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How to Map a Sandwich: Surfaces, Topological Existence Theorems and the Changing Nature of Modern Thematic Cartography, 1966-1972

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Abstract: This paper is meant to be the beginning of a project that examines the use of abstract mathematics and the changing ontology of mapmaking in the early years of the development of computer cartography. The history of the conceptual developments that took place during this revolutionary period in the history of mapmaking is both controversial and incomplete. Much of the primary source material has yet to be examined by historians, residing as it does in obscure journals, government archives and in obsolete software. This study provides a look at one example of this conceptual development in the early years of computer cartography through a close reading of two papers on existence theorems published by the Harvard Laboratory for Computer Graphics and Spatial Analysis. It attempts to highlight the changing conceptual and mathematical foundations of mapmaking during this period and in doing so provides a case study for the difficulties that historians of modern cartography face in researching this critical period in its history.

Keywords: Existence theorems; computer mapping; surfaces; topological data structures; Harvard Laboratory for Computer Graphics and Spatial Analysis.

Introduction: Abstraction in Early Computer Cartography

What does it mean to obtain a new concept of the surface of a sphere?
How is it then a concept of the surface of a sphere?
Only in so far as it can be applied to real spheres.[1].
--Wittgenstein

The science of cartography in the period after World War II saw revolutionary changes in its methods, in its data and in its conceptual foundations.[2] New sources of data from satellites, the development of new numerical and mathematical techniques and the creation of computers and the graphical displays that accompanied them, all changed the science of cartography in ways that historians are only just beginning to come to terms with.[3] In the 1960s and 1970s some of the most important work being accomplished in cartography from the mathematical standpoint had to do with the topological properties of surfaces, their relationship to geographical and spatial analysis, and the ontology of cartographic objects. The Harvard Laboratory for Computer Graphics and Spatial Analysis was a hotbed of such work and was led into new areas of research by the ideas of the theoretician William Warntz (1922-1988). During this critical period in the history of cartography Warntz's group and others at the Harvard Lab took a research path that essentially rethought the meaning of what it meant to create a map. In these years researchers there, and in other venues, let their imaginations run wild and experimented with formerly untapped areas of mathematics and computation, planting one of the many seeds that grew into modern Geographic Information Systems (GIS). It was an era that saw the melding of mathematics with new geographical concepts and in which increasing levels of geometric and mathematical abstraction would become an integral part of the visual and pragmatic science of cartography.

The history of the conceptual developments in cartography during this era are especially difficult to research, as much of the primary source material is still to be examined by historians, residing as it does in obscure journals, in government archives, in old computer programs and on obsolete hardware. The following paper is meant to be an example of those difficulties and makes no claim to completeness in the subject matter it examines, as the early history of computer mapmaking and its mathematical foundations is both controversial and incomplete.[4] Rather, it is meant as a case study, a beginning point, in the larger project of assessing the changing role of mathematical abstraction in the early years of computer cartography and role it played in how modern maps are created and perceived.

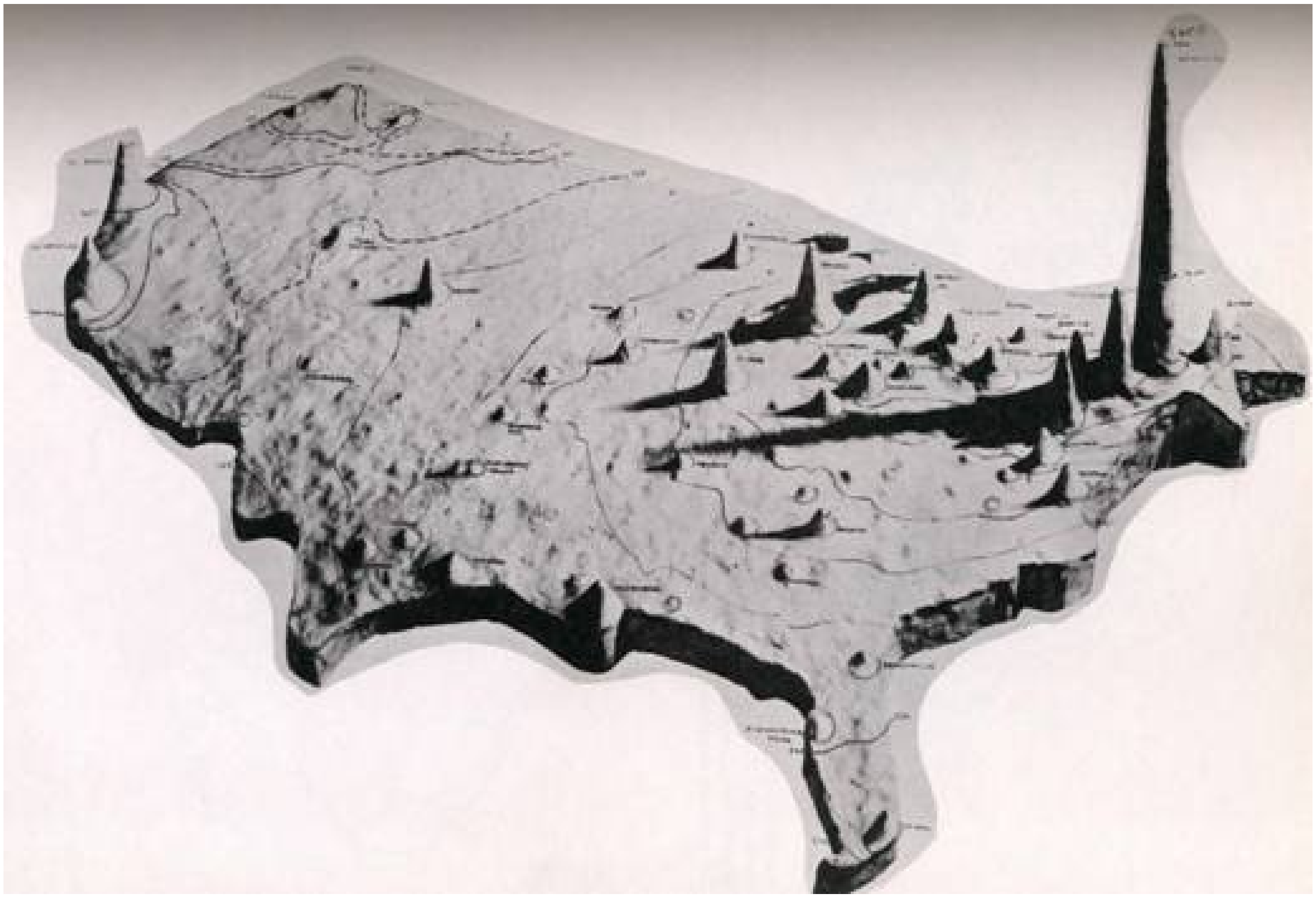

Figure 1: Surface model of Warntz's 1960 Population Map
U.S. Office of Naval Research, *Selected Projects Conference Notes, July 1972*.

During the 1960s and 1970s new levels of mathematical abstraction were especially evident in research that centered on thematic cartography.[5] While many researchers in the field were looking at the numerical properties of thematic maps and at the statistical relationships in the data they displayed, William Warntz of Harvard looked to understanding the topology of the surfaces that the data formed. He recognized that the most important properties of surfaces from a mathematical point of view had nothing to do with numbers and specific values, but rather with the surface's invariance under transformations. Warntz described the relationship of the topological properties of a surface to cartography in a number of papers that adopted a terminology and methodology built on the work of the mathematicians Arthur Cayley (1821-1895) [6] and James Clerk Maxwell (1831-1879).[7] Warntz generalized Cayley's vocabulary that described the contours on geographic and topographic maps for use in the description of surfaces. Cayley's lexicon categorized particular features on maps that he called summits, pits, immits, and other terms that Warntz used in describing the geometry of the surfaces of thematic maps. In Cayley's vocabulary Warntz found a natural and geometrically descriptive lexicon for pointing out particularly interesting features of surfaces that had analogues on topographic maps, resembling contours, singularities and extrema (maxima and minima).

In the preface to "Geography and an Existence Theorem" Warntz briefly wrote about the lack of attention given to surfaces in the fields of geography and cartography:

> The topological nature of the surface has not received as much attention and the requirements

> that this places upon the theories of spatial process in geography have not been recognized, to the disadvantage of those theories.[8]

Warntz was particularly interested in mapping and graphically displaying thematic surfaces and adopted a macro-geographical theoretical perspective that led not only to fundamental mathematical breakthroughs but also yielded philosophical insight into the nature of the objects described by the 'science' of cartography.[9]

In his 1966 paper on the topology of socio-economic terrain, Warntz expresses the new outlook towards thematic mapping by "modern" cartographers and geographers:

> Today geographers, regional scientists, and others are taking the geo in geometry literally, and the study of earth related surfaces and paths has now been expanded far beyond its original application to such things as land form, contour mapping, drainage patterns, temperatures, pressures, precipitation, and the like in physical geography alone. The modern scholar conceives of surfaces based also on social, economic, and cultural phenomena, portraying not only conventional densities but other things such as field quantity potentials and also probabilities, costs, times, and so on.[10]

Warntz goes on to state that all of these quantities can be mapped:

> Always however, these conceptual surfaces may be regarded as capable of overlying the surface of the real earth, and the geometric and topological characteristics of these surfaces, as transformed, could thus describe aspects of the geography of the real world.[11]

As an example of this type of thematic cartography, Warntz includes in the paper his map of the potentials of population as a three-dimensional model (figure 1) and also reproduces his "Map of the Potentials of Population" that he published with the American Geographical Society (figure 2). Warntz describes the map as showing a true "macrogeographic" quantity, one that varies continuously over the surface of the map. The fact that the surface could be portrayed as a continuous function, and not just composed of discrete values, opened up new areas of geographical analysis and research quickly developed around finding efficient algorithmic ways of smoothing surfaces and calculating their properties.[12]

Much of Warntz's research on this type of thematic cartographic analysis was inspired by the work of John Q. Stewart (1894-1972,) who attempted to formalize the study of population distribution and its cartographic nature in a series of papers in the late 1940s and early 1950s, which drew upon physical science models and potential theory.[13] Stewart, writing in the *American Journal of Physics*, described variables that could be mapped thematically as "demographic indices" that had attraction, interactance and influence on population at a distance. One article, aptly named "The Development of Social Physics," had some influence on geographers and thematic cartographers at the time, and suggested that thematic variables could be treated mathematically in cartography with the same sort of equations that scientists used to describe Newton's Law of Gravitation.[14] Warntz thought Stewart's contribution to thematic cartography to be important enough that he included him with other more recognizable geographers, like Ptolemy, in his book *Breakthroughs in Geography*, which he wrote with Peter Wolff in 1971.[15]

In producing his map of the population potentials shown in figure 2 and modeled in clay and photographed in figure 1, Warntz used the same type of physical potential analogue as was described by Stewart. The total potential of the population according to Warntz could be found by integrating the equation:

$$V = \iint \left(\frac{1}{r}\right) D da$$

where V is the total potential of the population at some point on the map, D is the population density and da is an infinitesimal unit of geographic area. In this way, Warntz formed contours on his map that showed areas that were attracting population, areas that were stable, and those with negative growth. Warntz did not directly solve the integral but used the summation for the potential that could easily be solved on digital computers:

$$V = \sum_{j=1}^{n} \frac{P_j}{r_{ij}}$$

where P is the population and r is the distance.

The use of geo-potentials is just one type of surface analysis that was pioneered by Warntz for use in thematic cartography. He and his students would exploit other mathematical and physical analogs as they rethought the types of surfaces and variables that could be thematically important for geographic analysis. These explorations would carry them into more and more abstract territory and, as we shall see, into more and more original cartographic applications.

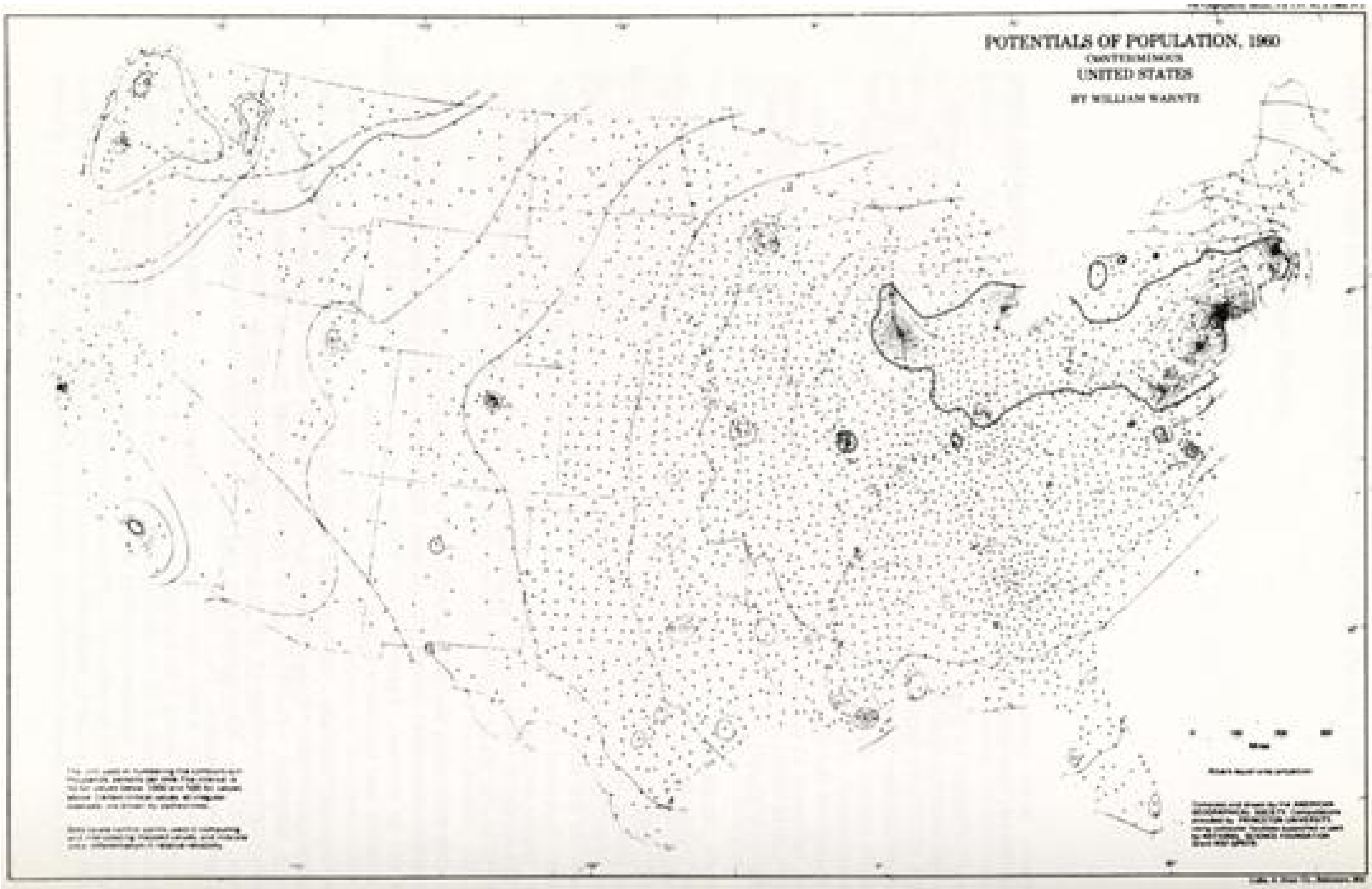

Figure 2: Potentials of Population by William Warntz, 1960.
U.S. Office of Naval Research, *Selected Projects Conference Notes, July 1972*.

One other mechanism through which the nature of cartographic objects changed at the time was the development of topological data structures. These data structures, formed around topological and combinatorial principles, would eventually form the basic unit of spatial information in the new computer cartography. In the history of cartography, the nature of geographic objects—those objects that were to be visualized and presented on a final printed map—took the form of simple objects. Maps showed identifiable things that were part of human life. Roads, mountains, bridges, cities, rivers and other basic elements interacted in a variety of conceptual and visual schemes that ranged from the medieval Mappa Mundi to the road map. This notion of objects interacting in cartographic space went through a revolution in the post-World War II era with the advent of computational systems that allowed for numerical and algebraic manipulation of cartographic objects. The use of abstract set theory and the development of topological data structures fundamentally changed what was actually being manipulated by the cartographer, and the topological structures presented a new conception of geographical space in which the well-known objects became highly abstracted. The choice of data structure for a mapping application implies a spatial theory and the debates surrounding these issues and the basic form that these new objects would take in cartographic space was hotly debated in geographic and cartographic circles throughout this critical period. [16]

Unlike the objects of everyday experience familiar to cartographers, these new objects were defined simply by their topological nature and dimension.[17] A topological object was defined in a way that gave it a different relationship to the other objects that are related to it in cartographic space than any that had previously existed in the history of cartography. In this new topological space there is no relationship that can exist between objects of the same dimension. The relationship of objects to each other in the new data structures is logically structured to allow only objects of different dimensions to interact. For example, a region on a map that had two-dimensional extent could not directly abut a region next to it without a one-dimensional boundary. The two regions do not therefore lie next two each other but are separated by a one-dimensional line. Lines do not connect directly but must meet at a node that has dimension zero.[18]

Although abstract, the topological model did however vastly improve the geometrical consistency of map making. Defining a map as a cellular structure of points, lines and polygons, as opposed to a mere tracing on a plane, allowed “every possible finitary geometric structure associated with a map to be described and manipulated algebraically.”[19] The relationships between these new topological objects were not developed as ad hoc or conventional systems but rather were formulated due to algorithmic, analytical and information theoretic reasons.[20]

It is apparent that during the early stages of computer cartography increasing levels of abstraction and new mathematical structures entered into the thinking of the cartographers and geographers who were at the forefront of its development. In the following section we focus on one particular aspect of topological abstraction and its effect on thematic cartography as found in the work of William Warntz and his students at the Harvard Laboratory: existence theorems. Existence theorems contain a statement of existential quantification such as “there is” and prove the existence of a particular set of mathematical objects. They do not, however, contain any directions as to how such objects might actually be constructed algorithmically or numerically.

Although there were perhaps more important developments that took place during this period of rapid growth in computer cartography, the papers on existence theorems written at the Harvard Lab struck this historian as an interesting way to look at the types of mathematical experimentation that were taking place concerning surfaces and their relationship to cartography, and to explore how the researchers themselves perceived the new mathematical methods they were employing. The Lab published two works on existence theorems in the now largely forgotten series the *Harvard Papers in Theoretical Geography*. We will provide a close reading of two of these papers, "The Sandwich Theorem: A Basic One for Geography," and "Geography and an Existence Theorem: A Cartographic Solution to the Localization of Sets of Equal-Valued Antipodal Points," in order to show how the Lab used a mathematical approach that was underexploited in cartography and in doing so helped change the accepted notions of the nature of cartographic objects and the analysis potential of thematic cartography. (Complete citations for these two papers can be found in notes 8 and 22-23.)

II. How to Map a Sandwich: Existence Theorems and the Nature of Thematic Cartography

From what rests on the surface we are led into the depths.[21]
--Edmund Husserl

Given any three sets in space, each of finite outer Lebesque measure, there exists a plane which bisects all three sets, in the sense that the part of each set which lies on one side of the plane has the same outer measure as the part of the same set which lies on the other side of the plane.[22]

This statement of the Sandwich Theorem at the beginning section of Warntz's introduction to "The Sandwich Theorem: A basic one for geography" [23] hardly seems at first reading to have import to the history of thematic cartography. Warntz however, thought it to be of "paramount importance."[24] In the introduction Warntz asks us to assume the earth to be a solid sphere and to picture the infinite number of planes that one could choose to bisect its volume. After presenting this rather straightforward set of bisecting planes, he then moves to discuss other types of partitioning of the earth's surface that are more abstract and employ less physically imaginable variables. He says that:

> It is certain that there exists the possibility of finding a small circle on the earth's surface such that two (unequal) parts into which it divides the earths surface contains half of the world's communists, half the world's income and half of the world's volcanoes. Another circle (presumably a different one than above) partitions the earth's surface into two equal shares of mosques, synagogues and cathedrals.[25]

The idea of partitioning geographic and social variables and to mapping their distribution on the surface of the earth has always been one of the mainstays of thematic cartography and the above theorem, though highly abstract, gives hope that any group of sets on a map could be partitioned into "regions" of equal size. The name of the Sandwich Theorem comes from the most straightforward example of its application. If one builds a sandwich of bread, cold cuts, cheese, and then spreads it with butter, is there a way to cut all of these elements that make up the sandwich precisely in half with a single cut of a plane? The theorem implies that the cut would produce two sections of the sandwich each of which has precisely half of the bread, half of the cold cuts and half of all of the other things at the same time no matter how they were originally oriented or distributed on the sandwich.

The Sandwich theorem as stated above was first fully explored in a paper by Hugo Steinhaus (1887-1972) published in *Fundamenta Mathematica* in 1945, under the title, "Sur la division des ensembles de l'espace par les plans et des esembles plans par les cercles" [26] and translated in full in the Sandwich Theorem paper. The theorem is part of a family of mathematical proofs called existence theorems and no matter how important Warntz thought it to be for geography and mapping, it immediately posed a problem for applications in thematic cartography. The theorem begins with the statement, "given any three sets there exists a plane", which is a statement of existential quantification; the theorem then goes on to prove that such a plane exists in the real world. The problem with the theorem and with all existence theorems is that they provide no way to actually calculate the mathematical object that the theorem claims existence for. In pure mathematics this is not usually an issue, but for actual applications to thematic cartography, one does need to find the partition that one is claiming existence for. Existence theorems are not only problematic for cartography, but there has been a standing debate in mathematics itself as to their philosophical foundations and usefulness.[27]

Warntz in his introduction calls attention to the fact that as an existence theorem the work of Steinhaus provided no analytical means for finding the partitions, but that he hoped that "numerical-graphical procedures may effect accurate approximations of the solution for given problems. Especially is computer graphic suited to such tasks."[28] The other parts of the Sandwich Theorem paper reflect Warntz's hope, and take up the numerical and cartographic solution to the problem culminating with a map of the United States that shows a partitioning of geographical area, population and income (figure 6).

In order to get to the map, however, the abstract notions described in the theorem had to be made calculable for the available computers of the time. The question of whether this was even possible was still to be answered as no one had ever looked for an algorithmic solution to the Sandwich Theorem and general applications of existence theorems in cartography were exceedingly rare. That the members of the Harvard Lab recognized the newness of using existence theorems and the associated problems with adopting them to cartography is shown quite explicitly in a statement by Stephen Selkowitz, author of the Harvard Existence Theorem paper:

> Even this relatively simple computer assistance in finding spatial solutions to the given existence theorem demonstrates the possibility of using computer mapping programs to find spatial solutions to that whole class of existence theorems having explicit or implicit spatial connotations. After initial problems related to adopting the particular existence theorem to a computer solution are resolved, computer mapping techniques are capable of quick and accurate spatial solutions with a minimum of human manipulation. This is another indication that the problem solving capabilities of computer mapping are at least as diverse and effective as are its proven capacities graphically to represent stores of spatially ordered information. These capabilities lay largely untapped, waiting to be exploited.[29]

In part I of the Sandwich Theorem paper, "A Geometric Analysis Concerning the Sandwich Theorem," C. Ernesto Lindgren restates the theorem in the form given by Stone and Turkey in their generalization of Steinhaus' work, "Given any three sets in space, each of finite Lebesque measure, there exists a plane which bisects all three sets, "[30] and begins to speculate on its possible algorithmic simplification. "Now, for more down-to-earth applications of this possibility, we had to look for simpler approaches, not involving the required advanced mathematics, because we lack the necessary background."[31] Lindgren realizes the mathematical complexity of the theorem and the difficulties that it would entail to solve it analytically. He goes on to say: "Consequently, we had to begin by reasoning that this existence theorem could be stated by basing its affirmation on other conclusions, preferably not mathematical."[32]

Lindgren in his analysis decides to take a surface-oriented approach to the problem and begins by looking at the geometry of the theorem, “It just happens that geometry provides such possibility and, if one wished to put claim on discovery, perhaps one should recognize this priority to geometry as a whole. As it turned out, the sandwich theorem, under the light of geometric synthesis, is only a conclusion.”[33] What Lindgren is describing here is an approach to problems that the Harvard Lab would often experiment with, namely, the development of algorithms and cartographic approaches to solving problems previously known only as results in pure mathematics.

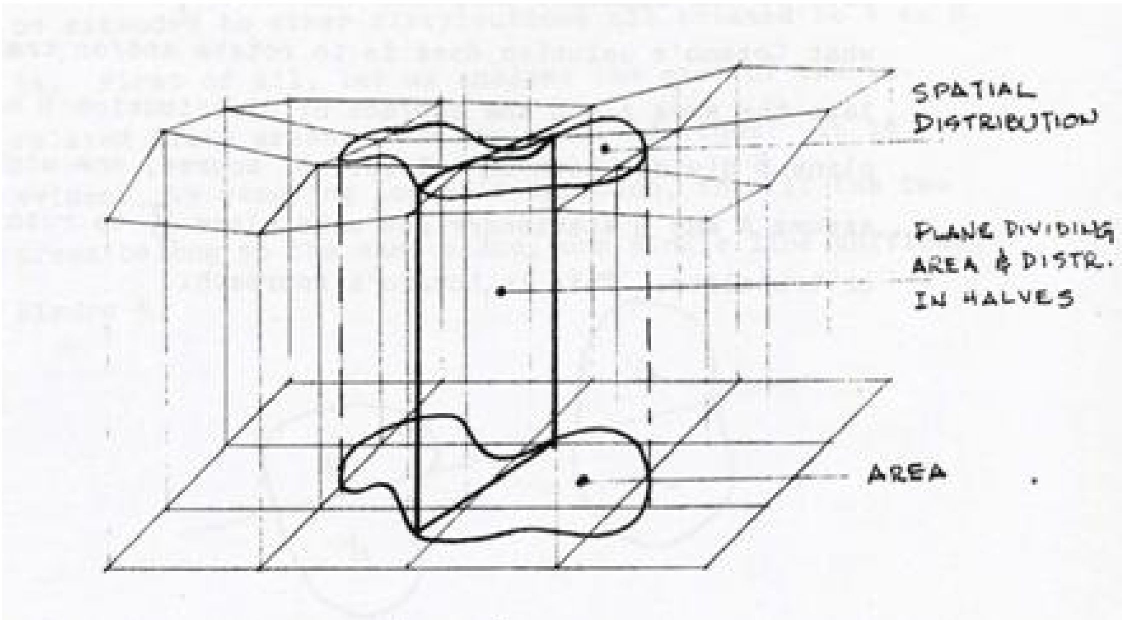

Figure 3: Bisection of two spatial distributions by a plane. Figure 3 from Sandwich Theorem Paper. Courtesy Fellows of Harvard University.

Lindgren, whose main research interest at the time centered on four-dimensional geometries, used a geometric analog of angle bisection to picture the partitioning of cartographic surfaces.[34] In figure 3 Lindgren describes the layering of two surface distributions and the plane that bisects them. The top layer represents schematically the surface distribution of population. The top surface in the diagram is not flat, but is curved in places to show the value of the population much like the potential surfaces in Warntz’s population potential map. The bottom distribution is geographic area, say, for example, of the United States or any finite region. What follows in the paper is a complicated treatment of the notion of angle and the possible iterative solutions to the problem of bisecting all of the distributions simultaneously (figure 4). Lindgren’s solution and that of Eduardo Lozano, who also writes in the Sandwich Theorem paper, corresponds to a series of geometric constructions that divide the various distributions in half. The solution to the overall problem was found using an iterative process that calculated various partitioning schemes for each distribution until one is found that corresponds to the partition for all of them.

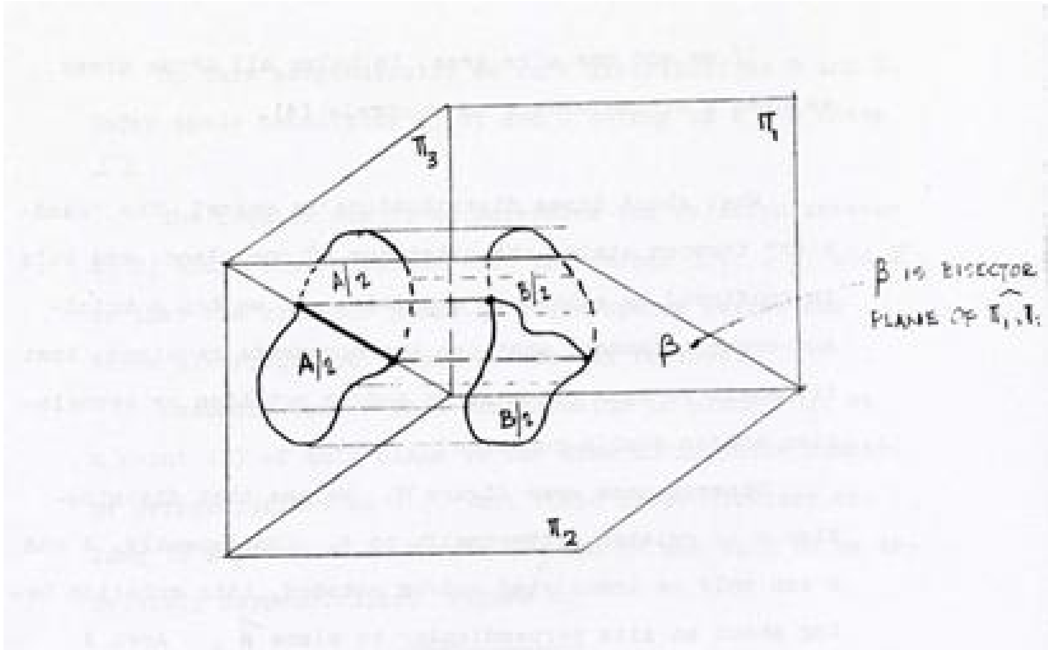

Figure 4 Lindgren's Schematic of the distribution as angular bisector
Courtesy Fellows of Harvard University

The iterations produced not an exact solution to the partitioning problem but an approximate solution that could be made more accurate depending on the amount of computer time one was willing to spend on the calculation. The actual computer code for the program to accomplish the partitioning was authored by Katherine Kiernan and used a common numerical programming technique of searching for a minimum by constantly reducing the space that was being analyzed. It is not known how much computer time was expended, but multiple iterations must have been necessary since there were 3,070 counties used in the calculation.[35] The approximate nature of the solution can be seen in figure 5, which is the computer printout from the Sandwich Theorem paper, and shows the percentages of the three thematic distributions that were partitioned. If this were an exact solution all the values for area, income and population would read 50%. Figure 6 shows the map of the line dividing the three thematic variables from the Sandwich Theorem paper.

SLOPE OF LINE (B)	INTERCEPT (A) POPULATION	INTERCEPT (A) AREA	INTERCEPT (A) INCOME	%ABOVE POPULATION	%BELOW POPULATION	%ABOVE AREA	%BELOW AREA	%ABOVE INCOME	%BELOW INCOME
0.017	38.286	37.322	38.682	49.81	50.19	49.97	50.03	49.53	50.47
0.105	30.946	28.617	31.489	49.51	50.49	50.26	49.74	49.94	50.06
0.194	23.095	19.733	23.881	50.12	49.88	49.91	50.09	50.13	49.87
0.287	14.840	10.340	15.843	50.74	49.26	50.47	49.53	50.01	49.99
0.384	6.574	0.502	7.627	50.16	49.84	50.04	49.96	49.86	50.14
0.488	-2.239	-10.180	-1.175	49.85	50.15	50.86	49.14	49.65	50.35
0.601	-12.068	-21.156	-11.259	50.49	49.51	50.05	49.95	50.97	49.03
0.727	-24.609	-32.939	-21.950	55.10	44.90	47.49	52.51	49.47	50.53
0.869	-35.221	-48.020	-34.523	50.06	49.94	50.17	19.83	50.90	49.10
1.036	-49.584	-64.336	-48.673	50.31	49.69	49.40	50.60	47.04	52.96
1.235	-66.589	-84.760	-66.082	49.56	50.44	50.40	49.60	47.68	52.32
1.483	-89.278	-109.867	-87.443	52.67	47.33	51.17	48.83	47.92	52.08
1.804	-116.163	-141.151	-116.163	48.28	51.27	49.88	50.12	49.95	50.05
2.246	-153.598	-184.366	-53.598	47.18	52.82	49.03	50.97	48.59	51.41
2.904	-210.890	-250.062	-209.305	49.84	50.16	49.20	50.80	49.32	50.68
4.011	-302.186	-362.848	-302.186	47.39	52.61	51.13	48.87	48.65	51.35
6.314	-498.805	-590.239	-498.805	48.64	51.36	49.96	50.04	49.72	50.28
14.300	-1185.599	-1384.044	-1185.600	50.32	49.68	50.08	49.92	51.20	48.80
-57.291	5046.898	5913.434	5046.898	45.13	54.87	41.85	58.15	44.76	55.24
-9.514	849.177	981.858	849.177	50.64	49.36	50.41	49.59	49.89	50.11
-5.145	477.717	549.578	475.883	50.14	49.86	49.83	50.17	50.23	49.77
-3.487	335.479	385.109	335.479	50.58	49.42	49.73	50.27	50.06	49.94
-2.605	261.286	296.692	261.286	49.33	50.67	50.48	49.52	49.10	50.90
-2.050	213.191	241.465	213.191	50.40	49.60	51.01	48.99	50.35	49.65
-1.664	180.480	203.643	181.011	51.48	48.52	50.22	49.78	48.57	51.43
-1.376	156.138	176.955	156.138	50.51	49.49	47.44	52.56	51.12	48.88
-1.150	136.919	152.973	137.248	49.63	50.37	49.94	50.06	50.11	49.89
-0.966	120.634	134.718	121.155	51.24	48.76	49.90	50.10	49.85	50.15
-0.810	108.305	120.553	108.305	45.97	54.03	47.09	52.91	47.57	52.43
-0.675	96.056	105.416	96.056	49.01	50.99	51.17	48.83	50.98	49.02
-0.554	85.801	93.756	86.133	49.88	50.12	50.00	50.00	50.10	49.90
-0.445	76.520	82.817	76.777	49.65	50.35	50.26	49.74	51.09	48.91
-0.344	68.057	72.725	68.459	50.31	49.69	50.45	49.55	50.25	49.75
-0.249	60.278	63.323	60.642	49.62	50.38	49.92	50.08	49.95	50.05
-0.158	52.779	54.477	52.779	47.91	52.09	50.03	49.97	50.54	49.46
-0.070	45.785	45.983	45.873	50.19	49.81	50.08	49.92	49.48	50.52

Figure 5: Computer printout from Kiernan's program showing percentages of each of the partitions. Courtesy Fellows of Harvard University.

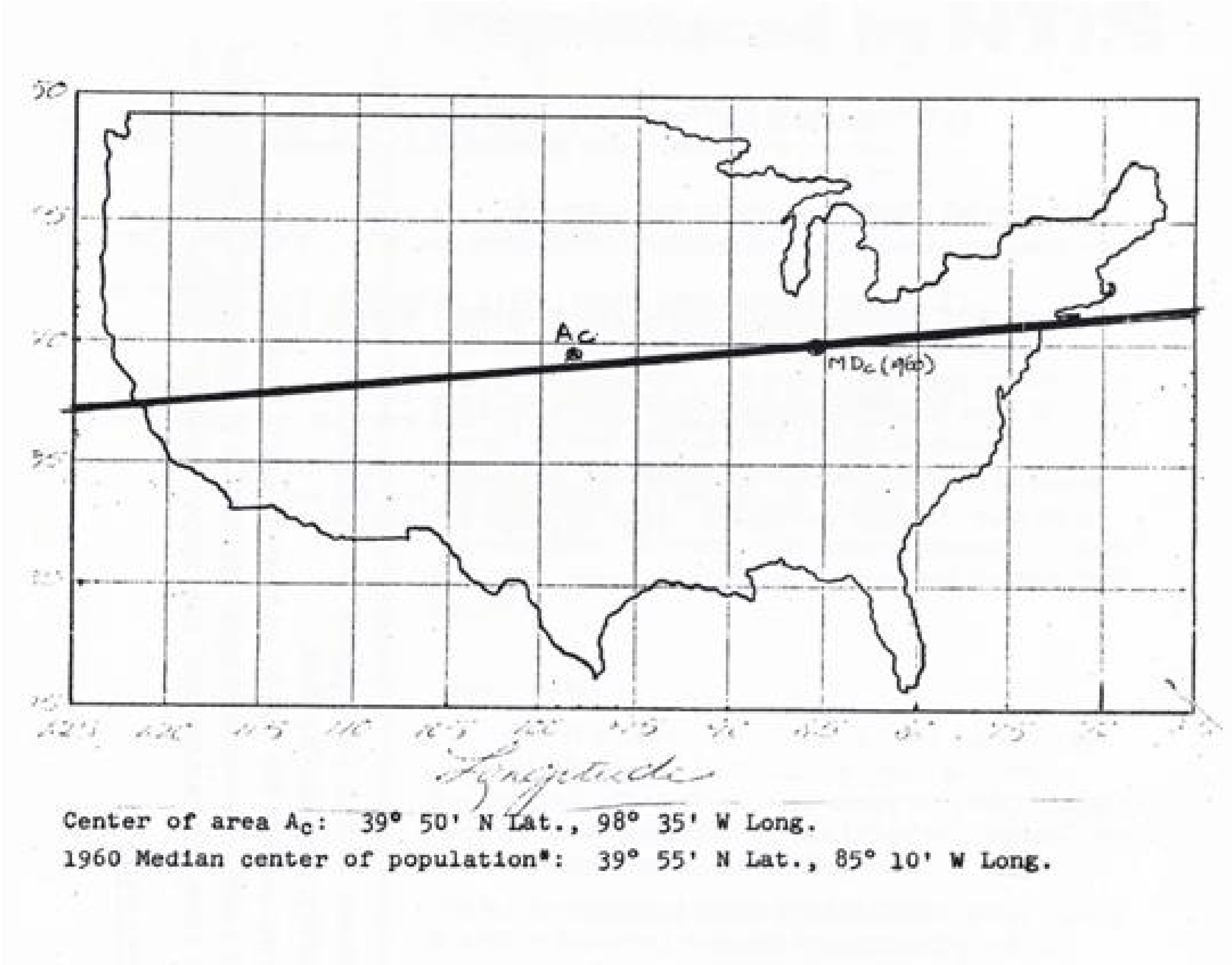

Figure 6: Map from the Sandwich Theorem Paper showing the line that simultaneous divides, area, population and income for the continental United States.
Courtesy Fellows of Harvard University.

Although much of what we have discussed in this paper so far may appear to the historian of cartography to have been a mere exercise in mathematics, it had much more effect on the future of cartography and the philosophical nature of what cartography would become than is generally realized.[36] Warntz, in the preface to the Existence Theorem paper discussed in some detail the philosophical and conceptual shifts that would be brought about by this new ability to import abstract ideas from mathematics into cartographic analysis. After a description of the Borsuk-Ulam theorem [37], another existence theorem, he writes about this new tool in cartography:

> Theoretical geography is a science of earth location and spatial relations. It describes, classifies, and predicts locations in the spatial sense. Cartographics stands to geographical science as graphics does to science generally. Mapping is a geographical concept in the theory of sets.[38]

This definition of mapping is radically innovative and extremely interesting. Cartography and mapping are no longer seen as just printed planar representations of the real world, but are algebraically defined and treated as applications of the theory of sets. Maps have become more abstract and axiomatic, but because of their ability to be algebraically manipulated, they have also become more useful tools for complex geographical analysis. This way of looking at maps was to have important implications for the development of GIS and other forms of computerized mapping. Warntz goes on:

> Cartography is the geographical example in the application of this concept. That which is ordinarily called a map by geographers and laymen is technically "a graphical image of a mapping." Whatever useful roles graphics plays in science generally also can be claimed for cartography with relation to geographical science. This is especially true now that most geographers increasingly employ geometry as an appropriate vehicle to carry their discipline. [39]

Thus, cartography was conceived of by Warntz in much the same way that the graph of an equation is conceived of in the mathematical sciences, as a tool for analysis. Warntz saw the map as an algebraically changeable image that is a visual representation of the abstract topology of the surfaces that made them up. For him this did not diminish the role of cartography but expanded it. Warntz described his conception of the nature of a map best when he said:

> There is already, of course, an accumulated stock of knowledge and experience concerning maps as stores of spatially stored information. We hope, however, to examine the expanded roles that mapping seems well suited to play in the sciences viewed from the standpoint of theoretical cartography and in the disciplines employing its models for decision making purposes.
>
> We recognize yet another role for maps. In the solution of certain problems for which mathematics, however elegantly stated, is intractable, graphical solutions are possible. This is especially true with regard to "existence" theorems. There are many cases in which the graphical solution to a spatial problem turns out to be a map in the full geographical sense of the term, "map." Thus a map is a solution to the problem.[40]

III. Conclusion: Just the Beginning

--If the intended application of mathematics is essential, how about parts of mathematics whose application —or at least what mathematicians take for their application—is quite fantastic?[41]
Wittgenstein

What we have tried to show in this paper, in what we must admit is a preliminary fashion, is the level to which abstract and pure mathematics played a role in the development of computer cartography and influenced the revolutionary conceptual changes that took place in the foundations of thematic map making during a period of extreme creativity at one lab in the 1960s and 1970s. For the history of cartography this is just one small example, using existence theorems, of the radical changes that future historians will have to deal with when trying to write that history. Challenges to the writing of that history are everywhere. Not only will historians of modern cartography have to confront and explain increasing levels of abstraction and technicality within the discipline of map making itself, but will also face preservation challenges to the material that makes up that history that have never before presented themselves in the long span of cartographic science. New technology in the form of computer hardware, software, algorithms and obsolete programming languages will all have to be preserved and archived in a way that gives future historians a chance to try to understand the full depth of the 20th century conceptual revolution in cartography, a small piece of which we have tried to explain here.[42] Most of the materials that will make up this history are ephemeral, composed on magnetic media, stored on computer printouts and buried in obscure journals whose long term viability is in doubt. Like those researchers at the forefront of the development of computer cartography in the 1960s, we historians of cartography are now also at an important beginning. We stand at a time when new tools and new methods must be brought to bare in our field in order to allow us, and those who will follow, to write the history of cartography of the 20th century and beyond.

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Notes

1. Ludwig Wittgenstein, *Remarks on the Foundations of Mathematics*, trans. G.E.M. Anscombe, (Cambridge, Mass..: MIT Press, 1979), 259.

2. The changes in cartography that occurred during this period fit in well with the philosophic model for the types of changes that occur during periods of rapid and radical epistemological change in the sciences discussed in Thomas Kuhn's *The Structure of Scientific Revolutions*, (Chicago: University of Chicago Press, 1962). Kuhn's philosophic models of paradigm shifts and lexical change are a good starting point for all those trying to discuss the nature of the changes that took place in cartography in the post-war period.

3. The research being accomplished currently for the 20th Century volume of *The History of Cartography*, edited by Mark Mommonier of Syracuse University and to be published by the University of Chicago Press,

should provide a starting point for future historians. The current author's article in that volume "Mathematics and Cartography" attempts to broadly deal with the more specific points highlighted in this study.

4. For a more complete look at the history of early GIS see Timothy W. Foresman editor, *The History of Geographic Information Systems* (Upper Saddle River, NJ: Prentice Hall, 1998). The contributors to this volume discuss the evolution of GIS at various University and Government agencies in Canada and the United States. One need only look at the diagram of the historical pathways and connections involved in the genesis of GIS on page 7 of Foresman's Introduction to understand the complex problems faced by historians in writing its history.

5. For purposes of this paper a thematic map will be defined as one that attempts to map the characteristics of a geographic phenomenon to reveal its spatial order and organization. See Judith Tyner, *Introduction to Thematic Cartography*, Englewood Cliff, NJ: Prentice Hall (1992), 10-11.

6. Arthur Cayley (1859) "On contour and slope lines,"*The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science* 18: 264-268.

7. James Clerk Maxwell (1870) "On hills and dales," *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science* 40: 421-427.

8.William Warntz, preface to Stephen Selkowitz, *Geography and an Existence Theorem: A Cartographic Solution to the Localization on a Sphere of Sets of Equal-Valued Antipodal Points for Two Continuous Distributions with Practical Applications to the Real Earth*. Harvard Papers in Theoretical Geography 21 (Cambridge, Mass. : Laboratory for Computer Graphics and spatial analysis, Center for Environmental Design Studies, Harvard University, 1968), i

9. William Warntz (1966) "The Topology of Socio-economic Terrain and Spatial Flows." *Papers, Regional Science Association* 17: 47-61.

10. Warntz, "Topology of Socio-Economic…", (see note 9), 51.

11. Warntz, "Topology of Socio-Economic…", (see note 9), 51.

12. There are many examples of this type of research being conducted at the Harvard Lab, including work by Frank Rens from the Harvard School of Architecture. Rens worked to find methods for creating smooth surfaces from data sets that had only discrete point observations. See Frank Rens "The Smoothing of Topographic Surfaces", in *Selected Projects: Harvard Laboratory for Computer Graphics and Spatial Analysis*, ed. Carl Steinitz, 1970.

13. John Q. Stewart, *Coasts, Waves and Weather* (Boston, Mass.: Ginn and Company,1945), and "Demographic Gravitation: Evidence and Application," *Sociometry* 11 (1948) 31-58.

14. John Q. Stewart, "The development of social physics," *American Journal of Physics* 84 (1950) 167-184.

15. William Warntz and Peter Wolff, *Breakthroughs in Geography*, (New York: New American Library, 1971).

16. Nicholas R. Chrisman, "Concepts of Space as a Guide to Cartographic Data Structures," in *First International Advanced Study Symposium on Topological Data Structures for Geographic Information Systems*, Volume 7, Spatial Semantics: Understanding and Interacting with Map Data. (Cambridge Mass.: Laboratory for Computer Graphics and Spatial Analysis, 1978) 1-19.

17. Chrisman, "Concepts of Space," (see 16).

18. This would spur a debate in philosophical circles regarding the nature of geographical objects that continues today. See Achille Varzi, "Introduction: Philosophical Issues in Geography", *Topoi* 20 (2001), 119-130, "Vagueness in Geography", *Philosophy & Geography* 4 (2001), 49-65, and "Fiat and Bona Fide Boundaries" (with Barry Smith), *Philosophy and Phenomenological Research* 60 (2000), 401-420. For an more geographically oriented take on these debates see J.F. Raper, "Spatial Representation: A Scientists Perspective", in Paul E. Longley et. al., *Geographic Information Systems: Principles and Technical Issues* (New York: John Wiley and Sons, 1999) 61-80; John Hessler, "From Topology to Mereology: Notes Toward a Philosophy of Cartography" *Topos* (forthcoming 2009).

19. James P. Corbett, *Topological Principles in Cartography* (Washington D.C.: US Census Bureau, 1979), 1.

20. Chrisman, "Concepts of Space" (see note 16).

21. Edmund Husserl, *The Crisis in the European Sciences and Transcendental Phenomenology*, trans. David Carr (Evanston, Il.: Northwestern University Press, 1970) 355.

22. William Warntz "Introduction" to Warntz et. al., *The Sandwich Theorem: A Basic One for Geography*. Harvard Papers in Theoretical Geography 44 (Cambridge, Mass.: Laboratory for Computer Graphics and Spatial Analysis, Harvard Univ.1971), i.

23. The Sandwich Theorem volume is divided into several parts. The first is Warntz's Introduction followed by a full translation of Hugo Steinhaus' original paper; "A Geometrical Analysis Concerning the Sandwich Theorem" by C. Ernesto Lindgren; "Solution to the Sandwich theorem with Two Distributions by Approximation" by Eduardo Lozano; "Euclidean Model for the Sandwich Theorem," by Luis Bonfiglioli; and "Implementation: Computer Program and One Example of the Halving of Three Distribution," by Ernesto Lindgren and Katherine Kiernan.

24. Warntz, "Introduction," *The Sandwich Theorem* (see note 22), ii.

25. Wartnz, "Introduction," *The Sandwich Theorem* (see note 22), i-ii.

26. Hugo Steinhaus, *Selected Papers* ed. Stanislaw Hartman (Warsaw: PWN-Polish Scientific Publishers,

1985), 533-548.

27. For historical background on this debate see Mathieu Marion , "Kronecker's Safe Haven of Real Mathematics" *Quebec Studies in the History of Science* 1 (1995): 189-215; David Corfield, *Toward a Philosophy of Real Mathematics* (Cambridge: Cambridge University Press, 2006); and Jeremy Gray, *Plato's Ghost: the Modernist Transformation of Mathematics* (Princeton, N.J.: Princeton University Press, 2008).

28. Warntz, "Introduction," *The Sandwich Theorem* (see note 22), iii.

29. Selkowitz, *Geography and an Existence Theorem* (see note 8), 60.

30. A.H. Stone and J.W. Turkey, "Generalized Sandwich Theorems," *Duke Mathematical Journal* 9 (1942), 356-9.

31. Lindgren, "Solution to the Sandwich Theorem," *The Sandwich Theorem* (see note 22), 3.

32. Lindgren, "Solution to the Sandwich Theorem," *The Sandwich Theorem* (see note 22), 3.

33. Lindgren, "Solution to the Sandwich Theorem," *The Sandwich Theorem* (see note 22), 3-4.

34. C. Ernesto S. Lindgren and Steve M. Blary, *Four-Dimensional Descriptive Geometry* (New York: McGraw-Hill Book Company, 1968).

35. Nick Chrisman, *Charting the Unknown: How Computer Mapping at Harvard Became GIS* (Redlands, Ca.: ERSI Press, 2006), 67.

36. The importance of the role of the Harvard Lab itself in the development of GIS is a controversial issue but the work accomplished at the Lab was very much in keeping with developments going on elsewhere in computer cartography. Applications of pure mathematics of the type we are discussing in this paper were attempted by many researchers and the discussion here is merely an example. Those interested in a broader look at primary research from a number of different labs should consult the publications found in the *Proceedings of the First International Advanced Study Symposium on Topological Data Structures for Geographic Information Systems*, 8 volumes, Geoffrey Dutton editor, (Cambridge, Mass.: Fellows of Harvard University, 1979).

37. For more information on this theorem see Jiri Matousek, *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry* (Berlin: Springer Verlag, 2003).

38.Warntz, preface to Selkowitz, *Geography and an Existence Theorem* (see note 8), ii.

39. Warntz, preface to Selkowitz, *Geography and an Existence Theorem* (see note 8), ii-iii.

40. Warntz, preface to Selkowitz, *Geography and an Existence Theorem* (see note 8), iii-iv.

41. Wittgenstein, *Remarks*, 260.

42. John Hessler, “Archive Fever: Challenges in Preserving Conceptual and Foundational Cartographic Materials from the Twentieth Century and Beyond,” *Journal of Map and Geography Libraries* 3 (2007): 79-95.