TATE COHOMOLOGY OF FINITE DIMENSIONAL HOPF ALGEBRAS

A Dissertation

by

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ABSTRACT

Let A be a finite dimensional Hopf algebra over a field $\mathbf{k}$. In this dissertation, we study the Tate cohomology $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ and Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(A, A)$ of A, and their properties. We introduce cup products that make them become graded-commutative rings and establish the relationship between these rings. In particular, we show $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ is an algebra direct summand of $\widehat{\mathrm{HH}}^*(A, A)$ as a module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$.

When A is a finite group algebra RG over a commutative ring R, we show that the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^*(RG, RG)$ of RG is isomorphic to a direct sum of the Tate cohomology rings of the centralizers of conjugacy class representatives of G. Moreover, our main result provides an explicit formula for the cup product in $\widehat{\mathrm{HH}}^*(RG, RG)$ with respect to this decomposition.

When A is symmetric, we show that there are finitely generated A-modules whose Tate cohomology is not finitely generated over the Tate cohomology ring $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ of A. It turns out that if a module in a connected component of the stable Auslander-Reiten quiver associated to A has finitely generated Tate cohomology, then so does every module in that component.

DEDICATION

To my parents, whose love has continuously given me strength and support.

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CHAPTER I

INTRODUCTION

Homological algebra has a large number of applications in differential geometry, algebraic topology, algebraic geometry, and commutative algebra. One of its major operations is **cohomology**, which can be viewed as a method of assigning invariant algebraic properties to an algebra. In most cases, homology and cohomology groups satisfy similar axioms. However, cohomology groups are contravariant functors while homology groups are covariant. This contravariant property can generate a multiplicative structure making cohomology into a ring. Because of this feature, cohomology provides strong invariant properties which can be used to differentiate between certain algebraic objects.

A **Hopf algebra** is an object whose rich structure makes it amenable to treatment by homological methods. Hopf algebras were originally observed in algebraic topology by Hopf in 1941. Many important examples of Hopf algebras appear in different fields of mathematics such as: algebraic geometry (affine group schemes), representation theory (group algebra, tensor algebra), Lie theory (universal enveloping algebra of a Lie algebra), quantum mechanics (quantum groups), graded ring theory, and combinatorics. We focus on the representations and cohomology of any finite dimensional Hopf algebra, denoted A, over a field $\mathbf{k}$. Being finite dimensional, A has additional features which are beneficial to understanding its modules and cohomology.

While the usual cohomology only involves positive degrees, the **Tate cohomology** (see Chapter III), however, is defined in both positive and negative degrees via a special construction. Tate cohomology was introduced by John Tate in 1952 for group cohomology arising from class field theory [38]. Others then generalized his theory to the group ring RG where R is a commutative ring and G is a finite group. Unlike the usual cohomology, this theory is based on complete resolutions, and hence, yields cohomology groups in both positive and negative degrees. Over the past several decades, a great deal of effort has gone into the study of this new cohomology. A summary may be found in [9, Ch. VI] or [14, Ch. XII]. In the early 1980's, through an unpublished work, Pierre Vogel extended Tate cohomology to any group and even to any ring using unbounded chain complexes. For finite groups, the Tate-Vogel cohomology coincides with the Tate cohomology. Accounts of Vogel's construction appeared, for examples, in a paper by Goichot [23] and in another paper by Benson and Carlson [6] in 1992. In the 1980's, Buchweitz introduced another construction of Tate cohomology of a two-sided Noetherian and Gorenstein ring, using the stable module category influenced by the work of Auslander and Bridger [10, §6]. Many authors have also considered the Hochschild analogue of Tate cohomology for Frobenius algebras. For instance, one of the first attempts was given in Nakayama's paper in 1957 on the complete cohomology of Frobenius algebras, using the complete standard complex (or complete bar resolution) [30]. The stable Hochschild cohomology of a Frobenius algebra, using the stable module category, was studied in various papers, e.g. [17]. More recently, using complete resolutions, Bergh and Jorgensen defined the Tate-Hochschild cohomology of an algebra A whose enveloping algebra A^e is two-sided Noetherian and Gorenstein over a field $\mathbf{k}$ [7]. If the Gorenstein dimension of A^e is 0, then this cohomology agrees with the usual Hochschild cohomology in positive degrees. It is noted in [7] that this Tate-Hochschild definition is equivalent to that using the stable module category in [17], at least in the finite dimensional case.

In this dissertation, we study the Tate and Tate-Hochschild cohomology for finite dimensional Hopf algebras A over a field $\mathbf{k}$. Since any finite dimensional Hopf algebra is a Frobenius algebra [29, Theorem 2.1.3], results from [7], [17], and [30] apply. The dissertation is organized as follows.

In Chapter II, we give the definition, examples, and properties of a (finite dimensional) Hopf algebra A over a field $\mathbf{k}$. We also recall basic concepts from homological algebra and define the cohomology ring of A. At the end of this chapter, we set general notation and conventions to be used through the rest of the dissertation.

In Chapter III, we introduce and construct the main objects of our study, the Tate cohomology $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ and Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(A, A)$ of A, using both the complete resolutions and the appropriate stable module categories. We then describe the product structures which turn $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ and $\widehat{\mathrm{HH}}^*(A, A)$ into graded-commutative rings. The next chapters display our efforts in studying properties of these two objects, inspired by the known results in the usual cohomology rings.

In Chapter IV, we establish the relationship between the two Tate cohomology rings. In particular, we show that for a finite dimensional Hopf algebra A over a field $\mathbf{k}$, $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ is a direct summand of $\widehat{\mathrm{HH}}^*(A, A)$ as a module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$. Hence, Tate cohomology shares the same relation as that of the usual cohomology. This similarity opens up many research questions in which one asks if it is possible to generalize results from the usual (Hochschild) cohomology to their Tate versions. This is still an ongoing project. However, we anticipate that some obstructions will occur in the Tate cohomology case. The construction of Tate cohomology is more complex, hence, some nice properties from the usual cohomology may not carry over to the Tate cohomology. To demonstrate this complexity, in this chapter, we explicitly compute the Tate and Tate-Hochschild cohomology for Taft algebras, in particular, for the Sweedler algebra H_4.

Chapter V focuses on the decomposition of the Tate-Hochschild cohomology ring of a finite group algebra RG, where R is the ring of integers $\mathbb{Z}$ or a field whose characteristic divides the order of the group G. Let G act on itself by conjugation. We show that $\widehat{\mathrm{HH}}^*(RG, RG)$ is a direct sum of the Tate cohomology rings of the centralizers of conjugacy representatives of G. Moreover, we establish a product formula with respect to this additive decomposition. This product structure implies that $\widehat{\mathrm{HH}}^*(RG, RG)$ decomposes not just as an R-module but as an $\widehat{\mathrm{H}}^*(G, R)$-module. The products in negative degrees of the Tate-Hochschild cohomology are also observed. By using the product formula and results from products in negative cohomology by [6], we can determine quickly when the products in $\widehat{\mathrm{HH}}^*(RG, RG)$ are 0 and obtain some information about the depth of the usual Hochschild cohomology ring $\mathrm{HH}^*(RG, RG)$. Finally, we use the results in this chapter to compute the Tate-Hochschild cohomology of the dihedral group of order 8 and of the symmetric group on three elements.

Many people have been interested in the finite generation question of the cohomology of a finite dimensional Hopf algebra A. If such property holds, one can apply algebraic geometry and commutative algebra in the study of A. One can also apply the theory of support varieties to the study of A-modules. Chapter VI addresses the finite generation question for the Tate cohomology of A when A is symmetric. We generalize some group cohomology results from [12] to show that there are finitely generated A-modules whose Tate cohomology is not finitely generated over the Tate cohomology ring $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ of A. To show this, we employ the boundedness conditions on finitely generated modules over Tate cohomology and the property that products in negative Tate cohomology of symmetric algebras are often zero [26, §8]. We also construct A-modules which have finitely generated Tate cohomology. It turns out that if a module in a connected component of the stable Auslander-Reiten quiver associated to A has finitely generated Tate cohomology, then so does every module in that component. In particular, all modules in the connected component of the quiver which contains $\mathbf{k}$ have finitely generated Tate cohomology. As applications, we show that an algebra defined by Radford [33] and the restricted universal enveloping algebra of the p-Lie algebra $\mathfrak{sl}_2$ have finitely generated usual cohomology rings but fail to do so for their Tate cohomology. These examples show that finite generation behaves differently in negative cohomology.

CHAPTER II

PRELIMINARIES

II.1 Hopf algebras

In this section, we define a Hopf algebra, our main object of study, and look at several examples of Hopf algebras that will occur throughout the dissertation. More information on Hopf algebras can be obtained, e.g. in [29]. We note that Hopf algebras can be defined over any commutative ring in general. However, for simplicity and for later use, we mainly consider our Hopf algebras to be over a field $\mathbf{k}$. Tensor products are assumed to be over $\mathbf{k}$ unless stated otherwise.

We say a $\mathbf{k}$-vector space A is an associative $\mathbf{k}$-algebra if A has two $\mathbf{k}$-linear maps, the multiplication map $m : A \otimes A \to A$ and the unit map $u : \mathbf{k} \to A$, satisfying:

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m),$$
$$m \circ (u \otimes \mathrm{id}_A) = 1_{\mathbf{k}}.\,\mathrm{id}_A,$$
$$m \circ (\mathrm{id}_A \otimes u) = \mathrm{id}_A .1_{\mathbf{k}},$$

where id_A is the identity map of A and $1_{\mathbf{k}}$ is the identity element of $\mathbf{k}$. The first condition is associativity and the last two conditions imply m is surjective. An algebra (A, m, u) is said to be **commutative** if $ab = ba$, for all $a, b \in A$.

We say a $\mathbf{k}$-vector space C is a coassociative coalgebra if it has two $\mathbf{k}$-linear maps, the comultiplication (coproduct) map $\Delta : C \to C \otimes C$ and the counit map $\varepsilon : C \to \mathbf{k}$, satisfying:

$$(\mathrm{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_C) \circ \Delta,$$
$$(\mathrm{id}_C \otimes \varepsilon) \circ \Delta = \mathrm{id}_C \otimes 1_{\mathbf{k}},$$
$$(\varepsilon \otimes \mathrm{id}_C) \circ \Delta = 1_{\mathbf{k}} \otimes \mathrm{id}_C .$$

The first condition is coassociativity and the last two conditions imply Δ is injective. We adopt

Sweedler's sigma notation for the coproduct: $\Delta(c) = \sum c_1 \otimes c_2$, for all $c \in C$. A coalgebra (C, Δ, ε) is called **cocommutative** if it is commutative with respect to the comultiplication, that is, $\sum c_1 \otimes c_2 = \sum c_2 \otimes c_1$.

Definition II.1. A **bialgebra** over a field $\mathbf{k}$ is a $\mathbf{k}$-vector space A endowed with an associative algebra structure (A, m, u) and a coassociative coalgebra structure (A, Δ, ε) such that it satisfies one of the following equivalent conditions:

1. Δ and ε are algebra morphisms
2. m and u are coalgebra morphisms.

A bialgebra is commutative (resp. cocommutative) if its underlying algebra (resp. coalgebra) is commutative (resp. cocommutative).

Let (A, m, u) be an algebra and (C, Δ, ε) be a coalgebra. Then $\mathrm{Hom}_{\mathbf{k}}(C, A)$ becomes an algebra under the **convolution product**:

$$(f \star g)(c) = m \circ (f \otimes g) \circ \Delta(c)$$

for all $f, g \in \mathrm{Hom}_{\mathbf{k}}(C, A)$ and $c \in C$. The unit element in $\mathrm{Hom}_{\mathbf{k}}(C, A)$ is $u \circ \varepsilon$. In sigma notation, $(f \star g)(c) = \sum f(c_1)g(c_2)$.

Definition II.2. A **Hopf algebra** is a bialgebra A together with a linear map $S : A \to A$, such that for all $a \in A$, S satisfies:

$$\sum S(a_1)\, a_2 \;=\; \varepsilon(a)1_A \;=\; \sum a_1\, S(a_2)$$

that is, S is a two-sided inverse of id_A under convolution product $\star$. The map S is called the **antipode map** of A.

We list here some examples of (finite dimensional) Hopf algebras. These examples will be reoccurring throughout the dissertation.

Example II.3. [Group algebra]

Let G be a (finite) multiplicative group. Let

$$\mathbf{k}G = \left\{ \sum_{g \in G} a_g g \mid a_g \in \mathbf{k} \right\}$$

be the associated group algebra over $\mathbf{k}$. It is a free $\mathbf{k}$-module with basis G. $\mathbf{k}G$ is a Hopf algebra with structure:

$$m(g \otimes h) = gh, \qquad u(1_{\mathbf{k}}) = 1_G,$$
$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1_{\mathbf{k}}, \qquad S(g) = g^{-1}$$

for all $g, h \in G$. The Hopf algebra $\mathbf{k}G$ is commutative if and only if the group G is abelian; it is always co-commutative by the above definition.

Example II.4. [Tensor algebra and its induced Hopf algebras]

Suppose V is a (finite dimensional) vector space over $\mathbf{k}$ and $T(V) := \bigoplus_{i \geq 0} V^{\otimes_{\mathbf{k}} i}$ is its tensor algebra, then $T(V)$ becomes a Hopf algebra with:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \varepsilon(x) = 0, \qquad S(x) = -x,$$

for all $x \in V$. Since all x in V generates $T(V)$ as an algebra, Δ is extended to the rest of $T(V)$ as an algebra homomorphism, not just as a $\mathbf{k}$-linear map. The tensor algebra gives rise to the following Hopf algebras (which are quotients of the tensor algebra) via the induced comultiplication, counit, and antipode:

(a) The symmetric algebra

$$\mathrm{Sym}(V) := T(V)/(v \otimes w - w \otimes v, \text{ for all } v, w \in V)$$

is a commutative, cocommutative Hopf algebra. If V is free over $\mathbf{k}$ of finite rank n, then the underlying $\mathbf{k}$-algebra of $\mathrm{Sym}(V)$ is isomorphic to the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_n]$.

(b) The exterior algebra

$$\Lambda(V) := T(V)/(v \otimes v, \text{ for all } v \in V)$$

is a cocommutative Hopf algebra. Denote $v_1 \wedge \ldots \wedge v_n$ to be the equivalence class of $v_1 \otimes \ldots \otimes v_n$ under the quotient. $\Lambda(V)$ is strictly graded-commutative in the sense that $v \wedge w = -(w \wedge v)$ and $v \wedge v = 0$, for all $v, w \in V$.

(c) Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{k}$ whose underlying $\mathbf{k}$-vector space is V. The universal enveloping algebra of $\mathfrak{g}$

$$U(\mathfrak{g}) := T(V)/(v \otimes w - w \otimes v - [v, w], \text{ for all } v, w \in V)$$

is a cocommutative Hopf algebra.

Example II.5. [Sweedler's 4-dimensional Hopf algebra]

Suppose $\mathbf{k}$ is a field of characteristic $\neq 2$. Sweedler defined H_4 to be the $\mathbf{k}$-algebra generated by g and x satisfying the relations : $g^2 = 1, x^2 = 0$, and $xg = -gx$. It is a Hopf algebra by defining:

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(x) &= 1 \otimes x + x \otimes g, \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0, \\ S(g) &= g^{-1} = g, & S(x) &= -xg. \end{aligned}$$

The underlying vector space is generated by $\{1, g, x, gx\}$ and thus H_4 has dimension 4. This is the smallest example of a Hopf algebra that is both non-commutative and non-cocommutative.

Example II.6. [Taft algebra]

More generally, let $N \geq 2$ be a positive integer. Assume the field $\mathbf{k}$ contains a primitive N-th root of unity ω. Let A be the algebra, called Taft algebra, generated over $\mathbf{k}$ by two elements g and x, subject to the relations: $g^N = 1, x^N = 0$, and $xg = \omega gx$. A is a Hopf algebra with structure given by:

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(x) &= 1 \otimes x + x \otimes g, \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0, \\ S(g) &= g^{-1}, & S(x) &= -xg^{-1}. \end{aligned}$$

A is of dimension N^2. It is non-semisimple, non-commutative, and non-cocommutative.

Example II.7. [Localized quantum plane]

Let q be a nonzero element in $\mathbf{k}$. The quantum plane is defined as:

$$\mathcal{O}_q(\mathbf{k}^2) = \mathbf{k}\langle x, y \mid xy = qyx\rangle.$$

We localize the quantum plane to obtain a Hopf algebra $A := \mathcal{O}_q(\mathbf{k}^2)[x^{-1}]$ with structure:

$$\begin{aligned} \Delta(x) &= x \otimes x, & \Delta(y) &= y \otimes 1 + x \otimes y, \\ \varepsilon(x) &= 1, & \varepsilon(y) &= 0, \\ S(x) &= x^{-1}, & S(y) &= -x^{-1}y. \end{aligned}$$

We note that in this example, the antipode map S has infinite order.

Example II.8. [NOT a Hopf algebra]

Let $B = \mathcal{O}(M_n(\mathbf{k})) = \mathbf{k}[x_{ij} \mid 1 \leq i, j \leq n]$, the polynomial functions on $n \times n$ matrices. As an algebra, B is simply the commutative polynomial ring in the n^2 indeterminates x_{ij}. For the coalgebra structure, think of x_{ij} as the coordinate function on the ij-th entry of the ring $M_n(\mathbf{k})$ of $n \times n$ matrices. Then Δ is the dual of matrix multiplication; that is, $\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}$. By setting $\varepsilon(x_{ij}) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, B becomes a bialgebra.

If we let $X = [x_{ij}]$ be the $n \times n$ matrix with ij-th entry x_{ij}, then one may check that $\det(X)$ is a **group-like element** (that is, $\Delta(\det(X)) = \det(X) \otimes \det(X)$ and $\varepsilon(\det(X)) = 1$). We see that B is not a Hopf algebra because $\det(X)$ is not invertible in B.

However, there are several Hopf algebras closely related to B:

$$\mathcal{O}(SL_n(\mathbf{k})) = \mathcal{O}(M_n(\mathbf{k}))/(\det(X) - 1)$$

$$\mathcal{O}(GL_n(\mathbf{k})) = \mathcal{O}(M_n(\mathbf{k}))[\det(X)^{-1}]$$

by defining $S(X) = X^{-1}$ on these bialgebras.

II.2 Homological algebra

Let R be a ring and M be a left R-module.

Definition II.9. A **projective resolution** of M, denoted by $P_\bullet = \{P_n, d_n\}_{n\geq 0}$, is an exact sequence of projective R-modules:

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon=d_0} M \to 0,$$

that is, each P_n is a projective R-module, and $\mathrm{Ker}(d_n) = \mathrm{Im}(d_{n+1})$, for all $n \geq 0$.

The **length** of a finite projective resolution is the first $n \geq 0$ such that $P_n \neq 0$ and $P_i = 0$ for all $i > n$. If M admits a finite projective resolution, the minimal length among all finite projective resolutions of M is called the **projective dimension of** M and denoted $\mathrm{pd}(M)$. If M does not admit a finite projective resolution, then by convention we say $\mathrm{pd}(M) = \infty$. We note that if $\mathrm{pd}(M) = 0$, then M has a projective resolution of the form $0 \to P_0 \xrightarrow{d_0} M \to 0$. By exactness of the sequence, this implies that d_0 is an isomorphism; and hence, M is itself projective. Conversely, if M is a projective module, then it is clear that $\mathrm{pd}(M) = 0$.

Definition II.10. An **injective resolution** of M, denoted by $I^\bullet = \{I^n, d^n\}_{n\geq 0}$, is an exact sequence of injective R-modules:

$$0 \to M \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \xrightarrow{d^2} I^2 \xrightarrow{d^3} \cdots,$$

that is, each I^n is an injective R-module, and $\mathrm{Ker}(d^{n+1}) = \mathrm{Im}(d^n)$, for all $n \geq 0$.

The **length** of a finite injective resolution is the first $n \geq 0$ such that $I^n \neq 0$ and $I^i = 0$ for all $i > n$. If a module M admits a finite injective resolution, the minimal length among all finite injective resolutions of M is called the **injective dimension of** M and denoted $\mathrm{id}(M)$. If M does not admit a finite injective resolution, then $\mathrm{id}(M) = \infty$. Similar observation as before, M has injective dimension 0 if and only if it is an injective module.

Projective and injective resolutions can be used to define derived functors such as the Ext functor. For consistency, we use projective resolutions throughout this dissertation.

Theorem II.11 (Comparison Theorem)**.** *Let M and M' be left R-modules. Let $P_\bullet$ be a projective resolution of M and $f : M \to M'$ be any map of modules. Then for every projective resolution $Q_\bullet$ of M', there is a chain map $f_\bullet = \{f_n\}_{n\geq 0} : P_\bullet \to Q_\bullet$ lifting f in the sense that $\varepsilon' \circ f_0 = f \circ \varepsilon$.*

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r, "d_n"] \arrow[d, "f_n"] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "d_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_n \arrow[r, "d'_n"] & Q_{n-1} \arrow[r, "d'_{n-1}"] & \cdots \arrow[r, "d'_1"] & Q_0 \arrow[r, "\varepsilon'"] & M' \arrow[r] & 0
\end{tikzcd}$$

The chain map $f_\bullet$ is unique up to chain homotopy equivalence. That is, given any two such maps $f_\bullet$ and $f'_\bullet$, there is a chain homotopy $h_\bullet : P_\bullet \to Q_{\bullet+1}$ such that $f_n - f'_n = d'_{n+1} \circ h_n + h_{n-1} \circ d_n$.

The proof of this theorem can be obtained in any homological algebra book, for example, see [6, Theorem 2.4.2] or [39, Theorem 2.2.6].

Definition II.12 (Ext^*_R)**.** Let M, N be left R-modules, apply $\mathrm{Hom}_R(-, N)$ to a projective resolution $P_\bullet$ of M and drop the last term $\mathrm{Hom}_R(M, N)$, we get:

$$0 \xrightarrow{0} \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, N) \to \cdots \to \mathrm{Hom}_R(P_n, N) \xrightarrow{d_n^*} \cdots$$

where $d_n^*(f) = f \circ d_n$, for all $n > 0$. The n-th homology of this (cochain) complex is

$$\mathrm{Ext}^n_R(M, N) := \mathrm{H}^n(\mathrm{Hom}_R(P_\bullet, N)) = \mathrm{Ker}(d^*_{n+1})/\,\mathrm{Im}(d^*_n),$$

and $\mathrm{Ext}^0_R(M, N) := \mathrm{Ker}(d^*_1)$.

By the Comparison Theorem, $\mathrm{Ext}^n_R(M, N)$ is independent of the choice of projective resolution of M.

Example II.13. [Cohomology of a cyclic group]

Let $G = \langle g \rangle$ be a cyclic group generated by g of order m and M be a $\mathbf{k}G$-module. The cohomology of G with coefficients in M is denoted as

$$\mathrm{H}^*(G, M) := \mathrm{H}^*(\mathbf{k}G, M) := \bigoplus_{n\geq 0} \mathrm{Ext}^n_{\mathbf{k}G}(\mathbf{k}, M).$$

When $M = \mathbf{k}$, we will compute $\mathrm{H}^n(G, \mathbf{k}) = \mathrm{Ext}^n_{\mathbf{k}G}(\mathbf{k}, \mathbf{k})$. Let

$$\cdots \xrightarrow{\cdot T} \mathbf{k}G \xrightarrow{\cdot(g-1)} \mathbf{k}G \xrightarrow{\cdot T} \mathbf{k}G \xrightarrow{\cdot(g-1)} \mathbf{k}G \xrightarrow{\varepsilon} \mathbf{k} \to 0$$

be a $\mathbf{k}G$-projective resolution of $\mathbf{k}$, where $T = 1 + g + g^2 + g^3 + ... + g^{m-1}$ and $\varepsilon(g^i) = 1$, for all $g^i \in G$. Apply $\mathrm{Hom}_{\mathbf{k}G}(-, \mathbf{k})$ and take the homology of the new complex, we get:

Case 1: When $\mathbf{k}$ is a field, $\mathrm{char}(\mathbf{k}) \mid m$

$$\mathrm{H}^n(G, \mathbf{k}) \cong \mathbf{k}, \text{ for all } n \geq 0$$

Case 2: When $\mathbf{k}$ is a field, $\mathrm{char}(\mathbf{k}) \nmid m$

$$\mathrm{H}^n(G, \mathbf{k}) = \begin{cases} \mathbf{k} & n = 0 \\ 0 & n > 0 \end{cases}$$

Case 3: When $\mathbf{k} = \mathbb{Z}$

$$\mathrm{H}^n(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/(m\mathbb{Z}) & n > 0, n \text{ is even} \\ 0 & n > 0, n \text{ is odd.} \end{cases}$$

II.3 Products in cohomology

Let M, N, L be left R-modules. Let $M_\bullet$ and $N_\bullet$ be projective resolutions of M and N, respectively. We can give $\mathrm{Ext}^*_R(M, M)$ the structure of a graded ring, as a special case of the natural multiplication:

$$\mathrm{Ext}^j_R(N, L) \times \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^{i+j}_R(M, L),$$

which can be described in several equivalent forms as follows:

II.3.1 Yoneda product

An i-**fold extension** of M by N is an exact sequence of R-modules

$$0 \to N \to M_{i-1} \to \cdots \to M_0 \to M \to 0,$$

beginning with N and ending with M, and with i intermediate terms. Two i-fold extensions are **equivalent** if there is a map of i-fold extensions such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & M_{i-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & M_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & M'_{i-1} \arrow[r] & \cdots \arrow[r] & M'_0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

We can show this defines an equivalence relation by checking symmetry and transitivity in the usual way.

An i-fold extension of M by N determines an element of $\mathrm{Ext}^i_R(M,N)$ by completing the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r] \arrow[d] & P_{i-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & M_{i-1} \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where the top row is a projective resolution of M.

From the discussion in [3, §2.6], we may interpret $\mathrm{Ext}^i_R(M,N)$ as the set of equivalence classes of i-fold extensions of M by N. Let

$$\alpha : 0 \to N \xrightarrow{\phi_2} M_{i-1} \to \cdots \to M_0 \to M \to 0,$$

represent an element $[\alpha] \in \mathrm{Ext}^i_R(M,N)$, and let

$$\beta : 0 \to L \to N_{j-1} \to \cdots \to N_0 \xrightarrow{\phi_1} N \to 0$$

represent $[\beta] \in \mathrm{Ext}_R^j(N, L)$. Then the **Yoneda product** (or **Yoneda composition**) $[\beta][\alpha] \in \mathrm{Ext}_R^{i+j}(M, L)$ is defined as the equivalence class of the exact sequence $\beta \circ \alpha$ formed by splicing α and β together at N:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & \cdots \arrow[r] & N_0 \arrow[rr, "\phi=\phi_2\circ\phi_1"] \arrow[dr, "\phi_1"'] & & M_{i-1} \arrow[r] & \cdots \arrow[r] & M \arrow[r] & 0 \\
& & & & N \arrow[ur, "\phi_2"'] \arrow[dr] & & & & \\
& & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

By this way, we obtain a bilinear map:

$$\mathrm{Ext}_R^j(N, L) \times \mathrm{Ext}_R^i(M, N) \to \mathrm{Ext}_R^{i+j}(M, L).$$

II.3.2 Cup product

Suppose $[\alpha] \in \mathrm{Ext}_R^i(M, N)$, so it can be represented by a homomorphism $\alpha : M_i \to N$ such that $(d_{i+1}^M)^*\alpha = \alpha \circ d_{i+1}^M = 0$, that is, $\alpha \in \mathrm{Ker}(d_{i+1}^M)^*$. There exists a chain map $\overline{\alpha} : M_\bullet \to N_\bullet$ of degree $-i$ induced by α. We can see that $\mathrm{Ext}_R^i(M, N)$ is isomorphic as an abelian group to the group of homotopy equivalence classes of chain maps of degree $-i$ from $M_\bullet$ to $N_\bullet$.

Let $[\alpha] \in \mathrm{Ext}_R^i(M, N)$ and $[\beta] \in \mathrm{Ext}_R^j(N, L)$. By above observation, let $\overline{\alpha} = \{\alpha_s : M_{i+s} \to N_s\}_{s\geq 0}$ and $\overline{\beta} = \{\beta_s : N_{j+s} \to L_s\}_{s\geq 0}$ be the induced chain maps of degrees $-i$ and $-j$, respectively. Define

$$[\beta][\alpha] = \{\beta_s\alpha_{j+s} : M_{i+j+s} \to L_s\}_{s\geq 0} \in \mathrm{Ext}_R^{i+j}(M, L)$$

Equivalently, we define $\beta \smile \alpha = \beta \circ \alpha_j$ up to chain homotopy. This operation $\smile$ induces a well-defined operation on Ext, called **cup product**. It can be shown that the Yoneda product agrees with this cup product. Hence, when $N = L = M$,

$$\mathrm{Ext}_R^*(M, M) = \bigoplus_{n\geq 0} \mathrm{Ext}_R^n(M, M)$$

is a graded ring.

II.3.3 Tensor product of projective resolutions

Let R be a Hopf algebra and M, M', N, N' be left R-modules. $M' \otimes M$ becomes a left R-module via Δ. Let $M'_\bullet, M_\bullet$ be projective resolutions of M', M, respectively. By the Künneth Theorem [39, Theorem 3.6.3], the total complex of $M'_\bullet \otimes M_\bullet$ is a projective resolution of $M' \otimes M$, with differential maps δ.

Let $\alpha \in \mathrm{Hom}_R(M_i, N)$ and $\beta \in \mathrm{Hom}_R(M'_j, N')$ represent elements of $\mathrm{Ext}^i_R(M, N)$ and $\mathrm{Ext}^j_R(M', N')$, respectively, then

$$\beta \otimes \alpha \in \mathrm{Hom}_R(M'_j \otimes M_i, N' \otimes N)$$

may be extended to an element of $\mathrm{Hom}_R\left(\bigoplus_{r+s=i+j}(M'_r \otimes M_s), N' \otimes N\right)$ by defining it to be the 0 map on all components other than $M'_j \otimes M_i$. One can check that

$$\delta(\beta \otimes \alpha) = \delta(\beta) \otimes \alpha + (-1)^j \beta \otimes \delta(\alpha).$$

So product of two cocycles is a cocycle, and the product of a cocycle with a coboundary is a coboundary. This induces a well-defined product on cohomology

$$\smile : \mathrm{Ext}^j_R(M', N') \times \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^{i+j}_R(M' \otimes M, N' \otimes N).$$

One may check that for a Hopf algebra R, these definitions of products are equivalent, e.g. [3, Prop. 3.2.1].

Example II.14. [Cohomology rings of a cyclic group and of an elementary abelian group]

Let G be a finite group and $\mathbf{k}$ be a field. By Maschke's Theorem, if the characteristic of $\mathbf{k}$ does not divide the order of G, then the group algebra $\mathbf{k}G$ is semisimple [29, §2.2] and its cohomology is then trivial except in the degree 0. We will only be interested in the case when the cohomology of $\mathbf{k}G$ is nontrivial; hence, throughout the discussion of group cohomology, we assume the characteristic of $\mathbf{k}$ divides the order of G. In this example, we assume the characteristic of $\mathbf{k}$ is $p > 0$ and let G be a cyclic group of order m such that p^c is the exact power of p dividing m. As computed before, we have

$$\mathrm{H}^n(G,\mathbf{k}) \cong \mathbf{k}, \text{ for all } n \geq 0.$$

A tedious calculation [18, §3.2] shows that:

$$\mathrm{H}^*(G,\mathbf{k}) := \mathrm{Ext}^*_{\mathbf{k}G}(\mathbf{k},\mathbf{k}) = \mathbf{k}[x, y \mid \deg x = 1, \deg y = 2, x^2 = 0]$$

if p is odd, or if $p = 2$ and $c > 1$; and

$$\mathrm{H}^*(G,\mathbf{k}) \cong \mathbf{k}[x \mid \deg x = 1]$$

if $p = 2$ and $c = 1$.

More generally, let G be an elementary abelian group $G = (\mathbb{Z}/p\mathbb{Z})^d$, for some integer $d \geq 1$, and let $\mathbf{k}$ be a field of characteristic $p > 0$. If p is odd, we have:

$$\mathrm{H}^*(G,\mathbf{k}) \cong \Lambda(x_1,\dots,x_d) \otimes \mathbf{k}[y_1,\dots,y_d],$$

where the first term on the right is an exterior algebra over $\mathbf{k}$ generated by x_i of degree 1, and the second term is a polynomial algebra generated by y_i of degree 2. If $p = 2$,

$$\mathrm{H}^*(G,\mathbf{k}) \cong \mathbf{k}[x_1,\dots,x_d]$$

a polynomial algebra generated by elements of degree 1 [18, §3.5].

Example II.15. [Cohomology ring of a polynomial ring]

Let $A = \mathbf{k}[x_1, x_2, \dots, x_t]$, where $\mathbf{k}$ is a field. A is an augmented algebra by defining $\varepsilon(x_i) = 0$ and $\varepsilon(r) = r$, for all $r \in \mathbf{k}$. Moreover, A is also a Hopf algebra with the coproduct $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ and antipode $S(x_i) = -x_i$, for all $i = 1, 2, \dots, t$.

Let M be a left A-module. The cohomology of A with coefficients in M is:

$$\mathrm{H}^*(A,M) = \bigoplus_{n\geq 0} \mathrm{H}^n(A,M) = \bigoplus_{n\geq 0} \mathrm{Ext}^n_A(\mathbf{k},M)$$

When $M=\mathbf{k}$, $\mathrm{H}^*(A,\mathbf{k})$ turns out to be a graded algebra under the cup product.

In particular, let $A=\mathbf{k}[x]$. Consider a projective resolution of $\mathbf{k}$:

$$0 \to A \xrightarrow{\cdot x} A \xrightarrow{\varepsilon} \mathbf{k} \to 0.$$

Apply $\mathrm{Hom}_A(-,\mathbf{k})$ to this resolution and delete the term $\mathrm{Hom}_A(\mathbf{k},\mathbf{k})$, we get:

$$0 \to \mathrm{Hom}_A(A,\mathbf{k}) \xrightarrow{(\cdot x)^*} \mathrm{Hom}_A(A,\mathbf{k}) \to 0,$$

which is equivalent to

$$0 \to \mathbf{k} \xrightarrow{0} \mathbf{k} \to 0,$$

since $\mathrm{Hom}_A(A,\mathbf{k}) \cong \mathbf{k}$. Thus:

$$\mathrm{Ext}^n_A(\mathbf{k},\mathbf{k}) = \begin{cases} \mathbf{k} & n=0,1 \\ 0 & n\geq 2. \end{cases}$$

Now let $A=\mathbf{k}[x,y]$. Note that $\mathbf{k}\cong A/(x,y)$. Consider a projective resolution of $\mathbf{k}$:

$$0 \to A \xrightarrow{\alpha} A\oplus A \xrightarrow{\beta} A \xrightarrow{\varepsilon} \mathbf{k} \to 0,$$

where $\alpha = \begin{pmatrix} y \\ -x \end{pmatrix}$ and $\beta = \begin{pmatrix} x & y \end{pmatrix}$. Apply $\mathrm{Hom}_A(-,\mathbf{k})$ and take the cohomology of the new complex, we get:

$$\mathrm{Ext}^n_A(\mathbf{k},\mathbf{k}) = \begin{cases} \mathbf{k}\oplus\mathbf{k} & n=1 \\ \mathbf{k} & n=0,2 \\ 0 & n>2. \end{cases}$$

One can compute that $\mathrm{H}^*(A,\mathbf{k}) := \mathrm{Ext}^*_A(\mathbf{k},\mathbf{k}) = \bigoplus_{n\geq 0}\mathrm{Ext}^n_A(\mathbf{k},\mathbf{k})$ is isomorphic to the exterior algebra $\Lambda^*(V)$, where V is a $\mathbf{k}$-vector space of dimension 2.

In general, let $A = \mathbf{k}[x_1, x_2, \ldots, x_t]$, we get:

$$\operatorname{Ext}^n_A(\mathbf{k},\mathbf{k}) \cong \mathbf{k}^{\binom{t}{n}}, \text{ for all } n \geq 0,$$

and,

$$\mathrm{H}^*(A,\mathbf{k}) \cong \bigoplus_{n\geq 0} \mathbf{k}^{\binom{t}{n}} \cong \Lambda^*(V),$$

where V is a $\mathbf{k}$-vector space of dimension t.

II.4 Notation and conventions

For the rest of this dissertation, unless specified otherwise, we let G be a group of finite order and $\mathbf{k}$ be a field. Tensor products $\otimes$ will be over $\mathbf{k}$. All modules are assumed to be finitely generated left modules. Let A be a finite dimensional Hopf algebra over $\mathbf{k}$. In this case, the antipode S of A is bijective [29, Theorem 2.1.3], and its inverse is denoted by $\overline{S}$. The $\mathbf{k}$-dual $\operatorname{Hom}_{\mathbf{k}}(-,\mathbf{k})$ is denoted by $D(-)$ and the ring-dual $\operatorname{Hom}_A(-,A)$ is denoted by $(-)^*$. This notation is unfortunate, because to a Hopf algebraist, $D(A)$ usually stands for the "Drinfeld double" of A. However, we adopt this notation to agree with our reference [7]. We observe that $D(A)$ is also a finite dimensional Hopf algebra, where the algebra structure of A becomes the coalgebra structure of $D(A)$, the antipode of A translates into an antipode $D(S)$ of $D(A)$ in a canonical fashion, and so on.

Let M, N be A-modules. The Hopf structure of A becomes advantageous in studying its homological properties. For instance, $\mathbf{k}$ is a trivial A-module with the action via the counit map ε, $a \cdot r = \varepsilon(a)r$, for all $a \in A$ and $r \in \mathbf{k}$. Tensor product of A modules $M \otimes N$ is again an A-module via the coproduct map Δ, $a \cdot (m \otimes n) = \sum (a_1 \cdot m) \otimes (a_2 \cdot n)$, for all $a \in A, m \in M$, and $n \in N$. The group $\operatorname{Hom}_{\mathbf{k}}(M,N)$ is also an A-module via the antipode map S, $(a \cdot f)(m) = f(S(a) \cdot m)$, for all $a \in A, m \in M$ and $f \in \operatorname{Hom}_{\mathbf{k}}(M,N)$. In particular, the $\mathbf{k}$-dual $D(M)$ is a left A-module. Since A is a finite dimensional Hopf algebra, it is **Frobenius**, that is $A \cong D(A)$ as left A-modules [29, Theorem 2.1.3]. Since $D(A)$ is an injective module over A, this implies that any Frobenius algebra is **self-injective** (injective as a module over itself). Therefore, projective A-modules coincide with injective A-modules.

The **opposite algebra** A^{op} has the same underlying set and linear operation as A but with multiplication performed in the reverse order: $a \cdot_{op} b = ba$, for all $a, b \in A$. Let $A^e := A \otimes A^{op}$ denote the **enveloping algebra of** A and define $\sigma : A \to A^e$ by $\sigma(a) = \sum a_1 \otimes S(a_2)$. Checking that σ is an injective algebra homomorphism, we may identify A with the subalgebra $\sigma(A)$ of A^e. Moreover, we can induce A^e-modules from A-modules as follows. Let M be a left A-module and consider A^e as a right A-module via right multiplication by $\sigma(A)$. Then $A^e \otimes_A M$ is a left A^e-module, with A^e-action given by $a \cdot (b \otimes_A m) = ab \otimes_A m,$ for all $a, b \in A^e$, and $m \in M$.

We use the following notation for the usual cohomology and Hochschild cohomology of A, respectively:

$$\mathrm{H}^*(A, M) := \mathrm{Ext}^*_A(\mathbf{k}, M) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(\mathbf{k}, M),$$
$$\mathrm{HH}^*(A, M) := \mathrm{Ext}^*_{A^e}(A, M) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_{A^e}(A, M),$$

where M denotes a left A-module in the former case and an A-bimodule in the latter case. From the discussion in [37], since A is a finite dimensional Hopf algebra, $\mathrm{H}^*(A, \mathbf{k})$ is a graded-commutative ring, that is, for $\alpha \in \mathrm{H}^i(A, \mathbf{k})$ and $\beta \in \mathrm{H}^j(A, \mathbf{k})$, $\alpha\beta = (-1)^{ij}\beta\alpha$. For any associative algebra A, $\mathrm{HH}^*(A, A)$ is always graded-commutative as a result by Gerstenhaber [21].

CHAPTER III

TATE AND TATE-HOCHSCHILD COHOMOLOGY *

As seen in the previous chapter, projective resolutions are used to compute the cohomology of an algebra. To define the Tate cohomology, we apply a more general resolution, which involves both positive and negative degrees.

First, we recall that a (not necessarily commutative) ring R is **Gorenstein** if R has finite injective dimensions both as a left R-module and as a right R-module. The readers may refer to, for example, [2, §2, 3] for a definition of **Gorenstein dimension** (or G**-dimension**) that was first introduced by Auslander and Bridger. For a two-sided Noetherian ring R, we say that R is Gorenstein of Gorenstein dimension d if the injective dimensions of R, both as a left and as a right module over itself, are equal to d.

Definition III.1. Let R be a ring. A **complete resolution** of a finitely generated R-module M is an exact complex $\mathbb{P} = \{P_i, d_i : P_i \to P_{i-1}\}_{i\in\mathbb{Z}}$ of finitely generated projective R-modules such that:

1. There exists a projective resolution $Q_\bullet \xrightarrow{\varepsilon} M$ of M and a chain map $\mathbb{P} \xrightarrow{\varphi} Q_\bullet$ where φ_n is bijective for $n \gg 0$.
2. The dual complex $\operatorname{Hom}_R(\mathbb{P}, R)$ is also exact.

The first condition says that for a sufficiently large degree, $\mathbb{P}$ coincides with a projective resolution of M. A resolution that satisfies the second condition is called **totally acyclic**. Unlike projective resolutions, complete resolutions, in general, do not always exist. However, if R is a two-sided Noetherian Gorenstein ring of Gorenstein dimension d, Theorems 3.1 and 3.2 in [2] guarantee the existence of such complete resolutions. In this case, the chain map φ_n is bijective for $n \geq d$.

In [38], Tate introduced a cohomology theory that can be defined by using complete resolutions as follows [4, §5.15]. Let G be a finite group and R be a commutative ring with G acting trivially on

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R. If

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} R \to 0$$

is an RG-projective resolution of R, then apply $\mathrm{Hom}_R(-, R)$ to get a dual sequence:

$$0 \to R \to \mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R) \to \mathrm{Hom}_R(P_2, R) \to \cdots$$

This is again an exact sequence of projective RG-modules. Splicing these two sequences together, one forms a doubly infinite sequence:

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to \mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R) \to \cdots$$

Introducing the notation $P_{-(n+1)} := \mathrm{Hom}_R(P_n, R)$, we arrive at a complete resolution of R:

$$\mathbb{P}: \qquad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to P_{-1} \to P_{-2} \to \cdots$$

Fix a (left) RG-module M. Applying $\mathrm{Hom}_{RG}(-, M)$ to $\mathbb{P}$ produces a new complex. The n-th homology of this new complex is the **n-th Tate cohomology** of RG:

$$\widehat{\mathrm{H}}^n(G, M) := \widehat{\mathrm{Ext}}^n_{RG}(R, M) = \mathrm{H}^n(\mathrm{Hom}_{RG}(\mathbb{P}, M)),$$

for all $n \in \mathbb{Z}$. We use the hat notation for Tate cohomology $\widehat{\mathrm{H}}^n(-)$ to distinguish from the usual cohomology $\mathrm{H}^n(-)$.

We note that Tate cohomology and its Hochschild version were later generalized by others and can be defined in a more general setting for Frobenius algebras e.g. [17, 30] or for two-sided Noetherian Gorenstein rings e.g. [2, 10]. The goal of this chapter is to specialize this cohomology theory for finite dimensional Hopf algebras A over a field $\mathbf{k}$. We recall the fact that a finite dimensional Hopf algebra A is a Frobenius algebra [29, Theorem 2.1.3], which is self-injective. Lemmas 3.1 and 3.2 in [7] show that A^{op}, A^e are also Frobenius, hence self-injective. In the context of the Definition III.1 and by the definition of Gorenstein rings, we observe that A, A^{op}, and A^e are Gorenstein of Gorenstein dimension $d = 0$. Using complete resolutions, we will introduce the Tate cohomology and Tate-Hochschild cohomology for A, their properties and product structures.

III.1 Tate cohomology for finite dimensional Hopf algebras

III.1.1 Definition of Tate cohomology

Generalizing the construction in [4, §5.15] and using the Hopf structure of A, we can explicitly form an A-complete resolution $\mathbb{P}$ of $\mathbf{k}$ from an A-projective resolution $P_\bullet$ of $\mathbf{k}$:

$$P_\bullet : \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon = d_0} \mathbf{k} \to 0.$$

This can be done by splicing $P_\bullet$ with its dual complex $D(P_\bullet) := \mathrm{Hom}_{\mathbf{k}}(P_\bullet, \mathbf{k})$, which is also an exact sequence of finitely generated projective A-modules, since a dual $D(P_i)$ of an A-module P_i is again an A-module and injective modules coincide with projective modules over a self-injective algebra. One can check that the resulting complex $\mathbb{P}$ is exact and satisfies the definition of a complete resolution of $\mathbf{k}$. The construction is described in the following diagram:

$$\begin{tikzcd}
\mathbb{P} : \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[rr, "D(\varepsilon)\circ\varepsilon"] \arrow[dr, "\varepsilon"'] & & P_{-1} \arrow[r] & P_{-2} \arrow[r] & P_{-3} \arrow[r] & \cdots \\
& & & & \mathbf{k} \arrow[ur, "D(\varepsilon)"'] \arrow[dr] & & & & \\
& & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

where we denote $P_{-1} := D(P_0)$, $P_{-2} := D(P_1)$, and so on.

Definition III.2. We define the **n-th Tate cohomology group of** A with coefficients in a left A-module M as:

$$\widehat{\mathrm{H}}^n(A, M) := \widehat{\mathrm{Ext}}_A^n(\mathbf{k}, M) = \mathrm{H}^n(\mathrm{Hom}_A(\mathbb{P}, M)), \text{ for all } n \in \mathbb{Z}.$$

The Tate homology groups $\widehat{\mathrm{H}}_n(A, M) := \widehat{\mathrm{Tor}}_n^A(\mathbf{k}, M)$ are defined analogously by applying $- \otimes_A M$ to $\mathbb{P}$ and taking the n-th homology of the new complex. Here, we are only interested in the Tate cohomology. Observe that in our context, naturally, the Tate (co)homology does not depend on the choice of the projective resolution of $\mathbf{k}$ (by the ordinary Comparison Theorem), and hence, is independent of the complete resolution of $\mathbf{k}$ [2, Theorem 5.2 and Lemma 5.3]. Moreover, from our construction of $\mathbb{P}$ described above, we see that the Tate cohomology groups agree with the usual cohomology groups in positive degrees:

$$\widehat{\mathrm{H}}^n(A, M) \cong \mathrm{H}^n(A, M), \text{ for all } n > 0.$$

Remark III.3. Instead of using complete resolutions, there is another formulation of the Tate cohomology for A via the stable module category [10, Lemma 6.1.2]. If M and N are finitely generated A-modules, we define $\underline{\mathrm{Hom}}_A(N, M)$ to be the quotient of $\mathrm{Hom}_A(N, M)$ by homomorphisms that factor through a projective module. Then for any integer n:

$$\widehat{\mathrm{Ext}}_A^n(\mathbf{k}, M) \cong \underline{\mathrm{Hom}}_A(\Omega^n \mathbf{k}, M) \cong \underline{\mathrm{Hom}}_A(\mathbf{k}, \Omega^{-n} M),$$

or equivalently [10, Prop. 6.5.1],

$$\widehat{\mathrm{Ext}}_A^n(\mathbf{k}, M) \cong \varinjlim_{k,\ k+n \geq 0} \underline{\mathrm{Hom}}_A(\Omega^{k+n} \mathbf{k}, \Omega^k M),$$

where Ω is the Heller operator, sending an A-module to the kernel of a projective cover of that module. This definition, which is equivalent to that using complete resolutions, is useful especially when proving some results on the cochain level. We will use these equivalent definitions of Tate cohomology interchangeably when it is convenient.

III.1.2 Properties of Tate cohomology

We compare the Tate cohomology and the usual cohomology of A. We note some important properties:

(a) For all $n > 0$, $\widehat{\mathrm{H}}^n(A, M) \cong \mathrm{H}^n(A, M)$.

(b) The group $\widehat{\mathrm{H}}^0(A, M)$ is a quotient of $\mathrm{H}^0(A, M)$.

These follow from the construction of complete resolutions.

(c) For all $n < -1$, we have isomorphisms: $\widehat{\mathrm{H}}^n(A, M) \cong \mathrm{H}_{-(n+1)}(A, M)$, by applying a similar argument as in [9, Prop. I.8.3c] to A.

(d) If $0 \to M \to M' \to M'' \to 0$ is a short exact sequence of (left) A-modules, then there is a doubly infinite long exact sequence of Tate cohomology groups, [2, Prop. 5.4] or see [30, Theorem 1] for the Tate-Hochschild version:

$$\cdots \to \widehat{\mathrm{H}}^n(A, M) \to \widehat{\mathrm{H}}^n(A, M') \to \widehat{\mathrm{H}}^n(A, M'') \to \widehat{\mathrm{H}}^{n+1}(A, M) \to \cdots$$

(e) If $(N_j)_{j\in J}$ is a finite family of (left) A-modules and $(M_i)_{i\in I}$ is any family of A-modules, then there are natural isomorphisms, for all $n \in \mathbb{Z}$ [2, Prop. 5.7]:

$$\widehat{\mathrm{Ext}}^n_A(\bigoplus_{j\in J} N_j, M) \cong \prod_{j\in J} \widehat{\mathrm{Ext}}^n_A(N_j, M),$$

$$\widehat{\mathrm{Ext}}^n_A(N, \prod_{i\in I} M_i) \cong \prod_{i\in I} \widehat{\mathrm{Ext}}^n_A(N, M_i).$$

III.2 Tate-Hochschild cohomology for finite dimensional Hopf algebras

Let A be a finite dimensional Hopf algebra over a field $\mathbf{k}$ and A^e be its enveloping algebra. Any bimodule M of A can be viewed as a left A^e-module by setting $(a\otimes b)\cdot m = amb$, for $a\otimes b \in A^e$ and $m \in M$. In particular, A is a left A^e-module. By [2, Theorems 3.1, 3.2], every finitely generated A^e-module admits a complete resolution. Hence, we obtain an A^e-complete resolution $\mathbb{X}$ for A.

Definition III.4. Let M be an A-bimodule. For any integer $n \in \mathbb{Z}$, the **n-th Tate-Hochschild cohomology group** of A is defined as:

$$\widehat{\mathrm{HH}}^n(A, M) := \widehat{\mathrm{Ext}}^n_{A^e}(A, M) = \mathrm{H}^n(\mathrm{Hom}_{A^e}(\mathbb{X}, M)), \text{ for all } n \in \mathbb{Z}.$$

Again, as A^e is Gorenstein of Gorenstein dimension 0, from the discussions in [2, 7], we see that the Tate-Hochschild cohomology groups of A agree with the usual Hochschild cohomology groups in all positive degrees:

$$\widehat{\mathrm{HH}}^n(A, M) \cong \mathrm{HH}^n(A, M), \text{ for all } n > 0.$$

Remark III.5. The Tate analog of the Hochschild cohomology (of a Frobenius algebra) is also considered in [30, §3] using a complete standard complex, or in [17, 2.1.11] as the stable Hochschild cohomology $\underline{\mathrm{Hom}}_{A^e}(\Omega^n A, M)$ using the stable module category of A^e. Eu and Schedler also showed that cup product and contraction structures extend to the stable $\mathbb{Z}$-graded setting for the Tate-Hochschild cohomology ring of A [17, Theorem 2.1.15].

In the next section, we observe that these two Tate-cohomology types obtain ring structures which can help us to develop a deeper understanding of their relation as algebras.

III.3 Multiplicative structures

III.3.1 Cup product on Tate cohomology

Suppose $\mathbb{P}$ is an A-complete resolution of $\mathbf{k}$. Based on the discussion in [9, §VI.5], we also note the following difficulties in constructing the cup product on Tate cohomology:

First of all, $\mathbb{P} \otimes \mathbb{P}$ is not a complete resolution of $\mathbf{k} \otimes \mathbf{k} \cong \mathbf{k}$, as $(\mathbb{P} \otimes \mathbb{P})_+$ is not the same as the tensor product of resolutions $\mathbb{P}_+ \otimes \mathbb{P}_+$, where $\mathbb{P}_+ = \{P_n\}_{n \geq 0}$. Consequently, using the map $\operatorname{Hom}_A(\mathbb{P}, M) \otimes \operatorname{Hom}_A(\mathbb{P}, N) \to \operatorname{Hom}_A(\mathbb{P} \otimes \mathbb{P}, M \otimes N)$ would not obviously induce a cohomology product in Tate cohomology as it does in the usual non-Tate cohomology. Secondly, when applying the diagonal approximation (a chain map that preserves augmentation) $\Gamma : \mathbb{P} \to \mathbb{P} \otimes \mathbb{P}$, for any $n \in \mathbb{Z}$, there are infinitely many (i, j) such that $i + j = n$, and the dimension-shifting property in Section III.1.2(d) suggests that the corresponding cup products should all be non-trivial. So Γ should have a non-trivial component Γ_{ij}, for all (i, j). Hence, the range of Γ should be the graded module which is $\prod_{i+j=n} P_i \otimes P_j$ in the dimension n, rather than $\bigoplus_{i+j=n} P_i \otimes P_j$. This discussion motivates us to the following definitions:

Let $\varepsilon : \mathbb{P} \to \mathbf{k}$ be an A-complete resolution of $\mathbf{k}$ and let d be the differentials in $\mathbb{P}$. We form the **complete tensor product** $\mathbb{P} \widehat{\otimes} \mathbb{P}$ by defining:

$$(\mathbb{P} \widehat{\otimes} \mathbb{P})_n = \prod_{i+j=n} P_i \otimes P_j, \text{ for all } n \in \mathbb{Z},$$

with the "total differential" $\partial_{i,j} = d^v_{i,j} + d^h_{i,j}$, where $d^h_{i,j} = d_i \widehat{\otimes} \mathbf{1}_{\mathbb{P}}$ and $d^v_{i,j} = (-1)^i \mathbf{1}_{\mathbb{P}} \widehat{\otimes} d_j$. It can be easily seen that $\mathbb{P} \widehat{\otimes} \mathbb{P}$ is an acyclic complex of A-modules. However, we note that $\mathbb{P} \widehat{\otimes} \mathbb{P}$ is not a complete resolution.

On the other hand, given graded modules B, B', C, C' and module homomorphisms $u : C \to B$ of degree r and $v : C' \to B'$ of degree s, there is a map $u\widehat{\otimes}v : C\widehat{\otimes}C' \to B\widehat{\otimes}B'$ of degree $r+s$ defined by:

$$(u\widehat{\otimes}v)_n = \prod_{i+j=n} (-1)^{is} u_i \otimes v_j : \prod_{i+j=n} C_i \otimes C'_j \to \prod_{i+j=n} B_{i+r} \otimes B'_{j+s}.$$

Definition III.6. A **complete diagonal approximation map** is a chain map $\Gamma : \mathbb{P} \to \mathbb{P}\widehat{\otimes}\mathbb{P}$ such that $(\varepsilon\widehat{\otimes}\varepsilon) \circ \Gamma_0 = \varepsilon$, that is, Γ is an augmentation-preserving chain map.

A similar argument to the proof given in [9, §VI.5] shows the existence of such a complete diagonal approximation map $\Gamma : \mathbb{P} \to \mathbb{P}\widehat{\otimes}\mathbb{P}$. Let M and N be left A-modules. Then $M \otimes N$ is also a left A-module via the coproduct: $a \cdot (m \otimes n) = \sum a_1 m \otimes a_2 n$, for all $a \in A, m \in M$, and $n \in N$. We define a cochain cup product:

$$\smile: \operatorname{Hom}_A(P_i, M) \otimes \operatorname{Hom}_A(P_j, N) \to \operatorname{Hom}_A(P_{i+j}, M \otimes N)$$

given by

$$f \smile g = (f\widehat{\otimes}g) \circ \Gamma,$$

where $f \in \operatorname{Hom}_A(P_i, M)$ and $g \in \operatorname{Hom}_A(P_j, N)$. One verifies that by the definition of differentials on the total complex, the usual coboundary formula holds:

$$\delta(f \smile g) = (\delta f) \smile g + (-1)^i f \smile (\delta g).$$

It follows from the formula that the product of two cocycles is again a cocycle and the product of a cocycle with a coboundary is a coboundary. Thus, this induces a well-defined product on Tate cohomology $\widehat{\mathrm{H}}^i(A, M) \otimes \widehat{\mathrm{H}}^j(A, N) \to \widehat{\mathrm{H}}^{i+j}(A, M \otimes N)$. Moreover, this cup product is unique, in the sense that: it is independent of the choice of $\mathbb{P}$ and Γ, it is associative at the chain level, that is,

$$(f \smile g) \smile h = f \smile (g \smile h),$$

and $1 \in \widehat{\mathrm{H}}^0(A, \mathbf{k})$ is an identity. One proves this using the dimension-shifting property in Section III.1.2 and exactness of tensor products over $\mathbf{k}$, similarly as in [9, §V.3.3 and Lemma VI.5.8] for group cohomology. It is immediate from the definitions that this product is natural with respect to coefficient homomorphisms. For example, an A-module homomorphism $M \otimes N \to Q$ yields products

$$\widehat{\mathrm{H}}^i(A, M) \otimes \widehat{\mathrm{H}}^j(A, N) \to \widehat{\mathrm{H}}^{i+j}(A, Q)$$

by composing the cup product and the induced map $\widehat{\mathrm{H}}^{i+j}(A, M \otimes N) \to \widehat{\mathrm{H}}^{i+j}(A, Q)$. In particular, when $M = N = Q = \mathbf{k}$, $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ is a graded ring. Moreover, by the construction of cup product, for f and g representing elements of $\widehat{\mathrm{H}}^i(A, \mathbf{k})$ and $\widehat{\mathrm{H}}^j(A, \mathbf{k})$, respectively:

$$f \smile g = (f \widehat{\otimes} g) \circ \Gamma = (-1)^{ij}(g \widehat{\otimes} f) \circ \Gamma = (-1)^{ij}(g \smile f),$$

proving $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ is graded-commutative. When $N = \mathbf{k}$ and $Q \cong M$, $\widehat{\mathrm{H}}^*(A, M)$ is a graded module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$.

III.3.2 Cup product on Tate-Hochschild cohomology

Let M and N be A-bimodules (which can be viewed as (left) A^e-modules). There is also a cup product on the Tate-Hochschild cohomology:

$$\widehat{\mathrm{HH}}^i(A, M) \otimes \widehat{\mathrm{HH}}^j(A, N) \to \widehat{\mathrm{HH}}^{i+j}(A, M \otimes_A N).$$

Before describing this cup product, let us recall some useful lemmas whose proofs can be found in [32]. We provide sketches of the proofs for completeness. Let $\sigma : A \to A^e$ be defined by $\sigma(a) = \sum a_1 \otimes S(a_2)$. Recall from Section II.4 that A^e may be viewed as a right A-module via right multiplication by elements of $\sigma(A)$.

Lemma III.7. *$A \cong A^e \otimes_A \mathbf{k}$ as left A^e-modules, where $A^e \otimes_A \mathbf{k}$ is the induced A^e-module.*

Proof. Let $f : A \to A^e \otimes_A \mathbf{k}$ be the function defined by:

$$f(a) = a \otimes 1 \otimes_A 1,$$

and let $g : A^e \otimes_A \mathbf{k} \to A$ be the function defined by:

$$g(a \otimes b \otimes_A 1) = ab,$$

for all $a, b \in A$. One can easily check that f and g are both A^e-module homomorphisms, and that they are inverses of each other. $\square$

Lemma III.8. *A^e is a (right) projective A-module.*

Proof. Since A is finite dimensional, its antipode map S is bijective. Moreover, S is an A-module map: for all $a, b \in A$, we have $S(ab) = S(b)S(a) = S(a) * S(b)$ in A^{op}. This implies $S : A \to A^{op}$ is an isomorphism of right A-modules, where A acts on A by right by multiplication and on A^{op} by multiplication by $S(A)$. This yields an isomorphism of right A-modules: $A \otimes A \to A \otimes A^{op} = A^e$. Since A is projective over itself and free over $\mathbf{k}$, $A \otimes A$ is a projective right A-module by [3, Prop. 3.1.5]. Therefore, A^e is a projective right A-module, where A acts on A^e by multiplying $\sigma(A)$. $\square$

Let $\mathbb{X}$ be any A^e-complete resolution of A. By the same argument as in Section III.3.1, $\mathbb{X}\widehat{\otimes}_A\mathbb{X}$ is an acyclic chain complex of A^e-modules. Since $\otimes_A$ is not an exact functor in general, the existence of a complete diagonal approximation map Γ does not follow trivially from [9] as before. We show it here in detail.

Lemma III.9. *There exists a complete diagonal approximation map $\Gamma : \mathbb{X} \to \mathbb{X}\widehat{\otimes}_A\mathbb{X}$.*

Proof. Observe that $A^e \otimes_A A^e = (A \otimes A^{op}) \otimes_A (A \otimes A^{op}) \cong A \otimes A^{op} \otimes A^{op} \cong A^e \otimes_{\mathbf{k}} A$. Since A^e acts only on the outermost two factors of A, dropping A in the third step does not change the A^e-module structure. Therefore, $A^e \otimes_A A^e \cong A^e \otimes_{\mathbf{k}} A$ is an A^e-module isomorphism, not just a $\mathbf{k}$-module isomorphism.

As A is a free (hence projective) $\mathbf{k}$-module, $A^e \otimes_{\mathbf{k}} A$ is also free as an A^e-module. Consequently, $A^e \otimes_A A^e$ is a free A^e-module. In general, tensor product over A of free A^e-modules is free. Since any projective module is a direct summand of a free module, this implies that for all $i, j \in \mathbb{Z}$,

$X_i \otimes_A X_j$ is projective as an A^e-module. Again, because A^e is self-injective, projective A^e-modules are also injective. Therefore, $X_i \otimes_A X_j$ is injective, implying the direct product $(\mathbb{X}\widehat{\otimes}_A\mathbb{X})_n$ is an injective A^e-module for all $n \in \mathbb{Z}$.

As remarked in [3, Theorem 2.4.2], to form the chain map $\Gamma_+ : \mathbb{X}_+ \to (\mathbb{X}\widehat{\otimes}_A\mathbb{X})_+$ in non-negative degrees, it suffices for the complex $\mathbb{X}_+$ to consist of projective modules but it need not be exact, and for the complex $(\mathbb{X}\widehat{\otimes}_A\mathbb{X})_+$ to be exact but not necessarily to consist of projective modules. Since $\mathbb{X}_+$ is a projective resolution of A, we can apply the ordinary Comparison Theorem to obtain a chain map Γ_+ that is augmentation-preserving. We then consider the projective A^e-modules in negative degrees of these complexes, which are (relatively) injective as discussed above. By a generalization of [9, Prop. VI.2.4], the family of maps Γ_+ extends to a complete chain map $\Gamma : \mathbb{X} \to \mathbb{X}\widehat{\otimes}_A\mathbb{X}$ in both positive and negative degrees. □

We may define a cup product on Tate-Hochschild cohomology as follows. Let M and N be A-bimodules, then $M \otimes_A N$ is also an A-bimodule which can be considered as a left A^e-module via $(a \otimes b) \cdot (m \otimes_A n) = am \otimes_A nb$, for $a \otimes b \in A^e, m \in M$ and $n \in N$. Let $f \in \mathrm{Hom}_{A^e}(X_i, M)$ represent an element of $\widehat{\mathrm{HH}}^i(A, M)$ and let $g \in \mathrm{Hom}_{A^e}(X_j, N)$ represent an element of $\widehat{\mathrm{HH}}^j(A, N)$. Then:

$$f \smile g = (f \widehat{\otimes} g) \circ \Gamma \in \mathrm{Hom}_{A^e}(X_{i+j}, M \otimes_A N)$$

represents an element of $\widehat{\mathrm{HH}}^{i+j}(A, M \otimes_A N)$. One can check that this product is independent of $\mathbb{X}$ and Γ and satisfies certain properties as in Section III.3.1. In particular, if $M = N = A$, then $\widehat{\mathrm{HH}}^*(A, A)$ is a graded-commutative ring. If $N = A$, then $\widehat{\mathrm{HH}}^*(A, M)$ is a graded $\widehat{\mathrm{HH}}^*(A, A)$-module.

CHAPTER IV

TATE COHOMOLOGY RELATION *

For a finite dimensional Hopf algebra A over a field $\mathbf{k}$, it is known that the usual cohomology $\mathrm{H}^*(A,\mathbf{k})$ of A is a direct summand of its Hochschild cohomology $\mathrm{HH}^*(A,A)$ [22, Prop. 5.6 and Cor. 5.6]. This motivates us to ask if the same assertion holds for the Tate cohomology version. We made an attempt to compare the Tate and the Tate-Hochschild cohomology groups of A in Section III.2. We approach a broader setting by establishing cup products for the two Tate cohomology types in Section III.3. These multiplication structures turn $\widehat{\mathrm{H}}^*(A,\mathbf{k})$ and $\widehat{\mathrm{HH}}^*(A,A)$ into graded-commutative rings. Using the ring structures, we will show that the Tate and Tate-Hochschild cohomology share the same relation as that of the usual cohomology. In this chapter, we also compute the Tate and Tate-Hochschild cohomology for the Taft algebra, in particular, the Sweedler algebra H_4, as seen in Examples II.6 and II.5. These examples demonstrate explicit computations using complete resolutions and help us to understand the above relation better.

IV.1 Relationship between the Tate and Tate-Hochschild cohomology rings of A

We begin with a lemma based on the original Eckmann-Shapiro Lemma but generalized to a complete resolution:

Lemma IV.1 (Eckmann-Shapiro). *Let B be a ring, let $C \subseteq B$ be a subring for which B is flat as a right C-module. Let M be a left C-module and let N be a left B-module. Consider N to be a left C-module via restriction of the action, and let $B \otimes_C M$ denote the induced B-module where B acts on the leftmost factor by multiplication. Then for all $i \in \mathbb{Z}$, there is an isomorphism of abelian groups:*

$$\widehat{\mathrm{Ext}}^i_C(M,N) \cong \widehat{\mathrm{Ext}}^i_B(B \otimes_C M, N).$$

If B and C are $\mathbf{k}$-algebras, then this is an isomorphism of vector spaces over $\mathbf{k}$.

*Reprinted from Journal of Pure and Applied Algebra, Vol 217, Van C. Nguyen, Tate and Tate-Hochschild cohomology for finite dimensional Hopf algebras, pp. 1967-1979. Copyright 2013, with permission from Elsevier.

Proof. Let $\varepsilon : \mathbb{P} \to M$ be a C-complete resolution of M. Since $B \otimes_C C \cong B$ as a left B-module, the induced modules $B \otimes_C P_i$ are projective B-modules, for all $i \in \mathbb{Z}$. The induced complex $B \otimes_C \mathbb{P}$ is exact as B is flat over C, with the "augmentation map" $\mathbf{1}_B \otimes_C \varepsilon : B \otimes_C \mathbb{P} \to B \otimes_C M$. So it is a complete resolution of $B \otimes_C M$ as a B-module.

It suffices to show that for all $i \in \mathbb{Z}$, $\mathrm{Hom}_C(P_i, N) \cong \mathrm{Hom}_B(B \otimes_C P_i, N)$ as abelian groups. This follows from the Nakayama relations [3, Prop. 2.8.3]. One can also check that these isomorphisms commute with the differentials. By the definition of Tate cohomology, these isomorphisms will comprise a chain map that induces an isomorphism on cohomology and give us the desired result. □

We consider A to be an A-module by the left adjoint action: for $a, b \in A$, $a \cdot b = \sum a_1 b S(a_2)$, and denote this A-module by A^{ad}. More generally, if M is an A-bimodule, denote by M^{ad} the left A-module with action given by $a \cdot m = \sum a_1 m S(a_2)$, for all $a \in A$ and $m \in M$. We now prove our main result:

Theorem IV.2. *([31, Theorem 7.2]) Let A be a finite dimensional Hopf algebra over a field $\mathbf{k}$. Then there exists an isomorphism of algebras:*

$$\widehat{\mathrm{HH}}^*(A, A) \cong \widehat{\mathrm{H}}^*(A, A^{ad}).$$

Proof. By Lemma III.8, A^e is a projective, hence, flat A-module. We then apply Lemmas III.7 and IV.1 with $B = A^e$, $C = A$ is identified as a subalgebra of A^e, $M = \mathbf{k}$ is a left A-module, and $N = A \cong A^e \otimes_A \mathbf{k}$ is an induced left A^e-module. We get $\widehat{\mathrm{Ext}}^*_A(\mathbf{k}, A^{ad}) \cong \widehat{\mathrm{Ext}}^*_{A^e}(A^e \otimes_A \mathbf{k}, A)$ as $\mathbf{k}$-modules, i.e. $\widehat{\mathrm{H}}^*(A, A^{ad}) \cong \widehat{\mathrm{HH}}^*(A, A)$ as $\mathbf{k}$-modules.

To show this is an algebra isomorphism, it remains to prove that cup products are preserved by this isomorphism. Let $\mathbb{P}$ denote an A-complete resolution of $\mathbf{k}$. Since A^e is a (right) projective A-module by Lemma III.8, $\mathbb{X} := A^e \otimes_A \mathbb{P}$ is an A^e-complete resolution of $A^e \otimes_A \mathbf{k} \cong A$.

Recall that A is acting on A^e on the left as well as on the right via σ. We define an A-chain map $\iota : \mathbb{P} \to \mathbb{X}$ by $\iota(p) = (1 \otimes 1) \otimes_A p$, for all $p \in P_i, i \in \mathbb{Z}$. Let $f \in \mathrm{Hom}_{A^e}(X_i, A)$ be a cocycle representing a cohomology class in $\widehat{\mathrm{Ext}}^i_{A^e}(A, A)$. The corresponding cohomology class in

$\widehat{\mathrm{Ext}}_A^i(\mathbf{k}, A^{ad})$ is represented by $f \circ \iota$.

Let $\Gamma : \mathbb{P} \to \mathbb{P}\widehat{\otimes}\mathbb{P}$ be a complete diagonal approximation map. Γ induces a cup product on cohomology as discussed in Section III.3.1. Γ also induces a chain map $\Gamma' : \mathbb{X} \to \mathbb{X}\widehat{\otimes}_A\mathbb{X}$ as follows. There is a map of A^e-chain complexes $\phi : A^e \otimes_A (\mathbb{P}\widehat{\otimes}\mathbb{P}) \to \mathbb{X}\widehat{\otimes}_A\mathbb{X}$ given by:

$$\phi((a \otimes b) \otimes_A (p \otimes q)) = ((a \otimes 1) \otimes_A p) \otimes_A ((1 \otimes b) \otimes_A q).$$

Γ induces a map from $A^e \otimes_A \mathbb{P}$ to $A^e \otimes_A (\mathbb{P}\widehat{\otimes}\mathbb{P})$. Let Γ' be the composition of this map with ϕ.

Let $f \in \mathrm{Hom}_{A^e}(X_i, A)$ and $g \in \mathrm{Hom}_{A^e}(X_j, A)$ be cocycles. The above discussions imply the following diagram commutes:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\Gamma'"] & \mathbb{X}\widehat{\otimes}_A\mathbb{X} \arrow[r, "f\widehat{\otimes}g"] & A \otimes_A A \arrow[r, "\sim"] & A \\
\mathbb{P} \arrow[u, "\iota"] \arrow[r, "\Gamma"] & \mathbb{P}\widehat{\otimes}\mathbb{P} \arrow[r, "(f\iota)\widehat{\otimes}(g\iota)"] & A \otimes A \arrow[r, "m"] & A \arrow[u, equal]
\end{tikzcd}$$

where m denotes the multiplication $a \otimes b \overset{m}{\longmapsto} ab,$ for all $a, b \in A$.

As described in Section III.3, the top row yields a product in $\widehat{\mathrm{Ext}}_{A^e}^*(A, A)$ and the bottom row yields a product in $\widehat{\mathrm{Ext}}_A^*(\mathbf{k}, A^{ad})$. Thus, cup products are preserved and $\widehat{\mathrm{HH}}^*(A, A)$ is isomorphic to $\widehat{\mathrm{H}}^*(A, A^{ad})$ as algebras. $\square$

As a consequence of Theorem IV.2, to determine the Tate-Hochschild cohomology of a finite dimensional Hopf algebra A, it suffices to compute its Tate cohomology with coefficients in the adjoint A-module. One may, therefore, apply known examples of Tate cohomology groups (such as, [14, §XII.7]) to compute the corresponding Tate-Hochschild cohomology. Furthermore, we arrive at the desired relation between the Tate and Tate-Hochschild cohomology rings of A:

Corollary IV.3. *([31, Cor. 7.3]) If A is a finite dimensional Hopf algebra over a field $\mathbf{k}$, then $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ is a direct summand of $\widehat{\mathrm{HH}}^*(A, A)$ as a module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$.*

Proof. Under the left adjoint action of A on itself, the trivial module $\mathbf{k}$ is isomorphic to the submodule of A^{ad} given by all scalar multiples of the identity 1. In fact, $\mathbf{k}$ is a direct summand of A^{ad} with complement the augmentation ideal $\mathrm{Ker}(\varepsilon)$, where $\varepsilon : A \to \mathbf{k}$ is the counit map.

From the property (e) in Section III.1.2, $\widehat{\mathrm{Ext}}_A^*(\mathbf{k}, -)$ is additive. By Theorem IV.2, we have:

$$\begin{aligned}\widehat{\mathrm{HH}}^*(A, A) \cong \widehat{\mathrm{H}}^*(A, A^{ad}) = \widehat{\mathrm{Ext}}_A^*(\mathbf{k}, A^{ad}) \\ \cong \widehat{\mathrm{Ext}}_A^*(\mathbf{k}, \mathbf{k}) \oplus \widehat{\mathrm{Ext}}_A^*(\mathbf{k}, \mathrm{Ker}(\varepsilon)) \\ \cong \widehat{\mathrm{H}}^*(A, \mathbf{k}) \oplus \widehat{\mathrm{Ext}}_A^*(\mathbf{k}, \mathrm{Ker}(\varepsilon)).\end{aligned}$$

Both $\mathbf{k}$ and $\mathrm{Ker}(\varepsilon)$ are closed under multiplication, and this multiplication induces multiplications on $\widehat{\mathrm{Ext}}_A^*(\mathbf{k}, -)$ which is also compatible with the ring structure on $\widehat{\mathrm{Ext}}_A^*(\mathbf{k}, A^{ad})$. Hence, this is in fact a direct summand, where $\widehat{\mathrm{HH}}^*(A, A) \cong \widehat{\mathrm{H}}^*(A, A^{ad})$ is considered as a (left) module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ with action via (left) multiplication. □

In the next section, we will compute the Tate and Tate-Hochschild cohomology for the Taft algebra, in particular, the Sweedler algebra. These simple examples should give us a rough procedure to produce other examples.

IV.2 Example: Taft algebra

IV.2.1 Tate cohomology of Taft algebra

Let $N \geq 2$ be an integer and $\mathbf{k}$ be a field containing a primitive N-th root of unity ω. We recall the Taft algebra A, which is a Hopf algebra of dimension N^2:

$$A = \mathbf{k}\langle g, x \mid g^N = 1, x^N = 0, xg = \omega gx\rangle$$

as described in Example II.6. It is known that as an algebra, Taft algebra is a smash product $A = B\#\mathbf{k}G$ (or a skew group algebra), with $B = \mathbf{k}[x]/(x^N)$, and G is a finite cyclic group generated by g of order N acting on B. Let $\chi : G \to \mathbf{k}^\times$ be the character, that is, a group homomorphism from G to the multiplicative group of $\mathbf{k}$, defined by $\chi(g) = \omega$. G acts by automorphisms on B via:

$$ {}^{g}x = \chi(g)x = \omega x. $$

Note that since G is generated by g, all G-actions are determined by the action of the generator g. By the definition of smash product, A is $B \otimes \mathbf{k}G$ as a vector space, with the multiplication:

$$ (b_1 \otimes g_1)(b_2 \otimes g_2) = b_1({}^{g_1}b_2) \otimes g_1 g_2, $$

for all $b_1, b_2 \in B$ and $g_1, g_2 \in G$. We abbreviate $b_i \otimes g_i$ by $b_i g_i$. Moreover, as the characteristic of $\mathbf{k}$ does not divide $|G| = N$, $\mathbf{k}G$ is semisimple and so all the cohomology of $\mathbf{k}G$ is trivial except in the degree 0. From [36, Cor. 3.4], the Taft algebra's usual cohomology is known as:

$$ \mathrm{H}^*(A, \mathbf{k}) \cong (\mathrm{Ext}^*_B(\mathbf{k}, \mathbf{k}))^G, $$

where the superscript G denotes the invariants under the action of G. Again, as the characteristic of $\mathbf{k}$ is relatively prime to $|G|$, G-invariants may be taken in a complex prior to taking the cohomology. We consider the following B-free resolution of $\mathbf{k}$:

$$ \cdots \xrightarrow{\cdot x} B \xrightarrow{\cdot x^{N-1}} B \xrightarrow{\cdot x} B \xrightarrow{\cdot x^{N-1}} B \xrightarrow{\cdot x} B \xrightarrow{\varepsilon} \mathbf{k} \to 0, \tag{IV.1} $$

with $\varepsilon(x) = 0$ is the augmentation map. This resolution could be extended to a projective A-resolution of $\mathbf{k}$ by giving B the following actions of G, for all $b \in B$ and $i > 0$:

- In degree 0, $g \cdot b := {}^{g}b$.
- In degree $2i$, $g \cdot b := \chi(g)^{iN}({}^{g}b) = {}^{g}b$, since $\chi(g)^{iN} = \omega^{iN} = 1^i = 1$.
- In degree $2i+1$, $g \cdot b := \chi(g)^{iN+1}({}^{g}b) = \omega({}^{g}b)$.

We check that this group action commutes with the differentials in (IV.1) in each degree. Thus, we may extend the differentials $\cdot x^{N-1}$ and $\cdot x$ in (IV.1) to be A-module maps. Moreover, since the characteristic of $\mathbf{k}$ does not divide $|G|$, an A-module is projective if and only if its restriction to B is projective. With these actions, (IV.1) is indeed an A-projective resolution of $\mathbf{k}$.

Take the $\mathbf{k}$-dual $\mathrm{Hom}_{\mathbf{k}}(-,\mathbf{k}) =: D(-)$ of (IV.1), we have:

$$0 \to \mathbf{k} \xrightarrow{D(\varepsilon)} D(B) \xrightarrow{D(\cdot x)} D(B) \xrightarrow{D(\cdot x^{N-1})} D(B) \xrightarrow{D(\cdot x)} D(B) \xrightarrow{D(\cdot x^{N-1})} \cdots \tag{IV.2}$$

which is an exact sequence of projective A-modules. Splicing (IV.1) and (IV.2) together at $\mathbf{k}$, we obtain an A-complete resolution of $\mathbf{k}$:

$$\mathbb{P}: \quad \cdots \xrightarrow{\cdot x} B \xrightarrow{\cdot x^{N-1}} B \xrightarrow{\cdot x} B \xrightarrow{\xi} D(B) \xrightarrow{D(\cdot x)} D(B) \xrightarrow{D(\cdot x^{N-1})} D(B) \xrightarrow{D(\cdot x)} \cdots \tag{IV.3}$$

where $\xi = D(\varepsilon)\circ\varepsilon$. To compute the Tate cohomology of A with coefficients in $\mathbf{k}$, apply $\mathrm{Hom}_A(-,\mathbf{k}) = \overline{(-)}$ to (IV.3):

$$\cdots \xrightarrow{\overline{D(\cdot x)}} \overline{D(B)} \xrightarrow{\overline{D(\cdot x^{N-1})}} \overline{D(B)} \xrightarrow{\overline{D(\cdot x)}} \overline{D(B)} \xrightarrow{\bar{\xi}} \overline{B} \xrightarrow{\overline{(\cdot x)}} \overline{B} \xrightarrow{\overline{(\cdot x^{N-1})}} \overline{B} \xrightarrow{\overline{(\cdot x)}} \cdots \tag{IV.4}$$

which is a complex of A-modules. Take the homology of this new complex, we will obtain the Tate cohomology of the Taft algebra A.

Let us compute explicitly for the case $N = 2$. Here, $G \cong \mathbb{Z}_2$, $B = \mathbf{k}[x]/(x^2)$, and $A = B\#\mathbf{k}G$ is the Sweedler algebra H_4 that we have seen in Example II.5. B has a basis $\{1, x\}$ and $D(B)$ has a dual basis $\{f_1, f_x\}$. By previously defined actions of G, B is an H_4-module with:

$$g \cdot x = -x, \quad g \cdot 1 = 1, \quad \text{in even degrees}$$
$$g \cdot x = x, \quad g \cdot 1 = -1, \quad \text{in odd degrees}$$

such that this action commutes with the differential maps in (IV.1). We denote $(-)_{ev}$ for objects in even degrees and $(-)_{odd}$ for objects in odd degrees, given the corresponding actions of G as H_4-modules; hence, (IV.3) becomes:

$$\mathbb{P}: \quad \cdots \xrightarrow{\cdot x} B_{ev} \xrightarrow{\cdot x} B_{odd} \xrightarrow{\cdot x} B_0 \xrightarrow{\xi} D(B)_{odd} \xrightarrow{D(\cdot x)} D(B)_{ev} \xrightarrow{D(\cdot x)} D(B)_{odd} \xrightarrow{D(\cdot x)} \cdots$$

as an H_4-complete resolution of $\mathbf{k}$. $D(B)$ is an H_4-module via the action: $(g \cdot f)(b) = f(S(g)\cdot b) = f(g \cdot b)$, for $f \in D(B), b \in B, g \in G$ and $S(g) = g^{-1} = g$ in H_4. Checking on the basis elements, we see that the G-actions on $D(B)$ can be carried along:

$$g \cdot f_1 = f_1, \quad g \cdot f_x = -f_x, \quad \text{in even degrees}$$
$$g \cdot f_1 = -f_1, \quad g \cdot f_x = f_x, \quad \text{in odd degrees.}$$

By identifying $f_1 \leftrightarrow x$ and $f_x \leftrightarrow 1$, we have $D(B)_{ev} \cong B_{odd}$ and $D(B)_{odd} \cong B_{ev}$ as H_4-modules. As a result, $\mathbb{P}$ can be written as:

$$\mathbb{P}: \quad \cdots \xrightarrow{\cdot x} B_{ev} \xrightarrow{\cdot x} B_{odd} \xrightarrow{\cdot x} B_0 \xrightarrow{\xi} B_{ev} \xrightarrow{D(\cdot x)} B_{odd} \xrightarrow{D(\cdot x)} B_{ev} \xrightarrow{D(\cdot x)} \cdots$$

Apply $\mathrm{Hom}_{H_4}(-, \mathbf{k}) = \overline{(-)}$ to $\mathbb{P}$, we get the complex of H_4-modules:

$$\cdots \xrightarrow{\overline{D(\cdot x)}} \overline{B_{ev}} \xrightarrow{\overline{D(\cdot x)}} \overline{B_{odd}} \xrightarrow{\overline{D(\cdot x)}} \overline{B_{ev}} \xrightarrow{\overline{\xi}} \overline{B_0} \xrightarrow{\overline{(\cdot x)}} \overline{B_{odd}} \xrightarrow{\overline{(\cdot x)}} \overline{B_{ev}} \xrightarrow{\overline{(\cdot x)}} \cdots$$

For all $f \in \mathrm{Hom}_B(B, \mathbf{k})$ and $b \in B$, $1 = f(b) = b \cdot f(1) = \varepsilon(b)f(1)$. We may identify f with a map in $\mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, \mathbf{k})$ and obtain an isomorphism $\mathrm{Hom}_B(B, \mathbf{k}) \cong \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, \mathbf{k})$ which is isomorphic to $\mathbf{k}$. We then observe that $\overline{B} = \mathrm{Hom}_{H_4}(B, \mathbf{k})$ is contained in $\mathrm{Hom}_B(B, \mathbf{k}) \cong \mathbf{k}$. One can check that under the corresponding group actions:

$$\overline{B} = \begin{cases} \mathbf{k} & \text{in 0 and even degrees} \\ 0 & \text{in odd degrees.} \end{cases}$$

This simplifies the above complex to:

$$\cdots \xrightarrow{\overline{D(\cdot x)}} \mathbf{k} \xrightarrow{\overline{D(\cdot x)}} 0 \xrightarrow{\overline{D(\cdot x)}} \mathbf{k} \xrightarrow{\overline{\xi}} \mathbf{k} \xrightarrow{\overline{(\cdot x)}} 0 \xrightarrow{\overline{(\cdot x)}} \mathbf{k} \xrightarrow{\overline{(\cdot x)}} \cdots$$

To compute the homology of this complex, we need to see what $\overline{\xi}$ looks like. $\overline{\xi} : \overline{D(B)} \to \overline{B}$ is defined as $\overline{\xi}(h)(b) = h(\xi(b))$ for $h \in \overline{D(B)}, b \in B$. By exactness of $\mathbb{P}$, $\mathrm{Im}(\xi) = \mathrm{Ker}(D(\cdot x)) = \langle f_1 \rangle$. As an H_4-module map, h sends $f_1 \mapsto 0$, and $f_x \mapsto 1$. Therefore, $\overline{\xi}(h)(b) = h(\xi(b)) = h(f_1) = 0$ and $\overline{\xi}$ is a 0-map.

Putting these together, we have computed the Tate cohomology for Sweedler algebra H_4:

$$\widehat{\mathrm{H}}^n(H_4, \mathbf{k}) = \begin{cases} \mathbf{k} & n < -1, n \text{ is odd} \\ 0 & n < -1, n \text{ is even} \\ \mathbf{k} & n = -1, 0 \\ 0 & n > 0, n \text{ is odd} \\ \mathbf{k} & n > 0, n \text{ is even.} \end{cases}$$

It follows that $\widehat{\mathrm{H}}^n(H_4, \mathbf{k}) \cong \widehat{\mathrm{H}}^{-(n+1)}(H_4, \mathbf{k})$, for all $n \in \mathbb{Z}$.

IV.2.2 Tate-Hochschild cohomology of Taft algebra

In order to compute the Tate-Hochschild cohomology of a general Taft algebra A, we use the following subalgebra of $A^e = A \otimes A^{op}$ as in [11]:

$$\mathcal{D} := B^e \# \mathbf{k}G \cong \bigoplus_{g \in G} (Bg \otimes Bg^{-1}) \subset A^e,$$

where the action of G on B^e is diagonal, that is, ${}^g(a \otimes b) = ({}^g a) \otimes ({}^g b)$. This isomorphism is given by $(a \otimes b)g \mapsto ag \otimes ({}^{g^{-1}} b)g^{-1}$, for all $a, b \in B$, and $g \in G$. Note that B is a $\mathcal{D}$-module under left and right multiplications. Since the characteristic of $\mathbf{k}$ does not divide $|G| = N$, the Hochschild cohomology $\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^e}(A, A)$ is known to satisfy:

$$\mathrm{HH}^*(A) \cong \mathrm{Ext}^*_{\mathcal{D}}(B, A) \cong \mathrm{Ext}^*_{B^e}(B, A)^G$$

as graded algebras. $\mathrm{Ext}^*_{B^e}(B, A)^G$ consists of invariants under the action induced from the action of G on a $\mathcal{D}$-module, see [11, (4.9)] or [36, Cor. 3.4] for more details. Observe the following B^e-free resolution of B, [39, Exercise 9.1.4]:

$$\cdots \to B^e \xrightarrow{\cdot v} B^e \xrightarrow{\cdot u} B^e \xrightarrow{\cdot v} B^e \xrightarrow{\cdot u} B^e \xrightarrow{m} B \to 0, \tag{IV.5}$$

where m is the multiplication map $a \otimes b \mapsto ab$,

$$u = x \otimes 1 - 1 \otimes x, \qquad \text{and } v = x^{N-1} \otimes 1 + x^{N-2} \otimes x + \cdots + 1 \otimes x^{N-1}.$$

Using this resolution, one computes $\mathrm{HH}^n(B) \cong B/(x^{N-1})$ and $(\mathrm{HH}^n(B))^G \cong \mathbf{k}$ for all $n > 0$. This resolution also becomes a $\mathcal{D}$-projective resolution of B by giving the following actions of G on B^e, for all $a \otimes b \in B^e$, $g \in G$, and integers $i > 0$:

- In degree 0, $g \cdot (a \otimes b) = ({}^g a) \otimes ({}^g b)$.
- In degree $2i$, $g \cdot (a \otimes b) = \chi(g)^{iN}({}^g a) \otimes ({}^g b) = ({}^g a) \otimes ({}^g b)$, since $\chi(g)^{iN} = \omega^{iN} = 1^i = 1$.
- In degree $2i+1$, $g \cdot (a \otimes b) = \chi(g)^{iN+1}({}^g a) \otimes ({}^g b) = \omega({}^g a) \otimes ({}^g b)$.

The differentials $\cdot u$ and $\cdot v$ commute with the group actions and turn out to be maps of $\mathcal{D}$-modules. Since $\mathrm{char}(\mathbf{k})$ does not divide $|G|$, a $\mathcal{D}$-module is projective if and only if its restriction to B^e is projective. With these actions, (IV.5) becomes a $\mathcal{D}$-projective resolution of B.

Because B^e is a left B-module by multiplying by the leftmost factor in B^e, we can take the dual $(-)^* := \mathrm{Hom}_B(-, B)$ of (IV.5). Since $B \cong \mathrm{Hom}_B(B, B)$, we have:

$$0 \to B \xrightarrow{m^*} (B^e)^* \xrightarrow{(\cdot u)^*} (B^e)^* \xrightarrow{(\cdot v)^*} (B^e)^* \xrightarrow{(\cdot u)^*} (B^e)^* \xrightarrow{(\cdot v)^*} \cdots \tag{IV.6}$$

One can show that this is an exact sequence of projective $\mathcal{D}$-modules. The dual $(B^e)^*$ is a left B^e-module via the action:

$$((a \otimes b) \cdot f)(c \otimes d) = f((a \otimes b)(c \otimes d)) = f(ac \otimes db),$$

for all $f \in \mathrm{Hom}_B(B^e, B)$, and $a, b, c, d \in B$. The differentials $(\cdot u)^*, (\cdot v)^*, m^*$ are B^e-module homomorphisms, since they are just the compositions of maps, $d^*(f) = f \circ d$.

For any $a \in B$ and $b \in B^{op}$, we may identify them with $a \otimes 1$ and $1 \otimes b$ in B^e, respectively. We observe that $(B^e)^* = \mathrm{Hom}_B(B^e, B) \cong \mathrm{Hom}_{\mathbf{k}}(B^{op}, B)$, since for all $f \in (B^e)^*$, we have $f(a \otimes b) = f(a(1 \otimes b)) = af(1 \otimes b)$. As f is determined by what it does on $a \otimes b$, we may identify $(B^e)^*$ with B^e under the correspondence $f_{a\otimes b} \mapsto a \otimes b$, where $f_{a \otimes b}(1 \otimes 1) = af(1 \otimes b)$. Hence, the maps $(\cdot u)^*$ and $(\cdot v)^*$ can be considered as the maps $\cdot u$ and $\cdot v$, respectively. Moreover, B^e is free over itself, so $(B^e)^* \cong B^e$ is a projective B^e-module. This implies (IV.6) is an exact sequence of projective B^e-modules; hence, an exact sequence of projective $\mathcal{D}$-modules.

Splicing (IV.5) and (IV.6) together at B, we form a $\mathcal{D}$-complete resolution of B:

$$\mathbb{X}: \qquad \cdots \xrightarrow{\cdot u} B^e \xrightarrow{\cdot v} B^e \xrightarrow{\cdot u} B^e \xrightarrow{\xi} B^e \xrightarrow{\cdot u} B^e \xrightarrow{\cdot v} B^e \xrightarrow{\cdot u} \cdots, \tag{IV.7}$$

where $\xi = m^* \circ m$. Due to the isomorphism $\mathrm{HH}^*(A) \cong \mathrm{Ext}^*_{\mathcal{D}}(B, A)$, we apply $\mathrm{Hom}_{\mathcal{D}}(-, A) = \widehat{(-)}$ to (IV.7):

$$\cdots \xrightarrow{\widehat{\cdot u}} \widehat{B^e} \xrightarrow{\widehat{\cdot v}} \widehat{B^e} \xrightarrow{\widehat{\cdot u}} \widehat{B^e} \xrightarrow{\widehat{\xi}} \widehat{B^e} \xrightarrow{\widehat{\cdot u}} \widehat{B^e} \xrightarrow{\widehat{\cdot v}} \widehat{B^e} \xrightarrow{\widehat{\cdot u}} \cdots, \tag{IV.8}$$

where $\widehat{B^e}$ denotes $\mathrm{Hom}_{\mathcal{D}}(B^e, A)$, and $\widehat{d}(f) = f \circ d$. It is easy to check that the composition of any two consecutive maps $\widehat{d} \circ \widehat{d}$ is equal to 0, making (IV.8) a complex.

For all $f \in \mathrm{Hom}_{B^e}(B^e, A), g \in G$, and $a \otimes b \in B^e$, we have $\mathrm{Hom}_{\mathcal{D}}(B^e, A) \cong \mathrm{Hom}_{B^e}(B^e, A)^G$, where G acts on $\mathrm{Hom}_{B^e}(B^e, A)$ by $(g \cdot f)(a \otimes b) = g \cdot f(g^{-1} \cdot (a \otimes b))$. Note that as a B^e-homomorphism, f is completely determined by its value on $1 \otimes 1$. We identify A with $\mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, A) \cong \mathrm{Hom}_{B^e}(B^e, A)$, under the correspondence $a \mapsto f_a$, where $f_a(1 \otimes 1) = a$, for all $a \in A$. The complex (IV.8) becomes:

$$\cdots \xrightarrow{\widehat{\cdot u}} A^G \xrightarrow{\widehat{\cdot v}} A^G \xrightarrow{\widehat{\cdot u}} A^G \xrightarrow{\widehat{\xi}} A^G \xrightarrow{\widehat{\cdot u}} A^G \xrightarrow{\widehat{\cdot v}} A^G \xrightarrow{\widehat{\cdot u}} \cdots$$

with the actions of G on A depending on the degrees as stated above. The maps $\widehat{\cdot u}$ and $\widehat{\cdot v}$ are:

$$(\widehat{\cdot u})(\mathbf{a}) = (\widehat{\cdot u})f_{\mathbf{a}}(1 \otimes 1) = f_{\mathbf{a}}(\cdot u(1 \otimes 1)) = x\mathbf{a} - \mathbf{a}x,$$
$$(\widehat{\cdot v})(\mathbf{a}) = x^{N-1}\mathbf{a} + x^{N-2}\mathbf{a}x + x^{N-3}\mathbf{a}x^2 + \cdots + \mathbf{a}x^{N-1},$$

for all $\mathbf{a} \in A^G$. We use an analogous argument as in [11, proof of Theorem 2.4] and apply the group actions on $\mathrm{Hom}_{B^e}(B^e, A)$ to take the invariants A^G. Since $\chi^{iN} = 1$, we find that in 0 and even degrees, $A^G = Z(\mathbf{k}G)$, the center of the group algebra $\mathbf{k}G$, which is $\mathbf{k}G$ itself because G is cyclic. Similarly, in odd degrees, the invariants are spanned by elements of the form Nxt, for $t \in G$. However, as G is cyclic generated by g, $t = g^j$ for some $j = 0, 1, \ldots, N-1$, we have A^G is spanned over $\mathbf{k}$ by $\{x, xg, xg^2, \ldots, xg^{N-1}\}$ in odd degrees. Thus, $\widehat{\cdot v}$ is the 0-map: $(\widehat{\cdot v})(xg^j) = 0$, as $x^N = 0$ in $A = B\#\mathbf{k}G$. We then have $\mathrm{Ker}(\widehat{\cdot v}) = A^G$ in odd degrees, and $\mathrm{Im}(\widehat{\cdot v}) = 0$. Similarly, in even degrees, $\operatorname{Ker}(\cdot\widehat{u}) = \mathbf{k}$. In odd degrees, $\operatorname{Im}(\cdot\widehat{u})$ is spanned over $\mathbf{k}$ by $\{xg, xg^2, \ldots, xg^{N-1}\}$.

We observe that:

$$\widehat{\xi} : A^G = \operatorname{Span}_{\mathbf{k}}\{x, xg, xg^2, \ldots, xg^{N-1}\} \to A^G = \mathbf{k}G$$

maps from degree -1 to degree 0. However, as there is no group element or field element in degree -1, x and its powers must be sent to 0. It follows that $\widehat{\xi}$ must be a 0-map. Putting these together, we obtain the Tate-Hochschild cohomology for the Taft algebra A, for any integer n:

$$\widehat{\mathrm{HH}}^n(A, A) = \begin{cases} \operatorname{Ker}(\cdot\widehat{u}) / \operatorname{Im}(\widehat{\xi}) = \mathbf{k}/0 = \mathbf{k}, & n = 0 \\ \operatorname{Ker}(\cdot\widehat{u}) / \operatorname{Im}(\cdot\widehat{v}) = \mathbf{k}/0 = \mathbf{k}, & n \text{ is even} \\ \operatorname{Ker}(\cdot\widehat{v}) / \operatorname{Im}(\cdot\widehat{u}) = \operatorname{Span}_{\mathbf{k}}\{x\}, & n \text{ is odd.} \end{cases}$$

In particular, since any two finite dimensional vector spaces over $\mathbf{k}$ having the same dimension are isomorphic, we have $\mathbf{k} \cong \operatorname{Span}_{\mathbf{k}}\{x\}$. We get a symmetric property for the Tate-Hochschild cohomology of Taft algebra:

$$\widehat{\mathrm{HH}}^n(A, A) \cong \widehat{\mathrm{HH}}^{-(n+1)}(A, A), \text{ for all } n \in \mathbb{Z}.$$

For the Sweedler algebra H_4, these isomorphisms can also be obtained without explicitly computing, as follows. Any Frobenius $\mathbf{k}$-algebra F is associated with a non-degenerate associative bilinear Frobenius form $\mathcal{B}(-,-) : F \times F \to \mathbf{k}$. The **Nakayama automorphism** $\nu : F \to F$ satisfies $\mathcal{B}(x, y) = \mathcal{B}(y, \nu(x))$, for all $x, y \in F$. Replacing $\mathcal{B}$ with a new Frobenius form $\mathcal{B}'$ defined by a unit element $u \in F$ gives us a new automorphism $\nu' = I_u \circ \nu$, where I_u is the inner automorphism $r \mapsto uru^{-1}$. The Nakayama automorphism ν is unique up to composition with an inner automorphism. Equivalently, it is a well-defined element of the group of outer automorphisms of F.

By [7, Cor. 3.8], if ν is the Nakayama automorphism of a Frobenius algebra F such that $\nu^2 = \mathbf{1}$, the identity map, then there is an isomorphism:

$$\widehat{\mathrm{HH}}^n(F, F) \cong \widehat{\mathrm{HH}}^{-(n+1)}(F, F), \text{ for all } n \in \mathbb{Z}.$$

Since H_4 is a Frobenius algebra, we can calculate the Nakayama automorphism ν of H_4 by applying the formula for ν given in [19, Lemma 1.5], we obtain $\nu^2 = \mathbf{1}$ on H_4. It follows that the Tate-Hochschild cohomology of H_4 has the property:

$$\widehat{\mathrm{HH}}^n(H_4, H_4) \cong \widehat{\mathrm{HH}}^{-(n+1)}(H_4, H_4), \text{ for all } n \in \mathbb{Z}.$$

CHAPTER V

TATE-HOCHSCHILD COHOMOLOGY OF A GROUP ALGEBRA

The theory of group cohomology is a well-studied yet ongoing research area. It has many applications to other areas such as representation theory, algebraic geometry, and commutative algebra. For an arbitrary commutative ring R and a group G, it is well-known that the Hochschild cohomology $\mathrm{HH}^*(RG, M) := \bigoplus_{n\geq 0} \mathrm{Ext}^n_{RG\otimes_R RG^{op}}(RG, M)$ with coefficients in an RG-bimodule M, is the same as the usual group cohomology ring, $\mathrm{H}^*(G, M) := \bigoplus_{n\geq 0} \mathrm{Ext}^n_{RG}(R, M)$ with coefficients in M under the diagonal action [16]. In particular, by considering RG as its own bimodule, $\mathrm{HH}^*(RG, RG)$ is isomorphic to $\mathrm{H}^*(G, RG)$, where RG is a left RG-module via conjugation. From this identification and the Eckmann-Shapiro Lemma, one can prove that $\mathrm{HH}^*(RG, RG)$ may be decomposed as a direct sum of the cohomology of the centralizers of conjugacy class representatives of G [4, Theorem 2.11.2]. In 1999, Siegel and Witherspoon described a formula for the products in $\mathrm{HH}^*(RG, RG)$ in terms of this additive decomposition [35]. When G is abelian, Holm [24] and Cibils and Solotar [15] proved that the Hochschild cohomology ring of G is (isomorphic to) the tensor product over R of RG and its usual cohomology ring.

As seen in Example II.3, a finite group algebra is a finite dimensional Hopf algebra. So our previous constructions and results hold for its Tate and Tate-Hochschild cohomology rings. In this chapter, we explore the structure of its Tate-Hochschild cohomology and generalize some known results in group cohomology to negative degrees.

V.1 Tate-Hochschild cohomology of a group algebra

Let G be a finite group. By Maschke's Theorem, if $\mathbf{k}$ is a field whose characteristic does not divide the order of G, then the group algebra $\mathbf{k}G$ is semisimple. So the cohomology of $\mathbf{k}G$ is trivial except in the degree 0 and nothing is interesting in such case (see also [14, Cor. XII.2.7]). Therefore, throughout this chapter, we let R be the ring of integers $\mathbb{Z}$ or a field $\mathbf{k}$ of prime characteristic $p > 0$ dividing the order of G (Tate group cohomology results over $\mathbb{Z}$ such as those in [14, Ch. XII] can also be proved for such a choice of field $\mathbf{k}$; hence, still hold for the group algebra RG). All rings and algebras are assumed to possess a unit; all modules are assumed to be left modules; and tensor products will be over R, unless stated otherwise. Consider the group algebra RG. Let $RG^e := RG \otimes RG^{op}$ be its enveloping algebra. If G is acting on a set X, we denote the action ${}^g x$, for all $g \in G$ and $x \in X$.

Using complete resolutions, the Tate cohomology $\widehat{\mathrm{H}}^*(G, M)$ and the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(RG, M)$ of RG, are the extensions of the usual cohomology and Hochschild cohomology, respectively, to negative degrees. There are also multiplicative structures making $\widehat{\mathrm{H}}^*(G) := \widehat{\mathrm{H}}^*(G, R)$ and $\widehat{\mathrm{HH}}^*(RG, RG)$ become graded-commutative rings, as seen in Section III.3 ([31, §3, 4, and 6]). Equivalently, using the stable module categories, one can also define the Tate cohomology, Tate-Hochschild cohomology, and their product structures, for example, see [17].

$$\widehat{\mathrm{H}}^*(G, M) := \bigoplus_{n \in \mathbb{Z}} \widehat{\mathrm{Ext}}^n_{RG}(R, M) \cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_{RG}(\Omega^n R, M),$$

$$\widehat{\mathrm{HH}}^*(RG, M) := \bigoplus_{n \in \mathbb{Z}} \widehat{\mathrm{Ext}}^n_{RG^e}(RG, M) \cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_{RG^e}(\Omega^n RG, M),$$

where M is a finitely generated RG-module in the former case and a finitely generated RG-bimodule in the latter case.

V.1.1 Agreement of products

Cup products (outer products, constructed via the tensor product of complexes, as seen in Section III.3 ([31, §6])) agree with the Yoneda products (compositions of maps, [17, §2]) in the following sense. Let M and N be RG-bimodules. For $[f] \in \widehat{\mathrm{HH}}^i(RG, M)$ and $[g] \in \widehat{\mathrm{HH}}^j(RG, N)$, we can write $[f]\widehat{\otimes}[g]$ as the composition of $([f]\widehat{\otimes}1_N) \in \widehat{\mathrm{HH}}^i(RG \otimes_{RG} N, M \otimes_{RG} N)$ and $(1_{RG}\widehat{\otimes}[g]) \in \widehat{\mathrm{HH}}^j(RG \otimes_{RG} RG, RG \otimes_{RG} N)$. In this way, the cup product agrees with the Yoneda product: The case when $i, j \geq 0$ is well-known and, for example, can be proved using a similar argument as in [3, Prop. 3.2.1]. For arbitrary integers i and j, we can take $f' \in \underline{\mathrm{Hom}}_{RG^e}(\Omega^{i+a} RG, \Omega^a M)$ and $g' \in \underline{\mathrm{Hom}}_{RG^e}(\Omega^{j+b} RG, \Omega^b N)$, such that $a, b, i+a, j+b \geq 0$, and apply a similar argument for $\Omega^b f'$ and $\Omega^a g'$. The desired agreement follows, using the following isomorphisms in the stable module category of RG (which can be generalized to those of RG^e):

$$\Omega^n(\Omega^m N) \xrightarrow{\sim} \Omega^{n+m} N \oplus (\text{projective}),$$

$$\Omega^n(N) \otimes \Omega^m M \xrightarrow{\sim} \Omega^{n+m}(N \otimes M) \oplus (\text{projective}),$$

$$\underline{\mathrm{Hom}}_{RG}(N, M) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{RG}(\Omega^n N, \Omega^n M).$$

To the author's knowledge, this agreement of products has not been done explicitly for the Tate-Hochschild cohomology.

V.1.2 Reduction to Tate cohomology and relations with subgroups

By Theorem IV.2 ([31, Theorem 7.2]), we have an algebra isomorphism:

$$\widehat{\mathrm{HH}}^*(RG, RG) \cong \widehat{\mathrm{H}}^*(G, RG), \tag{V.1}$$

where RG is considered as a module over itself via conjugation. Therefore, all properties of $\widehat{\mathrm{H}}^*(G, RG)$, as seen in Section III.1.2, transfer to those for the Tate-Hochschild cohomology.

Let H be a subgroup of G. By restricting the action, any RG-module N may be regarded as an RH-module and any RG-complete resolution $\mathbb{P}$ of R may also be considered as an RH-complete resolution of R. Sections XII.8 and XII.9 in [14] show that there are maps in the Tate cohomology with properties analogous to those in the usual group cohomology:

- The restriction map:

  $$\mathrm{res}^G_H : \widehat{\mathrm{H}}^*(G, N) \to \widehat{\mathrm{H}}^*(H, N),$$

  which is induced from the inclusion $\mathrm{Hom}_{RG}(\mathbb{P}, N) \subset \mathrm{Hom}_{RH}(\mathbb{P}, N)$.
- The corestriction map (or transfer):

  $$\mathrm{cor}^G_H : \widehat{\mathrm{H}}^*(H, N) \to \widehat{\mathrm{H}}^*(G, N),$$

which is given on the cochain level by defining:

$$(\operatorname{cor}_H^G f)(p) = \sum_{g \in \mathcal{G}} g f(g^{-1}p),$$

where $\mathcal{G}$ denotes a set of left coset representatives of H in G, $f \in \operatorname{Hom}_{RH}(P_i, N)$, and $p \in P_i$. One can check that this definition is independent of the choice of the representatives $g \in \mathcal{G}$.

- Moreover, for any $g \in G$, there is an isomorphism:

$$g^* : \widehat{\mathrm{H}}^*(H, N) \to \widehat{\mathrm{H}}^*(gHg^{-1}, N) = \widehat{\mathrm{H}}^*({}^gH, N)$$

defined on the cochain level as $(g^* f)(p) = g(f(g^{-1}p))$.

These maps and their algebraic relations will be the keys to our main results. We shall recall some properties of these maps without proving them. The proof goes through for our RG using similar arguments as in [14]. The readers can refer to [14] for more details.

Proposition V.1. *[14, §XII.8 (4)-(14) and §XII.9 (4)] Let $K \subseteq H \subseteq G$ be subgroups, and N_1, N_2 be RG-modules which may be regarded as RH-modules. Let $\alpha_i \in \widehat{\mathrm{H}}^*(G, N_i)$, $\beta_i \in \widehat{\mathrm{H}}^*(H, N_i)$, and $g_i \in G$, for $i = 1, 2$. Then the maps defined above satisfy:*

1. $g_1^* g_2^* = (g_1 g_2)^*$
2. *If $g \in H$, then $g^* = \mathbf{1}$, where $\mathbf{1}$ is the identity map on Tate cohomology*
3. $\operatorname{cor}_H^G \circ \operatorname{res}_H^G = (G : H)\mathbf{1}$
4. $\operatorname{res}_K^H \circ \operatorname{res}_H^G = \operatorname{res}_K^G$
5. $\operatorname{cor}_H^G \circ \operatorname{cor}_K^H = \operatorname{cor}_K^G$
6. $g^* \circ \operatorname{res}_K^H = \operatorname{res}_{{}^gK}^{{}^gH} \circ g^*$
7. $g^* \circ \operatorname{cor}_K^H = \operatorname{cor}_{{}^gK}^{{}^gH} \circ g^*$
8. $\operatorname{res}_H^G(\alpha_1 \smile \alpha_2) = (\operatorname{res}_H^G \alpha_1) \smile (\operatorname{res}_H^G \alpha_2)$
9. $\operatorname{cor}_H^G(\beta_1 \smile \operatorname{res}_H^G \alpha_2) = (\operatorname{cor}_H^G \beta_1) \smile \alpha_2$
10. $\operatorname{cor}_H^G(\operatorname{res}_H^G \alpha_1 \smile \beta_2) = \alpha_1 \smile (\operatorname{cor}_H^G \beta_2)$
11. $g^*(\beta_1 \smile \beta_2) = (g^*\beta_1) \smile (g^*\beta_2)$
12. *Let $H, K \subseteq G$ be subgroups and N be an RG-module which may be regarded as an RH (or*

RK)-module. The map $\operatorname{res}_K^G \circ \operatorname{cor}_H^G : \widehat{\mathrm{H}}^*(H,N) \to \widehat{\mathrm{H}}^*(K,N)$ is given by:

$$\operatorname{res}_K^G(\operatorname{cor}_H^G(\beta)) = \sum_{x \in D} \operatorname{cor}_{K\cap {}^xH}^K(\operatorname{res}_{K\cap {}^xH}^{{}^xH}(x^*\beta)),$$

where $\beta \in \widehat{\mathrm{H}}^*(H,N)$ and D is a set of double coset representatives such that $G = \bigcup_{x\in D} KxH$ is a disjoint union.

V.2 Generalized additive decomposition

For the rest of this chapter, we work on a more general setting by letting H be another finite group which acts as automorphisms on G. Via this action, RG becomes an RH-module. The multiplication map $RG \otimes RG \to RG$ is an RH-module homomorphism. Hence, it induces the ring structure on cohomology $\widehat{\mathrm{H}}^*(H,RG) := \widehat{\operatorname{Ext}}^*_{RH}(R,RG)$ by composing with the cup product. We will study the additive decomposition of this ring $\widehat{\mathrm{H}}^*(H,RG)$. The Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^*(RG,RG) \cong \widehat{\mathrm{H}}^*(G,RG)$ is a special case of $\widehat{\mathrm{H}}^*(H,RG)$ by letting $H = G$ act on itself by conjugation.

V.2.1 Decomposition of the Tate-Hochschild cohomology

We begin by generalizing Holm's [24] and Cibils and Solotar's result [15] to its Tate version.

Proposition V.2. *If H acts trivially on G, then $\widehat{\mathrm{H}}^*(H,RG) \cong RG \otimes_R \widehat{\mathrm{H}}^*(H,R)$ as graded R-algebras. In particular, if G is abelian, then*

$$\widehat{\mathrm{HH}}^*(RG,RG) \cong RG \otimes_R \widehat{\mathrm{H}}^*(G,R).$$

Proof. If G is abelian, $H = G$ acting on itself by conjugation yields the trivial action. Hence, the second statement follows from the first statement and the isomorphism (V.1):

$$\widehat{\mathrm{HH}}^*(RG,RG) \cong \widehat{\mathrm{H}}^*(G,RG) \cong RG \otimes_R \widehat{\mathrm{H}}^*(G,R).$$

To prove the first statement, let $\varepsilon : \mathbb{P} \to R$ be an RH-complete resolution of R. Since H acts trivially on G, RG is a trivial RH-module and is free as an R-module. We define the map $\gamma : RG \otimes \mathrm{Hom}_{RH}(\mathbb{P}, R) \to \mathrm{Hom}_{RH}(\mathbb{P}, RG)$ by sending $g \otimes f \mapsto \gamma(g \otimes f) = F$, where $F(p) = f(p)g$, for $f \in \mathrm{Hom}_{RH}(P_i, R), p \in P_i$, and $g \in G$. It can be checked that $F \in \mathrm{Hom}_{RH}(\mathbb{P}, RG)$ and γ is an isomorphism. Moreover, F is a cocycle when f is. Hence, passing to the homology, γ induces an isomorphism of graded R-modules:

$$\gamma_* : RG \otimes_R \widehat{\mathrm{H}}^*(H, R) \to \widehat{\mathrm{H}}^*(H, RG).$$

The definition of cup product corresponds to this map, making γ_* a ring homomorphism. □

Remark V.3. The proposition's statement for an abelian group G was observed in the proof of [25, Prop. 5.2] without an explicit verification. This proposition helps us to study the structure of the Tate-Hochschild cohomology ring of a finite abelian group algebra, given its Tate cohomology ring. For example, knowing the Tate cohomology of a cyclic group G, see [14, §XII.7], one can easily compute its Tate-Hochschild cohomology by applying Proposition V.2.

Now we return to the general case where H acts on G non-trivially and G is not necessarily abelian. Let $g_1 = 1, g_2, \ldots, g_t \in G$ be representatives of the orbits of the action of H on G. Let $H_i := \mathrm{Stab}_H(g_i) = \{h \in H \mid {}^h g_i = g_i\}$ be the stabilizer of g_i. For any $g \in G$, there are two $R(\mathrm{Stab}_H(g))$-module homomorphisms:

$$\theta_g : R \to RG \text{ via } r \mapsto rg,$$

$$\pi_g : RG \to R \text{ via } \sum_{a \in G} r_a a \mapsto r_g.$$

If V is any subgroup of $\mathrm{Stab}_H(g)$, then these maps induce maps on cohomology:

$$\theta_g^* : \widehat{\mathrm{H}}^*(V, R) \to \widehat{\mathrm{H}}^*(V, RG),$$

$$\pi_g^* : \widehat{\mathrm{H}}^*(V, RG) \to \widehat{\mathrm{H}}^*(V, R),$$

since $\widehat{\mathrm{Ext}}$ is covariant in the second argument. The following properties of θ_g^* and π_g^* will help us in proving the main result.

Lemma V.4. *Let $h \in H$ and $a, b \in G$.*

(a) If V is a subgroup of $Stab_H(a)$, then $h^* \circ \theta_a^* = \theta_{{}^h a}^* \circ h^*$ as maps from $\widehat{\mathrm{H}}^*(V)$ to $\widehat{\mathrm{H}}^*({}^hV, RG)$.

(b) Suppose $V \subseteq Stab_H(a) \cap Stab_H(b)$ and $\alpha, \beta \in \widehat{\mathrm{H}}^*(V)$. Then:

$$\theta_a^*(\alpha) \smile \theta_b^*(\beta) = \theta_{ab}^*(\alpha \smile \beta).$$

(c) Suppose $V' \subseteq V \subseteq Stab_H(a)$. Then θ_a^* and π_a^* commute with $\mathrm{res}_{V'}^V$ and $\mathrm{cor}_{V'}^V$.

(d) If $V \subseteq Stab_H(a) \cap Stab_H(b)$, then $\pi_a^* \circ \theta_b^* = \delta_{a,b}\mathbf{1}$, where $\mathbf{1}$ is the identity map on $\widehat{\mathrm{H}}^*(V)$ and $\delta_{a,b}$ is the Kronecker delta.

Proof. Lemma 5.2 in [35] showed these in positive degrees. We extend the proof to negative degrees and present it on the cochain level. The desired results are induced on cohomology.

(a) Let $\mathbb{P}$ be an RV-complete resolution of R, $f \in \mathrm{Hom}_V(P_i, R)$ be a cocycle representing an element of $\widehat{\mathrm{H}}^i(V)$, and $p \in P_i$. Then

$$h^*(\theta_a f)(p) = f(h^{-1}p)({}^h a) = \theta_{{}^h a}(h^*(f))(p).$$

(b) Let $m : RG \otimes RG \to RG$ be the multiplication map and $\Gamma : \mathbb{P} \to \mathbb{P}\widehat{\otimes}\mathbb{P}$ be a complete diagonal approximation map. Let $f, q \in \mathrm{Hom}_V(\mathbb{P}, R)$ represent $\alpha, \beta \in \widehat{\mathrm{H}}^*(V)$, respectively. Then on the cochain level:

$$m \circ ((\theta_a \circ f)\widehat{\otimes}(\theta_b \circ q)) \circ \Gamma = m \circ (\theta_a \otimes \theta_b) \circ (f\widehat{\otimes}q) \circ \Gamma = \theta_{ab} \circ (f\widehat{\otimes}q) \circ \Gamma,$$

where the left side represents $\theta_a^*(\alpha) \smile \theta_b^*(\beta)$ and the right side represents $\theta_{ab}^*(\alpha \smile \beta)$.

(c) Let $\mathbb{P}$ be an RV-complete resolution of R which can also be regarded as an RV'-complete resolution by restricting the action. Let $f \in \mathrm{Hom}_{V'}(P_i, RG)$ represent an element of $\widehat{\mathrm{H}}^i(V', RG)$, $q \in \mathrm{Hom}_{V'}(P_i, R)$ represent an element of $\widehat{\mathrm{H}}^i(V')$, and $p \in P_i$.

$$\begin{aligned}(\pi_a^* \mathrm{cor}_{V'}^V)(f)(p) = \pi_a\left(\sum_{v \in V/V'} ({}^v f(v^{-1}p))\right) &= \sum_v \pi_{v^{-1}a}(f(v^{-1}p)) \\ &= \sum_v (\pi_a \circ f)(v^{-1}p) = (\mathrm{cor}_{V'}^V \pi_a^*)(f)(p),\end{aligned}$$

since $v \in V \subseteq \mathrm{Stab}_H(a)$, we have ${}^{v^{-1}}a = a$, and V acts trivially on R. Similarly,

$$(\theta_a^* \operatorname{cor}_{V'}^V)(q)(p) = \theta_a\left(\sum_{v \in V/V'} q(v^{-1}p)\right) = \sum_v \theta_{v^{-1}a}(q(v^{-1}p))$$
$$= \sum_v (\theta_a \circ q)(v^{-1}p) = (\operatorname{cor}_{V'}^V \theta_a^*)(q)(p).$$

The other cases follow similarly by commutativity between π_a, θ_a and the inclusion map $\iota : \mathrm{Hom}_V(\mathbb{P}, N) \hookrightarrow \mathrm{Hom}_{V'}(\mathbb{P}, N)$, where $N = RG$ or R.

(d) Let $r \in R$.

$$\pi_a(\theta_b(r)) = \pi_a(rb) = \begin{cases} r, & \text{if } a = b \\ 0, & \text{else.} \end{cases}$$

$\square$

For $i = 1, 2, \ldots, t$, let $\psi_i : \widehat{\mathrm{H}}^*(H_i, R) \to \widehat{\mathrm{H}}^*(H, RG)$ be defined as $\psi_i = \operatorname{cor}_{H_i}^H \circ \theta_{g_i}^*$. We describe the additive decomposition of $\widehat{\mathrm{H}}^*(H, RG)$, generalizing from the usual cohomology [4, Theorem 2.11.2]:

Lemma V.5. *The map* $\widehat{\mathrm{H}}^*(H, RG) \to \bigoplus_{i=1}^t \widehat{\mathrm{H}}^*(H_i, R)$*, sending* $\zeta \mapsto (\pi_{g_i}^* \circ \operatorname{res}_{H_i}^H(\zeta))_i$*, is an isomorphism of graded* R*-modules. Its inverse sends* $\alpha \in \widehat{\mathrm{H}}^*(H_i, R)$ *to* $\psi_i(\alpha) \in \widehat{\mathrm{H}}^*(H, RG)$*.*

Proof. For $i = 1, 2, \ldots, t$, let M_i be the free R-module generated by elements of the orbit containing g_i. Then we have $RG = \bigoplus_{i=1}^t M_i$. Let $R\uparrow_{H_i}^H := RH \otimes_{RH_i} R$. There is an isomorphism $M_i \to R\uparrow_{H_i}^H$ given by $r({}^h g_i) \mapsto h \otimes r$. It induces an isomorphism in cohomology $\widehat{\mathrm{H}}^*(H, M_i) \cong \widehat{\mathrm{H}}^*(H, R\uparrow_{H_i}^H)$.

Since $\widehat{\mathrm{Ext}}$ is additive, $\widehat{\mathrm{H}}^*(H, RG) \cong \bigoplus_i \widehat{\mathrm{H}}^*(H, M_i)$. Apply the generalized Eckmann-Shapiro Lemma IV.1 ([31, Lemma 7.1]), we have $\widehat{\mathrm{H}}^*(H, RG) \cong \bigoplus_i \widehat{\mathrm{H}}^*(H_i, R)$. One can also check directly that the maps given in the statement of the lemma are inverses of each other by applying Proposition V.1 and Lemma V.4 to show that their compositions are the identity maps. $\square$

Remark V.6. If $H = G$ acts on itself by conjugation, then M_i is the free R-module generated by the conjugacy class of g_i. M_i is isomorphic to $R\uparrow_{C_G(g_i)}^G$, where $C_G(g_i)$ is the centralizer of g_i. Therefore, the isomorphism in Lemma V.5 gives an additive decomposition of the Tate-Hochschild cohomology of G as a direct sum of the Tate cohomology of the centralizers of conjugacy class representatives of G with coefficients in R:

$$\widehat{\mathrm{HH}}^*(RG, RG) \cong \bigoplus_i \widehat{\mathrm{H}}^*(C_G(g_i), R).$$

In 1999, Siegel and Witherspoon showed that there is a product formula for the usual Hochschild cohomology of G in terms of a similar additive decomposition [35, Theorem 5.1]. We will describe products in $\widehat{\mathrm{H}}^*(H, RG)$ with respect to the isomorphism in Lemma V.5. Motivated by the methods in [35], our work is a straightforward generalization of the usual group cohomology results.

Fix $i, j \in \{1, 2, \ldots, t\}$. Let D be a set of double coset representatives for $H_i \backslash H / H_j$. For each $x \in D$, there is a unique $k = k(x)$ such that

$$g_k = {}^y g_i \, {}^{yx} g_j \tag{V.2}$$

for some $y \in H$. One can rewrite the action on the right hand side and get $g_k = {}^y(g_i \, {}^x g_j)$ showing that k is independent of the choice of double coset representative x. Moreover, the set of all y satisfying (V.2) is also a double coset. To see this, let us fix $y = y(x)$ for which (V.2) holds. Let $y' \in H$ be another element such that $g_k = {}^{y'} g_i \, {}^{y'x} g_j$. Then ${}^{y'} g_i \, {}^{y'x} g_j = g_k = {}^y g_i \, {}^{yx} g_j$ implies ${}^{y'}(g_i \, {}^x g_j) = {}^y(g_i \, {}^x g_j)$. Let $h = y' y^{-1}$. We have ${}^h g_k = {}^h({}^y(g_i \, {}^x g_j)) = {}^{y'}(g_i \, {}^x g_j) = g_k$ showing $h \in H_k = \mathrm{Stab}_H(g_k)$. On the other hand, if $h \in H_k$, then let $y' = hy \in H$, we have ${}^{hy} g_i \, {}^{hyx} g_j = {}^h({}^y g_i \, {}^{yx} g_j) = {}^h g_k = g_k$. Putting together, we have shown:

$$\{y' \in H | g_k = {}^{y'} g_i \, {}^{y'x} g_j\} = H_k y = H_k y({}^x H_j \cap H_i) \in H_k \backslash H / ({}^x H_j \cap H_i),$$

where the last equality follows from (V.2) and $({}^{yx} H_j \cap {}^y H_i) \subseteq H_k$. We can now prove our main result which provides a formula for products in $\widehat{\mathrm{H}}^*(H, RG)$ with respect to Lemma V.5.

Theorem V.7. *Let $\alpha \in \widehat{\mathrm{H}}^*(H_i)$ and $\beta \in \widehat{\mathrm{H}}^*(H_j)$. Then*

$$\psi_i(\alpha) \smile \psi_j(\beta) = \sum_{x \in D} \psi_k(\mathrm{cor}_V^{H_k}(\mathrm{res}_V^{{}^y H_i} \, y^* \alpha \smile \mathrm{res}_V^{{}^{yx} H_j} (yx)^* \beta)),$$

where D is a set of double coset representatives for $H_i \backslash H / H_j$, $k = k(x)$ and $y = y(x)$ are chosen to satisfy (V.2), and $V = V(x) = {}^{yx} H_j \cap {}^y H_i \subseteq H_k$.

Proof. By Lemma V.5,

$$\begin{aligned}
\psi_i(\alpha) \smile \psi_j(\beta) &= \operatorname{cor}^H_{H_i}(\theta^*_{g_i}\alpha) \smile \operatorname{cor}^H_{H_j}(\theta^*_{g_j}\beta), \text{ by definition of } \psi_i, \psi_j \\
&= \operatorname{cor}^H_{H_i}(\theta^*_{g_i}\alpha \smile \operatorname{res}^H_{H_i} \operatorname{cor}^H_{H_j} \theta^*_{g_j}\beta), \text{ by Prop. V.1 (9)} \\
&= \sum_{x\in D} \operatorname{cor}^H_{H_i}(\theta^*_{g_i}\alpha \smile \operatorname{cor}^{H_i}_{{}^xH_j\cap H_i} \operatorname{res}^{{}^xH_j}_{{}^xH_j\cap H_i} x^*\theta^*_{g_j}\beta), \text{ by Prop. V.1 (12)} \\
&= \sum_{x\in D} \operatorname{cor}^H_{H_i}(\operatorname{cor}^{H_i}_{{}^xH_j\cap H_i}(\operatorname{res}^{H_i}_{{}^xH_j\cap H_i} \theta^*_{g_i}\alpha \smile \operatorname{res}^{{}^xH_j}_{{}^xH_j\cap H_i} x^*\theta^*_{g_j}\beta)), \text{ by Prop. V.1 (10)} \\
&= \sum_{x\in D} \operatorname{cor}^H_{{}^xH_j\cap H_i}(\operatorname{res}^{H_i}_{{}^xH_j\cap H_i} \theta^*_{g_i}\alpha \smile \operatorname{res}^{{}^xH_j}_{{}^xH_j\cap H_i} x^*\theta^*_{g_j}\beta), \text{ by Prop. V.1 (5)} \\
&= \sum_{x\in D} \operatorname{cor}^H_{{}^xH_j\cap H_i} \theta^*_{g_i\,{}^xg_j}(\operatorname{res}^{H_i}_{{}^xH_j\cap H_i} \alpha \smile \operatorname{res}^{{}^xH_j}_{{}^xH_j\cap H_i} x^*\beta), \text{ by Lemma V.4 (a)-(c)} \\
&= \sum_k \sum_x \psi_k \pi^*_{g_k} \operatorname{res}^H_{H_k}(\operatorname{cor}^H_{{}^xH_j\cap H_i} \theta^*_{g_i\,{}^xg_j}(\operatorname{res}^{H_i}_{{}^xH_j\cap H_i} \alpha \smile \operatorname{res}^{{}^xH_j}_{{}^xH_j\cap H_i} x^*\beta)),
\end{aligned}$$

by the isomorphism in Lemma V.5

$$= \sum_k \sum_{x,y} \psi_k \pi^*_{g_k} \operatorname{cor}^{H_k}_{V'} \operatorname{res}^{{}^{yx}H_j\cap {}^yH_i}_{V'} y^*\theta^*_{g_i\,{}^xg_j}(\operatorname{res}^{H_i}_{{}^xH_j\cap H_i} \alpha \smile \operatorname{res}^{{}^xH_j}_{{}^xH_j\cap H_i} x^*\beta),$$

by Prop. V.1 (12), where y runs over a set of representatives for $H_k\backslash H/{}^xH_j \cap H_i$

and $V' = H_k \cap {}^{yx}H_j \cap {}^yH_i$,

$$= \sum_k \sum_{x,y} \psi_k \operatorname{cor}^{H_k}_{V'} \pi^*_{g_k} \theta^*_{{}^yg_i\,{}^{yx}g_j} \operatorname{res}^{{}^{yx}H_j\cap {}^yH_i}_{V'} y^*(\operatorname{res}^{H_i}_{{}^xH_j\cap H_i} \alpha \smile \operatorname{res}^{{}^xH_j}_{{}^xH_j\cap H_i} x^*\beta),$$

by Lemma V.4 (a),(c)

$$= \sum_{x\in D} \psi_k \operatorname{cor}^{H_k}_{V'}(\operatorname{res}^{{}^yH_i}_{V'} y^*\alpha \smile \operatorname{res}^{{}^{yx}H_j}_{V'} (yx)^*\beta),$$

by Prop. V.1 (1), (4), (6) and Lemma V.4 (d).

By Lemma V.4 (d), the only terms that can be non-zero in the next to last step are those for which $g_k = {}^yg_i\,{}^{yx}g_j$. We have seen in the discussion prior to this theorem that each x determines a unique k and double coset $H_k y({}^xH_j \cap H_i)$ for which this holds. Therefore, we may take $y = y(x)$ and ${}^{yx}H_j \cap {}^yH_i \subseteq H_k$. Hence, $V' = V = {}^{yx}H_j \cap {}^yH_i$. □

Remark V.8. Since the cup product is well-defined and unique [14, Theorem XII.5.1], the sum in the statement of the theorem is independent of the choice of x and y. One can see this directly by replacing y with hy, for some $h \in H_k$. By Proposition V.1 (6), (7), and (11), h^* respects the cup product and commutes with the restriction and corestriction maps. Moreover, since H_k acts trivially on its own cohomology, any term of the sum in the theorem is unchanged by replacing y with hy. If x is multiplied on the right by an element of H_j, the terms are unchanged for similar reasons. If x is replaced by hx, for some $h \in H_i$, then we must replace y with yh^{-1} so that (V.2) holds:

$$^{(yh^{-1})}g_i \, ^{(yh^{-1})(hx)}g_j = {}^{y}g_i \, ^{yx}g_j = g_k,$$

and the terms remain unchanged.

We observe that when $i = 1$, $\psi_1 : \widehat{\mathrm{H}}^*(H, R) \to \widehat{\mathrm{H}}^*(H, RG)$ is an algebra monomorphism that is induced by the algebra homomorphism $R \to RG$ mapping $r \mapsto r1$. Alternatively, by letting $i = j = 1$ in Theorem V.7, we see that ψ_1 respects the cup product:

$$\psi_1(\alpha) \smile \psi_1(\beta) = \psi_1(\alpha \smile \beta),$$

where $\alpha, \beta \in \widehat{\mathrm{H}}^*(H)$. Hence, via ψ_1, we may view $\widehat{\mathrm{H}}^*(H, RG)$ as a (left) $\widehat{\mathrm{H}}^*(H)$-module with action via multiplying (on the left) by $\psi_1(\widehat{\mathrm{H}}^*(H))$. Each $\widehat{\mathrm{H}}^*(H_i)$ may also be regarded as an $\widehat{\mathrm{H}}^*(H)$-module via restriction. As a consequence, we obtain:

Corollary V.9. *The isomorphism in Lemma V.5 is an isomorphism of graded* $\widehat{\mathrm{H}}^*(H)$*-modules:*

$$\widehat{\mathrm{H}}^*(H, RG) \cong \bigoplus_{i=1}^{t} \widehat{\mathrm{H}}^*(H_i).$$

Proof. For $i = 1$, let $\alpha \in \widehat{\mathrm{H}}^*(H)$ and $\beta \in \widehat{\mathrm{H}}^*(H_j)$. Theorem V.7 reduces to:

$$\psi_1(\alpha) \smile \psi_j(\beta) = \psi_j(\mathrm{res}^H_{H_j}(\alpha) \smile \beta).$$

where the left hand side is considered as action of $\widehat{\mathrm{H}}^*(H)$ on $\widehat{\mathrm{H}}^*(H, RG)$ that corresponds to the action of $\widehat{\mathrm{H}}^*(H)$ on each $\widehat{\mathrm{H}}^*(H_j)$ on the right hand side, via the isomorphism in Lemma V.5. □

As noted in the remark following Lemma V.5, when $H = G$ acts on itself by conjugation, Theorem V.7 gives a formula for the multiplicative structure of $\widehat{\mathrm{HH}}^*(RG, RG) \cong \bigoplus_i \widehat{\mathrm{H}}^*(C_G(g_i), R)$ in terms of this decomposition. It reduces the computation of products in $\widehat{\mathrm{HH}}^*(RG, RG)$ to products within the Tate cohomology rings of certain subgroups of G.

V.2.2 Products in negative Tate-Hochschild cohomology

The (Tate) cohomology and (Tate) Hochschild cohomology rings of a finite group algebra $\mathbf{k}G$, over a field $\mathbf{k}$ of positive characteristic dividing the order of G, are graded-commutative. In fact, this is true for general finite dimensional Hopf algebras over $\mathbf{k}$, e.g. Section III.3 ([31, §6]). Hence, the usual concepts from commutative algebra apply. For example, one can talk about Krull dimension, depth, Gorenstein, and Cohen-Macaulay conditions. In this context, since $\mathrm{H}^*(G, \mathbf{k})$ is a finitely generated [5, Theorem 4.1.1] graded-commutative $\mathbf{k}$-algebra, we say $\mathrm{H}^*(G, \mathbf{k})$ is **Cohen-Macaulay** if there exist homogeneous elements of positive degree $x_1, \ldots, x_r$ forming a regular sequence, and $\mathrm{H}^*(G, \mathbf{k})/(x_1, \ldots, x_r)$ has finite rank as a $\mathbf{k}$-vector space [5, Prop. 2.5.1]. There are classes of groups for which $\mathrm{H}^*(G, \mathbf{k})$ is known to be Cohen-Macaulay [5, §6.1].

In 1992, D. J. Benson and J. F. Carlson [6] investigated the product structure of the Tate cohomology $\widehat{\mathrm{H}}^*(G, \mathbf{k})$. They showed that very often all products between elements of negative degrees vanish. In particular, this happens when the depth of the usual cohomology ring $\mathrm{H}^*(G, \mathbf{k})$ is greater than one [6, Theorem 3.1]. The existence of non-zero products in negative cohomology is also equivalent to the existence of non-zero products in mixed positive-negative degrees [6, Lemma 2.1]. We will analyze how cup products behave in the Tate-Hochschild cohomology, taking advantage of the product formula in Theorem V.7.

Assume the same setting as in Theorem V.7, with a finite group H acting non-trivially on G and H_i as before. To employ the results in [6], for the rest of this section, we will work over a field $\mathbf{k}$ of characteristic $p > 0$, where p divides the order of H. Let $\alpha \in \widehat{\mathrm{H}}^*(H_i) := \widehat{\mathrm{H}}^*(H_i, \mathbf{k})$ and $\beta \in \widehat{\mathrm{H}}^*(H_j)$. Observe that when $H = G$ acts on itself by conjugation, from the product formula in Theorem V.7, multiplying two elements of nonnegative degrees is the same as before for the usual Hochschild cohomology. We are interested in the products of negative degree elements, or products of a negative degree element and a positive degree element.

Case 1: Assume α and β are both of negative degrees.

Proposition V.10. *Assume the same setting as in Theorem V.7. Let* $\alpha \in \widehat{\mathrm{H}}^*(H_i)$ *and* $\beta \in \widehat{\mathrm{H}}^*(H_j)$ *both be of negative degrees. If for all* $x \in D$, $V = V(x) = {}^{yx}H_j \cap {}^{y}H_i$ *has p-rank at least* 2 *and* $\mathrm{H}^*(V, \mathbf{k})$ *is Cohen-Macaulay, then* $\psi_i(\alpha) \smile \psi_j(\beta) = 0$.

Proof. From Theorem V.7, we have:

$$\psi_i(\alpha) \smile \psi_j(\beta) = \sum_{x \in D} \psi_k(\mathrm{cor}_V^{H_k}(\mathrm{res}_V^{{}^yH_i} y^*\alpha \smile \mathrm{res}_V^{{}^{yx}H_j}(yx)^*\beta)).$$

It can be checked (on the cochain levels) that the maps $\mathrm{res}, \mathrm{cor}, y^*, (yx)^*$ preserve degrees. Since α and β are both of negative degrees, $\mathrm{res}_V^{{}^yH_i} y^*\alpha \smile \mathrm{res}_V^{{}^{yx}H_j}(yx)^*\beta$ is a product between two negative degree elements in V. It follows from the hypothesis and [6, Theorem 3.1] that this product is 0. Hence, $\psi_k(\mathrm{cor}_V^{H_k}(0)) = 0$. This holds for all $x \in D$, so the sum is 0, proving the statement. □

We note that if V has p-rank at least 2, then H also has p-rank at least 2, since $V \subseteq H$. Furthermore, if $i = j$ and assume that H_i has p-rank at least 2 and $\mathrm{H}^*(H_i, \mathbf{k})$ is Cohen-Macaulay, then $\alpha \smile \beta = 0$. As a consequence, their product in $\widehat{\mathrm{H}}^*(H, \mathbf{k}G)$ is:

$$\psi_i(\alpha) \smile \psi_i(\beta) = \sum_{x \in H \setminus H_i} \psi_k(\mathrm{cor}_{H_i}^{H_k}(\alpha \smile \beta)) = 0,$$

where $k = k(x)$ such that $g_k = g_i\,{}^xg_i$. Hence, by letting $H = G$ act on itself by conjugation and observing whether each $C_G(g_i)$ satisfies the above hypothesis, one may conclude that some products in the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G)$ will always be 0. Knowing this will speed up the computation. For the remaining products that could be nonzero, we have the product formula which generalizes what was known in nonnegative degrees.

Case 2: Assume α is in negative degree and β is in positive degree.

Let $\alpha \in \widehat{\mathrm{H}}^m(H_i)$ and $\beta \in \widehat{\mathrm{H}}^n(H_j)$, where $m < 0 < n$. Suppose $n + m \geq 0$. Let $V = V(x) = {}^{yx}H_j \cap {}^{y}H_i$. Then

$$\mathrm{res}_V^{{}^yH_i} y^*\alpha \smile \mathrm{res}_V^{{}^{yx}H_j}(yx)^*\beta \neq 0$$

in $\widehat{\mathrm{H}}^*(V)$ if and only if there exists a pair of negative integers $s, t < 0$ such that $\widehat{\mathrm{H}}^s(V) \smile \widehat{\mathrm{H}}^t(V) \neq 0$, by [6, Lemma 2.1]. It follows from [6, Theorem 3.3] that if there exists such a pair α and β, then $\mathrm{H}^*(V,\mathbf{k})$ has depth one, and the center of any Sylow p-subgroup of V has p-rank one. Similarly, for $i=j$, the same assertion holds for $\mathrm{H}^*(H_i,\mathbf{k})$ and the center of any Sylow p-subgroup of H_i.

Recently, Linckelmann studied the Tate duality and transfer maps in the Hochschild cohomology of symmetric algebras [26]. For the Tate and Tate-Hochschild cohomology rings (assuming they are graded-commutative) of such algebras, he also observed this behavior of the products in negative cohomology (detailed will the discussed in the next chapter). In particular, for a finite group G, as $\mathbf{k}G$ is symmetric and its Tate-Hochschild cohomology ring is graded-commutative, we obtain the following result from [26, §8]:

Proposition V.11. *Suppose there are negative integers $s,t<0$ such that $\widehat{\mathrm{HH}}^s(\mathbf{k}G)\smile\widehat{\mathrm{HH}}^t(\mathbf{k}G)\neq 0$, then the usual Hochschild cohomology $\mathrm{HH}^*(\mathbf{k}G,\mathbf{k}G)$ has depth at most one.*

Hence, without computing $\mathrm{HH}^*(\mathbf{k}G,\mathbf{k}G)$, we can find certain information about its depth. Using the product formula to compute the products in $\widehat{\mathrm{HH}}^*(\mathbf{k}G,\mathbf{k}G)$, if we know there is a non-zero product in negative degrees, then by Proposition V.11, we can conclude the depth of $\mathrm{HH}^*(\mathbf{k}G,\mathbf{k}G)$ is at most one.

V.3 Examples

In this section, we study the Tate-Hochschild cohomology of two non-abelian groups. In one example, by observing the p-rank of the Sylow p-subgroups, we can take advantage of the results in Section V.2.2 to simplify the calculations. In the other example, as the p-rank is at most one, we instead directly utilize the formula in Theorem V.7 to compute the products.

V.3.1 The dihedral group of order 8

Let $\mathbf{k}$ be a field of characteristic 2. Let $G=D_8=\langle a,b\mid a^4=1=b^2, aba=b\rangle$ denote the dihedral group of order 8. D_8 is defined as the group of all symmetries of the square, where a is a rotation and b is a reflection. Treating $\{1,2,3,4\}$ as the vertices of the square, this group can be regarded as the subgroup of the symmetric group S_4 (up to isomorphism) via setting $a=(1\ 2\ 3\ 4)$ and

$b = (1\ 3)$. Let G act on itself by conjugation. There are five conjugacy classes in G:

$$\begin{aligned}
\{1\} = \{e\} &\longleftrightarrow \text{ identity} \\
\{a^2\} = \{(1\ 3)(2\ 4)\} &\longleftrightarrow 180 \text{ degree rotation} \\
\{b, a^2b\} = \{(1\ 3), (2\ 4)\} &\longleftrightarrow \text{ vertex reflections} \\
\{ab, a^3b\} = \{(1\ 4)(2\ 3), (1\ 2)(3\ 4)\} &\longleftrightarrow \text{ edge reflections} \\
\{a, a^3\} = \{(1\ 2\ 3\ 4), (4\ 3\ 2\ 1)\} &\longleftrightarrow \text{ 90 degree rotations},
\end{aligned}$$

and the corresponding centralizers of conjugacy representatives:

$$\begin{aligned}
&H_1 = C_G(1) = G \\
&H_2 = C_G((1\ 3)(2\ 4)) = G \\
&H_3 = C_G((1\ 3)) = \langle (1\ 3), (2\ 4)\rangle \cong V_4, \text{ Klein-four group} \\
&H_4 = C_G((1\ 2)(3\ 4)) = \langle (1\ 2), (3\ 4)\rangle \cong V_4 \\
&H_5 = C_G((1\ 2\ 3\ 4)) = \langle (1\ 2\ 3\ 4)\rangle \cong \mathbb{Z}_4,
\end{aligned}$$

with H_3, H_4, H_5 are all of order 4 and normal subgroups of G.

For all $i = 1, 2, \ldots, 5$, we note that $\mathrm{H}^*(H_i, \mathbf{k})$ is Cohen-Macaulay by [5, (6.1.1) and (6.1.3)]. For $i \neq 5$, the 2-rank of H_i is at least 2. Therefore, by Proposition V.10, we see that the products in Tate-Hochschild cohomology arising from the elements of negative degrees in those $\widehat{\mathrm{H}}^*(H_i)$, $i \neq 5$, are all 0. That is, let $\alpha \in \widehat{\mathrm{H}}^*(H_i)$ and $\beta \in \widehat{\mathrm{H}}^*(H_j)$ be of negative degrees, we have $\psi_i(\alpha) \smile \psi_j(\beta) = 0$:

- for $i = j$ and $i \neq 5$, and
- for $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$.

Since H_5 is cyclic, by [14, Theorem XII.11.6] and [5, (4.1.3)], its cohomology ring $\widehat{\mathrm{H}}^*(H_5)$ is periodic and is of the form $\Lambda(x) \otimes_{\mathbf{k}} \mathbf{k}[y, y^{-1}]$, where $\Lambda(x)$ is the exterior $\mathbf{k}$-algebra on the element x of degree 1 and y, y^{-1} are of degrees $2, -2$, respectively, subject to the graded-commutative relations. Hence, when $i \in \{1, 2\}$ and $j = 5$,

$$\psi_i(\alpha) \smile \psi_5(\beta) = \psi_5(\mathrm{res}^G_{H_5}(\alpha) \smile \beta)$$

will depend on the product $\mathrm{res}^G_{H_5}(\alpha) \smile \beta$ in $\widehat{\mathrm{H}}^*(H_5)$, for example, see [14, §XII.7].

The dihedral 2-groups. The same analysis applies for more general groups. Let $n \geq 2$ be a power of 2 and $G = D_{4n} = \langle a, b \mid a^{2n} = 1 = b^2, aba = b\rangle$. The $n+3$ conjugacy classes of G and the centralizers of their representatives are:

$$\begin{aligned}
\{1\} &\longrightarrow H_1 = C_G(1) = G \\
\{a^n\} &\longrightarrow H_2 = C_G(a^n) = G \\
\{a^q b \mid q \text{ is even}\} &\longrightarrow H_3 = C_G(b) = \langle a^n, b\rangle \cong V_4 \\
\{a^q b \mid q \text{ is odd}\} &\longrightarrow H_4 = C_G(ab) = \langle a^n, ab\rangle \cong V_4 \\
\text{for } 1 \leq s \leq n-1, \{a^s, a^{-s}\} &\longrightarrow H_{s+4} = C_G(a^s) = \langle a\rangle \cong \mathbb{Z}_{2n}
\end{aligned}$$

For $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$, let $\alpha \in \widehat{\mathrm{H}}^*(H_i)$ and $\beta \in \widehat{\mathrm{H}}^*(H_j)$ be of negative degrees. Under similar reasons and by Proposition V.10, we have $\psi_i(\alpha) \smile \psi_j(\beta) = \psi_j(\mathrm{res}^G_{H_j}(\alpha) \smile \beta) = 0$.

This example demonstrates that one can compute products in the Tate-Hochschild cohomology of a finite group G by observing the centralizers of its conjugacy representatives $C_G(g_i)$ and applying Theorem V.7 and Proposition V.10. Knowing certain properties of $C_G(g_i)$ and $\widehat{\mathrm{H}}^*(C_G(g_i))$, one can quickly deduce that some products in $\widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G)$ will always be 0.

V.3.2 The symmetric group on three elements

Let $\mathbf{k}$ be a field of characteristic 3. Let $G = S_3 = \left\langle a, b \mid a^3 = 1 = b^2, ab = ba^2 \right\rangle$ act on itself by conjugation. Without loss of generality, we choose conjugacy class representatives $g_1 = 1$, $g_2 = a$, and $g_3 = b$ whose centralizers are $H_1 = G$, $H_2 = \langle a\rangle =: N$, and $H_3 = \langle b\rangle$, respectively. Observe that the 3-rank of all H_i is at most one. We will find the Tate-Hochschild cohomology ring of $\mathbf{k}G$ using elements of $\widehat{\mathrm{H}}^*(H_i)$ and the product formula given in Theorem V.7.

Let us examine each ring $\widehat{\mathrm{H}}^*(H_i)$. Since the characteristic of $\mathbf{k}$ is 3 and N is cyclic of order 3, the cohomology ring $\widehat{\mathrm{H}}^*(N)$ is periodic by [14, Theorem XII.11.6], and [5, (4.1.3)]. By direct computation from [14, §XII.7], $\widehat{\mathrm{H}}^*(N)$ is of the form $\Lambda(w_1) \otimes_{\mathbf{k}} \mathbf{k}[w_2, w_2^{-1}]$, where $\Lambda(w_1)$ is the exterior $\mathbf{k}$-algebra on the element w_1 of degree 1 and $\mathbf{k}[w_2, w_2^{-1}]$ is generated by the elements w_2 of degree 2 and w_2^{-1} of degree -2, subject to the graded-commutative relations and $w_2 w_2^{-1} = 1$. By

[14, XII.2.7], because the characteristic of $\mathbf{k}$ does not divide the order of H_3, we have $\widehat{\mathrm{H}}^*(H_3) = 0$.

We now compute $\widehat{\mathrm{H}}^*(G)$. It is easy to check that G is isomorphic to a semidirect product $N \rtimes \mathbb{Z}_2$ and every abelian subgroup of G is cyclic. It follows from [14, Theorem XII.11.6] and [5, (4.1.3)] that the Tate cohomology ring $\widehat{\mathrm{H}}^*(G)$ is periodic and Noetherian. One can directly compute $\widehat{\mathrm{H}}^*(G)$ by using an $\mathbf{k}N$-complete resolution of $\mathbf{k}$, imposing on it an action of $\mathbb{Z}_2$ to make it become a $\mathbf{k}G$-complete resolution of $\mathbf{k}$, computing the Tate cohomology groups from that resolution, and studying their products. Alternatively, following the discussion in [14, §XII.10], we see that for any G-module M, $\widehat{\mathrm{H}}^*(G, M)$ is a direct sum of $\widehat{\mathrm{H}}^*(G, M, p)$, where $\widehat{\mathrm{H}}^*(G, M, p)$ is the p-primary component of $\widehat{\mathrm{H}}^*(G, M)$ and p runs through all the prime divisors of $|G| = 6$. Here, $M = \mathbf{k}$ is a field of characteristic 3, so only the 3-primary component is non-zero. By [14, Theorem XII.10.1], G/N operates on $\widehat{\mathrm{H}}^*(N)$ and so $\widehat{\mathrm{H}}^*(G) = \widehat{\mathrm{H}}^*(G, \mathbf{k}, 3) \cong \left[\widehat{\mathrm{H}}^*(N)\right]^{G/N} \cong \Lambda(w_1 w_2) \otimes_{\mathbf{k}} \mathbf{k}[w_2^2, w_2^{-2}]$. Therefore, $\widehat{\mathrm{H}}^*(G)$ is of the form $\Lambda(x) \otimes_{\mathbf{k}} \mathbf{k}[z, z^{-1}]$, where x and z are of degrees 3 and 4, respectively, subject to the graded-commutative relations and $zz^{-1} = 1$.

By the decomposition Lemma V.5, $\widehat{\mathrm{H}}^*(G, \mathbf{k}G) \cong \widehat{\mathrm{H}}^*(G) \oplus \widehat{\mathrm{H}}^*(N)$ as graded $\mathbf{k}$-modules. We then can define elements of the Tate-Hochschild cohomology ring of $\mathbf{k}G$ as follows. Since ψ_1 is an algebra monomorphism, we may identify any element of $\widehat{\mathrm{H}}^*(G)$ with its image under ψ_1. Let $E_i = \psi_i(1)$, $W_i = \psi_2(w_i)$, for $i = 1, 2$, and $W_2^{-1} = \psi_2(w_2^{-1})$. For simplification, we will use $C := E_2 + 1$ in the following theorem.

Theorem V.12. *Let $\mathbf{k}$ be a field of characteristic 3 and S_3 be the symmetric group on three elements. Then the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(\mathbf{k}S_3, \mathbf{k}S_3)$ of S_3 is generated as an algebra by elements $x, z, z^{-1}, C, W_1, W_2,$ and W_2^{-1} of degrees $3, 4, -4, 0, 1, 2,$ and -2, respectively, subject to the following relations:*

$$xW_1 = 0, \;\; xW_2 = zW_1, \;\; z^{-1}W_1 = (xz^{-1})W_2^{-1},$$

$$C^2 = CW_2^{-1} = CW_i = 0 \; (i = 1, 2),$$

$$W_2^2 = zC, \;\; W_2^{-2} = z^{-1}C, \;\; W_1W_2 = xC, \;\; W_1W_2^{-1} = xz^{-1}C,$$

together with the graded-commutative relations. In particular, the algebra monomorphism ψ_1 :

$\widehat{\mathrm{H}}^*(S_3, \mathbf{k}) \to \widehat{\mathrm{HH}}^*(\mathbf{k}S_3, \mathbf{k}S_3)$ *induces an isomorphism modulo radicals.*

Proof. $\widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G) \cong \widehat{\mathrm{H}}^*(G, \mathbf{k}G)$ is a graded-commutative $\mathbf{k}$-algebra whose underlying $\mathbf{k}$-module is isomorphic to $\widehat{\mathrm{H}}^*(G) \oplus \widehat{\mathrm{H}}^*(N)$. Here, $\widehat{\mathrm{H}}^*(G)$ is a graded subalgebra of $\widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G)$ generated by x, z and z^{-1}. Additionally, $\psi_2(\widehat{\mathrm{H}}^*(N))$ is a graded $\widehat{\mathrm{H}}^*(G)$-submodule of $\widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G)$ generated by E_2, W_1, W_2 and W_2^{-1}. This follows from the discussion after the proof of Theorem V.7. Moreover, we will check that these generators satisfy the following conditions:

1. action on $\psi_2(\widehat{\mathrm{H}}^*(N))$ as an $\widehat{\mathrm{H}}^*(G)$-module, and
2. every product in $\psi_2(\widehat{\mathrm{H}}^*(N))$ can be expressed as the sum of an element of $\widehat{\mathrm{H}}^*(G)$ and a $\widehat{\mathrm{H}}^*(G)$-linear combination of the images under ψ_2 of the generators of $\widehat{\mathrm{H}}^*(N)$.

Therefore, it is clear that $\widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G)$ is generated as a $\mathbf{k}$-algebra by $x, z, z^{-1}, E_2, W_1, W_2$, and W_2^{-1}, subject to these conditions. The first line of the relations in the statement of the theorem satisfies the first condition. The second and third lines satisfy the second condition. We will check each of them in detail.

The restriction $\mathrm{res}^G_N : \widehat{\mathrm{H}}^*(G) \to \widehat{\mathrm{H}}^*(N)$, which sends $x \mapsto w_1 w_2$, $z \mapsto w_2^2$, and $z^{-1} \mapsto w_2^{-2}$, is injective. We also observe that by graded-commutativity of the Tate cohomology ring, every element of odd degree has square 0. In particular, $w_1 w_1 = -w_1 w_1$ implies $w_1^2 = 0$. One can check that $\widehat{\mathrm{H}}^*(N)$ is an $\widehat{\mathrm{H}}^*(G)$-module with action via res^G_N:

$$
\begin{aligned}
&x \cdot w_1 = w_1 w_2 w_1 = w_1^2 w_2 = 0,\\
&x \cdot w_2 = w_1 w_2 w_2 = (-1)^2 w_2 w_1 w_2 = (-1)^2 w_2 w_2 w_1 = z \cdot w_1,\\
&x \cdot w_2^{-1} = w_1 w_2 w_2^{-1} = w_1,\\
&z \cdot w_2 = w_2^2 w_2 = w_2^3,\\
&z \cdot w_2^{-1} = w_2^2 w_2^{-1} = w_2,\\
&z^{-1} \cdot w_1 = w_2^{-1} w_2^{-1} w_1 = (-1)^{-2} w_2^{-1} w_1 w_2^{-1} = (-1)^{-2} w_1 w_2^{-1} w_2^{-1} = (x z^{-1}) \cdot w_2^{-1},\\
&z^{-1} \cdot w_2 = w_2^{-1} w_2^{-1} w_2 = w_2^{-1},\\
&z^{-1} \cdot w_2^{-1} = (w_2^{-1})^3.
\end{aligned}
$$

Therefore, as an $\widehat{\mathrm{H}}^*(G)$-module, $\widehat{\mathrm{H}}^*(N)$ is generated by $1, w_1, w_2$ and w_2^{-1}, subject to the relation

$x \cdot w_1 = 0$, $x \cdot w_2 = z \cdot w_1$, and $z^{-1} \cdot w_1 = (xz^{-1}) \cdot w_2^{-1}$. By the isomorphism in Lemma V.5 and mapping through ψ_2, we obtain the first line of the relations.

To check the second and third lines of the relations, we recall the fact that the submodule of the invariants $(\mathbf{k}G)^G$ is the center $Z(\mathbf{k}G)$ of the group algebra $\mathbf{k}G$, which is generated by conjugacy class representatives of G. Therefore, we may identify the degree-0 Tate-Hochschild cohomology with a quotient of $Z(\mathbf{k}G)$, as $\widehat{\mathrm{HH}}^0(\mathbf{k}G, \mathbf{k}G) \cong \widehat{\mathrm{H}}^0(G, \mathbf{k}G)$ is a quotient of $\mathrm{H}^0(G, \mathbf{k}G)$. Under this identification, E_i corresponds to (a quotient of) the sum of the group elements conjugate to g_i. In particular,

$$E_2^2 = (a + a^{-1})^2 = a^2 + 2 + a^{-2} = a^{-1} - 1 + a = E_2 - 1$$

in characteristic 3, which implies

$$C^2 = (E_2 + 1)^2 = E_2^2 + 2E_2 + 1 = 3E_2 = 0.$$

For the rest of the relations, we utilize the product formula in Theorem V.7. Let α and β be elements of $\widehat{\mathrm{H}}^*(N)$, we have:

$$\psi_2(\alpha) \smile \psi_2(\beta) = \psi_2(b^*(\alpha\beta)) + \psi_1(\mathrm{cor}_N^G(\alpha b^*(\beta))).$$

Recall that $b^* : \widehat{\mathrm{H}}^*(N) \to \widehat{\mathrm{H}}^*({}^bN) = \widehat{\mathrm{H}}^*(N)$. By checking on the definition of b^* and the degrees of w_i, we see that $b^*(w_2^{-1}) = -w_2^{-1}$ and $b^*(w_i) = -w_i$, for $i = 1, 2$. Moreover, as there are no degree $1, 2$ and -2 elements in $\widehat{\mathrm{H}}^*(G)$, we have $\mathrm{cor}_N^G(w_1) = \mathrm{cor}_N^G(w_2) = \mathrm{cor}_N^G(w_2^{-1}) = 0$. Similarly, by checking on the cochain level and using Lemma V.1 (10), for all $n \in \mathbb{Z}$, we obtain:

$$\mathrm{cor}_N^G(w_2^n) = \begin{cases} 0, & n \text{ is odd} \\ -z^{n/2}, & n \text{ is even.} \end{cases}$$

Hence, using Lemma V.1 (10) again,

$$\mathrm{cor}_N^G(w_1 w_2^n) = \begin{cases} -xz^{(n-1)/2}, & n \text{ is odd} \\ 0, & n \text{ is even.} \end{cases}$$

Let $\alpha = 1$ and $\beta = w_1$, using the product formula in Theorem V.7, we obtain:

$$E_2 W_1 = \psi_2(1) \smile \psi_2(w_1) = \psi_2(b^*(w_1)) + \psi_1(\mathrm{cor}_N^G(b^*(w_1))) = \psi_2(-w_1) + 0 = -W_1.$$

So $CW_1 = (E_2 + 1)W_1 = E_2 W_1 + W_1 = -W_1 + W_1 = 0$. Similarly, let $\alpha = 1$ and $\beta = w_2$ or w_2^{-1}, we show that $CW_2 = 0 = CW_2^{-1}$. This proves the second line of the relations.

Let $\alpha = \beta = w_2$, we have:

$$\begin{aligned} W_2^2 = \psi_2(w_2) \smile \psi_2(w_2) &= \psi_2(b^*(w_2^2)) + \psi_1(\mathrm{cor}_N^G(w_2 b^*(w_2))) \\ &= \psi_2(\mathrm{res}_N^G z \smile 1) + \psi_1(z) \\ &= z \smile \psi_2(1) + z \\ &= z E_2 + z = zC. \end{aligned}$$

Similarly, for $\alpha = \beta = w_2^{-1}$, we acquire that $W_2^{-2} = z^{-1}C$.

Let $\alpha = w_1$ and $\beta = w_2^{-1}$:

$$\begin{aligned} W_1 W_2^{-1} = \psi_2(w_1) \smile \psi_2(w_2^{-1}) &= \psi_2(b^*(w_1 w_2^{-1})) + \psi_1(\mathrm{cor}_N^G(w_1 b^*(w_2^{-1}))) \\ &= \psi_2(\mathrm{res}_N^G xz^{-1} \smile 1) + \psi_1(xz^{-1}) \\ &= xz^{-1} \smile \psi_2(1) + xz^{-1} \\ &= xz^{-1}E_2 + xz^{-1} = xz^{-1}C. \end{aligned}$$

Using the same argument, for $\alpha = w_1$ and $\beta = w_2$, we obtain $W_1 W_2 = xC$. Thus, we have found all necessary relations for the generators of the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G)$. Furthermore, because it is a graded-commutative ring, its nilpotent elements all lie in its radical. We observe that $C^2 = 0 = W_1^2$, $W_2^3 = W_2^2 W_2 = zCW_2 = 0$, and $(W_2^{-1})^3 = W_2^{-2}W_2^{-1} = z^{-1}CW_2^{-1} = 0$. This implies that C, W_1, W_2, and W_2^{-1} are contained in the radical of $\widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G)$. Consequently, modulo radicals, the algebra monomorphism $\psi_1 : \widehat{\mathrm{H}}^*(G) \to \widehat{\mathrm{HH}}^*(\mathbf{k}G, \mathbf{k}G)$ induces an isomorphism. $\square$

CHAPTER VI

FINITE GENERATION OF TATE COHOMOLOGY

Many people have been interested in the finite generation of the cohomology of a finite dimensional Hopf algebra A. If such property holds, one can apply the theory of support varieties to the study of A-modules. It is known that there are several finite dimensional Hopf algebras whose cohomology over their base field $\mathbf{k}$ is finitely generated, among them are: group algebras of finite groups, finite group schemes or equivalently finite dimensional co-commutative Hopf algebras, small quantum groups, and certain pointed Hopf algebras (see, for example, [28, Introduction] for references). While the usual cohomology rings of such algebras are finitely generated, the same may not be true for their Tate cohomology rings. For example, it is shown in [12] that the only finite groups G having the property that every finitely generated $\mathbf{k}G$-module has finitely generated Tate cohomology have p-rank one or zero, where p is the characteristic of the field $\mathbf{k}$. The purpose of this chapter is to investigate the finite generation property for Tate cohomology of a finite dimensional symmetric Hopf algebra A. If M is a finitely generated A-module, we want to know whether $\widehat{\mathrm{H}}^*(A, M)$ is finitely generated as a graded module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$. While the methods we use here are mostly straightforward generalizations of those in [12], some additional assumption is needed to fit in the context. For instance, in Proposition VI.7, we need A to be a Hopf algebra so that tensor products of modules are again A-modules. Nonetheless, the author believes some of the results in this chapter hold for finite dimensional symmetric $\mathbf{k}$-algebras in general, not necessarily restricted to Hopf algebras.

Throughout this chapter, let A be a finite dimensional symmetric Hopf algebra over a field $\mathbf{k}$ with antipode S, coproduct Δ, and augmentation ε. Here, A is symmetric in the sense that A is isomorphic to its $\mathbf{k}$-dual $D(A) := \mathrm{Hom}_{\mathbf{k}}(A, \mathbf{k})$ as A-bimodules (equivalently, from the discussion at the end of Chapter IV, the bilinear form $\mathcal{B}(-,-)$ is symmetric, or the Nakayama automorphism $\nu = \mathbf{1}$). All modules are finitely generated left modules and tensor product is over $\mathbf{k}$ unless stated otherwise.

By Theorem IV.2 ([31, Theorem 7.2]), $\widehat{\mathrm{HH}}^*(A, A)$ is isomorphic to $\widehat{\mathrm{H}}^*(A, A^{ad})$, where A^{ad} is the left adjoint module of A. Using this relation, if the module A^{ad} has the required hypotheses as in the following Sections VI.1 and VI.2, then the corresponding finite generation results also hold for Tate-Hochschild cohomology of A.

Since $A \cong D(A)$ as A-bimodules, we obtain the Tate duality for symmetric algebras as a special case of Auslander-Reiten duality. Briefly stated, for any finitely generated left A-modules M and N, Tate duality for symmetric algebras says that for any integer n, there is an isomorphism:

$$\widehat{\mathrm{Ext}}_A^{n-1}(M, N) \cong D(\widehat{\mathrm{Ext}}_A^{-n}(N, M)),$$

which is natural in M and N. Equivalently, there is a natural nondegenerate bilinear form

$$\langle -, - \rangle : \widehat{\mathrm{Ext}}_A^{n-1}(M, N) \times \widehat{\mathrm{Ext}}_A^{-n}(N, M) \to \mathbf{k}.$$

The readers may refer to [26, §2] for more details. We will use this Tate duality throughout this chapter. There are many finite dimensional symmetric Hopf algebras that are of interest, such as, group algebras of finite groups, commutative Hopf algebras (this includes the $\mathbf{k}$-duals of cocommutative Hopf algebras), semisimple algebras, the Drinfield double of any Hopf algebra, the restricted universal enveloping algebra $V(\mathfrak{g})$ of a finite dimensional restricted p-Lie algebra $\mathfrak{g}$ when $\mathfrak{g}$ is nilpotent or semisimple, and an algebra defined by Radford in [33]. Therefore, our finite generation of Tate cohomology results will add to the study of these algebras.

VI.1 Modules with bounds in finitely generated submodules

In this section, we show that there are A-modules whose Tate cohomology is not finitely generated. The key ingredients in this section are the boundedness conditions on finitely generated modules over Tate cohomology and the property that products in negative Tate cohomology of symmetric algebras are often zero [26, §8]. We recall some definitions and properties that were proved in [12] for group algebras. The same proofs go through for any finite dimensional symmetric (Hopf) algebra A over a field $\mathbf{k}$. We present them here for completeness.

Definition VI.1. A graded module $C = \bigoplus_{n\in\mathbb{Z}} C^n$ over $\widehat{\mathrm{H}}^*(A,\mathbf{k})$ has **bounded finitely generated submodules** if for any m, there exists a number $N = N(m)$ such that the submodule D of C generated by $\bigoplus_{n>m} C^n$ is contained in $\bigoplus_{n>N} C^n$.

Lemma VI.2. *If a graded module $C = \bigoplus_{n\in\mathbb{Z}} C^n$ over $\widehat{\mathrm{H}}^*(A,\mathbf{k})$ has bounded finitely generated submodules and if $C^n \neq 0$ for arbitrary small values of n, then C is not a finitely generated module over $\widehat{\mathrm{H}}^*(A,\mathbf{k})$.*

Proof. This follows from the definition of bounded finitely generated submodules property. Any finitely generated submodule of C is contained in $\bigoplus_{n>N} C^n$ for some N, and hence, cannot generate all of C. □

For a graded module $C = \bigoplus_{n\in\mathbb{Z}} C^n$, $C[s] = \bigoplus_{n\in\mathbb{Z}} C^{n+s}$ denotes a shift in C by a degree s, for some integer s.

Lemma VI.3. *Suppose we have an exact sequence of A-modules:*

$$0 \to L \to M \to N \to 0$$

which represents an element $\xi \in \mathrm{Ext}^1_A(N,L)$. Multiplication by ξ induces a homomorphism m_ξ : $\widehat{\mathrm{H}}^*(A,N) \to \widehat{\mathrm{H}}^*(A,L)[1]$. Let $\mathcal{K}^*$ be the kernel of this map and $\mathcal{I}^*$ be the cokernel. Then we have an exact sequence of $\widehat{\mathrm{H}}^*(A,\mathbf{k})$-modules:

$$0 \to \mathcal{I}^* \to \widehat{\mathrm{H}}^*(A,M) \to \mathcal{K}^* \to 0.$$

Moreover, if $\mathcal{K}^*$ is not finitely generated over $\widehat{\mathrm{H}}^*(A,\mathbf{k})$, then neither is $\widehat{\mathrm{H}}^*(A,M)$.

Proof. By the naturality of the long exact sequence on Tate cohomology (III.1.2 (d), or [31, §3.2]), we have:

$$\cdots \xrightarrow{m_\xi} \widehat{\mathrm{H}}^n(A,L) \to \widehat{\mathrm{H}}^n(A,M) \to \widehat{\mathrm{H}}^n(A,N) \xrightarrow{m_\xi} \widehat{\mathrm{H}}^{n+1}(A,L) \to \cdots$$

The collection of the maps m_ξ in the long exact sequence is a map of degree 1 of $\widehat{\mathrm{H}}^*(A, \mathbf{k})$-modules

$$m_\xi : \widehat{\mathrm{H}}^*(A, N) \to \widehat{\mathrm{H}}^*(A, L)[1].$$

The last statement is a consequence of the fact that quotient modules of finitely generated modules are finitely generated. $\square$

Now for $d > 0$, let ξ be a non-zero element in $\widehat{\mathrm{H}}^d(A, k)$. Then ξ is represented by a homomorphism $\xi : \Omega^d \mathbf{k} \to \mathbf{k}$. Let L_ξ be the kernel of that map. If $\xi = 0$, we define $L_\xi := \Omega^d \mathbf{k} \oplus \Omega \mathbf{k}$. We have an exact sequence:

$$0 \to L_\xi \to \Omega^d \mathbf{k} \xrightarrow{\xi} \mathbf{k} \to 0.$$

In the corresponding long exact sequence on Tate cohomology

$$\cdots \to \widehat{\mathrm{H}}^{n-1}(A, \mathbf{k}) \to \widehat{\mathrm{H}}^n(A, L_\xi) \to \widehat{\mathrm{H}}^n(A, \Omega^d \mathbf{k}) \xrightarrow{m_\xi} \widehat{\mathrm{H}}^n(A, \mathbf{k}) \to \cdots$$

m_ξ is the multiplication map by ξ. It is the degree d map

$$m_\xi : \widehat{\mathrm{H}}^*(A, \mathbf{k})[-d] \to \widehat{\mathrm{H}}^*(A, \mathbf{k}).$$

Let $\mathcal{K}^*$ and $\mathcal{I}^*$ be the kernel and cokernel of m_ξ, respectively. As a result, as in Lemma VI.3, we have an exact sequence of $\widehat{\mathrm{H}}^*(A, \mathbf{k})$-modules:

$$0 \to \mathcal{I}^*[-1] \to \widehat{\mathrm{H}}^*(A, L_\xi) \to \mathcal{K}^*[-d] \to 0.$$

Lemma VI.4. *Suppose that $\xi \in \widehat{\mathrm{H}}^*(A, \mathbf{k})$, $d > 0$, is a regular element on the usual cohomology ring $\mathrm{H}^*(A, \mathbf{k})$. Then*

1. *$\mathcal{K}^t = 0$, for all $t \geq 0$, and*
2. *$\mathcal{I}^t = 0$, for all $t < 0$.*

Proof. Since ξ is regular on $\mathrm{H}^*(A, \mathbf{k})$, it is clear that $\mathcal{K}^t = 0$ for all $t \geq 0$. It remains to prove the second part of the lemma. We recall the Tate duality for symmetric algebras, see [26, §2 and

Lemmas 8.1, 8.2], equivalently, there is a natural nondegenerate bilinear form

$$\langle -,- \rangle : \widehat{\mathrm{H}}^{n-1}(A,\mathbf{k}) \times \widehat{\mathrm{H}}^{-n}(A,\mathbf{k}) \to \mathbf{k}$$

such that $\langle \zeta\eta, \tau \rangle = \langle \zeta, \eta\tau \rangle$. For $t < 0$, let $\alpha_1, \ldots, \alpha_r$ be a $\mathbf{k}$-basis for $\widehat{\mathrm{H}}^{-t-1}(A,\mathbf{k})$. Because multiplication by $\xi : \widehat{\mathrm{H}}^{-t-1}(A,\mathbf{k}) \to \widehat{\mathrm{H}}^{-t+d-1}(A,\mathbf{k})$ is a monomorphism by part (1), the elements $\xi\alpha_1, \ldots, \xi\alpha_r$ are linearly independent. So there must exist elements $\beta_1, \ldots, \beta_r$ in $\widehat{\mathrm{H}}^{t-d}(A,\mathbf{k})$ such that for all i and j, we have:

$$\langle \beta_i, \xi\alpha_j \rangle = \langle \beta_i\xi, \alpha_j \rangle = \delta_{ij}$$

where δ_{ij} is the usual Kronecker delta. Thus, the elements $\beta_1\xi, \ldots, \beta_r\xi$ must be linearly independent and hence must form a basis for $\widehat{\mathrm{H}}^{t}(A,\mathbf{k})$. This implies that multiplication by $\xi : \widehat{\mathrm{H}}^{t-d}(A,\mathbf{k}) \to \widehat{\mathrm{H}}^{t}(A,\mathbf{k})$ is a surjective map, for all $t < 0$. Hence, its cokernel $\mathcal{I}^t = 0$. $\square$

There are examples of algebras for which products between two elements in negative cohomology are zero. In particular, this holds for finite dimensional symmetric algebras whose usual cohomology has depth greater than or equal to 2. Recall that the Tate cohomology of a Hopf algebra is always graded-commutative. Hence, a homogeneous regular sequence must automatically be central and when the characteristic of $\mathbf{k}$ is not two, it must consist of elements in even degrees. Theorems 3.5 in [8] and 8.3 in [26] — both are generalizations of the group cohomology result in [6] — independently prove the following:

Theorem VI.5. *Let A be a finite dimensional symmetric algebra over a field $\mathbf{k}$. Let M be a finitely generated A-module. Assume $\mathrm{Ext}^*_A(M,M)$ is graded-commutative. If the depth of the usual cohomology (resp. Hochschild cohomology) of M is two or more, then the Tate cohomology (resp. Tate-Hochschild cohomology) of M has zero products in negative cohomology.*

We show that using this property, for some A-module M, $\widehat{\mathrm{H}}^*(A,M)$ is not finitely generated.

Proposition VI.6. *Suppose A is a finite dimensional symmetric Hopf algebra over a field $\mathbf{k}$ and $\widehat{\mathrm{H}}^*(A,\mathbf{k})$ has the property that the products in negative cohomology are zero. If $\xi \in \mathrm{H}^d(A,\mathbf{k})$, $d > 0$, is a regular element for $\mathrm{H}^*(A,\mathbf{k})$, then $\widehat{\mathrm{H}}^*(A,L_\xi)$ is not a finitely generated $\widehat{\mathrm{H}}^*(A,\mathbf{k})$-module.*

Proof. Let $\mathcal{K}^*$ be the kernel of the multiplication by ξ on $\widehat{\mathrm{H}}^*(A, \mathbf{k})$. By Lemma VI.4, we have shown that $\mathcal{K}^*$ has elements only in negative degrees. Moreover, products of elements in negative degrees are zero by assumption. By [26, Lemma 8.2] or a direct generalization of [6, Lemma 2.1] and by the fact that there is no bound on the dimensions of the spaces $\widehat{\mathrm{H}}^n(A, \mathbf{k})$ for negative values of n, it follows that $\mathcal{K}^*$ is not zero in infinitely many negative degrees. Thus, $\mathcal{K}^*$ has bounded finitely generated submodules and is not finitely generated over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ by Lemma VI.2. It follows from Lemma VI.3 that $\widehat{\mathrm{H}}^*(A, L_\xi)$ is not finitely generated over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$. □

We say that a cohomology element $\xi \in \widehat{\mathrm{H}}^d(A, \mathbf{k})$ annihilates the Tate cohomology of a module M if the cup product with ξ is the zero operator on $\widehat{\mathrm{Ext}}^*_A(M, M)$. We generalize the proof of [4, Prop. 5.9.5] to a finite dimensional (symmetric) Hopf algebra A.

Proposition VI.7. *Let A be a finite dimensional Hopf algebra over a field* $\mathbf{k}$*. Suppose M is a finitely generated A-module and $\xi \in \widehat{\mathrm{H}}^d(A, \mathbf{k})$, for some $d \in \mathbb{Z}$. Then ξ annihilates $\widehat{\mathrm{Ext}}^*_A(M, M)$ if and only if*

$$L_\xi \otimes M \cong \Omega(M) \oplus \Omega^d(M) \oplus (\mathrm{proj}),$$

where (proj) *denotes some projective A-module.*

Proof. We note here that it is necessary for A to be a Hopf algebra so that a tensor product of A-modules is again an A-module with action via the coproduct of A. By abuse of notation, let $\xi : \Omega^d\mathbf{k} \to \mathbf{k}$ be a cocycle representing the cohomology element $\xi \in \widehat{\mathrm{H}}^d(A, \mathbf{k})$. Let L_ξ be its kernel. The proposition is obvious for $\xi = 0$, as in this case, $L_\xi = \Omega^d\mathbf{k} \oplus \Omega\mathbf{k}$, and $\Omega^i(M) \cong \Omega^i\mathbf{k} \otimes M \oplus (\mathrm{proj})$ for any i.

Assume $\xi \neq 0$. As before, we have an exact sequence:

$$0 \to L_\xi \to \Omega^d\mathbf{k} \xrightarrow{\xi} \mathbf{k} \to 0.$$

By translating, we get the exact sequence:

$$0 \to \mathbf{k} \to \Omega^{-1}(L_\xi) \to \Omega^{d-1}\mathbf{k} \to 0$$

representing ξ in $\widehat{\operatorname{Ext}}_A^1(\Omega^{d-1}\mathbf{k}, \mathbf{k}) \cong \widehat{\mathrm{H}}^d(A, \mathbf{k})$. Let Id_M represent the identity homomorphism on M. Then $\xi \cdot Id_M$ in $\widehat{\operatorname{Ext}}_A^d(M, M) \cong \widehat{\operatorname{Ext}}_A^1(\Omega^{d-1}(M), M)$ is represented by the sequence:

$$0 \to M \to \Omega^{-1}(L_\xi) \otimes M \to \Omega^{d-1}\mathbf{k} \otimes M \to 0.$$

Now suppose ξ annihilates $\widehat{\operatorname{Ext}}_A^*(M, M)$, then $\xi \cdot Id_M = 0$ and the above sequence splits. Hence,

$$\Omega^{-1}(L_\xi) \otimes M \cong M \oplus (\Omega^{d-1}\mathbf{k} \otimes M),$$

the middle term is the direct sum of the two end terms. Equivalently,

$$\Omega^{-1}(L_\xi \otimes M) \cong M \oplus \Omega^{d-1}(M) \oplus (\text{proj}).$$

Now translate everything by Ω, we have:

$$L_\xi \otimes M \cong \Omega(M) \oplus \Omega^d(M) \oplus (\text{proj}).$$

Conversely, if $L_\xi \otimes M \cong \Omega(M) \oplus \Omega^d(M) \oplus (\text{proj})$, then the sequence

$$0 \to \Omega(M) \to L_\xi \otimes M \to \Omega^d(M) \to 0$$

splits. Translate everything by Ω^{-1}, we get the sequence that represents $\xi \cdot Id_M$ also splits. Hence $\xi \cdot Id_M = 0$ and ξ annihilates the Tate cohomology of M. $\square$

We are now ready to prove the main theorem of this section.

Theorem VI.8. *Suppose A is a finite dimensional symmetric Hopf algebra over a field $\mathbf{k}$ and $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ has the property that the products in negative cohomology are zero. Let $\xi \in \mathrm{H}^d(A, \mathbf{k})$, $d > 0$, be a regular element and M be a finitely generated A-module such that $\widehat{\mathrm{H}}^*(A, M) \neq 0$. If for some $t > 0$, ξ^t annihilates the Tate cohomology of M and of L_{ξ^t}, then $\widehat{\mathrm{H}}^*(A, M)$ is not finitely generated as a module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$.*

Proof. By assumption, $\widehat{\mathrm{H}}^*(A, M) \neq 0$, so by Lemma VI.2, it is enough to show that $\widehat{\mathrm{H}}^*(A, M)$ has bounded finitely generated submodules.

Now since ξ^t annihilates the Tate cohomology of M for some $t > 0$, it follows from Proposition VI.7 that

$$L_{\xi^t} \otimes M \cong \Omega(M) \oplus \Omega^{dt}(M) \oplus (\mathrm{proj}).$$

Thus, $\widehat{\mathrm{H}}^*(A, M)$ has bounded finitely generated submodules if and only if $\widehat{\mathrm{H}}^*(A, L_{\xi^t} \otimes M)$ also has this property.

We first recall that for left A-modules M and N, $\mathrm{Hom}_{\mathbf{k}}(M, N)$ is a left A-module via the action: $(a \cdot f)(m) = \sum a_1 f(S(a_2)m)$, for $a \in A, m \in M$, and $f \in \mathrm{Hom}_{\mathbf{k}}(M, N)$. When $N = \mathbf{k}$ is the trivial A-module, the above action simplifies to the action of A on $D(M) := \mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$: $(a \cdot f)(m) = f(S(a)m)$. Moreover, when M and N are finite dimensional as $\mathbf{k}$-vector spaces, $\mathrm{Hom}_{\mathbf{k}}(M, N) \cong N \otimes D(M)$ as left A-modules, [27, §2.1]. We let $\widehat{\mathrm{H}}^*(A, \mathbf{k}) \cong \widehat{\mathrm{Ext}}_A^*(\mathbf{k}, \mathbf{k})$ act on $\widehat{\mathrm{Ext}}_A^*(M, M)$ via $- \otimes M$. By [3, Cor. 3.1.6], Proposition VI.7, and the hypothesis that ξ^t annihilates the Tate cohomology of L_{ξ^t}, we have:

$$\begin{aligned}
\widehat{\mathrm{Ext}}_A^*(L_{\xi^t}, L_{\xi^t}) &\cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n(L_{\xi^t}), L_{\xi^t}) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n \mathbf{k} \otimes L_{\xi^t}, L_{\xi^t}) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n \mathbf{k}, \mathrm{Hom}_{\mathbf{k}}(L_{\xi^t}, L_{\xi^t})) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n \mathbf{k}, L_{\xi^t} \otimes D(L_{\xi^t})) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n \mathbf{k}, L_{\xi^t} \otimes \Omega^{-dt-1} L_{\xi^t}) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n \mathbf{k}, \Omega^{-dt-1}(L_{\xi^t} \otimes L_{\xi^t})) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n \mathbf{k}, \Omega^{-dt-1}(\Omega L_{\xi^t} \oplus \Omega^{dt} L_{\xi^t} \oplus (\mathrm{proj}))) \\
&\cong \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n \mathbf{k}, \Omega^{-dt} L_{\xi^t} \oplus \Omega^{-1} L_{\xi^t}) \\
&\cong \widehat{\mathrm{H}}^*(A, \Omega^{-dt} L_{\xi^t} \oplus \Omega^{-1} L_{\xi^t})
\end{aligned}$$

where $D(L_{\xi^t}) = \mathrm{Hom}_{\mathbf{k}}(L_{\xi^t}, \mathbf{k}) \cong \Omega^{-dt-1} L_{\xi^t}$ by a generalization of [13, Prop. 11.3.3].

As ξ is a regular element, it is not hard to see that ξ^t is also a regular element. By a similar argument as in Proposition VI.6, we have that $\widehat{\mathrm{Ext}}_A^*(L_{\xi^t}, L_{\xi^t})$ has bounded finitely generated submodules. By definition, there exists a number N such that

$$\widehat{\mathrm{H}}^*(A, \mathbf{k}) \cdot \bigoplus_{n \geq m} \widehat{\mathrm{Ext}}_A^n(L_{\xi^t}, L_{\xi^t}) \subseteq \bigoplus_{n \geq N} \widehat{\mathrm{Ext}}_A^n(L_{\xi^t}, L_{\xi^t}).$$

Now let m be any integer. Let

$$\mathcal{N} := \bigoplus_{n \geq m} \widehat{\mathrm{H}}^n(A, L_{\xi^t} \otimes M).$$

We observe that the action of $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ on $\widehat{\mathrm{H}}^*(A, L_{\xi^t} \otimes M)$ via $- \otimes L_{\xi^t} \otimes M$ factors through the map $\widehat{\mathrm{H}}^*(A, \mathbf{k}) \to \widehat{\mathrm{Ext}}_A^*(L_{\xi^t}, L_{\xi^t})$, and the target of that map has bounded finitely generated submodules. Thus, we have:

$$\begin{aligned}
\widehat{\mathrm{H}}^*(A, \mathbf{k}) \cdot \mathcal{N} &\subseteq \widehat{\mathrm{H}}^*(A, \mathbf{k}) \cdot \left(\bigoplus_{n \geq m} \widehat{\mathrm{Ext}}_A^n(L_{\xi^t}, L_{\xi^t}) \right) \left(\bigoplus_{n \geq m} \widehat{\mathrm{H}}^n(A, L_{\xi^t} \otimes M) \right) \\
&\subseteq \left(\bigoplus_{n \geq N} \widehat{\mathrm{Ext}}_A^n(L_{\xi^t}, L_{\xi^t}) \right) \left(\bigoplus_{n \geq m} \widehat{\mathrm{H}}^n(A, L_{\xi^t} \otimes M) \right) \\
&\subseteq \bigoplus_{n \geq m+N} \widehat{\mathrm{H}}^n(A, L_{\xi^t} \otimes M).
\end{aligned}$$

Therefore, $\widehat{\mathrm{H}}^*(A, L_{\xi^t} \otimes M)$ has bounded finitely generated submodules, and so does $\widehat{\mathrm{H}}^*(A, M)$. If follows from Lemma VI.2 that $\widehat{\mathrm{H}}^*(A, M)$ is not finitely generated over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$. $\square$

Remark VI.9. Suppose that the cohomology in even degrees $\mathrm{H}^{ev}(A, \mathbf{k})$ is finitely generated (so it is a finitely generated commutative algebra, since $\mathrm{H}^*(A, \mathbf{k})$ is graded-commutative) and for any finite dimensional A-module M, the $\mathrm{H}^{ev}(A, \mathbf{k})$-module $\mathrm{Ext}_A^*(M, M)$ is finitely generated. Then under this assumption, one can define the support varieties for modules as follows:

Let $I_A(M, M)$ be the annihilator of the action of $\mathrm{H}^{ev}(A, \mathbf{k})$ on $\mathrm{Ext}_A^*(M, M)$, a homogeneous ideal of $\mathrm{H}^{ev}(A, \mathbf{k})$, and let $\mathcal{V}_A(M) := \mathcal{V}_A(M, M)$ denote the maximal ideal spectrum of the finitely generated commutative $\mathbf{k}$-algebra $\mathrm{H}^{ev}(A, \mathbf{k})/I_A(M, M)$. As the ideal $I_A(M, M)$ is homogeneous, the variety $\mathcal{V}_A(M)$ is conical and is called the **support variety of** M.

Then the hypothesis "for some power ξ^t of ξ, ξ^t annihilates the Tate cohomology of M and of L_{ξ^t}" in Theorem VI.8 can be translated as $\mathcal{V}_A(M) \subseteq \mathcal{V}_A\langle\xi\rangle$ and $\mathcal{V}_A(L_{\xi^t}) \subseteq \mathcal{V}_A\langle\xi\rangle$, where $\mathcal{V}_A\langle\xi\rangle$ is the support variety of the ideal generated by ξ.

VI.2 Modules with finitely generated Tate cohomology

In this section, we study A-modules whose Tate cohomology is finitely generated. In particular, we will see that all modules in the connected component of the stable Auslander-Reiten quiver associated to A which contains $\mathbf{k}$ have this property.

It is obvious that any module M which is a direct sum of Heller translates $\Omega^i\mathbf{k}$ has finitely generated Tate cohomology, as in this case, its Tate cohomology is a direct sum of copies of $\widehat{\mathrm{H}}^*(A,\mathbf{k})$:

$$\begin{aligned}
\widehat{\mathrm{H}}^*(A,M) &\cong \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n\mathbf{k}, M) \cong \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n\mathbf{k}, \bigoplus_i \Omega^i\mathbf{k}) \\
&\cong \bigoplus_i \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^n\mathbf{k}, \Omega^i\mathbf{k}) \\
&\cong \bigoplus_i \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_A(\Omega^{n-i}\mathbf{k}, \mathbf{k}) \cong \bigoplus_i \widehat{\mathrm{H}}^*(A,\mathbf{k}).
\end{aligned}$$

We will show that in general, there are more modules with this property. First, we consider the Tate cohomology of a module M which can occur as the middle term of an exact sequence of the form:

$$0 \to \Omega^m\mathbf{k} \to M \to \Omega^n\mathbf{k} \to 0$$

for some $m, n \in \mathbb{Z}$. Such a sequence represents an element ξ in

$$\widehat{\mathrm{Ext}}_A^1(\Omega^n\mathbf{k}, \Omega^m\mathbf{k}) \cong \widehat{\mathrm{Ext}}_A^{n+1-m}(\mathbf{k},\mathbf{k}) \cong \widehat{\mathrm{H}}^{n+1-m}(A,\mathbf{k}).$$

Without loss of generality, we can apply the shift operator Ω^{-m} and assume that the sequence has the form

$$0 \to \mathbf{k} \to M \to \Omega^n\mathbf{k} \to 0$$

for some n, and that $\xi \in \widehat{\mathrm{H}}^{n+1}(A,\mathbf{k})$.

Theorem VI.10. *Suppose that for the module M and cohomology element ξ as above, the map $\xi : \widehat{\mathrm{H}}^*(A, \mathbf{k}) \to \widehat{\mathrm{H}}^*(A, \mathbf{k})$ given by multiplication by ξ has a finite dimensional image. Suppose that the usual cohomology ring $\mathrm{H}^*(A, \mathbf{k})$ is Noetherian. Then the Tate cohomology $\widehat{\mathrm{H}}^*(A, M)$ is finitely generated as a module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$.*

Proof. As in Lemma VI.3, we have an exact sequence of $\widehat{\mathrm{H}}^*(A, \mathbf{k})$-modules:

$$0 \to \mathcal{I}^* \to \widehat{\mathrm{H}}^*(A, M) \to \mathcal{K}^*[-n] \to 0$$

for $\xi \in \widehat{\mathrm{H}}^{n+1}(A, \mathbf{k})$, $\mathcal{K}^*$ is the kernel of multiplication by ξ on $\widehat{\mathrm{H}}^*(A, \mathbf{k})$, and $\mathcal{I}^*$ is its cokernel. By hypothesis, the image of multiplication by ξ has finite total dimension. Hence, in all but a finite number of degrees i, multiplication by ξ is the zero map. Clearly, $\mathcal{I}^*$ is finitely generated over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$. So, $\widehat{\mathrm{H}}^*(A, M)$ is finitely generated over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ if and only if $\mathcal{K}^*$ has the same property.

View $\mathcal{K}^*$ as a module over the usual cohomology ring $\mathrm{H}^*(A, \mathbf{k})$. The elements of $\mathcal{K}^*$ in non-negative degrees form a submodule $\mathcal{L}^* = \sum_{m \geq 0} \mathcal{K}^m$, which is finitely generated over $\mathrm{H}^*(A, \mathbf{k})$, since $\mathrm{H}^*(A, \mathbf{k})$ is Noetherian by assumption.

Let $\mathcal{M}^*$ be the $\widehat{\mathrm{H}}^*(A, \mathbf{k})$-submodule of $\mathcal{K}^*$ generated by $\mathcal{L}^*$. We want to show that $\mathcal{M}^* = \mathcal{K}^*$ therefore proving the finite generation of $\mathcal{K}^*$. For all $m \geq 0$, $\mathcal{K}^m \subseteq \mathcal{M}^*$ by construction. It remains to show $\mathcal{K}^m \subseteq \mathcal{M}^*$ for all $m < 0$.

Because the quotient of $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ by $\mathcal{K}^*$ is finite dimensional, we must have that $\widehat{\mathrm{H}}^j(A, \mathbf{k}) = \mathcal{K}^j$ for sufficiently large j. For some sufficiently large j, we can find an element $0 \neq \gamma \in \mathcal{K}^j$ which is a regular element for the usual cohomology ring $\mathrm{H}^*(A, \mathbf{k})$. By a generalized version of Lemma 3.5 in [6], we know that multiplication by γ is a surjective map:

$$\gamma : \widehat{\mathrm{H}}^{m-j}(A, \mathbf{k}) \to \widehat{\mathrm{H}}^m(A, \mathbf{k})$$

whenever $m < 0$. Hence, for all $m < 0$, we must have $\widehat{\mathrm{H}}^{m-j}(A, \mathbf{k})\gamma = \mathcal{K}^m$. Since $\gamma \in \mathcal{M}^*$, we get that $\mathcal{K}^m \subseteq \mathcal{M}^*$ for all $m < 0$. Therefore, $\mathcal{K}^* = \mathcal{M}^*$ is finitely generated as a module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$. This proves the theorem. $\square$

Recall that a sequence $0 \to P \to Q \to R \to 0$ of finitely generated left A-modules is called an **almost-split sequence** (or **Auslander-Reiten sequence**) if it has the following properties:

1. The sequence does not split.
2. R is indecomposable and any homomorphism from an indecomposable module to R that is not an isomorphism factors through Q.
3. P is indecomposable and any homomorphism from P to an indecomposable module that is not an isomorphism factors through Q.

Almost-split sequences were first introduced by Auslander and Reiten for an Artin algebra, see [1] for more details. From a result by Auslander and Reiten, for any finitely generated left module R that is indecomposable but not projective, there is an almost-split sequence $0 \to \mathrm{DTr}(R) \to Q \to R \to 0$, which is unique up to isomorphism, where D is the dual and Tr is the transpose. Similarly for any finitely generated left module P that is indecomposable but not injective, there is an almost-split sequence $0 \to P \to Q \to \mathrm{TrD}(P) \to 0$, which is unique up to isomorphism. The **Auslander-Reiten quiver** associated to A has a vertex for each finitely generated indecomposable A-module (up to isomorphism) and an arrow between vertices if there is an irreducible morphism between the corresponding A-modules. The map DTr is the translation from the non-projective vertices to the non-injective vertices.

Remark VI.11. There are many examples of sequences satisfying the condition in Theorem VI.10. In particular, it is often the case that multiplication by an element ξ in negative degree has a finite dimensional image. An example is the element in degree -1 which represents the almost-split sequence for the module $\mathbf{k}$. In addition, if the depth of $\mathrm{H}^*(A, \mathbf{k})$ is two or more, then all products in negative cohomology are zero; and the principal ideal generated by any element in negative cohomology contains no non-zero elements in positive degrees, for example, by [26, Lemma 8.2] or a direct generalization of [6, Lemma 2.1]. Hence, multiplication by any element ξ in negative cohomology has a finite dimensional image.

Corollary VI.12. *The middle term of the almost-split sequence*

$$0 \to \Omega^2\mathbf{k} \to M \to \mathbf{k} \to 0$$

ending with $\mathbf{k}$ *has finitely generated Tate cohomology.*

Proof. The almost-split sequence in the statement corresponds to an element $\xi \in \widehat{\mathrm{H}}^{-1}(A, \mathbf{k})$. One of the defining properties of the almost-split sequence is that for any module N, the connecting homomorphism δ in the corresponding sequence

$$\cdots \to \underline{\mathrm{Hom}}_A(N, M) \to \underline{\mathrm{Hom}}_A(N, \mathbf{k}) \xrightarrow{\delta} \widehat{\mathrm{Ext}}^1_A(N, \Omega^2 \mathbf{k}) \to \cdots$$

is non-zero if and only if $N \cong \mathbf{k}$ [1, Prop. V.2.2]. This connecting homomorphism is multiplication by ξ. Now any element $\eta \in \widehat{\mathrm{H}}^d(A, \mathbf{k})$ is represented by a map $\eta : \Omega^d \mathbf{k} \to \mathbf{k}$. Therefore, we have $\xi\eta = 0$ whenever $d \neq 0$. This implies that multiplication by ξ on $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ has a finite dimensional image, and it follows from the above theorem that $\widehat{\mathrm{H}}^*(A, M)$ is finitely generated as a module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$. □

Remark VI.13. For an almost-split sequence $0 \to P \to Q \to R \to 0$, we have $R \cong \mathrm{TrD}(P)$, equivalently $P \cong \mathrm{DTr}(R)$, see [1, Prop. V.1.14]. But for symmetric algebras, $\mathrm{TrD} \cong \Omega^{-2}$ and $\mathrm{DTr} \cong \Omega^2$. So for any indecomposable non-projective module N over a symmetric algebra, [1, Theorem V.1.15] shows the existence of an almost-split sequence

$$0 \to \Omega^2 N \to M \to N \to 0.$$

Proposition VI.14. *Let N be a finitely generated indecomposable non-projective A-module that is not isomorphic to $\Omega^i \mathbf{k}$ for any i. Consider the almost-split sequence*

$$0 \to \Omega^2 N \to M \to N \to 0.$$

If N has finitely generated Tate cohomology, then so does the middle term M.

Proof. For any $i \in \mathbb{Z}$, the connecting homomorphism δ in the corresponding sequence

$$\cdots \to \underline{\mathrm{Hom}}_A(\Omega^i \mathbf{k}, M) \to \underline{\mathrm{Hom}}_A(\Omega^i \mathbf{k}, N) \xrightarrow{\delta} \widehat{\mathrm{Ext}}^1_A(\Omega^i \mathbf{k}, \Omega^2 N) \to \cdots$$

is zero because $\Omega^i \mathbf{k} \ncong N$. Hence, δ induces the zero map on Tate cohomology. So the long exact sequence in Tate cohomology breaks into short exact sequences:

$$0 \to \widehat{\mathrm{H}}^*(A, \Omega^2 N) \to \widehat{\mathrm{H}}^*(A, M) \to \widehat{\mathrm{H}}^*(A, N) \to 0.$$

It follows that if $\widehat{\mathrm{H}}^*(A, N)$ is finitely generated, then $\widehat{\mathrm{H}}^*(A, M)$ is also finitely generated. $\square$

Combining the last two results, we have the following theorem:

Theorem VI.15. *If a module in a connected component of the stable Auslander-Reiten quiver associated to A has finitely generated Tate cohomology, then so does every module in that component. In particular, all modules in the connected component of the quiver which contains $\mathbf{k}$ have finitely generated Tate cohomology.*

VI.3 Finite generation examples

In this section, we apply some results from Section VI.1 on an algebra constructed by Radford in [33] and on the restricted universal enveloping algebra of $\mathfrak{sl}_2(\mathbf{k})$. By showing that these algebras have finitely generated usual cohomology but fail to do so for the Tate cohomology, these examples demonstrate that finite generation behaves differently in negative cohomology.

VI.3.1 Radford's algebra

The following Hopf algebra A is taken from [33, Example 1]. Let $N > 1$ and $\mathbf{k}$ be a field that contains a primitive N-th root of unity ω. Let A be an algebra generated over $\mathbf{k}$ by elements x, y, and g subject to the relations:

$$g^N = 1, \quad x^N = y^N = 0, \quad xg = \omega gx, \quad gy = \omega yg, \quad xy = \omega yx.$$

A is a symmetric algebra of dimension N^3 and has the Hopf structure:

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1},$$

$$\Delta(x) = x \otimes g + 1 \otimes x, \qquad \varepsilon(x) = 0, \qquad S(x) = -xg^{-1},$$

$$\Delta(y) = y \otimes g + 1 \otimes y, \qquad \varepsilon(y) = 0, \qquad S(y) = -yg^{-1}.$$

Let $Y = yg^{-1}$. Using the above relations, one can check that x and Y commute. Consider a subalgebra B of A generated by x and Y subject to the following relations:

$$x^N = Y^N = 0, \quad xY = Yx.$$

In particular, B is the truncated polynomial algebra which can be considered as the complete intersection $\mathbf{k}[x,Y]/(x^N, Y^N, xY - Yx) \cong \mathbf{k}[x]/(x^N) \otimes \mathbf{k}[Y]/(Y^N)$. Using the Künneth Theorem, the cohomology of B can be obtained by tensoring together the cohomology of $\mathbf{k}[x]/(x^N)$ and the cohomology of $\mathbf{k}[Y]/(Y^N)$. One can also construct a B-projective resolution of $\mathbf{k}$ by taking the tensor product of the following periodic resolutions:

$$\cdots \xrightarrow{\cdot x} \mathbf{k}[x]/(x^N) \xrightarrow{\cdot x^{N-1}} \mathbf{k}[x]/(x^N) \xrightarrow{\cdot x} \mathbf{k}[x]/(x^N) \xrightarrow{\varepsilon_x} \mathbf{k} \to 0,$$

and

$$\cdots \xrightarrow{\cdot Y} \mathbf{k}[Y]/(Y^N) \xrightarrow{\cdot Y^{N-1}} \mathbf{k}[Y]/(Y^N) \xrightarrow{\cdot Y} \mathbf{k}[Y]/(Y^N) \xrightarrow{\varepsilon_Y} \mathbf{k} \to 0,$$

where $\varepsilon_x(x) = 0$ and $\varepsilon_Y(Y) = 0$. This construction has been done in the literature, for example, in [28, §4]. Using the relations $xg = \omega gx$ and $gy = \omega yg$, we can see that Radford's algebra $A = B\#\mathbf{k}G$, where $G = \langle g \rangle$ acts on B by automorphisms for which x, Y are eigenvectors:

$$gxg^{-1} = {}^{g}x = \omega^{-1}x, \qquad {}^{g}Y = \omega Y.$$

Given basis elements $\{1_x, x\}$ of $\mathbf{k}[x]/(x^N)$ and $\{1_Y, Y\}$ of $\mathbf{k}[Y]/(Y^N)$, for $b = 1_x, 1_Y, x$, or Y, we define the action of g on the above resolutions as:

- In degree $2i$, $g \cdot b := {}^{g}b$.
- In degree $2i+1$, $g \cdot b := \begin{cases} \omega^{-1}({}^{g}b), & b = 1_x, x \\ \omega({}^{g}b), & b = 1_Y, Y. \end{cases}$

One checks that this group action commutes with the differentials in each degree.

The cohomology ring $\mathrm{H}^*(B,\mathbf{k})$ is generated by ξ_j, η_i, for $i,j \in \{1,2\}$, where $\deg(\xi_j) = 2$ and $\deg(\eta_i) = 1$, subject to the following relations:

$$\xi_1\xi_2 = \xi_2\xi_1, \quad \eta_1\eta_2 = -\eta_2\eta_1, \quad \eta_i\xi_j = \xi_j\eta_i, \quad (\eta_i)^2 = 0,$$

(see, for example, [28, Theorem 4.1]). We note that $\mathrm{H}^*(\mathbf{k}[x]/(x^N),\mathbf{k})$ is generated by ξ_1, η_1 and $\mathrm{H}^*(\mathbf{k}[Y]/(Y^N),\mathbf{k})$ is generated by ξ_2, η_2. If $N = 2$, ξ_i is a scalar multiple of η_i^2. As $A = B\#\mathbf{k}G$ and the characteristic of $\mathbf{k}$ does not divide the order of G, we have:

$$\mathrm{H}^*(A,\mathbf{k}) \cong \mathrm{H}^*(B,\mathbf{k})^G,$$

the invariant ring under the above G-action defined at the chain level. By (4.2.1) in [28], the induced action of G on generators ξ_j, η_i is given by:

$$g\cdot\xi_j = \xi_j, \qquad g\cdot\eta_1 = \omega\eta_1, \qquad g\cdot\eta_2 = \omega^{-1}\eta_2.$$

Thus, $\mathrm{H}^*(A,\mathbf{k}) \cong \mathbf{k}[\xi_1,\xi_2]$, where $\deg(\xi_j) = 2$.

The elements ξ_1, ξ_2 form a regular sequence on $\mathrm{H}^*(A,\mathbf{k})$. In fact, the depth of $\mathrm{H}^*(A,\mathbf{k})$ is 2. By Theorem VI.5, the Tate cohomology $\widehat{\mathrm{H}}^*(A,\mathbf{k})$ of A has zero products in negative cohomology. Therefore, since each ξ_j is a regular element on $\mathrm{H}^*(A,\mathbf{k})$, it follows from Proposition VI.6 that $\widehat{\mathrm{H}}^*(A,L_{\xi_j})$ is not finitely generated as a module over $\widehat{\mathrm{H}}^*(A,\mathbf{k})$, for $j = 1,2$.

VI.3.2 The restricted enveloping algebra of $\mathfrak{sl}_2(\mathbf{k})$

Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 3$. Let $\mathfrak{g} := \mathfrak{sl}_2(\mathbf{k})$ be the restricted p-Lie algebra of 2×2 trace-zero matrices over $\mathbf{k}$. It is generated over $\mathbf{k}$ by:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the Lie algebra structure:

$$[h, f] = -2f, \qquad [h, e] = 2e, \qquad [e, f] = h,$$

and the map $[p] : \mathfrak{g} \to \mathfrak{g}$ is given by:

$$e^{[p]} = f^{[p]} = 0, \qquad h^{[p]} = h.$$

Let $V(\mathfrak{g})$ be the restricted enveloping algebra of $\mathfrak{g}$. It is defined as the quotient algebra:

$$V(\mathfrak{g}) := T(\mathfrak{g}) / \left\langle X \otimes Y - Y \otimes X - [X, Y], X^{\otimes p} - X^{[p]} \mid X, Y \in \mathfrak{g} \right\rangle,$$

equivalently,

$$V(\mathfrak{g}) = U(\mathfrak{g}) / \left\langle X^{\otimes p} - X^{[p]} \mid X \in \mathfrak{g} \right\rangle,$$

where $T(\mathfrak{g})$ is the tensor algebra and $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. $V(\mathfrak{g})$ is a finite dimensional, co-commutative Hopf algebra over $\mathbf{k}$ with the Hopf structure:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) = 0, \qquad S(X) = -X,$$

for all $X \in \mathfrak{g}$. A **restricted $\mathfrak{g}$-module** is a module M of $\mathfrak{g}$ on which $X^{[p]}$ acts as the p-th iterate of X, for any $X \in \mathfrak{g}$. The category of restricted $\mathfrak{g}$-modules is equivalent to the category of $V(\mathfrak{g})$-modules. From here, all $\mathfrak{g}$-modules are assumed to be restricted and will be referred to as $V(\mathfrak{g})$-modules.

To a restricted Lie algebra $\mathfrak{g}$, we associate the **nullcone** $\mathcal{N} = \mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$, which is the closed subvariety of $\mathfrak{g}$ consisting of all nilpotent elements. We also define the **restricted nullcone** of $\mathfrak{g}$ to be the subvariety

$$\mathcal{N}_1(\mathfrak{g}) = \left\{ X \in \mathfrak{g} \mid X^{[p]} = 0 \right\}$$

of $[p]$-nilpotent elements in $\mathfrak{g}$.

The cohomology $\mathrm{H}^*(V(\mathfrak{g}), M)$ of $V(\mathfrak{g})$ with coefficients in a restricted $\mathfrak{g}$-module M is defined as the cohomology of the augmented algebra $V(\mathfrak{g})$ over $\mathbf{k}$. There is a close relationship between the nullcone $\mathcal{N}(\mathfrak{g})$ and the cohomology $\mathrm{H}^*(V(\mathfrak{g}), \mathbf{k})$ as described by Friedlander and Parshall in the following theorem:

Theorem VI.16 ([20])**.** *Let G be a simple, simply connected algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic $p > 0$. Assume that G is defined and split over the prime field $\mathbb{F}_p$. Let $\mathfrak{g}$ be the Lie algebra of G. If p is greater than the Coxeter number of G, then there is a G-equivariant isomorphism of algebras:*

$$\mathrm{H}^*(V(\mathfrak{g}), \mathbf{k}) \cong \mathbf{k}[\mathcal{N}]^{(1)},$$

where $\mathbf{k}[\mathcal{N}]$ is the coordinate ring of the nullcone $\mathcal{N}$ of $\mathfrak{g}$, and $\mathbf{k}[\mathcal{N}]^{(1)}$ means $\mathbf{k}[\mathcal{N}]$ to be regarded as a G-module by composing the usual conjugation action of G on $\mathbf{k}[\mathcal{N}]$ with the Frobenius morphism $f : G \to G$.

Recall that a basis for $\mathfrak{sl}_2(\mathbf{k})$ is $\{e, f, h\}$. Let $\{x, y, z\}$ be its $\mathbf{k}$-dual basis. As an affine space, $\mathfrak{sl}_2(\mathbf{k})$ can be identified with $\mathbb{A}^3$ and has coordinate ring $\mathbf{k}[x, y, z]$. Since every 2×2 nilpotent matrix squares to 0, we have

$$\begin{pmatrix} z & x \\ y & -z \end{pmatrix}^2 = (xy + z^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is zero whenever $xy + z^2$ is zero. Hence, independent of p, the nullcone $\mathcal{N}(\mathfrak{sl}_2)$ is a quadric in $\mathbb{A}^3$ defined by the equation $xy + z^2 = 0$. By the above theorem, the usual cohomology ring of $V(\mathfrak{sl}_2)$ is finitely generated:

$$\mathrm{H}^*(V(\mathfrak{sl}_2), \mathbf{k}) \cong \mathbf{k}[\mathcal{N}(\mathfrak{sl}_2)]^{(1)} \cong \mathbf{k}[x, y, z]/(xy + z^2)$$

and is concentrated in even degrees as a graded ring. One can observe that $\mathrm{H}^*(V(\mathfrak{sl}_2), \mathbf{k})$ has depth 2. Moreover, by [34], $V(\mathfrak{sl}_2)$ is symmetric. Therefore, we can apply Theorem VI.5 to conclude that the Tate cohomology $\widehat{\mathrm{H}}^*(V(\mathfrak{sl}_2), \mathbf{k})$ has zero products in negative cohomology. It then follows from Proposition VI.6 that for each regular element ξ of $\mathrm{H}^*(V(\mathfrak{sl}_2), \mathbf{k})$, the Tate cohomology $\widehat{\mathrm{H}}^*(V(\mathfrak{sl}_2), L_\xi)$ is not finitely generated as a module over $\widehat{\mathrm{H}}^*(V(\mathfrak{sl}_2), \mathbf{k})$.

CHAPTER VII

SUMMARY

We have seen several successful attempts in studying the Tate (Hochschild) cohomology, specified for finite dimensional Hopf algebras over a field. However, there are more open questions in this topic that are worth pursuing. Let us summarize some of what has been done in this work and point out problems that are still under investigation.

Known results for a finite dimensional Hopf algebra A over a field $\mathbf{k}$:

- $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ and $\widehat{\mathrm{HH}}^*(A, A)$ are graded-commutative rings.
- $\widehat{\mathrm{HH}}^*(A, A) \cong \widehat{\mathrm{H}}^*(A, A^{ad})$ as algebras, where A^{ad} is the adjoint module of A.
- $\widehat{\mathrm{H}}^*(A, \mathbf{k})$ is a direct summand of $\widehat{\mathrm{HH}}^*(A, A)$ as a module over $\widehat{\mathrm{H}}^*(A, \mathbf{k})$.
- Let $A = RG$, a finite group algebra over R, where R is the ring of integers $\mathbb{Z}$ or a field $\mathbf{k}$ whose characteristic divides the order of G. Let G act on itself via conjugation and let $C_G(g_i)$ be the centralizer of the conjugacy representative g_i of G. Then $\widehat{\mathrm{HH}}^*(RG, RG) \cong \bigoplus_i \widehat{\mathrm{H}}^*(C_G(g_i), R)$. There is a product formula with respect to this decomposition, making it become a decomposition as $\widehat{\mathrm{H}}^*(G, R)$-modules. Products in negative cohomology of $\widehat{\mathrm{HH}}^*(RG, RG)$ also depend on the $C_G(g_i)$'s and their cohomology rings.
- Let A be symmetric, that is, $A \cong D(A)$ as A-bimodules. Finite generation of Tate cohomology fails for some finitely generated modules of A. In particular, all modules in the connected component of the quiver which contains $\mathbf{k}$ have finitely generated Tate cohomology.

One can see the relations and results that we have established and studied in this dissertation are interesting in and of themselves, with potential applications in other subjects that remain to be seen. We close this dissertation with questions that are yet to be answered. We hope to consider them in our future work.

Open questions to consider in future research include the following:

1. Can Evens' norm map be defined for Tate cohomology of a finite group algebra RG? If so, what are the properties of the Tate-Evens norm map?

2. Let G be a finite group. Suppose the characteristic of $\mathbf{k}$ is $p > 0$ which divides the order of G. Let E be a (maximal) elementary abelian p-subgroup of G. What is the relation between $\widehat{\mathrm{H}}^*(E, \mathbf{k})$ and $\widehat{\mathrm{H}}^*(G, \mathbf{k})$?
3. Describe the construction and properties of Tate (Hochschild) cohomology for other classes of (more general, not necessarily Hopf) algebras.
4. Assume Tate cohomology can be defined for other algebras (not necessarily Hopf, not necessarily symmetric). What can one say about the products in negative cohomology of these algebras? What can one say about the finite generation of their Tate cohomology?
5. Working with complete resolutions: Can we reconstruct a spectral sequence of a double complex, for example, as those in [4] and [14]? A spectral sequence of a group extension?
6. What can one say about the vanishing of Tate (co)homology?

These questions may not have affirmative answers; however, they stimulate us to keep thinking in this direction. As there are more unknowns in this universe than we can discover, we can only look deeper and further, try to connect the dots, and enjoy the process in between.

"Some mathematician, I believe, has said that true pleasure lies not in the discovery of truth, but in the search for it." – Leo Tolstoy

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