Reconstruction of a multimode entangled state using a two-photon phase-sensitive linear amplifier

Mashhood Ahmad,[1] Shahid Qamar,[2,3] and M. Suhail Zubairy[1,2]
[1]*Department of Electronics, Quaid-i-Azam University, Islamabad, Pakistan*
[2]*Institute for Quantum Studies and Department of Physics, Texas A&M University, College Station, Texas 77843-4242*
[3]*Department of Physics and Applied Mathematics, Pakistan Institute of Engineering and Applied Sciences, Nilore, Islamabad, Pakistan*
(Received 22 November 2002; published 23 April 2003)

We propose a model for the measurement of an arbitrary multimode entangled state of the cavity field using two-photon correlated emission laser. We consider two cases: (a) The modes have different frequencies and are detected separately and (b) the modes consist of two orthogonal polarization states and are detected using a single balanced homodyne detector. The basic idea is to amplify the initial multimode state such that there is no-noise in the quadrature of interest and all the noise is fed into the conjugate quadrature component. The amplified noise-free quadrature is prepared in different phases and then corresponding quadrature distribution is measured. The Wigner function of the initial multimode entangled state is then reconstructed by using inverse Radon transformation. This scheme is insensitive to the noise associated with the nonunit efficiency of the detector in the homodyne detection measurement scheme.

DOI: 10.1103/PhysRevA.67.043815 PACS number(s): 42.50.Ct, 32.80.Qk

I. INTRODUCTION

Quantum entanglement is one of the most fascinating nonclassical properties of a composite quantum system [1]. Its nonlocal character lies at the heart of quantum information theory. The idea of teleportation [2], quantum computing [3], quantum error correction [4], cryptography [5] and many more [6] reside on the quantum entanglement. In this paper, we propose a scheme for the measurement of an arbitrary multimode entangled state of the cavity field.

Methods to measure the quantum state of the running field as well as the cavity field have been extensively studied in recent years. There are different proposals for the reconstruction of quantum state of the field based upon quantum tomography [7–9], absorption and emission spectroscopy [10], the conditional measurement of the atoms in a micromaser cavity [11], dispersive interaction of a single circular Rydberg atom with the cavity field [12], and others [13]. Recently, some pioneering experiments have also been done to measure the quantum state of the field [14] (for a review, see Refs. [15,16]). However, most of the recent studies are related to a single mode of the cavity field and there are only a few schemes for the measurement of multimode field inside a cavity [17].

Earlier, we proposed a scheme for the reconstruction of a single-mode quantum state of the cavity field [8]. The scheme is based upon the amplification of the field using a two-photon correlated emission laser (CEL) amplifier [18,19]. During the amplification, there is no noise in the quadrature of interest and all the noise is fed into the conjugate quadrature. The complete distribution for the noise-free field quadrature is measured via optical homodyne detection. The Wigner function of the initial quantum state is then reconstructed by using inverse radon transformation familiar in quantum state tomography. Following the same proposal, we study here the reconstruction of an arbitrary multimode entangled state of the cavity field. We consider two possible cases. In the first case, the cavity modes are defined in terms of different frequency component and detected separately using balanced homodyne detectors (BHD's) and in the other case the modes consist of two orthogonal polarization states and detected using a single balanced homodyne detector.

The case in which the cavity modes consist of different frequency components can be handled by separating n modes and sending them into separate BHD's. The joint quadrature distribution $\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)$ can be obtained by measuring the noise-free quadrature components $x_1(\theta_1)$, $x_2(\theta_2),\ldots,x_n(\theta_n)$ by varying the local oscillator phases θ_1, $\theta_2,\ldots,\theta_n$ from 0 to π, independently. Multimode amplification with reduced noise for different quadrature phases can be obtained by injecting multiple beams of three-level atoms inside the cavity, initially prepared in a coherent superposition of their upper and lower atomic levels using classical fields of different values of phases φ_1, $\varphi_2,\ldots,\varphi_n$, accordingly. The joint distribution function is then used to reconstruct the Wigner function of the multimode field using quantum state tomography.

In case when the cavity modes consist of two orthogonal polarization states of the cavity field, the noisefree amplification can be obtained by using the same two-photon CEL setup. The two beams of injected atoms independently amplify the two polarization modes such that there is no noise in the quadrature of interest. On the measurement side, we can use a single BHD setup to detect the cavity field in which the local oscillator field is in a linear superposition of the two polarization modes [20]. The field which is allowed to leak through the end mirror of the cavity is passed through a controllable phase shifter and then through a rotatable polarizer prior to its entry in the balanced homodyne detector. The generalized quadrature distribution $\omega(x,\theta_1,\theta_2,\psi)$ is calculated from the measured generalized quadrature

$$x(\theta_1,\theta_2,\psi)=\frac{1}{2}\{[a_1\exp(-i\theta_1)+a_1^\dagger\exp(i\theta_1)]\cos(\psi)+[a_2\exp(i\theta_2)+a_2^\dagger\exp(i\theta_2)]\sin(\psi)\}.$$

Here $\theta_2=\theta_1-\varphi$ is the difference between the phase θ_1 of the local oscillator and the phase difference φ introduced

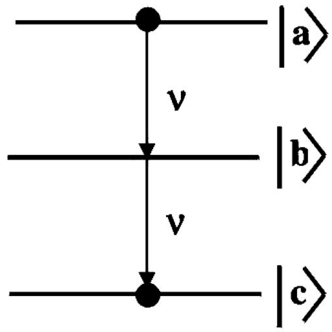

FIG. 1. Energy level scheme for three-level atoms.

between the modes of the field by the phase shifter. The angle ψ is controlled by a polarizer that determines the relative amplitude of the two modes that enter the BHD. To obtain noise-free amplification for a set of generalized quadrature phases given by θ_1 and θ_2, we prepare the amplifiers in different phases φ_1 and φ_2, accordingly. We have calculated the complete noise-free generalized quadrature distribution $\omega(x,\theta_1,\theta_2,\psi)$ for θ_1, θ_2, and ψ, all varying from 0 to π for an arbitrary two-mode entangled quantum state after its amplification through a two-photon CEL. The noise-free quadrature distributions are then used to reconstruct the Wigner function of initial state by employing quantum state tomography [7]. It may be pointed out that the proposed scheme overcomes the problems associated with the nonunit efficiency of the detector in homodyne detection measurement scheme.

The paper is organized as follows. In Sec. II, we present the model of a two-photon CEL, and in Sec. III and IV, we discuss the reconstruction scheme of a multimode cavity field for different frequency and polarization modes, respectively. Details of the calculations for multimode tomography are presented in the Appendix.

II. TWO-PHOTON PHASE-SENSITIVE LINEAR AMPLIFIER

For a phase-sensitive linear amplification, we consider the model of a two-photon CEL amplifier proposed by Scully and Zubairy [18]. In a CEL, atomic coherence is produced by considering a beam of three-level atoms in a cascade configuration (as shown in Fig. 1), initially prepared in a coherent superposition of their upper and lower atomic states $|a\rangle$ and $|c\rangle$, i.e.,

$$\rho_i=\rho_{aa}|a\rangle\langle a|+\rho_{ac}|a\rangle\langle c|+\rho_{ca}|c\rangle\langle a|+\rho_{cc}|c\rangle\langle c|. \quad (1)$$

The atoms are injected at a rate R inside the cavity where they interact with the cavity mode for a time τ, the injection rate and interaction time are such that there is not more than one atom at a particular time inside the cavity. The mode frequency ν of the cavity field is also considered to be resonant with the atomic transitions $|a\rangle-|b\rangle$ and $|b\rangle-|c\rangle$. The evolution of the reduced density matrix of the field is given by the following master equation:

$$\dot{\rho}_F=-\frac{A}{2}(N+1)[aa^\dagger\rho-2a^\dagger\rho a+\rho aa^\dagger]-\frac{A}{2}N[a^\dagger a\rho -2a\rho a^\dagger+\rho a^\dagger a]-\frac{A}{2}M^*[aa\rho-2a\rho a+\rho aa] -\frac{A}{2}M[a^\dagger a^\dagger\rho-2a^\dagger\rho a^\dagger+\rho a^\dagger a^\dagger], \quad (2)$$

where $A=Rg^2\tau^2(\rho_{aa}-\rho_{cc})$ is the gain coefficient. Here g is the atom-field interaction constant and ρ_{aa} (ρ_{cc}) is the density-matrix element corresponding to atoms in level $|a\rangle$ ($|c\rangle$). The constants N and M are defined as

$$N=\frac{\rho_{cc}}{(\rho_{aa}-\rho_{cc})}, \qquad M=\frac{\rho_{ac}}{(\rho_{aa}-\rho_{cc})}. \quad (3)$$

The terms proportional to M contain the phase sensitivity of the coherent atomic superposition, which play a crucial role during the amplification process. Two-photon phase-sensitive linear amplifier allows us to amplify the signal such that there is no noise in one of the quadrature components and all the noise is fed into the conjugate quadrature. Therefore, quantum features associated with the field remain intact in one of the quadrature components and can be measured using the balanced homodyne detection scheme. The measured quadrature components then can be used to reconstruct the initial quantum state of the cavity field.

Here we present an intuitive explanation of why the two-photon CEL serves as a phase-sensitive amplifier. During the amplification process, spontaneous emission event occurs, and as a result the atom undergoes a spontaneous transition from the upper level $|a\rangle$ to the middle level $|b\rangle$. The randomness of this transition leads to an arbitrary phase of $|b\rangle$ which is not determined by the atomic coherence. However, in the subsequent transition of the atom from level $|b\rangle$ to level $|c\rangle$, the atom remembers the arbitrary phase of the level $|b\rangle$ and as a result, the total phase coherence is preserved in a CEL. In other words, the noise which is created during the spontaneous transition from level $|a\rangle$ to $|b\rangle$ is compensated by a subsequent transition from $|b\rangle$ to $|c\rangle$ such that the combined field has the same phase. This phase is completely determined by the atomic coherence which is initially introduced between the levels $|a\rangle$ and $|c\rangle$. Therefore, the noise created by spontaneous emission events in a CEL is quenched and the process serves as a phase-sensitive amplification.

To reconstruct a multimode cavity field, we need noise-free amplification with respect to all the cavity modes. For different frequency modes we can inject multiple beams of three-level atoms inside the cavity, initially prepared in a coherent superposition of their upper and lower atomic states such that the transition frequency of each atom in a particular beam is resonant with a specific cavity mode. Under this condition, the individual frequency modes amplify independently and the noise-free amplification can be obtained. However, for two orthogonal polarization modes with the same frequency, we can consider two beams of atoms having the same transition frequency resonant with the cavity field. The atoms are again prepared in a coherent superposition of their upper and lower atomic states and injected inside the cavity. The injected atoms amplify the two polarizations modes independently such that there is no noise in the quadrature of interest.

The initial quantum state can be obtained by reconstructing the corresponding Wigner function. Here we study the evolution of the Wigner function for any arbitrary n-mode entangled state of the cavity field after amplification through a phase-sensitive linear amplifier. In the following sections, we show how it can be used to reconstruct the quantum state of the cavity field for different frequency or polarization modes. The evolution of the Wigner function for a single-mode field can be obtained using Eq. (2) (see Ref. [8]), which can be easily generalized for a multimode field and is given by the following:

$$W(\alpha_1,\alpha_2,\ldots,\alpha_n,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^2\beta_1 d^2\beta_2\cdots d^2\beta_n W(\beta_1,\beta_2,\ldots,\beta_n,0)\left[\prod_{j=1}^{n}W_c(\alpha_j,\beta_j;t)\right], \tag{4}$$

where n corresponds to the number of cavity modes and the conditional probability $W_c(\alpha_j,\beta_j;t)$ is given by

$$W_c(\alpha_j,\beta_j;t)=\frac{1}{\pi(G-1)\sqrt{(N_j+1/2)^2-|M_j|^2}}\exp\left[-\frac{[|\alpha_j|\cos(\vartheta_j-\varphi_j/2)-\sqrt{G}|\beta_j|\cos(\theta_{0_j}-\varphi_j/2)]^2}{[N_j+1/2-|M_j|](G-1)}\right.$$
$$\left.-\frac{[|\alpha_j|\sin(\vartheta_j-\varphi_j/2)-\sqrt{G}|\beta_j|\sin(\theta_{0_j}-\varphi_j/2)]^2}{[N_j+1/2+|M_j|](G-1)}\right]. \tag{5}$$

Here $G=\exp(At)$ is defined as the gain factor which is assumed to be the same for all the amplified modes. The complex quantities α_j and β_j are expressed in the polar forms as $\alpha_j=|\alpha_j|\exp(i\vartheta_j)$ and $\beta_j=|\beta_j|\exp(i\theta_{0_j})$. The phase φ_j corresponds to the coherent superposition of levels $|a_j\rangle-|c_j\rangle$, i.e., $\rho_{a_jc_j}=|\rho_{a_jc_j}|\exp(i\varphi_j)$ for the jth beam of the atoms.

In the case of perfect coherence, we have the relations $|\rho_{a_jc_j}|=\sqrt{\rho_{a_ja_j}\rho_{c_jc_j}}$. The squeezing parameters r_j are defined such that [21]

$$\tanh^2 r_j=\frac{\rho_{a_ja_j}}{\rho_{c_jc_j}}.$$

In terms of the squeezing parameters r_j, the constants N_j and $|M_j|$ are given by

$$N_j=\sinh^2(r_j),$$
$$|M_j|=\frac{\sinh(2r_j)}{2}. \tag{6}$$

It is clear from Eq. (4) that the initial state which is characterized by an arbitrary n-mode entangled state of the cavity field is amplified such that each individual mode amplifies independently.

III. QUANTUM STATE RECONSTRUCTION: *N*-CAVITY MODES WITH DIFFERENT FREQUENCIES

First we discuss the measurement of the Wigner function for a multimode cavity field which consists of different frequency components. Different frequency modes can be detected in separate balanced homodyne detection setup and the joint quadrature distribution can be obtained. The quadrature of each mode is given as

$$x_j(\theta_j)=\tfrac{1}{2}[a_j^\dagger\exp(i\theta_j)+a_j\exp(-i\theta_j)], \tag{7}$$

for $j=1,2,3,\ldots,n$, where n is the number of modes inside the cavity. Here phase θ_j can be varied independently by varying the phase of the jth local oscillator. A complete joint distribution for the measured quadratures is given by $\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots\theta_n)$. The joint quadrature distribution for the amplified multimode field, which bears a one to one correspondence with the n-mode Wigner function, can be obtained using Eqs. (A6) and (4) and is given by

$$\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^2\beta_1 d^2\beta_2\cdots d^2\beta_n W(\beta_1,\beta_2,\ldots,\beta_n,0)\left[\prod_{j=1}^{n}P(x_j,\theta_j;t)\right], \tag{8}$$

where $P(x_j,\theta_j;t)$ is given by [8]

$$P(x_j,\theta_j;t)=\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{(G-1)[N_j+1/2-|M_j|\cos(2\theta_j-\varphi_j)]}}\exp\left[-\frac{[x_j-\sqrt{G}(\beta_{x_j}\cos\theta_j+\beta_{y_j}\sin\theta_j)]^2}{(G-1)[N_j+1/2-|M_j|\cos(2\theta_j-\varphi_j)]}\right]. \tag{9}$$

It is clear from Eq. (9) that a one to one correspondence exists between the phases $\varphi_{j=1,2,\ldots,n}$ of the atomic coherence and the phases $\theta_{j=1,2,\ldots,n}$ of the local oscillator. To reconstruct the Wigner function, we need a complete set of joint quadrature distribution $\omega(x_1,x_2,\ldots,x_n;\ \theta_1,\theta_2,\ldots,\theta_n)$ for $\theta_{j=1,2,\ldots,n}$ all varying from 0 to π. The noise-free amplification for n modes can be obtained by injecting n beams of three-level atoms initially prepared in a coherent superposition of their upper and lower atomic states with a fixed phase φ_j and by adjusting the phase of separate local oscillators θ_j such that $\theta_j=\varphi_j/2$. The complete set of quadrature distribution for θ_j going from 0 to π can be obtained if we prepare the n beams of atoms for a set of values of atomic coherent superposition phases φ_j ranging from 0 to 2π.

Once the joint quadrature distribution $\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)$ is measured using separate balanced homodyne detection setup, then the corresponding Wigner function can be obtained by carrying out optical tomography (see the Appendix). On substituting Eqs. (8)–(9) into Eq. (A7), we obtain the Wigner function of the noise-free amplified quantum state in terms of the rescaled variables $\acute{\alpha}_{x_1}=\alpha_{x_1}/\sqrt{G},\ldots,\acute{\alpha}_{x_n}=\alpha_{x_n}/\sqrt{G}$, $\acute{\alpha}_{y_1}=\alpha_{y_1}/\sqrt{G},\ldots,\acute{\alpha}_{y_n}=\alpha_{y_n}/\sqrt{G}$, and $\acute{\eta}_1=\eta_1/\sqrt{G},\ldots,\acute{\eta}_n=\eta_n/\sqrt{G}$, which is given by the following:

$$W(\acute{\alpha}_1,\acute{\alpha}_2,\ldots,\acute{\alpha}_n,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{\pi}\cdots\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{\pi}d^2\beta_1 d\acute{\eta}_1 d\theta_1\cdots d^2\beta_n d\acute{\eta}_n d\theta_n|\acute{\eta}_1|\cdots|\acute{\eta}_n|$$

$$\times W(\beta_1,\beta_2,\cdots,\beta_n,0)\left[\prod_{j=1}^{n}Q(\acute{\alpha}_j,\beta_j;t)\right], \tag{10}$$

where

$$Q(\acute{\alpha}_j,\beta_j;t)=\frac{1}{(4\pi^2)}\exp\left[-\frac{(1-1/G)\exp(-2r_j)}{8}\acute{\eta}_j{}^2-i\acute{\eta}_j\{(\acute{\alpha}_{x_j}-\beta_{x_j})\cos(\theta_j)+(\acute{\alpha}_{y_j}-\beta_{y_j})\sin(\theta_j)\}\right]. \tag{11}$$

Here we have used the definition

$$(N_j+1/2-|M_j|)=\tfrac{1}{2}\exp(-2r_j). \tag{12}$$

It is clear from Eq. (11) that for sufficiently large squeezing, i.e., in the limit when $r_{j=1,2,\ldots,n}\to\infty$ and for any arbitrary value of the gain parameter $G>1$, we recover the Wigner function of initial multimode quantum state of the field. It is interesting to see that any arbitrary n-mode field which consists of different frequencies can be reconstructed using the proposed scheme, however, an appropriate rescaling of the measured distribution is required.

IV. QUANTUM STATE RECONSTRUCTION: TWO POLARIZATION MODES OF THE CAVITY FIELD HAVING THE SAME FREQUENCY

Here we consider the case when the two radiation modes have the same frequency but orthogonal polarizations. In order to measure such a two-mode field, we follow the scheme proposed in Ref. [20] which suggests the use of a single set of balanced homodyne detection. The cavity field which is allowed to leak through the end mirror is first passed through a phase shifter that produces a relative phase shift of φ in between the two modes of the field. The field is then passed through a polarizer that determines the relative amplitude of the two modes which then enter in a balanced homodyne detector. The field operator after the polarizer is given by the following:

$$a=a_1\cos(\psi)+a_2\exp(i\varphi)\sin(\psi), \tag{13}$$

where a_1 and a_2 correspond to modes one and two, respectively. The two-mode entangled state is then made incident on a balanced homodyne detector which measures the quadrature component

$$x(\theta_1,\theta_2,\psi)=x(\theta_1,\theta_2,\psi)^\dagger=\tfrac{1}{2}[(a_1^\dagger\exp^{i\theta_1}+a_1\exp^{-i\theta_1})\cos\psi$$
$$+(a_2^\dagger\exp^{i\theta_2}+a_2\exp^{-i\theta_2})\sin\psi], \tag{14}$$

where $\theta_2=\theta_1-\varphi$. In the balanced homodyne measurement, phase θ_1 can be varied by changing the phase of the local oscillator and phases φ and ψ are controlled by the phase shifter and polarizer, respectively. A complete distribution for $x(\theta_1,\theta_2,\psi)$ is given by the quadrature distribution $\omega(x,\theta_1,\theta_2,\psi)$, which bears a one to one correspondence with the two-mode Wigner function $W(\alpha_1,\alpha_2,t)$ and is given by Eq. (A11). On substituting for $W(\alpha_1,\alpha_2,t)$, i.e., n up to 2 from Eq. (4) into Eq. (A11), we obtain $\omega(x,\theta_1,\theta_2,\psi)$ for the amplified quantum state as

$$\omega(x,\theta_1,\theta_2,\psi)=\frac{1}{\sqrt{\pi(G-1)(\kappa_1\cos^2(\psi)+\kappa_2\sin^2(\psi))}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^2\beta_1 d^2\beta_2 W(\beta_1,\beta_2,0)\times\exp\Bigg[-\frac{1}{(G-1)[\kappa_1\cos^2(\psi)+\kappa_2\sin^2(\psi)]}\{x-\sqrt{G}([\beta_{x_1}\cos(\theta_1)+\beta_{y_1}\sin(\theta_1)]\cos(\psi)+[\beta_{x_2}\cos(\theta_2)+\beta_{y_2}\sin(\theta_2)]\sin(\psi))\}^2\Bigg], \tag{15}$$

where $\kappa_1=N_1+1/2-|M_1|\cos(2\theta_1-\varphi_1)$ and $\kappa_2=(N_2+1/2-|M_2|\cos(2\theta_2-\varphi_2)$. It is clear from Eq. (15) that the phases φ_1 and φ_2 of the atomic coherences and θ_1 and θ_2 of the field quadratures exhibit a one to one correspondence which is quite interesting. To reconstruct the Wigner function of the initial two-mode entangled state, we need a set of distribution functions $\omega(x,\theta_1,\theta_2,\psi)$ for different values of θ_1, θ_2, and ψ, all varying from 0 to π.

The two-mode Wigner function can be reconstructed by amplifying the signal such that there is no noise in the desired generalized quadrature measured by the balanced homodyne detector and all the noise is fed into the conjugate generalized quadrature. It follows from Eq. (15) that the amplified signal without the added noise in the quadrature $\omega(x,\theta_1,\theta_2,\psi)$ can be obtained if we choose $2\theta_1-\varphi_1=0$ and $2\theta_2-\varphi_2=0$. In order to obtain noise-free amplification for the two-mode entangled cavity field, we prepare the amplifier by injecting two beams of three-level atoms initially prepared in a coherent superposition of their upper and lower atomic states with fixed phases φ_1 and φ_2, respectively. The injected atoms amplify the two polarizatio*n* modes of the cavity field independently. The noise-free generalized quadrature can be obtained by adjusting the phase of the local oscillator θ_1 such that $\theta_1=\varphi_1/2$ and $\varphi=\theta_1-\varphi_2/2$ (where φ is the phase difference between the two modes of the field produced by the phase shifter). To find a complete set of distribution $\omega(x,\theta_1,\theta_2,\psi)$, we prepare the two beams of atoms for a set of values of atomic coherent superposition phases φ_1 and φ_2 ranging from 0 to 2π and obtain the noise-free amplification for the desired quadratures. The Wigner function of the initial two-mode quantum state can then be reconstructed from the measured values of noise-free generalized quadrature $\omega(x,\theta_1,\theta_2,\psi)$ by carrying out the inverse Radon transformation familiar in the tomographic imaging [7]. On substituting Eq. (15) into Eq. (A12) and after some simplification, we obtain the following expression for the Wigner function in terms of the rescaled variables $\acute{\alpha}_{x_1}=\alpha_{x_1}/\sqrt{G}$, $\acute{\alpha}_{y_1}=\alpha_{y_1}/\sqrt{G}$, $\acute{\alpha}_{x_2}=\alpha_{x_2}/\sqrt{G}$, and $\acute{\alpha}_{y_2}=\alpha_{y_2}/\sqrt{G}$:

$$W(\acute{\alpha}_1,\acute{\alpha}_2,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^2\beta_1 d^2\beta_2 W(\beta_1,\beta_2,0)\times\left[\prod_{j=1}^{2}R(\acute{\alpha}_j,\beta_j;t)\right], \tag{16}$$

where

$$R(\acute{\alpha}_j,\beta_j;t)=\frac{2}{\pi(1-1/G)\exp(-2(r_j))}\exp\left[-\frac{2}{(1-1/G)}\times\left\{\frac{(\acute{\alpha}_{x_j}-\beta_{x_j})^2+(\acute{\alpha}_{y_j}-\beta_{y_j})^2}{\exp(-2r_j)}\right\}\right]. \tag{17}$$

Here we have used

$$(N_1+1/2-|M_1|)=\tfrac{1}{2}\exp(-2r_1),$$
$$(N_2+1/2-|M_2|)=\tfrac{1}{2}\exp(-2r_2). \tag{18}$$

For sufficiently strong squeezing, i.e., in the limit when $r_{j=1,2}\to\infty$ and for any arbitrary value of the gain parameter $G>1$, the function $R(\acute{\alpha}_j,\beta_j;t)$ approaches a δ function and we obtain the same original two-mode entangled state. This clearly shows that any arbitrary two-mode entangled quantum state which is defined in terms of two orthogonal polarization states can be fully recovered after its amplification through a phase-sensitive linear amplifier and an appropriate rescaling of the measured distribution.

In conclusion, we propose a scheme for the measurement of multimode entangled state of the cavity field using phase-sensitive linear amplification. We consider two cases of interest: (i) modes are defined in terms of different frequencies and (ii) modes consist of two polarization states. It is shown that in both cases we can recover the original quantum state after amplification through a two-photon CEL amplifier, however, an appropriate rescaling of the measured distribution is required. For different frequency modes, the proposed scheme can be used to reconstruct any arbitrary n-mode field, which is quite interesting. It may be pointed out that, in a recent study, Santos, Lutterbach, and Davidovich have proposed an interesting scheme for the measurement of the Wigner function for n modes with different frequencies in the same or in different cavities [22].

Here we would like to mention that, during the measurement, cavity field leaks through the cavity. To ensure that the field leakage through the end mirrors does not occur during the amplification process, the time scales in the experiment have to be adjusted such that the total amplification time is very small as compared to the cavity decay time. The proposed scheme overcomes the problems associated with the nonunit efficiency of the detector.

ACKNOWLEDGMENTS

We would like to thank Tomas Opatrny for useful discussions. We gratefully acknowledge the support from the Air Force Research Laboratories (Rome, New York), DARPA-QuIST, TAMU Telecommunications and Informatic Task force (TITF) initiative, and the Office of Naval Research. One of us (S.Q.) would like to thank the Ministry of Science & Technology, Government of Pakistan, for financial support.

APPENDIX: MULTIMODE TOMOGRAPHY

In this appendix, we show how to determine the Wigner function in terms of probability distribution for the rotated quadrature phase for multimode field. We consider two cases of interest, one in which we assume the separate detection of different cavity modes and the other in which superposition of modes is detected.

1. Case-I: Separate detection of different cavity modes

The Wigner function may be defined as Fourier transforms of the characteristic function [23], which for the case of n modes is given by the following:

$$\tilde{W}(\xi_1,\xi_2,\ldots,\xi_n,t)=\mathrm{Tr}\{\exp[(\xi_1\hat{a}_1^{\dagger}-\xi_1^{*}\hat{a}_1)+(\xi_2\hat{a}_2^{\dagger}-\xi_2^{*}\hat{a}_2)+\ldots+(\xi_n\hat{a}_n^{\dagger}-\xi_n^{*}\hat{a}_n)]\hat{\rho}\}, \tag{A1}$$

i.e.,

$$W(\alpha_1,\alpha_2,\cdots,\alpha_n,t)=\frac{1}{\pi^{2n}}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}d^2\xi_1 d^2\xi_2\cdots d^2\xi_n \times\tilde{W}(\xi_1,\xi_2,\ldots,\xi_n,t)\exp\{\alpha_1\xi_1^{*}-\alpha_1^{*}\xi_1+\alpha_2\xi_2^{*}-\alpha_2^{*}\xi_2+\ldots+\alpha_n\xi_n^{*}-\alpha_n^{*}\xi_n\}, \tag{A2}$$

where ρ is the density operator.

If the n modes are separable then we can simply send them into n separate balanced homodyne detection systems and measure the joint statistics of their outputs. The homodyne detector measures the generalized quadrature-amplitude of the signal given by Eq. (7) in Sec. III. The complete information of the n-mode field can be obtained through its joint quadrature distribution $\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)$, which is defined as the Fourier transform of the characteristic function and is given by

$$\tilde{\omega}(\eta_1,\eta_2,\ldots,\eta_n;\theta_1,\theta_2,\ldots,\theta_n)=\mathrm{Tr}(\exp\{i[\eta_1\hat{x}_1(\theta_1)+\eta_2\hat{x}_2(\theta_2)+\cdots+\eta_n\hat{x}_n(\theta_n)]\}\hat{\rho}), \tag{A3}$$

i.e.,

$$\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)=\frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}d\eta_1 d\eta_2\cdots d\eta_n\tilde{\omega}(\eta_1,\eta_2,\ldots,\eta_n;\theta_1,\theta_2,\ldots,\theta_n)\times\exp\{-i(\eta_1x_1+\eta_2x_2+\cdots+\eta_nx_n)\}. \tag{A4}$$

A one to one correspondence can be established between $W(\alpha_1,\alpha_2,\ldots,\alpha_n,t)$ and $\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)$ which can be seen through Eqs. (A1) and (A3). On using the definition of quadrature phases given in Sec. III via Eq. (7), we immediately obtain

$$\tilde{\omega}(\eta_1,\eta_2,\ldots,\eta_n;\theta_1,\theta_2,\ldots,\theta_n)=\tilde{W}\left(i\frac{\eta_1}{2}\exp(i\theta_1),i\frac{\eta_2}{2}\exp(i\theta_2),\ldots,i\frac{\eta_n}{2}\exp(i\theta_n),t\right). \tag{A5}$$

If $\tilde{\omega}(\eta_1,\eta_2,\ldots,\eta_n;\theta_1,\theta_2,\ldots,\theta_n)$ is known for all $\eta_1,\eta_2,\ldots,\eta_n$ values in the range $-\infty<\eta_{i=1,2,\ldots,n}<\infty$ and for all $\theta_1,\theta_2,\ldots,\theta_n$ values in the range $0\leqslant\theta_{i=1,2,\ldots,n}<\pi$, then the characteristics, function $\tilde{W}(\xi_1,\xi_2,\ldots,\xi_n,t)$ is known in the whole complex plane $\xi_1,\xi_2,\ldots,\xi_n$. Therefore, there is a one to one correspondence between the characteristic functions given by Eqs. (A1) and (A3) and therefore, we also have a one to one correspondence between the Wigner function given by Eq. (A2) and the joint quadrature distribution given by Eq. (A4).

Using the Fourier transform of Eq. (A5) and inserting the inverse Fourier transform of Eq. (A2), we obtain

$$\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^2\alpha_1 d^2\alpha_2\cdots d^2\alpha_n d\eta_1 d\eta_2\cdots d\eta_n W(\alpha_1,\alpha_2,\ldots,\alpha_n,t)$$

$$\times\prod_{j=1}^{n}\frac{1}{(2\pi)^n}\exp\{-i\eta_j(x_j-\alpha_{x_j}\cos\theta_j-\alpha_{y_j}\sin\theta_j)\}. \tag{A6}$$

By similar steps we obtain from Eq. (A5) the inverse of Eq. (A6),

$$W(\alpha_1,\alpha_2,\ldots,\alpha_n,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{\pi}\cdots\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{\pi}dx_1 d\eta_1 d\theta_1\cdots dx_n d\eta_n d\theta_n|\eta_1|\cdots|\eta_n|$$

$$\times\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)\prod_{j=1}^{n}\frac{1}{(4\pi^2)^j}\exp\{i\eta_j[x_j-\alpha_{x_j}\cos(\theta_j)-\alpha_{y_j}\sin(\theta_j)]\}. \tag{A7}$$

It is clear that the Wigner function $W(\alpha_1,\alpha_2,\ldots,\alpha_n,t)$ for any arbitrary value of n can be determined from the quadrature distribution $\omega(x_1,x_2,\ldots,x_n;\theta_1,\theta_2,\ldots,\theta_n)$.

2. Case-II: Detection of superposition of cavity modes

Next, we consider the situation when the two modes cannot be separated, then it is not possible to measure the quadratures, independently. Under this situation, the two-mode field can be measured using a single set of balanced homodyne detection, as discussed in Sec. IV. The generalized quadrature distribution $\omega(x,\theta_1,\theta_2)$ for the two modes is defined as the Fourier transform of the characteristic function

$$\tilde{\omega}(\eta,\theta_1,\theta_2)=\mathrm{Tr}\{\exp[i\eta\hat{x}(\theta_1,\theta_2)]\hat{\rho}\}, \tag{A8}$$

i.e.,

$$\omega(x,\theta_1,\theta_2)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\eta\tilde{w}(\eta,\theta_1,\theta_2)\exp(-i\eta x). \tag{A9}$$

Again a one to one correspondence can be established in between $W(\alpha_1,\alpha_2,t)$ and $\omega(x,\theta_1,\theta_2)$ through Eq. (A1) (for $n=2$) and Eq. (A8). Using the definition [see Eq. (14)] given in Sec. IV, we obtain

$$\tilde{\omega}(\eta,\theta_1,\theta_2)=\tilde{W}\left(i\frac{\eta\cos(\psi)}{2}\exp(i\theta_1),i\frac{\eta\sin(\psi)}{2}\exp(i\theta_2),t\right). \tag{A10}$$

By taking the Fourier transform of Eq. (A10) and using the inverse Fourier transform from Eq. (A2) for $n=2$, we obtain

$$\omega(x,\theta_1,\theta_2,\psi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^2\alpha_1 d^2\alpha_2 d\eta W(\alpha_1,\alpha_2,t)$$

$$\times\exp(-i\eta\{x-[\alpha_{x_1}\cos(\theta_1)+\alpha_{y_1}\sin(\theta_1)]\cos(\psi)-[\alpha_{x_2}\cos(\theta_2)+\alpha_{y_2}\sin(\theta_2)]\sin(\psi)\}). \tag{A11}$$

Following the similar steps, we obtain

$$W(\alpha_1,\alpha_2,t)=\frac{1}{16\pi^4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}dx\,d\eta\,d\theta_1 d\theta_2 d\alpha|\eta||\eta\cos(\psi)||\eta\sin(\psi)|\omega(x,\theta_1,\theta_2,\psi)$$

$$\times\exp(i\eta\{x-[\alpha_{x_1}\cos(\theta_1)+\alpha_{y_1}\sin(\theta_1)]\cos(\psi)-[\alpha_{x_2}\cos(\theta_2)+\alpha_{y_2}\sin(\theta_2)]\sin(\psi)\}), \tag{A12}$$

which is the required expression for the two-mode Wigner function in terms of the joint quadrature distribution for two polarizatio*n* modes measured using a single balanced homodyne detector.

[1] E. Schrödinger, Naturwissenschaften **23**, 807 (1935); **23**, 823 (1935); **23**, 844 (1935); A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935); J.S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).

[2] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[3] D. Deutsch, Proc. R. Soc. London, Ser. A **400**, 97 (1985); D. Deutsch and R. Jozsa, *ibid.* **439**, 553 (1992).

[4] P.W. Shor, Phys. Rev. A **52**, R2493 (1995).

[5] C.H. Bennett, F. Bessete, G. Brassard, L. Salvail, and J. Smolin, J. Cryptogr. **5**, 3 (1992); J. Kempe, Phys. Rev. A **60**, 910 (1999).

[6] C.H. Bennett, C.A. Fuchs, and J.A. Smolin, in *Quantum Communication, Computing and Measurement*, edited by O. Hirota, A.S. Holevo, and C.M. Caves (Plenum, New York, 1997).

[7] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).

[8] M. Ahmad, S. Qamar, and M.S. Zubairy, Phys. Rev. A **62**, 043814 (2000).

[9] M. Ahmad, S. Qamar, and M.S. Zubairy, Phys. Rev. A **64**, 023811 (2001).

[10] M.S. Zubairy, Phys. Lett. A **222**, 91 (1996); Phys. Rev. A **57**, 2066 (1998); T. Azim and M.S. Zubairy, Phys. Lett. A **250**, 344 (1998); M. Mahmoudi, H. Tajalli, and M.S. Zubairy, J. Opt. B: Quantum Semiclassical Opt. **2**, 315 (2000); T. Azim, S. Qamar, and M.S. Zubairy, J. Mod. Opt. **49**, 245 (2002).

[11] P.J. Bardroff, E. Mayr, and W.P. Schleich, Phys. Rev. A **51**, 4963 (1995); W. Vogel, D.-G. Welsch, and L. Leine, J. Opt. Soc. Am. B **4**, 1633 (1987).

[12] L.G. Lutterbach and L. Davidovich, Phys. Rev. Lett. **78**, 2547 (1997).

[13] U. Leonhardt and H. Paul, Phys. Rev. Lett. **72**, 4086 (1994); K. Banaszek and K. Wodkiewicz, *ibid.* **76**, 4344 (1996); C.T. Bodendorf, G. Antesberger, M.S. Kim, and H. Walther, Phys. Rev. A **57**, 1371 (1998); A.M. Herkommer, V.M. Akulin, and W.P. Schleich, Phys. Rev. Lett. **69**, 3298 (1992); M. Freyberger and A.M. Herkommer, *ibid.* **72**, 1952 (1994); H. Paul, P. Törmä, T. Kiss, and I. Jex, *ibid.* **76**, 2464 (1996); M. Brune, S. Haroche, V. Lefevre, J.M. Raimond, and N. Zagury, *ibid.* **65**, 976 (1990); P.J. Bardroff, M.T. Fontenelle, and S. Stenholm, Phys. Rev. A **59**, R950 (1999).

[14] P. Bertet, A. Auffeves, P. Maioli, S. Osnaghi, T. Meunier, M. Brune, J.M. Raimond, and S. Haroche, Phys. Rev. Lett. **89**, 200402 (2002); D.T. Smithey, M. Beck, M.G. Raymer, and A. Faridani, *ibid.* **70**, 1244 (1993); S. Schiller, G. Breitenbach, S.F. Pereira, T. Müller, and J. Mlynek, *ibid.* **77**, 2933 (1996); D. Leibfried, D.M. Meekhof, B.E. King, C. Monroe, W.M. Itano, and D.J. Wineland, *ibid.* **77**, 4281 (1996); G. Nogues, A. Rauschenbeutel, S. Osnaghi, P. Bertet, M. Brune, J.M. Raimond, S. Haroche, L.G. Lutterbach, and L. Davidovich, Phys. Rev. A **62**, 054101 (2000); A.I. Lvovsky, H. Hansen, T. Aichele, O. Benson, J. Mlynek, and S. Schiller, Phys. Rev. Lett. **87**, 050402 (2001); G. Breitenbach, T. Müller, S.F. Pereira, J.-Ph. Poizat, S. Schiller, and J. Mlynek, J. Opt. Soc. Am. B **12**, 2304 (1995).

[15] D.-G. Welsch, W. Vogel, and T. Opatrny, in *Progress in Optics*, edited by E. Wolf (Elsevier Science, Amsterdam, 1999), Vol. 39, p. 63.

[16] U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, 1997).

[17] H. Kühn, D.-G. Welsch, and W. Vogel, Phys. Rev. A **51**, 4240 (1995); M.G. Raymer, D.F. Mc Alister, and U. Leonhardt, *ibid.* **54**, 2397 (1996); M.S. Kim and G.S. Agarwal, *ibid.* **59**, 3044 (1999); G.M. D'Ariano, M.F. Sacchi, and P. Kumar, *ibid.* **61**, 013806 (1999); J. Fiurasek, *ibid.* **63**, 033806 (2001); M. Ikram and M.S. Zubairy, *ibid.* **65**, 044305 (2002).

[18] M.O. Scully and M.S. Zubairy, Opt. Commun. **66**, 303 (1988).

[19] K. Zaheer and M.S. Zubairy, Opt. Commun. **69**, 37 (1988); M. Majeed and M.S. Zubairy, Phys. Rev. A **44**, 4688 (1991); K. Zaheer and M.S. Zubairy, in *Frontier of Quantum Electronics and Quantum Optics*, edited by A.O. Barut (Plenum Press, New York, 1990), p. 203; S. Qamar, S.-Y. Zhu, and M.S. Zubairy, Opt. Commun. **147**, 274 (1998); H. Huang, S.Y. Zhu, and M.S. Zubairy, Phys. Rev. A **53**, 1027 (1996); M.S. Kim, K.S. Lee, and V. Buzek, *ibid.* **47**, 4302 (1993); M.S. Zubairy and S. Qamar, Opt. Commun. **179**, 275 (2000).

[20] TH. Richter, J. Mod. Opt. **44**, 2385 (1997); M.G. Raymer and A.C. Funk, Phys. Rev. A **61**, 015801 (1999).

[21] J. Anwar and M.S. Zubairy, Phys. Rev. A **49**, 481 (1994).

[22] M. Franca Santos, L.G. Lutterbach, and L. Davidovich, J. Opt. B: Quantum Semiclassical Opt. **3**, S55 (2001).

[23] K.E. Cahill and R. Glauber, Phys. Rev. **177**, 1882 (1969).