MONOTONE SEQUENCES IN COMBINATORIAL STRUCTURES

A Dissertation

by

MITCHELL ALLEN PHILLIPSON

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Catherine Yan
Committee Members,	Eric Rowell
	Zoran Šunić
	Sergiy Butenko
Head of Department,	Emil Straube

May 2015

Major Subject: Mathematics

ABSTRACT

Symmetry of monotone sequences arise in many combinatorial structures, the classical examples being inversions and coinversions in permutations. Another example is crossings and nestings in matchings, partitions and permutations. These examples can be generalized to fillings of Ferrers diagrams and further generalized to moon polyominoes. This dissertation first introduces layer polyominoes, then extends the joint symmetry between northeast and southeast chains exhibited in moon polyominoes.

For a given structure it's not always true that symmetry of crossings and nestings holds. We introduce a type of matching, called an alternating matching, where the distribution of crossings and nestings is not symmetric. We prove a necessary and sufficient condition for an alternating matching to be non-nesting and use this to partially enumerate non-nesting alternating matchings.

Finally, we prove several results on crossings and nestings in graphs. First we show that the crossing number and nesting numbers are unrelated, i.e. there are families of graphs with no crossings and with nestings numbers that diverge and vice versa. Second we give a bijection between plane trees and bi-colored motzkin paths. Lastly, we provide a generating function for a special class of Ferrers diagrams, where each row a fixed length shorter than the previous row, and the filling of the diagram has no southeast chains.

TABLE OF CONTENTS

LIST OF TABLES

TABLE Page

LIST OF FIGURES

FIGURE Page

1. INTRODUCTION

This thesis is divided into four chapters. The first provides relevant definitions, context and history. The second contains new work in fillings of layer polyominoes. The third introduces alternating matchings and discusses results in alternating matchings with no nestings. The final chapter contains a collection of miscellaneous results such as crossings and nestings in graphs, a bijection from plane trees to Motzkin paths, and fillings of Ferrers diagrams with fixed step. This introductory chapter will detail the results of the final three chapters, without definitions or context.

We begin with the problem of enumerating chains in fillings of layer polyominoes. Unlike moon polyomoninoes or Ferrers diagrams, there are two distinct types of chains that can appear in a layer polyomino, strong chains and regular chains. Regular chains extend two results first proved for 01-fillings of moon polyominoes. The first results says for fixed row sum and at most one non-zero entry per column the rows of a layer polyomino can be arbitrarily permuted, preserving the number of northeast and southeast chains. Second, for fixed row sum the number of a 01-fillings with no northeast chains is the same as in any permutation of the rows. In addition to these, we give conditions on row and column sums so that $\mathbb{N}$-fillings of layer polyominoes exist and show that the filling with no northeast chains in unique. Unlike regular chains, strong chains are not necessarily preserved when the rows are permuted; however, symmetry results still hold. For fillings of layer polyomionoes we develop recursive bijection that breaks the layer polyomino into rectangles, allowing us to extend a maps on fillings of rectangles. This framework has allowed us to prove several results. First, in 01-fillings of a layer polyomino, with fixed row sum and at most one non-zero entry per column, the joint distribution of strong northeast and strong southeast chains is symmetric. Second, again in 01-fillings of layer polyominoes, with fixed row sum, the number of fillings with no strong northeast chains is the same as with no strong southeast chains. Finally, in $\mathbb{N}$-fillings, with fixed row and column sum, the joint distribution of strong northeast and strong southeast chains is symmetric. Unlike regular chains, it's not true that $\mathbb{N}$-fillings with no strong northeast chain are unique for given layer polyomino with fixed row and column sums.

Alternating matchings are a special type of matching where each part has both an even and odd element. Every non-crossing matching is also a non-crossing alternating matching; however, not every non-nesting matching is a non-nesting alternating matching. This leads to the interesting occurrence that crossings and nestings are not equi-distributed in alternating matchings. We modify a bijection between non-nesting permutations and bi-colored motzkin paths, originally stated by Sylvie Corteel [4], to a subset of alternating matchings that includes all non-nesting alternating matchings. We give two disjoint conditions on bicolored Motzkin paths that guarantee the corresponding alternating matching is non-nesting. These conditions are used to classify a subset of non-nesting alternating matchings, alternating matchings that correspond to single cycle permutations, which can be used to construct all non-nesting alternating matchings.

The final three results are smaller than the rest. The first is on crossings and nestings in graphs. An ordering on the vertices of a graph allows us to define crossings and nestings. The total number of crossings and nestings is highly dependent on the the ordering, to counteract this is to take the minimum over all orderings. Using this definition we show that trees and graphs with exactly one cycle have both crossing and nesting number zero. Additionally, the complete graph on n vertices, K_n, has crossing and nesting numbers $\sum_{i=1}^{n-2} \binom{n-2}{2}$. We also explore the relationship between the crossing and nesting numbers. Given a graph G with no crossings, we construct a family of graphs G_n so that each G_n has no crossings. Additionally, if G has at least one nesting then each G_n has at least n nestings. We explicitly construct two graphs, G and H, so that G has no crossing and one nesting and H has no nestings and one crossing. The graphs G and H imply that the crossing an nesting numbers are unrelated, as neither is bound by the other. The second is a bijection between plane trees and bicolored Motzkin paths. The final is on fillings, with no southeast chains, of Ferrers diagrams where each row is a fixed length shorter than the previous row. We give a recursive formula for the generating function enumerating over the number of non-zero entries and the number of rows. We explicitly solve this for step size 1 which corresponds to non-crossing (unordered) simple graphs.

2. HISTORY AND PRELIMINARIES

Monotone sequences originated in the study of inversions and co-inversions in permutations and was later extended to crossings and nestings in various structures. This chapter introduces key concepts and historical results on crossings and nestings in several combinatorial structures, then extends these results to monotone sequences (northeast and southeast chains) in polyominoes.

Section 2.1 defines important paths and a result due to Flajolet on continued fraction generating functions. Section 2.2 contains results on crossings and nestings in matchings. Section 2.3 extends the results on matchings to partitions. Section 2.4 further extends these results to crossings and nestings in permutations. Finally Section 2.5 defines polyominoes, fillings of polyominoes and two special types of polyominoes, Ferrers diagrams and moon polyominoes.

2.1 Paths

This section is dedicated to defining the types of paths that will be used throughout this document and state an extremely important result by Flajolet [8]. Flajolet's result gives a continued fraction generating function for classes of weighted Motzkin paths. In this section we define weighted Motzkin paths, state a simplified version of Flajolet's result and use this result to enumerate several different types of paths.

Definition 2.1. A *Motzkin* path of length n is a word, $u = u_1 u_2 \ldots u_n$ on the alphabet $\{\mathrm{N}, \mathrm{E}, \mathrm{S}\}$ so that for $1 \leq j \leq n$,

$$|u_1 \ldots u_j|_{\mathrm{N}} \geq |u_1 \ldots u_j|_{\mathrm{S}} \qquad\qquad |u_1 \ldots u_n|_{\mathrm{N}} = |u_1 \ldots u_n|_{\mathrm{S}}$$

where $|x|_c$ denotes the number of occurrences of c in x.

This definition is very abstract, a more concrete version of a Motzkin path of length n is path from $(0,0)$ to $(n+1,0)$ with steps N $=(1,1)$, E $=(1,0)$ and S $=(1,-1)$ so that the path stays above the $x-$axis. Figure 2.1 shows the Motzkin path $M=$ NENNSESS.

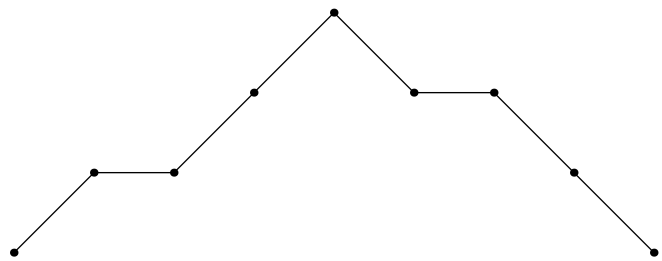


Figure 2.1: The path NENNSESS

Definition 2.2. The *height* at step i of a Motzkin path $M = m_1m_2\ldots m_n$ is

$$h_i = |m_1\ldots m_i|_{\mathrm{N}} - |m_1\ldots m_i|_{\mathrm{S}}$$

In a diagram of a Motzkin path the height of an edge is the y-coordinate of the left vertex, Figure 2.2 has a breakdown of the heights of the path in Figure 2.1.

Definition 2.3. A *weighted* Motzkin path is a pair (M,w) where M is a Motzkin path of length n and w is a word of length n, corresponding to weights on the steps of M. Figure 2.3 has an example of a weighted path.

Using these definitions we can state a simplified version of Flajolet's result. Flajolet's full theorem deals with generating functions in non-commutative variables, which is an unnecessary generalization for our concerns, as all our variables commute.

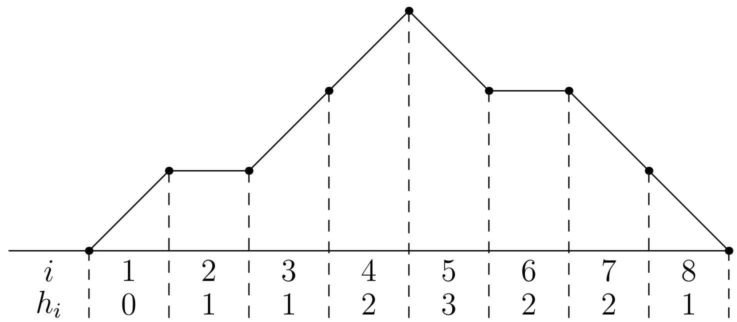


Figure 2.2: A breakdown of the heights of the path NENNSESS

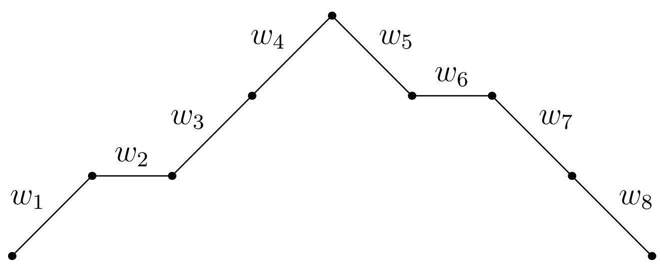


Figure 2.3: A Motzkin path with weights

Theorem 2.4 (Flajolet)**.** *Let $\{X_n\}_{n\geq 0}$ be a family of sets and for each $X \in X_n$ let $f(X)$ be the property we wish to enumerate. If for each n there is an injective map F_n from X_n to weighted Motzkin paths of length n, so that for each $X \in X_n$ the corresponding weight path $F_n(X) = (M, w = w_1 \dots w_n)$ satisfies*

$$\prod_{i=1}^{n} w_i = f(X).$$

Then, for all the paths in the image of this map, denote,

$$N_i = \textit{sum of all weights on N steps at height } i$$

$$S_i = \textit{sum of all weights on S steps at height } i$$

$$E_i = \textit{sum of all weights on E steps at height } i.$$

Then the continued fraction,

$$GF(x) = \sum_{n\geq 0} x^n \sum_{X\in X_n} f(X) = \cfrac{1}{1 - E_0 x - \cfrac{N_0 S_1 x^2}{1 - E_1 x - \cfrac{N_1 S_2 x^2}{\ddots}}} \tag{2.1}$$

converges and is well defined.

We now enumerate several simple examples as a demonstration of the previous Theorem.

2.1.1 Dyck Paths

Dyck paths, or Catalan paths, are a special case of Motzkin paths with no E steps. It's well known that the number of Dyck paths of length $2n$ is the n^{th} Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$ and the generating function is

$$D(x) = \sum_{n\geq 0} C(n)x^n = \frac{1-\sqrt{1-4x}}{2x} \tag{2.2}$$

However, we will use the method of Flajolet to arrive at this in a different way. Since we are only enumerating Dyck paths we can set the weights on each N and S step to 1 and E to 0, as there are no E steps. This makes $N_i = S_i = 1$ and $E_i = 0$.

Thus by Equation (2.1),

$$GF(x) = \cfrac{1}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{\ddots}}}$$

we can solve this explicitly to see

$$GF(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x^2}.$$

This means $GF(x) = D(x^2)$, where the x^2 comes from each Dyck path having even length, so the x^{2n-1} terms are all 0 in the continued fraction.

2.1.2 Motzkin paths

This is quite easy, we set all weigths to equal 1 so we have

$$GF(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - x - \cfrac{x^2}{\ddots}}}.$$

If we say M_n is the number of Motzkin paths of length n, then solving the above continued fraction explicitly, we see,

$$GF(x) = \sum_{n \geq 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

2.1.3 Bi-colored Motzkin Paths

A *bi-colored Motzkin Path* is a Motzkin path with an extra step $\bar{\mathrm{E}} = (1, 0)$ and the condition that $\bar{\mathrm{E}}$ can't appear at height 0. The weights will be given by $N_i = S_i = 1$ for all i, $E_0 = 1$ and $E_i = 2$ for $i > 0$ since for height $i > 0$ there are two possibilities for E steps. Then we have

$$GF(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - 2x - \cfrac{x^2}{1 - 2x - \cfrac{x^2}{\ddots}}}}.$$

If BM_n is the number of bi-colored Motzkin paths of length n, we see

$$GF(x) = \sum_{n \geq 0} BM_n x^n = \frac{2x^2}{-2x^3 + 2x^2 + 2x - 1 + \sqrt{1 - 4x}}.$$

This is significantly uglier than the previous generating functions, however it is still possible to solve.

2.1.4 Motzkin paths with k 'E' steps

Let $M_{n,k}$ be the number of Motzkin paths of length n with exactly k 'E' steps. For a given path set the weights so that $N_i = 1$, $S_i = 1$ and $E_i = u$. Then for any path the product of the weights is u^k, where k is the number of E steps. Then

$$GF(x,u) = \cfrac{1}{1 - ux - \cfrac{x^2}{1 - ux - \cfrac{x^2}{\ddots}}}$$

and when we solve explicitly,

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} M_{n,k} u^k x^n = \frac{1 - ux - \sqrt{(1-ux)^2 - 4x^2}}{2x^2}.$$

Notice that if we plug $u = 0$ into the above we get the generating function for Dyck paths, and if we use $u = 1$ we get the generating function for Motzkin paths. This should help confirm that our process is correct.

2.2 Matchings

A *matching* on the set $[n] = \{1, 2, \ldots, n\}$ is a decomposition of $[n]$ into subsets of size 2 so that all elements of $[n]$ are represented. This is actually a complete matching, however we only consider complete matchings. The set of all matchings on $[n]$ is denoted $\mathcal{M}_{2n}$.

Example 2.5. On [8], the set

$$\{(1,3), (2,4), (5,8), (6,7)\}$$

is a matching.

Definition 2.6. In a matching M a pair of vertices (i_1, j_1) and (i_2, j_2) is said to be a *crossing* if

$$i_1 < i_2 < j_1 < j_2,$$

and they are said to be a *nesting* if

$$i_1 < i_2 < j_2 < j_1.$$

The number of crossings in a matching M is denoted $\mathrm{cros}(M)$ and the number of nestings is $\mathrm{nest}(M)$.

One way to visualize this definition is to represent matchings as a graph on a number line. Given a matching M on $[n]$ create a graph on the number line with arcs the parts of the matching. By convention we draw the arcs above the number line. Figure 2.4 shows what this looks like using Example 2.5.

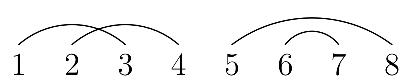
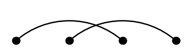


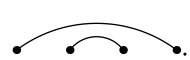
Figure 2.4: The graph of a matching

Using these graphs crossings and nestings are easily visualized. A crossing is an occurrence of



(2.3)

and a nesting is



(2.4)

Example 2.7. The matching M in Figure 2.4 has $\mathrm{cros}(M) = 1$ and $\mathrm{nest}(M) = 1$. And the matching M in Figure 2.5 has $\mathrm{cros}(M) = 2$ and $\mathrm{nest}(M) = 3$.

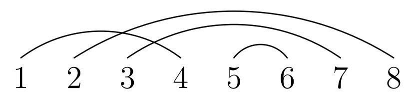


Figure 2.5: A second example of a matching.

These notions were introduced by De-Sainte Catherine in her 1983 Ph.D dissertation [7]. Using a bijective proof she showed the distribution of crossings and nestings in symmetric, or

$$\sum_{M\in\mathcal{M}_{2n}} p^{\mathrm{cros}(M)} = \sum_{M\in\mathcal{M}_{2n}} p^{\mathrm{nest}(M)}.$$

In 2006 Klazar [12] noticed that De-Sainte Catherine's proof implied the stronger result that the joint statistic $(\mathrm{cros}, \mathrm{nest})$ is symmetric. In other words, if

$$F(x,p,q) = \sum_{n\geq 0} x^n \sum_{M\in\mathcal{M}_{2n}} p^{\mathrm{cros}(M)} q^{\mathrm{nest}(M)}$$

then $F(x,p,q) = F(x,q,p)$. The remainder of this section applies the methods of Section 2.1 to find a continued fraction expansion of $F(x,p,q)$, which then implies this symmetry.

Let M be a matching and (i,j) be a pair in M with $i<j$, we call i an *opener* and j a *closer* in M. Construct a weighted Dyck path (D,w) with $D = d_1 \ldots d_{2n}$ and $w = w_1 w_2 \ldots w_{2n}$ so that

$$d_i = \begin{cases} \mathrm{N} & \text{if } i \text{ is an opener in } M \\ \mathrm{S} & \text{if } i \text{ is a closer in } M \end{cases}$$

and

$$w_i = \begin{cases} p^{\mathrm{cros}(M,i)} q^{\mathrm{nest}(M,i)} & \text{if } i \text{ is an opener in } M \\ 1 & \text{if } i \text{ is a closer in } M \end{cases}$$

where $\operatorname{cros}(M,i) = \#\{k \mid k < i < \ell < j, \text{ where } (i,j),(k,\ell) \in M\}$ and similarly $\operatorname{nest}(M,i) = \#\{k \mid k < i < j < \ell, \text{ where } (i,j),(k,\ell) \in M\}$. An example of this map is in Figure 2.6.

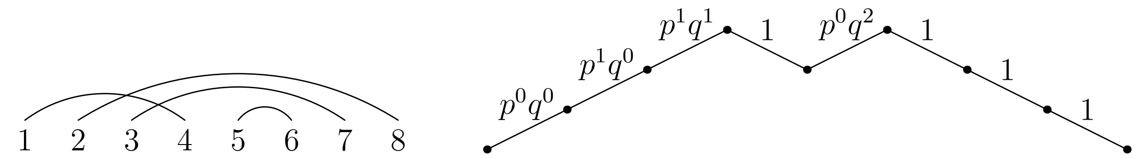


Figure 2.6: A matching with the corresponding Dyck path

Lemma 2.8. *For a matching M and the corresponding weighted Dyck path (D,w), if i is an opener in M and h_i is the height of the i^{th} step of the Dyck path, then*

$$h_i = \operatorname{cros}(i) + \operatorname{nest}(i)$$

This lemma implies that the image of this map is the set of $(D, w = w_1 \dots w_{2n})$ where $D = d_1 \dots d_{2n}$ is a Dyck path, if $d_i = \mathrm{N}$ then $w_i \in \{p^a q^b \mid a \geq 0, b \geq 0, a+b = h_i\}$ and if $d_i = \mathrm{S}$, $w_i = 1$. With a set image we can show the map is a bijection.

Let (D,w) be a weigthed Dyck path, construct a matching M as follows. Each N step in D corresponds to an opener in M and each S step a closer. To create the arcs read the weights right to left connecting each N step to the a^{th} available closer, where $w_i = p^a q^b$. Recall Figure 2.6 for an example.

Finally, it's clear that for a matching M and weighted Dyck path (D,w)

$$\prod_{i=1}^{n} w_i = p^{\operatorname{cros}(M)} q^{\operatorname{nest}(M)}.$$

Thus we have,

$$F(x,p,q) = \sum_{n\geq 0} x^n \sum_{M\in\mathcal{M}_{2n}} p^{\mathrm{cros}(M)} q^{\mathrm{nest}(M)} = \cfrac{1}{1-\cfrac{[1]_{p,q}x}{1-\cfrac{[2]_{p,q}x}{\ddots}}}.$$

Where $[n]_{p,q} = \dfrac{p^n - q^n}{p-q} = \sum_{i=0}^{n-1} p^i q^{n-i+1}$ and the x^2 has been changed to x, similar to the generating for Dyck paths in Section 2.1.1. Unlike the examples in the previous secton, this continued fraction has no known closed form.

This generating function exhibits several nice properties of crossing and nestings in matchings. First the joint distribution distribution $(\mathrm{cros}, \mathrm{nest})$ is symmetric, i.e. $F(x,p,q) = F(x,q,p)$. Second, the number of matchings with either no crossings or no nestings, i.e. $F(x,1,0)$ or $F(x,0,1)$, is $C(n)$, the n^{th} Catalan number. This is actually apparent from our bijection as for each Dyck path there is only one sequence of weights with either no p's or no q's.

2.3 Partitions

A *partition* is a collection of non-empty subsets of $[n]$, called blocks, which are mutually disjoint and have union equal to $[n]$. The set of partitions is denoted Π_n. If i and j are in the same block, we write $i \sim j$. A partition $\pi \in \Pi_n$ with k blocks is written $\pi = B_1 - B_2 - \cdots - B_k$ with the blocks ordered so that their minimal elements are increasing. Denote by $|\pi|$ the number of blocks in π, $\min(\pi)$ the minimal elements in each block of π and $\max(\pi)$ the maximal elements in each block of π.

Similar to matchings we can draw a partition as a graph on a number line. For a partition $\pi \in \Pi_n$ draw a graph on $[n]$ with arcs (i,j) where $i \sim j$ and $i < j$ so that there is no k with $i \sim k$ and $i < k < j$, and (i,i) if i is a singleton block. An example of the partition $\{\{1, 2, 4\}, \{3, 7\}, \{5, 8\}, \{6\}\}$ can be seen in Figure 2.7.

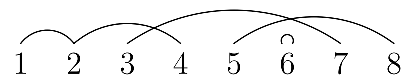
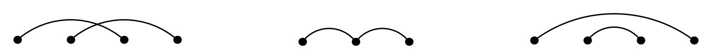


Figure 2.7: The graph of the partition $\{\{1, 2, 4\}, \{3, 7\}, \{5, 8\}, \{6\}\}$

For $\pi \in \Pi_n$ we say arcs (i_1, j_1) and (i_2, j_2) *cross* if $i_1 < i_2 \leq j_1 < j_2$ and *nest* if $i_1 < i_2 < j_2 < j_1$, these can be seen in Figure 2.8. The number of crossings in π is



(a) Cross type 1 (b) Cross type 2 (c) A nest

Figure 2.8: Crossings and nestings in partitions

denoted $\text{cros}(\pi)$ and the number of nestings is $\text{nest}(\pi)$.

For $\pi \in \Pi_n$ and $i \in [n] \setminus \max(\pi)$ define

$$\text{cros}(\pi, i) = \#\{j \mid j < i \leq a < b, \text{ where } i \sim b \text{ and } j \sim a\}$$

and

$$\text{nest}(\pi, i) = \#\{j \mid j < i < b < a, \text{ where } i \sim b \text{ and } j \sim a\}.$$

These are the number of crossings per arc in π.

Example 2.9. The partition, π, in Figure 2.7 has $\text{cros}(\pi) = 3$ and $\text{nest}(\pi) = 2$.

Our goal is determine the following generating function,

$$F(x, y, p, q) = \sum_{n\geq 0} x^n \sum_{\pi\in\Pi_n} y^{|\pi|} p^{\mathrm{cros}(\pi)} q^{\mathrm{nest}(\pi)}.$$

To find this generating function we are going to biject to weighted Motzkin paths. Let $\pi \in \Pi_n$ define a weighted Motzkin path (M, w) as follows. If $M = m_1 \dots m_n$ define

$$m_i = \begin{cases} \mathrm{N} & \text{if } i \in \min(\pi) \\ \mathrm{E} & \text{if } i \in \min(\pi) \cap \max(\pi) \text{ or } i \notin \min(\pi) \cup \max(\pi) \\ \mathrm{S} & \text{if } i \in \max(\pi) \end{cases}$$

and for $w = w_1 \dots w_n$,

$$w_i = \begin{cases} yp^{\mathrm{cros}(\pi,i)}q^{\mathrm{nest}(\pi,i)} & \text{if } i \in \min(\pi) \text{ or } i \in \min(\pi)\cap\max(\pi) \\ p^{\mathrm{cros}(\pi,i)}q^{\mathrm{nest}(\pi,i)} & \text{if } i \notin \min(\pi)\cup\max(\pi) \\ 1 & \text{if } i \in \max(\pi). \end{cases}$$

This implies that for each partition

$$\prod_{i=1}^{n} w_i = y^{|\pi|} p^{\mathrm{cros}(\pi)} q^{\mathrm{nest}(\pi)}.$$

Figure 2.9 has an example of this bijection.

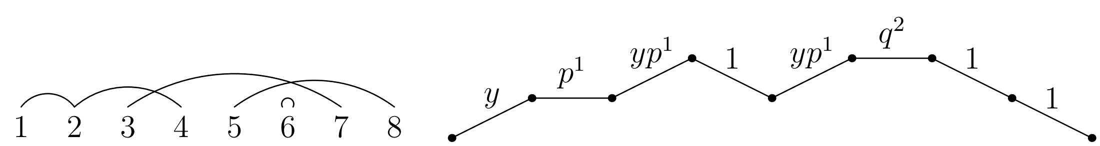


Figure 2.9: An example of the bijection from partitions to Motzkin paths

Lemma 2.10. *Let $\pi \in \Pi_n$ and (M, w) be the corresponding weighted Motzkin path. If h_i is the height of the Motzkin path at step i, then*

$$h_i = \mathrm{cros}(\pi, i) + \mathrm{nest}(\pi, i)$$

This lemma implies that the image of the map is the set of weighted Motzkin paths (M, w) so that if $m_i = \mathrm{N}$ then $w_i \in \{p^a q^b \mid a + b = h_i\}$, if $m_i = \mathrm{E}$ then $w_i \in \{yq^{h_i}, p^{a+1}q^b \mid a + b = h_i - 1\}$ and if $m_i = \mathrm{S}$ then $w_i = 1$.

The reverse map is almost identical to the reverse map for matchings. If (M, w) is a weighted Motzkin path, satisfying the conditions of the previous paragraph, define $\pi \in \Pi_n$ as follows. Reading right to left, connect each N or E step to the a^{th} available E or S step where $w_i = y^\epsilon p^a q^{h_i - a}$ ($\epsilon \in \{0, 1\}$).

Therefore, we have

$$\begin{aligned} F(x, y, p, q) &= \sum_{n \geq 0} x^n \sum_{\pi \in \Pi_n} y^{|\pi|} p^{\mathrm{cros}(\pi)} q^{\mathrm{nest}(\pi)} \\ &= \cfrac{1}{1 - (yq^0 + p[0]_{p,q})x - \cfrac{y[1]_{p,q}x^2}{1 - (yq + p[1]_{p,q})x - \cfrac{y[2]_{p,q}x^2}{\ddots}}}. \end{aligned}$$

This equation simplifies greatly if $y = 1$,

$$F(x, 1, p, q) = \cfrac{1}{1 - [0]_{p,q}x - \cfrac{[1]_{p,q}x^2}{1 - [1]_{p,q}x - \cfrac{[2]_{p,q}x^2}{\ddots}}}.$$

Further, the number of non-crossing partitions (and non-nesting) is the n^{th} Catalan number as can be seen from the generating function,

$$F(x,1,0,1) = \cfrac{1}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{\ddots}}}.$$

Crossings and nestings in partitions have been widely studied [1, 3, 10, 11, 15]. In particular Chen et al. [2] showed the joint statistic $(\mathrm{cros}, \mathrm{nest})$ is symmetric when restricted to be over the set of partitions with fixed minimal and maximal elements.

2.4 Permutations

The work in this section is due to Corteel [4]. We include the results and construction here for completeness and future use.

A *permutation* on the set $[n]$ is a bijection $\sigma : [n] \to [n]$. We denote the set of all permutations on $[n]$ by S_n.

Definition 2.11. For $\sigma \in S_n$ and $i, j \in [n]$ we say $(i, \sigma(i))$ and $(j, \sigma(j))$ *cross* if either

$$i < j \leq \sigma(i) < \sigma(j) \qquad\qquad \sigma(i) < \sigma(j) < i < j$$

and they *nest* if either

$$i < j \leq \sigma(j) < \sigma(i) \qquad\qquad \sigma(i) < \sigma(j) < j < i.$$

We denote $\mathrm{cros}(\sigma)$ to be the number of crossings in σ and $\mathrm{nest}(\sigma)$ the number of nestings.

Permutations can be represented on a line with arcs connecting i and $\sigma(i)$. We draw the arcs above the x-axis if $i \leq \sigma(i)$ and below otherwise. Figure 2.10 has an example of a permutation. Drawing permutations like this we can visualize crossings

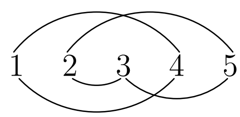
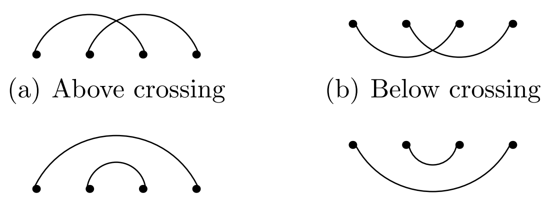


Figure 2.10: The graph of a permutation

and nestings easily. There are two cases for crossings and nestings, one above the x-axis and one below, Figure 2.11 contains an example each crossing and nesting.



(a) Above crossing (b) Below crossing

(c) Above nesting (d) Below nesting

Figure 2.11: The types of crossings and nestings in permutations

Example 2.12. The permutation, σ, in Figure 2.10 has $\text{cros}(\sigma) = 2$ and $\text{nest}(\sigma) = 1$.

The goal is to enumerate crossings and nestings over permutations. However, it turns out we are able to include one additional statistic for free, weak exceedances. For a permutation, σ, the number of *weak exceedances* is $WEX(\sigma) = \#\{j \mid j \leq$

$\sigma(j)\}$. Essentially this is the number of arcs above the x-axis. With this we can now define the generating function we want to find,

$$F(x,y,p,q) = \sum_{n\geq 0} x^n \sum_{\sigma\in S_n} y^{WEX(\sigma)} p^{\text{cros}(\sigma)} q^{\text{nest}(\sigma)}.$$

To find this generating function we are going to form a bijection with a collection of weighted bi-colored Motzkin paths. Before we do this we need to split the notion of crossings and nestings into above the x-axis and below versions.

For $\sigma \in S_n$ and $i \in [n]$ define the following,

$$\begin{aligned}
\text{cros}_+(\sigma,i) &= \{j \mid j < i \leq \sigma(j) < \sigma(i)\} \\
\text{cros}_-(\sigma,i) &= \{j \mid \sigma(i) < \sigma(j) < i < j\} \\
\text{nest}_+(\sigma,i) &= \{j \mid j < i \leq \sigma(i) < \sigma(j)\} \\
\text{nest}_-(\sigma,i) &= \{j \mid \sigma(j) < \sigma(i) < i < j\}.
\end{aligned}$$

It's important to observe that

$$\text{cros}(\sigma) = \sum_{i=1}^{n} \text{cros}_+(\sigma,i) + \text{cros}_-(\sigma,i) \qquad \text{nest}(\sigma) = \sum_{i=1}^{n} \text{nest}_+(\sigma,i) + \text{nest}_-(\sigma,i).$$

For $\sigma \in S_n$ define a weighted bi-colored Motzkin path (M,w) as follows, for $M = m_1 \dots m_n$ set

$$m_i = \begin{cases} \text{N} & \text{if } i < \sigma(i) \text{ and } i < \sigma^{-1}(i), \\ \text{E} & \text{if } i \leq \sigma(i) \text{ and } \sigma^{-1}(i) \leq i, \\ \bar{\text{E}} & \text{if } \sigma(i) < i \text{ and } i < \sigma^{-1}(i), \\ \text{S} & \text{if } \sigma(i) < i \text{ and } \sigma^{-1}(i) < i. \end{cases}$$

And for $w = w_1 \dots w_n$ set,

$$w_i = \begin{cases} yp^{\mathrm{cros}_+(\sigma,i)} q^{\mathrm{nest}_+(\sigma,i)} & \text{if } m_i \in \{\mathrm{N}, \mathrm{E}\}, \\ p^{\mathrm{cros}_-(\sigma,i)} q^{\mathrm{nest}_-(\sigma,i)} & \text{if } m_i \in \{\bar{\mathrm{E}}, \mathrm{S}\} \end{cases}$$

this implies that for any permutation,

$$\prod_{i=1}^{n} w_i = y^{WEX(\sigma)} p^{\mathrm{cros}(\sigma)} q^{\mathrm{nest}(\sigma)}.$$

An example of this bijection is in Figure 2.12.

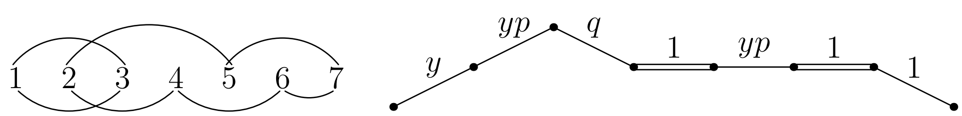


Figure 2.12: The bijection from permutations to bi-colored Motzkin paths

Lemma 2.13. *For a permutation $\sigma \in S_n$, the corresponding weighted bi-colored Motzkin path (M, w) and $i \in [n]$. If h_i is the height of M at step i then*

1. *If $i \leq \sigma(i)$, then $h_i = \mathrm{cros}_+(\sigma, i) + \mathrm{nest}_+(\sigma, i)$.*

2. *If $i > \sigma(i)$, then $h_i - 1 = \mathrm{cros}_-(\sigma, i) + \mathrm{nest}_-(\sigma, i)$.*

This lemma tells us the image of our map is the set of weighted bi-colored Motzkin paths where the weights satisfy

$$w_i \in \begin{cases} \{p^a q^b \mid a + b = h_i\}, & \text{if } m_i \in \{\mathrm{N}, \mathrm{E}\}, \\ \{p^a q^b \mid a + b = h_i - 1\}, & \text{if } m_i \in \{\bar{\mathrm{E}}, \mathrm{S}\}. \end{cases}$$

The inverse of this map is similar to the inverse in Section 2.3. For a weighted bi-colored Motzkin path M, the N and E steps give the top arcs and the $\bar{\text{E}}$ and S steps give the bottom arcs.

Therefore, we have

$$F(x,y,p,q) = \sum_{n\geq 0} x^n \sum_{\sigma\in S_n} y^{WEX(\sigma)} p^{\mathrm{cros}(\sigma)} q^{\mathrm{nest}(\sigma)} = \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cfrac{\lambda_2 x^2}{\ddots}}}$$

where,

$$b_n = y[n+1]_{p,q} + [n]_{p,q} \qquad\qquad \lambda_n = y[n]_{p,q}^2$$

Once again this generating function shows that the joint distribution of (cros, nest) is symmetric, $F(x,y,p,q) = F(x,y,q,p)$.

2.5 Polyominoes

A *polyomino* is a finite connected subset of $\mathbb{Z}^2$, where we regard an element of $\mathbb{Z}^2$ as a cell. A column of the polyomino is a set of cells along a vertical line, a row is the set of cells along a horizontal line. As convention we number rows top to bottoms and columns left to right.

The polyomino is row (column) convex if the intersection with any horizontal (vertical) line is convex. It is intersection–free if for any two rows the column coordinates of one are contained in the column coordinates of the other. For example, the polyomino in Figure 2.13 is row–convex but neither column–convex nor intersection–free.

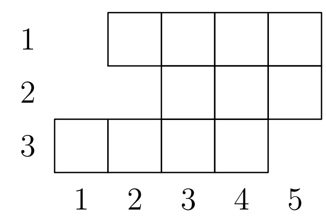


Figure 2.13: Row-convex polyomino, neither intersection–free nor column–convex

For $A \subset \mathbb{N}$ an A–filling of a polyomino $\mathcal{P}$ is an assignment of the elements of A to the cells of $\mathcal{P}$. The set of all A fillings of $\mathcal{P}$ is denoted

$$\mathcal{F}_A(\mathcal{P}).$$

A filling, P, of a polyomino has *row sums* $\vec{r} = \langle r_1, r_2, \ldots, r_n \rangle$ if the sum of the entries in the i^{th} row of P equals r_i, column sums are defined similarly. The set of all A-fillings of $\mathcal{P}$ with row sums $\vec{r}$ is denoted

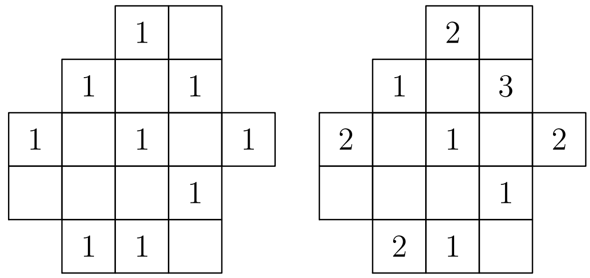
$$\mathcal{F}_A(\mathcal{P}, \vec{r})$$

and with row sums $\vec{r}$ and column sums $\vec{c}$ is denoted

$$\mathcal{F}_A(\mathcal{P}, \vec{r}, \vec{c}).$$

The two sets that are most common sets for fillings are $\{0,1\}$ and $\mathbb{N} = \{0, 1, \ldots\}$, for convenience we write 01-filling for $\{0,1\}$–filling.

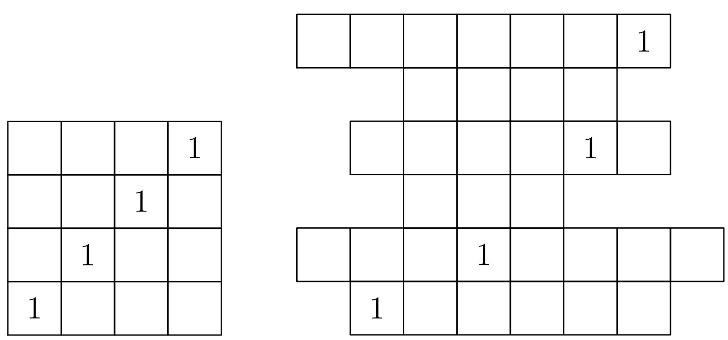
Example 2.14. Figure 2.14 has an example of both a 01-filling and an $\mathbb{N}$–filling. The filling in Figure 2.14(a) has row sums $\langle 1, 2, 3, 1, 2 \rangle$ and column sums $\langle 1, 2, 3, 2, 1 \rangle$, and the filling in Figure 2.14(b) has row sums $\langle 2, 4, 5, 1, 3 \rangle$ and column sums $\langle 2, 3, 4, 4, 2 \rangle$.



(a) A 01-filling (b) An $\mathbb{N}$-filling

Figure 2.14: Fillings in a polyomino

A northeast k–chain (ne$_k$–chain) in a filling, P, of a polyomino is k non-empty cells (i_a, j_a) for $1 \leq a \leq k$ with $i_1 > i_2 > \cdots > i_k$ and $j_1 < j_2 < \cdots < j_k$ so that the cells (i_1, j_a), (i_k, j_a), (i_a, j_1) and (i_a, j_k) are contained in P for all $1 \leq a \leq k$. In particular, a *northeast 2–chain* (ne$_2$–chain) is two non-empty cells (i_1, j_1) and (i_2, j_2) so that the cells (i_1, j_2) and (i_2, j_1) are contained in P. Southeast k–chains (se$_k$–chain) are defined similarly except we require $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. Figure 2.15 has examples of a ne$_k$–chain.



(a) A ne$_4$-chain in a rectangle (b) A ne$_4$-chain in a polyomino

Figure 2.15: A ne$_4$–chain in two polyominoes

For a filling, P, of a polyomino we denote the total number of ne$_k$–chains in P

as $\mathrm{ne}_k(P)$ and the number of se_k–chains as $\mathrm{se}_k(P)$. If P has no ne_k–chains we say P *avoids* ne_k–chains. For a collection of fillings of a polyomino, $\mathcal{F}(\mathcal{P})$, we denote by $\mathrm{Av}(\mathrm{ne}_k, \mathcal{F}(\mathcal{P}))$ the set of all $P \in \mathcal{P}$ that avoid ne_k–chains. Similarly define $\mathrm{Av}(\mathrm{se}_k, \mathcal{F}(\mathcal{P}))$.

2.5.1 Ferrers Diagrams

A *Ferrers Diagram* is a polyomino with rows $R_1, \ldots, R_n$ so that, if the length of R_i is $|R_i|$, $|R_1| \geq |R_2| \geq \cdots \geq |R_n|$ and the rows are left justified. Figure 2.16 has an example of Ferrers diagram.

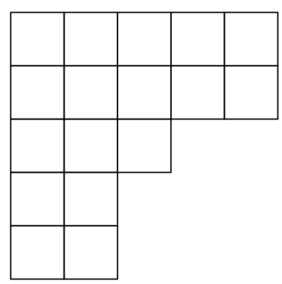


Figure 2.16: A Ferrers diagram

Ferrers diagrams arise naturally in partitions of natural numbers, with each part corresponding to a length of a row. However, we are mostly concerned with enumerating chains in fillings. It turns out many of the results in the previous sections can be described in terms of fillings of Ferrers diagrams.

This was first realized by Krattenthaler [13]. He was able to show symmetry in fillings over the longest ne_k-chain and se_k-chains.

Theorem 2.15 (Krattenthaler)**.** *For a Ferrers diagram $\mathcal{T}$ and positive integers n, s and t, the number of* 01*-fillings of $\mathcal{T}$ with exactly n 1's, such that there is at most* *one 1 in each column and in each row, with longest* ne-*chain of length s and longest* se-*chain of length t equals the number of fillings with longest* se-*chain of length s and longest* ne-*chain of length t.*

In the context of crossings and nestings in general graphs, de Mier [6] showed that graphs with no k-crossings are equidistributed as graphs with no k-nestings. She proved this using chains in Ferrers diagrams.

Theorem 2.16 (de Mier)**.** *For a Ferrers diagram $\mathcal{T}$, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$*

$$\mathrm{av}(\mathrm{ne}_k, \mathcal{F}_{\mathbb{N}}(\mathcal{T}, \vec{r}, \vec{c})) = \mathrm{av}(\mathrm{se}_k, \mathcal{F}_{\mathbb{N}}(\mathcal{T}, \vec{r}, \vec{c})).$$

2.5.2 Moon Polyominoes

A *moon polyomino* is a polyomino which is row and column convex and intersection–free. Figure 2.17 has an example of a moon polyomino.

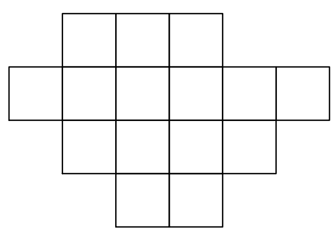


Figure 2.17: A moon polyomino

Fillings of moon polyominoes can be seen as a generalization of fillings of Ferrers diagrams. Kasraoui [9] showed several symmetry results in fillings of Ferrers diagrams have analogues in fillings of moon polyominoes.

Theorem 2.17 (Kasraoui). *For a moon polyomino $\mathcal{M}$, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \{0,1\}$, the joint statistic* $(\mathrm{ne}, \mathrm{se})$ *is symmetrically distributed over $\mathcal{F}_{01}(\mathcal{M}, \vec{r}, \vec{c})$, i.e.*

$$\sum_{M \in \mathcal{F}_{01}(\mathcal{M}, \vec{r}, \vec{c})} p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)} = \sum_{M \in \mathcal{F}_{01}(\mathcal{M}, \vec{r}, \vec{c})} p^{\mathrm{se}_2(M)} q^{\mathrm{ne}_2(M)}$$

Kasraoui also found the generating function for $(\mathrm{ne}_2, \mathrm{se}_2)$. The form of this generating function implies that the joint distribution of $(\mathrm{ne}_2, \mathrm{se}_2)$ is unaffected by a permutation of the rows of a moon polyomino, provided the resulting shape is a moon polyomino.

Theorem 2.18 (Kasraoui). *For any moon polyomino $\mathcal{M}$, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \{0,1\}^m$, the distribution of the joint statistic* $(\mathrm{ne}, \mathrm{se})$ *over $\mathcal{F}_{01}(\mathcal{M}, \vec{r}, \vec{c})$ is given by*

$$\sum_{M \in \mathcal{F}_{01}(\mathcal{M}, \vec{r}, \vec{c})} p^{\mathrm{ne}_2(M)} q^{\mathrm{se}_2(M)} = \prod_{j=1}^{n} \begin{bmatrix} h_j \\ r_j \end{bmatrix}_{p,q}$$

where h_j is the length of row j minus the row sums of all rows shorter than row j.

3. LAYER POLYOMINOES

Definition 3.1. Recall the notions of a polyomino and a filling of a polyomino from Section 2.5. A *layer polyomino* is a row–convex, intersection–free polyomino. A *moon polyomino* is a column–convex layer polyomino. Figure 3.1 shows examples of a layer polyomino.

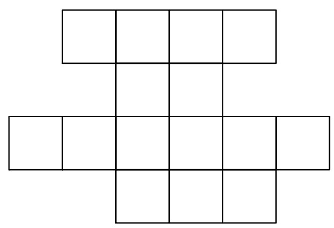


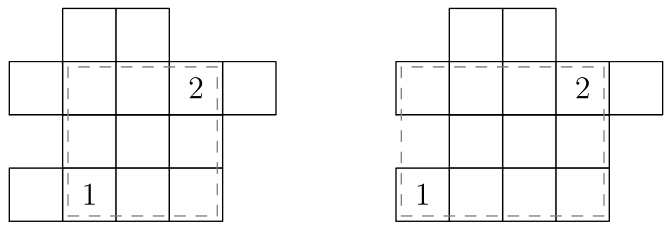
Figure 3.1: A layer polyomino.

In layer polyominoes there are two notions of ne-chains, regular chains and strong chains. A *regular* ne-chain is a ne_2-chain, as defined in Section 2.5. We now define strong chains. Figure 3.2 contains an example of how regular chains and strong chains differ.

Definition 3.2. A *strong northeast chain*, or $ne^{\square}$–chain, in a filling P of a polyomino $\mathcal{P}$, is a ne–chain so that the smallest rectangle containing the entries is contained in the polyomino. More precisely, the non-zero entries (r_1, c_1) and (r_2, c_2) in P form a $ne^{\square}$–chain if $r_2 < r_1$, $c_1 < c_2$, and all the cells in the set $\{(r, c) \mid r_2 < r <$

Part of the material in this chapter is reprinted with permission from "Chains of length 2 in fillings of layer polyominoes" by Mitchell Phillipson, Catherine Yan and Jean Yeh, The Electronic Journal of Combinatorics, 23(2), 2009

r_1 and $c_1 < c < c_2\}$ are contained in $\mathcal{P}$. We similarly define *strong southeast chains*, or $\mathrm{se}^{\square}$–chains.



(a) ne–chain and $\mathrm{ne}^{\square}$–chain (b) ne–chain, not $\mathrm{ne}^{\square}$–chain

Figure 3.2: Chains in a layer polyomino.

Figure 3.2 contains an example of how regular chains and strong chains differ. Note that strong and regular chains coincide on moon polyominoes. As we will see regular chains are a more natural generalization from chains in moon polyominoes as Kasraoui's [9] result on permuting rows in moon polyominoes extends to regular chains in layer polyominoes, this is the content of Subsection 3.1. This section also shows gives conditions when an $\mathbb{N}$-filings of a layer polyomino exists and shows the filling with no ne-chains in unique.

Subsection 3.2 contains results on strong chains, showing the joint distribution of $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ is symmetric for 01-fillings with fixed row sums and at most on non-zero entry per column, and for $\mathbb{N}$-fillings with fixed row and column sums. It concludes with several remarks on strong chains.

3.1 Regular Chains in Layer Polyominoes

In this section we prove several results about regular chains in fillings of layer polyominoes. For both 01- and $\mathbb{N}$- fillings we prove that an arbitrary permutation of the rows preserves the numbers of fillings with either no ne–chains or no se–chains.

Further, for $\mathbb{N}$-fillings we give necessary and sufficient conditions with which $\mathbb{N}$–fillings exist with given row and column sums, and prove that under those conditions, the filling with no ne-chains (reps, se-chains) is unique.

The first result shows that in fillings of layer polyominoes with at most 1 non-zero entry per column, the joint distribution $(\mathrm{ne}, \mathrm{se})$ is unaffected by an arbitrary permutation of rows. This result was originally published by Phillipson, Yan and Yeah in [14].

Theorem 3.3 (Phillipson,Yan,Yeah)**.** *Let $\mathcal{L}$ be a layer polyomino with rows $R_1, ..., R_n$ from top to bottom, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \{0,1\}^m$. For a permutation $\sigma \in S_n$, let $\mathcal{L}' = \sigma(\mathcal{L})$ be the layer polyomino whose rows are $R_{\sigma(1)}, \ldots, R_{\sigma(n)}$, and $\vec{r}' = \sigma(\vec{r}) = (r_{\sigma(1)}, \ldots, r_{\sigma(n)})$. Then*

$$\sum_{L \in \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})} p^{\mathrm{ne}(L)} q^{\mathrm{se}(L)} = \sum_{L' \in \mathcal{F}_{01}(\mathcal{L}', \vec{r}', \vec{c})} p^{\mathrm{ne}(L')} q^{\mathrm{se}(L')}.$$

Proof. It is sufficient to prove Theorem 3.3 for adjacent transpositions, that is, when $\sigma = (k, k+1)$ where $k \in \{1, \ldots, n-1\}$. Explicitly, let $\mathcal{L}'$ be the layer polyomino obtained from $\mathcal{L}$ by exchanging the rows R_k and R_{k+1}. We construct a bijection

$$\phi : \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c}) \to \mathcal{F}_{01}(\mathcal{L}', \vec{r}', \vec{c})$$

such that $\mathrm{ne}(L) = \mathrm{ne}(\phi(L))$ and $\mathrm{se}(L) = \mathrm{se}(\phi(L))$ for every $L \in \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$.

Let $\mathcal{R}$ be the largest rectangle contained in $R_k \cup R_{k+1}$ and $L \in \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$. Construct a filling L' from $\mathcal{L}'$ from L as follows

1. exchange rows R_i and R_{i+1} with their fillings,

2. fix the empty columns in $\mathcal{R}$, and

3. reverse the filling in each row of $\mathcal{R}'$, where $\mathcal{R}'$ is the rectangle consisting of all non-empty columns of $\mathcal{R}$.

See Figure 3.3 for an example of this process, the cells are labeled for clarity and the rectangle $\mathcal{R}$ is boxed. In this example we flip rows 2 and 3 and shade cells in $\mathcal{R}$ that we do not reverse.

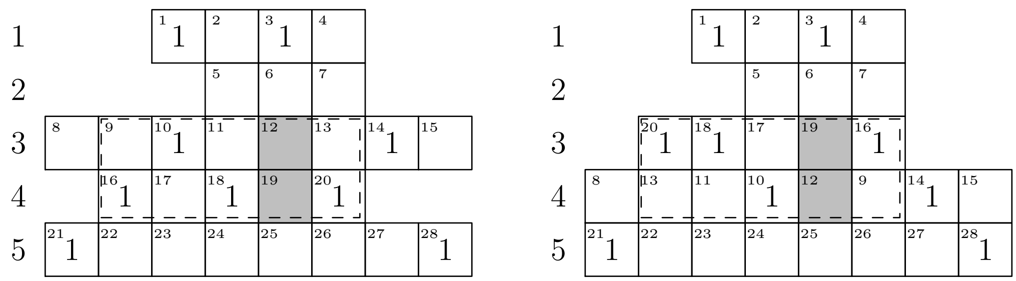


Figure 3.3: An example of the process in Theorem 3.4.

We claim that $\text{ne}(L) = \text{ne}(L')$ and $\text{se}(L) = \text{se}(L')$. We prove the first equation only as the second can be treated similarly. First, any ne-chain formed by two 1-cells outside $R_k \cup R_{k+1}$ is not changed.

Let C_ℓ be a column in $R_i \cup R_{i+1}$. If C_ℓ is empty or C_ℓ is outside $\mathcal{R}$, then any ne-chain formed with C_ℓ is preserved, as the column is unchanged. Suppose C_ℓ contains a non-zero enetry and is inside $\mathcal{R}$, then C_ℓ contains exactly one non-zero entry. Under the above bijeciton, C_ℓ moves and is replaces by a column that also has one non-zero entry. Any ne-chain that was form in L with C_ℓ is replaced in L' with the new column. Thus $\text{ne}(L) = \text{ne}(L')$. $\square$

If we consider 01-fillings with fixed row and column sums in $\mathbb{N}^*$, then the symmetry of (ne, se) may not hold. Moreover, for a layer polyomino $\mathcal{L}$ and $\sigma \in S_n$ it's no longer true that $\mathrm{av}(\mathrm{ne}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})) = \mathrm{av}(\mathrm{ne}, \mathcal{F}_{01}(\sigma(\mathcal{L}), \sigma(\vec{r}), \vec{c}))$. For example, for row sums $\vec{r} = \langle 1, 2\rangle$ and column sums $\vec{c} = \langle 1, 2\rangle$, there is only one possible filling of a 2×2 rectangle, Figure 3.4(a), and this filling has one ne–chain. On the other hand, transposing the rows givens $\sigma(\vec{r}) = \langle 2, 1\rangle$, but there is only one 01-filling with row sums $\sigma(\vec{r})$ and column sums $\vec{c}$, which is given in Figure 3.4(b) and has no ne–chains.

	1
1	1

(a) Row sums $\langle 1, 2\rangle$

1	1
	1

(b) Row sums $\langle 2, 1\rangle$

Figure 3.4: Fillings with fixed row sums and column sums

The next result states that the number of fillings with no ne–chain is preserved under permutations of rows if we only fix row sums, but have no restrictions on column sums.

Theorem 3.4. *Let $\mathcal{L}$ be a layer polyomino with rows $R_1, \ldots, R_n$ from top to bottom and $\vec{r} \in \mathbb{N}^n$. For a permutation $\sigma \in \mathcal{S}_n$, let $\mathcal{L}' = \sigma(\mathcal{L})$ be the polyomino with rows $R_{\sigma(1)}, \ldots, R_{\sigma(n)}$ and $\vec{r'} = \sigma(\vec{r}) = (r_{\sigma(1)}, \ldots, r_{\sigma(n)})$. Then*

$$\mathrm{av}(\mathrm{ne}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{ne}, \mathcal{F}_{01}(\mathcal{L}', \vec{r'})),$$

and

$$\mathrm{av}(\mathrm{se}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{se}, \mathcal{F}_{01}(\mathcal{L}', \vec{r'})).$$

Proof. The method used in this proof is identical to the proof of Theorem 3.3. The argument, including the bijection, is repeated here as the base set is different.

Any permutation can be obtained using a transposition of two adjacent rows, so we'll show that permuting two consecutive rows preserves the number of fillings with no ne–chains. The case for se-chains is similar.

Let R_i and R_{i+1} be two consecutive rows, $L \in \mathrm{Av}(\mathrm{ne}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$ and $\mathcal{R}$ be the largest rectangle contained in $R_i \cup R_{i+1}$. Construct L' from L by

1. exchanging R_i and R_{i+1} with their fillings,
2. fixing the empty columns of $\mathcal{R}$, and
3. reversing the filling of each row of $\mathcal{R}'$, where $\mathcal{R}'$ is the rectangle consisting of all the non-empty columns of $\mathcal{R}$.

Then $L' \in \mathrm{Av}(\mathrm{ne}, \mathcal{F}_{01}(\mathcal{L}', \vec{r'}))$ as the above operations preserve fillings on $\mathcal{L} - \mathcal{R}$, the empty columns of $\mathcal{R}$, and changes the row sum from $\vec{r}$ to $\sigma(\vec{r})$. This guarantees that no new ne–chains are created involving cells outside of $\mathcal{R}$. Additionally $\mathcal{R}$ will still not have ne–chains as flipping both the rows and columns preserves this property. $\square$

Corollary 3.5. *For a layer polyomino $\mathcal{L}$ with n rows and $\vec{r} \in \mathbb{N}^n$, then*

$$\mathrm{av}(\mathrm{ne}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{se}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})).$$

Proof. Let $\tilde{\mathcal{L}}$ be the polyomino obtained by reversing the rows of $\mathcal{L}$, that is, $\tilde{\mathcal{L}} = \sigma(\mathcal{L})$ where $\sigma = n \cdots 21$. For $L \in \mathcal{F}_{01}(\mathcal{L}, \vec{r})$, let $\tilde{L}$ be obtained from L by reversing the rows of L together with their fillings. Then $\tilde{L} \in \mathcal{F}_{01}(\tilde{\mathcal{L}}, \sigma(\vec{r}))$. It is clear that L has no se-chains if and only if $\tilde{L}$ has no ne-chains. Hence

$$\mathrm{av}(\mathrm{se}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{ne}, \mathcal{F}_{01}(\tilde{\mathcal{L}}, \sigma(\vec{r}))) = \mathrm{av}(\mathrm{ne}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})).$$

The last equation follows from Theorem 3.4. $\square$

Next we give necessary and sufficient conditions when $\mathbb{N}$–fillings of layer polyominoes exist with given row and column sums, and show that when such fillings do exist there is a unique one with no ne–chains. First we show the result for rectangles, then extend to Ferrers diagrams and finally to layer polyominoes by transforming $\mathbb{N}$-fillings of a layer polyomino to those of a Ferrers diagram with permutations of the rows and columns.

Lemma 3.6. *Let $\mathcal{R}$ be an $n \times m$ rectangle, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$. Then $\mathbb{N}$-fillings of $\mathcal{R}$ with row sums $\vec{r}$ and column sums $\vec{c}$ exist if and only if*

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{m} c_j. \tag{3.1}$$

Further, if $\vec{r}$ and $\vec{c}$ satisfy (3.1) then there is a unique filling of $\mathcal{R}$ with no ne*–chains (reps.* se*–chains).*

Proof. The necessity of (3.1) is clear. To prove sufficiency we use a greedy algorithm, implemented inductively. Let $\mathcal{R}$, $\vec{r}$ and $\vec{c}$ be as in the statement and satisfy (3.1). If $n = 1$ we can fill cell $(1, i)$ with c_i; similarly if $m = 1$ we fill cell $(i, 1)$ with r_i.

In general, we have three cases to consider $r_1 = c_1$, $r_1 < c_1$, and $r_1 > c_1$; however, the last two cases are similar. If $r_1 = c_1$, fill cell $(1, 1)$ with r_1, cell $(1, i)$, $(j, 1)$ with 0 for $1 < i \leq m$ and $1 < j \leq n$, and reduce the problem to an $(n - 1) \times (m - 1)$ rectangle.

Assume $r_1 < c_1$. Then we fill cell $(1, 1)$ with r_1, fill cell $(1, i)$ with 0 for $1 < i \leq m$, and reduce the problem to an $(n - 1) \times m$ rectangle with row sums $\langle r_2, \ldots, r_n \rangle$ and column sums $\langle c_1 - r_1, c_2, \ldots, c_m \rangle$. Continuing this process inductively produces a filling R with no ne–chains.

If f_{ij} is the entry in the (i, j) cell of the above constructed R, then each non-zero

f_{ij} has the property that either $\sum_{\ell=1}^{i} f_{\ell j} = c_j$ or $\sum_{\ell=1}^{j} f_{i\ell} = r_i$.

Now we show the filling R is the unique one with no ne-chains. Let R' be a different filling in $\mathcal{F}_{\mathbb{N}}(\mathcal{R}, \vec{r}, \vec{c})$ with entries f'_{ij} in cell (i,j). Find a cell i,j with minimal indices such that $f'_{ij} \neq f_{ij}$. Then we must have $0 \leq f'_{ij} < f_{ij}$ and hence

$$\sum_{\ell=1}^{i} f'_{\ell j} < c_j \qquad \text{and} \qquad \sum_{\ell=1}^{j} f'_{i\ell} < r_i. \tag{3.2}$$

Therefore there exist non-zero entries f_{kj} ne 0 with $i+1 \leq k \leq n$ and $f_{i\ell} \neq 0$ with $j+1 \leq \ell \leq m$, which means the entries f_{kj} and f_{il} form a ne–chain in R'. $\square$

Now we extend this result to Ferrers diagrams. The conditions for a Ferrers diagram are slightly more complex than for rectangles.

Theorem 3.7. *Given a Ferrers diagram $\mathcal{T}$ with n rows and m columns, vectors $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$, an $\mathbb{N}$-filling of $\mathcal{T}$, with row sums $\vec{r}$ and column sums $\vec{c}$, exists if and only if the following conditions hold.*

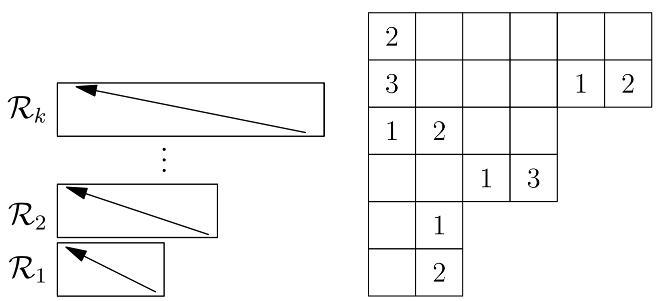
$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{m} c_j. \tag{3.3}$$

$$\forall S \subseteq [n], \quad \sum_{i \in S} r_i \leq \sum_{j : \exists i \in S ((i,j) \in \mathcal{T})} c_j. \tag{3.4}$$

Further, if $\vec{r}$ and $\vec{c}$ satisfy (3.3) and (3.4) then the filling of $\mathcal{T}$ with no ne–*chains (*se–*chains) is unique.*

Proof. Partition the Ferrers diagram into a collection of rectangles $\mathcal{R}_1, \mathcal{R}_2, \ldots \mathcal{R}_k$, where $\mathcal{R}_i$ is the union of the i^{th} shortest rows, see Figure 3.5(a). Starting with the rectangle $\mathcal{R}_1$, fill $\mathcal{R}_1$ using the greedy algorithm in the preceding proof from lower right to upper left, until all the cells of $\mathcal{R}_1$ are filled. By condition (3.4) this is possible, and in the resulting filling all the rows of $\mathcal{R}_1$ are saturated, i.e., the row sum of each row in $\mathcal{R}_1$ equals the desired row sum given by $\vec{r}$, and the column sums of $\mathcal{R}_1$ are no more than $\vec{c}$.

Subtract the column sums in $\mathcal{R}_1$ from the corresponding entries of $\vec{c}$ to get $\vec{c'}$. One checks that the conditions (3.3) and (3.4) still hold for the Ferrers diagram $\mathcal{T} \setminus \mathcal{R}_1$ with the row sums $\vec{r'}$ and column sums $\vec{c'}$, where $\vec{r'}$ is the restriction of $\vec{r}$ to the rows in $\mathcal{T} \setminus \mathcal{R}_1$. Then we continue inductively until we reach the last rectangle $\mathcal{R}_k$, which is filled and the row sums and columns sums are both saturate, by Lemma 3.6. Figure 3.5(b) has an example with row sums $\langle 2, 6, 3, 4, 1, 2 \rangle$ and column sums $\langle 6, 5, 1, 3, 1, 2 \rangle$.



(a) Fill with no ne-chains (b) No ne–chains

Figure 3.5: $\mathbb{N}$-filling of a Ferrers diagram

This filling has no ne–chains as if a column is saturated in $\mathcal{R}_i$, that column will remain empty in each subsequent $\mathcal{R}_j$, $j > i$. Additionally, the filling is unique by the same argument as in the proof of Lemma 3.6. $\square$

Note that if the Ferrers diagram is aligned at the top and the left as in English notation, (e.g, Figure 3.5(b)), then condition (3.4) is equivalent to the following set of inequalities: for each i such that the row R_i of $\mathcal{T}$ is the top row of some rectangle $\mathcal{R}_j$,

$$\sum_{j=i}^{n} r_i \leq \sum_{j:(i,j)\in\mathcal{F}} c_j.$$

In addition, conditions (3.3) and (3.4) imply that

$$\forall T \subseteq [m], \ \sum_{j\in T} c_j \leq \sum_{i:\exists j\in T((i,j)\in\mathcal{F})} r_i \tag{3.5}$$

Theorem 3.8. *Let $\mathcal{L}$ be a layer polyomino with rows $R_1, \ldots, R_n$ from top to bottom, m columns, $\vec{r} \in \mathbb{N}^n$, and $\vec{c} \in \mathbb{N}^m$. $R_{\sigma(1)}, \ldots, R_{\sigma(n)}$ and $\vec{r'} = \sigma(\vec{r}) = (r_{\sigma(1)}, \ldots, r_{\sigma(n)})$. Then*

$$\mathrm{av}(\mathrm{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) = \mathrm{av}(\mathrm{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}', \vec{r'}, \vec{c}))$$

and

$$\mathrm{av}(\mathrm{se}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) = \mathrm{av}(\mathrm{se}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}', \vec{r'}, \vec{c}))$$

Proof. Proceeding in a manner similar to Theorem 3.4, we'll show that we can permute any two adjacent rows while preserving the number of fillings with no ne–chains.

Let R_i and R_{i+1} be two consecutive rows, $L \in \mathrm{Av}(\mathrm{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ and $\mathcal{R}$ be the largest rectangle contained in $R_i \cup R_{i+1}$. Let $\sigma = (i, i+1)$ be a transposition. Define L' to be a filling of $\sigma(\mathcal{L})$ by the following operations.

1. Exchange R_i and R_{i+1} with their fillings,
2. Refill the rectangle $\mathcal{R}$ with the unique filling with no ne–chains (Lemma 3.6), preserving the row and column sums of $\mathcal{R}$.

We claim that the filling $\mathcal{L}'$ has no ne–chains. It is clear that any entry outside of $\mathcal{R}$ does not change, and $\mathcal{R}$ contains no ne–chains. Let α be a non-zero entry in column c_l outside $\mathcal{R}$. Assume α forms a ne–chain with entry $\beta \neq 0$ in $\mathcal{R}$.

- If $c_l \cap \mathcal{R} \neq \emptyset$, then in L there exists a nonzero entry in the same column as β's. This entry forms a ne-chain with α in L.

- If $c_l \cap \mathcal{R} = \emptyset$, then β is in the longer row of R_i, R_{i+1}. Hence in the filling L there exists a nonzero entry in the longer row of $\mathcal{R}$, which forms a ne-chain with α.

In either case we have a ne-chain in L, which is a contradiction. □

Corollary 3.9. *Given a layer polyomino $\mathcal{L}$ with n rows and m columns, vectors $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$, $\mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})$ is nonempty if and only if conditions* (3.3) *and* (3.4) *hold. Further, if $\vec{r}$ and $\vec{c}$ satisfy* (3.3) *and* (3.4) *then the filling of $\mathcal{L}$ with no* ne*–chains (*se*–chains) is unique.*

Proof. Given $\mathcal{L}$, rearrange the rows of $\mathcal{L}$ from large to small to get a polyomino $\mathcal{L}_1$. Then $\mathcal{L}_1$ can be viewed as a layer polyomino rotated 90°. Apply column permutations to transform $\mathcal{L}_1$ to a Ferrers diagram $\mathcal{L}_2$. By Theorem 3.8,

$$\mathrm{av}(\mathrm{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) = \mathrm{av}(\mathrm{ne}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}_2, \vec{r'}, \vec{c'})), \tag{3.6}$$

where $\vec{r'}$ and $\vec{c'}$ are obtained from $\vec{r}, \vec{c}$ in the same way when one permutes the rows and columns. From Theorem 3.7 the formulas in (3.6) is non-zero if and only if conditions (3.3) and (3.4) hold, in which case the filling is unique. □

3.2 Strong Chains in Layer Polyominoes

In this section we study strong chains as defined in Definition 3.2. We begin by introducing a framework for a bijection on fillings of layer polyominoes. We'll use this framework to prove three distinct results. The first is the equality of numbers of 01-fillings with no strong northeast chains and those with no strong southeast chains.

The second shows the symmetry of $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ when the column sum is restricted to $\{0,1\}$. The final result extends the first to $\mathbb{N}$-fillings, with the additional condition that both row and column sums are fixed.

For a polyomino $\mathcal{P}$, let $\mathcal{F}(\mathcal{P})$ be either $\mathbb{N}$- or 01-fillings of $\mathcal{P}$. Let f be an invertible operation so that for any $n \times m$ rectangle $\mathcal{R}$, f induces a bijection $f : \mathcal{F}(\mathcal{R}) \to \mathcal{F}(\mathcal{R})$. Let $\mathcal{L}$ be a layer polyomino with n rows, define $\rho_f : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ recursively as follows. If $\mathcal{L}$ is a rectangle then $\rho_f = f$, otherwise for $L \in \mathcal{F}(\mathcal{L})$,

1. Let $\mathcal{R}_1, \ldots, \mathcal{R}_k$ be the maximal blocks of the consecutive shortest rows, $\mathcal{B}_0, \ldots, \mathcal{B}_k$ be the layer polyominoes between each $\mathcal{R}_i$, and $\mathcal{L}_1$ be the maximal rectangle in $\mathcal{L}$ containing each $\mathcal{R}_i$. See Figure 3.6 for an illustration of these sets. Let L_1 be the filling of L restricted to $\mathcal{L}_1$.

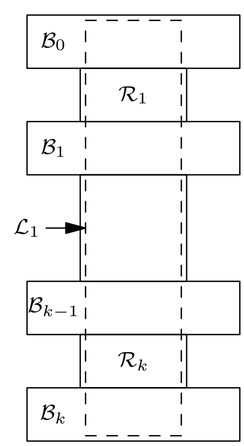


Figure 3.6: An example of the sets from step 1 of ρ_f.

2. Apply f to L_1.

3. For each i, apply f^{-1} to the current filling in $\mathcal{B}_i \cap \mathcal{L}_1$.

4. For each i, apply ρ_f to the current filling in $\mathcal{B}_i$.

The resulting filling of $\mathcal{L}$ is $\rho_f(L)$.

Proposition 3.10. *The map ρ_f is a well defined bijection. Additionally, if f preserves row (column) sums, so does ρ_f.*

Proof. By construction ρ_f does not modify the shape of $\mathcal{L}$, thus ρ_f is well defined. The map ρ_f can be inverted by performing each step in reverse, thus ρ_f is a bijection.

If f preserves row (column) sums, then in each step of ρ_f, the row (column) sums are preserved. Therefore, ρ_f preserves row (column) sums. $\square$

Example 3.11. For an $n \times m$ rectangle $\mathcal{R}$, define the map $f_1 : \mathcal{F}_{01}(\mathcal{R}) \to \mathcal{F}_{01}(\mathcal{R})$ so that for $R \in \mathcal{F}_{01}(\mathcal{R})$, $f_1(R)$ leaves empty columns unchanged and reverses each row of R in the non-empty columns. Figure 3.7 has an examples of f_1. Clearly f_1 preserves row sums.

$$f_1\left(\begin{array}{|c|c|c|c|c|}\hline {}^{1}1 & {}^{2} & {}^{3} & {}^{4}1 & {}^{5}1 \\ \hline {}^{6} & {}^{7} & {}^{8}1 & {}^{9} & {}^{10}1 \\ \hline {}^{11}1 & {}^{12} & {}^{13}1 & {}^{14} & {}^{15}1 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|c|}\hline {}^{5}1 & {}^{2} & {}^{4}1 & {}^{3} & {}^{1}1 \\ \hline {}^{10}1 & {}^{7} & {}^{9} & {}^{8}1 & {}^{6} \\ \hline {}^{15}1 & {}^{12} & {}^{14} & {}^{13}1 & {}^{11}1 \\ \hline \end{array}$$

Figure 3.7: An example of the map f_1.

Proposition 3.12. *The map ρ_{f_1} is a bijection from the set $\mathcal{F}_{01}(\mathcal{L}, \vec{r})$ to itself satisfying $(\rho_{f_1})^{-1} = \rho_{f_1^{-1}} = \rho_{f_1}$.*

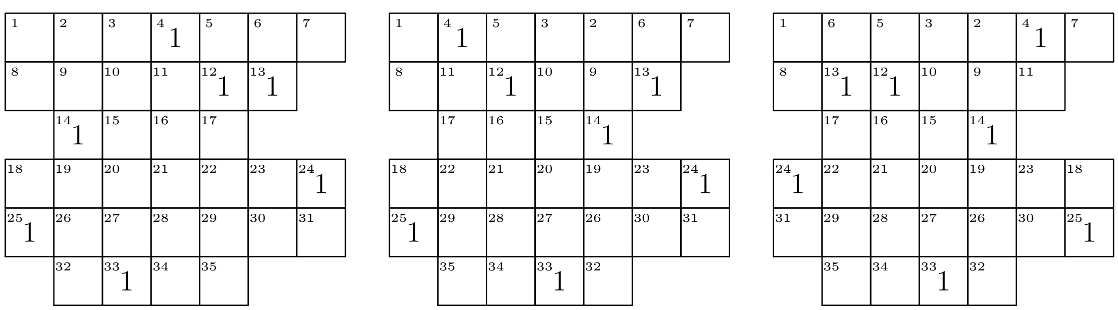
Proof. By definition $f_1 = f_1^{-1}$ on 01-fillings of any rectangle. Let $\mathcal{L}$ be a layer polyomino with n rows, $\vec{r} \in \mathbb{N}^n$, and $L \in \mathcal{F}_{01}(\mathcal{L}, \vec{r})$, we need to show $\rho_{f_1}(\rho_{f_1}(L)) = L$. It is clearly true when $\mathcal{L}$ is a rectangle.

Let $\mathcal{R}_i$, $\mathcal{B}_j$ and $\mathcal{L}_1$ be as in the definition of ρ_{f_1}. Ignoring empty columns of $\mathcal{L}_1$, the map ρ_{f_1} reverses fillings in each $\mathcal{R}_i$, so applying ρ_{f_1} twice leaves fillings in each $\mathcal{R}_i$ unaffected.

Each $\mathcal{B}_j$ is a layer polyomino. Applying f_1 to $\mathcal{L}_1$ and then to $\mathcal{B}_j \cap \mathcal{L}_1$ only changes the location of the empty columns of L in $\mathcal{B}_j$, but does not affect the fillings in the non-empty columns of $\mathcal{B}_j$. Applying ρ_{f_1} on $\mathcal{B}_j$ will not touch the empty columns. Now when we apply the ρ_{f_1} to the whole polyomino $\mathcal{L}$ again, steps 2 and 3 will move the empty columns in $\mathcal{B}_j$ back to their original places, while step 4 will map the current filling on $\mathcal{B}_j$ back to L on $\mathcal{B}_j$, by the inductive hypothesis.

$\square$

For a 01-filling L of a layer polyomino $\mathcal{L}$, Figure 3.8 shows one iteration of the map ρ_{f_1} and the final result. The cells in the polyomino are labeled so one can observe where each cell ends up.



(a) The filling of L (b) First iteration of ρ_{f_1} (c) Result of $\rho_{f_1}(L)$

Figure 3.8: A demonstration of the map ρ_{f_1}.

Theorem 3.13. *For a layer polyomino $\mathcal{L}$ with n rows and $\vec{r} \in \mathbb{N}^n$,*

$$\mathrm{av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) = \mathrm{av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})).$$

Proof. Let f_1 be as defined in Example 3.11. For an $n \times m$ rectangle $\mathcal{R}$ and $\vec{r} \in \mathbb{N}^n$ the restriction, $f_1 : \mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{R}, \vec{r})) \to \mathrm{Av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{R}, \vec{r}))$ is clearly well defined. For a layer polyomino $\mathcal{L}$ with n rows and $\vec{r} \in \mathbb{N}^n$, we show the restriction ρ_{f_1} : $\mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) \to \mathrm{Av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$ is also well defined. That is, if L has no $\mathrm{ne}^{\square}$-chains, then ρ_{f_1} has no $\mathrm{se}^{\square}$-chains.

We proceed by induction on the number of rows of $\mathcal{L}$. If $\mathcal{L}$ has only one row or is a rectangle, then the claim is true. In general, let $L \in \mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$ and $\mathcal{R}_i$, $\mathcal{B}_j$, $\mathcal{L}_1$ and L_1 be as in the definition of ρ_{f_1}. Within L_1, there are no $\mathrm{ne}^{\square}$-chains so that $f_1(L_1)$ has no $\mathrm{se}^{\square}$-chains. For any non-empty cell α in some $\mathcal{R}_i$ the cells to the upper left and lower right of α are empty in $f_1(L_1)$. Since f_1 fixes empty columns, the empty cells will remain empty in the final filling $\rho_{f_1}(L)$. Thus α forms no $\mathrm{se}^{\square}$-chains in $\rho_{f_1}(L)$.

For cells in $\mathcal{B}_j$ for some j, as observed before, applying f_1 to L_1 and then to $L_1 \cap \mathcal{B}_j$ will not change the filling $L \cap \mathcal{B}_j$ in the non-empty columns of $\mathcal{B}_j$. Hence there are no $\mathrm{ne}^{\square}$-chains after steps 2 and 3. By induction, applying ρ_{f_1} to $\mathcal{B}_j$ yields a filling with no $\mathrm{se}^{\square}$-chains.

Therefore, $\rho_{f_1}(\mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))) \subseteq \mathrm{Av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$, and hence

$$\mathrm{av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})) \leq \mathrm{av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})).$$

The reverse direction is proved similarly. In conclusion, ρ_{f_1} is a bijection from $\mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r})$ to $\mathrm{Av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}))$. $\square$

If one fixes both row sum $\vec{r}$ and column sum $\vec{c}$, then $\mathrm{av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c}))$ may not equal $\mathrm{av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c}))$, as shown in the polyomino in Figure 3.9 with $\vec{r} =$

$(1,1,2)$ and $\vec{c} = (1,2,1)$. It is easy to check that $\operatorname{av}(\mathrm{ne}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})) = 0$ and $\operatorname{av}(\mathrm{se}^{\square}, \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})) = 1$.

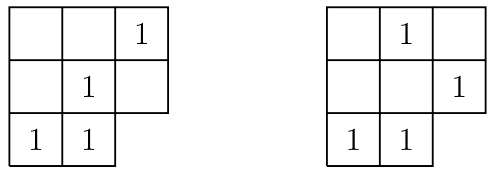


Figure 3.9: Fillings with $\vec{r} = (1,1,2)$ and $\vec{c} = (1,2,1)$.

The following theorem was first proved by Phillipson, Yan and Yeh [14] by induction on the generating functions. Now we give a bijective proof by using the map ρ_{f_1}. In the statement of Theorem 3.14 statistics $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ represent the numbers of strong ne- and se-chains.

Theorem 3.14. *For a layer polyomino $\mathcal{L}$ with n rows and m columns, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \{0,1\}^m$, the map ρ_{f_1} restricted to $\mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$ is a bijection that maps $\mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$ to itself and carries the statistics $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ to $(\mathrm{se}^{\square}, \mathrm{ne}^{\square})$, where f_1 is defined as in Example 3.11. Consequently, the distribution of the joint statistic $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ is symmetric in $\mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$.*

Proof. For an $n \times m$ rectangle $\mathcal{R}$, $\vec{r} \in \mathbb{N}^n$, and $\vec{c} \in \{0,1\}^m$, the restriction of f_1 to $\mathcal{F}_{01}(\mathcal{R}, \vec{r}, \vec{c})$ is well defined as f_1 fixes empty columns, and hence preserves the column sums when $\vec{c} \in \{0,1\}^m$. Also, f_1 exchanges $\mathrm{ne}^{\square}$ and $\mathrm{se}^{\square}$-chains in $\mathcal{R}$.

We'll show that ρ_{f_1} exchanges the numbers of $\mathrm{ne}^{\square}$ and $\mathrm{se}^{\square}$-chains for fillings in $\mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$. Again we proceed by induction. The claim is obvious if $\mathcal{L}$ has only one row or is a rectangle. Assume it is true for all layer polyominoes with less than n rows. For $L \in \mathcal{F}_{01}(\mathcal{L}, \vec{r}, \vec{c})$ set $\mathcal{R}_i$, $\mathcal{B}_j$ and $\mathcal{L}_1$, L_1 to be as in the definition of ρ_{f_1}.

First we show that each ne$^\square$-chain (resp, se$^\square$–chain) of L_1 not completely contained in some $\mathcal{B}_i$ has a corresponding se$^\square$-chain (resp, ne$^\square$–chain) in $\rho_{f_1}(L)$. Explicitly, let entries at cells α, β be such a ne$^\square$-chain, where the columns of α, β are the k_1^{th} and k_2^{th} nonempty columns of L_1, counting from left, then Step 2 of ρ_{f_1} maps them to a se$^\square$-chain with 1-cells γ and δ, in the k_1^{th} and k_2^{th}-th nonempty columns of $f_1(L_1)$, counting from right. We have three cases.

1. Both α and β are contained in (possibly different) $\mathcal{R}_i$'s.

2. One of α, β is contained in $\mathcal{R}_i$, and the other is in $\mathcal{B}_j$.

3. The cell α is in $\mathcal{B}_i$ and β in $\mathcal{B}_j$, with $i \neq j$.

For Case 1, γ and δ are in the same $\mathcal{R}_i$ as α and β, respectively, and are not changed further by steps 3,4 of ρ_{f_1}. Thus (γ, δ) remains a se$^\square$-chain.

For Case 2, without loss of generality, assume $\alpha \in \mathcal{R}_i$ and $\beta \in \mathcal{B}_j$. Then $\gamma \in \mathcal{R}_i$ and will not be changed further. The cell δ may be changed in Steps 3 and 4 of ρ_{f_1}. However, the operations on both steps 3 and 4 preserves the column sum, hence in the final filling there is a unique 1-cell in $\mathcal{B}_j$ that lies in the same column as δ. It forms a se$^\square$-chain with γ.

For Case 3, we have $\gamma \in \mathcal{B}_i$ and $\delta \in \mathcal{B}_j$. By the same argument as in Case 2, after steps 3 and 4 there is a unique 1-cell in $\mathcal{B}_i$ that lies in the same column as γ, and a unique 1-cell in $\mathcal{B}_j$ that lies in the same column as δ. These two 1-cells form a se$^\square$-chain.

Applying the same argument in reverse, we conclude that the ne$^\square$-chains (se$^\square$-chains) of L that are not completely in some $\mathcal{B}_i$ are in one-to-one-correspondence with the se$^\square$-chains (ne$^\square$-chains) of $\rho_{f_1}(L)$ not completely in some $\mathcal{B}_i$.

Finally we look at the strong chains inside $\mathcal{B}_i$ for some i. As observed before, steps 2 and 3 of ρ_{f_1} does not change the filling on the non-empty columns of $\mathcal{B}_i$.

Hence after these two steps, the number of strong chains in each $\mathcal{B}_i$ doesn't change. When one applies step 4, by inductive hypothesis, ρ_{f_1} on $\mathcal{B}_i$ is a bijection that maps the statistics $(\mathrm{ne}^{\square}, \mathrm{se}^{\square})$ of fillings of $\mathcal{B}_i$ to the statistics $(\mathrm{se}^{\square}, \mathrm{ne}^{\square})$ on $\mathcal{B}_i$.

Theorem 3.14 follows from combining all the above cases.

□

The final result is an analog of Theorem 3.13 on $\mathbb{N}$-fillings. We show that over $\mathbb{N}$–fillings of layer polyominoes with fixed row and column sums, the number of fillings with no $\mathrm{ne}^{\square}$–chains is the same as that with no $\mathrm{se}^{\square}$–chains.

Theorem 3.15. *For a layer polyomino $\mathcal{L}$ with n rows and m columns, $\vec{r} \in \mathbb{N}^n$, and $\vec{c} \in \mathbb{N}^m$,*

$$\mathrm{av}(\mathrm{ne}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) = \mathrm{av}(\mathrm{se}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})).$$

Proof. For an $n \times m$ rectangle $\mathcal{R}$, $\vec{r} \in \mathbb{N}^n$ and $\vec{c} \in \mathbb{N}^m$, define the map

$$f_2 : \mathrm{Av}(\mathrm{ne}, \mathcal{F}_{\mathbb{N}}(R, \vec{r}, \vec{c})) \to \mathrm{Av}(\mathrm{se}, \mathcal{F}_{\mathbb{N}}(R, \vec{r}, \vec{c}))$$

that maps the unique filling of $\mathcal{R}$ with no ne-chains to the unique filling with no se-chains, as in Lemma 3.6. For a layer polyomino $\mathcal{L}$ with row sums $\vec{r}$ and column sums $\vec{c}$, we show the restriction $\rho_{f_2} : \mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c})) \to \mathrm{Av}(\mathrm{se}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ is also well defined.

The proof is again by induction and analogous to that of Theorem 3.13. Let $L \in \mathrm{Av}(\mathrm{ne}^{\square}, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ and $\mathcal{R}_i$, $\mathcal{B}_j$ and $\mathcal{L}_1$, L_1 be as in the definition of ρ_{f_2}. Step 2 of ρ_{f_2} maps the filling L_1 to one with no $\mathrm{se}^{\square}$-chains. For any non-empty cell α in some $\mathcal{R}_i$, the columns to the upper right and lower left are empty. Since f_2 preserves column sums, these areas will remain empty in $\rho_{f_2}(L)$.

For each $\mathcal{B}_i$, we claim that after steps 2 and 3 of ρ_{f_2}, the filling on $\mathcal{B}_i$ contains on ne$^\square$-chains. To see this, note that by definition of f_2, there is no ne$^\square$-chain inside $\mathcal{L}_1 \cap \mathcal{B}_i$. Clearly there is no ne$^\square$-chains containing two cells in $\mathcal{B}_i \setminus \mathcal{L}_1$. If there exists a ne$^\square$-chain on $\mathcal{B}_i$ containing cells α, β with $\alpha \notin \mathcal{L}_1$ and $\beta \in \mathcal{L}_1$, since both operations in steps 2 and 3 preserves the row sums of $\mathcal{L}_1 \cap \mathcal{B}_i$, there must be a nonempty cell in L that lies in the same row as β in $\mathcal{L}_1 \cap \mathcal{B}_i$. Such a cell and α forms a ne$^\square$-chain in L, contradiction.

Thus ρ_{f_2} is well defined and is an injection from $\mathrm{Av}(\mathrm{ne}^\square, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ to $\mathrm{Av}(\mathrm{se}^\square, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$. The reverse inclusion is similarly proved, which implies Theorem 3.15. $\square$

Remarks:

1. Unlike 01-fillings, for general $\mathcal{L}$, $\mathrm{Av}(\mathrm{ne}^\square, \mathcal{F}_{\mathbb{N}}(\mathcal{L}, \vec{r}, \vec{c}))$ contains more than one element.

2. In general it's not true that for a given f, $(\rho_f)^{-1} = \rho_{f^{-1}}$. However, this is the case for the f_1 and f_2 we used. It's unclear what conditions on f would guarantee this, but it's an interesting occurrence.

3. It is natural to ask if we can flip adjacent rows while preserving the number of ne$^\square$–chains in layer polyominoes. The answer is no. Consider the polyominoes in Figure 3.10 with row and column sums $(1, 1, 1)$. The left polyomino has 2 fillings with no ne$^\square$–chains where as

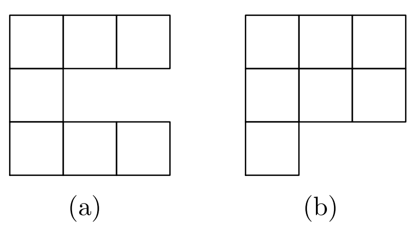


Figure 3.10: Layer polyominoes with one row flipped.

the right has only 1. In fact, this example shows that the joint distribution of $(\mathrm{ne}^\square, \mathrm{se}^\square)$ is dependent on the order of the rows in layer polyominoes.

4. CROSSINGS AND NESTINGS IN ALTERNATING MATCHINGS

A 01 sequence is a sequence of an equal number of 0's and 1's. A matching on a 01 sequence has the property that each part of the matching contains both a 0 and 1. In this chapter we focus on matchings of alternating sequences, which are 01 sequences that begin with 0 and alternate 0's and 1's. A matching on an alternating sequence can be viewed as a matching where each part of the matching contains both and even and an odd element. These matchings are interesting to study, from a combinatorial perspective, because they do not exhibit symmetry between crossings and nestings, which is very common in most combinatorial structures.

Section 4.1 provides key definitions and shows the relationship with permutations. Section 4.2 gives necessary and sufficient conditions when a matching on alternating sequences is non-nesting. Finally, Section 4.3 lays out how to explicitly enumerate non-nesting matchings on alternating sequences and concludes with a few conjectures.

4.1 Preliminaries

A matching, σ, on the alternating 01 sequence of length n is a matching on $[n]$ so that each part of σ has one even element and one odd. The set of all matchings on the alternating sequence of length n is denoted $\mathcal{A}_n$. For convenience, with refer to matchings on alternating sequences as *alternating matchings*. Unlike matchings, which are drawn on $[2n]$, alternating matchings are drawn on $(01)^n$ with each arc connected to both a 0 and 1. Figure 4.1 has an example of two matchings, one that is an alternating matching and one that is not.

Alternating matchings inherit the structure of crossings and nestings from matchings. Consider a matching, M, with part $\{2i, 2j\} \in M$. There are $|2j - 2i| - 1$ digits

(a) Alternating matching (b) Not an alternating matching

Figure 4.1: Two matchings related to alternating matchings

between $2i$ and $2j$, which implies there must be a crossing as this number is odd. This means that any non-crossing matching has parts with one even and one odd element so that each non-crossing matching is also an alternating matching. Thus there are an equal number of non-crossing matchings as alternating matchings. The same is not true for nestings, Figure 4.1(b) shows a non-nesting matching that is not alternating. This implies the usual symmetry between crossings and nestings does not hold for alternating matchings.

For an alternating matching $\sigma \in \mathcal{A}_n$ we say, for $i \in [n]$, $\sigma(i) = j$ if $\{i, j\} \in \sigma$. Using this identification, the conditions for crossings and nestings in alternating matchings can be restated. For $\sigma \in \mathcal{A}_n$, arcs $(i, \sigma(i))$ and $(j, \sigma(j))$ are *crossing* if one of the following holds,

$$\begin{array}{ll} i < j < \sigma(i) < \sigma(j) & \qquad i < \sigma(j) < \sigma(i) < j \\ \sigma(i) < j < i < \sigma(j) & \qquad \sigma(i) < \sigma(j) < i < j \end{array}$$

and are *nesting* if,

$$\begin{array}{ll} i < j < \sigma(j) < \sigma(i) & \qquad i < \sigma(j) < j < \sigma(i) \\ \sigma(i) < j < \sigma(j) < i & \qquad \sigma(j) < \sigma(i) < i < j. \end{array}$$

Alternating matchings are naturally identified with permutations, see Section 2.4 for relevant definitions of permutations. Given an alternating matching $\sigma \in \mathcal{A}_n$

define $\varphi \in S_n$ so that for $i \in [n]$, $\varphi(i) = \sigma(i)$. This is easily seen to be well defined; however, it does not preserve crossings and nestings. Figure 4.2 has an example of this bijection so that the permutation has no nestings whereas the alternating matching does.

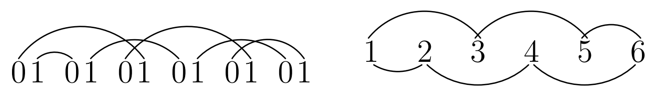


Figure 4.2: A comparison between alternating matchings and permutations.

Recall in Section 2.4 we stated a bijection between bi-colored motzkin paths and non-nesting permutations. This can be extended to alternating matchings, but the image will not necessarily be non-nesting.

4.2 Conditions for Top and Bottom Non-Nesting

In this section we prove two independent conditions for alternating matchings being non-nesting. Both theorems rely on the bijection between permutations and bicolored motzkin paths given in Section 2.4.

Theorem 4.1. *Let $M = m_1 m_2 \cdots m_n$ be a bicolored Motzkin path and $\sigma \in \mathcal{A}_n$ the corresponding alternating matching. Then σ is non-nesting if and only if for every i with $i < \sigma(i)$*

$$\#\{i < j < \sigma(i) \mid m_j = E\} + \delta_{m_{\sigma(i)}=E} = \#\{i < j < \sigma(i) \mid m_j = \bar{E}\} + \delta_{m_i=N}. \quad (4.1)$$

Theorem 4.2. *Let $M = m_1 m_2 \cdots m_n$ be a bicolored Motzkin path and $\sigma \in \mathcal{A}_n$ the corresponding alternating matching. Then σ is non-nesting if and only if for every i*

with $i > \sigma(i)$

$$\#\{\sigma(i) < j < i \mid m_j = E\} = \#\{\sigma(i) < j < i \mid m_j = \bar{E}\} + 1. \tag{4.2}$$

Before we prove these statements, we require a lemma. This lemma will allow us to adjust nestings so that they satisfy nicer conditions.

Lemma 4.3. *Let* $M = m_1 m_2 \dots m_n$ *be a bicolored motzkin path associated to the alternating matching* $\sigma \in \mathcal{A}_n$*. Suppose* σ *has a nesting. If* $i < \sigma(i)$ *is nested with* $\sigma(j) < j$ *then there exists* k *and* ℓ *so that*

1. *If* $\sigma(j) \leq i < \sigma(i) < j$ *then* $\sigma(k) \leq \sigma^{-1}(\ell) < \ell < k$ *with* $m_a = N$ *for all* $\ell < a < k$*.*

2. *If* $\sigma(j) < i < \sigma(i) \leq j$ *then* $\ell < k < \sigma(k) \leq \sigma^{-1}(\ell)$ *with* $m_a = S$ *for all* $\ell < a < k$*.*

3. *If* $i \leq \sigma(j) < j < \sigma(i)$ *then* $\sigma^{-1}(\ell) \leq \sigma(k) < k < \ell$ *with* $m_a = N$ *for all* $k < a < \ell$

4. *If* $i < \sigma(j) < j \leq \sigma(i)$ *then* $k < \ell < \sigma^{-1}(\ell) \leq \sigma(k)$ *with* $m_a = S$ *for all* $k < a < \ell$*.*

Proof. We'll prove the first statement, the others follow similarly. Choose ℓ so that $\sigma(i) \leq \ell < j$, $m_\ell \in \{\mathrm{E}, \mathrm{S}\}$ and $m_{\ell+1} \in \{\bar{\mathrm{E}}, \mathrm{N}, \mathrm{S}\}$. Such an ℓ exists as $m_{\sigma(i)} \in \{\mathrm{E}, \mathrm{S}\}$ and $m_j \in \{\bar{\mathrm{E}}, \mathrm{S}\}$. Let k be such that $\ell < k \leq j$, $m_k \in \{\bar{\mathrm{E}}, \mathrm{S}\}$ and for all a with $\ell < a < k$, $m_a = \mathrm{N}$. Then $\sigma(i) \leq \ell$ so $i \leq \sigma^{-1}(\ell)$ and $k \leq j$ so that $\sigma(k) \leq \sigma(j)$. Thus $\sigma(k) \leq \sigma(\ell) < \ell < k$. Figure 4.3 demonstrates this selection.

□

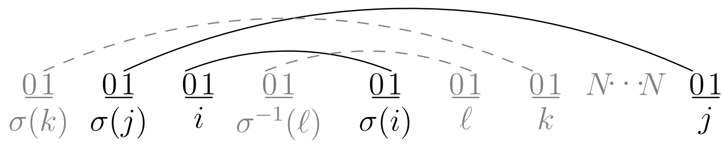


Figure 4.3: Lemma 4.3 part 1.

Note that for a non-nesting permutation, φ, for each arc $i \leq \varphi(i)$ and each $i < j < \varphi(i)$ we must have $\varphi^{-1}(j) < i$ and $\varphi(i) < \varphi(j)$, otherwise φ would have a nesting. If $M = m_1 \ldots m_n$ is the corresponding bicolored motzkin path, then we must have

$$\#\{j \leq \sigma(i) \mid m_j \in \{\mathrm{S}, \mathrm{E}\}\} = \#\{j \leq i \mid m_j \in \{\mathrm{N}, \mathrm{E}\}\}$$
$$\#\{j \leq \sigma(i) \mid m_j \in \{\mathrm{S}, \bar{\mathrm{E}}\}\} = \#\{j \leq i \mid m_j \in \{\mathrm{N}, \bar{\mathrm{E}}\}\}.$$

Proof of 4.1. Assume σ is a non-nesting alternating matching. Let $i < \sigma(i)$, since σ is non-nesting we must have the following equations.

$$\#\{j \leq \sigma(i) \mid m_j \in \{\mathrm{S}, \mathrm{E}\}\} = \#\{j \leq i \mid m_j \in \{\mathrm{N}, \mathrm{E}\}\}$$
$$\#\{j \leq \sigma(i) \mid m_j \in \{\mathrm{S}, \bar{\mathrm{E}}\}\} = \#\{j < i \mid m_j \in \{\mathrm{N}, \bar{\mathrm{E}}\}\}$$

Subtracting these equations and rearranging we obtain (4.1).

On the other hand, assume σ is a nested alternating matching corresponding to a bicolored motzkin path. By Lemma 4.3 and since the corresponding permutation is non-nesting, there are three cases to consider

1. There is i with $\sigma^2(i) = i$.

2. There are i, j so that $\sigma(i) < i < \sigma(i) \leq j$ with $m_a = S$ for all $\sigma(j) < a < i$.

3. There are i, j so that $i \leq \sigma(j) < j < \sigma(i)$ with $m_a = N$ for all $j < a < \sigma(i)$.

Case 4.1.1 ($\sigma^2(i) = i$)**.** Figure 4.4 shows this case. Since the permutation associated

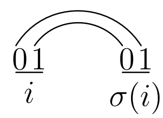


Figure 4.4: $i = \sigma^2(i)$

with σ is non-nesting, we have

$$\#\{\ell \leq \sigma(i) \mid m_\ell \in \{\mathrm{S}, \mathrm{E}\}\} = \#\{\ell \leq i \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\}$$
$$\#\{\ell \leq \sigma(i) \mid m_\ell \in \{\mathrm{S}, \bar{\mathrm{E}}\}\} = \#\{\ell \leq i \mid m_\ell \in \{\mathrm{N}, \bar{\mathrm{E}}\}\}$$

Subtracting these gives

$$\#\{i < \ell < \sigma(i) \mid m_\ell = \mathrm{E}\} = \#\{i < \ell < \sigma(i) \mid m_\ell = \bar{\mathrm{E}}\}.$$

But $m_i = \mathrm{N}$ and $m_{\sigma(i)} = \mathrm{S}$ so (4.1) cannot be satisfied.

Case 4.1.2 ($\sigma(i) < i < \sigma(i) \leq j$ with $m_a = S$ for all $\sigma(j) < a < i$)**.** Figure 4.5 shows this case. The following equations follow from the permutation corresponding

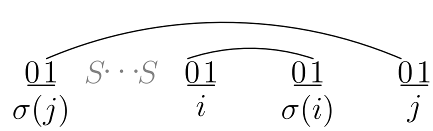


Figure 4.5: $\sigma(i) < i < \sigma(i) \leq j$ with $m_a = S$ for all $\sigma(j) < a < i$

to σ being non-nesting.

$$\#\{\ell \leq i \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\} = \#\{\ell \leq \sigma(i) \mid m_\ell \in \{\mathrm{S}, \mathrm{E}\}\} \tag{4.3}$$

$$\#\{\ell < \sigma(j) \mid m_\ell \in \{\mathrm{N}, \bar{\mathrm{E}}\}\} = \#\{\ell < j \mid m_\ell \in \{\mathrm{S}, \bar{\mathrm{E}}\}\} \tag{4.4}$$

$$h_i = h_{\sigma(j)} + \delta_{m_{\sigma(i)}=\mathrm{N}} \tag{4.5}$$

By manipulating equations (4.3) and (4.4) we have,

$$\begin{aligned} h_i &= -\delta_{m_i=\mathrm{N}} + \#\{i < \ell \leq \sigma(i) \mid m_\ell \in \{\mathrm{E}, \mathrm{S}\}\} \\ h_{\sigma(j)} &= \#\{\sigma(j) \leq \ell < j \mid m_\ell \in \{\mathrm{S}, \mathrm{E}\}\}. \end{aligned}$$

Plugging these into (4.5) we have,

$$\begin{aligned} \#\{i < \ell \leq \sigma(i) \mid m_\ell \in \{\mathrm{E}, \mathrm{S}\}\} =& \delta_{m_i=\mathrm{N}} + \#\{\sigma(j) \leq \ell < j \mid m_\ell \in \{\mathrm{S}, \bar{\mathrm{E}}\}\} \\ &+ \delta_{m_{\sigma(j)=N}}. \end{aligned}$$

Rearranging a bit we get,

$$\begin{aligned} \#\{i < \ell < \sigma(i) \mid m_\ell = \mathrm{E}\} + \delta_{m_{\sigma(i)}=\mathrm{E}} =& \delta_{m_i=\mathrm{N}} + \#\{i < \ell < \sigma(i) \mid m_\ell = \bar{\mathrm{E}}\} + \\ &+ 1 + \#\{\sigma(i) < \ell < j \mid m_\ell \in \{\mathrm{E}, \mathrm{S}\}\} \\ >& \delta_{m_i=\mathrm{N}} + \#\{i < \ell < \sigma(i) \mid m_\ell = \bar{\mathrm{E}}\} \end{aligned}$$

Thus equation (4.1) cannot hold.

Case 4.1.3 ($i \leq \sigma(j) < j < \sigma(i)$ with $m_a = N$ for all $j < a < \sigma(i)$)**.** Figure 4.6 shows this case. This is very similar to the previous case, except equation (4.5)

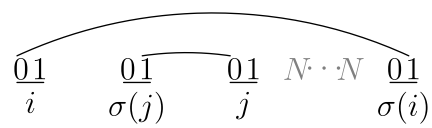


Figure 4.6: $i \leq \sigma(j) < j < \sigma(i)$ with $m_a = N$ for all $j < a < \sigma(i)$

becomes,

$$h_j = h_{\sigma(i)} + \delta_{m_j=\mathrm{S}}.$$

We rearrange equations (4.3) and (4.4) into,

$$\begin{aligned} h_{\sigma(i)} &= 1 + \#\{i < \ell < \sigma(i) \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\} \\ h_j &= \#\{\sigma(j) \leq \ell < j \mid m_\ell \in \{\mathrm{N}, \bar{\mathrm{E}}\}\}. \end{aligned}$$

Combine the previous equations and rearrange.

$$\begin{aligned} \#\{i < \ell < \sigma(i) \mid m_\ell = \bar{\mathrm{E}}\} + \delta_{m_i=N} =& \#\{i < \ell < \sigma(i) \mid m_\ell = \mathrm{E}\} + \delta_{m_{\sigma(i)}=\mathrm{E}} + 1 + \\ &+ \delta_{m_j=\mathrm{S}} + \delta_{m_j=\bar{\mathrm{E}}} + \delta_{m_i=\mathrm{N}} - \delta_{m_{\sigma(i)}=\mathrm{E}} + \\ &+ \#\{i < \ell < \sigma(j) \mid m_\ell \in \{\mathrm{N}, \bar{\mathrm{E}}\}\} \\ \geq& \#\{i < \ell < \sigma(i) \mid m_\ell = \mathrm{E}\} + \delta_{m_{\sigma(i)}=\mathrm{E}} + 1 \\ >& \#\{i < \ell < \sigma(i) \mid m_\ell = \mathrm{E}\} + \delta_{m_{\sigma(i)}=\mathrm{E}}. \end{aligned}$$

Once again equation (4.1) cannot hold.

□

Proof of 4.2. Assume σ is a non-nesting alternating matching. Let i be so that

$\sigma(i) < i$. We have

$$\#\{\ell \le \sigma(i) \mid m_\ell \in \{\mathrm{N}, \bar{\mathrm{E}}\}\} = \#\{\ell \le i \mid m_\ell \in \{\mathrm{S}, \bar{\mathrm{E}}\}\}$$
$$\#\{\ell \le \sigma(i) \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\} = \#\{\ell < i \mid m_\ell \in \{\mathrm{S}, \bar{\mathrm{E}}\}\}.$$

Subtract these,

$$\begin{aligned}\#\{\ell \le \sigma(i) \mid m_\ell = \bar{\mathrm{E}}\} - \#\{\ell \le \sigma(i) \mid m_\ell = \mathrm{E}\} =&\#\{\ell \le i \mid m_\ell = \bar{\mathrm{E}}\} + \delta_{(m_i=\mathrm{S})}+ \\ &+ \#\{\ell < i \mid \mathrm{E}\}.\end{aligned}$$

Continuing

$$\begin{aligned}\#\{i < \ell < \sigma(i) \mid m_\ell = \mathrm{E}\} =&\#\{i < \ell \le \sigma(i) \mid m_\ell = \bar{\mathrm{E}}\} + \delta_{(m_i=\mathrm{S})} \\ =&\#\{i < \ell < \sigma(i) \mid m_\ell = \bar{\mathrm{E}}\} + 1\end{aligned}$$

which is exactly equation (4.2).

On the other hand, assume σ is a nested alternating matching corresponding to a bicolored motzkin path. By Lemma 4.3 and since the corresponding permutation is non-nesting, there are three cases to consider

1. There is an i so that $\sigma^2(i) = i$.

2. There are i, j so that $j < \sigma(i) < i \le \sigma(j)$ with $m_a = \mathrm{S}$ for all $j < a < \sigma(i)$.

3. There are i, j so that $\sigma(i) \le j < \sigma(j) < i$ with $m_a = \mathrm{N}$ for all $\sigma(j) < a < i$.

Case 4.2.1 ($i = \sigma^2(i)$)**.** This is exactly the same as in the proof of Theorem 4.1.

Case 4.2.2 ($j < \sigma(i) < i \le \sigma(j)$ with $m_a = \mathrm{S}$ for all $j < a < \sigma(i)$)**.** Figure 4.7 shows this case. Since the permutation corresponding to σ is non-nesting, we have

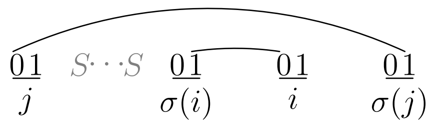


Figure 4.7: $j < \sigma(i) < i \leq \sigma(j)$ with $m_a = \mathrm{S}$ for all $j < a < \sigma(i)$

the equations

$$\#\{\ell < j \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\} = \#\{\ell < \sigma(j) \mid m_\ell \in \{\mathrm{S}, \mathrm{E}\}\} \tag{4.6}$$

$$\#\{\ell \leq \sigma(i) \mid m_\ell \in \{\mathrm{N}, \bar{\mathrm{E}}\}\} = \#\{\ell \leq j \mid m_\ell \in \{\mathrm{S}, \bar{\mathrm{E}}\}\} \tag{4.7}$$

$$h_{\sigma(i)} = h_j + \delta_{(m_j=\mathrm{N})} - \#\{j \leq \ell < \sigma(i) \mid m_\ell = \mathrm{S}\}. \tag{4.8}$$

Rewrite (4.6) and (4.7),

$$\begin{aligned} h_j &= \#\{j \leq \ell < \sigma(j) \mid m_\ell \in \{\mathrm{S}, \mathrm{E}\}\} \\ h_{\sigma(j)} &= \#\{\sigma(i) < \ell \leq i \mid m_\ell \in \{\mathrm{S}, \bar{\mathrm{E}}\}\} - \delta_{(m_{\sigma(i)}=\mathrm{N})} \end{aligned}$$

Plug both of these into (4.8),

$$\begin{aligned} \#\{\sigma(i) < \ell \leq i \mid m_\ell \in \{\mathrm{S}, \bar{\mathrm{E}}\}\} =& \delta_{(m_{\sigma(i)}=\mathrm{N})} + \#\{j \leq \ell < \sigma(j) \mid m_\ell \in \{\mathrm{S}, \mathrm{E}\}\} + \\ &+ \delta_{(m_j=\mathrm{N})} - \#\{j \leq \ell < \sigma(i) \mid \ m_\ell = \mathrm{S}\}. \end{aligned}$$

Simplifying,

$$\begin{aligned} \#\{\sigma(i) < \ell < i \mid m_\ell = \bar{\mathrm{E}}\} + 1 =& \#\{\sigma(i) < \ell < i \mid m_\ell = \mathrm{E}\} + \delta_{(m_j=E)} + \delta_{(m_j=\mathrm{N})} + \\ &+ \#\{i \leq \ell < \sigma(j) \mid m_\ell \in \{\mathrm{S}, \mathrm{E}\}\} + \delta_{(m_{\sigma(i)}=\mathrm{N})} \\ >& \#\{\sigma(i) < \ell < i \mid m_\ell = \mathrm{E}\} \end{aligned}$$

Thus equation (4.1) cannot hold.

Case 4.2.3 ($\sigma(i) \leq j < \sigma(j) < i$ with $m_a = \mathrm{N}$ for all $\sigma(j) < a < i$)**.** Figure 4.8 shows this case. Similar to the previous case, we have

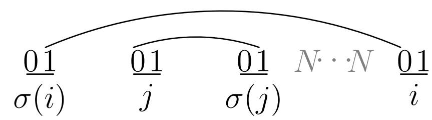


Figure 4.8: $\sigma(i) \leq j < \sigma(j) < i$ with $m_a = \mathrm{N}$ for all $\sigma(j) < a < i$

$$h_{\sigma(i)} = h_i + \delta_{(m_{\sigma(i)}=\mathrm{S})} - \#\{\sigma(j) \leq \ell < i \mid m_\ell = \mathrm{N}\} \tag{4.9}$$

Rewrite equations (4.6) and (4.7)

$$h_{\sigma(j)} = \#\{j \leq \ell < \sigma(j) \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\}$$
$$h_i = \#\{\sigma(i) < \ell < i \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\} + 1$$

Plug these equations into equation (4.9).

$$\begin{aligned}\#\{j \leq \ell < \sigma(j) \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\} =& \#\{\sigma(i) < \ell < i \mid m_\ell \in \{\mathrm{N}, \bar{\mathrm{E}}\}\} + 1 + \\ &+ \delta_{(m_{\sigma(j)}=\mathrm{S})} - \#\{\sigma(j) \leq \ell < i \mid m_\ell = \mathrm{N}\}\end{aligned}$$

Finally,

$$\begin{aligned}
\#\{\sigma(i) < \ell < i \mid m_\ell = \mathrm{E}\} =& \#\{\sigma(i) < \ell < i \mid m_\ell = \bar{\mathrm{E}}\} + 1 + \delta_{(m_{\sigma(j)}=\mathrm{S})} + \\
& + \#\{\sigma(i) < \ell < j \mid m_\ell \in \{\mathrm{N}, \mathrm{E}\}\} + \delta_{(m_{\sigma(j)}=\mathrm{S})} \\
>& \#\{\sigma(i) < \ell < i \mid m_\ell = \bar{\mathrm{E}}\} + 1.
\end{aligned}$$

Thus equation (4.1) cannot hold.

□

4.3 Primitive Permutations and Valid Paths

A primitive motzkin path has height $h_i > 0$ for all $i > 0$, in other words the path only touches the x-axis at the beginning and the end. A primitive alternating matching is one that corresponds to a primitive bicolored motzkin path.

Let $\mathcal{P}_n$ denote the set of primitive alternating matchings in $\mathcal{A}_n$ that are non-nesting. If we are able to classify all alternating matchings in $\mathcal{P}_n$ we can then construct all non-nesting alternating matchings. This process is quite simple, given $\sigma_i \in \mathcal{P}_{k_i}$, with $\sum_{i=1}^{\ell} k_i = n$, create $\sigma \in \mathcal{A}_n$ as follows, for $i \in [n]$ define

$$\sigma(i) = \sigma_\alpha(\beta) + \sum_{j=1}^{\alpha-1} k_j$$

where $\alpha = \max\{j \mid \sum_{m=1}^{j} k_m < i\} + 1$ and $\beta = i - \sum_{j=1}^{\alpha-1} k_j$.

Example 4.5. Figure 4.9 show an alternating matching that has been constructed using the above procedure. The vertical dashed line shows the split between the two primitive alternating matchings used.

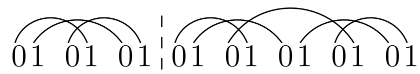


Figure 4.9: An alternating matching given by two primitive alternating matchings

Corollary 4.6. *The number of non-nesting alternating matchings is*

$$\sum_{m=1}^{n} \sum_{\substack{k_1+\cdots+k_m=n \\ k_i>0}} |\mathcal{P}_{k_1}| \cdots |\mathcal{P}_{k_m}| \,.$$

Due to these results we only need focus on primitive alternating matchings. Table 4.1 contains a list of sizes of $\mathcal{P}_n$. These values are not in OEIS and superseeker has no information.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\mathcal{P}_n\rvert$	1	0	1	0	2	1	5	6	15	29	53	121	213	580	934

Table 4.1: A table listing values of $|\mathcal{P}_n|$.

When viewed as permutations, elements of $\mathcal{P}_n$ have an interesting cycle structure. See example 4.7 for the first few $\mathcal{P}_n$. In the following, we write alternating matchings in a similar manner as we write permutations, in line and cycle notation. This is for convenience.

Example 4.7. The sets $\mathcal{P}_n$ for the first few n. The elements are written first as words, then as cycles and finally reduced cycles (if there are more than one cycle).

$\mathcal{P}_1$	$\mathcal{P}_3$	$\mathcal{P}_5$
1	$231 \rightarrow (123)$	$24153 \rightarrow (12453)$
		$34512 \rightarrow (13524)$

$\mathcal{P}_6$

$245163 \rightarrow (124)(356) \rightarrow (123)(123)$

$\mathcal{P}_7$	$\mathcal{P}_8$
$2416375 \rightarrow (2416375)$	$24167385 \rightarrow (12453)(123)$
$2516734 \rightarrow (2516734)$	$24517386 \rightarrow (123)(12453)$
$3461275 \rightarrow (3461275)$	$24617835 \rightarrow (123)(13524)$
$3561724 \rightarrow (3561724)$	$25671834 \rightarrow (123)(13524)$
$4567123 \rightarrow (4567123)$	$34617285 \rightarrow (13524)(123)$
	$35671284 \rightarrow (13524)(123)$

Example 4.7 is interesting because it implies that primitive alternating matchings form a basic cycle structure. It appears that for even n, $\mathcal{P}_n$ contains combinations of cycles of odd length, and for odd n, $\mathcal{P}_n$ contains single cycle elements. This is not true in general, for example $\mathcal{P}_9$ contains element $245178396 \rightarrow (123)(123)(123)$, but this is created from a cycles of odd length.

The remainder of this section will be devoted to attempting to prove the previous observations. The results that are unproven are left as conjectures.

4.3.1 Valid Paths

Definition 4.8. A *valid path* is a bicolored motzkin path, $M = m_1 \cdots m_{2n+1}$, so that for $n \geq 1$ we have $m_1 = \mathrm{N}$,

1. If $m_i = \mathrm{N}$ or $m_i = \bar{\mathrm{E}}$ then $m_{i+1} \in \{\mathrm{N}, \mathrm{E}\}$

2. If $m_i = \mathrm{S}$ or $m_i = \mathrm{E}$ then $m_{i+1} \in \{\bar{\mathrm{E}}, \mathrm{S}\}$

and for $k < 2n+1$

$$|m_1 \cdots m_k|_{\mathrm{N}} > |m_1 \cdots m_k|_{\mathrm{S}} \qquad \text{and} \qquad |m_1 \cdots m_{2n+1}|_{\mathrm{N}} = |m_1 \cdots m_{2n+1}|_{\mathrm{S}}$$

We claim that valid paths correspond to single cycle non-nesting alternating matchings. Notice that the horizontal steps of a valid path alternate $E\bar{E}E\bar{E}\cdots E$ so that $|M|_E = |M|_{\bar{E}} + 1$.

Example 4.9. Figure 4.10 contains both a valid path and the corresponding alternating matching. The alternating matching is non-nesting and its permutation in

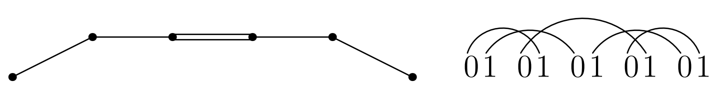


Figure 4.10: A valid path and the corresponding alternating matching

cycle notation is (12453), which is a single cycle.

First we will show that valid paths of length $2n+1$ are enumerated by the n^{th} Catalan number, we'll do this with a bijection to parentheses sequences. Next we'll show valid paths correspond to non-nesting permutations. And finally we'll state some conjectures.

Given a valid path M, define $f(M) = u_1 \dots u_{2n} = u$ so that

$$u_i = \begin{cases} \texttt{(} & \text{if } m_i \in \{\mathrm{N}, \bar{\mathrm{E}}\} \\ \texttt{)} & \text{if } m_i \in \{\mathrm{E}, \mathrm{S}\} \end{cases}$$

Note the parenthesis sequence is on length $2n$ whereas the motzkin path is of length $2n+1$. The final step in the path $m_{2n+1} = \mathrm{S}$ is unused.

Proposition 4.10. *For a valid path M, $f(M) = u$ is a closed parentheses sequence.*

Proof. We know $|M|_{\mathrm{N}} = |M|_{\mathrm{S}}$ and $|M|_{\mathrm{E}} = |M|_{\bar{\mathrm{E}}} + 1$, so that

$$|M|_{\mathrm{N}} - (|M|_{\mathrm{S}} - 1) + |M|_{\bar{\mathrm{E}}} - |M|_{\mathrm{E}} = 0$$

So there are an equal number of `(` and `)`. u is closed follows from the steps E and $\bar{\mathrm{E}}$ alternating, there being a greater number of N than S at any step, and $m_{2n} \in \{\bar{\mathrm{E}}, \mathrm{S}\}$ as $m_{2n+1} = \mathrm{S}$. □

Example 4.11.

$$M = \mathrm{NNE\bar{E}ES\bar{E}ES} \mapsto f(M) = \texttt{(()())()}$$

Thus valid paths map to closed parentheses sequences. Now we'll define the inverse map. Let $u = u_1 \cdots u_{2n}$ be a closed parentheses sequence. Define $\varphi(u) =$

$m_1 \cdots m_{2n+1}$ as follows. Set $m_1 = N$, $m_{2n+1} = S$ and for $1 < k < 2n+1$, set

$$m_k = \begin{cases} N & \text{if } u_k = \text{(} \text{ and } m_{k-1} \in \{\mathrm{N}, \bar{\mathrm{E}}\} \\ E & \text{if } u_k = \text{)} \text{ and } m_{k-1} \in \{\mathrm{N}, \bar{\mathrm{E}}\} \\ \bar{E} & \text{if } u_k = \text{(} \text{ and } m_{k-1} \in \{\mathrm{E}, \mathrm{S}\} \\ S & \text{if } u_k = \text{)} \text{ and } m_{k-1} \in \{\mathrm{E}, \mathrm{S}\} \end{cases}$$

Proposition 4.12. *$\varphi(u)$ is a valid path.*

Proof. It's clear that $\varphi(u) = M = m_1 \dots m_{2n+1}$ follows the structure of valid paths, the only this needing to be checked is that M is a primitive path. Let $k = \min\{i \mid |m_1 \cdots m_k|_{\mathrm{N}} = |m_1 \cdots m_k|_{\mathrm{S}}\}$. If $k < 2n+1$ we must have $m_{k+1} = \bar{\mathrm{E}}$, as $m_k = \mathrm{S}$. But, the steps E and $\bar{\mathrm{E}}$ alternate and we must have E before $\bar{\mathrm{E}}$ so $|m_1 \cdots m_k|_{\mathrm{E}} = |m_1 \dots m_k|_{\bar{\mathrm{E}}} + 1$ which implies $|u_1 \cdots u_k|_{)} = |u_1 \cdots u_k|_{(} + 1$ which is a contradiction. $\square$

It's easy to see that f and φ are inverses, which implies that there are Catalan number of valid paths.

Proposition 4.13. *Valid paths correspond to non-nesting alternating matchings.*

Proof. We are going to use Theorem 4.2. Let $M = m_1 m_2 \dots m_{2n+1}$ be a valid path and σ be the corresponding alternating matching. Let i be so that $\sigma(i) < i$. Then $m_i \in \{\bar{\mathrm{E}}, \mathrm{S}\}$ and $m_{\sigma(i)} \in \{\mathrm{N}, \bar{\mathrm{E}}\}$. There must be a set j with $\sigma(i) < j < i$ with $m_j = \mathrm{E}$, as neither $\bar{\mathrm{E}}$ nor S leads to $\bar{\mathrm{E}}$ or S in valid paths.

Let $j < i$ be so that $m_j \in \{\mathrm{E}, \bar{\mathrm{E}}\}$ and there is no ℓ so that $j < \ell < i$ with $m_\ell \in \{\mathrm{E}, \bar{\mathrm{E}}\}$. Similarly, let $k > \sigma(i)$ with $m_k \in \{\mathrm{E}, \bar{\mathrm{E}}\}$.

If $m_j = \bar{\mathrm{E}}$ then we must have $m_{j+1} = \mathrm{N}$, as $m_{j+1} = \mathrm{E}$ contradicts the choice of j. However this implies that starting at m_{j+1} we have a sequence of N steps. Since

$m_i \in \{\bar{\mathrm{E}}, \mathrm{S}\}$ we must have an E step before m_i, N steps only lead to N or E. But this contradicts the choice of j, as we've found a closer E step. Thus we must have $m_j = \mathrm{E}$. Similarly, we see that $m_k = \mathrm{E}$.

Therefore, the first step after $m_{\sigma(i)}$ and the last step before m_i to not be an N or S step is an E step. Since the steps E and Ē alternate in valid paths we have,

$$\#\{\sigma(i) < j < i \mid \mathrm{E}\} = \#\{\sigma(i) < j < i \mid \bar{\mathrm{E}}\} + 1$$

so that σ is non-nesting. $\square$

There is more work to be done in this area, the following are future goals.

Conjecture 4.14.

1. *These valid paths represent single cycle non-nesting permutations.*

2. *These are all (including evens) single cycle non-nesting permutations.*

3. *If $\varphi \in S_n$ is non-nesting and φ has cycles $\varphi_1, \cdots, \varphi_k$, then each φ_i is order isomorphic to a valid path.*

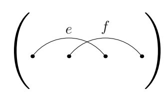
5. MISCELLANEOUS RESULTS

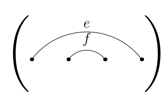
This chapter contains results in several different areas. First we define the notion of crossings and nestings in graphs and prove several results enumerating specific graphs. Then we show the crossing and nesting numbers are unrealated. Second we demonstrate a bijection between plane trees and motzkin paths. Finally we give a generating function for fillings of Ferrers diagrams with no se-chains.

5.1 Crossings and Nestings in Graphs

Recall a few notations from Graph Theory. Let G be a simple, finite undirected graph, $V(G)$ be the vertex set of G and $E(G)$ the edge set. Set $n = |V(G)|$ and $m = |E(G)|$. A vertex ordering is a bijection $\sigma : V(G) \to [n]$. We write $v <_\sigma w$ if $\sigma(v) < \sigma(w)$. The pair (G, σ) denotes the graph G with order σ. The order relation σ allows one to draw (G, σ) on the number line.

Given an edge $e \in E(G)$ define $L(e)$ and $R(e)$ as the endpoints of e with $L(e) <_\sigma R(e)$. Let $e, f \in E(G)$ with no common endpoint and $L(e) <_\sigma R(e)$, we say

1. e and f cross if $L(e) <_\sigma L(f) <_\sigma R(e) <_\sigma R(f)$ 

2. e and f nest if $L(e) <_\sigma L(f) <_\sigma R(f) <_\sigma R(e)$ 

Definition 5.1. Two arcs (v_1, w_1) and (v_2, w_2) in (G, σ) form a $2-$crossing if $v_1 < v_2 < w_1 < w_2$. The arcs form a $2-$nestings if $v_1 < v_2 < w_2 < w_1$.

Denote by $\text{cros}_2(G, \sigma)$ the number of $2-$crossings in (G, σ) and $\text{nest}_2(G, \sigma)$ the number of 2-nestings in (G, σ).

Example 5.2. Figure 5.1 show that $\text{cros}_2(G, \sigma)$ varies with σ. If σ_a gives the order for the left graph in Figure 5.1 and σ_b is the right, then $\text{cros}_2(G, \sigma_a) = 2$ and $\text{cros}_2(G, \sigma_b) = 0$.

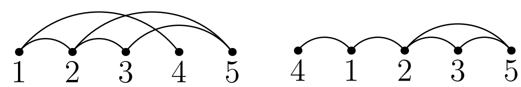


Figure 5.1: An example of cros_2 changing depending on σ

One way to make cros_2 not depend on σ is to take the minimal crossing number over S_n. For a graph G, the *crossing number* is

$$\text{cros}_2(G) = \min_{\sigma \in S_n}\{\text{cros}_2(G, \sigma)\}$$

and the *nesting number* is

$$\text{nest}_2(G) = \min_{\sigma \in S_n}\{\text{nest}_2(G, \sigma)\}$$

The remainder of this section will explore crossings and nestings in graphs. The first subsection computes the crossing and nesting numbers of special classes of graphs. The second shows that the crossing and nesting numbers are unrelated by demonstrating two families of graphs, one that always has crossing number 0 and the nesting numbers diverge and the other has the opposite.

5.1.1 Trees and other Simple Graphs

A tree is a connected graph containing no cycles. A rooted tree is a tree with one special vertex, called the root. For an order σ on T, we say the vertex v is the root if $\sigma(v) = 1$. Rooted trees are drawn in the plane with the root drawn above and the rest of the vertices descending. Figure 5.2 has an example of a tree with an order and the corresponding line graph.

Proposition 5.3. *For a tree T, $\text{cros}_2(T) = \text{nest}_2(T) = 0$.*

Proof. To see that $\text{cros}_2(T) = 0$, select one vertex to be the root of T. Construct the

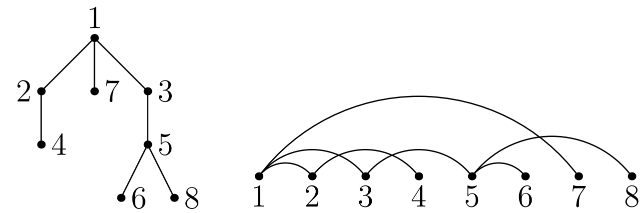


Figure 5.2: An example of an ordered together with its line graph

order σ so that the root has label 1 and the remainder of the vertices are labeled in a depth first search starting with the root. If there exists two vertices u, v in T so that $L(u) <_\sigma L(v) <_\sigma < R(u) <_\sigma R(v)$ (i.e. a cross) then we must have a path from $L(u)$ to $L(v)$, as we used a depth first search and $L(v) < R(u)$. Similarly, we must have a cycle from $L(v)$ to $R(u)$. However, this contradicts T being a tree as then we have a path from $L(u)$ to $L(v)$ to $R(u)$ to $L(u)$.

On the other hand, we have $\mathrm{nest}_2(T) = 0$ by a similar argument except the order is obtained using a breadth first search. □

Example 5.4. Figures 5.3 and 5.4 show a tree, the former with a depth first order and the later with a breadth first order and their corresponding line graphs.

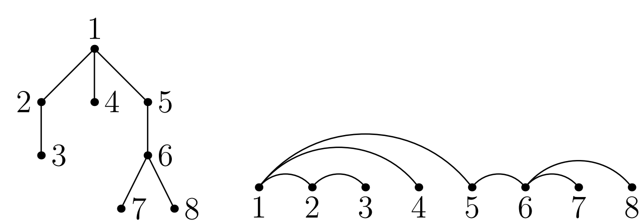


Figure 5.3: A depth first search in a tree

As can be seen the depth first order has no crossings and the breadth first has no nestings.

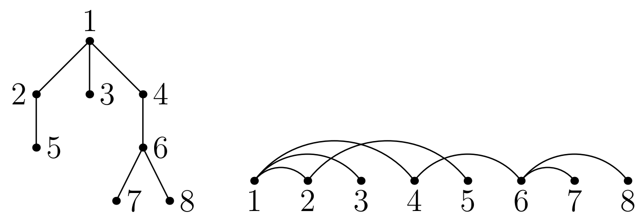


Figure 5.4: A breadth first search in a tree

Proposition 5.5. *Let C be a cycle of length n. Then* $\operatorname{cros}_2(C) = \operatorname{nest}_2(C) = 0$.

Proof. Choose one vertex in C and call it 1. Label the remaining vertices in a depth first pattern. Then for $1 < i < n$, there are arcs $(i-1, i)$ and $(i, i+1)$, which cause no crossing as there is no vertex between $i-1$, i and $i+1$. Additionally, the arc $(1, n)$ causes no crossings. Thus $\operatorname{cros}_2(C) = 0$.

The order with no nestings is given by a breadth first search. The argument is similar as almost every arc is of the form $(i, i+2)$, and there aren't enough vertices between i and $i+2$ to form a nest. □

Proposition 5.6. *Let G be a graph with* $\operatorname{cros}_2(G) = 0$ $(\operatorname{nest}_2(G) = 0)$ *and T be a rooted tree. If G' is the graph G with one vertex replaced by the tree T, then* $\operatorname{cros}_2(G') = 0$ $(\operatorname{nest}_2(G') = 0)$.

Proof. Let σ be an order so that $\operatorname{cros}_2(G, \sigma) = 0$. Call the vertex in G that is the root of the tree v. Create a new order σ' from σ so that $\sigma'(u) = \sigma(u)$ if $\sigma(u) \leq \sigma(v)$ and $\sigma'(u) = \sigma(u) + |T|$ if $\sigma(u) > \sigma(v)$. The remainder of σ' is defined as a depth first order on T, starting with $\sigma(v)$. Clearly we have $\operatorname{cros}_2(G', \sigma') = 0$.

A similar, albeit more complicated, argument holds for $\operatorname{nest}_2(G')$. The tree and graph are given a breadth first order, preserving the order of the original vertices in G. □

Proposition 5.7. *For complete graph on* $[n]$, K_n, $\text{cros}_2(K_n) = \text{nest}_2(K_n) = \displaystyle\sum_{k=2}^{n-2} \binom{k}{2}$

Proof. We'll prove the result for $\text{cros}_2(K_n)$ as the other result is similar. This is easy to prove inductively. A simple computation shows $\text{cros}_2(K_4) = 1$. Assume $\text{cros}_2(K_{n-1}) = \sum_{k=2}^{n-3} \binom{k}{2}$. Consider vertex n in K_n. For $1 < i < n-1$, the arc (i, n) in K_n crosses $i-1$ arcs in K_n so that the vertex n contributes $\sum_{i=2}^{n-2} i - 1 = \binom{n-2}{2}$ crossings. Combining this with all crossings in K_{n-1} we see

$$\text{cros}_2(K_n) = \sum_{k=2}^{n-2} \binom{k}{2}$$

□

5.1.2 The relationship between cros_2 and nest_2

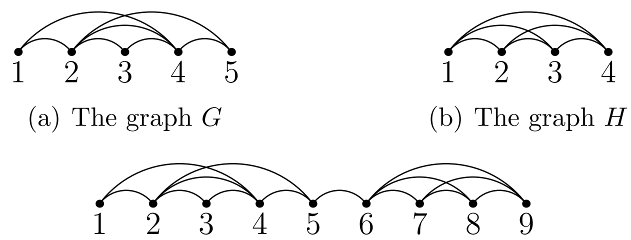
In this subsection we will construct two families of graphs that will show cros_2 and nest_2 are unrelated. But first we require a lemma.

Lemma 5.8. *For a graph* G *and an edge* e *in* G, $\text{cros}_2(G) \geq \text{cros}_2(G-e)$ $(\text{nest}_2(G) \geq \text{nest}_2(G-e)$.

Proof. Let σ be the permutation so that $\text{cros}_2(G) = \text{cros}_2(G, \sigma)$. It's easy to see that $\text{cros}_2(G, \sigma) \geq \text{cros}_2(G - e, \sigma) \geq \text{cros}_2(G - e)$. Thus $\text{cros}_2(G) \geq \text{cros}_2(G - e)$. □

For two graphs G and H and two orders σ and φ, define $G -_{(\sigma,\varphi)} H$ to be the graphs G and H together with an arc between the greatest vertex in G, under order σ, and smallest vertex in H, under order φ. If the orders are clear, we write $G - H$. Figure 5.5 contains an example of this process.

For a graph G on k vertices and permutation $\sigma \in S_k$, construct a family of graphs G_n so that $G_1 = G$ and $G_n = G_{n-1} -_{(\sigma',\sigma)} G$ where $\sigma'(i) = \sigma(i \mod k)$.



(a) The graph G

(b) The graph H

(c) The graph $G - H$

Figure 5.5: An example of the graphs G, H and $G - H$

Proposition 5.9. *If a graph G satisfies* $\operatorname{cros}_2(G) = 0$ *and* $\operatorname{nest}_2(G) > 0$*, then there is an order σ so that* $\operatorname{cros}_2(G_n) = 0$ *and* $\operatorname{nest}_2(G_n) > n \operatorname{nest}_2(G)$*.*

Similarly, if $\operatorname{nest}_2(G) = 0$ *and* $\operatorname{cros}_2(G) > 0$*, then there is an order σ so that* $\operatorname{nest}_2(G_n) = 0$ *and* $\operatorname{cros}_2(G_n) > n \operatorname{cros}_2(G)$*.*

Proof. We'll prove the first statement as the second is identical. Let σ be an order so that $\operatorname{cros}_2(G, \sigma) = 0$. If we assume $\operatorname{cros}_2(G_{n-1}) = 0$, then since $G_n = G_{n-1} - G$ and $\operatorname{cros}_2(G_{n-1}) = 0$ and $\operatorname{cros}_2(G) = 0$ we must have $\operatorname{cros}_2(G_n) = 0$ as the additional arc in G_n crosses no arcs in G_{n-1} or G.

Let e be the arc in G_n that connects G_{n-1} and G. Then $\operatorname{nest}_2(G_n) \geq \operatorname{nest}_2(G_n - e)$, but the graph $G_n - e$ has two components, so we can write $\operatorname{nest}_2(G_n - e) = \operatorname{nest}_2(G_{n-1}) + \operatorname{nest}_2(G)$. Continuing we obtain the desired result, or $\operatorname{nest}_2(G_n) \geq n \operatorname{nest}_2(G)$. $\square$

This proposition implies that if we can construct two graphs, G and H, with $\operatorname{cros}_2(G) = 0$, $\operatorname{nest}_2(G) > 0$, $\operatorname{nest}_2(H) = 0$ and $\operatorname{cros}_2(H) > 0$, then the statistic cros_2 and nest_2 will be unrelated as they will not bound each other.

Example 5.10. The graph, G, in Figure 5.6 is drawn with no crossings. However, it can be computed that $\operatorname{nest}_2(G) = 1$. The graph G turns out to be the smallest graph that satisfies this property.

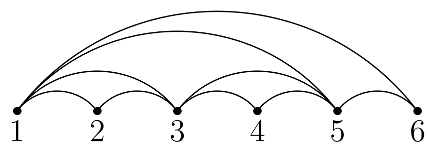


Figure 5.6: A graph with no crossings and one nesting

Example 5.11. The graph, $K_{2,3}$, in Figure 5.7 is the complete bipartite graph on the sets [2] and [3]. This graph can be seen to have $\mathrm{cros}_2(K_{2,3}) = 1$ and $\mathrm{nest}_2(K_{2,3}) = 0$.

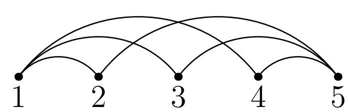


Figure 5.7: A graph with one crossing and no nestings

Examples 5.10 and 5.11 show that cros_2 and nest_2 have no relationship, i.e. there are two families of graphs G_n and H_n so that $\mathrm{cros}_2(G_n) = 0$ and $\mathrm{nest}_2(H_n) = 0$ for all n, but $\mathrm{nest}_2(G_n) \to \infty$ and $\mathrm{cros}_2(H_n) \to \infty$. Additionally, these examples show that cros_2 and nest_2 are unrelated over graphs with bounded degree sequence.

5.2 A Bijection from Motzkin Paths to Trees

Recall the definition of bicolored motzkin paths from Section 2.1. A *plane tree* is a rooted tree for which an ordering is specified for the children of each vertex. This is called a plane tree because an ordering of the children is equivalent to an embedding of the tree in the plane, with the root at the top and the children of each vertex lower than that vertex. For a plane tree T, a vertex v and edge (v, u) say (v, u) is a child of v if u is a child of v. In this section we will exhibit a bijection between bicolored motzkin paths and plane trees.

Let T be a plane tree. Label the edges of T with a depth first search, where left is prioritized before right. Say t_i is the i^{th} edge of the tree. Define a bicolored motzkin path $M = m_1 \dots m_n$ so that,

1. $m_i =$ N if t_i is the left most child of a vertex with at least two children.
2. $m_i =$ E if t_i is the only child of a vertex.
3. $m_i = \bar{\text{E}}$ if t_i is the middle child of a vertex with at least two children.
4. $m_i =$ S if t_i is the right most child of a vertex with at least two children.

Figure 5.8 has a demonstration of this procedure. Each edge is labeled showing the bijection in detail.

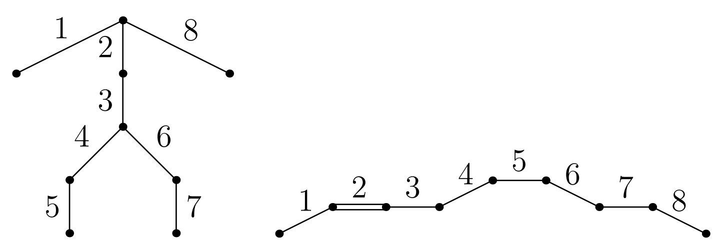


Figure 5.8: An example of the map from trees to motzkin paths

To prove this is a bijection we define the inverse. The inverse is described algorithmically. Given a bicolored motzkin path $M = m_1 \dots m_n$, construct a plane tree T starting with a single vertex, r. Initialize the algorithm setting the variable `current vertex` $= r$ and an ordered list `open` $= [r]$. For i from 1 to n do the following,

1. If $m_i \in \{\bar{\mathrm{E}}, \mathrm{S}\}$ then change `current vertex` to the last element of `open` that has a child (this must exist as m_i has positive height). Remove the vertices in `open` after the new `current vertex` from `open`.

2. Add a new vertex to T, denoted `child`, beneath the `current vertex` and to the right of any children `current vertex` may have.

3. If $m_i \in \{\mathrm{E}, \mathrm{S}\}$ then remove the `current vertex` from `open`.

4. Add `child` to the end of `open` and set `current vertex` to be `child`.

Example 5.12. Figure 5.9 shows the tree T obtained using the motzkin path from Figure 5.8, the labels on the vertices of the tree correspond to the order they are added to the tree. Table 5.1 shows the variables in each iteration of the algorithm.

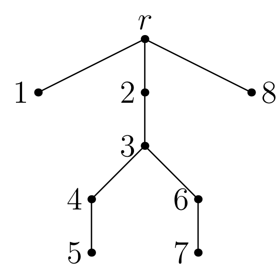


Figure 5.9: An example a tree given by a bicolored motzkin

These processes are easily seen to be inverses as both proceed in a depth first order on the tree and arcs are persevered between the two.

5.3 Ferrers Diagrams with Fixed Step

Recall the definitions of Ferrer's diagrams, fillings of polyominoes and chains in polyominoes from Section 2.5. In this section we enumerate the number of fillings

i	`current vertex`	m_i	`open`
0	-	-	$[r]$
1	r	N	$[r,1]$
2	1	Ē	$[r,2]$
3	2	E	$[r,3]$
4	3	N	$[r,3,4]$
5	4	E	$[r,3,5]$
6	3	S	$[r,6]$
7	6	E	$[r,7]$
8	1	S	$[8]$

Table 5.1: Each step from a bicolored motzkin path to a plane tree

with no se-chains in Ferrer's diagrams where each row is a fixed length shorter than the previous row.

For $s, k, n \in \mathbb{N}$ define $\mathcal{T}_{s,k,n}$ to be a Ferrer's diagram with k rows so that the bottom row has length n and each row is s shorter than the above row. Figure 5.11 has an example. For a filling $F \in \mathcal{F}_{01}(\mathcal{T}_{s,k,n})$ let $|F|$ be the number of non-zero entries in F.

For $n, k, s \in \mathbb{N}$, define the generating function,

$$A_{s,k,n}(t) = \sum_{F \in \mathrm{Av}(\mathrm{se}, \mathcal{F}_{01}(\mathcal{T}_{s,k,n}))} t^{|F|}. \tag{5.1}$$

This generating function counts the number of fillings of $\mathcal{T}_{s,k,n}$ that have $|F|$ non-zero entries and no se-chains. The following generating function iterates $A_{s,k,n}(t)$ over all k,

$$G_{s,n}(t,y) = \sum_{k=1}^{\infty} A_{s,k,n}(t) y^k. \tag{5.2}$$

We can write $A_{s,k,1}(t)$ recursively in terms of $A_{s,k-1,s+1}(t)$. In Figure 5.10 the highlighted cell is independent of the remainder of the Ferrer's diagram, its entry

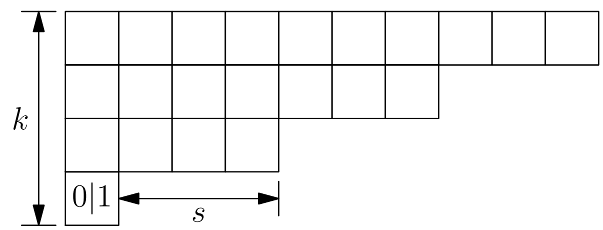


Figure 5.10: The bottom row has length 1 and doesn't affect the other rows

can not create any se-chains. Thus we can eliminate this cell so that,

$$A_{s,k,1}(t) = (1+t)A_{s,k-1,s+1}(t) \tag{5.3}$$

To recursively write $A_{s,k,n}(t)$, for $n > 1$, there are two distinct cases. First if the last column in each row is empty there are $A_{s,k,n-1}(t)$ fillings, as the last columns can be eliminated. Second, say i is the first row, from the bottom, that has a non-zero entry in its last column. Then we have two independent fillings of the remaining areas of the Ferrer's diagram. Figure 5.11 shows an example of this, the shaded cells are forced to be empty. This adds $A_{s,i,n-1}(t)A_{s,k-i,s+1}(t)$ fillings. Combining these

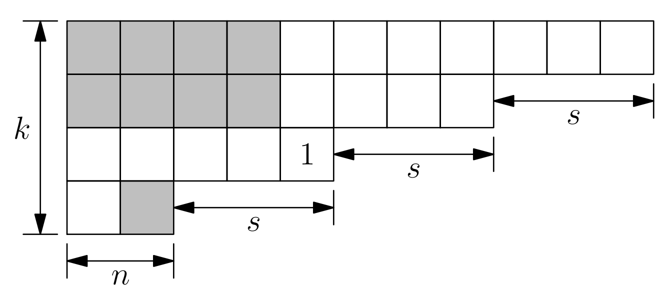


Figure 5.11: The shaded cells are required to be empty

and summing over i, we see

$$A_{s,k,n}(t) = A_{s,k,n-1}(t) + t\sum_{i=1}^{k} A_{s,i,n-1}(t)A_{s,k-i,s+1}(t).$$

Using (5.3) we can rewrite the previous,

$$A_{s,k,n}(t) = A_{s,k,n-1}(t) + \frac{t}{t+1}\sum_{i=1}^{k} A_{s,i,n-1}(t)A_{s,k-i+1,1}(t). \tag{5.4}$$

Using (5.4) we can expand $G_{s,n}(t,y)$.

$$\begin{aligned}
G_{s,n}(t,y) =& \sum_{k=1}^{\infty} A_{s,k,n-1}(t)y^k + \frac{t}{1+t}\sum_{k=1}^{\infty}\sum_{i=1}^{k} A_{s,i,n-1}(t)A_{s,k-i+1,1}(t)y^k \\
=& G_{a,n-1}(t,y) + \frac{t}{1+t}\sum_{i=1}^{\infty} A_{s,i,n-1}(t)y^{i-1}\sum_{k=1}^{\infty} A_{s,k,1}(t)y^k \\
=& G_{s,n-1}(t,y) + \frac{t}{y(1+t)}G_{s,n-1}(t,y)G_{s,1}(t,y) \\
=& G_{s,1}(t,y) + \left(1 + \frac{t}{y(1+t)}G_{s,1}(t,y)\right)^{n-1}
\end{aligned}$$

where the last step follows by recursive application of $G_{s,n-1}(t,y)$.

There is an alternate way to write $A_{s,k,1}(t)$ and it's similar to the derivation of (5.4). The difference being the bottom row (of length 1) disappears, so the new smallest row has length s. So,

$$A_{s,k,1}(t) = A_{s,k-1,s}(t) + \frac{t}{1+t}\sum_{i=1}^{k} A_{s,i-1,s}(t)A_{s,k-i+1,1}(t). \tag{5.5}$$

Using (5.5) we can make $G_{s,1}$ more explicit.

$$\begin{aligned}
G_{s,1}(t,y) =& \sum_{k=1}^{\infty} A_{s,k-1,s}(t)y^k + \frac{t}{1+t}\sum_{k=1}^{\infty}\sum_{i=1}^{k} A_{s,i-1,s}(t)A_{a,k-i+1,1}(t)y^k \\
=& y(1+G_{s,s}(t,y)) + \frac{t}{1+t}\sum_{i=1}^{\infty} A_{s,i-1,s}(t)y^{i-1}\sum_{k=1}^{\infty} A_{s,k,1}(t)y^k \\
=& y(1+G_{s,s}(t,y)) + \frac{t}{1+t}(1+G_{s,s}(t,y))G_{s,1}(t,y) \\
=& y + \frac{t}{1+t}G_{s,1}(t,y) + \frac{G_{s,1}(t,y)}{y^{s-1}}\left(y + \frac{t}{1+t}G_{s,1}(t,y)\right)^s
\end{aligned}$$

This turns out to be incredibly difficult to solve, but there are a few things we can do. First, for fixed $s \geq 0$ consider the generating function

$$F_s(t,y,x) = \sum_{n=1}^{\infty} G_{s,n}(t,y)x^n.$$

Then,

$$F_s(t,y,x) = \frac{G_{s,1}(t,y)x}{1-\left(1+\frac{t}{y(1+t)}G_{s,1}(t,y)\right)x}.$$

Second, for $s=0$, $G_{0,1}(t,y)$ represents fillings of a single column. This means the $y^m t^k$ coefficient of $G_{0,1}(t,y)$ should be $\binom{m}{k}$ as there are m cells and k 1's. Explicitly solving, we have

$$G_{0,1}(t,y) = \frac{(t+1)y}{1-(t+1)y} = \sum_{i=1}^{\infty}\sum_{j=0}^{i}\binom{i}{j}t^j y^i.$$

Which is exactly as predicted.

Finally, for $s=1$, the $y^m t^k$ coefficient of $G_{1,1}(t,y)$ represents the number of non-crossing simple graphs with $m+1$ vertices and k edges. We have

$$G_{1,1}(t,y) = \frac{y(t+1) + \sqrt{(y(t+1)-1)^2 - 4yt(t+1)}}{-2t}.$$

6. CONCLUSION

In conclusion, this dissertation attempted to accomplish several things. First to generalize symmetry results in Ferrers diagrams and moon polyominoes to layer polyominoes. Second, introduce a class of objects that did not satisfy symmetry between crossings and nestings. Third, define a notion of crossings and nestings on graphs and find if they bound each other. Fourth, demonstrate a bijection between plane trees and bi-colored motzkin paths. And finally, determine the generating function of fillings, with no southeast chains, of Ferrers diagrams with each row a fixed sized shorter than the previous.

Each of these goals was achieved; however there is still work to be carried out in each case. There are different types of polyominoes, such as skew polyominoes, that provide a clear representation of combinatorial structures. The fillings of these polyominoes would be interesting to study as they would imply new results in already established fields, such as pattern avoidance in permutations.

The study of alternating matchings has left much to be desired. Within this section we were unable to explicitly enumerate non-nesting alternating matchings, let alone find a generating function for the crossings and nestings. A continuation of this work would lead to a different order on the underlying set of the matching. Alternating matchings have a 0 on odd vertices and 1 on even vertices with each 0 connected to a 1. Modifying where 0's and 1's can occur change the problem, and is a natural generalization. However, the difficulties in the case of alternating matchings made these generalizations unattractive to study.

Crossings and nestings in graphs is quite interesting. There are many ways to generalize this problem, first for a given number of vertices, how many arc guarantee a cross or nest or how many arcs can you have guarantee no crossings or nestings? Second, what can one say about graphs with fixed degree sequence? Finally, what conditions on the graph (degree sequence, number of arcs) guarantee that the symmetry of crossings and nestings is symmetric.

Finally, fillings, with no southeast chains, of Ferrers diagrams with each row a fixed length shorter than the previous row. The main way to generalize this is change the function that determines row length. In these results the growth is linear, but what if the growth is governed by some sequence of natural numbers?

REFERENCES

[1] Mireille Bousquet-Mélou and Guoce Xin. On partitions avoiding 3-crossings. *Sminaire Lotharingien de Combinatoire*, B54e:21pp, 2006.

[2] William Y. C. Chen, Eva Y. P. Deng, Rosena R. X. Dur, Richard P. Stanley, and Catherine H. Yan. Crossings and nestings of matchings and partitions. *Transactions of the American Mathematical Society*, 359:1555–1575, 2007.

[3] William Y. C. Chen, Susan Y. J. Wu, and Catherine H. Yan. Linked partitions and linked cycles. *European Journal of Combinatorics*, 29(6):1377–1520, 2008.

[4] Sylvie Corteel. Crossings and alignments in permutations. *Advances in Applied Mathematics*, 38:149–163, 2007.

[5] Anna De Mier. On the symmetry of distribution of crossings and nestings in graphs. *Electronics Journal of Combinatorics*, 13, 2006. 21.

[6] Anna De Mier. k-noncrossings and k-nonnestings graphs and fillings of ferrers diagrams. *Combinatorica*, 27(6):699–720, 2007.

[7] Myriam De Sainte-Catherine. *Construction, manipulation et visualisation de processus sonores dans des environnements virtuels immersifs pour la performance musicale*. PhD thesis, Universit de Bordeaux, 1983.

[8] Philippe Flajolet. Combinatorial aspects of continued fractions. *Discrete Mathematics*, 32(2):125–161, 1980.

[9] Anisse Kasraoui. Ascents and descents in 01-fillings of moon polyominoes. *European Journal of Combinatorics*, 31(1):87 – 105, 2010.

[10] Anisse Kasraoui and Jiang Zeng. Distribution of crossing, nestings and alignments of two edges in matchings and partitions. *The Electronic Journal of Combinatorics*, 13(R33):12pp, 2006.

[11] Anisse Kasraoui and Jiang Zeng. Distribution of crossings, nestings and alignments of two edges in matchings and partitions. *Electronics Journal of Combinatorics*, 13(1)(R33), 2006.

[12] Martin Klazar. On identities concerning the number of crossings and nestings of two edges in matchings. *SIAM journal of discrete mathematics*, 20(4):960–976, 2006.

[13] C. Krattenthaler. Growth diagrams, and increasing and decreasing chains in fillings of ferrers shapes. *Advances in Applied Mathematics*, 37(3):404 – 431, 2006. Special Issue in honor of Amitai Regev on his 65th Birthday.

[14] Mitchell Phillipson, Catherine Yan, and Jean Yeh. Chains of length 2 in fillings of layer polyominoes. *The Electronic Journal of Combinatorics*, 20(3)(P51):14pp, 2013.

[15] Svetlana Poznanovik and Catherine Yan. Crossings and nestings of two edges in set partitions. *SIAM Journal of Discrete Mathematics*, 23(2), 2009.

[16] Martin Rubey. Increasing and decreasing sequences in fillings of moon polyominoes. *Advances in Applied Mathematics*, 47(1):57 – 87, 2011.

[17] Richard P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.