DERANGEMENTS IN A FERRERS BOARD

An Undergraduate Research Scholars Thesis

by

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TABLE OF CONTENTS

ABSTRACT

Derangements in a Ferrers Board. (May 2014)

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The classic derangement question of counting the number of derangements for n objects from some initial permutation of the objects was first considered by de Montfort in 1708. A particular recasting of a permutation allows us to place any permutation onto an $n \times n$ board, from which certain properties of derangements may be understood. This research extends the classic derangement question to the more general Ferrers board, which is an $n \times n$ board with a missing section λ in the lower-right corner. Various properties of the derangement numbers for these more general boards are stated and proven in the course of this work.

DEDICATION

I dedicate this research to my family, whose constant support is ever invigorating.

ACKNOWLEDGMENTS

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CHAPTER I

BACKGROUND AND INTRODUCTION

The study of rearrangements of objects is a classic exercise. I propose to consider the rearrangements of certain structures in a mathematically rigorous fashion. To explain the concepts involved in the question, I present some notation and definitions forthwith.

Definitions and Beginning Notions

We begin by letting S denote the set $S = \{1, 2, 3 \ldots, n\}$, where n is a positive integer. We now present two definitions.

Definition 1 A *permutation* of the set S is a rearrangement of the elements of S. Permutations can be considered to be *words* of length n, where a permutation $p = p_1 p_2 \ldots p_n$, and p_1, p_2, $\ldots$, p_n are the elements of S in some order.

Definition 2 A *derangement* of the set S is a permutation of S such that no element is in its original position. For example, 1 will not be in the first position of the permutation, 2 will not be in the second position, $\ldots$, n will not be in the nth position.

It is a classical question to ask: For a given positive integer n, how many derangements, D_n, does it have? We can see that $D_1 = 0$, $D_2 = 1$ (corresponding to the permutation 21), and $D_3 = 2$ (corresponding to the permutations 231 and 312). There is a general formula for D_n that can be attained by using the Inclusion-Exclusion Principle.

The classic derangement question can be formulated on an $n \times n$ board, where n is a positive integer. A *permutation* on such an $n \times n$ board is a placement of points on the board such that no two points are in the same row or column, and every row and column contains one point. This is equivalent to placing n nonattacking rooks onto the board, as a rook can only move along rows and columns, so if one has two rooks in the same row or column, they must necessarily be attacking one another. Given an initial permutation σ_0, a *derangement* is a permutation on the $n \times n$ board which shares no common squares with σ_0. From this formulation, the classic derangement properties can be easily computed.

The classic question is an example of the class of problems of forbidden position permutations, *i.e.* questions relating to permutations on chessboards, given that certain squares on the chessboard are forbidden.

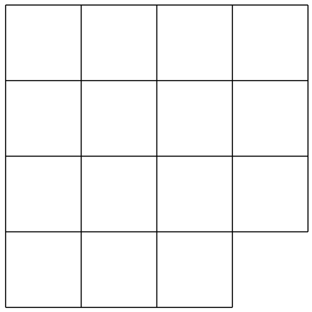
There exist results that are able to compute the number of derangements from a given permutation, if the forbidden positions of the board are known. However, these results are only computationally useful if certain properties of the forbidden positions are easier to calculate than the number of derangements themselves.

The Ferrers Board

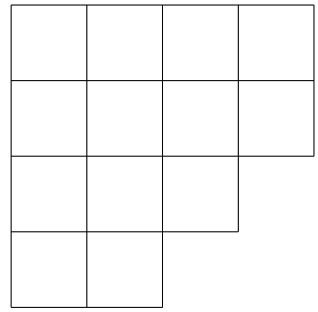
This document will investigate the derangement question on a more general class of boards known as *Ferrers Boards*. Given an $n \times n$ board as above, let $r_1, r_2, \ldots, r_i, \ldots, r_n$ denote the ith row of the board, where r_1 is the bottom row. Further, let $r'_1, r'_2, \ldots, r'_i, \ldots\ r'_n$ denote the number of squares in row r_i of the board. Then a Ferrers board B has the property that $1 \leq r'_1 \leq r'_2 \leq \ldots r'_n = n$. Thus, a Ferrers board is an $n \times n$ board with a section λ missing in the lower right corner, and so we may write $B = (n \times n) - \lambda$. Permutations and derangements on a Ferrers board are defined as in the classic case. If λ is a rectangle, then we shall refer to B as a *rectangular Ferrers board*; otherwise, B will be referred to as a *nonrectangular Ferrers Board* (see Figure I.2 below for examples).

With the above notation, note that if $\lambda = \emptyset$ (if no squares are forbidden), then the Ferrers board is equivalent to the board used in the classical case.

In the course of this document, it will be shown that, unlike the classic derangement, the enumeration of the derangements in the more general class of Ferrers board is dependent on σ (for a fixed $\lambda \neq \emptyset$). However, a particular bijection is obtained when λ is rectangular, showing that the derangement number for this class of boards is dependent on n, but independent of σ. It will also be shown that for *all* Ferrers boards, a particular injection is preserved,



A rectangular Ferrers Board



A nonrectangular Ferrers Board

Fig. I.2.: Examples of rectangular and nonrectangular Ferrers Boards

so that the number of derangements from any permutation can be bounded by the number of derangements resulting from two particular well-defined permutations on a Ferrers board. Finally, the derangement numbers will be enumerated in several ways for the case of a rectangular Ferrers board.

CHAPTER II

BOUNDING THE DERANGEMENT NUMBERS OF FERRERS BOARDS

Bounds for the Derangement Number on a Ferrers Board

In the classic derangement case, it is known that the derangement number is independent of the initial permutation and only dependent on n. This is reflected in the enumeration of the derangements as [4]:

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

In this paper, we will show that the derangement number for a general Ferrers Board is dependent on n, λ, and in certain cases the given permutation, σ. However, in any case, we will show that the derangement number is bounded by the derangement number of two well-defined permutations. To demonstrate this bound, we will define two operations on the initial permutation.

Left-Right and Right-Left Movements

Denote by $\mathcal{S}(B)$ the set of all permutations on a Ferrers board B (with a given λ). We label the rows of B ascendingly as r_1, r_2, ..., r_i, ..., r_n and the columns from left-to-right as c_1, c_2, ..., c_j, ..., c_n. Denote the square in the ith row and the jth column as (r_i, c_j). Note that $\mathcal{S}(B) = \emptyset$ unless the board B contains $\{(r_d, c_d) | 1 \leq d \leq n\}$, the diagonal from the lower left to the upper right corner. We subsequently assume that B does contain that diagonal.

Definition 1 Let σ be a permutation on B. Let $(r_m, c_i), (r_k, c_j) \in \sigma$, where r_m is above r_k (*i.e.* $m > k$) and c_i is to the left of c_j (*i.e.* $i < j$). Then the *left-right movement* is a transformation $LR : \sigma \to \sigma'$, with respect to (r_m, c_i), (r_k, c_j) where σ' is another permutation on B, defined by the following (see Figure II.1 on the next page):

1. $LR((r_m, c_i)) = (r_m, c_j)$

2. $LR((r_k, c_j)) = (r_k, c_i)$

3. $LR(x) = x$ for all other $x \in \sigma$

$$\begin{array}{c|cc} & c_i & c_j \\ \hline r_m & X & \\ r_k & & X \end{array} \quad \longrightarrow \quad \begin{array}{c|cc} & c_i & c_j \\ \hline r_m & & X \\ r_k & X & \end{array}$$

Fig. II.1.: An LR Movement (note that the rows and columns are not necessarily adjacent)

We can then define the transformation $RL : \sigma' \to \sigma$, the *right-left movement*, as the inverse function of LR, with the additional caveat that if the RL transformation would move a point of σ' to a point in λ, then RL is undefined for that particular set of points.

We will also make the simple observation that given two points in any permutation σ, one can perform either a left-right or a right-left movement on the points, but not both simultaneously, as it is necessarily true that either $i < j$ or $j < i$.

Noncrossing and Nonnesting Permutations

Given a Ferrers board B, we define two particular permutations using the notation of the previous section: the noncrossing permutation, denoted NC_B, and the nonnesting permutation, denoted NN_B.

Definition 2 The noncrossing permutation will be defined algorithmically:

1. Let (r_1, c_i) be the unique point such that $(r_1, c_i) \in B$, but $(r_1, c_{i+1}) \notin B$. Then, $(r_1, c_i) \in NC_B$.

2. For all subsequent rows r_m do the following:

   (a) If $r'_m > r'_{m-1}$, then find $(r_m, c_{i_m}) \in B$ such that $(r_m, c_{i_m+1}) \notin B$. Then, $(r_m, c_{i_m}) \in NC_B$.

(b) If $r'_m = r'_{m-1}$, and $(r_{m-1}, c_q) \in NC_B$, then find the maximal $p < q$ such that $(r_j, c_p) \notin NC_B$ for any $j \in \{1, 2, \ldots, m-1\}$. Then, $(r_m, c_p) \in NC_B$.

Definition 3 The nonnesting permutation is defined as $NN_B = \{(r_i, c_i) | 1 \leq i \leq n\}$. This corresponds to the diagonal from the bottom left to the upper right on B.

Remark These definitions come from the well-known noncrossing and nonnesting matchings. There is a well-known bijection between the set of Ferrers Boards of side length n and the set of matchings of cardinality $2n$; see [1] for instance.

Connecting LR, RL, NC_B and NN_B

We establish two lemmas that connect the previous notions together.

Lemma 1 The noncrossing permutation is a permutation such that initially only left-right movements can be performed on its elements. Similarly, the nonnesting permutation is a permutation such that initially only right-left movements can be performed on its elements.

Proof We begin with NC_B. Let r_m, r_k be two rows with $m > k$. If $r'_m = r'_k$, then there are c_i, c_j such that (r_m, c_i), $(r_k, c_j) \in NC_B$, with $i < j$ (by Definition 2). But this fulfills the requirements of Definition 1 for a LR movement, and so in this case only LR movements can be made. On the other hand, if $r'_m > r'_k$, and $(r_m, c_i), (r_k, c_j) \in NC_B$ then either $i < j$ or $j < i$. If $j < i$, then suppose that (r_k, c_i) is not in the forbidden area λ. Every row below r_k also contains a point in c_i by definition of a Ferrers board and by extension so does every column between c_j and c_i. By comparison to Definition 2, it follows that $(r_1, c_q) \in NC_B$ for some $q > i$. But since $(r_k, c_j) \in NC_B$ and $j < i$ by assumption, at some row below r_k the algorithmic definition of NC_B would yield c_i as the maximal column not already used. Hence, there would be some row r_p with $p < k$ such that $(r_p, c_i) \in NC_B$. But this is a contradiction to $(r_m, c_i) \in NC_B$, so we conclude that $(r_k, c_i) \in \lambda$. Since $(r_k, c_i) \notin B$, by Definition 1 for a RL movement, no RL movement can be performed, and hence no movement of any kind can be made on these points. If on the other hand, $j < i$, then clearly the conditions for Definition 1 of the LR movement are satisfied, so an LR movement can be performed on these points. Hence, only LR movements can be initially performed on the NC_B permutation.

For the NN_B permutation, let $(r_m, c_m), (r_n, c_n) \in NN_B$, with $m < n$. Then, by comparison to Definition 1, the conditions for a LR movement are nowhere satisfied, but the conditions for a RL movement are satisfied if $(r_m, c_n) \in B$. Thus, only RL movements can be initially performed on the NN_B permutation. ■

Lemma 2 Let M be any permutation on B. Then, M can be attained by a sequence of LR movements starting from NC_B. Similarly, M can also be attained by a sequence of RL movements starting from NN_B.

Proof We present an algorithm for NC_B to attain any other permutation $M = \{(r_m, c_{i_m}) | 1 \leq m \leq n\}$. This algorithm will operate from $m = 1$ to $m = n$. For r_1:

1. Let $(r_1, c_{j_1}) \in NC_B$ By definition, it is the rightmost cell of r_1. If $c_{i_1} = c_{j_1}$, then move onto r_2 as described below. Otherwise, go to step 2.

2. There is a set of columns C_{1j} between c_{i_1} and c_{j_1}. Take the maximal element less than j_1, say k, so that $(r_{i_k}, c_k) \in NC_B$ for some row r_{i_k}. Then, do:

    (a) $LR((r_1, c_{i_1})) = (r_1, c_k)$

    (b) $LR((r_{i_k}, c_k)) = (r_{i_k}, c_{i_1})$

3. With the new element (r_1, c_k), continue to form the set of columns to the right of c_{i_1} and to the left of c_k to find a maximal column and point in the new permutation σ' created following the first LR movement. As in step 2, perform the LR movement on the given two points. Continue to do this until the point (r_1, c_{i_1}) is achieved. then proceed to r_2 as described below.

For the rows r_m, $m \geq 2$:

1. In general, apply the algorithm as above. However, when computing between c_{j_m} and c_{i_m} (where (r_m, c_{j_m}) is a point attained at some time during the performance of the algorithm on the mth row), ignore columns c_{i_a}, where $a < m$, so that r_a is below r_m.
2. The LR movement for a general point (r_m, c_b) with point (r_c, c_d) where $d < b$ is the maximal (rightmost) column such that $m < c$ is given by:
   (a) $LR((r_m, c_b)) = (r_m, c_d)$
   (b) $LR((r_c, c_d)) = (r_c, c_b)$
3. Continue this procedure until a permutation with the points (r_1, c_{i_1}), …, (r_m, c_{i_m}) is attained. Then do the same procedure for the row r_{m+1}.

After a point (r_m, c_{i_m}) is attained in the above algorithm, it is subsequently fixed during the succeeding operations. Hence, since (r_1, c_{i_1}) can be initially attained, all points are subsequently attained, and so any permutation M can be generated from this algorithm. Further, by Lemma 1, NC_B is a permutation such that only LR movements can be initially performed on it, and every step in the algorithm is defined to be a LR movement, so that after one instance of the algorithm is performed, the points above the current row remain fixed so that only LR movements can be performed on those points in the upper rows. We therefore have that any permutation M can be generated from a sequence of LR movements starting from NC_B.

The algorithm for NN_B is similar. The algorithm computes a minimal leftmost column greater than the current column with a row greater than the current row at the given stage of the process. Then, an RL movement is performed on the two generated points, and the same process is applied onto the new generated permutation until the desired (r_m, c_{i_m}) is achieved for a given row r_m, and then as in the algorithm for NN_B, those points remain fixed. Hence, any permutation M can be achieved by a sequence of these RL movements from NN_B, as Lemma 1 gives that one can perform this operation and all rows above the current row remain so that they only perform RL movements with the other rows above the current one. ■

We easily attain the following corollary:

Corollary 1 The nonnesting permutation can be achieved following a sequence of left-right movements starting from the noncrossing permutation, and the noncrossing permutation can be achieved following a sequence of right-left movements from the nonnesting permutation.

Producing a Bound for the Derangement Number

Let $M \in \mathcal{S}(B)$. We use $d_B(M)$ to denote the number of derangements from a permutation M on a given board B. We now present and prove a general bound for $d_B(M)$.

Theorem 1 If $M \in \mathcal{S}(B)$, and NC_B and NN_B are the noncrossing and nonnesting permutations defined previously, then the following inequality holds:

$$d_B(NC_B) \leq d_B(M) \leq d_B(NN_B).$$

First, we will establish the following lemma:

Lemma 3 Let $(r_m, c_i), (r_k, c_j) \in B$ with $m > k$ and $i < j$. Let $M, M' \in \mathcal{S}(B)$ so that M contains (r_m, c_i) and (r_k, c_j), and M' be formed by $LR((r_m, c_i), (r_k, c_j))$. Let $D_B(M)$ denote the set of all derangements on B from the permutation M, and let $D_B(M')$ denote the set of all derangements on B from the permutation M'. Then, in general, there is an injection from $D_B(M)$ to $D_B(M')$. It follows that $d_B(M) \leq d_B(M')$.

Proof We intend to define an injective map $D_B(M) \to D_B(M')$, and demonstrate that in certain cases the map fails to be surjective.

We divide the derangements of $D_B(M)$ into three subcases: S_1, the set of all derangements which do not contain the points (r_k, c_i) and (r_m, c_j); S_2, the set of all derangements which contain both (r_k, c_i) and (r_m, c_j); and S_3, the set of all derangements which contain exactly one of these points. There are two subsets of S_3: S_{3a}, the subset of all derangements which contain (r_k, c_i), but not (r_m, c_j); and S_{3b}, the subset of all derangements which do contain (r_m, c_j), but not (r_k, c_i). We define a map $\phi : B - M \to B - M'$ as follows:

Set $\phi(\rho) = \rho$ for all derangements $\rho \in S_1$.

For $\rho_2 \in S_2$, set

$$\begin{cases} \phi((r_k, c_i)) = (r_m, c_i) \\ \\ \phi((r_m, c_j)) = (r_k, c_j) \\ \\ \phi(x) = x, \text{ for all other } x \in \rho_2. \end{cases}$$

Let $(r_m, c_e), (r_d, c_j)$ be in an arbitrary derangement ρ_3 of S_3. Consider derangements in S_{3a} which do not have corresponding derangements in S_{3b} (i.e. those where $(r_k, c_e) \in \lambda$). Then, we define a function

$$\begin{cases} \phi((r_m, c_e)) = (r_m, c_e) \\ \\ \phi((r_k, c_i)) = (r_d, c_j) \\ \\ \phi((r_d, c_j)) = (r_d, c_i). \end{cases}$$

Similarly, for those derangements where $\{(r_n, c_e), (r_d, c_i)\}$ is in S_{3b}, but $(r_d, c_j) \in \lambda$, then we set

$$\begin{cases} \phi((r_k, c_e)) = (r_m, c_e) \\ \\ \phi((r_d, c_i)) = (r_k, c_j) \\ \\ \phi((r_m, c_j)) = (r_d, c_i). \end{cases}$$

Now, consider those derangements where $(r_k, c_e), (r_d, c_i) \in B - M$. These contain derangements from both S_{3a} and S_{3b}. We write a function

$$\begin{cases} \phi((r_m, c_e)) = (r_m, c_i) \\ \\ \phi((r_k, c_i)) = (r_k, c_e) \\ \\ \phi((r_d, c_j)) = (r_d, c_j). \end{cases}$$

and

$$\begin{cases} \phi((r_k, c_e)) = (r_m, c_e) \\ \\ \phi((r_d, c_i)) = (r_k, c_j) \\ \\ \phi((r_m, c_j)) = (r_d, c_i). \end{cases}$$

For all other points $x \in \rho_3$ (in any of the subcases), set $\phi(x) = x$.

We show that ϕ defined in this way is injective. To see this, firstly note that the sets S_1, S_2, S_3 are disjoint, and that the ϕ function only operates on the row and column associated with the LR movement. Thus, it is enough to check that the restrictions of ϕ on S_1, S_2, and S_3 are themselves injective. But this is easily satisfied: Let ρ, $\rho' \in S_i$ for any i, so that $\rho \neq \rho'$. If ρ and ρ' differ in points other than those changed by the functional definitions, then those differences necessarily remain fixed, so that $\phi(\rho) \neq \phi(\rho')$. If, on the other hand, they differ in one of the changed points (in the case of S_3), then since the functional definitions all interchange among row and column, we conclude that the differences in row or column in the preimage are fixed in the image. Thus, $\phi(\rho) \neq \phi(\rho')$, and so ϕ is an injective map.

Because the RL movement is the inverse of the LR movement, we see that the right-left movement is injective with respect to these cases. However, this does not describe the entire space of possible derangements given the RL movement, as there is one other kind of derangement that cannot be created in the left-right structure. These are the derangements corresponding to the following: suppose that $(r_d, c_j), (r_d, c_e), (r_k, c_e) \in \lambda$. Then, there are derangements containing $(r_d, c_i), (r_k, c_j), (r_m, c_e)$ (refer to Figure II.2 below; note that the X's are points of M', and the bullets are points of the derangement). But, if a RL movement is performed, there is no possible way to rearrange the elements of the squares given to create a different derangement in $D_B(M')$ (since the only squares then available are $(r_d, c_i), (r_k, c_i), (r_m, c_j), (r_n, c_e)$). Thus, if we assign these derangements to derangements in $D_B(M)$, then its assignment will necessarily be equivalent to the assignment of a different derangement in $D_B(M')$. Thus, such a defined function would not be injective, and so the original LR movement is not surjective when the above case exists. We therefore conclude that $d_B(M) \leq d_B(M')$ for all cases, and that equality does not necessarily hold. ■

Proof of Theorem 1 By Lemma 2, any permutation M can be generated from a sequence of left-right moves beginning with NC_B, that $d_B(NC_B) \leq d_B(M)$, and again from Lemma 2 any permutation M can be achieved from a sequence of right-left moves starting from NN_B that $d_B(M) \leq d_B(NN_B)$. Thus, $d_B(NC_B) \leq d_B(M) \leq d_B(NN_B)$. ■

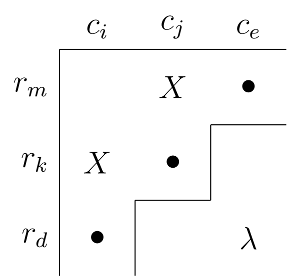


Fig. II.2.: The Failure of the Bijection of ϕ

There is now a corollary which will aid the enumeration of the derangement numbers.

Corollary 2 If λ is rectangular, then $d_B(M)$ is independent of the initial choice of M (corresponding to a property in the classic derangement case). The following converse also holds: if n is sufficiently large, and $d_B(M)$ is independent of M, then λ must be rectangular.

Proof The proof to Lemma 3 demonstrates that the function ϕ is bijective unless each of $(r_d, c_j), (r_d, c_e), (r_k, c_e) \in \lambda$. But if λ is rectangular, then also $(r_k, c_j) \in \lambda$, which is impossible, since by assumption $(r_k, c_j) \in B$. Thus, ϕ is bijective if λ is rectangular, and so $d_B(M)$ is independent of the choice of M.

To prove the converse, we require that $\text{card}(\mathcal{S}(B)) \geq 2$ (if there are only 0 or 1 permutations on B, then every permutation necessarily has the same number of derangements).

We first derive a condition for λ to be rectangular. Namely, we have that if $r'_1 = r'_2 = \cdots = r'_i < r'_{i+1} = r'_{i+2} = \cdots = r'_n$ for some i such that $1 \leq i \leq n$, then λ is rectangular. This is true, as the length of the bottom i rows is r'_1, and the length of the top $n-i$ rows is r'_n, whence $\lambda = i \times (n - r'_1)$, which is rectangular.

Suppose that λ is nonrectangular. The contrapositive of the above condition implies that if λ is nonrectangular, then there is some r_f such that $r'_1 < r'_f < r'_n$. For ease of notation, set $r'_f = g$. Consider the set of points $(r_f, c_g), (r_f, c_n), (r_1, c_g), (r_1, c_n)$. By construction, $(r_f, c_g) \in B$. However, since $r'_1 < r'_f$, we have that $(r_1, c_g), (r_1, c_n) \in \lambda$, and since $r'_f < r'_n$, we have that $(r_f, c_n) \in \lambda$. This set of points therefore satisfies the case where the bijection of ϕ in Lemma 3 fails to hold, so $d_B(M)$ cannot be constant for all M. $\blacksquare$

CHAPTER III

SPECIFIC ENUMERATION OF DERANGEMENTS IN FERRERS BOARDS WITH A MISSING RECTANGULAR SECTION

Enumeration of Derangements in Ferrers Board with a missing Rectangular Section

The exact enumeration of the derangement numbers may be described as a rook placement problem. For a Ferrers board as we have defined it, the forbidden positions are the squares in $\lambda \cup \sigma$, where λ is the forbidden area (since we will subsequently assume that λ is rectangular, we set $\lambda = r \times s$, where $r, s \geq 0$), σ the initial permutation. There is a well-known theorem connecting the rook coefficients r_k and the number of permutations avoiding the forbidden positions (the theorem can be found in [4]; the result is due to Kaplansky and Riordan in [3]).

Theorem 2 Let N_0 be the number of ways to avoid a forbidden position on an $n \times n$ board. Then, the following equality holds:

$$N_0 = \sum_{k=0}^{n} r_k (-1)^k (n-k)!$$

where the r_k are the aforementioned rook coefficients, *i.e* the number of ways to place k nonattacking rooks onto the forbidden positions of a board (that is, choose k of the forbidden positions such that no two so chosen are in the same row or the same column). This theorem follows from the Principle of Inclusion-Exclusion. We will investigate specifically the cases where λ is rectangular.

Rook Coefficients for a Rectangular λ

We will prove the following result regarding the rook coefficients.

Proposition 4 Let r_k be the kth rook coefficient as defined previously. Then,

$$r_k = \sum_{i=0}^{k} \binom{r}{i}\binom{s}{i} i! \binom{n-2i}{k-i}$$

Proof The two sections of the forbidden area, λ and σ, are disjoint, and so we may place points on λ first, and then points on σ. Suppose that we place i points on λ and $k-i$ points on σ. The number of ways to place i points on λ is denoted r_λ^i, analogously to the other rook coefficients. Then, note that each of the i points on λ "attacks" two points of σ (they attack a particular row and a particular column). Since no two of the i points on λ can be in the same row or column, there are then $2i$ points on σ that are attacked. The number of ways to choose $k-i$ points on σ from $n-2i$ possible points is given by the binomial coefficient $\binom{n-2i}{k-i}$. Hence, summing over all possible i, we have that

$$r_k = \sum_{i=0}^{k} r_\lambda^i \binom{n-2i}{k-i}$$

We now need to merely compute r_λ^i. But this is easily done. Since $\lambda = r \times s$ is rectangular, choose i of the r rows and i of the s columns on which to place points. If the rows are fixed, then there are $i!$ possible permutation of the columns with those rows. Hence, $r_\lambda^i = \binom{r}{i}\binom{s}{i} i!$, and so

$$r_k = \sum_{i=0}^{k} \binom{r}{i}\binom{s}{i} i! \binom{n-2i}{k-i}$$

as desired. ■

We will now compute the generating function for the rook coefficients $\sum_{k=0}^{n} r_k x^k$.

Proposition 5 Let $R(x) = \sum_{k=0}^{n} r_k x^k$ be the ordinary generating function for the rook coefficients as above. Then,

$$R(x) = \sum_{k=0}^{n} r_k x^k = (1+x)^n \sum_{i=0}^{\min(r,s)} \binom{r}{i}\binom{s}{i} i! \left(\frac{x}{(1+x)^2}\right)^i$$

for sufficiently large n.

Proof Using the formula for the rook coefficients and some well-known generating functions, we obtain

$$R(x) = \sum_{k=0}^{n} r_k x^k$$

$$= \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{r}{i}\binom{s}{i} i! \binom{n-2i}{k-i} x^k$$

$$= \sum_{i=0}^{n} \binom{r}{i}\binom{s}{i} i! x^i \sum_{k=i}^{n} \binom{n-2i}{k-i} x^{k-i}$$

$$= \sum_{i=0}^{n} \binom{r}{i}\binom{s}{i} i! x^i (1+x)^{n-2i}$$

$$= (1+x)^n \sum_{i=0}^{n} \binom{r}{i}\binom{s}{i} i! \left(\frac{x}{(1+x)^2}\right)^i$$

For $i > \min(r,s)$, note that $\binom{r}{i}\binom{s}{i} = 0$, so the sum is 0 for such i. Hence, the desired equality is attained. ■

The Direct Computation of the Derangement Numbers for Rectangular λ

Let $d_{n,r,s}$ denote the number of derangements on a Ferrers board defined as $n \times n - (r \times s)$. By direct comparison to Theorem 2, it is easy to see that

$$d_{n,r,s} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n-2i}{k-i} (-1)^k (n-k)!$$

Here $d_{n,r,s}$ is written out to emphasize that λ is rectangular. We will present an equivalent form of this sum. However, to do so, we introduce a new kind of derangement, analogous to the classical derangement.

For $q \leq p$, let $D_{p,q}$ denote the number of permutations of length p such that the first q numbers satisfy the derangement property: *i.e.* 1 is not in the first position, 2 is not in the second position, ..., q is not in the qth position. Then, we have the following lemma:

Proposition 6 $\quad D_{p,q} = \sum_{i=0}^{q} (-1)^i \binom{q}{i} (p-i)!$

Proof This is a direct computation from the principle of inclusion-exclusion, and closely resembles the computation for the classic derangement numbers. Let P_i be the property that $a_i = i$ (where each permutation is written $a_1 a_2 \dots a_p$). Suppose that j of the P_i are satisfied. Then, there are $\binom{q}{j}$ ways to select j properties, and $(p-j)!$ ways to permute the other elements. Hence, by PIE, summing over all possible j, we obtain

$$D_{p,q} = \sum_{j=0} (-1)^j \binom{q}{j} (p-j)!$$

as desired. ■

We will now establish an equivalent form of $d_{n,r,s}$ in terms of $D_{p,q}$.

Proposition 7 $\quad d_{n,r,s} = \sum_{i=0}^{\min(r,s)} (-1)^i i! \binom{r}{i} \binom{s}{i} D_{n-i,n-2i}$ for sufficiently large n.

Proof We compare to the form given above and show the two are equivalent. To wit:

$$d_{n,r,s} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{r}{i} \binom{s}{i} i! \binom{n-2i}{k-i} (-1)^k (n-k)!$$

$$= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! \sum_{k=i}^{n} \binom{n-2i}{k-i} (-1)^k (n-k)!$$

$$= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! \sum_{k=0}^{n-i} \binom{n-2i}{k} (-1)^{k+i} (n-(k+i))!$$

$$= \sum_{i=0}^{n} \binom{r}{i} \binom{s}{i} i! (-1)^i D_{n-i,n-2i}$$

As before, if $i > \min(r,s)$, then $\binom{r}{i}\binom{s}{i} = 0$, so the sum can be displayed in the desired form if $n \geq \min(r,s)$. ■

This form is interesting because the numbers $D_{p,q}$ have an easily computable exponential generating function, similar to the classical derangement numbers.

A Recurrence for the Derangement Numbers

In this subsection, we present a linear recurrence and partial differential equation satisfied by the derangements numbers $d_{n,r,s}$ associated with the Ferrers board $B = (n \times n) - (r \times s)$.

Proposition 8 The derangement numbers $d_{n,r,s}$ satisfy the following linear recurrence:

$$\begin{aligned} d_{n,r,s} &= (n-r-s-1)(d_{n-1,r,s} + d_{n-2,r,s}) \\ &\quad + s(d_{n-1,r,s} + d_{n-2,r,s-1}) \\ &\quad + r(d_{n-1,r-1,s} + d_{n-2,r-1,s}) \end{aligned}$$

with $d_{n,r,s} = 0$ if $n \leq 1$ or $r, s < 0$ or $n < r+s$.

Proof As λ is rectangular, by Corollary 2 the number of derangements from any starting position on B is constant. Therefore, we are free to choose any permutation on B from which to compute the derangement number, so we choose the permutation NC_B. Let n be sufficiently large so that $d_{n,r,s} \neq 0$ (it is enough to take $n \geq r+s$). In reference to Figure III.1, we know that $(r_n, c_1) \in NC_B$. By momentarily ignoring the row r_n and the column c_1, we can reduce the board B to a board $B' = (n-1) \times (n-1) - r \times s$, with the property that $NC_B = NC_{B'} \cup \{(r_n, c_1)\}$. Let ρ be any derangement on B. From the definition of a derangement, ρ necessarily has a point in the column c_1 and a distinct point in the row r_n. The other $n-2$ points of ρ are located on B'. To establish the recurrence, we will proceed by casework based on the location of the point in the c_1 column. Again, from Figure III.1, note that the noncrossing permutations NC_B and $NC_{B'}$ follow the same (definitional) pattern: there are r points proceeding diagonally (corresponding to the side of length r of λ); directly above the rows of these points are s points proceeding diagonally (corresponding to the side of length s of λ); and the remaining points are again diagonal and to the left of the first r points of the noncrossing permutations. Hence, when placing a point in the c_1 column, we may place the point in one of the first r rows; in one of the next s rows; or in one of the final $n-r-s-1$ rows. These are the three cases that we will now individually examine.

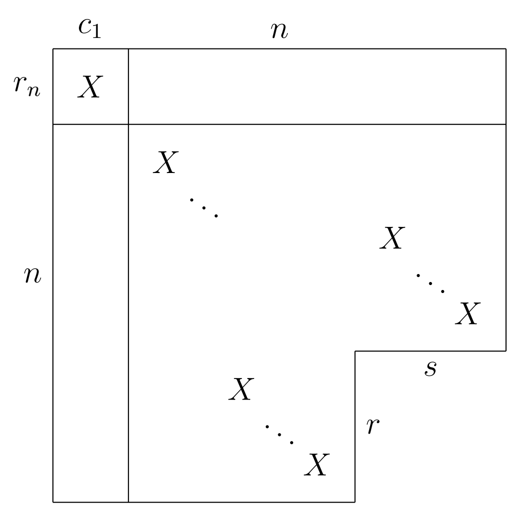


Fig. III.1.: NC_B on a board with forbidden area $r \times s$

In the first case, suppose that the point in the c_1 column is placed in one of the top $n-r-s-1$ rows, say r_α. We perform a reduction similar to the one to produce B'. In this case, we remove the c_1 column and the r_α row, leaving a $B_1 = (n-1) \times (n-1) - r \times s$ board with $n-2$ points of the permutation (see Figure III.2). We therefore have a Ferrers board of side length $n-1$ but $n-2$ permutation points. In particular, there is a column c_a which does not contain as an element a point of the noncrossing permutation NC_{B_1}, and we also have that $(r_\alpha, c_a) \in NC_B$. We investigate the point (r_n, c_a). We can do one of two (disjoint) things with this point: we can choose to place (r_n, c_a) in the derangement ρ (of B), or we can choose not to do so. If we place $(r_n, c_a) \in \rho$, then we can perform a similar reduction of the board, and attain a $(n-2) \times (n-2) - r \times s$ board with a permutation containing $n-2$ points. In this case, there are (by hypothesis) a total of $d_{n-2,r,s}$ derangements. On the other hand, should we choose to establish $(r_n, c_a) \notin \rho$, then we can treat this point as forbidden, and form a permutation with $n-1$ points on B_1. In this case, there are $d_{n-1,r,s}$ derangements, so in total we have $d_{n-1,r,s} + d_{n-2,r,s}$ derangements. As our initial choice r_α was arbitrary, and there are $n-r-s-1$ possible rows that could be selected, we have that there are $(n-r-s-1)(d_{n-1,r,s} + d_{n-2,r,s})$ derangements in this case.

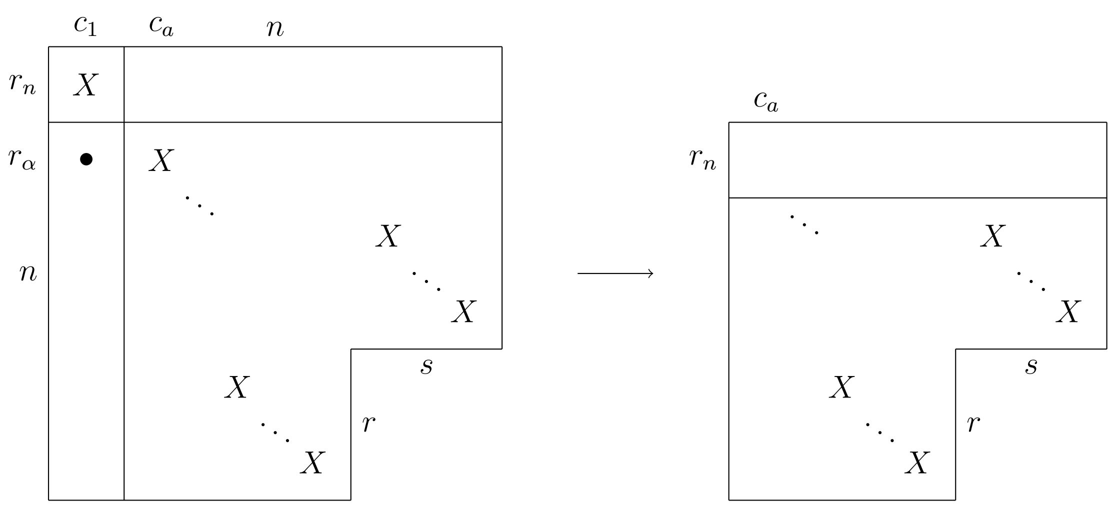


Fig. III.2.: The reduction described in the first case

Now, suppose that the point in the c_1 column is placed in one of the middle s rows, say r_β. We can again perform a reduction on the c_1 column and the r_β row, leaving a board $B_2 = (n-1)\times(n-1) - r\times s$ with $n-2$ points of the permutations. As above, if $(r_\beta, c_b) \in NC_B$, then we locate the point (r_n, c_b). The c_b column is one of the s columns placed on top of λ. We again consider two cases: those where $(r_n, c_b) \in \rho$, where ρ is a derangement of B, and those derangements ρ where $(r_n, c_b) \notin \rho$. If we let $(r_n, c_b) \in \rho$, then we can apply a similar reduction, with one additional caveat: we are also removing one of the s columns of λ, leaving a new forbidden area $r \times (s-1)$ (and a Ferrers board of side length $n-2$ with a permutation of $n-2$ points contained in it). Hence, there are $d_{n-2,r,s-1}$ derangements in this subcase. On the other hand, if we choose $(r_n, c_b) \notin \rho$, then we can treat (r_n, c_b) as forbidden (and part of the permutation), leaving B_2 with a permutation of $n-1$ points, or $d_{n-1,r,s}$ derangements in this case. Combining the subcases, we have a total of $d_{n-2,r,s-1} + d_{n-1,r,s}$ derangements for the row r_β, and since we are allowed to choose one of s rows, we have a total of $s(d_{n-2,r,s-1} + d_{n-1,r,s})$ derangements for this case.

Finally, suppose that the point in the c_1 column is placed in one of the bottom r rows, say r_γ. When we perform the standard reduction to a board B_3 of side length $n-1$, we also remove one of the r rows of λ, leaving a forbidden section $(r-1)\times s$, so that $B_3 = (n-1)\times(n-1) - (r-1)\times s$, with a permutation of $n-2$ points placed on B_3. If $(r_\gamma, c_c) \in NC_B$, then we locate the point (r_n, c_c). The c_c is one of the r columns to the immediate left of the forbidden section $(r-1)\times s$, so this case proceeds much like the first one. For derangements ρ of B, we have two choices: either $(r_n, c_c) \in \rho$, or $(r_n, c_c) \notin \rho$. If $(r_n, c_c) \in \rho$, then we can perform the standard reduction to a Ferrers board of side length $n-2$, with a forbidden section $(r-1)\times s$ and a permutation of $n-2$ points, giving $d_{n-2,r-1,s}$ derangements in this case. Similarly, if we choose $(r_n, c_c) \notin \rho$, then we can treat (r_n, c_c) as a forbidden point, and thus as the $(n-1)$th point of the permutation on B_3. In this case, there are $d_{n-1,r-1,s}$ possible derangements, for a total of $d_{n-2,r-1,s} + d_{n-1,r-1,s}$ derangement for the row r_γ. Hence, since there are r choices for the row, we have a total of $r(d_{n-2,r-1,s}+d_{n-1,r-1,s})$ derangements in this case.

Combining these three cases gives the desired recurrence. ■

Although the linear recurrence has been established, there does not seem to be a simple formula for the ordinary multivariate generating function $D(x, y, z) = \sum_{n,r,s\geq 0} d_{n,r,s} x^n y^r z^s$. Nevertheless, there is a partial differential equation which such a generating function would have to satisfy.

Proposition 9 Let $D(x, y, z)$ be defined as above. Then, $D(x, y, z)$ satisfies the following partial differential equation:

$$\begin{aligned} D(x,y,z) = &\left(x^2 + \frac{1}{xy} + \frac{1}{x^2 y} + \frac{1}{x^2 z}\right) D(x,y,z) \\ &+ \left(\frac{1}{x^2} + \frac{1}{x^3}\right) \frac{\partial D}{\partial x} \\ &+ \left(\frac{1}{xy^2} + \frac{1}{x^2 y^2} - \frac{1}{xy} - \frac{1}{xy^2}\right) \frac{\partial D}{\partial y} \\ &+ \left(\frac{1}{x^2 z^2} - \frac{1}{x^2 z}\right) \frac{\partial D}{\partial z} \end{aligned}$$

Sketch of Proof Rewrite the recurrence from Lemma 8 so that there are terms with the coefficients $n-1$, r and s. From the initial values of $d_{n,r,s}$, we compute

$$\frac{\partial D}{\partial x} = \sum_{n,r,s\geq 0} n d_{n,r,s} x^{n-1} y^r z^s = \sum_{n,r,s\geq 0} (n-1) d_{n-1,r,s} x^{n-2} y^r z^s$$

$$\frac{\partial D}{\partial y} = \sum_{n,r,s\geq 0} r d_{n,r,s} x^n y^{r-1} z^s = \sum_{n,r,s\geq 0} (r-1) d_{n,r-1,s} x^n y^{r-2} z^s$$

$$\frac{\partial D}{\partial z} = \sum_{n,r,s\geq 0} s d_{n,r,s} x^n y^r z^{s-1} = \sum_{n,r,s\geq 0} (s-1) d_{n,r,s-1} x^n y^r z^{s-2}$$

We also have that $D(x, y, z) = \sum_{n,r,s\geq 0} d_{n,r,s} x^n y^r z^s = \sum_{n,r,s\geq 0} d_{n-2,r,s} x^{n-2} y^r z^s$.

For each term in the recurrence from Lemma 8, we can then derive a corresponding value from among the four equations used above. The final result is equivalent to the desired form above. ■

CHAPTER IV

CONCLUSIONS AND FUTURE DIRECTIONS

In the course of this document, we have proven some interesting properties of derangements on Ferrers Board with a well-defined missing section λ. Specifically, we have been able to enumerate the derangement numbers in several interesting ways in the case that λ is rectangular, and to provide a well-defined bound that, in any case, describes the variation of the derangement numbers on a Ferrers board. This bound is very interesting, as it defines the extremal cases for the derangement numbers very well, with notions that are widely applicable and understood in the literature.

In the future, we may consider the permutations on the Ferrers board as a Markov process, or a process where new permutations may be created depending on the current permutation on the Ferrers board. Using this notion, we can compute the stationary distribution of the permutations, or specifically, the probability of a particular permutation being present on the Ferrers board at some time in the future. The stationary distribution of the classic case of permutations on an $n \times n$ board is already known in [2]; I hope to find an extension of the results in the case of a Ferrers board.

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