A NOVEL FEEDBACK DESIGN METHOD FOR MIMO QFT WITH APPLICATION TO THE X-29 FLIGHT CONTROL PROBLEM

A Dissertation

by

CHEN-YANG LAN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2008

Major Subject: Mechanical Engineering

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ABSTRACT

A Novel Feedback Design Method for MIMO QFT

with Application to the X-29 Flight Control Problem. (August 2008)

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Chair of Advisory Committee: Dr. Suhada Jayasuriya

Quantitative Feedback Theory (QFT) method employs a two degree of freedom control configuration that includes a feedback controller and a prefilter in the feed-forward path. When applied to multi-input multi-output (MIMO) systems, the QFT method calls for a special decomposition of the MIMO system. Specifically, the MIMO system is decomposed into multiple multi-input single-output (MISO) equivalent systems, and is followed by the single-input single-output (SISO) QFT design of each equivalent system. Depending on pole-zero structure of the equivalent SISO plants so obtained, the QFT design may become unnecessarily difficult/conservative or even infeasible. This situation is especially true for linear time invariant (LTI) systems with non-minimum phase (NMP) zero(s) and unstable pole(s).

This unnecessary design difficulty and the challenge of dealing with MIMO systems that have unstable poles and NMP transmission zeros in undesirable locations, when MIMO QFT is considered, is investigated and addressed in this research. A new MIMO QFT design methodology was developed using the generalized formulation. The key idea of the generalized formulation is to utilize appropriate modifications at the plant input and/or the output to obtain a better conditioned plant that in turn can be used to execute a standard MIMO QFT design. The formulation is based on a more general control structure, where input and output transfer function matrices (TFM) are included to provide additional degrees of freedom in the typical decentralised MIMO QFT feedback structure, which facilitates the exploitation of directions in MIMO QFT designs. The formulation captures existing design approaches for a fully populated MIMO QFT controller design and provides for a directional design logic involving the plant and controller alignment and the directional properties of their multivariable poles and zeros. As a case in point Horowitz's Singular-G design methodology is placed in the context of this generalized formulation, and the Singular-G design for the X-29 is analysed and redesigned using both non-sequential and sequential MIMO QFT demonstrating its utility.

The results highlight a fundamental trade-off between multivariable controller directions for stability and performance in classically formulated MIMO QFT design methodologies, which elucidate the properties of Singular-G designed controllers for the X-29 and validate the developed new MIMO QFT design method.

DEDICATION

To my parents and my wife.

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CHAPTER I

INTRODUCTION

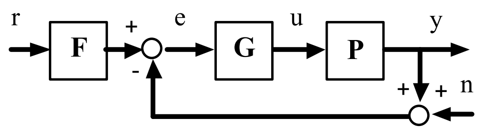

Figure 1: Two Degree of Freedom QFT Control Structure

Quantitative Feedback Theory (QFT), as a frequency domain control design method, provides the practical features of the classical methodologies, the ability to handle large structured/unstructured uncertainty, and an excellent balance between theory and practice. It has developed into a useful set of control system design tools for uncertain systems. The uncertainty is explicitly accommodated in the design in contrast to other robust control methods. QFT is based on the important principle that a system needs feedback only when there is uncertainty, which includes plant disturbance uncertainty [1-4] and the methodology yields a design that eliminates the necessity of feedback when uncertainty is reduced to zero. The QFT design methodology uses only

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output feedback and calls for a two degree of freedom (dof) control configuration (Fig. 1) that consists of a feedback loop and a prefilter, where the feedback controller is synthesized to account for any uncertainty and disturbances while the prefilter provides the necessary tracking performance [1-4].

In a QFT design, a feedback controller is synthesized through loop shaping of the nominal loop transmission so that sufficient loop gain is provided at each frequency to force the output of the system to be within the desired output variations. It is important to emphasize that the QFT technique looks for the minimum gain necessary to ensure that at each frequency the allowed variations in the closed-loop transfer function are guaranteed for the amount of plant uncertainty present. Thus the QFT method makes a special issue with the cost of feedback, especially in terms of loop bandwidth and sensor noise effects [2,5]. This is the importance of the term “Quantitative” in QFT.

For single-input single-output (SISO) systems, the efficacy of the QFT methods has been well established for linear, nonlinear, distributed, and time varying systems [2-6], and the early concerns stemming from potential deficiencies in its theoretical basis have been comprehensively addressed [2-8]. It is fair to say that SISO QFT is now an accepted and theoretically justified control system design methodology for uncertain systems.

As was the case with many of the classically formulated multivariable design methodologies [9-13], the multi-input multi-output (MIMO) QFT design methodologies [2,-5,14,15] grew out of a natural desire to extend a SISO design methodology to multivariable systems, while attempting to preserve the transparency and practicality present in the original SISO design method. Though early work on the multivariable QFT design methods was constructively validated through numerous design examples [5,16], the methods lacked mathematical rigor and were thus criticised [17], despite their utility in low dimensional systems. Some of these criticisms were valid, and can be seen to apply to much of the classically formulated multivariable design methodologies. Those are the poor scalability of the methods when applied to large dimensional systems, the need for design iteration, the lack of a measure of optimality, the limitation to diagonal controllers, and difficulties in enforcing robust stability and performance properties at all the potential signal (break) points in the system [17-21]. Despite this, recent work on MIMO QFT has validated the properties of the design methods and provides a sound theoretical basis for their application to low dimensional systems [3,19,20,22,23]. Among those, Kerr et al. [22, 23] has established the necessary and sufficient conditions of closed-loop stability for sequential and non-sequential MIMO QFT. The non-sequential MIMO QFT has even expanded its application to non-minimum phase (NMP) systems in theory. This theoretical basis has provided new insights to the MIMO QFT for control engineers. Not surprisingly, this work has also shown strong underling connections between the MIMO QFT design methods and other classical multivariable design methods based on dominance theory and sequential loop closure [22-26]. Despite the similarity to other classical multivariable design methodologies, the MIMO QFT still stands apart in its ability to quantitatively treat uncertainty and enforce frequency domain robust stability and performance specifications.

The main goal of this research is to demonstrate the utility of, and potential deficiencies in, the MIMO QFT design methods for low dimensional systems, when employed in conjunction with recent developments in stability theory. This is illustrated by applying the MIMO QFT design methodology to a notoriously difficult non-minimum phase (NMP) and unstable MIMO control problem: the X-29 aircraft [27,28]. A new design approach and control configuration, termed the generalized formulation, is developed to overcome the following deficiencies: spurious right-half-plane (RHP) poles in the SISO equivalent plants and the diagonal control structure. These issues are underscored in the design of the X-29 flight control system. When employing the proposed generalized formulation with MIMO QFT, the formulation conditions the equivalent SISO plants to be free of spurious RHP poles and permits a decentralised control structure (diagonal or non-diagonal) [29] thus alleviating inherent design conservatism in MIMO QFT. This decentralised control structure provides a capability to exploit directional information in the controller design, which is generally known to be important when considering performance limitations in multivariable systems [30-32] and general multivariable control system design [33].

The generalized formulation is employed with either non-sequential MIMO QFT or sequential MIMO QFT in the developed methodology herein. While design is straightforward with non-sequential MIMO QFT, it is more conservative compared to one using the sequential method, especially when $\mathbf{M}$ matrix is involved. However, the design of non-sequential method servers as a starting point for the sequential design which can then be fine tuned through further iteration.

Control of the X-29 presents a difficult design problem [27,28]. These difficulties arise from a combination of implementation issues and the NMP and unstable flight dynamics. The flight dynamics alone make it a difficult control design, especially if one fixed controller is to be designed to provide robust stability and tracking for a range of flight conditions. This design problem is a key focus in the present study. Previously published approaches to control the X-29 include [34,35], and within the QFT framework [36], where the Singular-G design method was employed. The design in [36] serves as a basis and benchmark for the MIMO QFT design developed herein, which considers the same X-29 control problem. The difficulties in controlling the X-29 should of course come as no surprise, as the design of control systems for NMP and unstable plants is known to be difficult [31,37,38]. As such, the X-29 control problem provides an opportunity to validate and investigate the properties of control system design methodologies, and it is in this spirit that the control of the X-29 aircraft is considered in this research. The effectiveness of the developed methodology will be demonstrated with the X-29 problem. We also study the features of the "Singular-G Method" by comparing it to the new MIMO QFT methodology to be developed in this research.

The dissertation is organised as follows. In Chapter II the background on MIMO QFT and the X-29 aircraft model to be considered are introduced. Chapter III presents the proposed generalized formulation. The new design procedure for MMO QFT using the generalized formulation is presented in Chapter IV. The Singular-G design and the

MIMO QFT design for the X-29 are presented and compared in Chapter V along with design insights. Conclusions and future research are presented in Chapter VI.

CHAPTER II

BACKGROUND

1. Introduction

In this chapter, a brief introduction for the linear multivariable (MIMO) QFT design methodology is first presented. The improved non-sequential MIMO QFT and sequential MIMO QFT, which are the design methods employed for the X-29 control problem in Chapter V, are then discussed with special focus on stability theory and limitations. The difficulty of applying MIMO QFT to NMP and unstable systems is then illustrated, which is the key motivation for this research. Lastly, the X-29 control problem under consideration in this research is introduced. Both the X-29 model and its challenges in control design are presented.

2. Quantitative Feedback Theory

2.1 MIMO QFT

To facilitate multivariable control system design, the SISO QFT design methodology has been extended to multivariable systems. This extension stems from the work of [1] and a survey of all the conceived methods attributed to Horowitz can be found in [5]. The multivariable QFT design methodologies are based on a decomposition of the multivariable design problem into a number of MISO design problems [2-5,14,15,22,23]. Consequently, the SISO QFT design methodology can be employed for each MISO design and as such serves as the kernel of all the MIMO QFT design methods. Several multivariable design methods have been developed for QFT, but the two methodologies that have received particular attention are the non-sequential MIMO QFT design methodology [2,14,23] and the sequential MIMO QFT design methodology [2,3,15,22], which are respectively termed the third and fourth QFT methods in [5] (see also [3,4], where in [4] non-sequential MIMO QFT is termed Method 1). The key difference between the non-sequential and sequential MIMO QFT design methodologies is that the multiple MISO designs are performed independently in the former, whereas explicit knowledge of the interactions between the MISO designs is favorably exploited in the latter. Subsequently, the sequential MIMO QFT design methodology is less conservative. Alternate MIMO QFT design methodologies were presented in [25,26] that employed dominance measures to guarantee acceptable closed-loop stability and performance properties. These approaches give fully populated controllers. Other approaches have also been proposed that employ fully populated controllers, such as [40] and the more recent contributions in [29,41-44].

The properties of MIMO QFT design methodologies have previously been a subject of contention [17,20,45]. However, the fundamental issue of robust stability has already been addressed in [3,22,23]. For the non-sequential MIMO QFT design methodology, [15] employed fixed-point theory to show that the methodology guaranteed robust stability and performance. The approach placed numerous conditions on the plant family, which include the requirement that the inverse-plant-domain (IPD) SISO plants employed in the design were minimum phase (MP). This generally requires the plant transmission zeros to also be MP. While the stability of the designed control system was questioned in [45], recently [23] re-affirmed the robust stability properties of a successful design when applied to a MP plant family and hence confirmed the stability properties of the design methodology as presented in [14]. More recently [46,47] extended the results of [23] such that it is broadly applicable to MP/NMP and stable/unstable plant families. This was achieved by exploiting the freedom offered by not stabilizing each MISO loop in the decomposed multivariable control problem, which has important implications on design conservatism [46,47].

The inherent conservatism in MIMO QFT designs was greatly reduced by using the sequential MIMO QFT design methodology [3-5,15]. Fixed-point theory was not required to justify robust stability and performance of the design methodology. However, no formal proof was presented for robust stability until [22]. Due to criticisms of the MIMO QFT design methodologies, and in particular the non-sequential MIMO QFT [17], Yaniv and Horowitz [48] highlighted through a 2×2 MIMO design example that it is not necessary to stabilize each loop in the sequential design methodology and that the ability of the non-sequential MIMO QFT design methodology to stabilize an uncertain system is dependent on alignment of the plant and controller. This property of the non-sequential MIMO QFT design procedure was shown to be related to the high frequency sign condition [20]. Nwokah and Thompson [20] also provided sufficient conditions for the stabilization of the control system using a diagonal controller. In the sequential MIMO QFT design methodology, Horowitz [2, p.403], stated that “except for very special (usually contrived) cases, the entire MIMO system is stable over the plant family” if the final SISO loop in the sequential design is robustly stable. Later Yaniv [3] provided sufficient conditions for robust stability in the sequential MIMO QFT design methodology employing a diagonal controller transfer function matrix (TFM). More recently [22], by developing necessary and sufficient conditions for stability, confirmed Horowitz's proposition that generally the closed-loop system is stable if the final loop is stable. This work also showed that designing for stability in the DPD or the IPD was equivalent, but the resulting properties of the closed-loop system, such as integrity, are different. In addition, recently [49] provided sufficient conditions for robust stability in the sequential MIMO QFT design methodology employing a non-diagonal controller TFM and sufficient conditions necessitating a non-diagonal controller for stability.

2.2 Improved Non-sequential MIMO QFT

The non-sequential MIMO QFT design methodology, as presented in [2,14], was intended for MP plant TFMs. Using the stability theory presented in [23], an improved non-sequential MIMO QFT method was presented in [46,47] to facilitate the application of the design method to general MP/NMP and stable/unstable plant TFMs. The improved method exploits the possibility of letting individual loops in the non-sequential MIMO QFT design be unstable while preserving the overall closed-loop stability and performance. The performance specifications in the improved method are equivalent to the standard non-sequential MIMO QFT specifications and the stability conditions are summarized in this section.

In non-sequential MIMO QFT, the typical SISO decomposition that is used can be illustrated through the following 2×2 system. In Fig. 1, let

$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$, and $\mathbf{G} = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix}$. Denote the TFM from the reference input to the output by $\mathbf{T}_r = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$. Then,

$$\mathbf{T}_r = (\mathbf{I} + \mathbf{PG})^{-1}\mathbf{PGF}, \tag{1}$$

which yields

$$(\mathbf{I} + \mathbf{PG})\mathbf{T}_r = \mathbf{PGF} \Rightarrow (\mathbf{P}^{-1} + G)\mathbf{T}_r = \mathbf{PGF}. \tag{2}$$

Denoting $\mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{q_{11}} & \frac{1}{q_{12}} \\ \frac{1}{q_{21}} & \frac{1}{q_{22}} \end{bmatrix} = \mathbf{\Lambda}^{-1} + \mathbf{B}$, where $\mathbf{\Lambda} = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & \frac{1}{q_{12}} \\ \frac{1}{q_{21}} & 0 \end{bmatrix}$.

Then Eqn. 2 can be rewritten as

$$(\mathbf{\Lambda}^{-1} + \mathbf{B} + \mathbf{G})\mathbf{T}_r = \mathbf{GF}$$
$$(\mathbf{\Lambda}^{-1} + \mathbf{G})\mathbf{T}_r = \mathbf{GF} - \mathbf{BT}_r$$

$$\begin{bmatrix} \frac{1}{q_{11}} + g_{11} & 0 \\ 0 & \frac{1}{q_{22}} + g_{22} \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{q_{12}} \\ \frac{1}{q_{21}} & 0 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} \frac{q_{11}}{1+g_{11}q_{11}}\left(g_{11}f_{11} - \frac{t_{21}}{q_{12}}\right) & \frac{q_{11}}{1+g_{11}q_{11}}\left(g_{11}f_{12} - \frac{t_{22}}{q_{12}}\right) \\ \frac{q_{22}}{1+g_{22}q_{22}}\left(g_{22}f_{21} - \frac{t_{11}}{q_{21}}\right) & \frac{q_{22}}{1+g_{22}q_{22}}\left(g_{22}f_{22} - \frac{t_{12}}{q_{21}}\right) \end{bmatrix}. \tag{3}$$

Eqn. 3 completes the decomposition with the two equivalent SISO plants being q_{11} and q_{22}. The decomposed MISO systems are shown in Fig. 2, where $c_{11} = -\frac{t_{21}}{q_{12}}$, $c_{12} = -\frac{t_{22}}{q_{12}}$, $c_{21} = -\frac{t_{11}}{q_{21}}$, and $c_{22} = -\frac{t_{12}}{q_{21}}$ are treated as input disturbances.

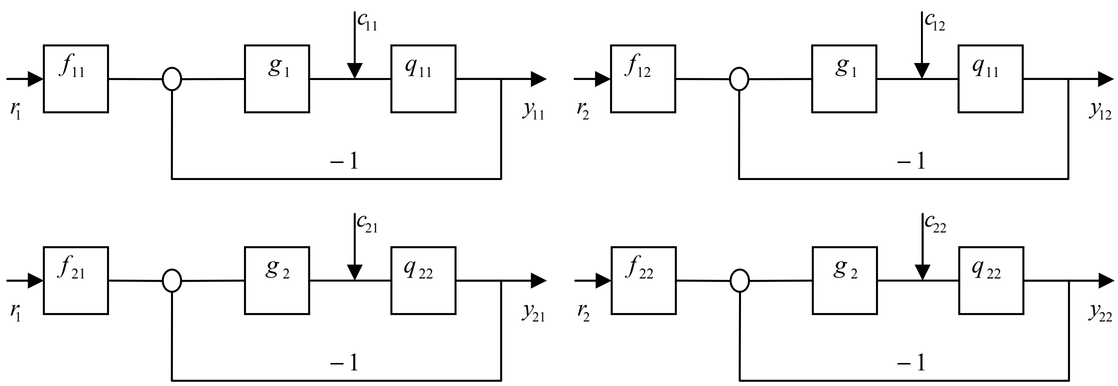

Figure 2: Effective MISO Loops for 2×2 System

When the non-sequential MIMO QFT approach is followed and a successful design for robust performance is achieved, the design ensures that [23,46,47],

$\rho(\mathbf{S}_{\Lambda}\mathbf{E}) < 1$, $\forall \omega$,

where $\mathbf{P}^{-1} = \mathbf{B} + \mathbf{\Lambda}^{-1}$, $\mathbf{P} = \lfloor p_{(i,j)} \rfloor$, $\mathbf{E} = \mathbf{\Lambda B}$, $\mathbf{\Lambda} = diag\lfloor q_{(i,i)} \rfloor$, and $\mathbf{S}_{\Lambda} = [\mathbf{I} + \mathbf{\Lambda G}]^{-1}$. When $\rho(\mathbf{S}_{\Lambda}\mathbf{E}) < 1$ is achieved, a necessary and sufficient condition for closed-loop stability of the true system states that

$$z_{\mathbf{P}} + z_{\phi} = z_{\mathbf{\Lambda}},$$

where $z_{\mathbf{\Lambda}}$, z_{ϕ}, and $z_{\mathbf{P}}$ are the number of RHP zeros in $\mathbf{\Lambda}$, $\phi = \det[\mathbf{I} + \mathbf{\Lambda G}]$ and $\mathbf{P}$, respectively. Furthermore, an obvious existence condition for a stable closed-loop system then becomes

$$z_{\mathbf{\Lambda}} \geq z_{\mathbf{P}}.$$

The resulting robust stability (RS) condition can be summarized as:

For a plant family satisfying the existence condition $z_{\mathbf{\Lambda}} \geq z_{\mathbf{P}}$, synthesize a fixed diagonal controller such that $z_{\mathbf{P}} + z_{\phi} = z_{\mathbf{\Lambda}}$ and $\rho(\mathbf{S}_{\mathbf{\Lambda}}\mathbf{E}) < 1$, $\forall \omega$, are satisfied for all plants in the family.

The improved method facilitates the application of the non-sequential MIMO QFT approach to NMP systems provided the existence condition $z_{\mathbf{\Lambda}} \geq z_{\mathbf{P}}$ is satisfied by all plants in the plant family. However, due to independent loop design employed, the implicit conditions enforced in the non-sequential MIMO QFT design and the use of only sufficient RS conditions, the resulting design can be conservative [23,46,47]. Some of these conservatisms are summarized below for later reference.

(NS L1) The performance limitation from a NMP transmission zero in the plant TFM which may appear in a particular loop is not shared in the design.

(NS L2) Spurious RHP poles can appear in the diagonal equivalent SISO plants. These can place unnecessary constraints on the controller bandwidth in the associated SISO loop design.

(NS L3) The satisfaction of the stability conditions in NS MIMO QFT requires that $\rho(\mathbf{S}_\Lambda \mathbf{E}) < 1$ for all frequencies [23]. While this condition can be violated at the frequencies where $\rho(\mathbf{L}) < 1$, with $\mathbf{L} = \mathbf{PG}$, $\rho(\mathbf{S}_\Lambda \mathbf{E}) < 1$ is typically necessary at the low and intermediate frequencies. Hence $\rho(\mathbf{E}) < 1$ is required when $\mathbf{S}_\Lambda \approx \mathbf{I}$ or $\rho(\mathbf{S}_\Lambda) \approx 1$. These conditions must arise in the design when the SISO loop transmissions are all rolled-off, and hence the design method requires that $\rho(\mathbf{E}) < 1$ over some frequency range in the design, with $\mathbf{L}$ not rolled-off until $\rho(\mathbf{E}) < 1$.

2.3 Sequential MIMO QFT

The sequential MIMO QFT, presented by Yaniv and Horowitz [2,3,15], is the other approach of extending MISO QFT to MIMO system and removing some of the conservatism associated with the employment of the non-sequential MIMO QFT for multivariable control system synthesis such as NS L1. It also considers a two degree of freedom control configuration which includes a diagonal controller $\mathbf{G}$ and a prefilter $\mathbf{F}$. Just as in the non-sequential MIMO QFT, the sequential method also decomposes the original design problem into multiple MISO design problems. However, in contrast to the non-sequential MIMO QFT, the design information that becomes available from previous stages is explicitly considered in the following design stages. In other word, the equivalent SISO plant used in a subsequent design stage is updated with the information from previously designed loops. Once all loops in the system have been closed, individual loops can be opened if necessary for redesign with information from all other loops included. Through this iterative redesign process, the overall control design can be fine tuned and the control effort shared among loops, resulting in a less conservative design compared to the non-sequential MIMO QFT. This is the most important feature of the sequential method.

In sequential MIMO QFT, the design requirements are met by satisfying robust stability and robust performance specifications on the sequentially updated MISO QFT design problems. The necessary and sufficient conditions for closed-loop stability are stated in [22]. The stability of the closed-loop system is guaranteed if the final loop in the sequential design is stable, with the exception being when the plant TFM has unstable decentralized fixed modes. The stability theorem highlights the freedom in a sequential design where it is not necessary to stabilize the internal loops. This greatly reduces the conservatism (NS L2) that is inherent in the non-sequential MIMO QFT, but is fully removed as can be seen in the X-29. Compared with the non-sequential MIMO QFT, the conservatism due to NS L1 and NS L3 does not apply to the sequential method which necessarily means that a solution from the sequential method is less conservative. However, the limitations that result from RHP pole-zero structures and the use of a diagonal controller remain. Although the non-sequential MIMO QFT is more conservative, its design procedure is more straightforward and its solution can serve as a starting point for executing a sequential design.

In the standard sequential MIMO QFT design methodology [2,3,15,22], the SISO equivalent plants are based on the elements of the plant inverse $\mathbf{\Pi} = \left[\pi_{ij}\right] = \mathbf{P}^{-1}$. This we call the inverse plant domain (IPD) methodology. Later, the sequential MIMO QFT design methodology is also presented in the direct plant domain (DPD), where the SISO equivalent plants considered in the design is based on the elements of the plant $\mathbf{P} = \lfloor p_{ij} \rfloor$.

Consequently, the appropriate plant elements are updated according to

$$p_{(i,j)}^{k} = p_{(i,j)}^{k-1} - \frac{p_{(j,k-1)}^{k-1} p_{(k-1,j)}^{k-1} g_{(k-1,k-1)}}{1 + p_{(k-1,k-1)}^{k-1} g_{(k-1,k-1)}} \text{ (in DPD), or} \tag{4}$$

$$\pi_{(i,j)}^{k} = \pi_{(i,j)}^{k-1} - \frac{\pi_{(j,k-1)}^{k-1} \pi_{(k-1,j)}^{k-1}}{\pi_{(k-1,k-1)}^{k-1} \left(1 + g_{(k-1,k-1)} \big/ \pi_{(k-1,k-1)}^{k-1} \right)} \text{(in IPD).} \tag{5}$$

The superscript k refers to the update of the plant TFM from the $(k-1)$ state assuming that the loops are closed in the ascending order. We note that $p_{(i,i)}^{i}$ and $q_{(i,i)}^{i} = 1 \big/ \pi_{(i,i)}^{i}$ are the equivalent SISO plants used in design DPD and IPD respectively.

For a two-input two-output (TITO) plant, the following formulae are employed for IPD sequential MIMO QFT.

Let the plant be $\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} z_{11}/\phi & z_{12}/\phi \\ z_{21}/\phi & z_{22}/\phi \end{bmatrix}$. (6)

Then the determinant of the plant is $\det(\mathbf{P}) = \frac{z_{11}z_{22} - z_{12}z_{21}}{\phi^2} = \frac{Z}{\phi}$, where Z is the zero polynomial and ϕ is the pole polynomial of the plant.

The inverse of plant is $\mathbf{\Pi} = \mathbf{P}^{-1} = \begin{bmatrix} \pi_{11}^{1} & \pi_{12}^{1} \\ \pi_{21}^{1} & \pi_{22}^{1} \end{bmatrix} = \begin{bmatrix} z_{22}/Z & -z_{12}/Z \\ -z_{21}/Z & z_{11}/Z \end{bmatrix}$. (7)

With $\mathbf{Q}$ matrix given by $\mathbf{Q} = \begin{bmatrix} q_{11}^{1} & q_{12}^{1} \\ q_{21}^{1} & q_{22}^{1} \end{bmatrix} = \begin{bmatrix} 1/\pi_{11}^{1} & 1/\pi_{12}^{1} \\ 1/\pi_{21}^{1} & 1/\pi_{22}^{1} \end{bmatrix} = \begin{bmatrix} Z/z_{22} & -Z/z_{12} \\ -Z/z_{21} & Z/z_{11} \end{bmatrix}$. (8)

In IPD sequential method, assuming the first loop closed is $q_{11}^{1}=\frac{1}{\pi_{11}^{1}}$, the updated equivalent SISO plant becomes $q_{22}^{2}=\frac{1}{\pi_{22}^{2}}$, where $\pi_{22}^{2}=\pi_{22}^{1}-\frac{\pi_{12}^{1}\pi_{21}^{1}}{\pi_{11}^{1}\left(1+g_{11}q_{11}^{1}\right)}$. (9)

The following necessary and sufficient conditions apply for closed-loop stability [22] of the sequential MIMO QFT designs in both the IPD and DPD. Specifically, the closed-loop system is guaranteed to be stable if the last MISO loop closed is stable and several reasonable conditions hold true for plants in the plant family as stated below.

Let the characteristic polynomial for the closed-loop system be given by $\phi=\det\left[\mathbf{I}+\mathbf{PG}\right]$ which can shown to be equivalent to the following.

$$\phi=\det\left[\mathbf{P}\right]\det\left[\mathbf{G}+\mathbf{\Pi}\right]=\delta_{\mathbf{P}(1,n-1)}+g_{(n,n)}\delta_{o\mathbf{P}}\delta_{\mathbf{\Pi}(1,n-1)} \quad (10)$$

where $\delta_{o\mathbf{P}}=\det\left[\mathbf{P}\right]$, $\delta_{\mathbf{P}(1,n-1)}=\Pi_{i=1}^{n-1}\left(1+p_{(i,i)}^{i}g_{(i,i)}\right)$, and $\delta_{\mathbf{\Pi}(1,n-1)}=\Pi_{i=1}^{n-1}\left(g_{(i,i)}+\pi_{(i,i)}^{i}\right)$. The plant is assumed to have RHP zeros at $\mathbf{z}=\lfloor z_j\rfloor$, where $\mathbf{z}$ is $1\times l_z$. Each z_j is of multiplicity m_j, with output directions $\mathbf{Y}_j$ and input directions $\mathbf{U}_j$. With this notation the stability theorem can be stated as follows [22].

Theorem 1: (Nominal Plant Stability) Consider the internal stability of the nominal closed-loop system with plant $\mathbf{P}$ and loops $i=1,\cdots,n-1$ closed with $g_{(i,i)}$. Let $l_{Q(n,n)}$ have $p_{ol(n)}$ poles in the RHP. Let D be the usual closed-contour (Nyquist contour) in the complex plane enclosing all finite poles and zeros in the RHP. Let $l_{Q(n,n)}$ map D into $\Gamma_{(n)}$ and let $\Gamma_{(n)}$ encircle the point $(-1,0)$ $\beta_{(n)}$ times clock-wise. Then the nominal closed-loop is internally stable if and only if :

(i) $\beta_{(n)} = -p_{ol(n)}$.

(ii) There are no RHP pole-zero cancellations in $q^{n}_{(n,n)} g_{(n,n)}$.

(iii) For $g_{(i,i)} \neq \infty$, $i \in 1,\cdots,n-1$, $\delta_{o\mathbf{P}}\delta_{\mathbf{\Pi}(1,n-1)}$ and $\delta_{\mathbf{P}(1,n-1)}$ have no common RHP zeros.

(iv) For all $j = 1,\cdots,l_z$, $\forall k = 1,\cdots,m_j$, $g_{(i,i)} \neq \infty$ for at least one i, where $\mathbf{Y}_{j(i,k)} \neq 0$.

(v) For all $j = 1,\cdots,l_z$, $\forall k = 1,\cdots,m_j$, $g_{(i,i)} \neq \infty$ for at least one i, where $\mathbf{U}_{j(i,k)} \neq 0$.

Proof: see reference [22].

2.4 Difficulty with NMP and Unstable Systems

The difficulty of applying MIMO QFT to NMP and unstable systems stems from the decomposition employed in the MIMO QFT. When applying QFT for control synthesis both methods (non-sequential and sequential) decompose the MIMO design problem into multiple MISO design problems, so that SISO QFT design can be employed on each equivalent MISO system. It should be emphasized that the unfavorable equivalent plants affect both sequential and non-sequential MIMO QFT approaches. However, it is less severe in the sequential method due to the different stability requirements and the different loop closure procedure employed.

In the stability theorems developed by Zhao and Jayasuriya [45], Kerr and Jayasuriya[50], and Kerr et. al. [22, 23], the equivalent plants from the nominal case is required to be stabilized as a first step regardless of which stability theorem is employed. However, this step can be quite challenging when the pole-zero structure of the equivalent SISO plants is unfavorable, especially when the plant is NMP and unstable with the unstable pole to the right of the NMP zeros. In a MISO control design problem, when the SISO plant has unstable poles lying to the right of NMP zeros, i.e., RHP dipoles, this SISO QFT design problem is very difficult if not solvable. The reason for this is the conflicting bandwidth requirements imposed by the NMP zero and the unstable pole. In frequency domain design a NMP zero limits the maximum allowable loop cross-over frequency (ω_c) while an unstable pole limits the minimum ω_c. Thus, if the NMP zero is smaller than the unstable pole, there may be no stabilizing solution for such a SISO system. Consequently, the MIMO QFT design can not be executed.

Although the improved non-sequential MIMO QFT method does not necessarily require all SISO equivalent plants to be stabilized, there still is a necessary and sufficient condition, $z_{\mathbf{p}} + z_{\phi} = z_{\mathbf{\Lambda}}$, that is placed on the number of the RHP zeros ($z_{\mathbf{P}}$) of the MIMO plant, the equivalent plant's RHP zero ($z_{\mathbf{\Lambda}}$) and the closed-loop RHP poles of the equivalent plants (z_{ϕ}). Even with this result which allows some freedom to not necessarily stabilize all SISO equivalent plants, the required condition on the number of RHP roots can still be difficult to satisfy for some NMP and unstable systems. Consequently, realizing a successful design can be quite a challenge. In sequential MIMO QFT, although the internal loops are not necessarily stable after closure, the undesired spurious RHP pole and zeros can still affect the design and could prevent a successful design as seen in the X-29 case study. Thus a good pole-zero structure is still desirable in the improved non-sequential and sequential MIMO QFT.

Due to the decomposition used in the MIMO QFT, sometimes a MP and stable MIMO system could also end up having at least one of the SISO equivalent systems unstable. Even a NMP stable MIMO system could end up with a NMP and unstable plant among its equivalent SISO systems. In such a case, re-numbering of the inputs and/or outputs is worth a try and usually gives a different structure for the equivalent SISO plants, i.e., transforming from unstable to stable and/or from NMP to MP. Thus, it may be possible to transform the equivalent SISO systems from an apparent un-stabilizable situation to a stabilizable one. This difficulty due to the decomposition is further illustrated using the X-29 problem presented in section 3.3 of this chapter.

Furthermore, the typical MIMO QFT assumes or considers only diagonal controllers although it is entirely possible to use a fully populated controller. The restriction to diagonal controller obviously limits the ability of the MIMO QFT to deal with systems having RHP dipoles. It is important to remember that in MIMO control design problems the directions play a significant role as opposed to SISO designs. The use of a diagonal controller in MIMO QFT limits the control to only certain orthogonal directions and input-output control relations. As was alluded to earlier re-numbering the inputs gives a non-diagonal but still a geometrically orthogonal controller for the original ordered inputs. The idea of renumbering provides a starting point for this research.

3. The X-29 Flight Control Problem

The X-29 aircraft (Fig. 3) was a technology demonstrator built by Grumman Aerospace. The aircraft was designed to test the advantages of exploiting forward swept wing configurations for improved aircraft aerodynamic characteristics, such as

Figure 3: Photo of the X-29 (Photo Courtesy of NASA)

maneuverability and control at high angles of attack [27,28]. As discussed in [28], the use of a forward swept wing configuration resulted in static instability, which was most dominant at lower flight velocities (subsonic). While providing aerodynamic advantages, this instability made the aircraft particularly difficult to control which was due to the difficulty in balancing the need to stabilize the unstable modes of the aircraft while satisfying strict bandwidth constraints on the control system arising from the control system hardware, actuator and sensor limitations, and the structural modes of the airframe [28]. In addition to this, although not noted in [27,28] but seen in [36], for some flight conditions the aircraft dynamics exhibit a NMP phenomena, resulting in additional bandwidth constraints that must be accommodated in the control design. This combination of unstable and NMP aircraft dynamics makes the control problem challenging and hence ideally suited for testing the limits of control system design methods.

3.1 X-29 Longitudinal Flight Control Model

The X-29 model considered here is taken from [36]. In [36], the aircraft is modeled as a four plants family that is linearized about four flight conditions. The model is represented by 6 inputs and 2 outputs, capturing both the longitudinal and lateral flight dynamics. These dynamics are approximately decoupled. As was done in [36], a simplification of the model is considered herein, with only the decoupled longitudinal dynamics considered for the designs. The plant TFMs then have two inputs and outputs, being the canard angle (degree) and flap angle (degree), and vertical acceleration (g's) and pitch rate (degree/second), respectively. The linear model has four states, the forward velocity, angle of attack, pitch rate and pitch angle. The plant TFMs for four flight conditions are given below:

$$\mathbf{P}_1 = \begin{bmatrix} \dfrac{0.1172(s-0.0001867)(s+0.06333)(s^2+3.534s+558.4)}{(s-6.066)(s+11.59)(s^2+0.06203s+0.005368)} & \dfrac{0.3481(s-0.0001553)(s-13.82)(s+0.06116)(s+14.62)}{(s-6.066)(s+11.59)(s^2+0.06203s+0.005368)} \\ \dfrac{0.5577\,s(s+0.0645)(s+3.758)}{(s-6.066)(s+11.59)(s^2+0.06203s+0.005368)} & \dfrac{-0.3264\,s(s+0.06172)(s+6.903)}{(s-6.066)(s+11.59)(s^2+0.06203s+0.005368)} \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} \dfrac{0.009072(s-0.01364)(s+0.01776)(s^2+2.126s+127.3)}{(s-2.371)(s+3.872)(s^2+0.01758s+0.01056)} & \dfrac{0.07102(s-0.01304)(s-4.308)(s+0.01593)(s+4.645)}{(s-2.371)(s+3.872)(s^2+0.01758s+0.01056)} \\ \dfrac{0.08576s(s+0.9733)(s+0.01503)}{(s-2.371)(s+3.872)(s^2+0.01758s+0.01056)} & \dfrac{-0.05019s(s+0.008433)(s+2.046)}{(s-2.371)(s+3.872)(s^2+0.01758s+0.01056)} \end{bmatrix}$$

$$\mathbf{P}_3 = \begin{bmatrix} \dfrac{0.021(s-0.002717)(s+0.01434)(s^2+2.06s+201.2)}{(s-4.251)(s+6.082)(s^2+0.01363s+0.00349)} & \dfrac{0.145(s-0.002521)(s-6.986)(s+0.01288)(s+7.341)}{(s-4.251)(s+6.082)(s^2+0.01363s+0.00349)} \\ \dfrac{0.1589s(s+0.01542)(s+1.157)}{(s-4.251)(s+6.082)(s^2+0.01363s+0.00349)} & \dfrac{-0.118s(s+0.01245)(s+2.743)}{(s-4.251)(s+6.082)(s^2+0.01363s+0.00349)} \end{bmatrix}$$

$$\mathbf{P}_4 = \begin{bmatrix} \dfrac{0.004423(s-0.002035)(s+0.01136)(s^2+0.9399s+135.9)}{(s-2.45)(s+3.027)(s^2+0.02945s+0.06591)} & \dfrac{0.04373(s-0.001655)(s-4.083)(s+0.008558)(s+4.197)}{(s-2.45)(s+3.027)(s^2+0.02945s+0.06591)} \\ \dfrac{0.05476s(s+0.01654)(s+0.4047)}{(s-2.45)(s+3.027)(s^2+0.02945s+0.06591)} & \dfrac{-0.03643s(s+0.006622)(s+0.7542)}{(s-2.45)(s+3.027)(s^2+0.02945s+0.06591)} \end{bmatrix} \tag{11}$$

The plants for the four flight conditions are clearly unstable, with RHP poles as detailed in Table 1. The plants all possess a differentiator aligned (pinned) to output two. The transmission zeros for flight condition 1 are in the open left-half-plane (LHP), while flight conditions 2, 3 and 4 have one NMP transmission zero, as detailed in Table 1. The directions of the RHP poles and transmission zeros are also given in Table 1.

Table 1: X-29 Plant RHP Pole and Zero Properties

	Plant	1	2	3	4
NMP Zero	Location	-	1.08e-2	6.89e-4	1.17e-3
	Input Vector	-	[-0.67, -0.74]	[-0.82, -0.57]	[-0.48, -0.88]
	Output Vector	-	[0.24, 0.97]	[-0.16, -0.99]	[-0.07, -0.99]
Unstable Pole	Location	6.07	2.37	4.25	2.45
	Input Vector	[-0.79, 0.61]	[-0.79, 0.61]	[0.72, -0.69]	[-0.80, 0.60]
	Output Vector	[-0.99, -0.08]	[0.97, 0.22]	[-0.98, -0.18]	[-0.97, -0.24]

Considering the poles and zeros of the plant TFMs and their directional properties, it is evident that, while the RHP pole and zero are highly undesirable, with plant cases 2, 3 and 4 possessing a RHP dipole, the directional properties appear to be favorable, as they are approximately orthogonal. It is well known [31,32,37,38] that for MIMO plants with both the RHP poles and zeros, the relative direction of the RHP poles and zeros affects the severity of the resulting performance limitations on the system. Hence, while the presence of such a RHP dipole in a SISO plant would result in extreme performance limitations, the favorable directional properties in the MIMO plant reduce this limitation. However, when considered from the perspective of a classical multivariable design, the orthogonality of the directions of the RHP pole and zero might appear to be of little assistance.

3.2 Challenges in the X-29 Flight Control

The longitudinal flight control of the X-29 ([36]), which is a 2×2 NMP and unstable system with uncertainty, falls into the category that cannot be stabilized by using the standard MIMO QFT methodology. The X-29 was however robustly stabilized by Walke using the so called “Singular-G Method” ([36]) proposed by Horowitz. This is a case in point which shows that although a NMP and unstable MIMO system may not be stabilized by the standard MIMO QFT method it does not necessarily mean that the original MIMO system cannot be stabilized. It is worth pointing out, however, that although it appears possible to stabilize the X-29 with other competing robust control methods, to date no such design has been realized with the required stability specifications. In other words, this class of problems is extremely difficult to solve and posses unexpected challenges.

Considering the elements of the plant TFMs in Eqn. 11, namely the direct plant domain (DPD) SISO plants [22], it is evident from the perspective of a classical MIMO controller design, that the properties of the SISO DPD plants are highly undesirable. All the SISO DPD plants are unstable for all flight conditions, possessing a differentiator for elements (2,1) and (2,2), and NMP zeros for elements (1,1) and (1,2). The plants also possess a high level of coupling over the intermediate (~10 rad/s) and high frequency

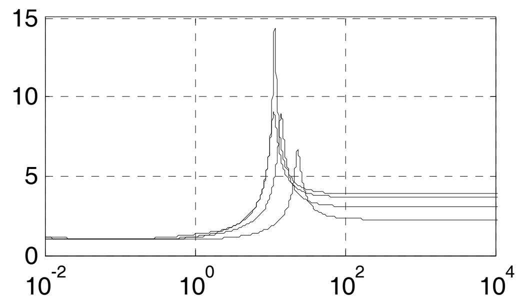

Figure 4: Spectral Radius of $\mathbf{E}_p = \mathbf{E}_Q$ ([dB] vs [rad/s])

ranges (>100 rad/s), being off-diagonally dominant over these frequencies, as seen in Fig. 4, where $\mathbf{E}_P = \left(\mathbf{P} - \mathbf{P}_D\right)\mathbf{P}_D^{-1}$, $\mathbf{P}_D$ the diagonal of the plant. The DPD SISO plants, however, only tell part of the story. Considering the elements of the plant inverse, namely the inverse plant domain (IPD) SISO plants [22], it is again evident that the properties are highly undesirable. All the SISO IPD plants are NMP for plant cases 2, 3 and 4, with all plant cases possessing a differentiator and unstable poles for elements (1,2) and (2,2). These plants have the same level of coupling as the DPD plants, which are quantified by $\mathbf{E}_Q = \mathbf{\Lambda B}$ as defined in Section 2.2 (see also [23]), with $\mathbf{E}_Q = \mathbf{E}_P$ for 2×2 systems. Irrespective of the input-output pairing, even initializing the design will be problematic since (2,2) elements of the four Q-matrices have a RHP dipole, which are extremely difficult if not impossible to be stabilized by SISO QFT.

$$\mathbf{Q}_1 = \left[\frac{0.71(s+0.05899)(s+0.003335)}{(s+6.903)(s+0.06172)} \quad \frac{0.42(s+0.05899)(s+0.003335)}{(s+3.758)(s+0.0645)} \quad \frac{0.67s(s+0.05899)(s+0.003335)}{(s+14.62)(s-13.82)(s+0.06116)(s-0.0001553)} \quad \frac{-1.98s(s+0.05899)(s+0.003335)}{(s+0.06333)(s-0.0001867)(s^2+3.534s+558.4)} \right]',$$

$$\mathbf{Q}_2 = \left[\frac{0.13(s+0.01874)(s-0.01078)}{(s+2.046)(s+0.008433)} \quad \frac{0.076(s+0.01874)(s-0.01078)}{(s+0.9733)(s+0.01503)} \quad \frac{0.092s(s+0.01874)(s-0.01078)}{(s+4.645)(s-4.308)(s+0.01593)(s-0.01304)} \quad \frac{-0.72s(s+0.01874)(s-0.01078)}{(s+0.01776)(s-0.01364)(s^2+2.126s+127.3)} \right]',$$

$$\mathbf{Q}_3 = \left[\frac{0.22(s+0.01331)(s-6.885e-4)}{(s+2.743)(s+0.01245)} \quad \frac{0.16(s+0.01331)(s-6.885e-4)}{(s+1.157)(s+0.01542)} \quad \frac{0.18s(s+0.01331)(s-6.885e-4)}{(s+7.341)(s-6.986)(s+0.01288)(s-0.002521)} \quad \frac{-1.2s(s+0.01331)(s-6.885e-4)}{(s+0.01434)(s-0.002717)(s^2+2.06s+201.2)} \right]'$$

$$\mathbf{Q}_4 = \begin{bmatrix} \dfrac{0.07(s+0.00935)(s-0.001167)}{(s+0.7542)(s+0.006622)} & \dfrac{0.058s(s+0.00935)(s-0.001167)}{(s+4.197)(s-4.083)(s+0.008558)(s-0.001655)} \\ \dfrac{0.047(s+0.00935)(s-0.001167)}{(s+0.4047)(s+0.01654)} & \dfrac{-0.58s(s+0.00935)(s-0.001167)}{(s+0.01136)(s-0.002035)(s^2+0.9399s+135.9)} \end{bmatrix}. \quad (12)$$

The DPD plant elements all possess the RHP poles of their respective plant TFMs. However, they also possess NMP zeros, some of which are spurious, such as the NMP zeros in elements (1,1) and (1,2) for plant case one, which is a MP plant, and the two NMP zeros in element (1,2) for plant cases 2, 3 and 4, when these plant possess only one NMP transmission zero. A similar situation occurs in the IPD plant elements, where all the IPD SISO plants possess the transmission zeros for the respective plant (MP for case 1), but possess unstable poles in addition to those present in their respective plant TFMs. These spurious RHP poles and zeros exacerbate the difficulties in the design and the alleviation of this effect is a key focus of the work presented herein using the generalized formulation that can exploit directions.

3.3 Failure of MIMO QFT

To apply the improved non-sequential MIMO QFT, it is first necessary to check certain conditions, as follows: Is the existence condition $z_{\mathbf{\Lambda}} \geq z_{\mathbf{P}}$ is satisfied by all plants in the plant family? Is the necessary high frequency sign condition for the robust stabilization is satisfied? And, is $\rho(\mathbf{E}) < 1$ over some frequency range in the design (NS

L3)? For the X-29, first two conditions are satisfied, and $\rho(\mathbf{E}) < 1$ for some frequency ranges, as seen in Fig. 4. Hence, a non-sequential MIMO QFT design is not possible for this input-output pairing. But, simply switching the input-output pairing resolves this problem. This input-output switched plant satisfies all the conditions above. The modified equivalent plants then become:

$$\mathbf{\Lambda}_1 = diag\begin{bmatrix} \dfrac{0.41669(s+0.003335)(s+0.05899)}{(s+3.758)(s+0.0645)}, \\ \dfrac{0.66759s(s+0.003335)(s+0.05899)}{(s-0.0001553)(s-13.82)(s+0.06116)(s+14.62)} \end{bmatrix},$$

$$\mathbf{\Lambda}_2 = diag\begin{bmatrix} \dfrac{0.076329(s-0.01078)(s+0.01874)}{(s+0.01503)(s+0.9733)}, \\ \dfrac{0.092171s(s-0.01078)(s+0.01874)}{(s-0.01304)(s-4.308)(s+0.01593)(s+4.645)} \end{bmatrix},$$

$$\mathbf{\Lambda}_3 = diag\begin{bmatrix} \dfrac{0.16059(s-0.0006885)(s+0.01331)}{(s+0.01542)(s+1.157)}, \\ \dfrac{0.17599s(s-0.0006885)(s+0.01331)}{(s-0.002521)(s-6.986)(s+0.01288)(s+7.341)} \end{bmatrix},$$

$$\mathbf{\Lambda}_4 = diag\begin{bmatrix} \dfrac{0.046672(s-0.001167)(s+0.00935)}{(s+0.01654)(s+0.4047)}, \\ \dfrac{0.058445s(s-0.001167)(s+0.00935)}{(s-0.001655)(s-4.083)(s+0.008558)(s+4.197)} \end{bmatrix}. \tag{13}$$

Considering the equivalent plants above, the condition that $z_{\mathbf{P}} + z_{\phi} = z_{\mathbf{\Lambda}}$ be satisfied in the improved non-sequential MIMO QFT design necessitates that $z_{\phi} = 0$ for plant case 1, and for the remaining plant cases $z_{\phi} = 1$ is required. However, due to this inconsistent property among the equivalent plants it is very difficult to achieve this condition simultaneously for all plant cases. In fact, even for a particular plant case this requirement is hard to accomplish. For example, in plant case 1, both loops have to be stabilized. This is difficult because the second loop possesses two RHP poles and one differentiator with one of the unstable poles very close to the differentiator. For other plant cases, the requirement and the pole-zero structures are similar. In these plant cases, one of the loops is allowed to be closed loop unstable with precisely one unstable closed-loop pole. Since the second loop has RHP poles and zeros, it is natural to choose the first loop to be closed stable. Again, the design is difficult because of the RHP poles and zeros in the second loop and the requirement to have precisely one closed-loop unstable pole. Eventually, only the plant family with plant cases 2, 3 and 4 could be robustly stabilized with the condition $z_{\mathbf{P}} + z_{\phi} = z_{\mathbf{\Lambda}}$ not simultaneously satisfied for plant case 1, which results in an unstable controller. Nonetheless, this result does validate the recently developed improved non-sequential MIMO QFT.

In sequential MIMO QFT, the closed-loop stability is guaranteed if the last loop closed is stable. Although the diagonal dominance is not a constraint in sequential method, the pole-zero structure of the X-29 plant makes the design difficult in using design formulae either from DPD or IPD. In the IPD, the second loop equivalent SISO plant used in second stage is $q_{22}^{2} = \dfrac{1}{\pi_{22}^{2}}$, assuming without loss of generality that the first loop closed is $q_{11}^{1} = \dfrac{1}{\pi_{11}^{1}}$ with controller $g_1 = \dfrac{g_{z1}}{g_{d1}}$. The characteristic equation of the first closed loop is $1+q_{11}^{1}g_{1}=1+\frac{Z}{z_{22}}\frac{g_{z1}}{g_{d1}}=\frac{\phi_{1}}{z_{22}g_{d1}}$, where $\phi_{1}=z_{22}g_{d1}+Zg_{z1}$ is the closed-loop pole polynomial of q_{11}^{1}. The updated equivalent plant for the second loop is further deduced as

$$q_{22}^{2}=\frac{1}{\pi_{22}^{2}}=q_{22}^{1}\left(\frac{\phi_{1}}{\phi_{1}-\left(z_{12}z_{21}/z_{11}\right)g_{d1}}\right)=\frac{Z\phi_{1}}{z_{11}\phi_{1}-z_{12}z_{21}g_{d1}}.$$

It is important to note that the closed-loop poles of the first loop becomes portions of the zeros in updated second equivalent plant q_{22}^{2}. From Eqn. 13, first loop (q_{11}^{1}) is easily stabilized with small gain. However, the updated second loop equivalent plant q_{22}^{2} remains NMP and unstable because of the use of small gain such that $q_{22}^{2}\approx p_{22}$.

If one starts the design with q_{22}^{1}, then the updated equivalent plant is

$$q_{11}^{2}=\frac{1}{\pi_{11}^{2}}=q_{11}^{1}\left(\frac{\phi_{2}}{\phi_{2}-\left(z_{12}z_{21}/z_{22}\right)g_{d2}}\right)=\frac{Z\phi_{2}}{z_{22}\phi_{2}-z_{12}z_{21}g_{d2}},$$

where $\phi_{2}=z_{11}g_{d2}+Zg_{z2}$ is the closed-loop pole polynomial of q_{22}^{1} and $g_{2}=\frac{g_{z2}}{g_{d2}}$. Assuming the first closed-loop is unstable because of the difficult pole-zero structure in q_{22}^{1}, then q_{11}^{2} is NMP. Moreover, $z_{22}\phi_{2}-z_{12}z_{21}g_{d2}$ is generally unstable and therefore q_{11}^{2} is NMP and unstable. If g_{2} has only small gain, then $q_{11}^{2}\approx p_{11}$ is surely NMP and unstable. Thus, no matter which loop is chosen to be closed first, at least one SISO equivalent plant having RHP zeros and poles must be handled. This design difficulty due to the RHP zeros and poles in the equivalent plants, either using sequential or non-sequential MIMO QFT, is the main motivation for this research.

4. Chapter Summary

In this chapter, the MIMO QFT design methodology was introduced. Both the non-sequential MIMO QFT and sequential MIMO QFT were discussed. Although the two methods differ in the loop closure procedure; they both suffer the same difficulty when applied to NMP and unstable systems, despite the recent improvement and the new understanding gained from new result in stability theory. This difficulty comes from the spurious RHP poles and zeros in the equivalent SISO plants, which are a manifestation of the decomposition employed in the MIMO QFT.

One important NMP and unstable systems of practical significance is the longitudinal flight control of the X-29. The X-29 model considered in this research is taken from [36]. This X-29 control problem has never been succefully resolved using the MIMO QFT due to the resulting NMP and unstable SISO equivalent plants. Thus, it serves as a challenging example to test the new approach developed in this research for MIMO QFT.

CHAPTER III

GENERALIZED FORMULATION

1. Motivations

As shown in section 2.4 and 3.3 of Chapter II, the non-sequential MIMO QFT design methodologies are not ideally suited for the control of NMP and unstable systems such as the X-29 due to its NMP and unstable plant dynamics and large level of coupling. In the X-29 problem, the spurious RHP poles in the diagonal equivalent plants (NS L2) and the fact that $\rho(\mathbf{E}) > 1$ over a frequency range prohibits $\rho(\mathbf{S}_{\Lambda}\mathbf{E}) < 1$ (NS L3) prevents a successful design using the non-sequential MIMO QFT. Although NS L3 is not a limitation in the sequential MIMO QFT, the difficulty due to spurious RHP poles and zeros in the equivalent plants still holds in the sequential MIMO QFT as illustrated in section 3.3 of Chapter II.

This difficulty of handling NMP and unstable systems using MIMO QFT guides the research herein to develop a better approach for applying MIMO QFT to NMP and unstable systems. The commonly used trick of switching input-output pairing, which may be possible to transform the equivalent SISO plants from an apparent un-stabilizable situation to a stabilizable one, provides a starting point for the generalized formulation. For example, such an inputs order switching corresponds to the use of $\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the generalized formulation for a TITO system. Thus, when equivalent SISO plants obtained are undesirable, the generalized formulation is recommended and presents a feasible approach. The original MIMO plant is conditioned under the generalized formulation with two transfer function matrices $\mathbf{M}$ and $\mathbf{N}$ such that the conditioned equivalent SISO plants have desirable pole-zero structure and are free of spurious RHP poles and zeros, thus facilitating a successful MIMO QFT design. In fact, the $\mathbf{M}$ and $\mathbf{N}$ matrices are chosen to condition the plant, and hence the control problem for the modified diagonal controller TFM $\mathbf{G}_m$, in a variety of ways. For instance, the matrices can also be employed to provide for improved dominance or to exploit the design freedom provided by a fully populated controller and permit the synthesis of fully populated controllers in MIMO QFT. Since one major limitation of the non-sequential and sequential MIMO QFT is the employment of a diagonal controller, the generalized formulation for MIMO QFT is employed to permit a less conservative design for both non-sequential and sequential MIMO QFT.

While the advantages of using the generalized formulation are clear, the existence of such a pair of $\mathbf{M}$ and $\mathbf{N}$ matrices is not clear. So it would be useful to have a result that can characterize the class of MIMO plants for which the generalized formulation is effective. Studied in section 3 is the fundamental issue of whether such a pair of $\mathbf{M}$ and $\mathbf{N}$ matrices exists so that the plant can be conditioned in order to result in desirable equivalent SISO plants. Developed are some conditions that are based on the nominal plant parameters that can be used prior to executing a QFT design.

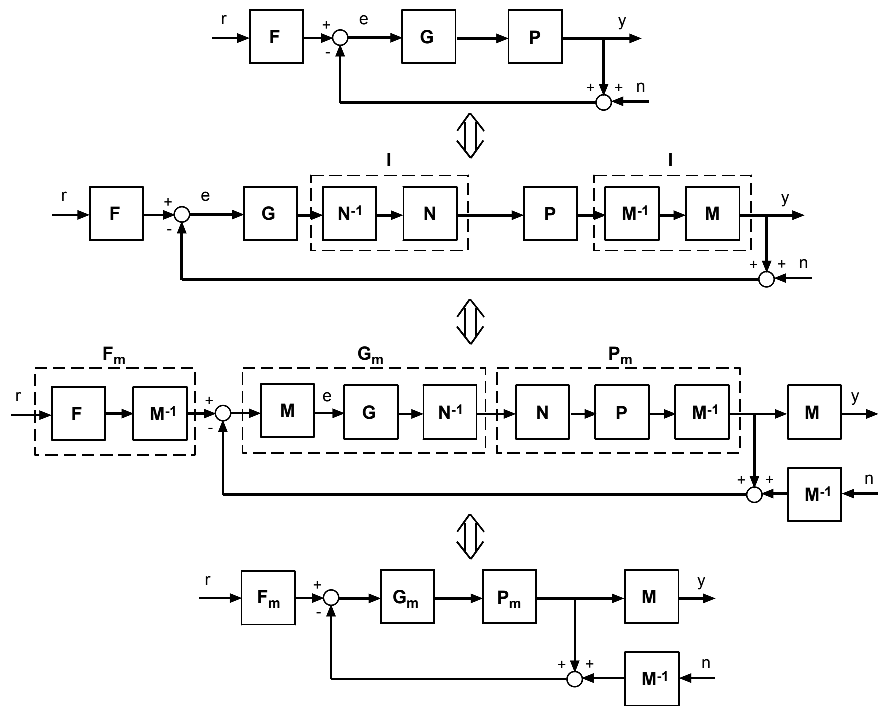

Figure 5: Schematic of the Generalised Formulation - Showing the Relationship between the Modified and True Control Systems

2. Generalized Formulation in MIMO QFT

The generalized formulation is depicted in Fig. 5, where the TFMs $\mathbf{M}$ and $\mathbf{N}$ are added to provide additional degrees of freedom in the design. The formulation permits the conditioning of the plant, prior to or during the design process, via the choice of $\mathbf{M}$ and $\mathbf{N}$, to reduce the effects of design limitations and permit a successful design. The formulation effectively facilities decentralized design of diagonal or non-diagonal MIMO controllers, while permitting the exploitation of directional properties of the plant and controller.

Two identity matrices $\mathbf{M}^{-1}\mathbf{M}$ and $\mathbf{N}^{-1}\mathbf{N}$ are added in the feedback path, with the non-singular matrices $\mathbf{M}=\left[m_{ij}\right]$ and $\mathbf{N}=\left[n_{ij}\right]$ applied to both the plant and controller. This results in a modified design problem involving the modified plant $\mathbf{P}_m=\left[p_{m(i,j)}\right]=\mathbf{M}^{-1}\mathbf{PN}$ and the associated modified controller $\mathbf{G}_m=\left[g_{m(i,j)}\right]=\mathbf{N}^{-1}\mathbf{GM}$ and prefilter $\mathbf{F}_m=\left[f_{m(i,j)}\right]=\mathbf{M}^{-1}\mathbf{F}$. The stability of the modified system guarantees the stability of the true system, which immediately follows by noting that

$$\begin{aligned}&\det(\mathbf{I}+\mathbf{P}_m\mathbf{G}_m)=\det(\mathbf{I}+\mathbf{M}^{-1}\mathbf{PNN}^{-1}\mathbf{GM})\\&=\det\left(\mathbf{M}^{-1}\left(\mathbf{I}+\mathbf{PG}\right)\mathbf{M}\right)=\det\left(\mathbf{I}+\mathbf{PG}\right).\end{aligned} \tag{14}$$

Hence, both the non-sequential and sequential MIMO QFT design theory can be applied directly to the transformed control problem involving $\mathbf{P}_m$, $\mathbf{\Lambda}_m=\mathrm{diag}\left[q_{m(i,i)}\right]$, $\mathbf{G}_m$ and $\mathbf{F}_m$. In a design, the matrices can be employed to condition that IPD or DPD SISO plants [29] or provide for improved dominance [26]. The Singular-G design approach proposed by Horowitz can be also presented within the framework of the generalized formulation as shown in section 2 Chapter V.

It should be noted, however, that employing the $\mathbf{M}$ and $\mathbf{N}$ matrices generally changes the closed-loop system under consideration, as the inputs and outputs of the system are redefined, and in the simplest case of diagonal $\mathbf{M}$ and $\mathbf{N}$ the system is scaled. This change can be seen from the following relationships that hold between the true system and the modified system after the application of $\mathbf{M}$ and $\mathbf{N}$:

$$\mathbf{L}_{om}=\mathbf{M}^{-1}\mathbf{L}_o\mathbf{M},\ \mathbf{L}_{im}=\mathbf{N}^{-1}\mathbf{L}_i\mathbf{N},\ \mathbf{T}_{om}=\mathbf{M}^{-1}\mathbf{T}_o\mathbf{M},\ \mathbf{T}_{im}=\mathbf{N}^{-1}\mathbf{T}_i\mathbf{N}, \tag{15}$$

where $\mathbf{L}_{om} = \mathbf{P}_m \mathbf{G}_m$, $\mathbf{L}_o = \mathbf{PG}$, $\mathbf{L}_{im} = \mathbf{G}_m \mathbf{P}_m$, $\mathbf{L}_i = \mathbf{GP}$, $\mathbf{T}_{om} = \mathbf{L}_{om}\left(\mathbf{I} + \mathbf{L}_{om}\right)^{-1}$, $\mathbf{T}_{im} = \mathbf{L}_{im}\left(\mathbf{I} + \mathbf{L}_{im}\right)^{-1}$, and analogous equations can be defined for other system TFMs. Considering the above relationships, which define similarity transformations on the system, the eigenvalues are preserved but the singular values of the modified and true TFMs will only be preserved in general if the $\mathbf{M}$ and $\mathbf{N}$ matrices are both unitary (orthonormal). In this case $\sigma_i\left(\mathbf{X}\right) = \sigma_i\left(\mathbf{X}_m\right)$, $\forall i \in \{1, \ldots, n\}$, for any closed-loop TFM $\mathbf{X}$ defined at either the input or output side of the plant.

In addition to the issue of internal stability in multivariable control, there is an additional issue of RHP pole-zero cancellation between $\mathbf{G}$, $\mathbf{P}$ and the $\mathbf{M}$ and $\mathbf{N}$ matrices in the construction of Eqn. 14. However, as QFT designs are performed manually, this issue does not present a problem in practice. Additionally, there is the practical constraint that the $\mathbf{M}$ and $\mathbf{N}$ matrices are chosen to be MP and stable to ensure that $\mathbf{G}$ is MP and stable whenever $\mathbf{G}_m$ is designed to be MP and stable. When this constraint prevents a suitable choice of the $\mathbf{M}$ and $\mathbf{N}$ matrices, an easy sufficient condition to ensure internal stability and correct encirclement counting can be enforced in addition to those in multivariable control. This is for the pole and zero sets of $\mathbf{M}^{-1}$, $\mathbf{P}$ and $\mathbf{N}$ to be mutually distinct (ignoring directions) and also for the pole-zero sets of $\mathbf{M}$, $\mathbf{G}_m$ and $\mathbf{N}^{-1}$.

3. Existence for M and N Matrices

The generalized formulation was presented in the previous section to be used along with the MIMO QFT when the control of NMP and unstable systems such as the X-29 is considered. In the control design scheme, the generalized formulation is mainly used to condition the plant such that equivalent SISO plants used in the MIMO QFT design are desirable in terms of the pole and zero structure.

In this section, the question under study is whether a pair of $\mathbf{M}$ and $\mathbf{N}$ matrices exists so that the conditioned SISO equivalent plants have a desirable pole-zero structure. It is then shown that such matrices exist for a nominal MIMO plant that has no unstable blocking poles which leads to all its SISO equivalent plants being stable.

Definition: A complex number $z_0 \in \mathbf{C}$ is called a blocking zero of $\mathbf{P}(s)$ if $\mathbf{P}(z_0)$ is the null matrix. And a complex number $p_0 \in \mathbf{C}$ is called a blocking pole of $\mathbf{P}(s)$ if $\mathbf{P}^{-1}(p_0)$ is the null matrix.

Let $A_{(i,:)}$ denote the i-th row of $\mathbf{A}$ and $A_{(:,j)}$ denote j-th column of $\mathbf{A}$. Let $\mathbf{A}_{(i',j')}$ be a partitioned matrix of $\mathbf{A}$ with i-th row and j-th column deleted. Further, $\det(\mathbf{A})$ denote the determinant of the $\mathbf{A}$ and $adj(\mathbf{A})$ denote the classical adjoint of $\mathbf{A}$. The inverse of $\mathbf{A}$ is denoted as $\mathbf{A}^{-1} = \hat{\mathbf{A}} = \left[\hat{a}_{ij}\right]$.

3.1 Motivation

Consider a TITO plant of the form $\mathbf{P}=\begin{bmatrix} p_{11} & 0 \\ 0 & 1 \end{bmatrix}$ with $\mathbf{P}^{-1}=\begin{bmatrix} \frac{1}{p_{11}} & 0 \\ 0 & 1 \end{bmatrix}$ and

$\mathbf{Q}=\left[\frac{1}{\hat{p}_{ij}}\right]=\mathbf{P}$. It is obvious that the unstable pole and the NMP zero are pinned to the input 1 and the output 1 because the input/output direction of the unstable pole and the NMP the zero are $u_p=y_p=\begin{bmatrix}1 & 0\end{bmatrix}^T$ and $u_z=y_z=\begin{bmatrix}1 & 0\end{bmatrix}^T$, respectively. Stabilization of this TITO plant using MIMO QFT is difficult due to the pole-zero structure of $q_{11}=p_{11}$, which has a RHP dipole.

Let $\mathbf{N}=\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$ and $\mathbf{M}^{-1}=\begin{bmatrix} \hat{m}_{11} & \hat{m}_{12} \\ \hat{m}_{21} & \hat{m}_{22} \end{bmatrix}$. Then, $\mathbf{P}_m=\mathbf{M}^{-1}\mathbf{P}\mathbf{N}=\begin{bmatrix} p_{m11} & p_{m12} \\ p_{m21} & p_{m22} \end{bmatrix}$, where

$$P_{m11}=\frac{\hat{m}_{11}n_{11}(s-2)+\hat{m}_{12}n_{21}(s+1)(s+2)(s-3)}{(s+1)(s+2)(s-3)},$$

$$P_{m12}=\frac{\hat{m}_{11}n_{12}(s-2)+\hat{m}_{12}n_{22}(s+1)(s+2)(s-3)}{(s+1)(s+2)(s-3)},$$

$$P_{m21}=\frac{\hat{m}_{21}n_{11}(s-2)+\hat{m}_{22}n_{21}(s+1)(s+2)(s-3)}{(s+1)(s+2)(s-3)} \text{ and}$$

$$P_{m22}=\frac{\hat{m}_{21}n_{12}(s-2)+\hat{m}_{22}n_{22}(s+1)(s+2)(s-3)}{(s+1)(s+2)(s-3)}.$$

Furthermore, $\mathbf{Q}_m=\left[\frac{1}{\hat{p}_{m\,ij}}\right]=\begin{bmatrix} q_{m11} & q_{m12} \\ q_{m21} & q_{m22} \end{bmatrix}$, where

$$q_{m11}=\frac{(s-2)\det(\mathbf{N})\det(\mathbf{M}^{-1})}{\hat{m}_{21}n_{12}(s-2)+\hat{m}_{22}n_{22}(s+1)(s+2)(s-3)} \text{ and}$$

$$q_{m22} = \frac{(s-2)\det(\mathbf{N})\det(\mathbf{M}^{-1})}{\hat{m}_{11}n_{11}(s-2) + \hat{m}_{12}n_{21}(s+1)(s+2)(s-3)}.$$

In order to have well conditioned equivalent SISO plants, a simple thing to do is at least make the denominator of q_{m22} (or equivalently the numerator of p_{m11}) Hurwitz while keeping $p_{m22} = p_{22}$. This is equivalent to un-pinning the NMP zero from input 1 and output 1 while keeping the unstable pole pinned to input 1 and output 1 in the modified plant. If this is achieved, then q_{m22} is NMP and stable and q_{m11} is NMP and unstable, while p_{m11} is minimum phase (MP) and unstable and p_{m22} remains unchanged in the direct plant domain. Now by using the improved non-sequential MIMO QFT approach, the stability criterion $z_{\mathbf{p}} + z_{\Phi} = z_{\mathbf{\Lambda}}$ can be easily satisfied by assigning a large gain to g_1 and a small gain to g_2. Note that the requirement of making the numerator of p_{m11} Hurwitz is essentially the same as stabilizing p_{11} itself. Therefore, no benefit is actually gained by using the generalized formulation on this expanded TITO plant. This appears to be so because the NMP zero and unstable pole are pinned together.

From the Smith-McMillan form of the expanded plant,

$\mathbf{UPV} = \mathbf{Mc} = diag\left\{\frac{1}{(s+1)(s+2)(s-3)}, \quad (s-2)\right\}$, we can find a pair of matrices that can un-pin the NMP zero and the unstable pole of the plant which at the same time decouples the plant. Consider $\mathbf{N} = \mathbf{V} = \begin{bmatrix} 1 & \frac{1}{12}(s+1)(s+2)(s-3) \\ 1 & \frac{1}{12}(s-1)(s-2)(s-3) \end{bmatrix}$ and

$$\mathbf{M}^{-1} = \mathbf{U} = \begin{bmatrix} \frac{1}{12}(s+3)(s-1) & -\frac{1}{12} \\ -(s+1)(s+2)(s-3) & (s-2) \end{bmatrix}. \text{ Thus,}$$

$$\mathbf{P}_m = \mathbf{Q}_m = diag\left\{ \frac{1}{(s+1)(s+2)(s-3)}, \quad (s-2) \right\}$$ has a good pole-zero structure.

From the TITO plant, it is seen that pinning the NMP zeros and unstable poles could potentially reduce the benefits of the generalized formulation. However, this pinning effect dose not necessarily prevent the existence of a pair of $\mathbf{M}$ and $\mathbf{N}$ matrices.

Remark 1: Although $\mathbf{M}$ and $\mathbf{N}$ contain unstable roots in their elements, there is no RHP pole-zero cancellation between the controller $\mathbf{G}$ and the plant $\mathbf{P}$ because $\mathbf{M}$ and $\mathbf{N}$ are unimodular. Consequently, the true controller $\mathbf{G} = \mathbf{N}\mathbf{G}_m\mathbf{M}^{-1}$ is MP and stable if $\mathbf{G}_m$ is MP and stable.

3.2 Existence of $\mathbf{M}$ and $\mathbf{N}$ for a Fully-Populated Nominal Plant

We use the Smith-McMillan form for general $r \times r$ nominal plant as given in the following well known Lemma.

Lemma 1: (Smith-McMillan Form [51]) Let $\mathbf{P}(s) \in \mathbf{R}_P[s]$ be any proper real rational matrix, then there exist unimodular matrices $\mathbf{U}(s)$, $\mathbf{V}(s) \in \mathbf{R}[s]$ such that

$$\mathbf{U}(s)\mathbf{P}(s)\mathbf{V}(s)=\mathbf{Mc}(s):=\begin{bmatrix} \frac{\alpha_1(s)}{\beta_1(s)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\alpha_2(s)}{\beta_2(s)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\alpha_r(s)}{\beta_r(s)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \text{ and } \alpha_i(s) \text{ divides } \alpha_{i+1}(s),$$

and $\beta_{i+1}(s)$ divides $\beta_i(s)$.□

It is true that for any invertible plant $\mathbf{Mc}(s)$ has full rank because $\mathbf{U}(s)$ and $\mathbf{V}(s)$ are invertible. The roots of all the polynomials $\beta_i(s)$ in the McMillan from for $\mathbf{P}(s)$ are the poles of $\mathbf{P}(s)$ and the pole polynomial $\phi(s)=\prod_{i=1}^{r}\beta_i$. The roots of all the polynomials $\alpha_i(s)$ are the transmission zeros of $\mathbf{P}(s)$ and the zero polynomial $\varphi(s)=\prod_{i=1}^{r}\alpha_i$. Furthermore, if $z_0 \in \mathbf{C}$ is a blocking zero, the dimension of the null space of $\mathbf{P}(z_0)$ is r. The dimension of the null space of $\mathbf{Mc}(z_0)$ is also r because $\mathbf{U}(s)$ and $\mathbf{V}(s)$ are unimodular matrices. Therefore, all $\alpha_i(s)$ contain the term $(s-z_0)$ signifying the root at z_0, i.e., the geometric multiplicity of $z_0 \in \mathbf{C}$ is r. Moreover, $\alpha_1(s)$ only contains blocking zeros because of the property that $\alpha_i(s)$ divides $\alpha_{i+1}(s)$. Therefore it follows that $\alpha_1(s)$ is Hurwitz if and only if $\mathbf{P}(s)$ has only MP blocking zeros. Similarly, the dimension of the null space of $\mathbf{Mc}^{-1}(p_0)$, where $p_0 \in \mathbf{C}$ is a pole of the plant, is r and $\beta_r(s)$ is Hurwitz if and only if $\mathbf{P}(s)$ has only stable blocking poles.

Now consider the transfer function matrix (TFM) of a $r \times r$ MIMO nominal plant given by $\mathbf{P} = \frac{\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1r} \\ z_{21} & z_{22} & \cdots & z_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rr} \end{bmatrix}}{\phi} = \frac{\mathbf{Z}}{\phi}$ which is derived from its minimal realization. It now follows that $\det\left(\mathbf{P}\right) = \frac{\det\left(\mathbf{Z}\right)}{\phi^r} = \frac{\varphi}{\phi}$, and

$\mathbf{P}^{-1} = \left[\hat{p}_{ij}\right] = \frac{adj\left(\mathbf{P}\right)}{\det\left(\mathbf{P}\right)} = \frac{\phi^r \, adj\left(\mathbf{P}\right)}{\det\left(\mathbf{Z}\right)} = \frac{\left[\left(-1\right)^{i+j} \det\left(\mathbf{Z}_{(j',i')}\right)\right]}{\varphi\phi^{r-2}} \triangleleft$, where $\phi\left(s\right)$ is the pole polynomial and $\varphi\left(s\right)$ is the zero polynomial of this MIMO system. The **Q**-matrix, needed for the MIMO QFT design, is given by $\mathbf{Q} = \left[q_{ij}\right] = \left[\frac{1}{\hat{p}_{ij}}\right] = \frac{\varphi\phi^{r-2}}{\left[\left(-1\right)^{i+j} \det\left(\mathbf{Z}_{(j',i')}\right)\right]}$.

Following the generalized formulation, the modified plant

$$\mathbf{P}_m = \mathbf{M}^{-1}\mathbf{P}\mathbf{N}$$

$$= \frac{\begin{bmatrix} \hat{M}_{(1,:)}\mathbf{Z}N_{(:,1)} & \hat{M}_{(1,:)}\mathbf{Z}N_{(:,2)} & \cdots & \hat{M}_{(1,:)}\mathbf{Z}N_{(:,r)} \\ \hat{M}_{(2,:)}\mathbf{Z}N_{(:,1)} & \hat{M}_{(2,:)}\mathbf{Z}N_{(:,2)} & \cdots & \hat{M}_{(2,:)}\mathbf{Z}N_{(:,r)} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{M}_{(r,:)}\mathbf{Z}N_{(:,1)} & \hat{M}_{(r,:)}\mathbf{Z}N_{(:,2)} & \cdots & \hat{M}_{(r,:)}\mathbf{Z}N_{(:,r)} \end{bmatrix}}{\phi}$$

$$= \frac{\begin{bmatrix} z_{m11} & z_{m12} & \cdots & z_{m1r} \\ z_{m21} & z_{m22} & \cdots & z_{m2r} \\ \vdots & \vdots & \ddots & \vdots \\ z_{mr1} & z_{mr2} & \cdots & z_{mrr} \end{bmatrix}}{\phi} = \frac{\mathbf{Z}_m}{\phi}, \tag{16}$$

where $\mathbf{M}^{-1} = \left[\widehat{m}_{ij}\right] = \begin{bmatrix} \hat{M}_{(1,:)} \\ \hat{M}_{(2,:)} \\ \vdots \\ \hat{M}_{(r,:)} \end{bmatrix}$ and $\mathbf{N} = \left[n_{ij}\right] = \left[N_{(:,1)} \quad N_{(:,2)} \quad \cdots \quad N_{(:,r)}\right]$. The elements of

$\mathbf{N}$ and $\mathbf{M}^{-1}$ are polynomials of s and $\mathbf{N}$ and $\mathbf{M}^{-1}$ are invertible. The sub-index m denotes elements associated with the modified design problem. Moreover, the determinant of the modified plant is

$$\det(\mathbf{P}_m) = \frac{\det\left(\mathbf{M}^{-1}\right)\det(\mathbf{Z})\det(\mathbf{N})}{\phi^r} = \frac{\det\left(\mathbf{M}^{-1}\right)\det(\mathbf{N})\varphi}{\phi}.$$

Eqn. 16 yields the following for the modified $\mathbf{P}_m$ and $\mathbf{Q}_m$

$$\mathbf{P}_m^{-1} = \left[\hat{p}_{mij}\right] = \frac{\phi\, adj(\mathbf{P}_m)}{\det\left(\mathbf{M}^{-1}\right)\det(\mathbf{N})\varphi} = \frac{\left[(-1)^{i+j}\det\left(\mathbf{Z}_{m(j',i')}\right)\right]}{\det\left(\mathbf{M}^{-1}\right)\det(\mathbf{N})\varphi\phi^{r-2}}$$

$$= \frac{\begin{bmatrix} \det\left(\mathbf{Z}_{m(1',1')}\right) & -\det\left(\mathbf{Z}_{m(2',1')}\right) & \cdots & (-1)^{1+r}\det\left(\mathbf{Z}_{m(r',1')}\right) \\ -\det\left(\mathbf{Z}_{m(1',2')}\right) & \det\left(\mathbf{Z}_{m(2',2')}\right) & \cdots & (-1)^{2+r}\det\left(\mathbf{Z}_{m(r',2')}\right) \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{r+1}\det\left(\mathbf{Z}_{m(1',r')}\right) & (-1)^{r+2}\det\left(\mathbf{Z}_{m(2',r')}\right) & \cdots & (-1)^{2r}\det\left(\mathbf{Z}_{m(r',r')}\right) \end{bmatrix}}{\det\left(\mathbf{M}^{-1}\right)\det(\mathbf{N})\varphi\phi^{r-2}}, \tag{17}$$

and $\mathbf{Q}_m = \left[q_{mij}\right] = \left[\frac{1}{\hat{p}_{mij}}\right] = \frac{\det\left(\mathbf{M}^{-1}\right)\det(\mathbf{N})\varphi\phi^{r-2}}{\left[(-1)^{i+j}\det\left(\mathbf{Z}_{m(j',i')}\right)\right]}$

$$
= \begin{bmatrix}
\frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{\det(\mathbf{Z}_{m(1',1')})} & \frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{-\det(\mathbf{Z}_{m(2',1')})} & \cdots & \frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{(-1)^{1+r}\det(\mathbf{Z}_{m(r',1')})} \\
\frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{-\det(\mathbf{Z}_{m(1',2')})} & \frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{\det(\mathbf{Z}_{m(2',2')})} & \cdots & \frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{(-1)^{2+r}\det(\mathbf{Z}_{m(r',2')})} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{(-1)^{r+1}\det(\mathbf{Z}_{m(1',r')})} & \frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{(-1)^{r+2}\det(\mathbf{Z}_{m(2',r')})} & \cdots & \frac{\det(\mathbf{M}^{-1})\det(\mathbf{N})\varphi\phi^{r-2}}{(-1)^{2r}\det(\mathbf{Z}_{m(r',r')})}
\end{bmatrix}. \tag{18}
$$

It is difficult to derive sufficient conditions directly from Eqns. 17 and 18. However, by developing a set of formulae in the inverse plant domain, a better handle on the question can be achieved.

Since $\mathbf{P}_m = \mathbf{M}^{-1}\mathbf{P}\mathbf{N}$, $\mathbf{P}_m^{-1} = \left[\hat{p}_{mij}\right] = \left(\mathbf{M}^{-1}\mathbf{P}\mathbf{N}\right)^{-1} = \mathbf{N}^{-1}\mathbf{P}^{-1}\mathbf{M} = \frac{\mathbf{N}^{-1}adj(\mathbf{Z})\mathbf{M}}{\varphi\phi^{r-2}}$

$$
= \frac{\begin{bmatrix}
\hat{N}_{(1,:)}adj(\mathbf{Z})M_{(:,1)} & \hat{N}_{(1,:)}adj(\mathbf{Z})M_{(:,2)} & \cdots & \hat{N}_{(1,:)}adj(\mathbf{Z})M_{(:,r)} \\
\hat{N}_{(2,:)}adj(\mathbf{Z})M_{(:,1)} & \hat{N}_{(2,:)}adj(\mathbf{Z})M_{(:,2)} & \cdots & \hat{N}_{(2,:)}adj(\mathbf{Z})M_{(:,r)} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{N}_{(r,:)}adj(\mathbf{Z})M_{(:,1)} & \hat{N}_{(r,:)}adj(\mathbf{Z})M_{(:,2)} & \cdots & \hat{N}_{(r,:)}adj(\mathbf{Z})M_{(:,r)}
\end{bmatrix}}{\varphi\phi^{r-2}}, \tag{19}
$$

and $\mathbf{Q}_m = \frac{\varphi\phi^{r-2}}{\mathbf{N}^{-1}adj(\mathbf{Z})\mathbf{M}}$

$$
= \frac{\varphi\phi^{r-2}}{\begin{bmatrix}
\hat{N}_{(1,:)}adj(\mathbf{Z})M_{(:,1)} & \hat{N}_{(1,:)}adj(\mathbf{Z})M_{(:,2)} & \cdots & \hat{N}_{(1,:)}adj(\mathbf{Z})M_{(:,r)} \\
\hat{N}_{(2,:)}adj(\mathbf{Z})M_{(:,1)} & \hat{N}_{(2,:)}adj(\mathbf{Z})M_{(:,2)} & \cdots & \hat{N}_{(2,:)}adj(\mathbf{Z})M_{(:,r)} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{N}_{(r,:)}adj(\mathbf{Z})M_{(:,1)} & \hat{N}_{(r,:)}adj(\mathbf{Z})M_{(:,2)} & \cdots & \hat{N}_{(r,:)}adj(\mathbf{Z})M_{(:,r)}
\end{bmatrix}}. \tag{20}
$$

Remark 2: Since the MIMO QFT is a design methodology based on the plant inverse, it is more intuitive to derive the formulae in the inverse plant domain using Eqns. 19 and 20. Consequently, we synthesize $\mathbf{N}^{-1}$ and $\mathbf{M}$ instead of $\mathbf{N}$ and $\mathbf{M}^{-1}$.

Similarly, the elements of $\mathbf{N}^{-1}$ and $\mathbf{M}$ are selected as polynomials of s and are invertible. Moreover, since $\mathbf{G} = \mathbf{N}\mathbf{G}_m\mathbf{M}^{-1} = \frac{adj\left(\mathbf{N}^{-1}\right)\mathbf{G}_m adj\left(\mathbf{M}\right)}{\det\left(\mathbf{N}^{-1}\right)\det\left(\mathbf{M}\right)}$, $\det\left(\mathbf{N}^{-1}\right)$ and $\det\left(\mathbf{M}\right)$ are chosen to be stable polynomials which in turn ensure a stable controller. Because $\mathbf{N}^{-1}$ and $\mathbf{M}$ are chosen as polynomial matrices, no multi-variable zeros are present in the controller that results from the synthesis of $\mathbf{N}^{-1}$ and $\mathbf{M}$. Therefore, possible RHP pole-zero cancellation when using $\mathbf{N}$ and $\mathbf{M}$ matrices is eliminated.

Remark 3: Since any nominal LTI, SISO, MP and unstable plant of $P = \frac{n_p}{d_p}$ is always stablizable with an LTI, MP and stable controller $G = \frac{n_g}{d_g}$, there exists a set of Hurwitz polynomials, n_g and d_g, such that the closed-loop characteristic equation

$\phi_{closed} = n_p n_g + d_p d_g$, a Diophantine equation, is Hurwitz. (21)

Now we can state a sufficient condition as follows.

Lemma 2: (Sufficient Condition) For a $r \times r$ nominal plant, if one of the $\varphi_{(i',j')} = \frac{\det\left(\mathbf{Z}_{(i',j')}\right)}{\phi^{r-2}}$ has only stable roots, there exist $\mathbf{N}^{-1}$ and $\mathbf{M}$ matrices such that all so-conditioned equivalent SISO plants are stable. Moreover, the synthesized $\mathbf{N}^{-1}$ and $\mathbf{M}$ matrices are invertible and have stable determinants.

Proof: The pole polynomial of the plant is the least common denominator of all non-zero minors of all orders of the plant, given by $\det\left(\mathbf{Z}_{(i',j')}\right)=\phi^{r-2}\varphi_{(i',j')}$, where $\varphi_{(i',j')}$ is the zero polynomial of $\mathbf{P}_{(i',j')}$. Therefore the condition that one of the $\varphi_{(i',j')}=\dfrac{\det\left(\mathbf{Z}_{(i',j')}\right)}{\phi^{r-2}}$ has only stable roots is equivalent to the condition that one of the $(r-1)\times(r-1)$ sub-systems $\mathbf{P}_{(i',j')}=\dfrac{\left[\mathbf{Z}_{(i'j')}\right]}{\phi}$ has no NMP zeros.

From Eqn. 20, the modified SISO equivalent plants used in MIMO QFT design are $q_{mii}=\dfrac{\varphi\phi^{r-2}}{\hat{N}_{(i,:)}adj\left(\mathbf{Z}\right)M_{(:,i)}}$ for $i=1,\cdots r$. Note that each q_{mii} is independently conditioned by a row of $\mathbf{N}^{-1}$, $\hat{N}_{(i,:)}$, and a column of $\mathbf{M}$, $M_{(:,i)}$.

Without loss of generality, assume that $\varphi_{(1',1')}=\dfrac{\det\left(\mathbf{Z}_{(1',1')}\right)}{\phi^{r-2}}$ is the only stable polynomial. This is achievable by re-arranging the order of plant inputs and outputs.

Since $q_{mii}=\dfrac{\varphi}{d_{mii}}=\dfrac{\varphi\phi^{r-2}}{\hat{N}_{(i,:)}adj\left(\mathbf{Z}\right)M_{(:,i)}}$, its denominator is

$$
\begin{aligned}
d_{mii}&=\frac{\phi^{r-2}}{\hat{N}_{(i,:)}adj\left(\mathbf{Z}\right)M_{(:,i)}}\\
&=\left[\hat{N}_{(i,:)}\frac{\phi^{r-2}}{adj\left(\mathbf{Z}\right)_{(:,1)}}\quad \hat{N}_{(i,:)}\frac{\phi^{r-2}}{adj\left(\mathbf{Z}\right)_{(:,2)}}\quad\cdots\quad \hat{N}_{(i,:)}\frac{\phi^{r-2}}{adj\left(\mathbf{Z}\right)_{(:,r)}}\right]M_{(:,i)}\\
&=\left[f_1^i\quad f_2^i\quad\cdots\quad f_r^i\right]M_{(:,i)}=m_{1i}f_1^i+m_{2i}f_2^i+\cdots+m_{ri}f_r^i,
\end{aligned}
$$

where $f_j^i=\hat{n}_{i1}\varphi_{(j',1')}-\hat{n}_{i2}\varphi_{(j',2')}+\cdots+(-1)^{j+r}\hat{n}_{ir}\varphi_{(j',r')}$, $\forall j=1,\cdots r$.

$$\text{Let, } \mathbf{N}^{-1} = \begin{bmatrix} \hat{n}_{11} & 0 & \cdots & 0 & 0 \\ \hat{n}_{21} & \hat{n}_{22} & 0 & & 0 \\ \hat{n}_{31} & 0 & \hat{n}_{33} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \hat{n}_{r1} & 0 & \cdots & 0 & \hat{n}_{rr} \end{bmatrix} \text{ and } \mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1r} \\ 0 & m_{22} & 0 & \cdots & 0 \\ \vdots & 0 & m_{33} & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & m_{rr} \end{bmatrix}.$$

Consequently, for $i=1$, $f_j^1 = \hat{n}_{11}\varphi_{(j',1')}$, $\forall j = 1,\cdots r$, and

$d_{m11} = m_{11}f_1^1 + m_{21}f_2^1 + \cdots + m_{r1}f_r^1 = m_{11}f_1^1 = m_{11}\hat{n}_{11}\varphi_{(1',1')}$. For $i = 2,\cdots r$,

$f_j^i = \hat{n}_{i1}\varphi_{(j',1')} + \left(-1\right)^{j+i}\hat{n}_{ii}\varphi_{(j',i')}$, $\forall j = 1,\cdots r$. Thus, $d_{mii} = m_{1i}f_1^i + m_{ii}f_i^i$ for $i = 2,\cdots r$.

As a result, m_{11} and $\hat{n}_{11}$ are selected as stable polynomials such that d_{m11} is Hurwitz. For $i = 2,\cdots r$, because $f_1^i = \hat{n}_{i1}\varphi_{(1',1')} + \left(-1\right)^{1+i}\hat{n}_{ii}\varphi_{(1',i')}$ is a Diophantine equation in the form of Eqn. 21, there exist Hurwitz $\hat{n}_{i1}$ and $\hat{n}_{ii}$ such that f_1^i is Hurwitz. Using the same reasoning, there exist stable m_{1i} and m_{ii} such that $d_{mii} = m_{1i}f_1^i + m_{ii}f_i^i$ is Hurwitz. Furthermore, $\det\left(\mathbf{N}^{-1}\right) = \prod_{i=1}^{r}\hat{n}_{ii}$ and $\det\left(\mathbf{M}\right) = \prod_{i=1}^{r} m_{ii}$ are stable polynomials because $\hat{n}_{ii}$ and m_{ii} are all stable polynomials.

As a result, there exist $\mathbf{N}^{-1}$ and $\mathbf{M}$ matrices, which have stable determinants and are invertible, such that all conditioned SISO equivalent plants are stable in MIMO QFT design if one of the $\varphi_{(i',j')}$ has only stable roots. □

Remark 4: If none of the $\varphi_{(i',j')}$ is stable, there is actually no benefit gained by using the generalized formulation directly from the formulae. This is seen from the example in section 3.1.

Remark 5: Although the $\mathbf{N}^{-1}$ and $\mathbf{M}$ matrices have a specified structure for this constructive proof, there are other possible structures for its selection.

Remark 6: A MIMO system $\mathbf{P}$ possessing a blocking unstable pole, which is a blocking NMP zero in the inverse plant $\mathbf{P}^{-1}$, violates the sufficient condition in Lemma 2. A blocking NMP zero from the plant $\mathbf{P}$ however will not prevent a solution as it is cancelled by the NMP zeros of φ in Eqn. 20.

Remark 7: Using the simultaneous stability theory from [52], the result from Lemma 2 can be further developed for $r \times r$ uncertain plants with some extra conditions added on $\varphi_{(i',j')}$ among the plant family.

Lemma 3: (Necessary and Sufficient Condition) For a $r \times r$ nominal plant, there exist $\mathbf{N}^{-1}$ and $\mathbf{M}$ matrices such that all so-conditioned equivalent SISO plants used in the MIMO QFT design are stable if and only if that the plant has no unstable blocking poles. Moreover, the synthesized $\mathbf{N}^{-1}$ and $\mathbf{M}$ matrices are invertible with stable determinants.

Proof: Let Smith-McMillan form of $\mathbf{P}$ be

$\mathbf{Mc} = \mathbf{UPV} = diag\left\{ \frac{\alpha_1(s)}{\beta_1(s)}, \quad \frac{\alpha_2(s)}{\beta_2(s)}, \quad \cdots \quad \frac{\alpha_r(s)}{\beta_r(s)} \right\}$, where $\mathbf{U}$ and $\mathbf{V}$ are unimodular.

Thus, the pole and the zero polynomials are, $\phi = \prod_{i=1}^{r} \beta_i$ and $\varphi = \prod_{i=1}^{r} \alpha_i$ respectively. From the properties of the Smith-McMillan form, β_r is a Hurwitz polynomial if and only if there are no unstable blocking poles.

Sufficiency:

Let $\mathbf{N}^{-1} = \mathbf{N}_2\mathbf{N}_1 = \mathbf{N}_2\mathbf{V}^{-1}$ and $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2 = \mathbf{U}^{-1}\mathbf{M}_2$. Since the plant inverse is $\mathbf{P}^{-1} = \mathbf{VMc}^{-1}\mathbf{U}$, then $\mathbf{P}_m^{-1} = \left(\mathbf{M}^{-1}\mathbf{PN}\right)^{-1} = \mathbf{N}^{-1}\mathbf{P}^{-1}\mathbf{M} = \mathbf{N}_2\mathbf{V}^{-1}\mathbf{P}^{-1}\mathbf{U}^{-1}\mathbf{M}_2 = \mathbf{N}_2\mathbf{Mc}^{-1}\mathbf{M}_2$

$$= \frac{\mathbf{N}_2\begin{bmatrix} \beta_1\delta_1 & 0 & \cdots & 0 & 0 \\ 0 & \beta_2\delta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \beta_r \end{bmatrix}\mathbf{M}_2}{\alpha_r}, \text{ where } \delta_i = \frac{\alpha_\gamma}{\alpha_i}.$$

Repeating the argument used in the Lemma 2, there exist nonsingular $\mathbf{N}_2$ and $\mathbf{M}_2$ matrices such that the diagonal numerators of $\mathbf{P}_m^{-1}$ are all MP. Moreover, $\mathbf{N}_2$ and $\mathbf{M}_2$ have stable determinants. Thus, there exist $\mathbf{N}^{-1}$ and $\mathbf{M}$ matrices such that all so-conditioned SISO equivalent plants are stable if a plant has no blocking unstable poles. Furthermore, $\det\left(\mathbf{N}^{-1}\right) = \det\left(\mathbf{N}_2\right)\det\left(\mathbf{N}_1\right)$ and $\det\left(\mathbf{M}\right) = \det\left(\mathbf{M}_1\right)\det\left(\mathbf{M}_2\right)$ are both stable polynomials.

Necessity:

For a plant having an unstable blocking pole, $\left(s - p_u\right)$, assume that there exist $\mathbf{N}^{-1}$ and $\mathbf{M}$ matrices such that all so-conditioned equivalent SISO plants used in the MIMO QFT design are stable.

Thus, $\mathbf{P}^{-1} = \mathbf{VMc}^{-1}\mathbf{U}$

$$= \mathbf{V}\begin{bmatrix} \frac{(s-p_u)\beta_1'(s)}{\alpha_1(s)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{(s-p_u)\beta_2'(s)}{\alpha_2(s)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{(s-p_u)\beta_r'(s)}{\alpha_r(s)} \end{bmatrix}\mathbf{U}$$

$$= (s-p_u)\left(\mathbf{V}\begin{bmatrix} \frac{\beta_1'(s)}{\alpha_1(s)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\beta_2'(s)}{\alpha_2(s)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{\beta_r'(s)}{\alpha_r(s)} \end{bmatrix}\mathbf{U} \right).$$

Furthermore, $\mathbf{P}_m^{-1} = \mathbf{N}^{-1}\mathbf{P}^{-1}\mathbf{M}$

$$= (s-p_u)\left(\mathbf{N}^{-1}\mathbf{V}\begin{bmatrix} \frac{\beta_1'(s)}{\alpha_1(s)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\beta_2'(s)}{\alpha_2(s)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{\beta_r'(s)}{\alpha_r(s)} \end{bmatrix}\mathbf{MU} \right)$$

. Since the diagonal elements of $\mathbf{P}_m^{-1}$ are all NMP containing the unstable pole, $(s-p_u)$, all the $\mathbf{N}^{-1}$ and $\mathbf{M}$ conditioned SISO equivalent plants are unstable, which contradicts the assumption. Therefore, the condition that a plant possesses no unstable blocking poles is necessary and sufficient condition. □

A similar condition can be developed in the direct plant domain. It states that there exist $\mathbf{N}$ and $\mathbf{M}^{-1}$ matrices such that the diagonal elements in the modified plant are conditioned MP if and only if the plant has no NMP blocking zeros. Moreover, $\varphi_{(i',j')}$ are reduced to individual z_{ij} of the plant in TITO systems. Thus, there exist $\mathbf{N}$ and $\mathbf{M}^{-1}$ matrices for TITO plants such that all SISO equivalent plants are conditioned stable if and only if the TITO plant has no NMP blocking zeros. Consequently, one could either choose to synthesize $\mathbf{N}^{-1}$ and $\mathbf{M}$ or $\mathbf{N}$ and $\mathbf{M}^{-1}$ depending on the unstable blocking pole or NMP blocking zero in a TITO plant.

In summary, an existence condition for MIMO plants was established for when it might be feasible to adopt the generalized formulation in the MIMO QFT design. If a nominal MIMO plant does not possess any unstable blocking poles, there is a pair of $\mathbf{M}$ and $\mathbf{N}$ matrices such that so-conditioned nominal SISO equivalent plants in IPD are all stable. Consequently, the limitation of using decentralized control is minimized and this nominal plant is easily stabilized provided that the modified plant is also diagonally dominant. Although this pair of $\mathbf{M}$ and $\mathbf{N}$ matrices are synthesized with respect to the nominal case and not for the entire plant family it is still effective for a small range of uncertainty.

The important interpretation of this result is that if one plant in the family possesses any unstable blocking poles, then at least for that plant it is not possible to use $\mathbf{M}$ and $\mathbf{N}$ matrices to condition it so that its SISO equivalent plants are stable. This type of systems is inherently difficult to handle as a MIMO QFT design even with the generalized formulation.

CHAPTER IV

MIMO QFT DESIGN USING GENERALIZED FORMULATION

1. Introduction

In Chapter III, a generalized formulation was proposed to alleviate design difficulties and conservatisms in MIMO QFT and to permit the synthesis of fully populated controllers if necessary. When the design difficulty due to the RHP poles and zeros in the equivalent SISO plants is eliminated, the plant is said to have good pole-zero alignment as will be defined in section 2.1 of this chapter. Therefore, the key guideline for the synthesis of $\mathbf{M}$ and $\mathbf{N}$ matrices is to condition a plant with respect to this pole-zero alignment. In particular, this is done while persevering or improving the diagonal dominance property of the plant.

The plant/controller alignment, which was originally proposed by Freudenberg for single-input two-output (SITO) systems [55] and extended to TITO system in this chapter, is used to analyze the properties of uncertain, ill-conditioned, TITO systems. This extension was motivated by the work of [53, 54] where the generalized formulation is implicitly considered. Consequently, $\mathbf{M}$ and $\mathbf{N}$ matrices can also be used to improve the plant/controller alignment and produce a better closed-loop performance for ill-conditioned TITO systems. This plant/controller alignment is discussed in section 2.2 and used as a guideline in the design steps.

In section 3, it is shown how the generalized formulation is employed in MIMO QFT to condition the SISO equivalent plants and the plant/controller alignment. The exploitation of the formulation and alignment concepts is later demonstrated in Chapter V via the X-29.

Notation: $\|\cdot\|$ is the standard vector 2-norm, with $(\bar{\cdot})$ denoting a vector of norm 1. Subscripts $(i\cdot)$ and $(\cdot j)$ correspond to a row or column of the TFM, respectively.

2. Plant/Controller Alignment and Pole-Zero Alignment in TITO Systems

When generalized formulation is employed in MIMO QFT, $\mathbf{M}$ and $\mathbf{N}$ matrices are synthesized for the improvement of the pole-zero alignment, diagonal dominance, and the plant/controller alignment specially for ill-conditioned plants. These two alignments are discussed in this section.

2.1 Pole-Zero Alignment

In the generalized formulation, the $\mathbf{M}$ and $\mathbf{N}$ matrices are chosen to condition the poles and zeros of the DPD and IPD SISO plants considered in the MISO loop designs. In the case there are row and column permutation matrices, they correspond to input-output swapping. This is termed herein to be an adjustment of the pole-zero alignment of the plant relative to the controller [56,57]. The following qualitative description is provided:

Pole-zero alignment: The modified plant possesses favorable pole-zero alignment if the spurious zeros in the DPD SISO plants and poles in the IPD SISO plants do not prohibit a successful design.

For an IPD SISO plant, $q_{(i,i)}$ this refers to the choice of the $\mathbf{M}$ and $\mathbf{N}$ matrices such that the principal cofactor (i,i) of the plant TFM is Hurwitz, in the case that the $q_{(i,i)}$ needs to be stable or such that the poles of $q_{(i,i)}$ are not design prohibitive. For a DPD SISO plant $p_{(i,i)}$, good pole-zero alignment refers to the choice of the $\mathbf{M}$ and $\mathbf{N}$ matrices such that the zeros of the plant dynamics between input i and output i are MP, in the case that $p_{(i,i)}$ need be MP or else possesses zeros that are not design prohibitive. Of course, this should be done for the entire plant family as demonstrated in the designs in Chapter V. Consequently, if a plant has any blocking unstable poles, there is no such pair of $\mathbf{M}$ and $\mathbf{N}$ matrices that its IPD SISO equivalent plants are conditioned to be stable simultaneously for the entire plant family. Similarly, for a plant with any blocking NMP zero, its DPD equivalent plants can not be conditioned to be MP simultaneously for the entire plant family.

It should be stated here that pole-zero alignment is presented only as a concept for design consideration, and not a formal, quantitative characterization of the system. That being said, the design problem to condition these poles and zeros is quite straightforward for low dimensional systems. The (robust) synthesis of the $\mathbf{M}$ and $\mathbf{N}$ matrices to provide for favorable pole-zero alignment for low dimensional (uncertain) systems can simply be seen as a (robust) pole placement problem (as seen in section 3 of Chapter III) for which several methods are appropriate, such as Routh Hurwitz, [52] and [58]. The use of the concept of pole-zero alignment in Chapter V, and the simple approach for assuring favorable pole-zero alignment, serves to constructively justify its consideration. In the designs presented in Chapter V, Routh's stability criterion is adequate to place conditions on the elements of the $\mathbf{M}$ and $\mathbf{N}$ matrices for a chosen structure to robustly condition the properties of the DPD and IPD SISO plants. The effectiveness of this simple approach for assuring favorable pole-zero alignment is demonstrated in the MIMO QFT designs for the X-29, which serves to constructively justify the consideration of pole-zero alignment.

2.2 Plant/Controller Alignment

In this section the theory for plant/controller alignment [55] is presented within the generalized formulation, and the theory is extended to TITO systems that are strongly ill-conditioned due to sensor or actuator redundancy, or more generally due to excessive bandwidth constraints arising from plant dynamics, such as NMP zeros or unstable poles.

Consider a feedback system with SITO plant $\mathbf{P} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^T$ and TISO controller $\mathbf{G} = \begin{bmatrix} g_1 & g_2 \end{bmatrix}$. The plant/controller alignment can be defined as [55]:

$$\phi(j\omega) = \arccos\left(\frac{\left|\mathbf{G}(j\omega)\mathbf{P}(j\omega)\right|}{\left\|\mathbf{G}(j\omega)\right\|\left\|\mathbf{P}(j\omega)\right\|} \right). \tag{22}$$

Evidently, ϕ is bounded between 0° and 90° with perfect alignment being $\phi = 0^\circ$. In [55], it is proven that this angle relates the properties of the TITO closed-loop output TFMs to the SISO closed-loop input TFMs. For example, the output complementary sensitivity and sensitivity TFMs $\mathbf{T}_o$ and $\mathbf{S}_o$ are related to their SISO input counterparts $\mathbf{T}_i$ and $\mathbf{S}_i$, via

$$\bar{\sigma}(\mathbf{S}_o) \geq \max\left\{ |\mathbf{S}_i|, \sqrt{1+|\mathbf{T}_i|^2 \tan(\phi)^2} \right\} \tag{23}$$

and

$$\bar{\sigma}(\mathbf{T}_o) = \frac{|\mathbf{T}_i|}{\cos(\phi)}. \tag{24}$$

The plant/controller alignment for the true system can be analogously defined (non-uniquely) within the generalized formulation with $\mathbf{G} = \mathbf{N}\mathbf{G}_m\mathbf{M}^{-1}$, $\mathbf{G}_m = \begin{bmatrix} g_1 & 0 \end{bmatrix}$, $\mathbf{M} = \left[m_{ij} \right]$, a 2×2 matrix, and $\mathbf{N}$ which is simply a scalar due to the consideration of a SITO plant. This gives

$$\phi(j\omega) = \arccos\left(\frac{\left|\hat{\mathbf{M}}_{1\bullet}(j\omega)\mathbf{P}(j\omega)\right|}{\left\|\hat{\mathbf{M}}_{1\bullet}(j\omega)\right\| \left\|\mathbf{P}(j\omega)\right\|} \right), \tag{25}$$

where $\hat{\mathbf{M}} = \mathbf{M}^{-1}$. Evidently good plant/controller alignment can be assured by properly selecting $\hat{\mathbf{M}}_1$, which defines the input direction of the controller. The relationships between the input and output closed-loop transfer functions can then be defined analogously using $\mathbf{M}$. While the plant/controller alignment theory can be evidently presented within the generalized formulation in a straight-forward way, it is the use of the generalized formulation to extend the results to ill-conditioned TITO systems that is of importance here. This development is motivated by [54], where it is evident from Fig.

13 therein that the generalized formulation with only $\mathbf{M}$ is useful for plant/controller alignment improvement.

Following analogous derivations to [55] and employing the generalized formulation with the $\mathbf{M}$ and $\mathbf{N}$ matrices restricted to unitary matrices to ensure the singular values in the modified and true systems match, the following approximate relationships can be derived between the properties of the input and output TFMs of an ill-conditioned system (see Appendix A):

$$\bar{\sigma}\left(\mathbf{S}_o\right)=\bar{\sigma}\left(\mathbf{S}_{om}\right)\geq \max\left\{\left|\mathbf{S}_{im(1,1)}\right|,\sqrt{1+\left|\mathbf{T}_{im(1,1)}\right|^2\tan\left(\phi\right)^2}\right\}, \tag{26}$$

and

$$\bar{\sigma}\left(\mathbf{T}_o\right)=\bar{\sigma}\left(\mathbf{T}_{om}\right)=\frac{\left|\mathbf{T}_{im(1,1)}\right|}{\cos\left(\phi\right)}. \tag{27}$$

Considering the derivation in Appendix A and the assumptions employed, the above relationships are seen to generally hold for any system with highly ill-conditioned controller, plant or more generally loop transmission. When the $\mathbf{M}$ and $\mathbf{N}$ matrices are identity, this simply means that the redundancy that is quantified by the lower singular value direction is a basis direction, implying that the plant or controller is input or output redundant, as considered in [53], [55], or more generally the loop transmission. However, the freedom offered by the generalized formulation is that for systems which possess redundancy that is not in a basis direction, for instance that arises from the satisfaction of the bandwidth limitation from a NMP zero, the $\mathbf{M}$ and $\mathbf{N}$ matrices can be employed to rotate the system directions to ensure that in the modified system the direction is a basis direction. As the relationship is algebraic, this can always be achieved by choosing $\mathbf{M}$ and $\mathbf{N}$ matrices to be the unitary input or output direction matrix given by the singular-value decomposition (SVD) of the system at that frequency.

3. Design Steps

The proposed concept of pole-zero alignment and the developed plant/controller alignment theory are employed in concert with the generalized formulation, which provide a classically formulated MIMO QFT design procedure that facilitates the decentralized design of multivariable controllers. The procedure aims to exploit the directional features of the design by casting the problem within the generalized formulation and providing favorable decentralized design features, such as dominance, plant/controller alignment and pole-zero alignment. The design procedure assumes that the (uncertain) plant and frequency domain objectives or specifications are given. Additionally, any structural constraints on $\mathbf{G}$ or $\mathbf{F}$ should be known.

The design procedure (Fig. 6) amounts to three steps: (i) design initialization, (ii) direct controller synthesis and (iii) direct redesign. However, only the first two steps are applicable to the non-sequential MIMO QFT since all MISO loops are closed independently in the non-sequential design. Moreover, when the non-sequential MIMO QFT is considered, employing $\mathbf{M}$ requires a transformation of the closed-loop tracking specifications or a conservative treatment on the input disturbance of MISO loops, which are not desirable. Hence, where possible, only the $\mathbf{N}$ matrix should be employed for non-sequential MIMO QFT to avoid this conservatism in the performance design.

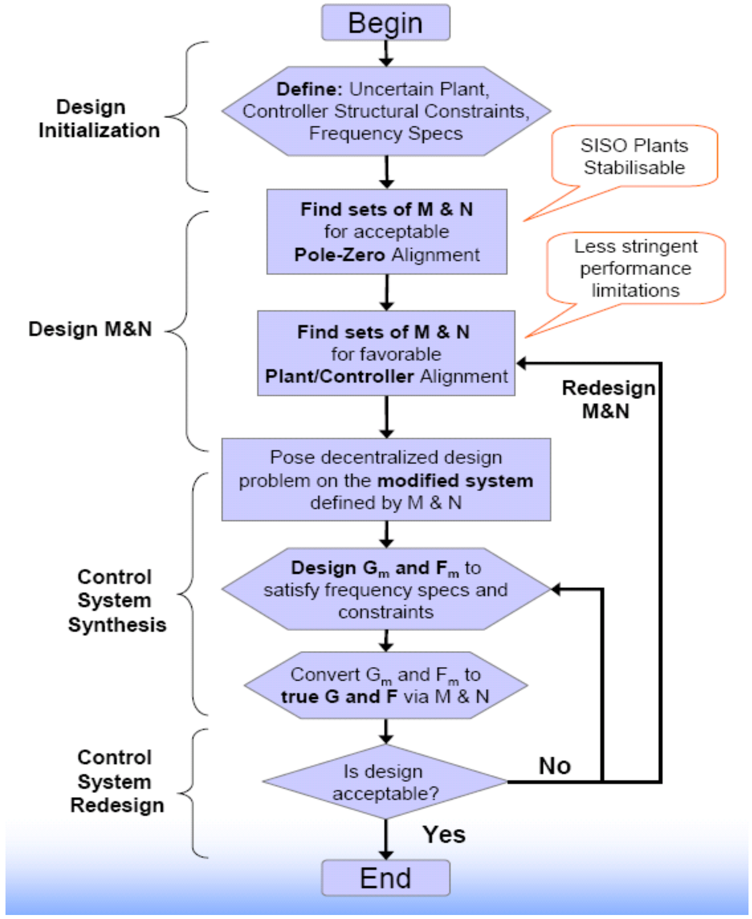

Figure 6: Design Steps

Despite this conservatism, a stabilized closed-loop solution from the non-sequential MIMO QFT is still beneficial since the non-sequential MIMO QFT design is generally straightforward and such a solution serves as a starting point for step (iii) for sequential redesign. This approach of using the non-sequential MIMO QFT for stabilization and then sequential MIMO QFT for performance iteration is utilized in the X-29 design.

Ideally, the matrices should be chosen such that $\rho(\mathbf{E})$ is minimized in the design while simultaneously ensuring the equivalent plants contain no spurious RHP poles (NS L2). In addition, the matrices should also be chosen to provide favorable plant/controller alignment for ill-conditioned plants. And this should be achieved for all plants in the plant family.

The design steps are explained below.

3.1 Design Initialization

The set of $\mathbf{M}$ and $\mathbf{N}$ matrices that provide acceptable pole-zero alignment for the entire plant family is determined, via an appropriate method, while also satisfying any structural constraints on $\mathbf{G}$ and $\mathbf{F}$. This must be sufficient to permit simultaneous stabilization of the plant family. If the design is performed independently using non-sequential MIMO QFT, or integrity properties are required, this generally requires the SISO plant families to be simultaneously stabilizable. For sequential designs the constraints are less severe, but the cost of unstabilizable SISO plants in unstable internal loop closures makes the properties of the SISO plants more severe in the subsequent loop designs [22, 59].

From this set, $\mathbf{M}$ and $\mathbf{N}$ matrix pairs that give favorable dominance properties are chosen such that $\rho(\mathbf{E})$ is minimized. Additionally, if the plant is ill-conditioned, the plant/controller alignment properties should also be considered when selecting the matrices.

Based on the chosen $\mathbf{M}$ and $\mathbf{N}$ matrix pair(s), the generalized formulation is employed to pose the decentralized design problem on the modified system where the non-sequential or sequential MIMO QFT is directly applied with the mapped frequency specifications in the modified system. Note that the mapped performance specification is very conservative if $\mathbf{M}$ matrix is employed. On the other hand, facilitating direct design in the true plant domain using the non-sequential MIMO QFT requires a conservative treatment on the input disturbance of MISO loops, which is also undesirable.

Thus, to overcome this conservatism of mapping the specifications and to facilitate direct design in the true plant domain, the true closed-loop system TFMs are expressed as a function of the $\mathbf{M}$ and $\mathbf{N}$ matrices, the elements of the diagonal controller $\mathbf{G}_m$ in the modified plant domain, and the true plant $\mathbf{P}$ and the sequential MIMO QFT is employed for design with the representative equations presented in Appendix B.

3.2 Direct Controller Synthesis

The diagonal elements of $\mathbf{G}_m$ are designed for the satisfaction of the frequency domain specifications in the true plant domain while attempting to satisfy implementation constraints, such as controller bandwidth and order. The design procedure amounts to the synthesis of the diagonal controller $\mathbf{G}_m$ and the prefilter $\mathbf{F}_m$, which may be diagonal or fully populated. Using the chosen $\mathbf{M}$ and $\mathbf{N}$ matrix pair, the control system designed in the modified plant domain is mapped back into the true plant domain. The properties of the resulting design can then be assessed.

3.3 Direct Redesign

If the properties of the design in the true plant domain are not desirable, the design can be directly modified, including the control elements, $\mathbf{M}$ and $\mathbf{N}$ matrices and the performance specifications using the sequential MIMO QFT. As the design process is highly transparent and importantly posed on the true plant domain transfer functions, design improvements can be made with minimal iteration. The design problem in Chapter V serves to demonstrate this feature.

4. Design Trade-Offs

The design trade-off is clearly seen during the synthesis of the $\mathbf{M}$ and $\mathbf{N}$ matrices. One objective of the $\mathbf{M}$ and $\mathbf{N}$ matrices is to provide good pole-zero alignment, which allows stabilization for difficult systems, while on the other hand the $\mathbf{M}$ and $\mathbf{N}$ matrices are also synthesized for good dominance property and good plant/controller alignment such that the system has improved performance. A tradeoff thus exists between stabilization and performance when these two features require different properties of the $\mathbf{M}$ and $\mathbf{N}$ matrices.

Because stabilization is a precursor for system performance, a design may have bad performance due to the selected $\mathbf{M}$ and $\mathbf{N}$ matrices especially when they are chosen for stabilization and not for the plant/controller alignment. The conflict of these two alignments has a huge effect in the design as seen in the design example of Chapter V.

5. Limitation

The generalized formultion permits multivariable controller synthesis using classically fomulated MIMO QFT and thus the limitations due to decentralized controller structure are alleviated. The advantage and improvement are clearly seen through the design example in Chapter V where the X-29 problem becomes stabilizable under proper $\mathbf{M}$ and $\mathbf{N}$ matrices and the performance is improved after the plant/controller alignment is tuned via $\mathbf{M}$ and $\mathbf{N}$ matrices.

However, the inherent conservertism in large order systems still remains. As the system dimension increases, the design complexity increases drastically. This poor scalability is aggravated with the generalized formulation because of the additional $\mathbf{M}$ and $\mathbf{N}$ matrices.

Secondly, the plant/controller alignment theory presented herein is only for TITO systems. There still lacks a clear extension to general MIMO systems. Depite this shortcoming, improved dominance generally implies good plant/control alignment and generally improves the closed-loop system performance. Hence, the $\mathbf{M}$ and $\mathbf{N}$ matrices can be synthesized for improved dominance property in general MIMO systems.

CHAPTER V

A CASE STUDY OF THE X-29 LONGITUDINAL FLIGHT CONTROL PROBLEM

1. Introduction

In this section three designs for the X-29 are presented using (i) the Singular-G design developed in [36], (ii) an improved non-sequential MIMO QFT design, and (iii) a sequential MIMO QFT design. Both the improved non-sequential MIMO QFT designs and the sequential MIMO QFT designs are seen to employ the generalized formulation to provide good pole-zero alignment such that the design problem become feasible.

The Singular-G method is discussed first and followed by its capture in the frame work of the generalized formulation as an extreme case. The non-sequential design represents an improvement on the Singular-G design, as it employs the directional properties of the Singular-G design as a basis for the choice of the $\mathbf{M}$ and $\mathbf{N}$ matrices and synthesizes a non-singular controller. The sequential MIMO QFT design, which uses the solution from the non-sequential MIMO QFT design as a starting point for re-design, provides an improved performance over the non-sequential MIMO QFT design, in which the plant/controller alignment is improved through the re-designing process.

It should be noted that the control problem considered in the Singular-G and MIMO QFT designs presented herein is made more challenging by trying to stabilize all four flight conditions simultaneously, with associated performance levels, and using only output feedback.

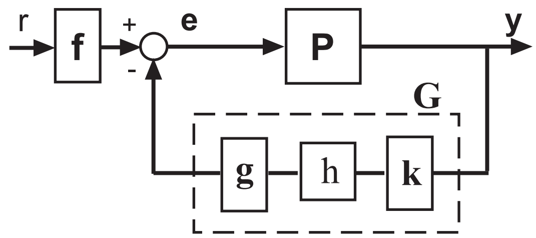

Figure 7: Singular-G Control Structure [36]

2. The Singular-G Design

The Singular-G method was employed in [36] for the control of the X-29, where the design problem is reduced to a SISO control problem through the employment of a singular controller. The method is illustrated below for a TITO plant, with the control structure depicted in Fig. 7. The closed-loop input-output TFM for this system is $\mathbf{T} = (\mathbf{I} + \mathbf{PG})^{-1}\,\mathbf{PF}$, where $\mathbf{g} = [g_1 \quad g_2]^T$, $\mathbf{k} = [k_1 \quad k_2]$, $\mathbf{f} = [f_1 \quad f_2]^T$ and

$$\mathbf{G} = h\,\mathbf{g}\,\mathbf{k} = h\begin{bmatrix} g_1k_1 & g_1k_2 \\ g_2k_1 & g_2k_2 \end{bmatrix}. \tag{28}$$

The stability of the closed-loop system is determined by

$$\begin{aligned} \det(\mathbf{I} + \mathbf{PG}) &= 1 + h\{(p_{11}g_1k_1) + (p_{12}g_2k_1) + (p_{21}g_1k_2) + (p_{22}g_2k_2)\} \\ &= 1 + h\,\mathbf{k}\,\mathbf{P}\,\mathbf{g} \end{aligned}. \tag{29}$$

Note that from Eqn. 29 it is evident that the poles of the effective SISO plant $\mathbf{k}\,\mathbf{Pg}$ considered in the Singular-G design cannot be modified using the Singular-G approach, unless undesirable pole-zero cancellation is employed. The first step in the

Singular-G design is to choose the free elements k_1, k_2, g_1, and g_2 such that the resulting effective SISO plant has desirable properties for stabilization. This is equivalent to providing improved pole-zero alignment. This search can be simplified by letting $g_1 = ag_2$, and $k_2 = bk_1$, which results in

$$\det\left(\mathbf{I}+\mathbf{PG}\right) = 1 + k_1 g_2 h\left(p_{11}a + p_{12} + p_{21}ab + p_{22}b\right) = 1 + p_e g_e, \tag{30}$$

where $g_e = k_1 g_2 h$ and $p_e = p_{11}a + p_{12} + p_{21}ab + p_{22}b$ is now the effective SISO plant. This is precisely the same as equation II-43 in [36]. Once the degrees of freedom a and b are chosen, which characterize the non-singular direction of the controller, the control problem is to design g_e (ie h) such the closed-loop system is stable. Hence, the Singular-G method can be seen as designing a SISO controller, with designed input and output directions, to control the multivariable plant. Evidently, the Singular-G controller has control in only one loop (direction) in the closed-loop system. Therefore, for unstable systems the directions of the singular controller cannot be orthogonal to those of the RHP pole(s). Here we show that in a general sense, the Singular-G method can be considered to be an extreme case of the generalized formulation. Note that the compensator for the Singular-G method was originally presented in the feedback path. However, when considering only the stabilization problem, it can equivalently be considered to be in the feed-forward path.

In the generalized formulation, two identity matrices are inserted before and after the plant as shown in Fig. 5. These are decomposed into $\mathbf{N}$, $\mathbf{N}^{-1}$, $\mathbf{M}$ and $\mathbf{M}^{-1}$, such that the $\mathbf{N}$ and $\mathbf{M}^{-1}$ matrices condition the modified plant $\mathbf{P}_m$ used for design. The controller $\mathbf{G}_m$ is designed based on $\mathbf{P}_m$ and then transformed back to the true controller $\mathbf{G}$ via $\mathbf{N}$ and $\mathbf{M}^{-1}$. The Singular-G method can be put in the same structure where the modified controller $\mathbf{G}_m$ has only one non-zero diagonal element g_e and is therefore singular as shown in Eqn. 32. Notably, the realization of the Singular-G controller as given by Eqns 31 and 32 is not unique, but importantly Eqn. 33 is, and is consistent with [36]. This can be seen by selecting the generalized formulation matrices as follows:

$$\mathbf{M} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}, \mathbf{M}^{-1} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \ \mathbf{N} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{N}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -a \end{bmatrix}. \tag{31}$$

Using the relationships for the generalized formulation with

$$\mathbf{G}_m = \begin{bmatrix} g_e & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{N}^{-1}\mathbf{G}\mathbf{M}, \tag{32}$$

gives $$\mathbf{G} = \mathbf{N}\mathbf{G}_m\mathbf{M}^{-1} = \begin{bmatrix} g_e a & g_e ab \\ g_e & g_e b \end{bmatrix}. \tag{33}$$

The reason that g_e is on the (1,1) entry rather than (2,2) in Eqn. 32 is the choice of the input and output direction pair selected in $\mathbf{M}$ and $\mathbf{N}$ matrices, which are the first row and first column, respectively. The other input and output direction pair is irrelevant as the control is zero. With the above choice of controller and $\mathbf{M}$ and $\mathbf{N}$ matrices, the closed-loop characteristic equation becomes

$$\det\left(\mathbf{I} + \mathbf{P}\mathbf{G}\right) = 1 + g_e\left(p_{11}a + p_{12} + p_{21}ab + p_{22}b\right), \tag{34}$$

which is equal to Eqn. 30. Hence the equivalence is established. The effective SISO modified plant is then given by $p_e = \left(p_{11}a + p_{12} + p_{21}ab + p_{22}b\right)$ where the variables a and b are to be selected such that p_e is MP.

The Singular-G design of [36] is presented in the following. In [36] the ranges of a and b that provide MP zeros for the effective SISO plants p_e, and hence good pole-zero alignment, were calculated using Routh's stability theory under the constraints on scalars a and b. The ranges for MP zeros were found to be:

$$-3 < a < 0 \text{ and } b < -30. \tag{35}$$

Note that, although not discussed in [36], it is clear that this range of values for a and b effectively constrains the alignment of the plant and controller to be within a fixed range. In the design the values were chosen to be $a = -1.5$ and $b = -100$. The resulting SISO modified plants for design are

$$\{p_{e1}, p_{e2}, p_{e3}, p_{e4}\} = \left\{ \begin{array}{l} \dfrac{0.17(s+7.7e-5)(s+6.7e-2)(s+3.2)(s+6.7e2)}{(s-6.0)(s+11.6)(s+3.1e-2\pm 6.8e-2)}, \\ \dfrac{5.7e-2(s+3.8e-3)(s+1.0e-2)(s+1.1)(s+3.1e2)}{(s-2.4)(s+3.9)(s+1.2e-2\pm 0.1)}, \\ \dfrac{0.11(s+7.6e-4)(s+1.4e-2)(s+1.2)(s+3.1e2)}{(s-3.6)(s+5.4)(s+7.0e-e\pm 5.7e-2)}, \\ \dfrac{3.7e-2(s+4.5e-3)(s+1.2e-2)(s+0.37)(s+3.2e2)}{(s-2.5)(s+3.0)(s+1.6e-2\pm 8.5e-2)} \end{array} \right\} \tag{36}$$

Two control system designs were presented in [36] based on this modified family of SISO plants; one a static controller and the other dynamic. The resulting closed-loop properties for both designs were similar and therefore for simplicity only the static controller is presented here, which has $g_e = 710$. The true controller can then be found by mapping this controller using the relationship $\mathbf{G} = \mathbf{N}\mathbf{G}_m\mathbf{M}^{-1}$, which yields

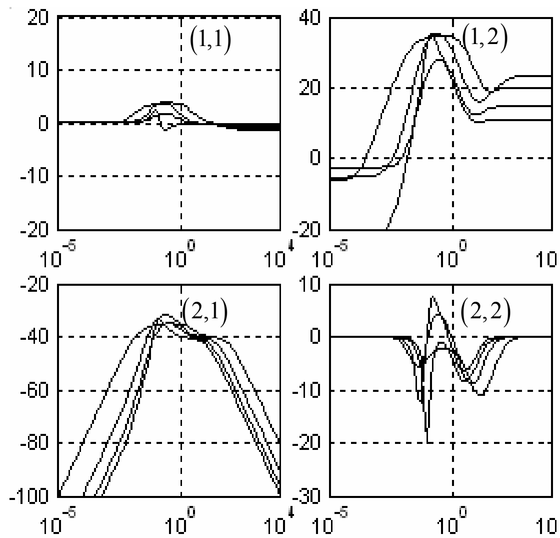

Figure 8: Frequency Response of the TFM **S** for Sing-G Design ([dB] vs [rad/s])

$$\mathbf{G} = 710\begin{bmatrix} 1 & a \\ b & ab \end{bmatrix} = 710\begin{bmatrix} 1 & -1.5 \\ -100 & 150 \end{bmatrix}. \quad (37)$$

This is clearly a singular controller and hence effectively provides for tracking and sensitivity reduction in only one loop (direction) in the closed-loop system. Notably, the singular controller design stabilized the four plant family. However, due to the use of a singular controller, the resulting closed-loop performance levels are very poor, with tracking and sensitivity reduction in only one direction and large levels of coupling in the closed-loop system, as shown in Figs. 8 and 9, which present the frequency response of the closed-loop output sensitivity and complementary sensitivity TFMs. This is in spite of the SISO loop in the modified system comprised of g_m and p_e possessing

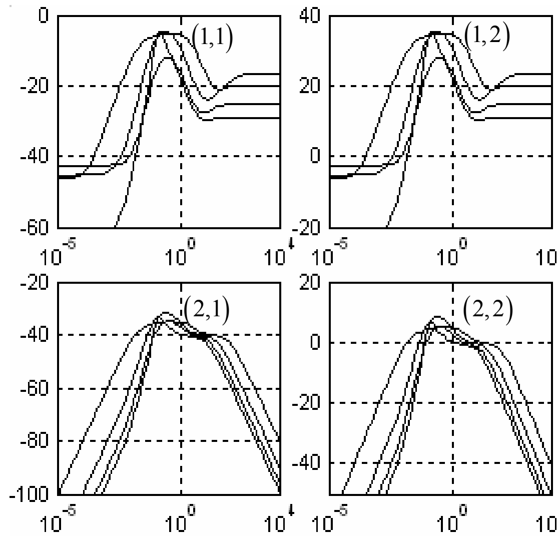

Figure 9: Frequency Response of the TFM **T** for Sing-G Design ([dB] vs [rad/s])

acceptable sensitivity properties. In addition to the controller, one prefilter was designed for each plant case. The frequency response of the resulting single-input two-output system representing the single reference to output responses is shown in Fig. 10, with the system only tracking inputs in one input direction. Using the scheduled prefilter, the tracking response in the one direction was good for all flight conditions, but this hides the underlying problems which are the high level of coupling and unacceptable sensitivity levels.

The reason for the high level of coupling can be seen from the generalized formulation equations that relate the modified and true system closed-loop TFMs in Eqn. 15. With the **M** and **N** matrices not unitary, the TFMs will not have the same singular

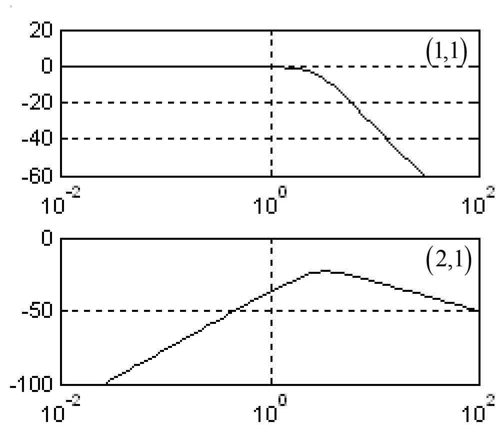

Figure 10: Frequency Response of the (1.1) and (2,1) Prefiltered Output Transfer Functions for Sing-G Design, Plant Case 4 ([dB] vs [rad/s])

values, and therefore the properties of the true system will differ from the modified system and could be much worse. Considering that $\mathbf{M}$ and $\mathbf{N}$ are as given in Eqn. 31, with $a=-1.5$ and $b=-100$, neither matrix is unitary and $\mathbf{M}$ is poorly conditioned. With this $\mathbf{M}$ employed, the (1,2) elements of the output TFMs are,

$s_{o(1,2)} = s_{om(1,2)} + b\left(s_{om(2,2)} - s_{om(1,1)} - s_{om(2,1)}b\right)$, noting that $s_{om(1,2)} = 0$ and at DC $s_{om(2,2)} = 1$,

and $t_{o(1,2)} = t_{om(1,2)} + b\left(t_{om(2,2)} - t_{om(1,1)} - t_{om(2,1)}b\right)$, noting $t_{om(1,2)} = 0$ and at DC $t_{om(2,2)} = 0$.

With the choice of $b=-100$, this necessitates a large $s_{o(1,2)}$ and $t_{o(1,2)}$, and hence a high level of sensitivity and coupling in the tracking response, respectively. This problem can only be alleviated by changing the directional properties of the controller via a change in

a and b, with b desirably reduced. This freedom is exploited in the non-sequential design and sequential design.

3. Non-sequential MIMO QFT Design using Generalized Formulation

The Singular-G design for the X-29 presented in the previous section does robustly stabilize the four plant family. However, due to the limitation of a singular controller, it can only track inputs and reject disturbances in one direction, with the response in the other directions being highly unsatisfactory. Additionally, high levels of cross coupling exist due to the choice of $\mathbf{M}$ and $\mathbf{N}$. To overcome this limitation, in this section a non-sequential MIMO QFT non-singular controller is designed to build on the Singular-G design. The improved non-sequential MIMO QFT design method is employed to permit stabilization of the combined NMP and MP plant family and the generalized formulation is employed to reduce the conservatisms associated with the employment of only sufficient conditions for robust stability.

3.1 Non-sequential Design

To permit the stabilization of all four plant cases, the generalized formulation was employed using both the $\mathbf{M}$ and $\mathbf{N}$ matrices. The properties of the $\mathbf{M}$ and $\mathbf{N}$ matrices are based on those for the Singular-G design, as the first row of the $\mathbf{M}$ matrix and the first column of the $\mathbf{N}$ matrix effectively perform the same function as those in the Singular-G design, being to provide a MP (1,1) element of the modified plant to permit stabilization of the plant family. Considering the necessary conditions for

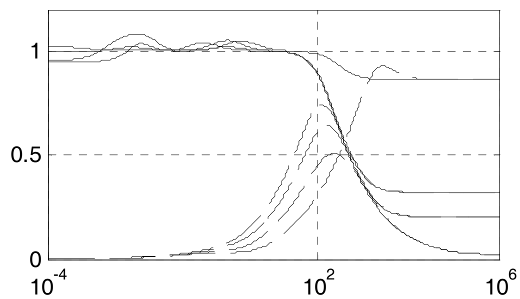

Figure 11: Spectral Radius of $\mathbf{E}$ (solid) and $\mathbf{S}_\Lambda\mathbf{E}$ (dashed) ([dB] vs [rad/s]).

stabilization of the plant using improved non-sequential MIMO QFT, this was the only way seen possible to achieve stability using a stable $\mathbf{G}$ (NS L1-NS L3), and hence the design satisfies similar constraints on the relative plant and controller directions to the Singular-G design. The elements of the first row of the $\mathbf{M}$ matrix and the first column of the $\mathbf{N}$ matrix were therefore fixed such that $3 < n_{11}/\mathrm{n}_{21} < 0$ and $m_{12}/m_{11} > 30$ (equivalently $-3 < a < 0$ and $b < -30$) for good pole-zero alignment, with b desirably small to reduce the effects of coupling, as seen in the Singular-G design. The remaining matrix elements were then manipulated to provide acceptable dominance levels, to satisfy the requirement that $\rho\left(\mathbf{E}_Q\right) < 1$ over some frequency range (NS L3), and desirably over as large a range as possible to reduce the cross-coupling and the associated conservatism. The matrices were therefore chosen to be

$$\mathbf{N} = \begin{bmatrix} -1.5 & 6.9 \\ 1 & -1 \end{bmatrix} \text{ and } \quad \mathbf{M} = \begin{bmatrix} 1 & 40 \\ 0 & 1 \end{bmatrix}. \tag{38}$$

Note that this corresponds to $b = -40$ rather than -100 in the Singular-G design and a the same. The resulting dominance levels are now favorable for all plants except plant case 1, for which $\rho\left(\mathbf{E}_Q\right) < 1$ is only slightly less than 1 at high frequencies, as seen in Fig. 11. This choice of $\mathbf{M}$ and $\mathbf{N}$ also gave $p_{m(2,2)}$ to be MP and hence $q_{m(1,1)}$ to be stable, but this was not necessary as loop 1 will have high gain. The resulting diagonal IPD plant TFMs are:

$$\mathbf{\Lambda}_{m1} = \text{diag}\begin{bmatrix} \dfrac{0.3(s+.059)(s+.003)}{(s+4)(s+0.064)}, \\ \dfrac{7.283\,s(s+0.003)(s+0.059)}{(s+5.8e-4)(s+6.7e-2)(s+1.026)(s+266.962)} \end{bmatrix},$$

$$\mathbf{\Lambda}_{m2} = \text{diag}\begin{bmatrix} \dfrac{0.055(s-0.011)(s+0.019)}{(s+0.014)(s+1.058)}, \\ \dfrac{0.616\,s(s-0.011)(s+0.019)}{(s+0.836)(s+123.656)(s^2+0.016s+1.2e-4)} \end{bmatrix},$$

$$\mathbf{\Lambda}_{m3} = \text{diag}\begin{bmatrix} \dfrac{0.113(s-6.885e-4)(s+0.013)}{(s+0.015)(s+1.311)}, \\ \dfrac{1.214\,s(s-6.885e-4)(s+0.013)}{(s+3.35e-3)(s+0.143)(s+0.717)(s+124.742)} \end{bmatrix},$$

$$\mathbf{\Lambda}_{m4} = \text{diag}\begin{bmatrix} \dfrac{0.033(s-1.17e-3)(s+9.35e-3)}{(s+0.015)(s+0.436)}, \\ \dfrac{0.372\,s(s-1.17e-3)(s+9.35e-3)}{(s+2.28e-3)\,(s+1.857e-2)(s+0.157)\,(s+127.648)} \end{bmatrix}. \tag{39}$$

With all of the IPD SISO plants for design stable, and hence no spurious unstable poles in the IPD SISO plants or zeros in the DPD SISO plants (NS L2), the design for stability is straightforward provided the interaction levels in the modified plant are not too large (NS L3). As seen in Fig. 11, the interaction levels are around one over the lower frequency range and less than one at high frequencies, with plant case 1 being only slightly less than one at 0.86. The latter will be shown to cause a problem in the design associated with NS L3, in that the bandwidth of $g_{m(1,1)}$ is unnecessarily large. Note that the desirable properties of the SISO IPD and DPD plants were only provided for a small range of values in the $\mathbf{M}$ and $\mathbf{N}$ matrices when only scalar entries were employed. If dynamic entries in $\mathbf{M}$ and $\mathbf{N}$ were permitted, improved properties could possibly be achieved.

Table 2: X-29 Plant RHP Pole and Zero Properties for Non-sequential Design

	Plant	**1**	**2**	**3**	**4**
NMP Zero	Location	-	1.08e-2	6.89e-4	1.17e-3
	Input Vector	-	[-0.96, -0.30]	[0.94, 0.33]	[0.96, 0.27]
	Output Vector	-	[0.02, 0.99]	[-0.01, -0.99]	[-0.02, -0.99]
Unstable Pole	Location	6.07	2.37	4.25	2.45
	Input Vector	[0.28, -0.96]	[0.28, -0.96]	[-0.30, 0.95]	[0.28, -0.96]
	Output Vector	[0.99, -0.04]	[-0.99, 0.03]	[0.99, -0.03]	[-0.99, -0.03]

Table 2 shows the directional properties of the modified plant used in the design. Comparing these with the directional properties in the original plant given in Table 1, the effect of the $\mathbf{M}$ and $\mathbf{N}$ matrices on the directions of the RHP poles and zeros is evident

(Note that $\mathbf{M}$ only affects the output directions and $\mathbf{N}$ only the input directions). Compared to the directions in the original plant, it is clear that all the RHP poles and zeros are now almost aligned (pinned) to one input and output, being input 1 and output 2 for the NMP zeros and input 2 and output 1 for the unstable poles. It is these directional properties of the poles and zeros that appear to provide favorable pole-zero alignment and limited the effect of NS L2. However, this favorable conditioning of the IPD and DPD SISO plants was found to be quite sensitive to changes in the directions, which is probably due to the ill-conditioned nature of the plant and/or the different directional properties of plant case 1 relative to the NMP plant cases. It is also not clear whether these directional properties are necessary or sufficient for favorable conditioning in general. It is, however, now evident that the orthogonal nature of the RHP poles and zeros of the plant can be exploited to give favorable properties of the SISO IPD (and DPD) SISO plants, and hence favorable pole-zero alignment.

Considering the DPD and IPD SISO plants, the condition that $z_{\mathbf{P}} + z_{\phi} = z_{\mathbf{\Lambda}}$ be satisfied in the non-sequential MIMO QFT design necessitates that $z_{\phi} = 0$ for plant case 1, and $z_{\phi} = 1$ for the remaining plant cases. A stable controller was realized that provided a robustly stable closed-loop system for the entire four plant family. This was achieved by designing loop 1 to have one unstable zero for plant cases 2, 3 and 4 satisfying $z_{\phi} = 1$. This was easily achieved due to the integrator and NMP zero in $g_{m(1,1)} q_{m(1,1)}$ for these plant cases, when $g_{m(1,1)}$ has an integrator. However, only limited tracking and sensitivity properties were achieved.

To achieve the steady state tracking in one direction, an integrator was added to the design of loop 1 for the modified system, and the interpolation constraint and the NMP performance limitation were both handled in loop 2. While the design permitted the stabilization of all four plant cases, the use of a non-unitary $\mathbf{M}$ in the design makes the achievement of quantitative robust performance (RP) specifications on the original closed-loop system difficult, as seen in the Singular-G design. In general, it is difficult to design for tracking and sensitivity properties on the modified system so as to ensure favorable properties on the true system. In this design only limited tracking and sensitivity properties were obtained. Notably, with loop 2 rolling-off at low frequencies to satisfy the NMP zero constraint, the high value of $\rho\left(\mathbf{E}_Q\right)$ for plant case 1 limited the roll-off for loop 1 to be slow to preserve stability, with a cross-over of at least approximately 1000 rad/s, else $\rho\left(\mathbf{S}_\Lambda \mathbf{E}_Q\right) > 1$ would arise and stability would no longer be assured (NS L3). Attempts to significantly reduce $\rho\left(\mathbf{E}_Q\right)$ for plant 1, while preserving the properties of the equivalent plants, were unsuccessful using static $\mathbf{M}$ and $\mathbf{N}$.

In addition to robust stability, the design called for the following.

Specifications: The RP specifications are to be satisfied for all plants in the modified plant family and $\forall\, i, j \in \{1,2\}$:

RP: $$\left| 1 \big/ 1 + q_{m(i,i)}\left(j\omega\right) g_{m(i,i)}\left(j\omega\right) \right| \le 3\,\text{dB},\ \forall \omega \in [0,\infty), \tag{40}$$

$$\left|\alpha_{ij}\left(j\omega\right)\right| \le \left|t_{m(i,j)}\left(j\omega\right)\right| \le \left|\beta_{ij}\left(j\omega\right)\right|,\ \forall \omega \in [0,30), \tag{41}$$

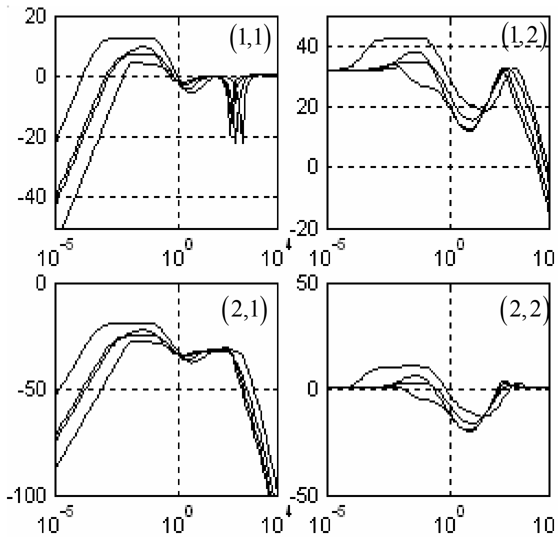

Figure 12: Frequency Response of the TFM $\mathbf{S}_o$ for Non-sequential Design ([dB] vs [rad/s])

with $\beta_{11} = \dfrac{1}{(s/30+1)^2}$, $\alpha_{11} = \dfrac{0.8}{(s/30+1)^3}$, $\alpha_{12} = \alpha_{21} = 0$, $\beta_{12} = \dfrac{0.01(s/90+1)}{(s/30+1)^2}$,

$\beta_{22} = \dfrac{s}{(s/0.01+1)(s/10+1)^3}$, $\beta_{21} = \dfrac{0.01(s/1e-5+1)(s/15+1)(s/30+1)^4}{(s/2e-4+1)(s/10+1)^4(s/100+1)}$ and

$$\alpha_{22} = \frac{0.01s(s/1e-3+1)(s/8+1)}{(s/0.01+1)(s/0.03+1)(s/5+1)(s/10+1)^4}.$$

The following modified controller and prefilter were designed to satisfy the above RS and RP specifications:

$$\mathbf{G}_m = \text{diag}\begin{bmatrix} \dfrac{7.7e5\left(s/0.8+1\right)}{s\left(s/0.0035+1\right)\left(s/4000+1\right)}, \\ \dfrac{1.5e2\left(s/260+1\right)}{\left(s/2.5+1\right)\left(s/1000+1\right)\left(s/4000+1\right)} \end{bmatrix}, \tag{42}$$

$$\mathbf{F}_m = \text{diag}\begin{bmatrix} \dfrac{1}{\left(s/15+1\right)\left(s/60+1\right)}, \\ \dfrac{5e-2\left(s/2.5e-3+1\right)\left(s/0.25+1\right)}{\left(s/0.01+1\right)\left(s/0.015+1\right)\left(s/5.8+1\right)\left(s/13+1\right)} \end{bmatrix}. \tag{43}$$

The resulting controller and prefilter for the true plant are given by $\mathbf{G} = \mathbf{N}\mathbf{G}_m\mathbf{M}^{-1}$ and $\mathbf{F} = \mathbf{M}\mathbf{F}_m$ respectively, and due to the off-diagonal entries in $\mathbf{N}$ and $\mathbf{M}$, both $\mathbf{G}$ and $\mathbf{F}$ will be fully populated TFMs. The condition that $\rho\left(\mathbf{S}_\Lambda\mathbf{E}\right) < 1$ is satisfied in the design, as shown in Fig. 11. The resulting close-loop system was confirmed to be robustly internally stable. However, as expected, the closed-loop performance in the true system was unsatisfactory. The closed-loop sensitivity and complementary sensitivity functions are shown in Figs. 12 and 13. Evidently, the sensitivity and complementary sensitivity functions are again large in the element (1,2). This is because, for the $\mathbf{M}$ employed, $s_{o(1,2)} = s_{om(1,2)} + 40s_{m(1,1)} - 40s_{m(2,2)} - 1600s_{m(2,1)}$ and similarly for $t_{o(1,2)}$. With $s_{om(2,2)} = 1$ at DC due to the differentiator, an integrator employed in $g_{m(1,1)}$ and consequently $t_{om(1,1)} = 1$ at DC, $s_{o(1,2)}$ and $t_{o(1,2)}$ are *necessarily* large. However, the choice of $m_{(1,2)}/m_{(1,1)} = 40$ (ie $b = -40$) did reduce this level of coupling compared to the Singular-G design. Additionally, the need for a high gain in loop 1 to satisfy the $\rho\left(\mathbf{S}_\Lambda\mathbf{E}\right) < 1$ condition, (NS L3), made the first loop highly over designed, with an excessive bandwidth. Hence, while the use of the generalized

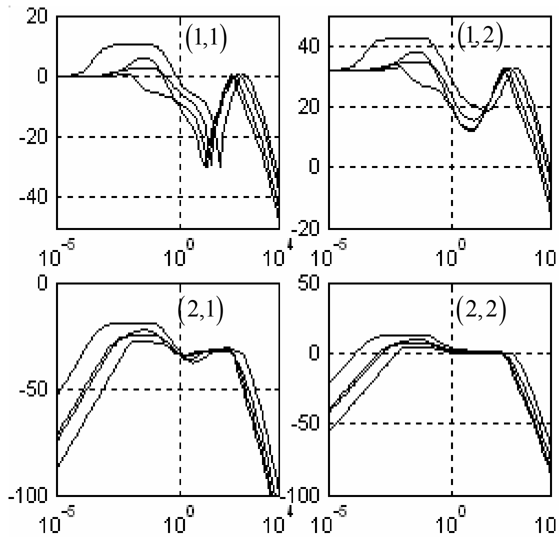

Figure 13: Frequency Response of the TFM $\mathbf{T}_o$ for Non-sequential Design ([dB] vs [rad/s])

formulation simplified the design for stability, the large cross-coupling made NS L3 dominant in the design, with plant case 1 the most difficult to stabilize due to the high value for $\rho(\mathbf{E})$ at high frequencies. The prefilter closed-loop TFMs are shown in Fig. 14, where the mapping from the modified to the true plant domain provided for acceptable coupling levels but the tracking performance in loop 2 was reduced.

3.2 Comparison with Singular-G Design

The non-sequential MIMO QFT and Singular-G designs for the X-29 highlighted important features of classical control when applied to multivariable systems with high

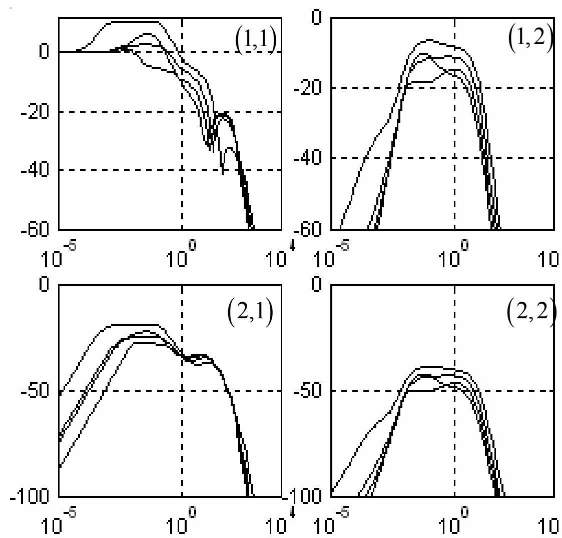

Figure 14: Frequency Response of the TFM $\mathbf{T}_o\mathbf{F}$ for Non-sequential QFT Design ([dB] vs [rad/s])

levels of coupling, combined with NMP and/or unstable plants cases. It is effectively seen that the freedom to adjust the directions of the multivariable controller must be exploited if the design is to be successful (or even stable). This is especially important if independent (dominance) based classical designs, as $z_\Lambda - z_P$ IPD or n DPD SISO plants must be stabilized in an $n \times n$ control system design. Hence, a design framework that presents the directional properties of the control system, and facilitates their trade-off, would appear to be necessary if classically formulated multivariable design methods are to be successful when applied to these difficult classes of systems.

Specifically considering each design, it is evident that the Singular-G design methodology, while providing for a stabilized closed-loop system, will *inherently* possess poor properties, in terms of high levels of coupling, unless the directional properties are well chosen. For NMP and unstable plants, the restriction on the non-singular controller direction for favorable pole-zero alignment may prevent such an alignment, as seen in the current design. The improved non-sequential MIMO QFT design methodology is better suited for the control of these difficult systems, as the use of a non-singular controller provides for additional control authority that can be used to provide for performance and improved sensitivity properties in all directions. When used in conjunction with the generalized formulation, the methodology is better suited to this class of systems, as the alleviation of the conservatism associated with the use of sufficient conditions for stability is strongly dependent on good pole-zero alignment and dominance. However, when the directional properties for acceptable pole-zero alignment conflict with those for good dominance (coupling) levels, the design will be *inherently poor*, as seen in the present design. Effectively managing this trade-off between closed-loop coupling and favorable pole-zero alignment is seen to be paramount in a successful design.

The application of the improved non-sequential MIMO QFT did however validate the properties of this recent improvement in the non-sequential methodology, facilitating its application to a NMP control problem. The employment of the generalized formulation was the key to the successful application of the non-sequential design methodology to the X-29. It should be stated that, while the non-sequential MIMO QFT design presented was carefully executed, no tuning of the design was performed because the objective was simply to confirm and highlight the properties of a non-sequential design for difficult problems, such as the X-29. An improved design could definitely be achieved by tuning the specifications in the modified plant domain. However, the *fundamental* constraint on large cross-coupling at low frequencies will remain, and dominate the design, unless dynamic $\mathbf{M}$ and $\mathbf{N}$ are employed, which is the case in the following sequential design.

Overall, the design problem highlights an intuitive trade-off between controller complexity and performance in a more general sense. It is evident that employing a robust controller for a large operating envelope will restrict the set of simultaneously stabilizing controllers, which in some cases will exclude the set of controllers that provide for acceptable performance. In the current setting, this can be seen as a trade-off between a simplified simultaneously stabilizing robust control system and achieving good pole-zero alignment and dominance levels in the closed-loop system. This effectively highlights a limitation of diagonal (decentralized) control with its directional properties fixed.

4. Sequential MIMO QFT Design

The effectiveness of the proposed design procedure from Chapter IV and the consideration of alignments are demonstrated here. The non-sequential design from section 3 exploited the concept of pole-zero alignment to stabilize the uncertain plant family and attempted to provide good dominance properties with implicit good plant-controller alignment, although the plant/controller alignment theory was not employed. In section 3, the design was limited to static $\mathbf{M}$ and $\mathbf{N}$ matrices for simplicity while it is improved herein by explicitly considering both pole-zero and plant/controller alignment and the permitting of **M** and **N** matrices to be dynamic.

The sensitivity properties of the non-sequential MIMO QFT design are shown in Fig. 12. As discussed in the previous section, the closed-loop system possesses strong coupling that arises from a conflict between good pole-zero alignment and acceptable dominance levels for the design. Notably, at DC the loop transmission is necessarily ill-conditioned due to the presence of a differentiator and the need to satisfy the bandwidth constraint from NMP zero. At higher frequencies ill-conditioning arises from the need to roll-off loop two before loop one in the non-sequential MIMO QFT design to satisfy the sufficient conditions for closed-loop stability. A less conservative sequential MIMO QFT design has been attempted for the non-sequential design from section 3 with the same static **M** and **N** matrices, but little design improvement was attained, with similar coupling levels at DC and low frequencies. This appeared to indicate that the coupling levels were inherent in the design problem with the choice of **M** and **N** matrices and it was confirmed by applying the plant/controller alignment theory.

Based on this observation, two approaches can be employed to improve the design. First, dynamic **M** and **N** matrices can be used to reduce the stringency of the trade-off between alignments and improve the plant/controller alignment. This particularly helps at low frequencies. Second, the design can be deliberately performed so as to invalidate the plant/controller alignment theory, such that the associated limitations do not apply. This is helpful at the higher frequency range where the theory applies due to the difference in loop bandwidths and removed the second peak in the sensitivity response at high frequencies.

4.1 Design Initialization

Specifications on the true closed-loop were given. It was assumed that there are no structural constraints on $\mathbf{G}$, so multivariable $\mathbf{M}$ and $\mathbf{N}$ can be employed. For simplicity $\mathbf{M}$ and $\mathbf{N}$ were designed to possess static elements except for $\mathbf{M}_{(1,2)}$, which was identified to be important to the low frequency plant/controller alignment. Using Routh Hurwitz theory, with $\mathbf{M}_{(1,2)}$ a lead-lag element, parameter ranges for elements of $\mathbf{M}$ and $\mathbf{N}$ for good pole-zero alignment were determined, so that the DPD and IPD SISO plants were unstable and MP, and stable and NMP, respectively. Hence no spurious RHP poles or zeros shows up in the SISO plants. This gave a set of values for the gains in $\mathbf{M}$ and $\mathbf{N}$ and the pole and zero in $\mathbf{M}_{(1,2)}$. From this set the $\mathbf{M}$ and $\mathbf{N}$ matrices were then chosen so as to give the most favorable plant/controller alignment, giving:

$$\mathbf{M} = \begin{bmatrix} 1 & \left(\dfrac{\dfrac{s}{1e-7}+1}{\dfrac{s}{1e-5}+1} \right) \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{N} = \begin{bmatrix} -1.5 & 6.9 \\ 1 & -1 \end{bmatrix}. \tag{44}$$

4.2 Direct Controller Synthesis

Based on the choice of $\mathbf{M}$ and $\mathbf{N}$, the diagonal controller $\mathbf{G}_m$ in the modified plant domain was designed using the sequential MIMO QFT design method [3, 22], to directly satisfy the performance specifications on the true closed-loop system. An integrator was employed in g_{m11} and an attempt was made to roll-off the loops at a reasonable frequency range. The designed modified plant domain controller is given in Eqn. 45. The true controller $\mathbf{G}$ can then be obtained and is multivariable (fully populated).

$$\mathbf{G}_m = diag\left[\frac{1.5e4\left(s/95+1\right)\left(s/610+1\right)\left(s/840+1\right)}{s\left(s/26+1\right)\left(s/250+1\right)\left(s/1300+1\right)\left(s/1800+1\right)\left(s/3400+1\right)}, \frac{8e4}{\left(s/0.4+1\right)\left(s/20+1\right)}\right] \tag{45}$$

$$\mathbf{G}_m = diag\left[\frac{7.8e3\left(s/1750+1\right)}{s\left(s/910+1\right)\left(s/6800+1\right)\left(s/1.98e4+1\right)}, \frac{2.27e4\left(s/2.8e-3+1\right)\left(s/1.8e-2+1\right)\left(s/11+1\right)}{\left(s/1.1e-3+1\right)\left(s/5.7e-3+1\right)\left(s/0.5+1\right)\left(s/20+1\right)\left(s/62+1\right)}\right] \tag{46}$$

4.3 Direct Re-design

The design provided significant improvement in the coupling levels at low frequencies. To further improve the plant/controller alignment the DC gain of the $\mathbf{M}_{(1;2)}$ element was modified to be 0.3 and a further attempt to reduce the bandwidths was made. The resulting controller is given in Eqn. 46. The improvement from the use of dynamic $\mathbf{M}$ and the improved alignment is evident from the sensitivity frequency

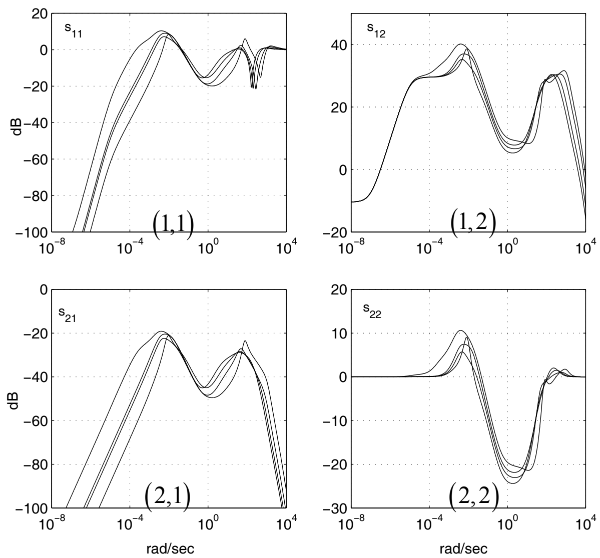

Figure 15: Magnitude Bode Plots of the Output Sensitivity TFMs for the Sequential Design.

response in Fig. 15. The improvement in plant/controller alignment at low frequencies, while providing good pole-zero alignment, has reduced the low frequency sensitivity level in $\mathbf{S}_{12}$ from approximately 40dB to -10dB. This is a significant improvement and is consistent with the improvement in the plant/controller alignment over this frequency range. Fig. 16 shows the alignment levels for the present choice of $\mathbf{M}$ and $\mathbf{N}$, the only difference from the previous choice of $\mathbf{M}$ and $\mathbf{N}$ in section 3 being $\mathbf{M}_{(1;2)} = 40$ in that design.

Further improvement could not be achieved in the design over the low frequency range where $\mathbf{S}_{12} \cong 35dB$. Higher order $\mathbf{M}_{(1;2)}$ may have helped here, along with other dynamic elements in $\mathbf{M}$ and $\mathbf{N}$. As previously noted, over the higher frequency range

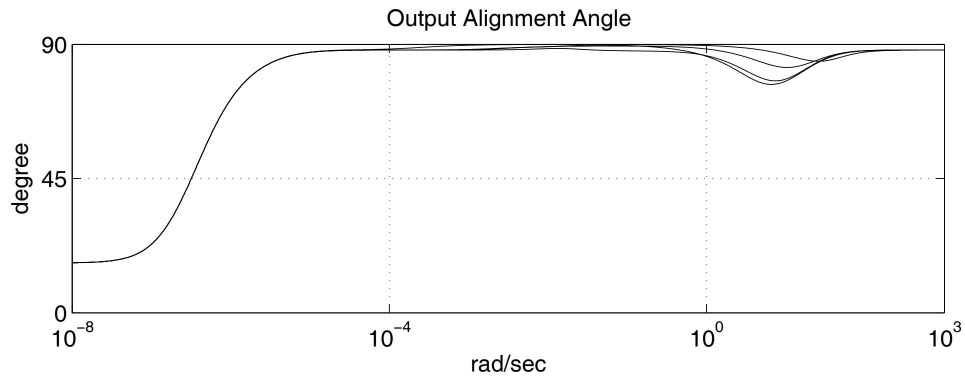

Figure 16: Plant/Controller Alignment Angle.

where $\mathbf{S}_{12} \cong 30dB$, the design could also be improved by invalidating the alignment theory via a lower bandwidth loop one. Evidently, while the closed-loop system still possesses poor sensitivity levels over some frequency ranges, the exploitation of the directional information within the generalized formulation has provided an effective and transparent method to improve the design.

5. Chapter Summary

This chapter presented a design example, X-29, in which multiple designs were performed and compared. The theory for plant/controller alignment and pole-zero alignment issues were employed in the design procedure to facilitate more effective MIMO QFT designs for ill-conditioned, NMP and unstable systems. The results of this chapter provide insights into decentralized control and associated performance limitations. It is evident that for difficult systems, such as the X-29, decentralized control will be inadequate due to the limited directional freedom in the design of the controller. For such systems the explicit exploitation of the directional design features is seen to be critical to the success of the design. The generalized formulation provided an important handle on MIMO QFT with respect to this latter.

CHAPTER VI

SUMMARY AND CONCLUSIONS

1. Summary

In this dissertation, the properties of multivariable QFT design methodologies, when applied to systems with undesirable RHP poles and zeros and/or large coupling, have been elucidated. A classically formulated directional design procedure using MIMO QFT and the generalized formulation was presented in the dissertation. The theory for plant/controller alignment was partially extended to TITO systems and the concept of pole-zero alignment proposed. These alignment issues were employed in the design procedure to facilitate more effective classically formulated MIMO QFT designs for ill-conditioned, NMP and unstable systems. In the new design method, the generalized formulation is used to condition the equivalent SISO plants to attain a desired pole-zero structure, favourable plant/controller alignment, and dominance. The design procedure was demonstrated on the X-29 flight control problem for its utility. The results provided insights into decentralized control and associated performance limitations.

The X-29 provided a challenging control problem through which limitations of control design methods were clearly exposed. It is evident that for difficult systems, such as the X-29, decentralized control will be inadequate due to the limited directional freedom in the design of the controller. For such systems, the explicit exploitation of the directional design features is seen to be critical to the success of the design and the generalized formulation provided such a feature for MIMO QFT.

Control design for the X-29 highlighted the utility of the proposed generalized formulation for MIMO control system design. The formulation facilitates both diagonal and non-diagonal controller design with improved transparency that comes from the presentation of directional information within the design framework. Hence, it effectively provides a directional MIMO QFT design procedure. Horowitz's Singular-G method was also revisited and its properties and limitations elucidated by placing the design approach within the proposed formulation. An analysis of the Singular-G X-29 design showed that, while the design stabilized the plant family, the directional properties of the controller inherently gave poor performance. The proposed formulation also alleviated some of the conservatisms associated with non-sequential and sequential MIMO QFT, facilitating acceptable pole-zero alignment and improved performance relative to the Singular-G design.

The results also highlight the salient role and importance of directions in classically formulated MIMO QFT control designs. The employment of sufficient stability conditions in the classical design techniques necessitates a trade-off between good pole-zero alignment, plant/controller alignment, and favourable dominance (coupling) levels in the closed-loop system. The explicit consideration of directions using generalized formulation is thus seen to be necessary to achieve this trade-off in more difficult control system designs.

The application of new design methodology to the X-29 control problem validated the developments in the design methodology. The methodology was shown to be applicable to control problems consisting of a family of mixed MP and NMP unstable systems, albeit with conservatism arising from the design methodology's specific limitations and those inherent in the design problem.

2. Contributions

The contributions of this research are the following:

a. The unnecessary design difficulty and the challenge of applying MIMO QFT to NMP and/or unstable MIMO systems, which have unstable poles and NMP zeros in undesirable locations, was investigated and addressed.

b. A new MIMO QFT design methodology was developed using a generalized formulation, which alleviates limitations and conservatisms of standard MIMO QFT. The formulation provides additional degrees of freedom in the decentralized MIMO QFT feedback structure, which facilitates the exploitation of directions in MIMO QFT designs and fully-populated controller.

c. A concept of pole-zero alignment was proposed and the plant/controller alignment was expanded to TITO systems. The fundamental trade-off between multivariable controller directions for stability and performance was then explicitly presented with plant/controller alignment and pole-zero alignment, which provide directional design logic in the developed design procedure.

d. A multivariable controller was successfully synthesized for the X-29 problem using the developed design method. The Singular-G design was then compared with this MIMO QFT design. Singular-G design methodology were analysed and placed in the context of the generalized formulation. This also demonstrated the utility of the generalized formulation.

3. Future Research

The original contributions of the dissertation were described in section 2. In achieving these, some directions for further research have been identified. These are summarized below.

a. A large order MIMO system case study should be considered in the future. This will introduce additional problems associated with scalability of MIMO QFT design method and expose required contribution in this area. In fact, the poor scalability of MIMO QFT is exacerbated with the generalized formulation.

b. The design procedure herein is highly complex and time consuming, even for low order systems, due to the employment of the generalized formulation. The development of an automated MIMO QFT design procedure would potentially reduce the design time taken to execute a design.

c. The developed method is intended for a square system. Thus, an extension of the generalized formulation to non-square plant TFM would be desirable.

d. The prior condition for $\mathbf{M}$ and $\mathbf{N}$ matrices is only chosen for the nominal plant parameter case. Although a sufficient condition for uncertain plants can be derived from Lemma 2 using the simultaneous stability theory from [52], the question of whether there exists a pair of suitable $\mathbf{M}$ and $\mathbf{N}$ matrices conditioning all plants in the family simultaneously remains an open question worth studying.

e. The proposed concept of pole-zero alignment remains as a qualitative description. The fundamental relationships among the NMP zero direction, the unstable pole direction, and desired SISO equivalent plants are still not clear and require further investigation in the future.

f. The plant/controller alignment is only extended to TITO systems. For general MIMO systems, it would be very useful to have guidelines for the selection of the $\mathbf{M}$ and $\mathbf{N}$ matrices based on the directionality of plant and controller to produce designs with better performance.

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APPENDIX A

EXTENSION OF SITO TO TITO SYSTEMS

This section presents an extension of the SITO theory to TITO systems, with the extension of the TISO theory following analogously. The design equations are presented in the true plant domain. To facilitate the analysis of TITO systems using the SITO theory, the TITO plant and controller TFMs are decomposed into their rows and columns as follows:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} P_{\cdot 1} & P_{\cdot 2} \end{bmatrix} = \begin{bmatrix} P_{1\cdot} \\ P_{2\cdot} \end{bmatrix}, \qquad (47)$$

and similarly for $\mathbf{G}$. The associated TITO loop transmissions at the plant output and input are

$$\mathbf{L}_o = \begin{bmatrix} P_{1\cdot}G_{\cdot 1} & P_{1\cdot}G_{\cdot 2} \\ P_{2\cdot}G_{\cdot 1} & P_{2\cdot}G_{\cdot 2} \end{bmatrix} = \begin{bmatrix} L_{o\cdot 1} & L_{o\cdot 2} \end{bmatrix} = \begin{bmatrix} L_{o1\cdot} \\ L_{o2\cdot} \end{bmatrix}, \qquad (48)$$

and similarly for $\mathbf{L}_i$. Let $\det(\mathbf{L}) \equiv \det(\mathbf{L}_i) = \det(\mathbf{L}_o)$. The standard relationship between the sensitivity and complementary sensitivity TFMs apply such that $\mathbf{S}_o + \mathbf{T}_o = \mathbf{I}$ and $\mathbf{S}_i + \mathbf{T}_i = \mathbf{I}$. The following additional relationships hold for the TITO system:

$$G_{1\cdot}P_{\cdot 1} + G_{2\cdot}P_{\cdot 2} = P_{1\cdot}G_{\cdot 1} + P_{2\cdot}G_{\cdot 2}, \qquad (49)$$

$$\mathbf{L}_o = P_{\cdot 1}G_{1\cdot} + P_{\cdot 2}G_{2\cdot}, \qquad (50)$$

$$\mathbf{L}_i = G_{\cdot 1}P_{1\cdot} + G_{\cdot 2}P_{2\cdot}. \qquad (51)$$

Based on the above relationships, one can derive:

$$\mathbf{T}_o = \mathbf{L}_o \left(\frac{1}{1 + P_{1.}G_{.1} + P_{2.}G_{.2} + \det(\mathbf{L})} \right) - \mathbf{I} \left(\frac{\det(\mathbf{L})}{1 + P_{1.}G_{.1} + P_{2.}G_{.2} + \det(\mathbf{L})} \right), \tag{52}$$

and similarly for $\mathbf{S}_o$. To develop the relationship between the input and output TFMs of the closed-loop systems, two assumptions are made. The validity of these assumptions is discussed at the end of this section. Assumption 1 is that $\det(\mathbf{L})$ can be neglected in Eqn. 52. Applying this assumption, along with Eqns. 49 and 50 to Eqn. 52 gives:

$$\mathbf{T}_o = \frac{P_{.1}G_{1.} + P_{.2}G_{2.}}{1 + G_{1.}P_{.1} + G_{2.}P_{.2}}, \tag{53}$$

and similarly for $\mathbf{S}_o$. The second assumption is that (a) $P_{.2}G_{2.}$ and (b) $G_{2.}P_{.2}$ can be neglected in Eqn. 53. Applying this assumption gives:

$$\mathbf{T}_o = \frac{P_{.1}G_{1.}}{1 + G_{1.}P_{.1}}, \tag{54}$$

and

$$\mathbf{S}_o = \mathbf{I} - \frac{P_{.1}G_{1.}}{1 + G_{1.}P_{.1}}. \tag{55}$$

Notably Eqn. 55 is in the same form as Eqn. 3 in [55]. Hence these equations can be employed to develop relationships between the input and output TFMs of the closed-loop TITO system. The derivation follows analogously to [55], where here the projections are onto the range and null space of $P_{.1}$ and the relationship is to the $(1,1)$ element of he input TFMs. The derivation is only shown for the output complementary TFM. From Eqns. 48 and 54,

$$\mathbf{T}_o = \frac{\left\|G_{1\cdot}\right\|\left\|P_{\cdot1}\right\|\left(\vec{G}_{1\cdot}\vec{P}_{\cdot1}\right)\left(\vec{G}_{1\cdot}\vec{P}_{\cdot1}\right)^{-1}\left(\vec{P}_{\cdot1}\vec{G}_{1\cdot}\right)}{1+\left\|G_{1\cdot}\right\|\left\|P_{\cdot1}\right\|\left(\vec{G}_{1\cdot}\vec{P}_{\cdot1}\right)} = \mathbf{T}_{i(1,1)}\frac{\vec{P}_{\cdot1}\vec{G}_{1\cdot}}{\vec{G}_{1\cdot}\vec{P}_{\cdot1}}. \quad (56)$$

Noting that $\vec{G}_{1\cdot}\vec{P}_{\cdot1} = \cos(\phi)$, $\bar{\sigma}\left(\vec{P}_{\cdot1}\vec{G}_{1\cdot}\right) = 1$, and $\bar{\sigma}(\mathbf{T}_o) = \bar{\sigma}(\mathbf{T}_{om})$, gives Eqn. 27.

Assumption Validity:

Assumption 1 will generally be true over the frequency ranges where the system is strongly ill-conditioned. The validity of this assumption can be checked by considering the relative magnitudes of the terms in Eqn. 52. Assumption 2 (a) holds whenever $\left\|P_{\cdot1}\right\|\left\|G_{1\cdot}\right\| >> \left\|P_{\cdot2}\right\|\left\|G_{2\cdot}\right\|$. Assumption 2 (b) holds whenever the magnitude and/or projections are favorable. Clearly Assumption 2 need not apply, even in a strongly ill-conditioned system. However, if the system is strongly ill-conditioned, the generalized formulation can be employed to rotate the output directions of the singular values of the system such that they are basis vectors, and in doing so improve the validity of the assumption. This is the utility of the generalized formulation here. The simplest case is when G and P is strongly ill-conditioned. In this case a unitary transformation can be performed to make the lower singular value controller output (plant input) direction that of the row of G (column of P). Subsequently, $G_{2\cdot}$ ($P_{\cdot2}$) is negligible, and the analysis holds. Both assumptions, and hence the validity of the plant/controller alignment relationships, can be considered prior to design based on the knowledge of interpolation constraints, performance specifications, likely directional performance limitations arising NMP zeros or unstable poles, and information on actuator and sensor redundancies. The validity of the relationships can easily be assessed a posteriori, using the generalized formulation, and if desired the effect of approximation errors can be included in the calculation of the relationships.

Remark 9: The synthesis of orthonormal $\mathbf{M}$ and $\mathbf{N}$ such that the modified system for analysis best satisfies assumption 2 has not been resolved in general. However, if the ill-conditioning arises specifically from either gain or redundancy the choice is simpler. For the design problem in section 4 Chapter V, $\mathbf{M}$ and $\mathbf{N}$ were designed to minimize $P_{2.}G_{.2}$ and $G_{2.}P_{.2}$, respectively. This can and was done for each frequency and plant case individually.

APPENDIX B

DIRECT DESIGN EQUATIONS FOR SEQUENTIAL MIMO QFT

In this section, the TITO output sensitivity TFM is presented as a function of the modified plant domain elements, with $\mathbf{M}$ free and $\mathbf{N} = \mathbf{I}$ for simplicity. Analogous equations hold for the input side, and similar relationships hold when both $\mathbf{M}$ and $\mathbf{N}$ are employed.

$$\mathbf{S}_o = \mathbf{M}\left(I + \mathbf{P}_m \mathbf{G}_m\right)^{-1} \mathbf{M}^{-1}$$

$$= \frac{1}{\alpha}\begin{bmatrix} \det\left(\Pi\right) + \pi_{11} g_{m22} \hat{m}_{22} - \pi_{21} g_{m11} \hat{m}_{12} & \pi_{12} g_{m22} \hat{m}_{22} - \pi_{22} g_{m11} \hat{m}_{12} \\ \pi_{21} g_{m11} \hat{m}_{11} - \pi_{11} + g_{m22} \hat{m}_{21} & \det\left(\Pi\right) + \pi_{22} g_{m11} \hat{m}_{11} - \pi_{12} + g_{m22} \hat{m}_{21} \end{bmatrix}. \qquad (57)$$

With the controller element g_{m11} isolated in the denominator, this gives Eqn. 57, where

$\Pi = \left[\pi_{(i,j)}\right] = \mathbf{P}^{-1}$, $M^{-1} = \left[\hat{m}_{(i,j)}\right]$,

and

$\alpha = \left(\det\left(\Pi\right) + g_{m22}\left(\hat{m}_{22}\pi_{11} - \hat{m}_{21}\pi_{12}\right)\right) + g_{m11}\left(g_{m22}\det\left(\mathbf{M}^{-1}\right) + \hat{m}_{11}\pi_{22} - \hat{m}_{12}\pi_{21}\right)$. The design problem is then to directly synthesize g_{m11} for desirable properties in the true plant domain.

VITA

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