MINIMUM TIME/MINIMUM FUEL CONTROL OF AN AXISYMMETRIC RIGID BODY

A Thesis

by

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ABSTRACT

Many times it is necessary to reorient an aerial vehicle during flight in a minimum time or minimum fuel fashion. This thesis will present a minimum time/fuel control solution to reorienting an axisymmetric rigid body using eigenaxis maneuvers. Any fixed desired attitude can be achieved by rotating the rigid body about its eigenaxis. While an eigenaxis is not a time-optimal maneuver, it will produce the shortest angular trajectory between the rigid bodys current attitude and the desired attitude. Using the eigenaxis, a reference frame will be defined with the third unit vector direction parallel to the eigenaxis. In this reference frame, the controls and the equations of motion will be developed. A control weight will dictate whether the controller will drive the vehicle to the desired orientation in minimum time, minimum fuel, or a hybrid between minimum time and fuel. The controls can then be translated to the body frame through an attitude matrix that relates the body frame to the eigenaxis frame. After a minimum time/fuel controller has been developed, a minimum time/energy controller will then be designed. This minimum time/energy controller will then be compared against the minimum time/fuel controller by examining two fuel performance indices. Comparing these two controllers results in the most efficient controller being dependent on the cost function that describes the actuator type. Therefore, it is not feasible to select one controller over another independent of the fuel cost function. A minimum time/fuel controller has been designed using an eigenaxis maneuver in order to reorient itself. A comparison between the minimum time/fuel and minimum time/energy controller has been investigated using the two cost functions resulting in neither controller being the most efficient independent of the fuel cost function.

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1. INTRODUCTION

1.1 Introduction

When a space vehicle is reorienting itself, it is imperative that the vehicle achieve the desired orientation using the minimum amount of fuel, the minimum amount of time, or a hybrid between minimum fuel and time. The objective of this thesis is to design a tunable controller so that the vehicle can reorient itself in a minimum time/minimum fuel fashion. The vehicle will be characterized as an axisymmetric rigid body where quaternions will be used as an attitude description. An optimal control method approach will be used to find a minimum time/minimum fuel controller. This minimum time/minimum fuel controller will then be compared against a minimum time/minimum energy controller.

1.1.1 Relevance

In some instances, mission objectives may require a spacecraft to reorient itself during the mission. These attitude changes may require the maneuver to be performed within a certain amount of time or while utilizing the least amount of fuel. According to NASA, the price of all fuel expelled during liftoff is approximately 1,380,000 dollars [12]. With the price of propellants being so costly, it would be beneficial for a mission that requires a space shuttle to travel a long distance without any time requirements to utilize minimum fuel maneuvers when reorienting. Any deep space mission or exploratory mission would meet the requirement of a vehicle that must travel a long distance without any time requirements. Minimum time maneuvers are beneficial when an aerial vehicle's mission objective is to execute quick maneuvers. Some aerial vehicles require maneuvers that are time critical and some require maneuvers that are not. Therefore, it would be advantageous to have one controller that would be able to be adjusted so that the vehicle's maneuvers could be executed in a minimum amount of time or using a minimum amount of fuel. In order for one controller to have the capability to execute minimum time and minimum fuel maneuvers, a tunable parameter will enable the controller to reorient the vehicle to a specific orientation using a minimum amount of time or a minimum amount of fuel. Introducing a tunable parameter eliminates the need for two different controllers for each type of maneuver.

1.1.2 Literature

The following literature demonstrates successful minimum time and mimimum fuel controllers for reorienting an aerial vehicle. Vadali (1995) et. al. presented a near-minimum-time solution to reorienting an aerial vehicle. The vehicle is assumed to be a rigid body that utilizes thrusters for attitude control. Quaternions were chosen to be the attitude description of the vehicle. The moment of inertia matrix in the body frame contained nonzero values in the upper and lower diagonal; therefore, the vehicle is non symmetric. A feedback law was successfully designed for large angle reorientation using the Lyapunov Stability Criterion [14].

Foy solved a fuel minimization attitude control problem using a bang-off-bang control method. The objective was to design a control law for a single-axis rigid body. Solving this type of problem enables a fuel minimization control method when the body is in "cruise mode". Cruise mode is when it is not necessary for the vehicle's attitude angles and angular rates to be precise, but the amount of fuel output should be minimized. The states for the optimal control problem will be x_θ and x_ω, where x_θ is the difference between the desired angle and the actual angle and x_ω is the derivative of x_θ. The cost function will be as follows:

$$P(\bar{x_0}, u) = \frac{1}{2}\int_0^{t_f} \left\{c_1 x_\theta^2(t) + c_2 x_\omega^2(t)\right\} \mathrm{d}t + \int_0^{t_f} |u(t)|\mathrm{d}t$$

The control will also be bounded between one and negative one. Pontryagin's Minimums Principle can then be applied in order to design a bang-off-bang feedback control for vehicle reorientation[4].

Another minimimum fuel controller was designed by Topcu (2007) et. al. to land an aerial vehicle within 100 meters of a selected area on Mars using a minimum amount of fuel. When landing, the vehicle may experience strong wind drifts that would cause the vehicle to move horizontally. A controller was designed so that the vehicle could counteract against the strong gusts of wind. The vehicle was considered to be a rigid body in this analysis. The cost function was as follows:

$$J = -C(t_f)$$

where C is the characteristic velocity defined as

$$\dot{C} = \Gamma$$

Γ is defined as the magnitude of the specific thrust. From this cost function, a bang-bang feedback controller can be designed that would satisfy the mission objective[13].

Fleming (2010) et. al. proposed a bang-bang solution to the minimum time attitude reorientation problem. The body was characterized to be a rigid axisymmetric body. In order to form a bang-bang feedback solution, a maneuver about the eigenaxis would have to be executed. The cost function of the optimal control problem was as follows:

$$J = \int_0^{t_f} \mathrm{d}t$$

The states of the optimal control problem would be the angular velocities in a ax-isymmetric rigid body's equation of motion and the quaternions used for attitude description. A Hamiltonian can then be formed and Pontryagin's principle can then be applied to form a feedback control law [3].

Jahangir and Howe investigated a minimum time attitude control solution for a spinning missile. The missile was described as a rigid body that was symmetric about its spin axis. The body had two angular velocities orthogonal to the spin axis. The objective was to eliminate the orthogonal angular velocities while reorienting the vehicle to a specific orientation with the vehicle spinning at a constant rate about the spin axis. There will only be one thruster orthogonal to the spin axis. In order for the missile to achieve the correct orientation, thrusters will be active and inactive during certain times when the vehicle is rolling. A optimal control method will be utilized to find these times when the missile's thrusters will be active and inactive. Since this is a minimimum time maneuver, the cost function will the the same as Flemming's (2010) et. al. cost function shown above. The feedback control law will use Euler Angles for the attitude description. The control will also be constrained between zero and a maximum value. Since the angular velocity about the spin axis is constant, the states of the system will be described by two dynamical equations consisting of the orthogonal angular velocities and the rotational differential equations consisting of the Euler Angles. When solving this optimal control problem, a Two Point Boundary Value Problem (TBVP) approach is necessary to find initial conditions of the costates [8].

Since solving a TBVP is an iterative process, problems arise when implementing this control law on a real-time system. When a TBVP approach is employed, the thruster times are calculated offline and stored in an onboard computer. The thruster switching times can then be calculated in real-time by using the stored data as a look-up table. In order to implement this control law on a real-time system, another approach is taken. The method used to derive the appropriate thruster switching times assumed a final state vector and integrated backward in time. This approach also produces a time optimal solution [8].

Lee (2008) et. al also derives a minimum time solution using the rotation matrix as the attitude description. The body is a rigid body that has control constraints. The rotation matrix is used instead of an attitude parameter set to avoid singularities and other undesirable effects quaternions present. Since a rotation matrix is used, the equations of motion for the optimal control problem will change. The dynamical equations will be the same for a rigid body; however, Lee presents that the rotational differential equation will be as follows:

$$\dot{R} = R\Omega$$

where R and Ω represent the rotation matrix and the rigid body's angular velocities respectively. The same cost function will be used in Jahangir and Howe's work. A minimum time feedback solution can then be solved [10].

In conclusion, throughout all of the literature reviewed, a minimum time/minimimum fuel controller that utilizes an eigenaxis rotation and quaternions as the attitude description has not yet been designed.

1.1.3 Definition of an Axisymmetric Rigid Body

A rigid body is defined as a collection of point masses with constant relative distances from each other[2]. The rigid body has translational and rotational movement, meaning it has six degrees of freedom. A rigid body's position in space is described by a vector, and its orientation can be described by several different parameter sets [7]. An axisymmetric body is a body that is symmetric about two axis. If a body is symmetric about the x and y axis, the moment of inertia about the x-z plane would be as follows:

$$\int\int xz\mathrm{d}m$$

Since the body is symmetric about the x axis, the integral would be summed to zero. So, I_{xy}, I_{yz}, and I_{zx} will all be zero. Therefore, the inertia matrix will only have nonzero elements along its diagonal when a body has two axis of symmetry [9].

1.1.4 Attitude Descriptions

There are four common mathematical descriptions used to describe an aircraft's attitude: Classic Rodrigues Parameters (CRPs), Modified Rodrigues Parameters (MRPs), Euler Angles, and Quaternions. At a certain orientation, a CRP or MRP attitude description can result in a orientation singularity. When using Euler angles, certain orientations can result in non unique parameter values. While the other attitude descriptions have three parameter values, quaternions have four. Any orientation can be described using quaternions without an orientation singularity [7].

1.1.5 Implementation

In order to design a minimimum fuel/minimum time controller, the body will be rotated about the eigenaxis. A single rotation about the eigenaxis will prove to be the only necessary maneuver to achieve the desired orientation. Therefore, all feedback control laws will be designed in the eigenframe and then transformed back to the body frame. The equations of motion for an axisymmetric rigid body will be represented in the egin-frame, with the spin axis being the eigenvector. It will be demonstrated that in the eigen-frame, one of the equations will be a double integrator; therefore, a feedback control solution will be presented dependent on the cost function. Two cost functions will be implemented. One cost function will produce a minimum time/minimum fuel controller, and the other cost function will produce a minimum time/minimum energy controller. This control solution can then be substituted into the remaining equations to obtain a feedback control law for the reorientation of an axisymmetric rigid body. The two cost functions will be compared by observing their control cost. The control cost will be determined by investigating the control subject to the following cost functions:

$$C_1 = \int_0^{t_f} u^2 \mathrm{d}t$$

and

$$C_2 = \int_0^{t_f} |u| \mathrm{d}t$$

2. DYNAMICS

2.1 Dynamics

2.1.1 Eigenaxis Maneuvers

Wie and Bilimoria focused on reorienting a body by designing a minimum time controller based on an eigenaxis rotation [2]. They proved that an eigenaxis rotation is not a time optimal maneuver; however, the eigenaxis maneuver can be proven to be close to a time optimal maneuver. In their paper, they demonstrated an eigenaxis maneuver about the body's yaw axis. By performing this maneuver, they were able to show that an optimal maneuver will attain the desired orientation in 8.514% less time than an eigenaxis maneuver. While the eigenaxis may not necessarily be a time-optimal maneuver, it does provide the shortest angular trajectory between two orientations.

2.1.2 Notation

Since an eigenaxis maneuver yields the shortest angular path between two orientations, the minimum time eigenaxis maneuver will be investigated [6]. Three reference frames will be introduced in order to demonstrate a minimum-time eigenaxis maneuver. The frames will be the inertial frame, the body frame, and the eigenaxis reference frame. The inertial frame, body-frame, and eigenaxis reference frame will be denoted as $\mathbf{n}^+$, $\mathbf{b}^+$, and $\mathbf{e}^+$ respectively. In order to define the inertial and body frame, the following notation will be utilized.

$$[\mathbf{v}]_b = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T \tag{2.1}$$

$$[\mathbf{v}]_n = \begin{bmatrix} \acute{v}_1 & \acute{v}_2 & \acute{v}_3 \end{bmatrix}^T \tag{2.2}$$

The variable $\mathbf{v}$ denotes a vector and $[\mathbf{v}]_b$ describes the body's location in the body frame. Similarly, $[\mathbf{v}]_n$ is a vector that describes the body's location in the inertial frame.

2.1.3 Defining the Eigenaxis

The vectors $[\mathbf{v}]_b$ and $[\mathbf{v}]_n$ are related through an attitude matrix, $[C]$.

$$[\mathbf{v}]_b = [C][\mathbf{v}]_n \tag{2.3}$$

Since $[C]$ is a square matrix with orthonormal columns, the relation between the body frame and inertial frame can also be written as

$$[\mathbf{v}]_n = [C]^T[\mathbf{v}]_b \tag{2.4}$$

If the body's desired attitude is represented in the inertial reference frame, the eigenaxis that will relate the current attitude with the desired attitude will be the eigenvector of $[C]$ that has an eigenvalue of one associated with it [6].

$$[e] = [C][e] \tag{2.5}$$

This eigenvector contains the same vector components in the inertial and body frames.

$$[e]_b = [e]_n = [e] \tag{2.6}$$

The attitude matrix, C, can be paramertized using several different attitude descriptions. The chosen attitude description will be quaternions due to the quaternions non-singular nature. Therefore, C will be written in terms of the quaternions,

$(\beta_0, \beta_1, \beta_2, \beta_3)$, as

$$[C] = [1] - 2\beta_0 [\beta^x] + 2 [\beta^x] [\beta^x] \tag{2.7}$$

where

$$[B^x] = \begin{bmatrix} 0 & -\beta_3 & \beta_2 \\ \beta_3 & 0 & \beta_1 \\ -\beta_2 & \beta_1 & 0 \end{bmatrix} \tag{2.8}$$

with the property

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \tag{2.9}$$

The eigenvector of C associated with the eigenvalue of one can then be calculated.

$$[\mathbf{e}] = \begin{bmatrix} \frac{\beta_1}{\beta_3} & \frac{\beta_2}{\beta_3} & 1 \end{bmatrix} \tag{2.10}$$

Since the eigenaxis has unit length, the eigenvector is then normalized to compute the eigenaxis.

$$[\mathbf{e}] = \begin{bmatrix} \frac{\beta_1}{\beta} & \frac{\beta_2}{\beta} & \frac{\beta_3}{\beta} \end{bmatrix} \tag{2.11}$$

The variable β will be defined as

$$\beta = \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \tag{2.12}$$

2.1.4 The Eigenaxis Reference Frame

Now that the eigenaxis has been calculated a new reference frame can be defined, the eigenaxis reference frame. This eigenaxis reference frame will consist of a third unit vector direction, $\mathbf{e}_3$, parallel to the eigenaxis. All the control laws will be formulated in the eigenaxis reference frame. In order to relate the body frame to the eigenaxis reference frame, a new attitude matrix, $[D]$, is introduced.

$$[v]_e = [D][v]_b \tag{2.13}$$

The rows of $[D]$ will represent the eigenaxis reference frame axes along the body frame axes.

$$[D]^T = \begin{bmatrix} [e_1]_b & [e_2]_b & [e_3]_b \end{bmatrix} \tag{2.14}$$

Since the third row of $[D]$ is the eigenaxis, Equation 2.11 can be substituted into Equation 2.14.

$$[D]^T = \begin{bmatrix} & & \frac{\beta_1}{\beta} \\ [e_1]_b & [e_2]_b & \frac{\beta_2}{\beta} \\ & & \frac{\beta_3}{\beta} \end{bmatrix} \tag{2.15}$$

In order to form a right-handed coordinate system, the cross product of the eigenaxis and the 3-axis of the body frame will be executed to obtain $[\mathbf{e}_2]_b$.

$$[e_2]_b = \begin{bmatrix} \frac{\beta_2}{\beta_*^2} & -\frac{\beta_1}{\beta_*^2} & 0 \end{bmatrix}^T \tag{2.16}$$

where

$$\beta_* = \sqrt{\beta_1^2 + \beta_2^2} \tag{2.17}$$

The cross product can then be performed between the eigenaxis and $[\mathbf{e}_2]_b$ to form $[\mathbf{e}_1]_b$.

$$[e_1]_b = \begin{bmatrix} \frac{\beta_1\beta_3}{\beta_*^2\beta^2} & \frac{\beta_2\beta_3}{\beta_*^2\beta^2} & -\frac{\beta_*^2}{\beta^2} \end{bmatrix}^T \tag{2.18}$$

Equations 2.16 and 2.18 can then be substituted into Equation 2.15 to form the attitude matrix $[D]$.

$$[D] = \begin{bmatrix} \frac{\beta_1\beta_3}{\beta_*\beta} & \frac{\beta_2\beta_3}{\beta_*\beta} & -\frac{\beta_*}{\beta} \\ \frac{-\beta_2}{\beta_*} & \frac{\beta_1}{\beta_*} & 0 \\ \frac{\beta_1}{\beta} & \frac{\beta_2}{\beta} & \frac{\beta_3}{\beta} \end{bmatrix} \tag{2.19}$$

If the inertia, control, angular velocity, and angular momentum is known in the body frame, the $[D]$ attitude matrix can be applied to represent each parameter in the eigenaxis reference frame [6].

$$[\mathbf{I}]_e = [D][\mathbf{I}]_b[D]^T \tag{2.20}$$

$$[\mathbf{u}]_e = [D][\mathbf{u}]_b \tag{2.21}$$

$$[\mathbf{h}]_e = [D][\mathbf{h}]_b \tag{2.22}$$

$$[\boldsymbol{\omega}]_e = [D][\boldsymbol{\omega}]_b \tag{2.23}$$

2.1.5 Equations of Motion in the Eigenaxis Frame

The eigenaxis vector componenets will be denoted as follows:

$$\mathbf{v} = \tilde{v}_1\mathbf{e}_1 + \tilde{v}_2\mathbf{e}_2 + \tilde{v}_3\mathbf{e}_3 \tag{2.24}$$

Therefore, the governing rigid body equations of motion in the eigenaxis reference frame are

$$\dot{\tilde{h}}_1 + \tilde{\omega}_2\tilde{h}_3 - \tilde{\omega}_3\tilde{h}_2 = \tilde{u}_1 \tag{2.25}$$

$$\dot{\tilde{h}}_2 + \tilde{\omega}_3\tilde{h}_1 - \tilde{\omega}_1\tilde{h}_3 = \tilde{u}_2 \tag{2.26}$$

$$\dot{\tilde{h}}_3 + \tilde{\omega}_1\tilde{h}_2 - \tilde{\omega}_2\tilde{h}_1 = \tilde{u}_3 \tag{2.27}$$

However, since the body is required to spin about the eigenaxis, the angular velocities $\tilde{\omega}_1$ and $\tilde{\omega}_2$ must equal zero.

$$\dot{\tilde{h}}_1 - \tilde{\omega}_3 \tilde{h}_2 = \tilde{u}_1 \tag{2.28}$$

$$\dot{\tilde{h}}_2 - \tilde{\omega}_3 \tilde{h}_1 = \tilde{u}_2 \tag{2.29}$$

$$\dot{\tilde{h}}_3 = \tilde{u}_3 \tag{2.30}$$

Since the equation for angular momentum is

$$\mathbf{h} = \mathbf{I}\boldsymbol{\omega} \tag{2.31}$$

and the body is an axisymmetric body, the inertia matrix will be

$$[\mathbf{I}]_b = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{2.32}$$

However since ω_1 and ω_2 both equal zero, the angular momentum in the eigenaxis frame can be written as

$$[\mathbf{h}]_e = \begin{bmatrix} \tilde{h}_1 \\ \tilde{h}_2 \\ \tilde{h}_3 \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} \tilde{\omega}_3 \tag{2.33}$$

The components $[J_1, J_2, J_3]^T$ can be calculated by creating the matrix $[\mathbf{I}]_e$ using Equation 2.20. The third column of $[\mathbf{I}]_e$ will be the values of $[J_1, J_2, J_3]^T$ in terms of the quaternions and elements of the inertia matrix.

$$J_1 = \frac{I_1\beta_1^2\beta_3}{\beta_*\beta^2} + \frac{I_2\beta_2^2\beta_3}{\beta_*\beta^2} - \frac{I_3\beta_*\beta_3}{\beta^2} \tag{2.34}$$

$$J_2 = (I_1 - I_2)\frac{\beta_1\beta_2}{\beta_*\beta} \tag{2.35}$$

$$J_3 = \frac{I_1\beta_1^2}{\beta^2} + \frac{I_2\beta_2^2}{\beta^2} + \frac{I_3\beta_3^2}{\beta^2} \tag{2.36}$$

Therefore, the equations of motion in the eigenaxis frame written in terms of the J inertia values are

$$J_1\dot{\tilde{\omega}}_3 - J_2\tilde{\omega}_3^2 = \tilde{u}_1 \tag{2.37}$$

$$J_2\dot{\tilde{\omega}}_3 + J_1\tilde{\omega}_3^2 = \tilde{u}_2 \tag{2.38}$$

$$J_3\dot{\tilde{\omega}}_3 = \tilde{u}_3 \tag{2.39}$$

Utilizing Equation 2.39, a minimum time/fuel feedback controller will be developed. The control developed for $\tilde{u}_3$ will then be used to develop the controls, $\tilde{u}_1$ and $\tilde{u}_2$.

3. MIN TIME-FUEL CONTROL DESIGN

3.1 Forming the State-Space

The plant in Equation 2.39 can be characterized as a double integrator transfer function. An optimal control approach can be applied to solve the minimimum time/fuel controller. The plant and cost function will be of the following form [11]:

$$\dot{x} = f(x, u, t) \tag{3.1}$$

$$J = \int_0^{t_f} L(x, u, t)\mathrm{d}t \tag{3.2}$$

If an intermediate variable, ϕ, is introduced denoting the angular position of the body along the eigenaxis reference frame's $\mathbf{e}_3$ axis, Equation 2.39 can be reduced to

$$x_1 = \phi \tag{3.3}$$

$$x_2 = \dot{\phi} = \tilde{\omega}_3 \tag{3.4}$$

From Equations 3.3 and 3.4, the following state-space model can be formed from Equation 2.39:

$$\dot{x}_1 = x_2 \tag{3.5}$$

$$\dot{x}_2 = u \tag{3.6}$$

The control will also be constrained so that

$$|u(t)| \leq 1 \tag{3.7}$$

3.2 Optimal Control

The cost function for a minimum time/fuel controller is as follows [5]:

$$min\ J = \int_0^{t_f} (1 + b|u(t)|)\mathrm{d}t \tag{3.8}$$

where b will be the control weight. If b is set to zero, the cost function will represent a minimum time optimal control problem; if b is set to infinity, the cost function will represent a minimimum fuel problem. The cost function can also be written as [1]:

$$J = \int_0^{t_f} (k + |u(t)|)\mathrm{d}t \tag{3.9}$$

where k is equal to $\frac{1}{b}$. The Hamiltonian can then be formed [11].

$$H(x, u, t) = L(x, u, t) + \lambda^T f(x, u, t) \tag{3.10}$$

$$H(x, u, t) = k + |u(t)| + \lambda_1(t)x_2(t) + \lambda_2(t)u(t) \tag{3.11}$$

From the Hamiltonian, the co-state differential equations can now be composed.

$$\dot{\lambda}_1(t) = -\frac{dH}{dx_1(t)} = 0 \tag{3.12}$$

$$\dot{\lambda}_2(t) = -\frac{dH}{dx_2(t)} = -\lambda_1(t) \tag{3.13}$$

Therefore, the algebraic equations for the co-states are as follows:

$$\lambda_1(t) = c_1 \tag{3.14}$$

$$\lambda_2(t) = -c_1 t + c_2 \tag{3.15}$$

where c_1 and c_2 are constants.

Applying Pontrayagin's Minimum Principle [11] to the Hamiltonian, the open-loop control is as follows:

$$u_(t) = 0 \qquad \text{if } |\lambda_2(t)| < 1 \tag{3.16}$$

$$u_(t) = -sgn|\lambda_2(t)| \qquad \text{if } |\lambda_2(t)| > 1 \tag{3.17}$$

$$0 \leq u(t) \leq 1 \qquad \text{if } \lambda_2(t) = -1 \tag{3.18}$$

$$-1 \leq u(t) \leq 0 \qquad \text{if } \lambda_2(t) = 1 \tag{3.19}$$

Since there is no condition when $|\lambda_2(t)|$ equals one during some time interval, it can be demonstrated that this condition will never exist. In order to prove this, suppose $\lambda_2(t) = -1$ at time t within the interval $[t_1, t_2]$. Substituting these terms into the Hamiltonian yields:

$$H = k + |u(t)| + c_1 x_2(t) - u(t) \tag{3.20}$$

Since $\lambda_2(t)$ is equal to negative one, u(t) will be between zero and one; therefore, the Hamiltonian can be simplified.

$$H = k + c_1 x_2(t) \tag{3.21}$$

Since $\lambda_2(t) = -1$, then using Equation 3.15

$$c_1 = 0 \tag{3.22}$$

$$c_2 = -1 \tag{3.23}$$

Substituting the values for c_1 and c_2, the Hamiltonian becomes

$$H = k \tag{3.24}$$

with $k > 0$. However, since the cost function and state-space are not explicit functions of time and the final time is not fixed, the Hamiltonian must equal zero. Therefore, the Hamiltonian can never equal k. The same logic can be applied for $\lambda_2(t)$ equal to one. In conclusion, $|\lambda_2(t)|$ will never equal one during a certain time interval.

3.3 Valid Control Sequences

In order to design an optimal controller, nine control sequences will be considered. The control sequences will be

$$[0], [1], [-1], \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \tag{3.25}$$

Since the body will perform a rest-to-rest maneuver, $(0, 0)$ are the desired final states, resulting in only six of the nine sequences being valid. In order to optimally drive the states to the origin, it will be shown that the sequences $[\ 0\ ]$, $\begin{bmatrix} 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} -1 & 0 \end{bmatrix}$ are invalid.

Each of the invalid control sequences contains one common characteristic which is

$$u(t_f) = 0 \tag{3.26}$$

Therefore, at the final time the Hamiltonian will be

$$H(t_f) = k + |u(t_f)| + x_2(t_f)c_1 + u(t_f)\lambda_2(t_f) \tag{3.27}$$

Since at the final state

$$x_2(t_f) = 0 \qquad\qquad u(t_f) = 0 \tag{3.28}$$

the Hamiltonian will be

$$H(t_f) = k \tag{3.29}$$

These sequences are invalid because the Hamiltonian does not equal zero. The principle of optimality [1] states that if a control sequence is optimal, then all of its subsequences are also optimal. Since [1] and $[\begin{array}{cc} 0 & 1 \end{array}]$ are subsequences of $[\begin{array}{ccc} -1 & 0 & 1 \end{array}]$ and [-1] and $[\begin{array}{cc} 0 & -1 \end{array}]$ are subsequences of $[\begin{array}{ccc} 1 & 0 & -1 \end{array}]$, the principle of optimality proves that all subsequences are valid optimal sequences.

3.4 Switching Curves

3.4.1 Switching Curves for [-1 0 1] Control Sequence.

Assuming the following conditions:

$$c_1 > 0 \qquad\qquad c_2 > 1 \tag{3.30}$$

The costate $\lambda_2(t)$ with corresponding control sequence plot is generated in Figure 3.1. From the function $\lambda_2(t)$, a control sequence of $[\begin{array}{ccc} -1 & 0 & 1 \end{array}]$ is generated. Therefore, the control as a function of time is

$$u(t) = \begin{cases} -1, & 0 \leq t < t_1 \\ 0, & t_1 \leq t < t_2 \\ 1, & t_2 \leq t \leq t_f \end{cases} \tag{3.31}$$

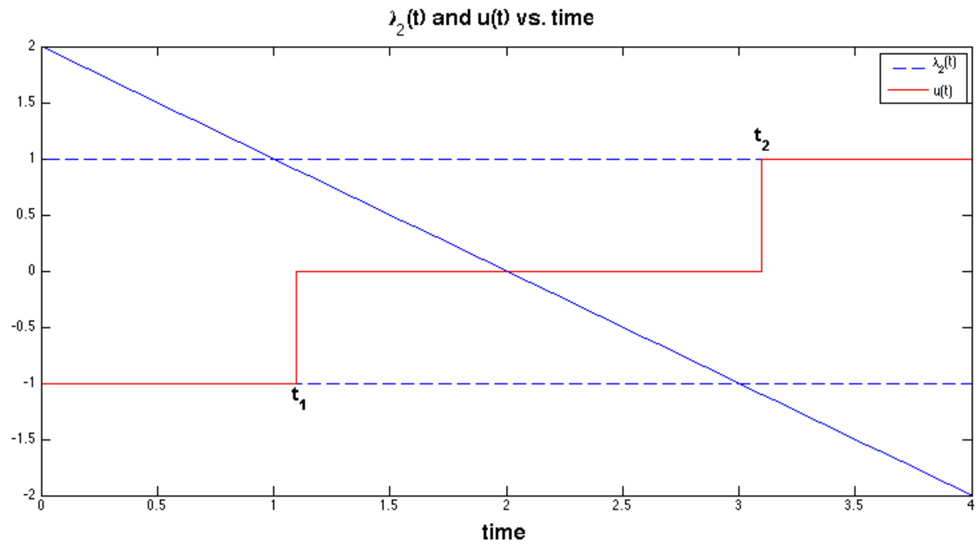


Figure 3.1: $\lambda_2(t)$ with corresponding u(t)

Also, at the switching times, t_1 and t_2, the costates can be simplified to

$$\lambda_2(t_1) = 1 = c_2 - c_1 t_1 \tag{3.32}$$

$$\lambda_2(t_2) = -1 = c_2 - c_1 t_2 \tag{3.33}$$

Suppose the state $\{\zeta_1, \zeta_2\}$ represents an initial state and Equation 3.4.1 is applied. The state $\{z_1, z_2\}$ will be at the switching time t_1, and the state $\{w_1, w_2\}$ will be at the switching time t_2. Since the control is constant over a specific time interval, Equation 3.1 can be solved with control equal to a constant and the initial conditions

$$x_1(0) = \zeta_1 \qquad\qquad x_2(0) = \zeta_2 \tag{3.34}$$

From zero to t_1 the control will equal negative one. Therefore, the states at the switching time, t_1, will be represented as

$$\int_{\zeta_2}^{z_2} \mathrm{d}x_2 = -\int_0^{t_1} \mathrm{d}t$$

$$\boxed{z_2 = \zeta_2 - t_1} \tag{3.35}$$

$$\int_{\zeta_1}^{z_1} \mathrm{d}x_1 = \int_0^{t_1} \{\zeta_2 - t\}\mathrm{d}t$$

$$\boxed{z_1 = \zeta_1 + \zeta_2 t_1 - \frac{1}{2}t_1^2} \tag{3.36}$$

The control from t_1 to t_2 will be equal to zero. The states at the switching time, t_2, can be expressed as

$$\int_{z_2}^{w_2} \mathrm{d}x_2 = \int_{t_1}^{t_2} 0\mathrm{d}t$$

$$\boxed{w_2 = z_2} \tag{3.37}$$

$$\int_{z_1}^{w_1} \mathrm{d}x_1 = \int_{t_1}^{t_2} z_2 \mathrm{d}t$$

$$\boxed{w_1 = z_1 + z_2(t_2 - t_1)} \tag{3.38}$$

From t_2 to the final time, the control will be one. Expressions for the final states are

$$\int_{w_2}^{0} \mathrm{d}x_2 = \int_{t_2}^{t_f} \mathrm{d}t$$

$$\boxed{0 = w_2 + t_f - t_2} \tag{3.39}$$

$$\int_{w_1}^{0} \mathrm{d}x_1 = \int_{t_2}^{t_f} [w_2 + t_f - t]\mathrm{d}t$$

$$\boxed{0 = w_1 + w_2(t_f - t_2) + \frac{1}{2}(t_f - t_2)^2} \tag{3.40}$$

The resulting relation from substituting Equation 3.39 into 3.40 is

$$w_1 = \frac{1}{2}w_2^2 \tag{3.41}$$

Utilizing Equation 3.38 and 3.37, the following realtion is obtained:

$$\frac{1}{2}z_2^2 = z_1 + z_2(t_2 - t_1) \tag{3.42}$$

When time equals the first switching time the states, costates, and control shall be

$$x_1(t_1) = z_1 \tag{3.43}$$

$$x_2(t_1) = z_2 \tag{3.44}$$

$$\lambda_2(t_1) = 1 \tag{3.45}$$

$$u(t_1) \leq 0 \tag{3.46}$$

Since the Hamiltonian must equal zero over all time, substituting Equations 3.43 through 3.46 into the Hamiltonian yields c_1 as a function of z_2.

$$H(t_1) = k + |u(t_1)| + \lambda_2(t_1)x_2(t_1) + \lambda_2(t_1)u(t_1) \tag{3.47}$$

$$= k + z_2c_1 = 0 \tag{3.48}$$

$$\boxed{c_1 = -\frac{k}{z_2}} \tag{3.49}$$

Subtracting Equation 3.33 from Equation 3.32 results in the following equation:

$$2 = c_1(t_2 - t_1) \tag{3.50}$$

Substituting Equation 3.50 and Equation 3.49 into Equation 3.42 results in

$$\frac{1}{2}z_2^2 = z_1 - \frac{2z_2^2}{k} \tag{3.51}$$

which simplifies to

$$z_1 = g_k z_2^2 \tag{3.52}$$

where

$$g_k = \frac{k+4}{2k} \tag{3.53}$$

The set of states (z_1, z_2) that describe the state trajectory from the control sequence $\{\ -1 \quad 0 \quad -1\ \}$ up to and including when the control switches from negative one to zero are displayed in Equations 3.52 and 3.53.

3.4.2 Switching Curves for a [1 0 -1] Control Sequence

In order to derive the equation that describes the control that switches from one to zero from the control sequence $\{\ 1 \quad 0 \quad -1\ \}$, new states, $(\zeta_1', \zeta_2', w_1', w_2', z_1', z_2')$, will be introduced. From the costate $\lambda_2(t)$, the control can be generated from the control sequence $[\ 1 \quad 0 \quad -1\ ]$. This can be seen graphically in Figure 3.2.

$$u(t) = \begin{cases} 1, & 0 \leq t < t_1 \\ 0, & t_1 \leq t < t_2 \\ -1, & t_2 \leq t \leq t_f \end{cases} \tag{3.54}$$

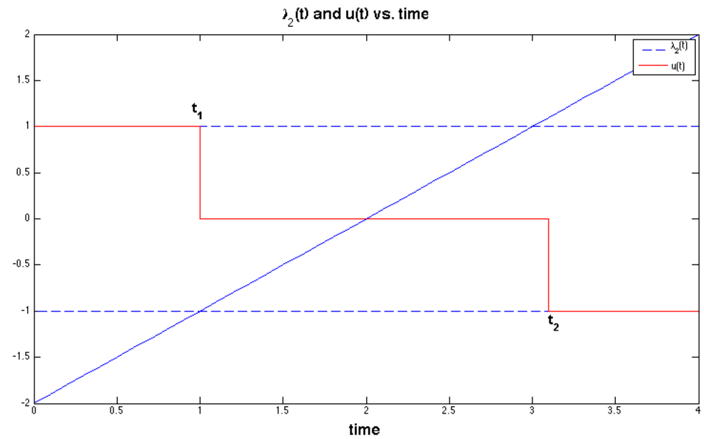


Figure 3.2: Costate $\lambda_2(t)$ and corresponding control

From the Figure 3.2, the slope and y-intercept of $\lambda_2(t)$ at the switching can be simplified to Equations 3.150 and 3.151.

$$\lambda_2(t_1) = -1 = c_2 - c_1 t_1 \tag{3.55}$$

$$\lambda_2(t_2) = 1 = c_2 - c_1 t_2 \tag{3.56}$$

Substituting the control sequence $\{\begin{array}{ccc} 1 & 0 & -1 \end{array}\}$ into the state space results in the following equations for the intermediate states. Since $u = 1$ from the interval $[\begin{array}{cc} 0 & t_1 \end{array}]$, z_1' and z_2' can be represented as a function of the initial conditions, ζ_1' and ζ_2'.

$$\int_{\zeta_2'}^{z_2'} \mathrm{d}x_2 = \int_0^{t_1} \mathrm{d}t$$

$$\boxed{z_2' = \zeta_2' + t_1} \tag{3.57}$$

$$\int_{\zeta_1'}^{z_1} \mathrm{d}x_1 = \int_0^{t_1} \{\zeta_2' + t\}\mathrm{d}t$$
$$\boxed{z_1' = \zeta_1' + \zeta_2' t_1 + \frac{1}{2}t_1^2} \tag{3.58}$$

From the interval [t_1 t_2], the control will be equal to zero. As can be seen in Equations 3.60 and 3.59, w_1 and w_2 can be represented as functions of the intermediate states z_1 and z_2.

$$\int_{z_2'}^{w\prime_2} \mathrm{d}x_2 = \int_{t_1}^{t_2} 0\mathrm{d}t$$
$$\boxed{w_2' = z_2'} \tag{3.59}$$

$$\int_{z_1'}^{w_1'} \mathrm{d}x_1 = \int_{t_1}^{t_2} z_2'\mathrm{d}t$$
$$\boxed{w_1' = z_1' + z_2'(t_2 - t_1)} \tag{3.60}$$

From time interval [t_2 t_f], the control is equal to negative one, and the states w_1 and w_2 can be written in terms of the final time and the second switching time.

$$\int_{w_2'}^{0} \mathrm{d}x_2 = -\int_{t_2}^{t_f} \mathrm{d}t$$
$$\boxed{0 = w_2' - t_2 + t_f} \tag{3.61}$$

$$\int_{w_1'}^{0} \mathrm{d}x_1 = \int_{t_2}^{t_f} \{w_2 + t_2 - t\}\mathrm{d}t$$
$$\boxed{0 = w_1' + w_2'(t_2 - t_f) - \frac{1}{2}(t_f - t_2)^2} \tag{3.62}$$

In order to find w_1 as a function of w_2, Equation 3.59 will be substituted into Equation 3.60.

$$w_1' = -\frac{1}{2}w_2' \tag{3.63}$$

Utilizing Equations 3.60 and 3.59, the following relation can be obtained.

$$-\frac{1}{2}z_2' = z_1' + z_2'(t_2 - t_1) \tag{3.64}$$

Since the Hamiltonian must be zero at all times, the following conditions will be applied when t equals t_1.

$$x_1(t_1) = z_1' \qquad x_2(t_1) = z_2' \tag{3.65}$$

$$\lambda_2(t_1) = -1 \qquad u(t_1) \geq 0 \tag{3.66}$$

These conditions can now be substituted into the Hamiltonian in Equation 3.20.

$$H(t_1) = k + |u(t_1)| + \lambda_1(t_1)x_2(t_1) + \lambda_2(t_1)u(t_1) \tag{3.67}$$

$$= k + c_1 z_2' = 0 \tag{3.68}$$

$$\boxed{c_1 = -\frac{k}{z_2'}} \tag{3.69}$$

Equation 3.151 can then be subtracted from Equation 3.150 in order to obtain Equation 3.70.

$$-2 = c_1(t_2 - t_1) \tag{3.70}$$

Equations 3.69 and 3.70 can be substituted into Equation 3.52 in order to form Equation 3.71.

$$z_1' = -g_k z_2'^2 \tag{3.71}$$

where g_k is defined in Equation 3.53.

3.5 Forming the Feedback Control Law

Equations 3.71 and 3.52 can then be combined to form the curve Γ_k which is defined in Equation 3.72.

$$\Gamma_k: \; x_1 = -g_k x_2 |x_2| \tag{3.72}$$

The next curve that is necessary to define in order to obtain the feedback control law is the curve γ. In order to form this curve, the state space will be solved with $u(t)$ equal to one and negative one. In the first case let

$$u(t) = 1 \tag{3.73}$$

which results in states of the form

$$x_2(t) = t + c_3 \tag{3.74}$$

$$x_1(t) = \frac{1}{2}t^2 + c_3 t + c_4 \tag{3.75}$$

where c_3 and c_4 are constants. Since x_1 and x_2 both equal zero at the final time, the constants c_3 and c_4 are able to be solved. Substituting the condition, $x_2(t_f) = 0$ into Equation 3.74

$$0 = t_f + c_3$$

$$\boxed{c_3 = -t_f} \tag{3.76}$$

and applying the condition, $x_1(t_f) = 0$ to Equation 3.75

$$\frac{1}{2}t_f^2 - t_f^2 + c_4 = 0$$

$$\boxed{c_4 = \frac{1}{2}t_f^2} \tag{3.77}$$

Both values, c_3 and c_4, will be substituted into the states $x_1(t)$ and $x_2(t)$.

$$x_1(t) = \frac{1}{2}(t - t_f)^2 \tag{3.78}$$

$$x_2(t) = t - t_f \tag{3.79}$$

Substituting 3.79 into 3.78 results in a description of the trajectory in the x_1-x_2 plane that is completely independent of the tunable parameter k.

$$x_1(t) = \frac{1}{2}x_2(t)^2 \tag{3.80}$$

Next, take into the cosideration the case where

$$u(t) = -1 \tag{3.81}$$

Therefore, the state-space will be of the form

$$\dot{x}_1 = x_2 \tag{3.82}$$

$$\dot{x}_2 = -1 \tag{3.83}$$

Solving the state-space results in Equations 3.84 and 3.85.

$$x_1(t) = -t + c_3 \tag{3.84}$$

$$x_2(t) = -\frac{1}{2}t^2 + c_3 t + c_4 \tag{3.85}$$

Since, at the final time

$$x_1(t_f) = 0 \qquad\qquad x_2(t_f) = 0 \tag{3.86}$$

the unknown constants c_3 and c_4 are able to be solved.

$$0 = -t_f + c_3$$

$$\boxed{c_3 = t_f} \tag{3.87}$$

$$0 = -\frac{1}{2}t_f^2 + c_3 t_f + c_4$$

$$\boxed{c_4 = -\frac{1}{2}t_f^2} \tag{3.88}$$

Substituting the final conditions into the state equations yields Equations 3.89 and 3.90.

$$x_1(t) = -\frac{1}{2}t^2 + t_f t - \frac{1}{2}t_f^2 = -\frac{1}{2}(t - t_f)^2 \tag{3.89}$$

$$x_2(t) = -t + t_f \tag{3.90}$$

Equation 3.90 can be substituted into Equation 3.89, to yield another result in the

x_1-x_2 plane that is independent of the tunable parameter k.

$$x_1(t) = -\frac{1}{2}x_2(t) \tag{3.91}$$

Equations 3.80 and 3.91 can then be combined to form the curve γ.

$$x_1(t) = -\frac{1}{2}x_2|x_2| \tag{3.92}$$

From Equation 3.72 and Equation 3.92, four different regions can be defined. Figure 3.3 shows the curves γ and Γ_k with four different values of k. The four different regions, H_1, H_2, H_3, and H_4, will be displayed in Figure 3.4.

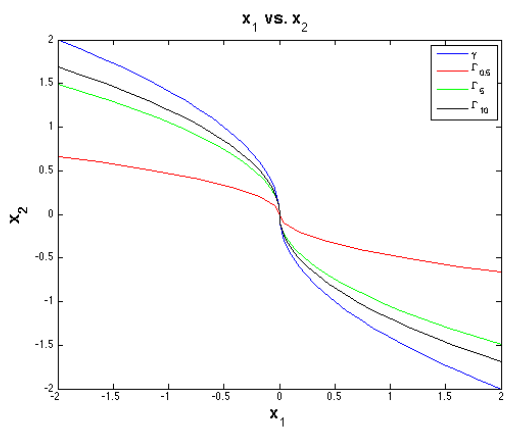


Figure 3.3: Switching curves for γ and Γ_k for varying k values

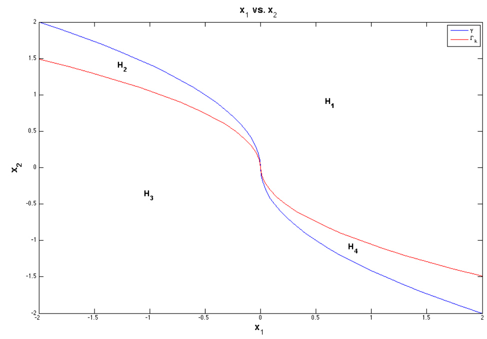


Figure 3.4: Regions of H set by the curves γ and Γ_k

Therefore, the four regions will be algebraically defined in Equation 3.93.

$$\begin{aligned}
H_1 &= \{(x_1, x_2) : x_1 \geq -\frac{1}{2}x_2|x_2| \text{ and } x_1 \geq -g_k x_2|x_2|\} \\
H_2 &= \{(x_1, x_2) : x_1 < -\frac{1}{2}x_2|x_2| \text{ and } x_1 \geq -g_k x_2|x_2|\} \\
H_3 &= \{(x_1, x_2) : x_1 \leq -\frac{1}{2}x_2|x_2| \text{ and } x_1 < -g_k x_2|x_2|\} \\
H_4 &= \{(x_1, x_2) : x_1 > -\frac{1}{2}x_2|x_2| \text{ and } x_1 \leq -g_k x_2|x_2|\}
\end{aligned} \tag{3.93}$$

The feedback control law can then be formed in Equation 3.94

$$u^*(x_1, x_2) = \begin{cases} -1, & (x_1, x_2) \in H_1 \\ 1, & (x_1, x_2) \in H_3 \\ 0, & (x_1, x_2) \in H_2 \cup H_4 \end{cases} \tag{3.94}$$

In order to prove the validity of the control feedback law in Equation 3.94, states will be chosen in the regions H_1, H_2, H_3, and H_4. The Hamiliton will be zero along the optimal trajectory and all control sequences in Equation 3.25 will be attempted. Equation 3.94 is the same as Athans' and Falb's feedback control solution to the optimal fuel bounded control problem [1], Equations 8.216 through 8.219, with the exception that m_β is substituted with g_k. This can be verified by the following limits:

$$\lim_{k\to+\infty} g_k = \frac{1}{2} \qquad \lim_{k\to 0} g_k \to \infty \tag{3.95}$$

These limits prove that

$$\lim_{k\to\infty} \Gamma_k \to \gamma \tag{3.96}$$

$$\lim_{k\to 0} \Gamma_k \to x_1 axis \tag{3.97}$$

From the cost function in Equation 3.9, as k $\to \infty$, the feedback control in Equation 3.94 transforms into a time-optimal control law. Let R_- and R_+ be defined in Equations 3.98 and 3.99.

$$R_- = \{(x_1, x_2) : x_1 > -\frac{1}{2}x_2|x_2|\} \tag{3.98}$$

$$R_+ = \{(x_1, x_2) : x_1 < -\frac{1}{2}x_2|x_2|\} \tag{3.99}$$

From Athans' and Falb's time optimal control law [1], Equation 3.94 is reduced to

$$\lim_{k\to\infty} H_1 = R_- \cup \gamma_- \tag{3.100}$$

$$\lim_{k\to\infty} H_3 = R_+ \cup \gamma_+ \tag{3.101}$$

$$\lim_{k\to\infty} H_2 \cup H_4 = \emptyset \tag{3.102}$$

However, if k $\to$ 0, then Equation 3.9 results in a fuel-optimal controller. This is evident by letting k $\rightarrow 0$ in Equation 3.94. Let R_1 through R_4 be defined as

$$R_1 = \{(x_1, x_2) : x_2 \geq 0; x_1 > x_1', \text{ where } (x_1', x_2) \in \gamma_-\} \tag{3.103}$$

$$R_2 = \{(x_1, x_2) : x_2 > 0; x_1 < x_1', \text{ where } (x_1', x_2) \in \gamma_-\} \tag{3.104}$$

$$R_3 = \{(x_1, x_2) : x_2 \leq 0; x_1 \leq x_1', \text{ where } (x_1', x_2) \in \gamma_+\} \tag{3.105}$$

$$R_4 = \{(x_1, x_2) : x_2 < 0; x_1 > x_1', \text{ where } (x_1', x_2) \in \gamma_+\} \tag{3.106}$$

With the limits

$$\lim_{k \to 0} H_1 = R_1 \tag{3.107}$$

$$\lim_{k \to 0} H_3 = R_3 \tag{3.108}$$

$$\lim_{k \to 0} H_2 \cup H_4 = R_2 \cup R_4 \tag{3.109}$$

the control law defined in Equation 3.94 is equivalent to the optimal fuel control law developed by Athans and Falb defined in Equation 8.2. Therefore, the control law in Equation 3.94, will result in a fuel-optimal controller as k $\rightarrow 0$. Hence, proving the validity at the extremities of k, results in a valid control law detailed in Equation 3.94. Since the feedback control law has been designed, the only objective remaining is deriving a solution for the final time.

3.6 Final Time Calculation

3.6.1 Final Time Calculation for [-1 0 1] Control Sequence

In order to derive this result, the cost function in Equation 3.2 will be considered [5]. The Hamiltonian can then be formed in Equation 3.110 from the cost function in Equation 3.2 with the same state-space, Equations 3.5 and 3.6.

$$H(\mathbf{x}, u, t) = 1 + b|u(t)| + \lambda_1(t)x_2(t) + \lambda_2(t)u(t) \tag{3.110}$$

The costates will remain the same form as described in Equations 3.14 and 3.15. Utilizing Pontrayagin's Minimimum Principle, the open loop control will be

$$u(t) = \begin{cases} -1, & b < \lambda_2(t) \\ 0, & -b < \lambda_2(t) < b \\ 1, & \lambda_2(t) < -b \cup H_4 \end{cases} \tag{3.111}$$

Therefore, if c_1 is greater than zero, the costate as a function of time is displayed in Figure 3.5.

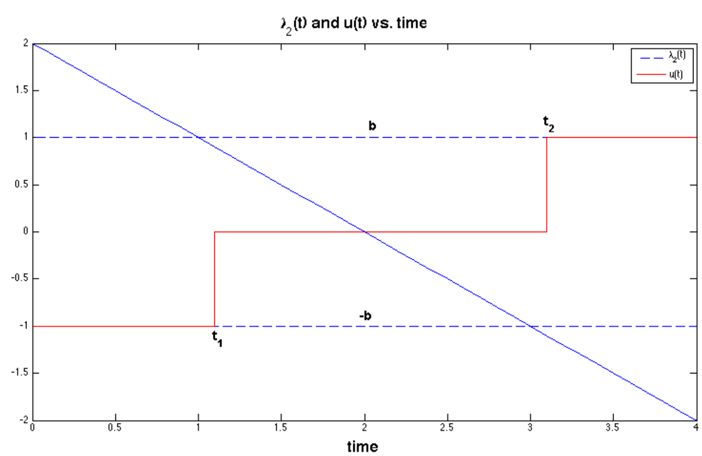


Figure 3.5: $\lambda_2(t)$ and u(t) using cost function Equation 3.2

The Hamiltonian will then be analyzed at the final time.

$$H(\mathbf{x}, u, t_f) = 1 + b|u(t_f)| + \lambda_1(t_f)x_2(t_f) + \lambda_2(t_f)u(t_f) \tag{3.112}$$

$$= 1 + b + \lambda_2(t_f) = 0 \tag{3.113}$$

$$\boxed{\lambda_2(t_f) = -(1 + b)} \tag{3.114}$$

Substituting Equation 3.114 and analyzing the $\lambda_2(t)$ in Equation 3.15 at the final time will result in the following relation.

$$\lambda_2(t_f) = c_2 - c_1 t_f \tag{3.115}$$

$$= c_2 - c_1 t_f = -(1 + b) \tag{3.116}$$

$$\boxed{c_2 = c_1 t_f - 1 - b} \tag{3.117}$$

From Figure 3.5, the costate λ_2 at the switching times are as follows:

$$\lambda_2(t_1) = b \tag{3.118}$$

$$\lambda_2(t_2) = -b \tag{3.119}$$

Substituting Equations 3.118 and 3.117 into Equation 3.15 analyzed at t_1, the switching time can be calculated as a function of t_f and c_1.

$$\lambda_2(t_1) = b = c_2 - c_1 t_1 \tag{3.120}$$

$$= b = c_1 t_f - 1 - b - c_1 t_1 \tag{3.121}$$

$$\boxed{t_1 = t_f - \frac{2b + 1}{c_1}} \tag{3.122}$$

Also, Equations 3.119 and 3.117 can be substituted into Equation 3.15 analyzed at

t_2 to calculate t_2 as a function of t_f and c_1.

$$\lambda_2(t_2) = -b = c_2 - c_1 t_2 \tag{3.123}$$

$$-b = c_1 t_f - 1 - b - c_1 t_2 \tag{3.124}$$

$$\boxed{t_2 = t_f - \frac{1}{c_1}} \tag{3.125}$$

Assuming the control sequence $[\begin{array}{ccc} -1 & 0 & 1 \end{array}]$, from t_2 to t_f $u(t)$ will equal one. Equation 3.78 can be used to calculate $x_1(t)$ at t_2.

$$x_1(t_2) = \frac{1}{2}(t_2 - t_f)^2 \tag{3.126}$$

Equation 3.125 can be applied so that $x_1(t_2)$ is in terms of c_1 at time t_2.

$$x_1(t_2) = \frac{1}{2}(t_2 - t_f)^2 = \frac{1}{2c_1^2} \tag{3.127}$$

Equation 3.125 can also be applied so that $x_2(t_2)$ is in terms of c_1 at time t_2.

$$x_2(t_2) = t_2 - t_f = -\frac{1}{c_1} \tag{3.128}$$

During the coasting phase, $x_2(t)$ at the switching times are equal.

$$x_2(t_1) = x_2(t_2) = -\frac{1}{c_1} \tag{3.129}$$

Since the control from t_0 to t_1 be negative one, Equation 3.90 analyzed at t_1 can be set equal to Equation 3.129.

$$x_2(t_1) = -t_1 + x_2(0) = -\frac{1}{c_1} \tag{3.130}$$

Substituting Equation 3.122 into Equation 3.130 results in the following expression for c_1.

$$-(t_f - \frac{2b+1}{c_1}) + x_2(0) = -\frac{1}{c_1}$$
$$\boxed{c_1 = \frac{2(b+1)}{t_f - x_2(0)}} \tag{3.131}$$

When the control is switching from zero to one, the following relation can be made by solving for $(t_2 - t_f)$ in Equation 3.125 and substituting it into Equation 3.84 analyzed at t_2.

$$x_1(t_2) = \frac{1}{2}(t_2 - t_f)^2 = \frac{1}{2c_1^2} \tag{3.132}$$

A indefenite integral can be formed from Equation 3.128 and the unknown constant can be solved by the relation given in Equation 3.132.

$$x_1(t_1) = \frac{1}{c_1}(t_2 - t_1) + \frac{1}{2c_1^2} \tag{3.133}$$

Equations 3.122 and 3.125 can then be substituted into Equation 3.132 to form a relation between x_1 and x_2 that describes the curve when the control switches from zero to one.

$$x_1(t) = (2b + \frac{1}{2})x_2^2(t) \tag{3.134}$$

The expression $(2b + \frac{1}{2})$ will be named α. Substituting Equation 3.131 into Equation 3.122 yields the following result:

$$t_1 = t_f - \frac{2b+1}{2(b+1)(t_f - x_2(0))} \tag{3.135}$$

t_1 can now be expressed in terms of t_f.

$$t_2 = \frac{1}{2(b+1)} t_f + \frac{2b+1}{2(b+1)} x_2(0) \tag{3.136}$$

A final time quadratic equation can be formed using Equation 3.134 analyzed at t_1 and substituting the expressions for $x_1(t_1)$ and $x_2(t_2)$. Since control is negative one from t_0 to t_1, the states at t_1 will be of the form.

$$x_1(t_1) = -\frac{1}{2} t_1^2 + x_2(0) t_2 + x_1(0) \tag{3.137}$$

$$x_2(t_1) = -t_2 + x_2(0) \tag{3.138}$$

Equation 3.139 can then be substituted into Equations 3.137 and 3.138 to form expressions for $x_1(t_1)$ and $x_2(t_1)$ in terms of t_f. Equation 3.136 can then be substituted into Equation 3.84 to form the following expression.

$$x_1(t_1) = \frac{1}{8(b+1)^2} \left(A t_f + B t_f + C \right) \tag{3.139}$$

where

$$A = -1 \tag{3.140}$$

$$B = 2x_2(0) \tag{3.141}$$

$$C = 4b^2 + x_2(0)^2 + 8x_1(0)b^2 + 8bx_2(0)^2 + 16x_1(0)b + 3x_2(0)^2 + 8x_1(0) \tag{3.142}$$

$$x_2(t_1) = -t_f + \frac{2b+1}{2(b+1)(t_f - x_2(0)} + x_2(0) \tag{3.143}$$

Substituting Equations 3.139 and 3.143 into Equation 3.134 analyzed at t_1 yields a quadratic equation in terms of t_f.

$$\begin{aligned} x_1(t_1) - \alpha x_2(t_1)^2 &= 0 \\ \frac{1}{4(b+1)^2}(At_f^2 + Bt_f + C) &= 0 \end{aligned} \tag{3.144}$$

where

$$A = -(2b+1) \tag{3.145}$$

$$B = 2x_2(0)(2b+1) \tag{3.146}$$

$$C = 2b^2x_2(0)^2 + 4x_1(0)b^2 + 2bx_2(0)^2 + 8x_1(0)b + x_2(0)^2 + 4x_1(0) \tag{3.147}$$

The solution to the quadratic equation will produce one negative and one positive root. The positive root will be taken to be the final time. However, this solution for t_f is only valid if the initial states are in the $H_2 \cup H_3$ regions. This is due to the assumption that the control sequence will be $[\ -1 \quad 0 \quad 1\ ]$. The same methodology to derive the numerical solution for t_f when the initial states are in the $H_1 \cup H_4$ regions will be applied to compute the final time when the initial states are in the $H_2 \cup H_3$ regions. Assume the control sequence is now $[\ -1 \quad 0 \quad 1\ ]$. The costate, $\lambda_2(t)$, and the corresponding control will be displayed in Figure 3.6. Therefore, the Hamiltonian evaluated at the final time will be

$$\begin{aligned} H(\mathbf{x}, u, t_f) = 1 + b|u(t_f)| + \lambda_1(t_f)x_2(t_f) + \lambda_2(t_f)u(t_f) &= 0 \\ \boxed{\lambda_2(t_f) = 1 + b} \end{aligned} \tag{3.148}$$

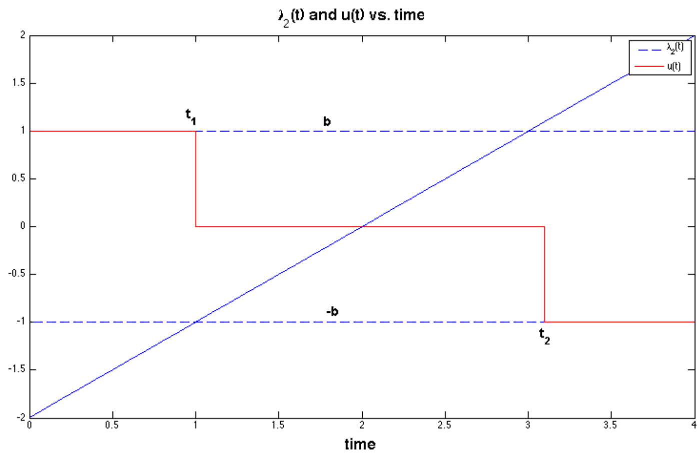


Figure 3.6: Costate, $\lambda_2(t)$, and corresponding control

Substituting Equation 3.148 into Equation 3.15 analyzed at the final time will result in Equation 3.149

$$c_2 = 1 + b + c_1 t_f \tag{3.149}$$

Referring to Figure 3.6, the costate, $\lambda_2(t)$, at the switching times is

$$\lambda_2(t_1) = -b \tag{3.150}$$

$$\lambda_2(t_2) = b \tag{3.151}$$

Equation 3.149 can be substituted into Equation 3.15 analyzed at t_1 and t_2 to form the following expressions for $\lambda_2(t_1)$ and $\lambda_2(t_2)$.

$$\lambda_2(t_1) = -c_1 t_1 + 1 + b + c_1 t_f \tag{3.152}$$

$$\lambda_2(t_2) = -c_1 t_2 + 1 + b + c_1 t_f \tag{3.153}$$

Applying Equations 3.150 and 3.151 to Equations 3.152 and 3.153 respectively results in the following expression for t_1 and t_2.

$$t_1 = t_f + \frac{2b+1}{c_1} \tag{3.154}$$

$$t_2 = t_f + \frac{1}{c_1} \tag{3.155}$$

Equation 3.89 analyzed at t_2 can be expressed in terms of c_1 by utilizing Equation 3.155.

$$x_1(t_2) = -\frac{1}{2}(t_2 - t_f)^2 = -\frac{1}{2c_1^2} \tag{3.156}$$

Also, Equation 3.155 can be applied to express Equation 3.90 analyzed at t_2 in terms of c_1.

$$x_2(t_1) = -t_2 + t_f = -\frac{1}{c_1} \tag{3.157}$$

During the coast phase $x_2(t)$ will be constant.

$$x_2(t_1) = x_2(t_2) = -\frac{1}{c_1} \tag{3.158}$$

Evaluating the indefinite integral for $x_1(t)$ during the coast phase and applying the condition in Equation 3.156 results in the following expression for $x_1(t_1)$

$$x_1(t) = \frac{t - t_1}{c_1} + x_1(t_1) \tag{3.159}$$

$$x_1(t_1) = \frac{t_2 - t_1}{c_1} - \frac{1}{2c_1^2} \tag{3.160}$$

Substituting Equations 3.154 and 3.155 into Equation 3.160 results in an equation in the $x_1(t)/x_2(t)$ plane.

$$x_1(t) = -\left(2b + \frac{1}{2}\right) x_2(t)^2 \tag{3.161}$$

From t_0 to t_1 the control will be one. Therefore, the states, $x_1(t)$ and $x_2(t)$, will be

$$x_1(t) = \frac{1}{2}t^2 + x_2(0)t + x_1(0) \tag{3.162}$$

$$x_2(t) = t + x_2(0) \tag{3.163}$$

Analyzing Equation 3.163 at t_1 and applying Equation 3.158, c_1 can be solved in terms of t_1 and $x_2(0)$. Equation 3.154 can then be applied to compute c_1

$$x_2(t_1) = t_1 + x_2(0) = -\frac{1}{c_1} \tag{3.164}$$

$$= t_f + \frac{2b+1}{c_1} + x_2(0) = -\frac{1}{c_1} \tag{3.165}$$

$$\boxed{c_1 = -\frac{2(b+1)}{t_f + x_2(0)}} \tag{3.166}$$

The first switching time, t_1, can be written as a function of t_f by substituting Equation 3.166 into Equation 3.154.

$$t_1 = \frac{t_f - (2b+1)x_2(0)}{2(b+1)} \tag{3.167}$$

In order to form a quadratic equation expressed in terms of the final time, the switching curve that describes the trajectory in x_1/x_2 space when the control switches from one to zero can be utilized. This curve can be analyzed at t_1.

$$x_1(t_1) - \alpha x_2(t_1)^2 = 0 \tag{3.168}$$

The state $x_1(t_1)$ can be expressed in terms of the final time by substituting Equation 3.167 into Equation 3.162. The variables A, B, and C will represent the coefficients of the final time quadratic formula.

$$
\begin{aligned}
x_1(t_1) &= \frac{1}{2}t_1^2 + x_2(0)t_1 + x_1(0) \\
x_1(t_1) &= At_f^2 + Bt_f + C && (3.169) \\
A &= \frac{1}{8(b+1)^2} && (3.170) \\
B &= \frac{x_2(0)}{4(b+1)^2} && (3.171) \\
C &= \frac{-4b^2x_2(0)^2 + 8x_1(0)b^2 - 8bx_2(0)^2 + 16x_1(0)b - 3x_2(0)^2 + 8x_1(0)}{8(b+1)^2} && (3.172)
\end{aligned}
$$

The state $x_2(t_1)$ can be written as a function of t_f by subsituting Equation 3.167 into the state Equation 3.163 evaluated at t_1.

$$
\begin{aligned}
x_2(t_1) &= t_1 + x_2(0) \\
x_2(t_1) &= \frac{1}{2(b+1)}t_f + \frac{x_2(0)}{2(b+1)} && (3.173)
\end{aligned}
$$

Equations 3.169 and 3.129 can be substituted into Equation 3.168 to form a quadratic equation in terms of t_f.

$$
\begin{aligned}
0 &= At_f^2 + Bt_f + C && (3.174) \\
A &= \frac{2b+1}{4(b+1)^2} && (3.175) \\
B &= \frac{x_2(0)(2b+1)}{2(b+1)^2} && (3.176) \\
C &= \frac{-2b^2x_2(0)^2 + 4x_1(0)b^2 - 2bx_2(0)^2 + 8x_1(0)b - x_2(0)^2 + 4x_1(0)}{4(b+1)^2} && (3.177)
\end{aligned}
$$

Equation 3.174 will produce two roots of opposite sign. The positive root will be the final time. This final time calculation enables the initial states, $x_1(0)$ and $x_2(0)$, to be in the H_2 and H_3 regions. A final time can now be calculated for all H-regions.

Therefore, the minimum time/fuel controller can now be implemented.

With initial conditions, $[x_1(0), x_2(0)] = [2, 3]$, Figure 3.7 displays the states $x_1(t)$ and $x_2(t)$ in the $x_1(t)/x_2(t)$ plane. Since the initial conditions were in the $H_1 \cup H_2$ region, the controller executes a $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ control sequence in order to drive both states to the origin, $(0, 0)$. Both states as a function of time are displayed in Figures 3.8a and 3.8b. The control sequence can be observed in Figure 3.9.

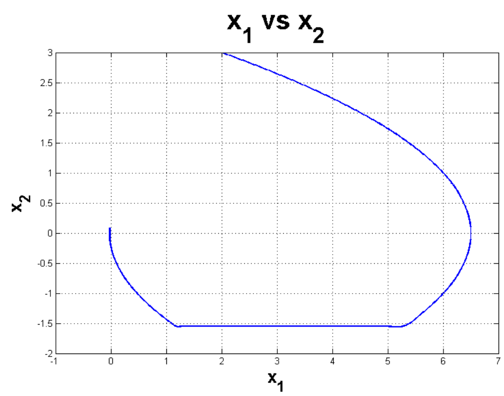
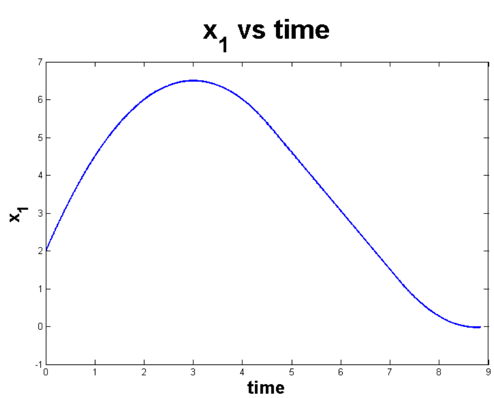
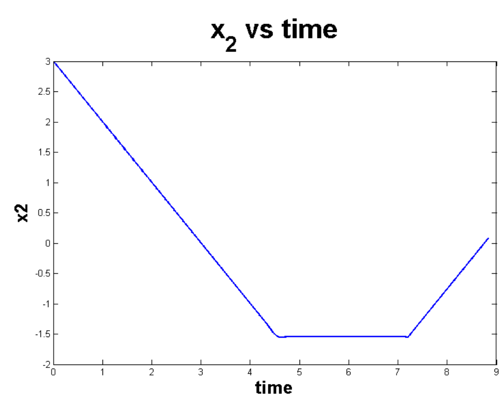


Figure 3.7: States $x_1(t)$ and $x_2(t)$ in the x_1/x_2 plane according to the control sequence { -1 0 1 }



(a) $x_1(t)$ vs time



(b) The state $x_2(t)$ vs. time

Figure 3.8: The states as a function of time for { -1 0 1 } control sequence

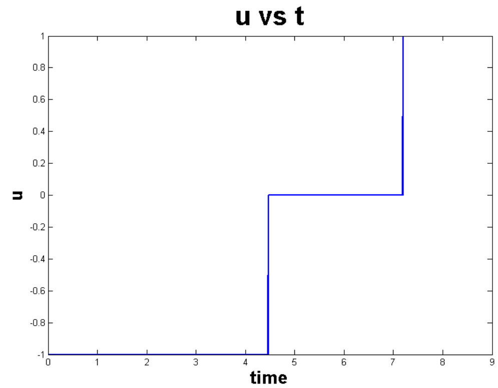


Figure 3.9: Necessary control to drive the states $x_1(t)$ and $x_2(t)$ to zero when initial conditions in $H_1 \cup H_4$ region

If the initial conditions are in the $H_2 \cup H_3$ region and $[x_1(0), x_2(0)] = [-2, -3]$, then the control sequence will be $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ as shown in Figure 3.10.

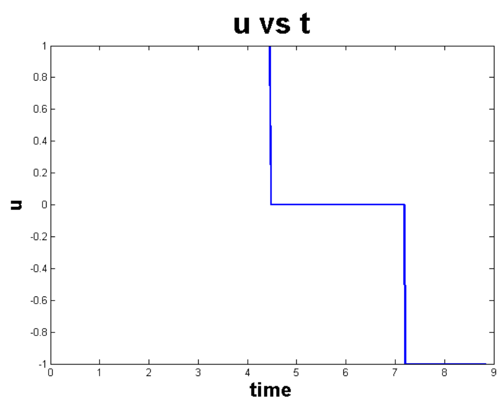


Figure 3.10: Necessary control to drive the states $x_1(t)$ and $x_2(t)$ to zero when initial conditions in $H_2 \cup H_3$ region

The states in the x_1/x_2 plane are displayed in Figure 3.11.

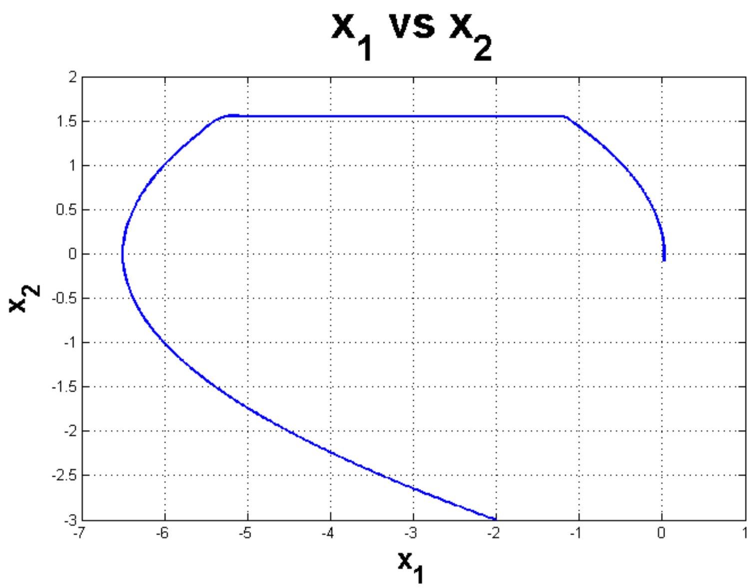
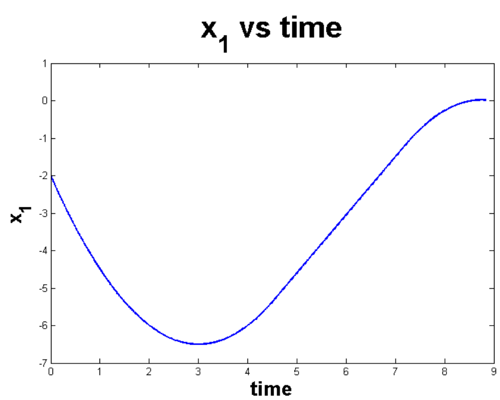
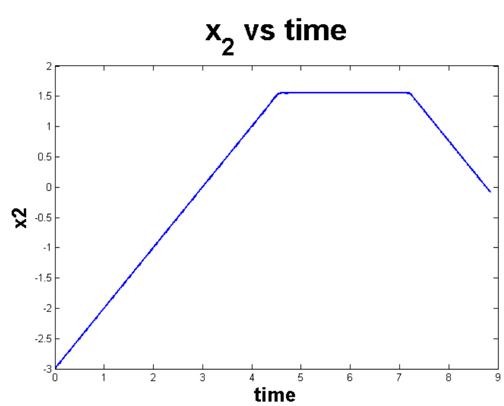


Figure 3.11: States $x_1(t)$ and $x_2(t)$ in the x_1/x_2 plane according to the control sequence { 1 0 -1 }

These states are also shown as functions of time in Figures 3.12a and 3.12b.



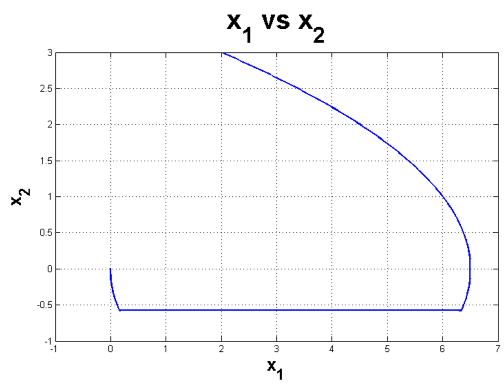
(a) $x_1(t)$ vs time for initial $x_1(0)$ in $H_2 \cup H_3$



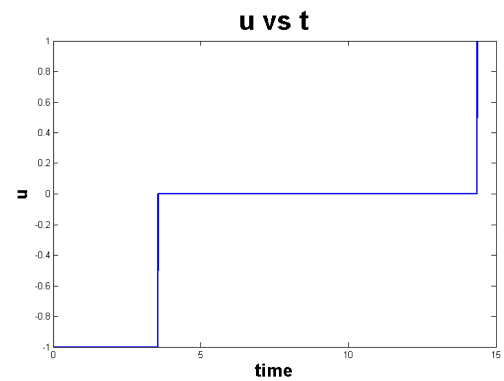
(b) The state $x_2(t)$ vs. time for initial $x_2(0)$ in ' $H_1 \cup H_4$

Figure 3.12: The states as a function of time for { 1 0 -1 } control sequence

For the two demonstrations mentioned above, the controller was implemented with the tunable parameter, b, set to one. Figures 3.13a and 3.13b present the states and control respectively with the same initial conditions but with an increased control weight. The parameter b is now set to ten. In these figures there is a noticeable difference in the time necessary to reach the desired final state.



(a) $x_1(t)$ vs $x_2(t)$ for initial $x_1(0)$ in $H_2 \cup H_3$.The tunable parameter b now equals ten.



(b) The control when b is equal to ten.

Figure 3.13: The resulting states and control when b is increased to ten

To gain a broader perspective, Figure 3.14 represents a range of increasing control weights and their corresponding final times. In addition, the cost is also affected by increasing the tunable parameter b. In Figure 3.15, the cost function represented in Equation 3.8 is shown as the control weight increases.

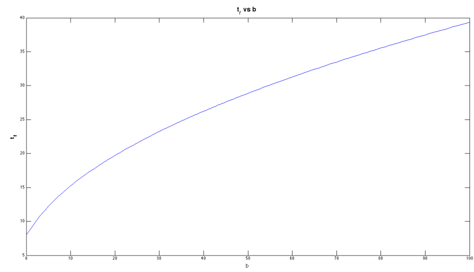


Figure 3.14: The control weight versus the time necessary to reach the desired final state

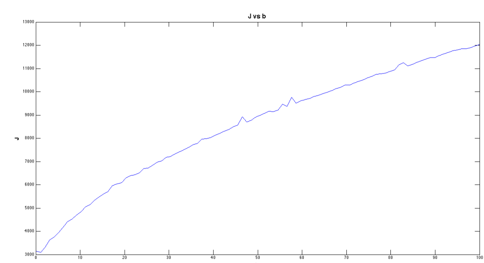


Figure 3.15: The cost function $J = \int_0^{t_f} |u(t)| \mathrm{d}t$, versus the control weight b

3.7 Implementing the Minimum Time/Fuel Controller

Since a minimum time/fuel controller can now be designed, the control for $\tilde{u}_3$ can be solved. The control $\tilde{u}_3$ can then be used to solve for the remaining controls $\tilde{u}_2$ and $\tilde{u}_1$. The necessary state equations to solve are the rotational and translational dynamics. In order to solve for the translational dynamics, the minimum time/fuel controller will be implemented to solve for $\dot{\tilde{w}}_3$ in Equation 2.39 by dividing the resulting control from the minimum time/fuel controller by J_3. The controls, $\tilde{u}_1$ and $\tilde{u}_2$, can now be solved by using the resulting $\dot{\tilde{\omega}}_3$ and Equation 3.4.

The rotational equations of motion for the body will be as follows [7]:

$$[\dot{\beta}_0, \dot{\beta}_1, \dot{\beta}_2, \dot{\beta}_3] = [A][\boldsymbol{\omega}]_e \tag{3.178}$$

where

$$[A] = \frac{1}{2}\begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & -\beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \tag{3.179}$$

Since the controls are being developed in the eigenaxis reference frame, the angular velocities $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are zero; therefore, the rotational equations of motion will simplify to

$$\dot{\beta}_0 = -\frac{\beta_3\omega_3}{2} \tag{3.180}$$

$$\dot{\beta}_1 = \frac{\beta_2\omega_3}{2} \tag{3.181}$$

$$\dot{\beta}_2 = -\frac{\beta_1\omega_3}{2} \tag{3.182}$$

$$\dot{\beta}_3 = \frac{\beta_0\omega_3}{2} \tag{3.183}$$

The initial state vector for the translational and rotational equations of motion will be

$$\mathbf{x}(0) = [\ \beta_0(0) \quad \beta_1(0) \quad \beta_2(0) \quad \beta_3(0) \quad x_1(0) \quad x_2(0)\ ] \tag{3.184}$$

From the initial state vector, $\beta_1(0)$, $\beta_2(0)$, and $\beta_3(0)$ can be used to form the inertia components, J_1, J_2,and J_3 in Equations 2.34 to 2.36. The initial condition for x_2 or $\tilde{\omega}_3$ will be zero since a rest-to-rest eigneaxis maneuver is to be performed. Therefore, the initial condition vector will be

$$\mathbf{x}(0) = [\ 1 \quad 1 \quad 0 \quad 1 \quad 2 \quad 0\ ] \tag{3.185}$$

With the initial conditions and a body frame inertia matrix of

$$\mathbf{I}_e = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{3.186}$$

the minimimum time/fuel control feedback solution for $\tilde{u}_3$ is shown in Figure 3.16.

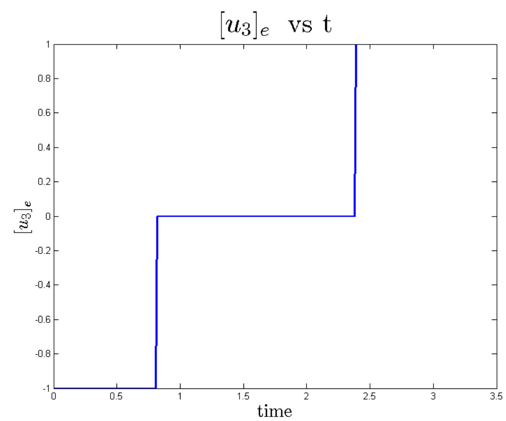
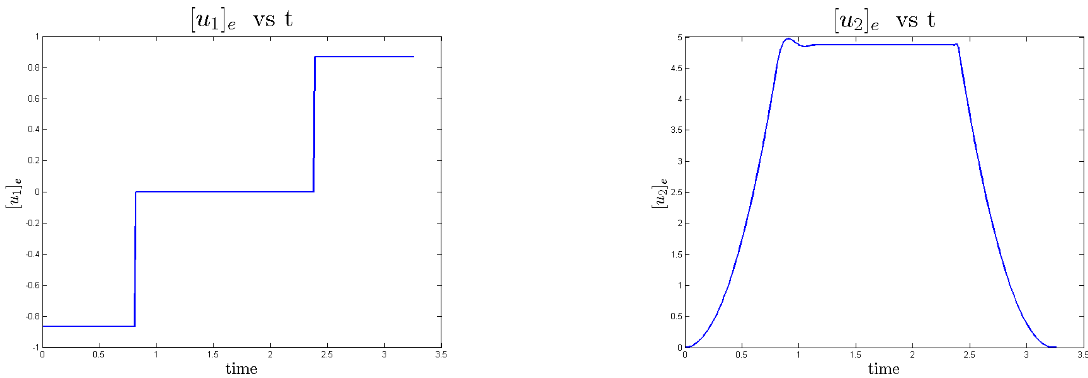


Figure 3.16: The control solution, in the eigenaxis frame, from the minimum time/fuel feedback controller

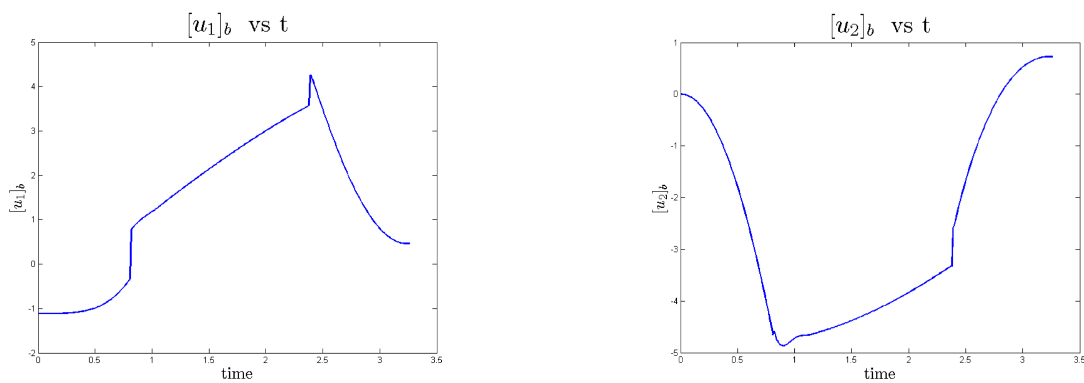
The control $\tilde{u}_3$ can now be substituted into Equation 2.39 to solve for $\dot{\tilde{\omega}}_3$. The angular velocity $\dot{\tilde{\omega}}_3$ can then be substituted into Equations 2.37 and 2.38 to solve for the controls $\tilde{u}_1$ and $\tilde{u}_2$, which are shown in Figures 3.17a and 3.17b. In order to solve for the controls in the body frame, Equation 2.21 will be utilized. The resulting control for $[\mathbf{u}]_b$ is displayed in Figures 3.18a, 3.18b, and 3.19.



(a) The control, $\tilde{u}_1$, in the eigenaxis reference frame

(b) The control, $\tilde{u}_2$, in the eigenaxis reference frame

Figure 3.17: The controls, $\tilde{u}_1$ and $\tilde{u}_2$, in the eigenaxis frame when the minimum time/fuel feedback solution is substituted into the remaining equations of motion



(a) The control, $[u_1]_b$, in the body frame.

(b) The control, $[u_2]_b$, in the body frame.

Figure 3.18: The controls, $[u_1]_b$ and $[u_2]_b$, developed by transforming the controls in the eigenaxis reference frame to the body frame using Equation 2.21

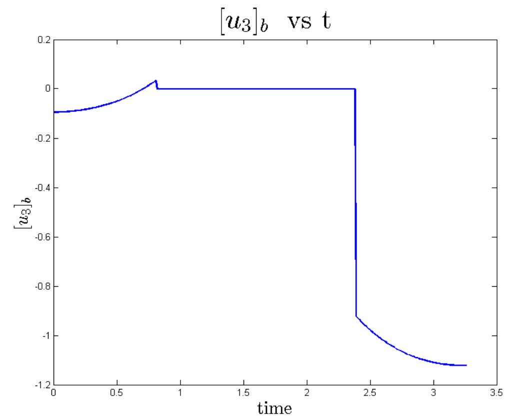
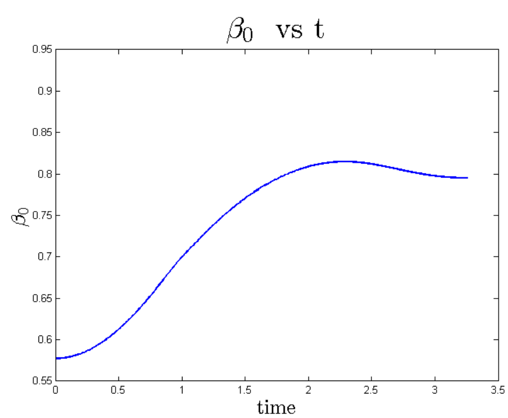
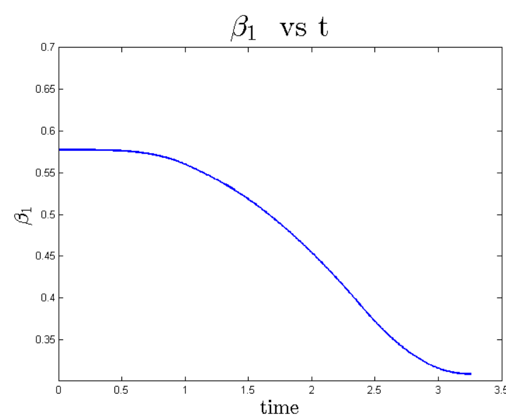


Figure 3.19: The control, $[u_3]_b$, in the body frame

The quaternion time histories in Figures 3.20a through 3.21b signify that the body achieved a new desired attitude by performing an eigenaxis maneuver.

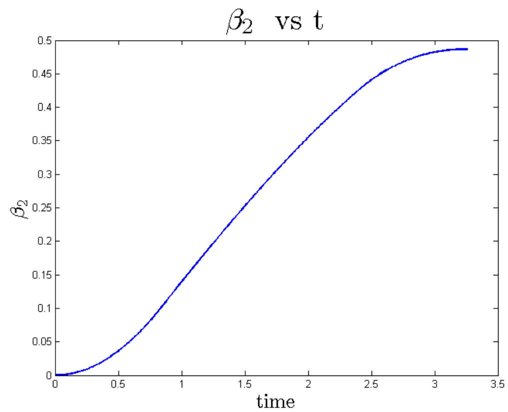


(a) The time history of the quaternion, β_0

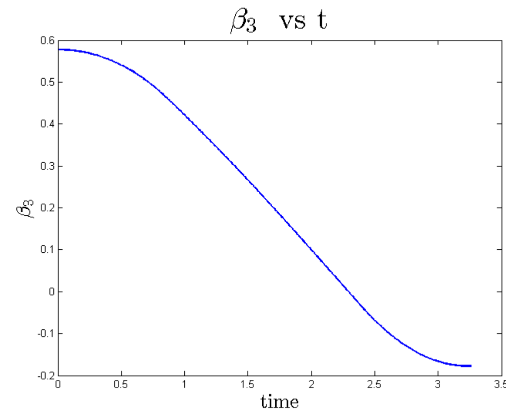


(b) The time history of the quaternion, β_1

Figure 3.20: The time histories of quaternions, β_0 and β_1, from solving the system of first order differential equations noted in Equation 3.178



(a) The time history of the quaternion, β_2



(b) The time history of the quaternion, β_3

Figure 3.21: The time histories of quaternions, β_2 and β_3, from solving the system of first order differential equations detailed in Equation 3.178

Since $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are already zero, it is only necessary to check that the angular velocity $\tilde{\omega}_3$ is zero at the final time to ensure a rest-to-rest eigenaxis maneuver. Figures 3.22, 3.23b, and 3.23a show that the minimum time/fuel controller drives both of the states, $[\omega_3]_e$ and ϕ, to the origin ensuring a rest-to-rest eigenaxis maneuver.

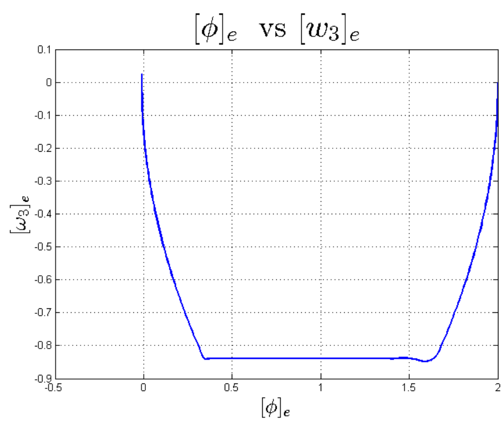
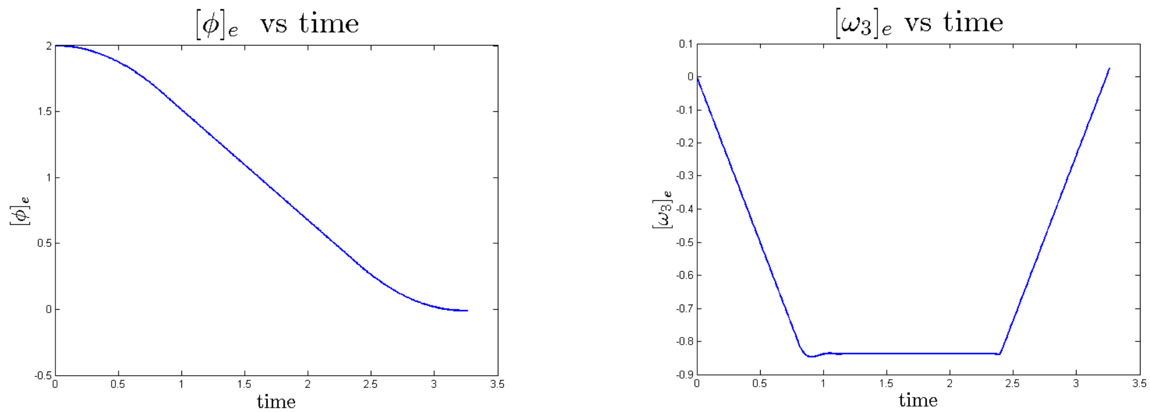


Figure 3.22: The angular velocity, $[\omega_3]_e$, vs the angular position about the $\mathbf{e}_3$ axis



(a) The time history of the angular position, $[\phi]_e$, about the $\mathbf{e}_3$ axis.

(b) The time history of the angular velocity, $[\omega_3]_e$.

Figure 3.23: The time histories for the states in the minimimum time/fuel control problem

From the controls shown in the eigenaxis frame, it can be seen that $[\omega_2]_e$ does not honor the minimum time/fuel control constraints. However, the controls $[\omega_1]_e$ and $[\omega_2]_e$ are not expected to honor the minimum time/fuel control constraints because these controls are being calculated algebraically by utilizing the results of the minimum time/fuel control. The control $\tilde{u}_3$ was only calculated due to the simplicity of the plant. There is no way of ensuring that calculating this control results in the most optimal output for the remaining controls. In order to guarantee that the best choice for all three controllers was chosen, a minimum time/fuel controller would have to be designed for the controls $\tilde{u}_2$, with the remaining controls calculated algebraically. Likewise, a minimimum time/fuel controller would also have to be designed for the control $\tilde{u}_1$, with the remaining controls calculated algebraically. Once all three sets of controls have been solved, the optimal set can then be chosen. Dependent upon the initial states, the eigenaxis control variable that corresponds to the minimum time/fuel control solution will vary.

If it is desired that the control be constrained in the body frame, a different approach would have to be applied. If the absolute value of the control in the body frame cannot exceed one, a smaller value may be chosen in the eigenaxis frame; therefore, when the control is translated to the body frame, all three control variables will not exceed one. Dependant on the intial states, the control constraint value will vary when the control is being solved in the eigenaxis frame.

The minimum time/fuel controller has been successfully implemented in order to reorient a vehicle using a rest-to-rest eigenaxis maneuver. However, this controller can also be applied to a spin-to-rest maneuver. If the initial condition for x_2 is nonzero, the implication is that the body is initially spinning at some angular velocity about the eigenaxis. Since the states, $x_1(t)$ and $x_2(t)$, are both being driven to zero as seen in the example for the minimum time/fuel solution, the body will perform a spin-to-rest eigenaxis maneuver. Next, a minimum time/energy controller will be designed using the same control constraints as the minimum time/fuel controller. Both controllers will then then be compared based on the required time needed to complete a maneuver and two cost functions commonly used to measure fuel cost.

4. DESIGNING A MINIMUM TIME/ENERGY CONTROLLER

Since a rest-to-rest eigenaxis maneuver is still being performed, the dynamics will be the same resulting in the same state space as the minimum time/energy controller.

$$\dot{x}_1(t) = x_2(t) \tag{4.1}$$

$$\dot{x}_2(t) = u \tag{4.2}$$

However, the minimimum time/energy cost function used will be

$$J = \int_0^{t_f} \left\{ 1 + \frac{r}{2} u^2 \right\} \mathrm{d}t \tag{4.3}$$

Therefore, the Hamiltonian will be

$$H = 1 + \frac{r}{2} u(t)^2 + \lambda_1(t) x_2(t) + \lambda_2(t) u(t) \tag{4.4}$$

The costate differential equations pertaining to the Hamiltonian in Equation 4.4 is

$$\dot{\lambda}_1(t) = 0 \tag{4.5}$$

$$\dot{\lambda}_2(t) = -\lambda_1(t) \tag{4.6}$$

Solving both of the costate differential equations will yield costate equations of the form

$$\lambda_1(t) = \lambda_1(t_f) \tag{4.7}$$

$$\lambda_2(t) = -\lambda_1(t_f) t + \lambda_2(t_f) \tag{4.8}$$

where $\lambda_1(t_f)$ and $\lambda_2(t_f)$ represent each costate's value at the final time. Taking the derivative of the Hamiltonian with respect to the control will result in the following optimal control law:

$$ru + \lambda_2 = 0$$
$$\boxed{u = -\frac{\lambda_2}{r}} \tag{4.9}$$

The control law can then be substituted into the state equation 3.6.

$$\dot{x}_2(t) = \frac{1}{r}[\lambda_1(t_f)t - \lambda_2(t_f)] \tag{4.10}$$

An expression for $x_2(t)$ can now be formed as a function of each costate's final state and the initial condition, $x_2(0)$.

$$x_2(t) = \frac{1}{2r}\lambda_1(t_f)t^2 - \frac{\lambda_2(t_f)}{r}t + x_2(0) \tag{4.11}$$

The expression for $x_2(t)$ can then be substituted into Equation 3.5 to solve for $x_1(t)$.

$$x_1(t) = \frac{1}{6r}\lambda_1(t_f)t^3 - \frac{\lambda_2(t_f)}{2r}t^2 + x_2(0)t + x_1(0) \tag{4.12}$$

The final constraints for $x_1(t)$ and $x_2(t)$ can then be used to solve for the costates at the final time in Equations 4.12 and 4.11.

$$\lambda_1(t_f) = \frac{6r}{t_f^3}(2x_1(0) + t_f x_2(0)) \tag{4.13}$$

$$\lambda_2(t_f) = \frac{r}{t_f^2}(6x_1(0) + 4t_f x_2(0)) \tag{4.14}$$

Equations 4.13 and 4.14 can then be substituted into Equations 4.9, 4.12, and 4.12 to obtain a closed-form solution for the states and control.

$$u(t) = \frac{6}{t_f^3}\left(2x_1(0) + t_f x_2(0)\right)t - \frac{1}{t_f^2}\left(6x_1(0) + 4t_f x_2(0)\right) \tag{4.15}$$

$$x_1(t) = \frac{3}{t_f^3}\left(2x_1(0) + t_f x_2(0)\right)t^3 - \frac{6x_1(0) + 4t_f x_2(0)}{2t_f^2}t^2 + x_2(0)t + x_1(0) \tag{4.16}$$

$$x_2(t) = \frac{6x_1(0) + 3t_f x_2(0)}{t_f^3}t^2 - \left(\frac{x_2(0)}{t_f} + \frac{6x_1(0) + 3t_f x_2(0)}{t_f^2}\right)t + x_2(0) \tag{4.17}$$

When comparing the minimum time/fuel controller to the minimum time/energy controller, the final time equation corresponding to the bang-off-bang control sequence of $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ will be used. The following control constraints will be imposed on the minimum time/energy controller in order to compare it with the minimum time/fuel controller.

$$|u(t)| \leq 1 \tag{4.18}$$

$$u(0) = -1 \tag{4.19}$$

The constraint in Equation 4.19 can be applied to Equation 4.15 in order to find an expression for the final time.

$$u(0) = -1 = -\frac{1}{t_f^2}(6x_1(0) + 4t_f x_2(0) \tag{4.20}$$

$$\boxed{t_f = 2x_2(0) \pm \sqrt{4x_2(0)^2 + 6x_1(0)}} \tag{4.21}$$

Equation 4.21 expresses the final time as a function of the initial conditions. Notice that none of the states or control are dependent on the control weight, r. This is evident by further simplifying the cost function shown in Equation 4.3.

$$J = \int_0^{t_f} (1 + \frac{r}{2}u^2)\mathrm{d}t \tag{4.22}$$

$$2\left(\frac{J - t_f}{r}\right) = \int_0^{t_f} u^2 \mathrm{d}t \tag{4.23}$$

From Equation 4.23, it can be seen that r is only a scalar multiplier of the cost integral and has no impact on the states or control. The minimum time/energy controller can now be implemented. Figures 4.1, 4.3, 4.2a, and 4.2b show the states and control of the minimum time/energy controller given the initial conditions

$$\left[\begin{array}{cc} x_1(0) & x_2(0) \end{array}\right] = \left[\begin{array}{cc} 4 & 10 \end{array}\right] \tag{4.24}$$

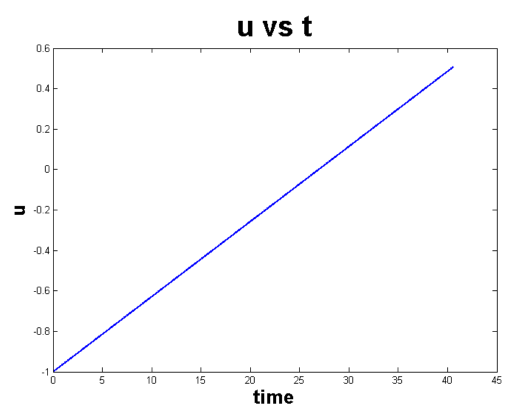
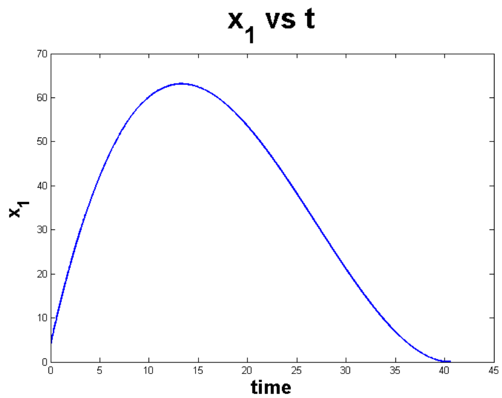
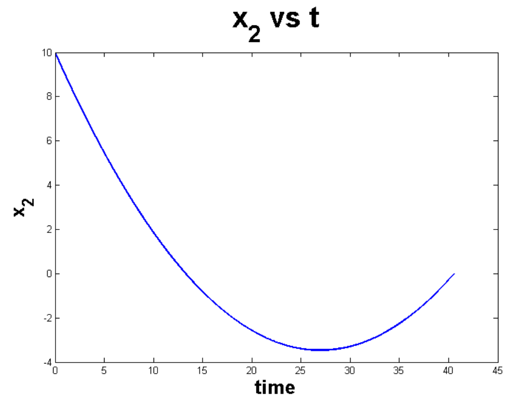


Figure 4.1: The control with the control constraints noted in Equations 4.18 and 4.19



(a) The time history of the state, $x_1(t)$



(b) The time history of the state. $x_2(t)$

Figure 4.2: The time histories of both states, $x_1(t)$ and $x_2(t)$

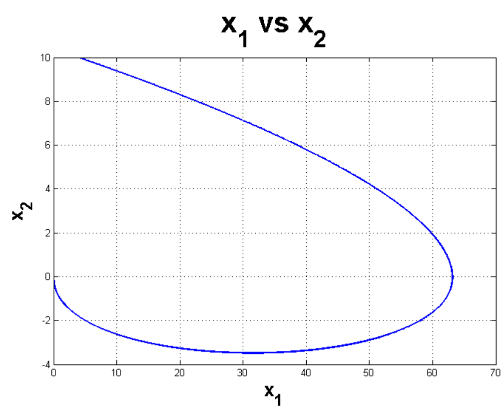


Figure 4.3: The states, $x_1(t)$ and $x_2(t)$, both being driven to the origin

4.1 Comparing the Minimum Time/Fuel and Minimum Time/Energy Controllers

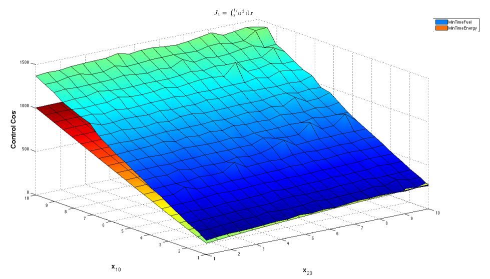
In order to compare the Minimum Time/Fuel and Minimum Time/Energy Controllers two cost functions will be considered.

$$J_1 = \int_0^{t_f} u^2 \mathrm{d}t \tag{4.25}$$

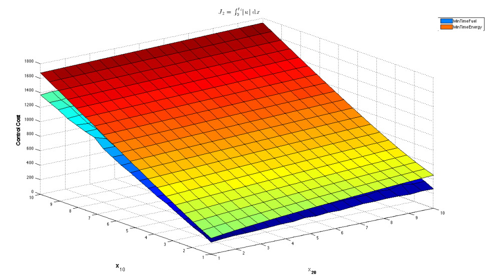
$$J_2 = \int_0^{t_f} |u| \mathrm{d}t \tag{4.26}$$

In order to accurately compare the two controllers, the control weight b in the minimum time/fuel controller will be chosen so that both controllers' final times match given the same set of initial state conditions. The correct control weight will be computed by choosing a set of initial conditions and calculating the final time for the minimum time/energy controller. The final time will then be calculated for the control weight, b, starting at zero. If the final times do not match, b will then be incremented until a matching final time is found.

After the control weight is calculated, several sets of initial conditions in $H_2 \cup H_4$ space will be considered. For each set both the cost functions in Equations 4.26 and 4.25 will be computed. The total cost for each set of initial conditions will be calculated post-processing by using a trapezoidal integration method. Given several sets of initial conditions, each controller can be compared by plotting the total cost for each cost function for different sets of initial conditions. Figures 4.4a and 4.4b show each cost function and their corresponding initial conditions.



(a) Several sets of initial conditions and the resulting total cost from the cost function, $\int_0^{t_f} u^2 \mathrm{d}t$



(b) Several sets of initial conditions and the resulting total cost from the cost function, $\int_0^{t_f} |u| \mathrm{d}t$

Figure 4.4: Both controllers final time and their corresponding cost function noted in Equations 4.26 and 4.25.

As can be seen in Figures 4.4a and 4.4b, the desired controller will be dependent on the type of cost function used to describe the type of actuator. If the cost function described in Equation 4.26 is utilized, the desired controller will be a minimum time/fuel controller. However, if the cost function described in Equation 4.25 is used, the desired controller will be a minimum time/energy controller.

5. CONCLUSION AND FUTURE IMPROVEMENTS

5.1 Conclusion

A minimum time/fuel controller was designed using an eigenaxis maneuver. The eigenaxis was developed in terms of a quaternion attitude description. This calculation was performed by normalizing the eigenvector corresponding to the eigenvalue of one of the attitude matrix that related the body frame to the desired attitude inertial frame. After the eigenaxis was formed, it was used to create a reference frame in which the governing equations of motion would be defined. From the equations of motions, a minimum time/fuel controller was developed in the eigenaxis reference frame. The control generated was then substituted into the equations of motion to find the remaining two controls in the eigenaxis frame. All three controls were translated to the body frame by utilizing the attitude matrix that related the eigenaxis reference frame to the body frame. A minimimum time/energy controller was then developed to compare the fuel cost, using two different cost functions, against the minimum time/fuel controller. It was discovered that the optimal controller would be dependent on the cost function that best described the actuator type.

5.2 Further Research

The minimum time/fuel controller was designed for a rest-to-rest eigenaxis maneuver. Different types of maneuvers can be investigated for the minimum time/fuel controller such as a rest-to-spin maneuver and a spin-to-spin maneuver. As discussed earlier, the minimum time/fuel controller for the plants corresponding to the $\tilde{u}_1$ and $\tilde{u}_2$ controls will need to be solved. Also, a method needs to be developed to calculate the correct control constraint value in the eigenaxis reference frame so that when the controls are translated to the body frame all controls are bounded from negative one to one.

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