $S_{\ddot{t}+1} \backslash S'_{\ddot{t}+1} \subseteq C_i\left(R', \ddot{t}\right)$ because $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$, $S_{\ddot{t}} \backslash S'_{\ddot{t}} \subseteq C_i\left(R', \ddot{t}-1\right)$, any cycle that occurs at step $\ddot{t}$ under R' also occurs under R and any cycle that occurs at step $\ddot{t}$ under R but not under R' must consist entirely of agents in $C_i\left(R', \ddot{t}\right)$.

Now, we show that $N'_{\ddot{t}+1} \backslash N_{\ddot{t}+1} \subseteq C_i\left(R', \ddot{t}\right)$. Suppose $j \in N'_{\ddot{t}+1} \backslash N_{\ddot{t}+1}$ but $j \notin C_i\left(R', \ddot{t}\right)$. Note that if $j \in N'_{\ddot{t}}$, then $j \in N_{\ddot{t}}$ because $N'_{\ddot{t}} \backslash N_{\ddot{t}} \subseteq C_i\left(R', \ddot{t}-1\right)$ and $C_i\left(R', \ddot{t}-1\right) \subseteq C_i\left(R', \ddot{t}\right)$ by *persistence*. Suppose that $j \in S_{\ddot{t}}$. It cannot be that $j \in S'_{\ddot{t}}$ since then $j \in S'_{\ddot{t}+1}$ which contradicts $S'_{\ddot{t}+1} \subseteq S_{\ddot{t}+1}$. So, $j \notin S'_{\ddot{t}}$. But then, $j \in S_{\ddot{t}} \backslash S'_{\ddot{t}} \subseteq C_i\left(R', \ddot{t}-1\right)$ and $C_i\left(R', \ddot{t}-1\right) \subseteq C_i\left(R', \ddot{t}\right)$ by *persistence*.

Now, suppose that $j \notin S_{\ddot{t}}$. Note that (2) at step $\ddot{t}$ is equivalent to $U_{\ddot{t}} \subseteq U'_{\ddot{t}}$ and $U'_{\ddot{t}} \backslash U_{\ddot{t}} \subseteq C_i\left(R', \ddot{t}-1\right)$. Moreover, $j \notin S'_{\ddot{t}+1}$ because $S'_{\ddot{t}+1} \subseteq S_{\ddot{t}+1}$. So, $j \in U'_{\ddot{t}+1}$ and $j \notin N_{\ddot{t}+1}$ which implies that $j \in C_i\left(R', \ddot{t}\right)$ because $U'_{\ddot{t}+1} \backslash U_{\ddot{t}+1} \subseteq C_i\left(R', \ddot{t}\right)$.

Now, we show that $O'_{\ddot{t}+1} \backslash O_{\ddot{t}+1} \subseteq h_{\ddot{t}+1}\left(C_i\left(R', \ddot{t}\right)\right)$. Suppose $b \in O'_{\ddot{t}+1} \backslash O_{\ddot{t}+1}$ such that $b \notin h_{\ddot{t}+1}\left(C_i\left(R', \ddot{t}\right)\right)$. It must be that $b \notin O'_{\ddot{t}} \backslash O_{\ddot{t}}$ because $O'_{\ddot{t}} \backslash O_{\ddot{t}} \subseteq h_{\ddot{t}}\left(C_i\left(R', \ddot{t}-1\right)\right)$ and $C_i\left(R', \ddot{t}-1\right) \subseteq C_i\left(R', \ddot{t}\right)$ by *persistence*. Then, some agent, say j, departed with b at step $\ddot{t}+1$, $h_{\ddot{t}+1}(j) = b$, under R but not under R'. Then, $j \in N'_{\ddot{t}+1} \backslash N_{\ddot{t}+1}$ and hence, $j \in C_i\left(R', \ddot{t}\right)$. Then, it must be the case that $b \in h_{\ddot{t}+1}\left(C_i\left(R', \ddot{t}\right)\right)$ which is a contradiction.

Now, we prove (3) for step $\ddot{t}+1$ i.e. for each $j \in N'_{\ddot{t}+1} \backslash C_i\left(R', \ddot{t}+1\right)$, $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$. Take any $j \in N'_{\ddot{t}+1} \backslash C_i\left(R', \ddot{t}+1\right)$. If agent j is *persistently* pointing at someone at step $\ddot{t}+1$ under R', we have $p'_{\ddot{t}+1}(j) = p'_{\ddot{t}}(j)$. Notice that by *persistence*, $j \in N'_{\ddot{t}} \backslash C_i\left(R', \ddot{t}\right)$ and, by (3) at step $\ddot{t}$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$ so that $p'_{\ddot{t}+1}(j) = p_{\ddot{t}}(j)$. Moreover, $p_{\ddot{t}}(j) \notin C_i\left(R', \ddot{t}\right)$ and so, $p_{\ddot{t}}\left(p_{\ddot{t}}(j)\right) = p'_{\ddot{t}}\left(p'_{\ddot{t}}(j)\right)$ and so on. Hence, the same unsatisfied agent, say agent k, is chosen for agent j in pointing phase of step $\ddot{t}$ under R and R'. Then, it must be the case that $k \notin N'_{\ddot{t}+1} \backslash N_{\ddot{t}+1}$ otherwise, $k \in C_i\left(R', \ddot{t}\right)$ and $j \in C_i\left(R', \ddot{t}\right)$. Moreover, it cannot be the case that agent k is part of a trading cycle at step $\ddot{t}$ under R but not under R' since then $k \in C_i\left(R', \ddot{t}\right)$ and $j \in C_i\left(R', \ddot{t}\right)$. Since agent j is *persistently* pointing at step $\ddot{t}+1$ under R', it must be that $k \in U'_{\ddot{t}+1}$ and since agent k cannot be part of a trading cycle at step $\ddot{t}$ under R without being part of a trading cycle at step $\ddot{t}$ under R', it must be that $k \in U_{\ddot{t}+1}$. Hence, agent j must be *persistently* pointing at step $\ddot{t}+1$ under R as well i.e. $p_{\ddot{t}+1}(j) = p_{\ddot{t}}(j)$. Therefore, $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$.

Now, suppose that agent j is not *persistently* pointing at step $\ddot{t}+1$ under R'. Note that it cannot be the case that agent j is *persistently* pointing at step $\ddot{t}+1$ under R. Since $j \notin C_i\left(R', \ddot{t}\right)$ it cannot be the case that $k \in C_i\left(R', \ddot{t}\right)$ where k is any agent on a chain starting from agent j. Then, for each agent k on a chain from agent j, we have $p_{\ddot{t}}(k) = p'_{\ddot{t}}(k)$ from (3) at step $\ddot{t}$. Moreover, we have $U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$. So, if agent j is *persistently* pointing at step $\ddot{t}+1$ under R, it must be that j is *persistently* pointing at step $\ddot{t}+1$ under R' as well.

Note that it cannot be the case that in pointing phase of step $\ddot{t}+1$ under R', the path chosen for agent j includes some agent in $C_i\left(R', \ddot{t}+1\right)$ since then $j \in C_i\left(R', \ddot{t}+1\right)$. So, no agent on a chain starting from agent j at step $\ddot{t}+1$ under R' can be a member of $C_i\left(R', \ddot{t}+1\right)$. Let $k \in U_{\ddot{t}+1}$ be the unsatisfied agent chosen for agent j at step $\ddot{t}+1$ under R. Notice that it cannot be the case that $k \in C_i\left(R', \ddot{t}+1\right)$ because k is highest priority unsatisfied agent in $U_{\ddot{t}+1}$ that agent j can reach. Also, $U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and $U'_{\ddot{t}+1} \backslash U_{\ddot{t}+1} \subseteq C_i\left(R', \ddot{t}\right)$, so that in pointing phase of step $\ddot{t}+1$ under R', either agent k is chosen for agent j or some unsatisfied agent from $U'_{\ddot{t}+1} \backslash U_{\ddot{t}+1}$ is chosen. In both cases, it would mean that $j \in C_i\left(R', \ddot{t}+1\right)$. So, $k \notin C_i\left(R', \ddot{t}+1\right)$ and it must be that agent k is the highest priority unsatisfied agent in $U'_{\ddot{t}+1}$ that can be reached by agent j at step $\ddot{t}+1$ under R'. Moreover, for each agent $k \in N'_{\ddot{t}+1} \backslash C_i\left(R', \ddot{t}+1\right)$, we have $k \in N'_{\ddot{t}} \backslash C_i\left(R', \ddot{t}\right)$ and, so, by (4) at step $\ddot{t}$, $h_{\ddot{t}+1}(k) = h'_{\ddot{t}+1}(k)$. So, it must be the case that $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$.

Now, we show that (4) is true for step $\ddot{t}+1$ i.e. for each $j \in N'_{\ddot{t}+1} \backslash C_i\left(R', \ddot{t}+1\right)$, $h_{\ddot{t}+2}(j) = h'_{\ddot{t}+2}(j)$. Notice that for each $j \in N'_{\ddot{t}+1} \backslash C_i\left(R', \ddot{t}+1\right)$ we have $j \in N'_{\ddot{t}} \backslash C_i\left(R', \ddot{t}\right)$ so that $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$ and $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$. So, a trading cycle consisting of agents only in $N'_{\ddot{t}+1} \backslash C_i\left(R', \ddot{t}+1\right)$ must occur under both R and R' at step $\ddot{t}+1$. Moreover, if any trading cycle, C, occurs under R but not under R' it must be the case that $C \subseteq C_i\left(R', \ddot{t}+1\right)$. So, we have $h_{\ddot{t}+2}(j) = h'_{\ddot{t}+2}(j)$ for all $j \in N'_{\ddot{t}+1} \backslash C_i\left(R', \ddot{t}+1\right)$. This completes the proof of the Post-trade Inclusion claim. □

First, suppose that $\alpha(i) \in O'_{t'}$. Since agent i is part of a trading cycle at step t' under R', it must be the case that $\alpha'(i) R'_i \alpha(i)$. Now, suppose that $\alpha(i) \notin O'_{t'}$. Since agent i is assigned $\alpha(i)$ under R, there is some $\tilde{t}$ such that $h_{\tilde{t}+1}(i) = \alpha(i)$. By post-trade inclusion, $\tilde{t} < t'$, because $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$ for all $\ddot{t} \in \{t, \cdots, t'\}$. Moreover, it must be the case that $\tau\left(R'_i, O'_{\tilde{t}}\right) = \{\alpha(i)\}$ because $O_{\tilde{t}} \subseteq O'_{\tilde{t}}$, $O'_{\tilde{t}} \backslash O_{\tilde{t}} \subseteq h_{\tilde{t}}\left(C_i\left(R', \tilde{t}-1\right)\right)$ and $\alpha(i) R_i \alpha'(i)$. So, agent i points at some agent, say agent j, holding $\alpha(i)$ at step $\tilde{t}$ under R' i.e. $p'_{\tilde{t}}(i) = j$ and $h'_{\tilde{t}}(j) = \alpha(i)$. Since $\alpha'(i) \neq \alpha(i)$, $j \notin C_i\left(R', \tilde{t}\right)$ and so, by post-trade inclusion, $h_{\tilde{t}}(j) = h'_{\tilde{t}}(j)$. Then, the chain initiating from agent j at step $\tilde{t}$ under R' cannot consist of any agent in $C_i\left(R', \tilde{t}\right)$. So, for any agent k on a chain from agent j at step $\tilde{t}$

under R', we have $k \notin C_i\left(R', \tilde{t}\right)$. By post-trade inclusion, $p_{\tilde{t}}(k) = p'_{\tilde{t}}(k)$. Then, it cannot be the case that agent i is part of a trading cycle consisting of agent j at step $\tilde{t}$ under R. This contradicts the assumption that $h_{\tilde{t}+1}(i) = \alpha(i)$.

Step 4: $\alpha'(i) = \alpha(i)$. Note that whenever $\alpha(i) R_i \alpha'(i)$ and $\alpha'(i) R'_i \alpha(i)$ it must be the case that $\alpha'(i) = \alpha(i)$. Suppose not. Then, by definition of R'_i, $\alpha'(i) P'_i \alpha(i)$ because $\alpha'(i) R'_i \alpha(i)$. Hence, $\alpha'(i) P_i \alpha(i)$ which is a contradiction. So, MTC satisfies *local invariance*. ■

By *local invariance* of MTC rule, Theorems 3.1 and 3.2 (presented in Chapter III), MTC rule is *strategy proof*. This result is stated in the following corollary.

Corollary 2.1. For each priority ordering $\prec$ and $(R, \omega) \in \bar{\mathcal{R}}^N \times A$, $MTC^{\prec}(R, \omega)$ is *strategy proof*.

Now, I show that MTC satisfies *no justified-envy for agents with identical endowments.*

Proposition 2.8. For each priority ordering $\prec$ and $(R, \omega) \in \bar{\mathcal{R}}^N \times A$, $MTC^{\prec}(R, \omega)$ satisfies *no justified-envy for agents with identical endowments.*

Proof. On contrary, there is $(R, \omega) \in \bar{\mathcal{R}}^N \times A$, priority ordering $\prec$ and $i, j \in N$ such that $i \prec j$, $\omega(i) I_k \omega(j)$ for all $k \in N$ and $\alpha(j) P_i \alpha(i)$ where $\alpha \equiv MTC^{\prec}(R, \omega)$. It cannot be the case that $\alpha(j) = \omega(j)$ because then $\omega(i) P_i \alpha(i)$ which contradicts *individual rationality* of $MTC^{\prec}$. So, agent j must have been part of a trading cycle at least once.

Let t be the first step agent j becomes part of a trading cycle so that $\alpha(j) \in O_t$. It cannot be that agent i is part of a trading cycle at some $t' \leq t$ since then, $\alpha(i) \in \tau(R_i, O_{t'})$ and hence, $\alpha(i) R_i \alpha(j)$ because $O_t \subseteq O_{t'}$. Moreover, it cannot be that agent i departed with $\omega(i)$ at some step $t' \leq t$ because then $\omega(i) R_i b$ for all $b \in O_{t'}$ which is a contradiction. So, we have $h_t(i) = \omega(i)$ and $h_t(j) = \omega(j)$. Let $k \in N_t$ be the agent pointing at agent j in the trading cycle containing agent j. Then, $\omega(j) \in \tau(R_k, O_t)$ and so, $\omega(i) \in \tau(R_k, O_t)$. Since agent k is not pointing at agent i, it must be the case that agent k was pointing at agent j in step $t-1$ and is pointing at agent j in step t owing to *persistence*. Let t_k be the first step agent k points at agent j. Since agent k is not *persistently* pointing at step t_k and $i \prec j$, it cannot be the case that agent k points at agent j in step t_k. This completes the proof. ■

The next result shows that whenever priority ordering of agents reflects social ranking of agents in the housing market problem, MTC satisfies *prioritized treatment of market-equal unsatisfied agents*.

Proposition 2.9. For each priority ordering $\prec$ and $(R, \omega) \in \bar{\mathcal{R}}^N \times A$, $MTC^{\prec}(R, \omega)$ satisfies *prioritized treatment of market-equal unsatisfied agents.*

Proof. Consider any priority ordering $\prec$, $(R, \omega) \in \bar{\mathcal{R}}^N \times A$ and any step of $MTC^{\prec}$. Let $\alpha \equiv MTC^{\prec}(R, \omega)$. Since we are considering a general step, we drop sub-script of step in the notation. Let $AS = (N', O')$ be any absorbing set at this step, where N' and O' are sets of agents and objects in the absorbing set AS, respectively, such that AS satisfies *strategy proofness compliance.* Let $U \subseteq N'$ be the set of unsatisfied agents in this absorbing set. Let agent i be the highest priority unsatisfied agent in the absorbing set.

Since AS satisfies *strategy proofness compliance*, there is at least one cycle which has at most two unsatisfied agents (including agent i). First, suppose each such cycle has agent i as the unique unsatisfied agent. Let this cycle be $(i_1, \cdots, i_m)$ such that there is an arc from each i_l to i_{l+1} in the absorbing set, $i_{m+1} = i_1$, and $i_1 = i$. Then, i_m is either *persistently* pointing at agent i or she is pointing at agent i under the pointing phase because agent i owns one of her most preferred objects (among the remaining ones) and agent i is the highest priority unsatisfied agent in the absorbing set. Then, i_{m-1} is either *persistently* pointing at i_m, points directly at agent i, points at i_m or a higher priority satisfied agent who has a path to agent i under the pointing phase because agent i is the highest priority unsatisfied agent. In this manner, it can be concluded that agent i_2 is either *persistently* pointing at agent i_3 (who has a path to agent i via pointing), points at agent i_3 or a higher priority satisfied agent who has a path to agent i via pointing. Once unique pointees of satisfied agents have been determined, we determine unique pointees of unsatisfied agents. Either agent i is *persistently* pointing at agent i_2 or points at an agent holding one of her most preferred objects with a path (via pointing) to the highest priority unsatisfied agent reachable. Since i_2 holds one of the most preferred objects for agent i, agent i either points at i_2 or a higher priority satisfied agent with a path to herself. So, $\alpha(i) \in \tau(R_i, O')$.

Now, suppose there is a cycle in the *strategy proofness compliant* absorbing set which has two unsatisfied agents; agent i and j. By assumption, $i \prec k$ and $j \prec k$ for each $k \in U \backslash \{i, j\}$. Additionally, suppose agent i is the highest priority unsatisfied agent so that $i \prec j$. Let this cycle in the absorbing set be as follows: $(i_1, \cdots, i_k, \cdots, i_m)$ such that there is an arc from each i_l to i_{l+1} in the absorbing set, $i_{m+1} = i_1$, $i_1 = i$ and $i_k = j$. By same reasoning as for the previous case, each agent in $\{i_{k+1}, \cdots, i_m\}$ has a path (via pointing) to agent i. If there is a path from some agent in $\{i_2, \cdots, i_{k-1}\}$ to agent i via pointing, then $\alpha(i) \in \tau(R_i, O')$. So, suppose there is no path from an agent in $\{i_2, \cdots, i_{k-1}\}$ to agent i via pointing. Then, in the same manner, each agent in

$\{i_2, \cdots, i_{k-1}\}$ forms a path to agent j via pointing. Now, we determine unique pointees for agents i and j. For the agents holding one of the most preferred objects of agent i (among the remaining ones), agent j has to be the highest priority unsatisfied agent who can be reached via pointing. So, agent i either points at agent i_2 or a higher priority agent with a path to agent j. Similarly, agent i has to be the highest priority unsatisfied agent that can be reached by agents holding one of the most preferred objects of agent j. So, agent j either points at agent i_{k+1} or a higher priority agent with a path to agent i via pointing. In either case, we get $\alpha(i) \in \tau(R_i, O')$. ■

Based on Propositions 2.6-2.9, following theorem can be stated.

Theorem 2.2. For housing market problem under weak preferences and no indifferences with endowments, there exist rules which satisfy *Pareto efficiency*, *weak-core selection* (hence, *individual rationality*), *strategy proofness*, *no justified-envy for agents with identical endowments* and *prioritized treatment of market-equal unsatisfied agents*.

I conclude this chapter in the next section.

Conclusion

When housing market problem is considered under weak preferences, several mechanisms have been shown to satisfy desirable properties like *Pareto efficiency*, *individual rationality*, *weak core selection* and *strategy proofness*. I consider some additional properties. I show that three of the existing rules satisfy *no justified-envy for agents with identical endowments*. Additionally, I provide sufficient conditions for a rule to satisfy *weak group strategy proofness*. Finally, I consider a criterion which prioritizes how unsatisfied agents are treated in the problem. I show that this property is not satisfied by TC, $TTAS$ and HPO rules. I present a rule, MTC, which does satisfy this procedural enhancement along with additional desirable properties.

In deriving this new rule, I was able to identify an oversight in the paper of Saban & Sethuraman [27]. This oversight pertains to their sufficient conditions for *strategy proofness*. In the next chapter, I explain that oversight and provide conditions which rectify this issue.

In going forward, it might be of interest to show if any existing rules other than $TTAS$ satisfy *weak group strategy proofness* especially because $TTAS$ has an exponential running time in the worst case. However, this chapter does prove existence of rules which satisfy *Pareto efficiency*, *individual rationality*, *weak core selection*, *strategy proofness*, *weak group strategy proofness* and *no* *justified-envy for agents with identical endowments.* Additionally, relationship between *weak group strategy proofness* and *prioritized treatment of market-equal unsatisfied agents* is unclear which might require further investigation.

CHAPTER III

A NOTE ON "HOUSE ALLOCATION WITH INDIFFERENCES: A GENERALIZATION AND A UNIFIED VIEW"

Introduction

Saban & Sethuraman [27] consider the housing market problem while allowing for indifferences. In this problem, each agent initially owns at most one object and each object is initially owned by at most one agent. The goal is to reallocate these resources in a way that the final allocation satisfies some desirable properties; *Pareto efficiency* (no agent can be made better-off without making someone else worse-off), *individual rationality* (each agent receives something at least as good as her endowment), *weak core* (no subset of agents can trade among themselves, each using her endowment, such that each agent in the subset gets something better) and *strategy proofness* (no agent has an incentive to misreport her preferences). They provide a class of rules for which each member is *Pareto efficient*, *weak core selecting* (hence, *individually rational*) and *strategy proof*. Mechanisms satisfying such properties for housing market problem with weak preferences had already been proposed (independently) by Alcalde-Unzu & Molis [3] and Jaramillo & Manjunath [14].

Major contributions of Saban & Sethuraman [27] include unifying the existing mechanisms and establishing sufficient conditions for *strategy proofness*. They provide sufficient conditions for *strategy proofness* and hence, narrow down the class of rules provided by Aziz & de Keijzer [6]. They use their sufficient conditions to come up with mechanisms that are computationally more efficient than the mechanisms provided in earlier works. The sufficient conditions are derived by establishing equivalence of *strategy proofness* with *local invariance*. While *strategy proofness* requires that no agent has an incentive to misrepresent her preferences (in any manner), *local invariance* only eliminates very specific misrepresentations. *Local invariance* is defined in Saban & Sethuraman [27] as follows:

> ***Local Invariance:*** *Let $R = (R_{-i}, R_i)$ be the preference lists of the agents, where $R_i = (p_1, \cdots, p_r)$ and p_m represents the set of objects corresponding to agent i's m^{th} indifference class*[18]*. Suppose that agent i obtains object $a \in p_m$ (a is in agent i's m^{th}*

[18]For each $a, b \in p_m$, aI_ib. Moreover, for each $a \in p_m$ and $b \in p_{\tilde{m}}$, aP_ib if $m < \tilde{m}$.

indifference class) when mechanism M is applied to the preference profile R, and suppose $aP_i\omega(i)$. Let $R' = (R_{-i}, R'_i)$, where $R'_i = (p_1, \cdots, p_{k-1}, a, p_k \backslash \{a\}, p_{k+1}, \cdots, p_r)$. Then, when mechanism M is applied to R', agent i still obtains a.

Saban & Sethuraman [27] show that for mechanisms satisfying *independence of unsatisfied agents* and *persistence*, *local invariance* and *strategy proofness* are equivalent. Following notation of Saban & Sethuraman [27], let F be a selection rule which determines unique pointees at each step of a mechanism. That is, let G be any graph of a housing market problem in which agents are represented as vertices and for each vertex representing an agent, say i, edges are extended to all vertices representing agents holding one of the most preferred objects (among the remaining ones) of agent i. Then, $F(G)$ represents a subgraph of G in which each vertex has a unique outgoing edge. In any such graph G, agent i is said to be *satisfied* if agent i owns one of her most preferred objects (among the remaining ones) and *unsatisfied* if she is not *satisfied*. Additionally, $(a_1, \cdots, a_k)$ is said to be a path in $F(G)$ if there is an edge from vertex a_l to vertex a_{l+1} for each $l \in \{1, \cdots, k-1\}$. The aforementioned properties are defined by Saban & Sethuraman [27] as follows:

Independence of unsatisfied agents: *The selection rule F satisfies independence of unsatisfied agents if for any unsatisfied agent i, and any two graphs G_1 and G_2 that differ only in the outgoing edges from i, $F(G_1)$ and $F(G_2)$ can differ only in the outgoing edge from agent i.*

Persistence: *Let $p = (a_1, \cdots, a_k)$ be a path in $F(G)$. Then, path p is said to be persistent if p appears in all the successive steps of the algorithm until agent a_k trades her object or leaves the problem.*

Saban & Sethuraman [27] present the equivalence of *strategy proofness* and *local invariance* in Theorem 3 which is reproduced below.

Theorem 3. *A mechanism M satisfying the "Independence of Unsatisfied Agents" and the "Persistence" properties is strategy proof if and only if it satisfies local invariance.*

In the next section, I describe why the result proposed in Theorem 3 of Saban & Sethuraman [27] may not hold, in general, and then present a mechanism which was shown to satisfy *local invariance* in the previous chapter for a restricted class of preferences but, in general, is not *strategy proof*. Then, I present additional restrictions under which result of the aforementioned theorem holds true.

Results

To prove converse of Theorem 3, Saban & Sethuraman [27] proceed as follows: Let M be a mechanism that satisfies *local invariance* but not *strategy proofness*. Then, there is an agent i who can report $R_i' \in \mathcal{R}$ to obtain a strictly better object, say a. Let $\alpha \equiv M(R,\omega)$ and $\alpha' \equiv M(R',\omega)$ where $R' = (R_{-i}, R_i')$. Then, $\alpha'(i)\, P_i \alpha(i)$ where $\alpha'(i) = a$. The authors then state that, by *local invariance*, it can be assumed that a is the only object in its indifference class for misreported preferences R_i' of agent i. However, by definition, *local invariance* requires the assigned object of the agent to be strictly better than her endowment under the reported preferences i.e. the proof implicitly assumes that $aP_i'\omega(i)$ which may not be the case in general.

As a counter-example, I consider the *Modified Top Cycles* (MTC) rule which was presented in the previous chapter. It was shown that this rule satisfies *independence of unsatisfied agents*, *persistence* and *local invariance*. However, the following example shows that this rule does not satisfy *strategy proofness* for weak preferences.

Example 3.1: Consider the following housing market problem: $N = \{1,2,3,4,5\}$, $O = \{a,b,c,d,e\}$, $\omega = (a,b,c,d,e)$, $1 \prec 2 \prec 3 \prec 4 \prec 5$ and the preference profile:

R_1	R_2	R_3	R_4	R_5	R_4'
e	ab	b	b	cde	bd
a		c	d		

Then, MTC outcome under R and R' is (e,a,b,d,c) and (e,a,c,b,d), respectively. Agent 4 is able to receive a better outcome by feigning to be satisfied. That is, under *local invariance* agents might still have an opportunity to manipulate the rule by pretending to be satisfied i.e. reporting their endowment as one of their most preferred objects (among the remaining ones).

Now I present a property, *local push-up invariance*, which is equivalent to *strategy proofness* under the aforementioned restrictions on the mechanism. This property was used by Alcalde-Unzu & Molis [3] and Jaramillo & Manjunath [14] to establish *strategy proofness* of their rules.

Local Push-up Preference: Consider any $R \in \mathcal{R}^N$. Let α be outcome of some mechanism M under preferences R. $R_i' \in \mathcal{R}$ is said to be *local push-up preference* of R_i for mechanism M if:

1. $R_i|_{O\setminus\{\alpha(i)\}} = R_i'|_{O\setminus\{\alpha(i)\}}$,
2. $aP_i\alpha(i)$ if and only if $aP_i'\alpha(i)$ for all $a \in O\setminus\{\alpha(i)\}$, and

3. $\alpha(i) R_i a$ if and only if $\alpha(i) P'_i a$ for all $a \in O \setminus \{\alpha(i)\}$.

Local Push-up Invariance: A mechanism M is said to satisfy *local push-up invariance* if for each $R \in \mathcal{R}^N$ and $i \in N$, $\alpha(i) = \alpha'(i)$ where $\alpha \equiv M(R, \omega)$, $\alpha' \equiv M(R', \omega)$, $R' = (R_{-i}, R'_i)$ and R'_i is *local push-up preference* of R_i for mechanism M.

Theorem 3.1 states that *local push-up invariance* is equivalent to *strategy proofness* for mechanisms satisfying *independence of unsatisfied agents* and *persistence* properties.

Theorem 3.1. A mechanism M satisfying the *independence of unsatisfied agents* and *persistence* properties is *strategy proof* if and only if it satisfies *local push-up invariance*.

Proof. Consider mechanism M which satisfies *independence of unsatisfied agents* and *persistence*. First, suppose that M satisfies *strategy proofness* but not *local push-up invariance*. Then, there is $i \in N$ and $R, R' \in \mathcal{R}^N$, where $R' = (R_{-i}, R'_i)$ and R'_i is *local push-up preference* of R_i, such that $\alpha(i) \neq \alpha'(i)$ where $\alpha \equiv M(R, \omega)$ and $\alpha' \equiv M(R', \omega)$. Since M is *strategy proof*, it must be the case that $\alpha'(i) R'_i \alpha(i)$. Since $\alpha(i) \neq \alpha'(i)$, by definition of *local push-up preference*, $\alpha'(i) P'_i \alpha(i)$ which implies that $\alpha'(i) P_i \alpha(i)$. This contradicts *strategy proofness* of M.

To show the converse, suppose that M satisfies *local push-up invariance* but not *strategy proofness*. Then, there is $i \in N$ and $R, R' \in \mathcal{R}^N$, where $R' = (R_{-i}, R'_i)$, such that $\alpha'(i) P_i \alpha(i)$ where $\alpha \equiv M(R, \omega)$ and $\alpha' \equiv M(R', \omega)$. Then, by *local push-up invariance*, we can assume that $\alpha'(i)$ is the only object in its indifference class for preferences R'_i. The remaining proof follows from Theorem 3 of Saban & Sethuraman [27]. ■

It should be noted that *local push-up invariance* implies *local invariance* but not vice versa. Next, I present additional restrictions under which *local invariance* is equivalent to *local push-up invariance* so that result of Theorem 3 from Saban & Sethuraman [27] holds true under these conditions. One possible solution is to restrict attention to only those preference profiles for which endowment is the only object in its indifference class for each agent. I refer to this class of preferences as *no indifferences with endowment*. For this restricted class of preferences, *local invariance* and *local push-up invariance* are obviously equivalent. Before proceeding further, I provide some additional notation.

Some Notation: Consider any $R, R' \in \mathcal{R}^N$. Let $N_{\tilde{t}}$ (resp. $N'_{\tilde{t}}$) and $O_{\tilde{t}}$ (resp. $O'_{\tilde{t}}$) be set of agents and objects, respectively, remaining at step $\tilde{t}$ under R (resp. R'). Let $h_{\tilde{t}}$ (resp. $h'_{\tilde{t}}$) denote objects held by agents at step $\tilde{t}$ under R (resp. R'). For $i \in N_{\tilde{t}}$ (resp. $N'_{\tilde{t}}$), let $p_{\tilde{t}}(i)$ (resp. $p'_{\tilde{t}}(i)$) be the agent pointed at by agent i at step $\tilde{t}$ under R (resp. R'). Let t (resp. t') be the first step agent i becomes satisfied or part of a trading cycle under R (resp. R'). For any $R \in \mathcal{R}^N$ and step $\tilde{t}$, denote all agents having a path to agent i via pointing (including agent i) as $C_i\left(R,\tilde{t}\right)$ i.e.

$$C_i\left(R,\tilde{t}\right) = \begin{cases} & j = i \\ j \in N_{\tilde{t}} : & p_{\tilde{t}}(j) = i \\ & p_{\tilde{t}}(p_{\tilde{t}}(j)) = i \\ & \vdots \end{cases}$$

Before presenting additional restrictions, I return to the housing market example to provide some intuition for the following restrictions. Under MTC, agent 4 had an opportunity to benefit by pretending to be satisfied because she was able to get an additional agent, agent 5, to point at her. Intuitively, such incentives need to be eliminated in order to achieve *strategy proofness*. Consider the following requirement:

Ineffective Status Change (Agents): At each step $t \leq \tilde{t} < t'$, any $R \in \mathcal{R}^N$, $R' = (R_{-i}, R'_i)$ where R'_i is *local push-up preference* of R_i under rule M, if $j \in N'_{\tilde{t}} \backslash C_i\left(R',\tilde{t}\right)$, then $j \in N_{\tilde{t}}$. Moreover, if $p_{\tilde{t}}(j) = k$, then $p'_{\tilde{t}}(j) = k$.

Ineffective status change (agents) makes sure that no agent is able to attain additional "pointers" by misreporting her preferences. Another possible way to achieve this would be to use an object variant of the above property:

Ineffective Status Change (Objects): At each step $t \leq \tilde{t} < t'$, any $R \in \mathcal{R}^N$, $R' = (R_{-i}, R'_i)$ where R'_i is *local push-up preference* of R_i under rule M, if $j \in N'_{\tilde{t}} \backslash C_i\left(R',\tilde{t}\right)$, then $j \in N_{\tilde{t}}$. Moreover, if $p_{\tilde{t}}(j) = k$, then $h'_{\tilde{t}}\left(p'_{\tilde{t}}(j)\right) = h_{\tilde{t}}(k)$.

Under *ineffective status change (objects)*, instead of each agent pointing at the same agent under both preference profiles, each agent may point at different agents under the two preference profiles but the object held by these agents are the same. In fact, the two variants of *ineffective status change* are same as two of the sufficient conditions of *local invariance* provided in Theorem 4 of Saban & Sethuraman [27]. I use these conditions to show that class of rules presented in Saban & Sethuraman [27] are still *strategy proof.* The next result shows that under each of aforementioned conditions, *local invariance* is equivalent to *local push-up invariance* and hence, *strategy proofness* from Theorem 3.1.

Theorem 3.2. For a mechanism M satisfying *independence of unsatisfied agents* and *persis-*

tence properties, *local invariance* is equivalent to *local push-up invariance* (equivalently, *strategy proofness*) if:

1. there are no *indifferences with endowment*,
2. M satisfies *ineffective status change (agents)*, or
3. M satisfies *ineffective status change (objects)*.

Proof. Under (1), equivalence of *local invariance* and *local push-up invariance* is obvious.

Since *local push-up invariance* implies *local invariance*, for (2) and (3), we need to prove that *local invariance* implies *local push-up invariance*. Suppose that M satisfies *local invariance* but not *local push-up invariance*. Then, there is $i \in N$ and $R, R' \in \mathcal{R}^N$, $R' = (R_{-i}, R'_i)$ and R'_i is *local push-up preference* of R_i for mechanism M, such that $\alpha(i) \neq \alpha'(i)$ where $\alpha \equiv M(R, \omega)$ and $\alpha' \equiv M(R', \omega)$. Since M satisfies *local invariance*, it must be the case that $\alpha(i) I_i \omega(i)$. Moreover, by definition of *local push-up preference*, we have either $\alpha'(i) P'_i \alpha(i)$ or $\alpha(i) P'_i \alpha'(i)$. Since $R_i|_{O\setminus\{a(i)\}} = R'_i|_{O\setminus\{a(i)\}}$ and $R_j = R'_j$ for all $j \in N\setminus\{i\}$, steps of M under R and R' are identical until all $a \in O$ such that $aP_i\alpha(i)$ have been removed[19]. Then, it cannot be the case that $\alpha'(i) P'_i \alpha(i)$ because, by definition of *local push-up preference*, $\alpha'(i) P_i \alpha(i)$ and so, agent i should be able to receive $\alpha'(i)$ under R as well. Also, note that if $\alpha(i) = \omega(i)$, then $\alpha'(i) = \omega(i)$. So, suppose that $\alpha(i) \neq \omega(i)$.

Now, suppose that $\alpha(i) P'_i \alpha'(i)$. Then, it must be the case that $\alpha'(i) I'_i \omega(i)$ and $\alpha'(i) I_i \omega(i)$. Again, steps of M under R and R' are identical until all $a \in O$ such that $aP_i\alpha(i)$ have been removed. Let $\bar{t}$ be the first step such that each $a \in O$ with $aP_i\alpha(i)$ has been removed from the problem under R and R'. Then, $N_{\bar{t}} = N'_{\bar{t}}$, $O_{\bar{t}} = O'_{\bar{t}}$ and $h_{\bar{t}} = h'_{\bar{t}}$[20]. Moreover, for all $a \in O_{\bar{t}}\setminus\{\alpha(i)\}$, $\alpha(i) R_i a$ and $\alpha(i) P'_i a$. Since $\alpha(i) I_i \omega(i)$, $\alpha(i) P'_i \omega(i)$ and $h_{\bar{t}}(i) = \omega(i)$, $i \in S_{\bar{t}}$ and $i \in U'_{\bar{t}}$ i.e. $\bar{t} = t < t'$ where t (resp. t') is the first step agent i becomes satisfied or part of a trading cycle under R (resp. R') for mechanism M.

We show that following is true for M under both *ineffective status change (agents)* and *ineffective status change (objects)* for all $\tilde{t} \in \{t, \cdots, t'-1\}$:

1. $$\begin{array}{c} N_{\tilde{t}} \subseteq N'_{\tilde{t}} \\ N_{\tilde{t}}\setminus N'_{\tilde{t}} \subseteq C_i\left(R', \tilde{t}-1\right) \end{array} \quad \text{and} \quad \begin{array}{c} O_{\tilde{t}} \subseteq O'_{\tilde{t}} \\ O'_{\tilde{t}}\setminus O_{\tilde{t}} \subseteq h_{\tilde{t}}\left(C_i\left(R', \tilde{t}-1\right)\right) \end{array},$$

[19] Until all such objects have been removed, at each step, set of remaining agents and objects are same, pointing decisions are same for all agents and hence, same cycles occur under R and R'.

[20] Since $\alpha(i) \neq \omega(i)$, it must be the case that $\alpha(i) \in O_{\bar{t}}$.

2. for each $j \in N'_{\tilde{t}} \backslash C_i\left(R', \tilde{t}\right)$, $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$, and

3. for each $j \in N'_{\tilde{t}} \backslash C_i\left(R', \tilde{t}\right)$, $h_{\tilde{t}+1}(j) = h'_{\tilde{t}+1}(j)$.

Additionally, (1) is true for step t'.

First let $\tilde{t} = t$. Since $\bar{t} = t$, we have $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$ and $h_{\tilde{t}} = h'_{\tilde{t}}$. So, (1) holds. Now we show that (2) is true under both; *ineffective status change (agents)* and *ineffective status change (objects)*. By *ineffective status change (agents)*, we have $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for all $j \in N'_{\tilde{t}} \backslash C_i\left(R', \tilde{t}\right)$ and so, (2) holds at step $\tilde{t} = t$. Since $h_{\tilde{t}} = h'_{\tilde{t}}$, we have $p_{\tilde{t}}(j) = p'_{\tilde{t}}(j)$ for all $j \in N'_{\tilde{t}} \backslash C_i\left(R', \tilde{t}\right)$ under *ineffective status change (objects)* as well. By $h_{\tilde{t}} = h'_{\tilde{t}}$ and (2) at step $\tilde{t}$, we get (3) at step $\tilde{t}$.

Now, suppose that (1)-(3) hold for each $\tilde{t}$ such that $t \leq \tilde{t} < t' - 1$. We want to show that (1)-(3) are true for step $\tilde{t} + 1$. Note that any cycle that occurs at step $\tilde{t}$ under R but not under R' must consist entirely of agents in $C_i\left(R', \tilde{t}\right)$. So, (1) is true at step $\tilde{t} + 1$. Now, we show that (2) holds for both; *ineffective status change (agents)* and *ineffective status change (objects)*. By *ineffective status change (agents)*, we have (2) at step $\tilde{t}+1$. For *ineffective status change (objects)*, consider any $j \in N'_{\tilde{t}+1} \backslash C_i\left(R', \tilde{t}+1\right)$. Let $p'_{\tilde{t}+1}(j) = k$. Then, it must be the case that $k \in N'_{\tilde{t}+1} \backslash C_i\left(R', \tilde{t}+1\right)$ and, by *persistence*, $k \in N'_{\tilde{t}} \backslash C_i\left(R', \tilde{t}+1\right)$. Then, by (3) at step $\tilde{t}$, $h_{\tilde{t}+1}(k) = h'_{\tilde{t}+1}(k)$ and so, $p_{\tilde{t}+1}(j) = p'_{\tilde{t}+1}(j)$ by *ineffective status change (objects)*. Then, by (3) at step $\tilde{t}$ and (2) at step $\tilde{t}+1$, we have (3) at step $\tilde{t}+1$. At step $t'-1$, any cycle that occurs under R but not under R' must consist entirely of agents in $C_i(R', t'-1)$. So, we have (1) at step t'.

Now, let t_i be the step agent i departs under R. It cannot be the case that $t_i > t'$ because $O_{t_i} \subseteq O_{t'}$ and $O_{t'} \subseteq O'_{t'}$. So, $\alpha(i) \in O'_{t'}$. Agent i becomes satisfied or part of trading cycle at step t' under R' and so, $\alpha'(i) R'_i \alpha(i)$ which is a contradiction. Now suppose that $t_i \leq t'$. Let $\ddot{t}_i$ be the last step agent i becomes part of a trading cycle under R. Denote this cycle as C. It must be the case that $t \leq \ddot{t}_i < t_i$. Since $t_i \leq t'$, agent i is not part of a trading cycle at step $\ddot{t}_i$ under R' so that C is not a trading cycle at step $\ddot{t}_i$ under R'. Then, it must be the case that C consists entirely of agents in $C_i\left(R', \ddot{t}_i\right)$. Then, it must be the case that $\alpha(i) \in h'_{\ddot{t}_i}\left(C_i\left(R', \ddot{t}_i\right)\right)$. If not, we have $\alpha(i) \in h'_{\ddot{t}_i}\left(N'_{\ddot{t}_i} \backslash C_i\left(R', \ddot{t}_i\right)\right)$ i.e. there is $j \in N'_{\ddot{t}_i} \backslash C_i\left(R', \ddot{t}_i\right)$ such that $h'_{\ddot{t}_i}(j) = \alpha(i)$. Since $j \in N'_{\ddot{t}_i} \backslash C_i\left(R', \ddot{t}_i\right)$, by *persistence*, it must be the case that $j \in N'_{\ddot{t}_i - 1} \backslash C_i\left(R', \ddot{t}_i - 1\right)$. Then, by (3) at step $\ddot{t}_i - 1$, we have $h_{\ddot{t}_i}(j) = h'_{\ddot{t}_i}(j)$. This implies that $j \in C$ which is a contradiction. Since $\alpha(i) \in h'_{\ddot{t}_i}\left(C_i\left(R', \ddot{t}_i\right)\right)$, by *persistence*, it must be the case that $\alpha'(i) R'_i \alpha(i)$ which is a contradiction. This completes the proof. ■

An immediate corollary of the above theorem is that MTC is *strategy proof* when considering class of preferences with *no indifferences with endowment* because in that case, *local invariance* is equivalent to *local push-up invariance.* Let $\bar{\mathcal{R}}$ denote the set of preferences for which there are no indifferences with endowment, A be the set of all possible assignments and $\prec$ be priority ordering over agents.

Corollary 3.1. For any $(R, \omega) \in \bar{\mathcal{R}}^N \times A$ and priority ordering $\prec$ over agents, $MTC^{\prec}(R, \omega)$ is *strategy proof.*

Conclusion

In this chapter, I identify an oversight in Saban & Sethuraman [27]. I provide conditions under which results of Saban & Sethuraman [27] hold true. An earlier version of this note was made available to Daniela Saban and Jay Sethuraman. Appropriate revisions have been made in their paper to rectify the oversight.

CHAPTER IV

FRACTIONAL HOUSING MARKET WITH SINGLE AND DISCRETE ENDOWMENTS

Introduction

In this chapter, I study the problem of reallocating goods to agents in a manner which satisfies some desirable properties. In particular, I consider the problem where each agent owns an object and has preferences over the set of objects and each object is owned by one agent. The goal is to reassign these objects in a manner which satisfies some desirable properties such as: *Pareto efficiency* (not possible to make someone better-off without making someone worse-off), *individual rationality* (each agent gets something at least as good as her endowment), *strategy proofness* (truth-telling is a weakly dominant strategy) and *core stability* (not possible for any group of agents to achieve a better assignment by trading among themselves). In literature, such models of exchange economy of goods are commonly referred to as the *housing market*. Housing market is widely used in the kidney exchange markets [26].

Shapley & Scarf [28] proposed *Top Trading Cycles* (TTC) rule which is attributed to David Gale. Under strict preferences, outcome of TTC is the unique allocation in the *core* and it is also the unique *competitive allocation* [24]. Moreover, TTC rule is *strategy proof [23]* and it is the only rule which satisfies *Pareto efficiency*, *individual rationality* and *strategy proofness* [17].

Considerable amount of work has been done to extend the housing market problem. Alcalde-Unzu & Molis [3] and Jaramillo & Manjunath [14] independently proposed generalizations of TTC rule; *Top Cycles* (TC) rule and *Top Trading Absorbing Sets* ($TTAS$) rule, respectively, which allow for indifferences in preferences for the housing market problem. These rules are *Pareto efficient*, *weak core selecting* (hence, *individually rational*), *strategy proof* and *core selecting* (when *core* is non-empty) [3, 6, 14]. Saban & Sethuraman [27] present sufficient conditions to achieve *strategy proofness* when considering housing market with indifferences. In doing so, they are able to present a class of rules; *common ordering on agents, individual ordering on objects* ($CAIO$), for which each member is *Pareto efficient*, *weak core selecting* (hence, *individually rational*) and *strategy proof*[21]. Moreover, they propose a member of $CAIO$, *Highest Priority Object* (HPO)

[21] TC and $TTAS$ rules are members of $CAIO$.

rule, which is computationally quicker than TC and $TTAS$. Each member of $CAIO$ is an iterative algorithm where each iteration has three phases: *departure phase*, a set of agents are chosen to depart if they hold one of their most preferred objects (among the remaining ones) and all their most preferred objects (among the remaining ones) are held by that set of agents; *pointing phase*, priority orderings[22] over agents and/or objects are used to determine a unique pointee for each agent; *trading phase*, trades occur based on the cycles formed in the pointing phase.

Another extension to the housing market problem is to consider random assignment solutions rather than deterministic assignment solutions. In a deterministic assignment solution, an object is assigned to an agent or not whereas for a random assignment solution, an agent can be assigned fractions of an object. Random assignment solutions are of significance because such solutions allow to consider fairness properties in a meaningful manner which may not be possible for deterministic assignment solutions. For a random assignment solution, fractions of assigned objects can be interpreted as partial ownership or probability of receiving the object. Random assignment solutions to housing market problem have been studied in various settings.

Abdulkadiroglu & Sonmez [1] and Bogomolnaia & Moulin [9] consider the problem where agents do not own objects. Such problems are referred to as *random assignment* problem in literature. Abdulkadiroglu & Sonmez [1] present a mechanism which is *strategy proof*, *ex-post Pareto efficient* and satisfies *equal treatment of equals*. Bogomolnaia & Moulin [9] propose a mechanism which is *ordinally efficient*, *envy-free* and satisfies *weak strategy proofness*. Katta & Sethuraman [15] extend this problem to weak preferences and show that *weak strategy proofness* conflicts with *ordinal efficiency* and *envy-freeness*.

Recently, several papers have considered random assignment solution to the housing market problem. Yilmaz [33] explores this problem while allowing for indifferences. They present a rule which is *individually rational*, *ordinally efficient* and satisfies *no justified-envy*. *No-envy* is a central notion of fairness in economics. It requires that no agent likes (envies) assignment of another agent better than her own. However, when agents own endowments, *no-envy* is incompatible with *individual rationality*. Yilmaz [32] introduced the notion of *no justified-envy*. *No justified-envy* requires that an agent can envy another agent only if her assignment is *individually rational* for the other agent. So, *no justified-envy* is weaker than *no-envy* but is compatible with *individual rationality*. However, *individual rationality*, *strategy proofness* and *no justified-envy* are incompatible in this

[22] These are complete, transitive and antisymmetric orderings.

setting [32, 33].

Athanassoglou & Sethuraman [4] study the housing market problem, while allowing for indifferences, where agents may initially own fractions of objects. They present a mechanism which satisfies *ordinal efficiency*, *individual rationality* and *no justified-envy*. They also prove three impossibility results: (1) for at least 3 agents, any mechanism which satisfies *individual rationality*, *ordinal efficiency* and *no justified-envy* cannot satisfy *weak strategy proofness*[23], (2) for at least 4 agents, there is no mechanism which satisfies *individual rationality*, *ordinal efficiency* and *strategy proofness*, and (3) for at least 5 agents, any mechanism satisfying *individual rationality* and *ordinal efficiency* cannot simultaneously satisfy *no justified-envy* and *no-envy for agents with identical endowments*. Additionally, these impossibility results hold even under the restricted case of strict preferences.

Aziz [5] generalizes *Top Trading Cycles* rule for the housing market problem with fractional endowments while allowing for indifferences. One particular contribution of Aziz [5] is to define *core* using stochastic dominance relation which they refer to as *SD-core stability*. They present a mechanism, *Fractional Top Trading Cycles* ($FTTC$) rule, which is *SD-core stable* and *ordinally efficient*. They show that $FTTC$ satisfies maximal set of desirable results by presenting two impossibility results: (1) there is no mechanism that satisfies *SD-core stability* and *no justified-envy*, and (2) there is no mechanism which satisfies *individual rationality*, *ordinal efficiency* and *weak strategy proofness*. Additionally, these results hold even for strict preferences and single unit allocations and endowments [5]. However, $FTTC$ is not in general a fractional solution to the housing market problem. In fact, when endowments are single and discrete, $FTTC$ reduces to Plaxton's mechanism which is a class of deterministic rules for the housing market problem with indifferences [5].

Notably, several impossibility results have been established for random assignment solution to the housing market problem and several of these results hold even when endowments are single and discrete and preferences are strict. In this chapter, I consider random assignment solution to the housing market problem with single and discrete endowments for a restricted class of preferences; trichotomous preferences. Under trichotomous preferences, each agent considers an object as being acceptable or unacceptable. Additionally, I assume that each agent finds each acceptable object to be better than her endowment and her endowment to be better than each unacceptable object i.e. there are three indifference classes for each preference ordering. So, trichotomous pref-

[23] This impossibility result holds even for the case of single endowments.

erences, like strict preferences, are an extreme case of the full preference domain. I consider this class of preferences for several reasons. First, multiple impossibility results have been established for random assignment solutions to the housing market problem even for strict preferences and single and discrete endowments. So, it seems natural to consider another extreme case of the full preference domain to determine whether these impossibility results still hold. As we show later on, the aforementioned impossibility results do not hold for this setting. Secondly, for trichotomous preferences, probability of being assigned acceptable objects can be used as the canonical utility representation. Thirdly, trichotomous preferences can arise in real-life situations; housemates assigning rooms, preferences over dorm-rooms in hostels, etc. More importantly, kidney exchange can be argued to have trichotomous preferences. Some of the most important criteria to determine whether a kidney is acceptable for a patient are: blood type, cross-matching and tissue type. Of these three criteria, blood type and cross-matching are binary i.e. based on these a kidney is either acceptable or unacceptable for a patient. Tissue type, on the other hand, compares six basic tissue antigens between donor and the patient where the best match would be to have all six antigens matching. However, successful transplants are possible even if there are no matching tissue antigens between donor and the patient.

I present the model and some relevant notation in the next section.

Model

I consider the fractional housing market problem with single and discrete endowments. In this problem, each agent initially owns one object, each object is initially owned by one agent and each agent is to be assigned fractions of objects adding up to one. These fractions of objects can be considered as probability of receiving an object or a part-time assignment of an object. Let N and O denote set of agents and objects, respectively. Let ω denote the vector of endowment of agents in N and ω_i represent endowment of agent i. For each $i \in N$, I represent preferences as R_i, R_i', etc. Strict and indifference relations associated with R_i are denoted as P_i and I_i, respectively. Moreover, $R = (R_i)_{i \in N}$ represents preference profile. In this chapter, I assume agents to have *trichotomous* preferences where each agent considers an object to be acceptable or unacceptable. Agents are indifferent between acceptable objects and also between unacceptable objects. Moreover, I assume that each agent prefers every acceptable object to her endowment and prefers her endowment to each unacceptable object. Let $\mathcal{T}$ be the set of all possible trichotomous preferences[24]. For any $R_i \in \mathcal{T}$, let $A(R_i)$ denote the set of acceptable objects under R_i. Then, for any $a, b \in A(R_i)$, aI_ib. Moreover, if $a, b \notin A(R_i)$ and $\omega_i \notin \{a, b\}$, then agent i considers objects a and b to be unacceptable and aI_ib. Finally, for each $a \in A(R_i)$ and $b \notin A(R_i) \cup \{\omega_i\}$, we have $aP_i\omega_i$, aP_ib and ω_iP_ib. An interesting aspect of these preferences is that preferences of any agent can be described completely by the set of objects that she finds acceptable. Additionally, probability of being assigned acceptable objects can be used as the canonical utility representation.

An allocation is said to be a *deterministic assignment* if each agent receives exactly one object and each object is allocated to exactly one agent. Each deterministic assignment can be represented as a $n \times n$ permutation matrix $[\alpha_{i,a}]_{i \in N, a \in O}$ such that for each $i \in N$ and $a \in O$, $\alpha_{i,a} \in \{0, 1\}$[25], $\sum_{i \in N} \alpha_{i,a} = 1$ and $\sum_{a \in O} \alpha_{i,a} = 1$. When assignment α is deterministic, with a slight abuse of notation, I use α_i to denote the object assigned to agent i. Let $\mathcal{A}$ be the set of all possible deterministic assignments. I use α, β, etc. to represent deterministic assignments. As mentioned already, endowment is represented as ω and by assumption, $\omega \in \mathcal{A}$. Then, a *deterministic assignment mechanism* for the housing market problem with single and discrete endowments is a mapping from $\mathcal{T}^N \times \mathcal{A}$ to $\mathcal{A}$ for trichotomous preferences.

A *random assignment* $x = [x_{i,a}]_{i \in N, a \in O}$ is a $n \times n$ stochastic matrix satisfying the following:

1. $x_{i,a} \in [0, 1]$ for each $i \in N$ and $a \in O$,

2. $\sum_{a \in O} x_{i,a} = 1$ for each $i \in N$, and

3. $\sum_{i \in N} x_{i,a} = 1$ for each $a \in O$.

For random assignment x, $x_{i,a}$ represents probability with which agent i is assigned object a. Moreover, agent i's assignment under random assignment x is represented as $x_i = (x_{i,a})_{a \in O}$. Let $\mathcal{X}$ be the set of all random assignments. Then, $\mathcal{X}$ is the convex hull of $\mathcal{A}$. I use x, y, z, etc. to represent random assignments. Additionally, for any $i \in N$, random allocation x_0 and preferences $R_0 \in \mathcal{R}$, let $U(R_0, x_0) = \sum_{a \in A(R_0)} x_{0,a}$ i.e. $U(R_0, x_0)$ represents the probability of receiving an acceptable object for preferences R_0 under x_0. Moreover, let:

$$W_i(R_0, x_0) = U(R_0, x_0) + x_{0,\omega_i}$$

[24]Trichotomous preferences are identical to the preferences considered in Bogomolnaia & Moulin [10].

[25]$\alpha_{i,a}$ represents allocation of object a to agent i under α.

That is, $W_i(R_0, x_0)$ represents the probability with which agent i receives an object at least as good as her endowment according to preferences R_0 under the random assignment x_0. Then, for the housing market problem $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$ and random assignment $x \in \mathcal{X}$, $U(R_i, x_i)$ represents probability with which agent i receives an acceptable object under allocation x_i and $W_i(R_i, x_i)$ represents probability with which agent i receives an object at least as good as her endowment under allocation x_i. Additionally, let $U(R, x) = (U(R_i, x_i))_{i \in N}$ and $W(R, x) = (W_i(R_i, x_i))_{i \in N}$ for any $R \in \mathcal{T}^N$ and $x \in \mathcal{X}$.

Then, a *random assignment mechanism* for the housing market problem with single and discrete endowments is a mapping from $\mathcal{T}^N \times \mathcal{A}$ to $\mathcal{X}$ for trichotomous preferences. I refer to the problem of finding a random assignment solution to the housing market problem with single and discrete endowments as the *fractional housing market problem with single and discrete endowments*. Moreover, each such problem can simply be represented as $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$ for trichotomous preferences.

Properties

Before describing appropriate properties for random assignment mechanisms, I extend preferences of agents over O to preferences over $\mathcal{X}$. Standard method of comparing random assignments is to use the stochastic dominance relation. For any $i \in N$, $R_i \in \mathcal{R}$ and $x, y \in \mathcal{X}$, agent i likes x_i at least as much as y_i if and only if x_i stochastically dominates y_i with respect to R_i i.e.

$$\sum_{aR_ib} x_{i,a} \geq \sum_{aR_ib} y_{i,a} \ \forall b \in O$$

Moreover, when simply comparing assignment of agent i for random assignments x and y, I denote x at least as good as y for agent i as $x \succeq_i y$. Additionally, to show that agent i likes assignment of agent j at least as much as that of agent k under random assignment x, I use the notation of $x_j \succeq_i x_k$. Additionally, if $\sum_{aR_ib} x_{i,a} \geq \sum_{aR_ib} y_{i,a}$ for each $b \in O$ and $\sum_{aR_ib} x_{i,a} > \sum_{aR_ib} y_{i,a}$ for some $b \in O$, then agent i strictly prefers x to y which is denoted as $x \succ_i y$. If $x \succeq_i y$ and $y \succeq_i x$, then agent i is indifferent between random assignments x and y. I denote this as $x \sim_i y$.

Following Bogomolnaia & Moulin [10], for any reported preferences, I restrict attention to assignments under which each agent receives only those objects which are at least as good as her endowment i.e. for any $i \in N$, $R_i \in \mathcal{T}$ and $x \in \mathcal{X}$, $x_{i,a} > 0$ if and only if $a \in A(R_i) \cup \{\omega_i\}$.

Additionally, I assume that agents have an aversion to receiving an unacceptable object with positive probability. Formally, for any $i \in N$, $R_i \in \mathcal{T}$ and $x, y \in \mathcal{X}$ such that $W_i(R_i, x_i) = 1$ and $W_i(R_i, y_i) < 1$, $x_i \succ_i y_i$. Then, for any random assignments satisfying these assumptions, say $x, y \in \mathcal{X}$, $x \succeq_i y$ if and only if $U(R_i, x_i) \geq U(R_i, y_i)$[26].

A deterministic assignment α is said to be *Pareto efficient* if there is no $\beta \in \mathcal{A}$ such that $\beta_i R_i \alpha_i$ for each $i \in N$ and $\beta_i P_i \alpha_i$ for some $i \in N$. Random assignment x is said to be *SD-efficient* if there is no $y \in \mathcal{X}$ such that $y \succeq_i x$ for each $i \in N$ and $y \succ_i x$ for some $i \in N$. A mechanism is *SD-efficient* if it chooses a *SD-efficient* assignment for every $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$. A random assignment is said to be *ex-post efficient* if it can be represented as a probability distribution over *Pareto efficient* deterministic assignments. A mechanism is *ex-post efficient* if, for every $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, it chooses an assignment which is *ex-post efficient.* A random assignment is said to be *ex-ante efficient* if for every profile of utility functions consistent with preference profile of agents, the expected utility vector is *Pareto efficient.* Since probability of receiving an acceptable object can be used as the canonical utility representation for trichotomous preferences, random assignment x is *ex-ante efficient* if there is no $y \in \mathcal{X}$ such that $U(R_i, y_i) \geq U(R_i, x_i)$ for each $i \in N$ and $U(R_i, y_i) > U(R_i, x_i)$ for some $i \in N$. A mechanism is said to be *ex-ante efficient* if, for every $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, it chooses an *ex-ante efficient* assignment.

A deterministic assignment α is said to be *individually rational* if $\alpha_i R_i \omega_i$ for each $i \in N$ i.e. each agent receives an object at least as good as her endowment. A random assignment x is said to be *SD-individually rational* (*SD-IR*) if $W_i(R_i, x_i) = 1$ for each $i \in N$. A mechanism is said to be *SD-IR* if it chooses *SD-IR* assignment for every $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$. A random assignment is said to be *ex-post individually rational* (*ex-post IR*) if it can be represented as a probability distribution over *individually rational* deterministic assignments. A mechanism is *ex-post IR* if, for every $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, it chooses an *ex-post IR* assignment. A random assignment x is said to be *ex-ante individually rational* (*ex-ante IR*) if $W(R_i, x_i) = 1$ for each $i \in N$. A mechanism is *ex-ante IR* if, for each $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, it chooses an *ex-ante IR* assignment.

A random assignment mechanism is said to be *SD-strategy proof* if truthful revelation of preferences is a weakly dominant strategy for each agent i.e. $\varphi : \mathcal{R}^N \times \mathcal{A} \to \mathcal{X}$ is *SD-strategy proof* if for each $R \in \mathcal{T}^N$, $i \in N$ and $R'_i \in \mathcal{T}$, $\varphi(R, \omega) \succeq_i \varphi(R', \omega)$ where $R' = (R_{-i}, R'_i)$. A random assignment mechanism is said to be *weakly SD-strategy proof* if for every agent, allocation received

[26]This is because I am assuming $W_i(R_i, x_i) = W_i(R_i, y_i) = 1$.

by truthful revelation is not dominated by allocation obtained by reporting any other preference i.e. $\varphi : \mathcal{R}^N \times \mathcal{A} \to \mathcal{X}$ is *weakly SD-strategy proof* if for each $R \in \mathcal{T}^N$, $i \in N$ and $R'_i \in \mathcal{T}$, $\varphi(R', \omega) \not\succeq_i \varphi(R, \omega)$ where $R' = (R_{-i}, R'_i)$[27]. Moreover, rule $\varphi : \mathcal{R}^N \times \mathcal{A} \to \mathcal{X}$ is *ex-ante strategy proof* if for each $R \in \mathcal{T}^N$, $i \in N$ and $R'_i \in \mathcal{T}$, $U(R_i, \varphi_i(R, \omega)) \geq U(R_i, \varphi_i(R', \omega))$ where $R' = (R_{-i}, R'_i)$.

For any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, a deterministic assignment $\alpha \in \mathcal{A}$ is said to be *blocked* by coalition $S \subseteq N$ if there is $\beta \in \mathcal{A}$ such that $\{\beta_i : i \in S\} = \{\omega_i : i \in S\}$ and $\beta_i P_i \alpha_i$ for each $i \in S$. An allocation is said to be in the *weak core* if it is not blocked by any subset of N. A random assignment is said to be in the *ex-post weak core* if it can be represented as a probability distribution over deterministic assignments in the *weak core*. A random assignment $x \in \mathcal{X}$ is said to be in *SD-core* if there is no coalition $S \subseteq N$ and $y \in \mathcal{X}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$ and $y \succ_i x$ for each $i \in S$. Additionally, a random assignment $x \in \mathcal{X}$ will be said to be in *ex-ante core* if there is no coalition $S \subseteq N$ and $y \in \mathcal{X}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$ and $U(R_i, y_i) > U(R_i x_i)$ for each $i \in S$.

A random assignment is said to be *envy-free* if each agent prefers her allocation to every other agent's allocation. That is, a random assignment $x \in \mathcal{X}$ is said to be *envy-free* if for each $i, j \in N$, $x_i \succeq_i x_j$. However, for the housing market problem, *envy-freeness* conflicts with *individual rationality*. This can be established by the following example:

Example 4.1: Let $N = \{1, 2\}$, $O = \{a, b\}$, $\omega = (a, b)$ and preference profile be:

R_1	R_2
a	a
b	b

In this housing market problem, any assignment for which agent 2 does not envy agent 1 would not be *individually rational* for agent 1.

So, it would be reasonable to consider *no-envy for agents with identical endowments* (*NEIE*) i.e. a random assignment $x \in \mathcal{X}$ satisfies *NEIE* if $x_i \succeq_i x_j$ whenever $\omega_i I_h \omega_j$ for each $h \in N \backslash \{i, j\}$[28] and $\omega_j \in A(R_i)$ if and only if $\omega_i \in A(R_j)$. The latter condition states that there is some symmetry in preferences of agents whose endowments are identical for every other agent. Moreover, if this

[27]Note that, for deterministic assignments these notions are equivalent to truthful revelation being a weakly dominant strategy for each agent.

[28]It should be noted that no agent is indifferent between her endowment and any other object. I consider $h \in N \backslash \{i, j\}$ instead of the usual definition because otherwise this property is trivially satisfied.

condition is not imposed, *NEIE* would be incompatible even with *individual rationality*. Consider the housing market problem in Example 4.1. Agents 1 and 2 have identical endowments trivially and any assignment for which agent 2 does not envy agent 1 would not be *individually rational* for agent 1. Another fairness notion usually considered in literature is *equal treatment of equals* (*ETE*). A random assignment is said to satisfy *ETE* if two agents with identical endowments and preferences receive identical allocations. Formally, a random assignment $x \in \mathcal{X}$ satisfies *ETE* if for any $i, j \in N$ such that $\omega_i I_h \omega_j$ for all $h \in N \backslash \{i, j\}$ and $A(R_i) = A(R_j)$[29], then $x_i \sim_i x_j$. I require $A(R_i) = A(R_j)$ instead of $R_i = R_j$, as is usually the case, because there are no indifferences with endowment. Clearly, *NEIE* implies *ETE* but converse may not be the case in general. I say a random assignment mechanism satisfies *NEIE* (resp. *ETE*) if it always finds an assignment which satisfies *NEIE* (resp. *ETE*).

Yilmaz [33] introduced an alternative notion of fairness, *no justified-envy* (*NJE*). This notion of fairness allows an agent to envy another's assignment only if her assignment is *SD-IR* for the latter. Formally, for a random assignment $x \in \mathcal{X}$, agent i *justifiably envies* agent j if $x_i \not\succeq_i x_j$ and $x_i \succeq_j \omega_j$ i.e. agent i does not like her allocation at least as much as agent j's and x_i is *SD-IR* for agent j. A random assignment mechanism is said to satisfy *NJE* if it finds an assignment for which no agent *justifiably envies* any other agent. However, this criterion conflicts with *Pareto efficiency*, *strategy proofness* and *individual rationality* even for the case of strict preferences and single and discrete endowments. Consider the following example:

Example 4.2: Let $N = \{1, 2, 3\}$, $\omega = (a, b, c)$ and preference profile be:

R_1	R_2	R_3
b	a	b
c	b	c
a	c	a

Outcome of TTC would be (b, a, c) for this housing market problem. Note that agent 3 *justifiably envies* agent 1 because bP_3c and cP_1a. However, TTC is the only *Pareto efficient*, *strategy proof* and *individually rational* mechanism for housing market problem with strict preferences [17]. As it turns out, however, in setting of this chapter, this conflict does not arise. Additionally, I show that *no justified-envy* is satisfied even when agent i is allowed to *justifiably envy* agent j if $x_{j,a} > 0$ for

[29] By definition of trichotomous preferences, $\omega_j \notin A(R_i)$ and $\omega_i \notin A(R_j)$ because $A(R_i) = A(R_j)$.

some $a \in A(R_i)$, $x_{i,\omega_i} > 0$ and $x_i \succeq_j \omega_j$. This variation of *NJE* is clearly stronger than the notion introduced above and is similar to *ex-ante stability* as defined by Kesten & Unver [16] for random assignment solution to school choice problem.

Rules

In the recent years, several rules have been proposed for housing market problem while allowing for indifferences. Alcalde-Unzu & Molis [3] and Jaramillo & Manjunath [14] independently proposed *Top Trading Absorbing Sets* ($TTAS$) rule and *Top Cycles* (TC) rule, respectively. Both these rules are *Pareto efficient*, *weak core selecting* and *strategy proof.* Moreover, these rules select from the *core* whenever *core* is non-empty[3, 6]. Saban and Sethuraman [27] present a class of mechanisms: *common ordering on agents, individual ordering on objects* ($CAIO$). Each member in this class is *Pareto efficient*, *weak core selecting* and *strategy proof.* From this class of rules, they propose *Highest Priority Object Rule* (HPO) and show that it is computationally quick. Moreover, $TTAS$ and TC are members of $CAIO$.

Each member of $CAIO$ is an iterative algorithm where each iteration proceeds in three phases; *departure*, *pointing* and *trading.* In *departure phase*, a set of agents, N', and objects, O', may be chosen to depart, where each agent in N' holds some unique object in O' and each object in O' is held by some unique agent in N', if each agent in N' holds one of her most preferred objects (among the remaining ones) and all such objects are in O'. Each departing agent is assigned the object she is holding at that iteration of the algorithm. This phase ensures *Pareto efficiency* and *weak core selection.* The *departure phase* is repeated until no more agents and objects can be chosen to depart. In *pointing phase*, each agent i points at a unique agent holding one of her most preferred objects (among the remaining ones). In presence of indifferences, it is possible that more than one agents hold one of the most preferred objects (among the remaining ones) for agent i. For agent i, unique agent to point at is determined by using a linear ordering over agents and/or objects while satisfying some properties to ensure *strategy proofness*[30]. In *trading phase*, trades occur based on the cycles formed in the *pointing phase.*

Each member of $CAIO$ uses linear orderings over agents and/or objects[31]. Additionally, Saban

[30] Saban and Sethuraman [27] provide sufficient conditions (selection criterion for determining unique pointees) for *strategy proofness.*

[31] TC uses priority ordering of agents whereas $TTAS$ and HPO use priority ordering of objects

and Sethuraman [27] also show that these priority orderings do not need to be fixed and can be changed in a particular manner while still maintaining *strategy proofness*. However, I restrict attention to rules in $CAIO$ with fixed priority ordering structure. These rules use an ordering over agents and for each agent, an ordering over objects. The pointing phase of these rules proceeds as follows: Each unsatisfied agent points at the agent holding one of her most preferred objects (among the remaining ones). If there are more than one such objects, the unsatisfied agent points at whoever holds the highest priority object among such objects. Among satisfied agents who are not pointing yet and have one of their most preferred objects (among the remaining ones) held by some agent who is already pointing, the highest priority agent points at the agent holding one of her most preferred objects (among the remaining ones). If there are more than one such objects, the agent points at whoever holds the highest priority object among such objects[32].

Let $CAIO^f$ denote these rules from the class of $CAIO$ rules and let $\lhd$ be the set of all possible priority orderings. Each member $\prec \in \lhd$ consists of common ordering over agents where $i \prec j$ is interpreted as agent i having a higher priority than agent j and individual orderings over objects where $a \prec_i b$ represents that object a has a higher priority than object b for agent i. Then, for any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and $\prec \in \lhd$, outcome of rule φ associated with priority ordering $\prec$ is an $n \times n$ permutation matrix represented as $\varphi^{\prec}(R, \omega)$.

The rule I propose in this chapter takes a lottery over specific outcomes that are achievable for priority orderings in $\lhd$. For any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$ and $\varphi \in CAIO^f$, let $\varphi(R, \omega)$ be the set of all assignments achieved under the rule φ for priority orderings in $\lhd$. Formally,

$$\varphi(R, \omega) \equiv \left\{\varphi^{\prec}(R, \omega) : \prec \in \lhd\right\}$$

Among the assignments in $\varphi(R, \omega)$, I take a lottery over outcomes in $\varphi(R, \omega)$ which maximize number of agents receiving an acceptable object. For any $\alpha \in \mathcal{A}$, let $|\alpha|$ represent number of agents receiving an acceptable object under α i.e. $|\alpha| = |\{i \in N : \alpha_i \in A(R_i)\}|$. Let set of assignments with maximal number of agents receiving an acceptable object be represented as $\bar{\varphi}(R, \omega)$. Then,

$$\bar{\varphi}(R, \omega) \quad \equiv \quad \{\alpha \in \varphi(R, \omega) : |\alpha| \geq |\beta| \, \forall \beta \in \varphi(R, \omega)\}$$

The random assignment solution to the fractional housing market problem with single and

[32] Details of pointing phase can be found in Saban & Sethuraman [27].

discrete endowments, $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, is a lottery over assignments in $\bar{\varphi}(R,\omega)$ for some weights $\pi = (\pi_\alpha)_{\alpha \in \bar{\varphi}(R,\omega)}$ such that $\pi_\alpha \geq 0$ for each $\alpha \in \bar{\varphi}(R,\omega)$ and $\sum_{\alpha \in \bar{\varphi}(R,\omega)} \pi_\alpha = 1$. Then, random assignment rule for $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, rule $\varphi \in CAIO^f$ and lottery π is represented as $RA^{\varphi,\pi}(R,\omega)$ where:

$$RA^{\varphi,\pi}(R,\omega) = \sum_{\alpha \in \bar{\varphi}(R,\omega)} \pi_\alpha \alpha$$

Since each member of $CAIO$ is *Pareto efficient*, *individually rational* and *weak-core selecting*, RA is *ex-post efficient*, *ex-post IR* and belongs in *ex-post weak core*. I say a lottery π is a uniform distribution if for each $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, each assignment in $\bar{\varphi}(R,\omega)$ is given an equal weight under π. Formally, if π is a uniform distribution, then for each $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, $\pi_\alpha = \pi_\beta$ for each $\alpha, \beta \in \bar{\varphi}(R,\omega)$.

Results

In this section, I provide results for the random assignment rule proposed in the previous section. The rule proposed in the previous section takes a lottery over only specific outcomes of the rule from $CAIO^f$. It might be argued that taking lotteries over priority orderings could be a desirable solution. Since rules in $CAIO^f$ are *Pareto efficient*, this rule would be *ex-post efficient*. However, as shown in the next example, *ex-post efficiency* does not imply *SD-efficiency*, in general, even for the case of trichotomous preferences[33]. This is contrary to the result of Bogomolnaia & Moulin [10] for the two-sided matching problem under similar preference structure where they establish equivalence between *ex-post efficiency* and *ex-ante efficiency*.

Example 4.3. Let $N = \{1,2,3,4,5,6,7\}$, $O = \{a,b,c,d,e,f,g\}$ and $\omega = (a,b,c,d,e,f,g)$. Consider the following preferences:

R_1	R_2	R_3	R_4	R_5	R_6	R_7
be	c	df	a	f	cg	a
a	b	c	d	e	f	g
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then, the following assignments are *Pareto efficient* (even, *individually rational*): $\alpha^1 = (b,c,d,a,e,f,g)$,

[33] I would like to thank Vikram Manjunath for this counter-example.

$\alpha^2 = (e,b,c,d,f,g,a)$, $\alpha^3 = (c,b,d,a,f,c,g)$ and $\alpha^4 = (b,c,f,d,e,g,a)$. The utility vectors associated with these assignments, respectively, are as follows: $U^1 = (1,1,1,1,0,0,0)$, $U^2 = (1,0,0,0,1,1,1)$, $U^3 = (1,0,1,1,1,1,0)$ and $U^4 = (1,1,1,0,0,1,1)$. Now, consider the following random assignments: $x = 0.5\alpha^1 + 0.5\alpha^2$ and $y = 0.5\alpha^3 + 0.5\alpha^4$. The utility vectors corresponding to x and y are $U(R,x) = (1,0.5,0.5,0.5,0.5,0.5,0.5)$ and $U(R,y) = (1,0.5,1,0.5,0.5,1,0.5)$. It is obvious that random assignment x is not *SD-efficient* because it is dominated by the assignment y even though x is *ex-post efficient*.

As illustrated in Example 4.3, *ex-post efficiency* does not ensure *SD-efficiency*. This happens because number of agents receiving an acceptable object may differ for different *Pareto efficient* assignments. This is not the case for the two-sided matching model considered in Bogomolnaia & Moulin [10]. However, I establish that *RA* rule is *SD-efficient* because it is a lottery over *Pareto efficient* and *individually rational* assignments with most number of agents receiving an acceptable object. To that end, I first show that $\varphi \in CAIO^f$ can find every *Pareto efficient* and *individually rational* assignment for some priority ordering in $\lhd$.

Lemma 4.1. For any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$ and $\varphi \in CAIO^f$, if α is *Pareto efficient* and *individually rational*, there is $\prec \in \lhd$ such that $\varphi^{\prec}(R,\omega) = \alpha$.

Proof. Consider any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$. Let α be any *Pareto efficient* and *individually rational* assignment. Let G^α be a graph associated with α where each agent is represented by a node. Moreover, for each agent $i \in N$, there is a unique arc extending to agent j such that $\omega_j = \alpha_i$. Then, each agent receiving an acceptable object under α is part of a cycle. To reproduce α using $\varphi \in CAIO^f$, I generate priority orderings $\prec$ such that improving cycles[34] in G^α are generated in the first step of $\varphi^{\prec}(R,\omega)$. For each $i \in N$, let $\alpha_i \prec_i a$ for each $a \in O \backslash \{\alpha_i\}$. Then, each agent in an improving cycle in G^α points at the same agent as in G^α in the first step of $\varphi^{\prec}(R,\omega)$ since each agent in N is unsatisfied. So, all improving cycles in G^α occur in the first step of $\varphi^{\prec}(R,\omega)$. Note that once trades happen in accordance with the improving cycles, no other beneficial trades can occur because α is *Pareto efficient*. Remaining priority orderings can be arbitrarily assigned.■

Since every *Pareto efficient* and *individually rational* rule can be found using $\varphi \in CAIO^f$ and some priority ordering, the result regarding *SD-efficiency* of *RA* rule can be established.

Proposition 4.1. For any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and lottery π, $RA^{\varphi,\pi}(R,\omega)$ is *SD-efficient*.

[34] Cycles where at least one agent is receiving an acceptable object.

Proof. Consider any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and lottery π. Let $x \equiv RA^{\varphi,\pi}(R,\omega)$. On contrary, suppose that x is not *SD-efficient*. Then, there is $y \in \mathcal{X}$ such that $y \succeq_i x$ for each $i \in N$ and $y \succ_j x$ for some $j \in N$. Then, $\sum_{i\in N} U(R_i, y_i) > \sum_{i\in N} U(R_i, x_i)$. Note that for each $\alpha, \beta \in \bar{\varphi}(R,\omega)$, $|\alpha| = |\beta|$. Suppose $|\alpha| = m$ for each $\alpha \in \bar{\varphi}(R,\omega)$. Then, $\sum_{i\in N} U(R_i, x_i) = m$. Hence, $\sum_{i\in N} U(R_i, y_i) > m$. Then, it must be the case that y gives positive probability to some *individually rational* assignment $\beta \in \mathcal{A}$ such that $|\beta| > m$. By Lemma 4.1, it must be the case that $\beta \in \bar{\varphi}(R,\omega)$ which is a contradiction. ■

Next, I show that *SD-IR* is equivalent to *ex-post IR* for the fractional housing market problem with single and discrete endowments under trichotomous preferences.

Proposition 4.2. For any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, a random assignment is *SD-IR if and only if* it is *ex-post IR*.

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Now, we show that the converse is true as well. Consider any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$. Let x be a random assignment that satisfies *ex-post IR*. Then, x can be represented as a lottery over deterministic *individually rational* assignments. Then:

$$x = \sum_{l=1}^{T} \pi_l \alpha^l$$

where for each l we have $\pi_l \geq 0$, $\alpha^l \in \mathcal{A}$ such that $\alpha_i^l R_i \omega_i$ for each $i \in N$ and $\sum_l \pi_l = 1$. In other words, $W_i\left(R_i, \alpha_i^l\right) = 1$ for each $i \in N$ and each $l \in \{1, \cdots, T\}$. Then, we have $W_i(R_i, x_i) = 1$ for each $i \in N$ which completes the proof. ■

Since $RA^{\varphi,\pi}(R,\omega)$ is *ex-post IR* for each lottery π over assignments in $\bar{\varphi}(R,\omega)$, following result can be stated:

Corollary 4.1. For any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and lottery π over assignments in $\bar{\varphi}(R,\omega)$, $RA^{\varphi,\pi}(R,\omega)$ is *SD-IR*.

Now, I establish that RA rule is *SD-strategy proof* for any lottery π. The proof utilizes the fact that no agent can decrease the number of agents receiving an acceptable object under RA rule and whenever she successfully increases the number of agents receiving an acceptable object, it must be the case that she receives an unacceptable object.

Proposition 4.3. For any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and uniform distribution π, $RA^{\varphi,\pi}(R,\omega)$ is *SD-strategy proof*.

Proof. On contrary, suppose there is $i \in N$, $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, $R'_i \in \mathcal{T}$, $\varphi \in CAIO^f$ and uniform distribution π such that $x \not\succeq_i x'$ where $x \equiv RA^{\varphi,\pi}(R, \omega)$, $x' \equiv RA^{\varphi,\pi}(R', \omega)$ and $R' = (R_{-i}, R'_i)$. For each $\alpha \in \bar{\varphi}(R, \omega)$ and $\alpha' \in \bar{\varphi}(R', \omega)$, let $|\alpha| = m$ and $|\alpha'| = m'$. It should be obvious that m' is the number of agents receiving an acceptable object under preference profile R'.

Since $x \not\succeq_i x'$, it must be the case that $U(R_i, x_i) < 1$ i.e. there is $\alpha \in \bar{\varphi}(R, \omega)$ such that $\alpha_i = \omega_i$. If α is *Pareto efficient* under R', then $m' \geq m$. Moreover, if α is not *Pareto efficient* under R', then there is $\alpha' \in \mathcal{A}$ such that α' Pareto dominates α under R' i.e. $m' > m$. Now consider the following cases:

Case 1. $m' > m$.

Consider any $\alpha' \in \bar{\varphi}(R', \omega)$. It cannot be the case that $\alpha'_i \in A(R_i)$ because α' is *individually rational* under R and so, $m \geq m'$. Then, for each $\alpha' \in \bar{\varphi}(R', \omega)$, $\alpha'_i \notin A(R_i)$ so that $x \succ_i x'$ which is a contradiction.

Case 2. $m' = m$.

Then, for any $\alpha' \in \bar{\varphi}(R', \omega)$ such that $\alpha'_i \in A(R_i)$, it must be the case that $\alpha' \in \bar{\varphi}(R, \omega)$, by Lemma 4.1, because α' is *Pareto efficient*[35] and *individually rational* under R. Let $S \equiv \{\alpha \in \bar{\varphi}(R, \omega) : \alpha_i \in A(R_i)\}$ and $S' \equiv \{\alpha' \in \bar{\varphi}(R', \omega) : \alpha'_i \in A(R_i)\}$. Then, $|S| \geq |S'|$. Moreover, for each $\alpha \in \bar{\varphi}(R, \omega)$ such that $\alpha_i = \omega_i$, it must be the case that $\alpha \in \bar{\varphi}(R', \omega)$, by Lemma 4.1, because α is *Pareto efficient*[36] and *individually rational* under R'. Similarly, for each $\alpha' \in \bar{\varphi}(R', \omega)$ such that $\alpha'_i = \omega_i$, it has to be the case that $\alpha' \in \bar{\varphi}(R, \omega)$, by Lemma 4.1, because α' is *Pareto efficient* and *individually rational* under R. Let $\tilde{S} \equiv \{\alpha \in \bar{\varphi}(R, \omega) : \alpha_i = \omega_i\}$ and $\tilde{S}' \equiv \{\alpha' \in \bar{\varphi}(R', \omega) : \alpha'_i = \omega_i\}$. Then, $\left|\tilde{S}\right| = \left|\tilde{S}'\right|$. Since each assignment in $\bar{\varphi}(R, \omega)$ is *individually rational*, $S \cup \tilde{S} = \bar{\varphi}(R, \omega)$. If all assignments in $\bar{\varphi}(R', \omega)$ are not *individually rational* for agent i, $x'_{i,a} > 0$ for some $a \notin A(R_i) \cup \{\omega_i\}$. Then, by assumption, $x \succ_i x'$ which is a contradiction. If all assignments in $\bar{\varphi}(R', \omega)$ are *individually rational* for agent i, $S' \cup \tilde{S}' = \bar{\varphi}(R', \omega)$. Since $|S| \geq |S'|$, $\left|\tilde{S}\right| = \left|\tilde{S}'\right|$ and π is a uniform distribution over assignments in $\bar{\varphi}(R, \omega)$ and $\bar{\varphi}(R', \omega)$ under preferences R and R', respectively, it must be the case that $x \succeq_i x'$. This concludes the proof. ■

Next, I show that any *ex-post IR* and *SD-efficient* random assignment belongs in the *SD-core* for trichotomous preferences. If there is a group of agents who can trade among themselves such that

[35] If α' is not *Pareto efficient* under R, it must be dominated by some assignment in $\mathcal{A}$. However, this suggests that $m' < m$.

[36] If α is not *Pareto efficient* under R', it must be dominated by some assignment in $\mathcal{A}$. This suggests that $m' > m$.

each agent in that group gets something better-off, then, by *ex-post IR*, each agent in this group receives a positive fraction of her endowment. Then, these agents could trade among themselves using fractions of their endowments so that each agent becomes better-off. This would contradict *SD-efficiency*.

Proposition 4.4. For any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$, any *ex-post IR* and *SD-efficient* random assignment is in *SD-core*.

Proof. Consider any $(R,\omega) \in \mathcal{T}^N \times \mathcal{A}$. Let $x \in \mathcal{X}$ be any *ex-post IR* (hence, *SD-IR*) and *SD-efficient* random assignment. On contrary, suppose x is not in *SD-core*. Then, there is $S \subseteq N$ and $y \in \mathcal{X}$ such that $\sum_{i\in S} y_i = \sum_{i\in S} \omega_i$ and $y \succ_i x$ for each $i \in S$. Since $y \succ_i x$, we have $U(R_i, x_i) < 1$ for each $i \in S$. Hence, $x_{i,\omega_i} > 0$ for each $i \in S$. Note that $\sum_{i\in S} y_{i,a} = 1$ for each $a \in \omega(S)$ and $\sum_{a\in\omega(S)} y_{i,a} = 1$ for each $i \in S$.

Let $c = \min_{i\in S} x_{i,\omega_i}$. Additionally, let ϵ be an $n \times n$ matrix such that for each $i \in N \backslash S$, $\epsilon_i = 0$ and for each $i \in S$, $\epsilon_i = c(y_i - \omega_i)$.

Let $z = x + \epsilon$. We show that $z \in \mathcal{X}$. Consider the following for any $a \in O$:

$$\begin{aligned}
\sum_{i\in N} z_{i,a} &= \sum_{i\in N\backslash S} z_{i,a} + \sum_{i\in S} z_{i,a} \\
&= \sum_{i\in N\backslash S} x_{i,a} + \sum_{i\in S} x_{i,a} + \sum_{i\in S} \epsilon_{i,a} \\
&= 1 + c\sum_{i\in S} (y_{i,a} - \omega_{i,a})
\end{aligned}$$

Consider the second term in the above expression. If $a \notin \omega(S)$, then $y_{i,a} = 0$ and $\omega_{i,a} = 0$ for each $i \in S$. Additionally, if $a \in \omega(S)$, then $\sum_{i\in S} y_{i,a} = \sum_{i\in S} \omega_{i,a} = 1$. So, we get $\sum_{i\in N} z_{i,a} = 1$ for each $a \in O$. Note that for any $i \in N\backslash S$, we have $z_{i,a} = x_{i,a}$ for each $a \in O$. So, $\sum_{a\in O} z_{i,a} = 1$ for each $i \in N\backslash S$. Now consider the following for any $i \in S$:

$$\begin{aligned}
\sum_{a\in O} z_{i,a} &= \sum_{a\notin\omega(S)} z_{i,a} + \sum_{a\in\omega(S)} z_{i,a} \\
&= \sum_{a\notin\omega(S)} x_{i,a} + \sum_{a\in\omega(S)} (x_{i,a} + \epsilon_{i,a}) \\
&= 1 + c\sum_{a\in\omega(S)} (y_{i,a} - \omega_{i,a})
\end{aligned}$$

For the second term in the above expression, $\sum_{a\in\omega(S)} y_{i,a} = \sum_{a\in\omega(S)} \omega_{i,a} = 1$. So, $\sum_{a\in O} z_{i,a} =$

1 for each $i \in N$. Now, we just need to show that $z_{i,a} \geq 0$ for each $i \in N$ and $a \in O$. By construction of z, $z_{i,a} = x_{i,a}$ for each $i \in N \backslash S$. So, $z_{i,a} \geq 0$ for each $i \in N \backslash S$ and $a \in O$. For any $i \in S$ and $a \notin \omega(S)$, $z_{i,a} = x_{i,a}$. So, $z_{i,a} \geq 0$ for each $i \in S$ and $a \notin \omega(S)$. For any $i \in S$ and $a \in \omega(S)$, consider the following:

$$\begin{aligned} z_{i,a} &= x_{i,a} + c\left(y_{i,a} - \omega_{i,a}\right) \\ &= x_{i,a} - c\omega_{i,a} + cy_{i,a} \end{aligned}$$

If $a \neq \omega_i$, then $\omega_{i,a} = 0$ and $z_{i,a} = x_{i,a} + cy_{i,a} \geq 0$. If $a = \omega_i$, $c\omega_{i,a} = c$ and $x_{i,\omega_i} \geq c$ because $c = \min_{j \in S} x_{j,\omega_j}$. Then, $z_{i,a} \geq 0$ for each $i \in S$ and $a \in \omega(S)$.

Therefore, $z_{i,a} \geq 0$ for each $i \in N$ and $a \in O$. Moreover, it cannot be the case that $z_{i,a} > 1$ for some $i \in N$ and $a \in O$ because $\sum_{i \in N} z_{i,a} = 1$ for each $a \in O$ and $\sum_{a \in O} z_{i,a} = 1$ for each $i \in N$. Hence, $z \in \mathcal{X}$.

Since $z_{i,a} = x_{i,a}$ for each $i \in N \backslash S$ and $a \in O$, $z \sim_i x$ for each $i \in N \backslash S$. Moreover, for each $i \in S$, we have the following:

$$\begin{aligned} U\left(R_i, z_i\right) &= U\left(R_i, x_i + c\left(y_i - \omega_i\right)\right) \\ &= U\left(R_i, x_i + cy_i\right) \\ &= U\left(R_i, x_i\right) + cU\left(R_i, y_i\right) \\ &> (1+c)\, U\left(R_i, x_i\right) \end{aligned}$$

We have $U\left(R_i, x_i + c\left(y_i - \omega_i\right)\right) = U\left(R_i, x_i + cy_i\right)$ because $\omega_i \notin A\left(R_i\right)$ for any $i \in N$. Moreover, since $c > 0$, $U\left(R_i, z_i\right) > U\left(R_i, x_i\right)$ for each $i \in S$ i.e. $z \succ_i x$ for each $i \in S$. This contradicts *SD-efficiency* of x and completes the proof. ■

Following is an immediate corollary of Proposition 4.4.

Corollary 4.2. For any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and lottery π over assignments in $\bar{\varphi}(R, \omega)$, $RA^{\varphi,\pi}(R, \omega)$ is in *SD-core*.

Proposition 4.4 is reminiscent of the result by Bogomolnaia and Moulin [10] where they show that a matching is *core stable if and only if* it is *efficient* and *individually rational* for two-sided matching under "dichotomous" preferences. The preferences in their paper are identical to the trichotomous preferences considered in this chapter. However, this equivalence result does not hold for the fractional housing market problem under single and discrete endowments for trichotomous preferences. Consider the following example:

Example 4.4: Let $N = \{1, 2, 3\}$, $O = \{a, b, c\}$ and $\omega = (a, b, c)$. Consider the following preferences:

R_1	R_2	R_3
bc	ac	ab
a	b	c

Let $R = (R_1, R_2, R_3)$ and $\alpha = (b, a, c)$. Note that α belongs in the *SD-core* because $U(R_1, \alpha_1) = U(R_2, \alpha_2) = 1$ i.e. agents 1 and 2 cannot be made better-off. However, α is clearly not *ex-post efficient* because $\beta = (c, a, b)$ Pareto dominates α.

Now, I consider the fairness notions introduced in Section 3. The next result shows that any *ex-post IR* and *SD-efficient* random assignment satisfies the stronger notion of *NJE* mentioned earlier.

Proposition 4.5. Consider any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$. An *ex-post IR* and *SD-efficient* random assignment satisfies *NJE*.

Proof. On contrary, suppose there are $i, j \in N$ such that $x_{j,a} > 0$ for some $a \in A(R_i)$, $x_{i,\omega_i} > 0$ and $x_i \succeq_j \omega_j$. Since $x_i \succeq_j \omega_j$, it must be the case that $bR_j\omega_j$ for each $b \in O$ such that $x_{i,b} > 0$. Specifically, we have $\omega_i \in A(R_j)$. Let $y_{h,b} = x_{h,b}$ for each $h \in N \setminus \{i, j\}$ and $b \in O$. Let $c = \min\{x_{i,\omega_i}, x_{j,a}\}$. Then, set $y_{i,b} = x_{i,b}$ and $y_{j,b} = x_{j,b}$ for each $b \in O \setminus \{a, \omega_i\}$ and set $y_{i,a} = x_{i,a} + c$, $y_{i,\omega_i} = x_{i,\omega_i} - c$, $y_{j,a} = x_{j,a} - c$ and $y_{j,\omega_i} = x_{j,\omega_i} + c$. Then, $y \sim_h x$ for each $h \in N \setminus \{i, j\}$, $y \succeq_j x$ and $y \succ_i x$ which contradicts *SD-efficiency* of x. ■

Since *RA* rule satisfies *ex-post IR* and *SD-efficiency*, following result can be stated.

Corollary 4.3. For any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and lottery π over assignments in $\bar{\varphi}(R, \omega)$, $RA^{\varphi,\pi}(R, \omega)$ satisfies *NJE*.

Next, I show that *RA* rule satisfies *NEIE*. In the proof, I show that whenever agents have identical endowments, the rule satisfies *NEIE*.

Proposition 4.6. For any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and uniform distribution π over assignments in $\bar{\varphi}(R, \omega)$, $RA^{\varphi,\pi}(R, \omega)$ satisfies *NEIE*.

Proof. Consider any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$, uniform lottery π over assignments in $\bar{\varphi}(R, \omega)$ and $i, j \in N$ such that $\omega_i I_h \omega_j$ for each $h \in N \setminus \{i, j\}$ and $\omega_j \in A(R_i)$ *if and only if* $\omega_i \in A(R_j)$. Let $x \equiv RA^{\varphi,\pi}(R, \omega)$. We want to show that $x_i \succeq_i x_j$. Suppose $\omega_i \in A(R_j)$. Then, for each $\alpha \in \bar{\varphi}(R, \omega)$, $\alpha_i R_i \alpha_j$ by *Pareto efficiency*. So, $x_i \succeq_i x_j$.

Now, suppose that $\omega_j \notin A(R_i)$ and $\omega_i \notin A(R_j)$. If $x_{j,a} > 0$ for some $a \notin A(R_i) \cup \{\omega_i\}$, then $x_i \succ x_j$. Now suppose that $x_{j,a} > 0$ only if $a \in A(R_i) \cup \{\omega_i\}$ and $U(R_i, x_j) > U(R_i, x_i)$. Then, it has to be the case that there is $\alpha \in \bar{\varphi}(R, \omega)$ such that $\alpha_i = \omega_i$ and $\alpha_j \in A(R_i)$. Then, $\alpha_j \neq \omega_j$ and so, $\alpha_j \in A(R_j)$. We construct priority orderings, $\prec$, such that $\beta_i \in A(R_i)$ and $\beta_j = \omega_j$ where $\beta = \varphi^{\prec}(R, \omega)$. Suppose $|\gamma| = m$ for each $\gamma \in \bar{\varphi}(R, \omega)$. Let G^{α} be the graph associated with α where each agent is represented by a node. Moreover, for each agent $i' \in N$ a unique arc is extended to $j' \in N$ such that $\omega_{j'} = \alpha_{i'}$. Construct individual priority orderings over objects as follows: for each $i' \in N \backslash \{i, j\}$ with an arc extending to an agent in $N \backslash \{j\}$, $\alpha_{i'} \prec_{i'} a$ for each $a \in O \backslash \{\alpha_{i'}\}$. Then, each improving cycle which does not contain agent j occurs in pointing phase of step 1 of $\varphi^{\prec}(R, \omega)$. Now, consider the improving cycle containing agent j. For the agent pointing at agent j in G^{α}, say agent $j' \notin \{i, j\}$[37], $\omega_i \prec_{j'} a$ for each $a \in O \backslash \{\omega_i\}$. Then, agent j' points at agent i in step 1 because $\omega_j \in A(R_{j'})$ and $\omega_i I_h \omega_j$ for each $h \in N \backslash \{i, j\}$. Finally, for agent i, $\alpha_j \prec_i a$ for each $a \in O \backslash \{\alpha_j\}$. Then, agent i points at the agent whose endowment is α_j in the pointing phase of step 1 of $\varphi^{\prec}(R, \omega)$ because $\alpha_j \in A(R_i)$.

Note that the only difference in cycles formed under $\varphi^{\prec}(R, \omega)$ and improving cycles of G^{α} is the cycle containing agent j in G^{α}. Moreover, that cycle differs only in replacement of agent j with agent i. So, exactly m agents must be receiving an acceptable object in step 1 of $\varphi^{\prec}(R, \omega)$ and, by construction of $\bar{\varphi}(R, \omega)$, no more beneficial trades can occur after step 1 of $\varphi^{\prec}(R, \omega)$. Hence, $\varphi^{\prec}_j(R, \omega) = \omega_j$. So, for each $\alpha \in \bar{\varphi}(R, \omega)$ such that $\alpha_j \in A(R_i)$ and $\alpha_i = \omega_i$, there is $\beta \in \bar{\varphi}(R, \omega)$ such that $\beta_h = \alpha_h$ for each $h \in N \backslash \{i, j, j'\}$, $\beta_{j'} = \omega_i$, $\beta_i = \alpha_j$ and $\beta_j = \omega_j$ where $j' \in N$ is such that $\alpha_{j'} = \omega_j$. Additionally, for each $\gamma \in \bar{\varphi}(R, \omega)$ either $\gamma_j \notin A(R_i)$ or $\gamma_i \in A(R_i)$ i.e. $\gamma_i R_i \gamma_j$. Since π is a uniform distribution over assignments in $\bar{\varphi}(R, \omega)$, it must be the case that $x_i \succeq_i x_j$. This completes the proof. ■

Since *NEIE* implies *ETE*, following corollary can be stated.

Corollary 4.4. For any $(R, \omega) \in \mathcal{T}^N \times \mathcal{A}$, $\varphi \in CAIO^f$ and uniform distribution π over assignments in $\bar{\varphi}(R, \omega)$, $RA^{\varphi, \pi}(R, \omega)$ satisfies *ETE*.

The results presented in this section can be summarized in the following theorem.

Theorem 4.1. For fractional housing market problem with single and discrete endowments under trichotomous preferences, there exist rules which are *SD-IR*, *SD-efficient*, *SD-strategy proof*,

[37] This is the case because $\alpha_j \neq \omega_j$ and agent i is not part of an improving cycle in G^{α}.

SD-core stable and satisfy *NEIE*, *NJE* and *ETE*.

This states that under trichotomous preferences, the impossibility results of Yilmaz [33]; incompatibility of *individual rationality*, *no justified-envy* and *strategy proofness*, and Athanassoglou & Sethuraman [4]; any mechanism which satisfies *individual rationality*, *SD-efficiency* and *no justified-envy* cannot satisfy even *weak strategy proofness*, which hold under strict preferences and single and discrete endowments, do not hold. Neither do impossibility results of Aziz [5], (1) incompatibility of *SD-core stability* with *NJE*, and (2) *SD-IR*, *SD-efficiency* and *weak SD-strategy proofness*, which hold for strict preferences even for single endowments and allocations. Moreover, by relaxing assumption of fractional endowments and full preference domain to single and discrete endowments and trichotomous preferences, the following impossibility results of Athanassoglou and Sethuraman [4] no longer hold; (1) incompatibility of *individual rationality*, *SD-efficiency* and *strategy proofness*, and (2) any *individually rational* and *SD-efficient* mechanism cannot simultaneously satisfy *NJE* and *NEIE*.

Conclusion

I use rules in *common ordering on agents*, *individual ordering on objects* under the assumption of trichotomous preferences and single and discrete endowments, to show that several impossibility results for fractional housing market problem with single and discrete endowments, which hold even under strict preferences, can be avoided.

A particular concern would be the computational complexity of the proposed rule because RA rule requires solving for all possible priority orderings and then selecting allocations with maximal number of agents receiving an acceptable object. Additionally, for some housing market problems, RA rule may not provide a random assignment solution. This could occur when there is a unique *Pareto efficient* assignment with maximal number of agents receiving an acceptable object. However, the goal of this chapter was to determine whether some impossibilities can be avoided for a random assignment solution to the housing market problem under a restricted setting.

A possible way to resolve computational complexity might be to design this problem as an *assignment problem* from linear programming. Details of *assignment problems* in various settings can be found in [11, 20]. However, additional constraints would be required for *SD-IR* and *NEIE*. Since this chapter provides evidence of existence of solutions satisfying these properties, the *assignment* *problem* with appropriate constraints would have a solution for the setting studied in this chapter. Moreover, by Propositions 4.4 and 4.5, this solution also satisfies *SD-core stability* and *NJE*. However, *strategy proofness* might be difficult to achieve using linear programming. Another possible solution could be to design the fractional housing market problem with single and discrete endowments as a multi-objective optimization problem. Marler & Arora [18] present a brief survey on multi-objective optimization methods which is a good starting point for designing such problems.

CHAPTER V

SUMMARY

In this chapter, I summarize the research presented in this dissertation. I consider housing market problem in various settings. In housing market problem, each agent owns at most one object and each object is owned by at most one agent. Agents have preferences over objects. I consider the case where agents may be indifferent among objects.

In Chapter II, I show that some existing rules, in addition to several desirable properties, also satisfy *no justified-envy for agents with identical endowments*. I also provide sufficient conditions for *weak group strategy proofness*. Using one of these conditions, I show that *Top Trading Absorbing Sets* [3] rule satisfies *weak group strategy proofness*. This result shows that even though *Pareto efficiency* and *group strategy proofness* are incompatible for weak preferences [12], *Pareto efficiency* and *weak group strategy proofness* can be achieved simultaneously. Then, I propose a procedural enhancement, *prioritized treatment of market-equal unsatisfied agents*. This property directs how trading cycles are selected at each step of the algorithm. It requires that when certain conditions are satisfied[38], the highest priority unsatisfied agent among *market-equal unsatisfied agents* receives one of her most preferred objects (among the remaining ones). This property might be of importance when agents need to be treated in accordance with their social ranking. I show that some existing rules do not, in general, satisfy this property. So, I propose a new rule, *Modified Top Cycles* rule, which satisfies this property in addition to other desirable results.

Chapter III is a note on Saban & Sethuraman [27]. Using *Modified Top Cycles* rule, I was able to identify an oversight in sufficient condition for *strategy proofness* provided in their paper. I present conditions which rectify this issue.

In Chapter IV, I explore random assignment solution to the housing market problem. For general and strict preferences, several impossibility results have been established. I consider a restricted class of preferences, trichotomous preferences. Under these preferences, each agent finds an object (other than her endowment) to be acceptable or unacceptable. Agents are indifferent between all acceptable objects and also unacceptable objects. Each agent prefers each acceptable object to her endowment and prefers her endowment to each unacceptable object. These preferences are identical to dichotomous preferences considered by Bogomolnaia & Moulin [10]. I show that for this class

[38]These conditions ensure that *strategy proofness* is not violated.

of preferences, there are rules which satisfy *efficiency*, *strategy proofness*, *core stability*, *no-envy for agents with identical endowments*, *no justified-envy* and *equal treatment of equals*. Hence, several impossibility results do not hold under the assumption of trichotomous preferences for fractional housing market problem where endowments are single and discrete.

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