JOINT PRODUCTION AND ECONOMIC RETENTION QUANTITY DECISIONS IN CAPACITATED PRODUCTION SYSTEMS SERVING MULTIPLE MARKET SEGMENTS

A Thesis

by

ABHILASHA KATARIYA

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2008

Major Subject: Industrial Engineering

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Approved by:

Chair of Committee,	Eylem Tekin
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ABSTRACT

Joint Production and Economic Retention Quantity
Decisions in Capacitated Production Systems Serving
Multiple Market Segments. (August 2008)
Abhilasha Katariya, B.E., Pune University
Chair of Advisory Committee: Dr Eylem Tekin

In this research, we consider production/inventory management decisions of a firm that sells its product in two market segments during a finite planning horizon. In the beginning of each period, the firm makes a decision on how much to produce based on the production capacity and the current on-hand inventory available. After the production is made at the beginning of the period, the firm first satisfies the stochastic demand from customers in its primary market. Any primary market demand that cannot be satisfied is lost. After satisfying the demand from the primary market, if there is still inventory on hand, all or part of the remaining products can be sold in a secondary market with ample demand at a lower price. Hence, the second decision that the firm makes in each period is how much to sell in the secondary market, or equivalently, how much inventory to carry to the next period.

The objective is to maximize the expected net revenue during a finite planning horizon by determining the optimal production quantity in each period, and the optimal inventory amount to carry to the next period after the sales in primary and secondary markets. We term the optimal inventory amount to be carried to the next period as "economic retention quantity". We model this problem as a finite horizon stochastic dynamic program. Our focus is to characterize the structure of the optimal policy and to analyze the system under different parameter settings. Conditioning on given parameter set, we establish lower and upper bounds on the optimal policy parameters. Furthermore, we provide computational tools to determine the optimal policy parameters. Results of the numerical analysis are used to provide further insights into the problem from a managerial perspective.

To my Parents

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

TABLE Page

CHAPTER I

INTRODUCTION

The objective of the present research work is to investigate sequential decision making models for firms, which sell their products in multiple markets and periodically determine optimal production quantities and desirable inventory levels to maximize expected net revenues.

In this research, we consider production/inventory management decisions of a firm that sells its product in two market segments during a finite planning horizon. In the beginning of each period, the firm makes a decision on how much to produce based on the production capacity available for that period and the current inventory available on hand. After the production is made at the beginning of the period, the firm first satisfies the stochastic demand from customers in its primary market. Any primary market demand that cannot be satisfied is lost. After satisfying the demand from the primary market, if there is still inventory on-hand, all or part of the remaining products can be sold in a secondary market with ample demand at a lower price. Hence, the second decision that the firm makes in each period is how much to sell in the secondary market, or equivalently, how much inventory to carry to the next period if there is positive inventory after satisfying the primary market demand.

The objective is to maximize the expected net revenue earned during a finite planning horizon by determining the optimal production quantity in each period, and the optimal inventory amount to carry to the next period after the sales in primary and secondary markets. We term the optimal inventory amount to be carried to the next

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The journal model is *IEEE Transactions on Automatic Control.*

period as "economic retention quantity". The economic retention quantity represents the maximum level above which the marginal expected value of the product in future periods is less than its current value in the secondary market. Even if selling in the secondary market is profitable, after satisfying the demand in the primary market, it may be more profitable not to sell all of the on-hand inventory in the secondary market. Instead, if one reserves a inventory equal to the economic retention quantity for future periods, in anticipation of selling it in the primary market, higher profits could be earned. At economic retention quantity the cost of carrying a unit to the next period is equal to the profit loss incurred by selling at a cheaper rate in the secondary market. Any decision made in the current period affects the expected profits in the subsequent periods. Therefore, we model this problem as a finite horizon stochastic dynamic program. All of the system parameters such as the cost structure, the available capacity in each period and the demand distribution contribute to the complex structure of the optimal policy. We characterize the structure of the optimal policy and determine the bounds for computing the optimal policy parameters. In addition, these bounds also provide managerial insights about the optimal policy parameters. Furthermore, we present a numerical study that computes the optimal policy parameters, and analyze the system under different parameter settings.

More specifically, we focus on the following problem: We consider a finite planning horizon of T periods where each period has a finite production capacity of C_t units. In the beginning of each period, the optimal production quantity is determined based on the maximum production capacity available, and a production cost of p is incurred for each unit produced. For modeling purposes, we assume that the production is instantaneous. This is valid for systems where production time is short with respect to the length of the period. After the production is completed, demand from the primary market is satisfied. The revenue from selling one unit in the primary market is r_1. Any demand that cannot be satisfied from the primary market is lost and results in a lost sales penalty of b per unit. After satisfying the demand from the primary market, if the remaining inventory is in excess of the economic retention quantity, then the excess units are sold in the secondary market at a unit price of $r_2 \leq r_1$. As a result, economic retention quantity is carried to the next period. On the other hand, after satisfying the demand from the primary market, if the remaining inventory does not exceed the economic retention quantity, all remaining inventory is carried to the next period. We consider a holding cost h for each unit carried to the next period.

The assumption that the production time is short with respect to the length of the period is true for assembly operations, for which the production lead times are short. For example, consider a final product that is assembled from two or more components. The assembly operation receives components from various machining centers and sub-assembly operations which stage inventory of ready-to-assemble components. These supply operations generally have long lead times. The maximum number of assemblies that can me manufactured is limited by the capacity of the assembly line. The inventory of ready-to-assemble components puts an additional constraint on the number of assemblies that can be produced in a given period. Thus, in the beginning of each period, the firm determines how many assemblies to produce based on the available capacity for that period and the current inventory available on hand.

The motivation for this research is based on the practices in electronics industry. Many consumer electronics companies have restructured their manufacturing operations (servers, computers, data storage system, digital cameras etc.) into two stages. In the first stage, components with long manufacturing lead times, such as printed circuit boards, are built in a make-to-stock fashion. In the second stage, these components are assembled into the final product, in response to production decisions made in accordance to the demand forecast or in a make-to-order fashion. The short assembly lead time helps the firm to use a better forecast for these decisions. The demand for these consumer products arises from corporate customers as well as individual buyers. The corporate customers are given higher priority, in accordance with contracts with the manufacturing firm, and they are willing to pay more for priority over other customers. In many cases, there is a penalty of lost sales if all the demand from the high priority customers is not satisfied in any given period. In contrast, there is no contract between the individual buyers and the manufacturing firm. These customers are considered secondary, and they may not be willing to pay a price as high as the corporate customers. In addition, as new developments in technology make better products available, consumer electronic products become outdated soon. At times, this may lead the firms to sell at a lower unit price in the secondary market. For such systems, jointly managed production and retention policies can result in higher profits. We show through the results of a numerical study that for such firms the proposed policy is more profitable than the policies where either no units are sold in the secondary market or all units are sold in the secondary market with no units carried to the next period.

The remainder of the thesis is as follows: In Chapter II, we provide a brief literature review. In Chapter III, we present the mathematical model and characterize the single-period problem. In Chapter IV, we discuss the structure of the optimal policy for multi-period problems. We also establish lower and upper bounds for the optimal policy parameters. Chapter V discusses the results of a numerical study and presents the sensitivity of the optimal policy parameters to the cost parameters. In Chapter VI, we offer a brief summary and conclude with possible extensions to this research.

CHAPTER II

LITERATURE REVIEW

This research is closely related to two streams of literature on production/inventory management models. The first one is the literature on inventory management models that consider disposal of excess stock. The second stream is the literature on the periodic review production/inventory systems with priority and price differentiated customer classes, and with production capacity constraints. Below we summarize the related papers from both streams of research as follows:

Most research in the area of disposal of excess stock focuses on systems with constant and known demand rate, and either aims to determine the optimal disposal quantity or the optimal economic retention quantity (i.e., the maximum number of units that should be retained at the end of a sales period), after satisfying the customer demand in current period. Silver and Willoughby [8] considered such problem faced in the construction industry where there is excess stock at the end of a construction phase. The stock is to be used in the on-going projects and there is also an opportunity to dispose off some units. They considered a problem such that the unit disposal or salvage value is a function of the number of units disposed. They developed a systematic scheme to aid in decision making for cases where the demand rate is known and constant and there is no item obsolescence. In their scheme, they first computed the present value of salvage revenue, inventory carrying costs and costs for replenishments made in the future. Showing that the present value of sum of these costs is a convex function of the number of units disposed, they developed a procedure to evaluate the optimal disposal quantity that minimizes the total cost. Simpson [9] also considered such a multi-period problem with known and constant demand rate. He proposed a formula to determine the economic retention quantity, in case of a finite probability of deterioration of the product under consideration. The formula takes into account the trade-off between: (a) the cost of storing the product, considering the probability that it may become obsolete and worthless, and (b) the cost of purchasing it again in future when and if needed, if the present surplus inventory is disposed. The approach was based on determining a break-even point of time (i.e., economic retention period) so that the inventory that is sufficient to meet the expected needs up to that time should be retained, and any excess over that amount should be disposed.

Tersine and Toelle [12] considered the problem of determination of economic retention quantity for a known and constant demand rate, where stock-outs are not permitted. They showed that the total net benefit in any given period is a concave parabolic function of the time for which the retained inventory lasts. They provided derivations and closed form solutions to compute this economic time supply, which is the same as the economic retention period considered by Simpson [9]. In addition, they also provided a detailed analysis summarized in the form of a flow chart to calculate the optimal amount of inventory to be retained. As pointed out by Simpson [9], the assumptions made by all the above papers such as the known and constant demand rate and constant cost parameters over the planning horizon are too restrictive and unrealistic for practical application.

Fukuda [5] considered a multi-period stochastic demand problem for a single product where the decision to be made in the beginning of each period is one of the following: ordering, disposal, or do nothing. Under backlogging assumption, he formulated the optimal ordering and disposal policy taking into account the cost of ordering, disposal, holding and shortage and provided a detailed analysis of the two period and three period problems. He showed that the for any given period the total expected cost until the end of the horizon is a convex function of the on-hand inventory available in the beginning of a period, and hence, there is a finite optimal inventory level to start each period that minimizes the total expected costs. He assumed no restrictions on the amount that can be ordered in any period.

Hart [6] considered a multi-period, stochastic demand problem for items having low deterioration and obsolescence rates for a case with lost sales. Along with the variable costs, he also considered a fixed setup cost for every procurement made. He provided a heuristic procedure to determine a procurement schedule, i.e., the number of units to be procured each period. He showed that the total expected cost is a unimodal function of the quantity retained at the end of every period and provided a sequential search procedure based on Golden Section method to find the retention quantity which gives minimum expected cost. Via a computational study, he conducted a sensitivity analysis and showed that the retention decision is reasonably robust. Rosenfield [7] considered the disposal problem for slow moving inventory. He assumed a renewal process for the stochastic demand realization and that there is no penalty in case of a stock-out. He proposed that when a specific product is ordered, each successive unit is worth less, since it is expected to sell at a later time. Hence, some units may not be worth saving. On the other hand, a unit is worth disposing if its immediate value (what it presently can be sold for) exceeds its expected discounted sales value from a future sale if it is maintained in inventory, less the holding costs until the time of sale. Comparing the value of each item in the inventory with its disposal value, he presented a derivation to compute the number of units to be retained. Both Hart and Rosenfield, examined the problem of disposal of excess inventory in the face of demand uncertainty, however they did not account for the scenarios where it is optimal to dispose at the end of one period and at the same time place a fresh order at the beginning of the very next period.

Willoughby [13] analyzed procurement and disposal decisions for a multi-period stochastic demand problem where there is only one single procurement opportunity at the beginning of the planning horizon and no capacity constraints. He formulated a dynamic program that determines the optimal quantity to be procured at the beginning of the planning horizon and the optimal quantity to be retained at the end of every period that minimize the expected total discounted costs. Teisberg [11] developed a dynamic programming cost minimization model for a multi-period, stochastic demand system to determine joint procurement and sales policy, the number of units to be sold in a given period, for the United States strategic petroleum reserves. In addition to the stochastic nature of the demand, he also considered the effect of the world oil market on the total cost. The dynamic program generates a set of optimal procurement and sales rates for each time period, for each possible size of procurement and for each possible state of world oil market in that period. Assuming a sequence of future oil market states for the purpose of numerical analysis, he determined the optimal policy parameters for every time period. He examined sensitivity of the optimal policy to variations in the world oil market. Using these results, he showed that when such a policy is used it is possible to offset, at least in part, the effect of increase in world oil prices on procurement and total costs of the petroleum reserves and large benefits are obtainable as a result of using the optimal policy.

The second stream of literature related to this research is on the periodic review inventory systems with priority and price differentiated demand classes. The economic retention quantity we consider in this study is equivalent to rationing inventory to the secondary demand class, and hence, capacity rationing or allocation policies developed for firms selling their products to two or more customer classes are closely related to our work. Below we summarize some of the related papers in this area:

Frank et al. [4] considered a multi-period problem with two demand classes, one deterministic and one stochastic. In each period the deterministic demand must be satisfied completely, stochastic demand may not be satisfied completely. The revenue earned from both demand classes is the same. They considered a fixed setup cost but no capacity constraints. They showed that the stochastic demand may not be satisfied in a period in order to save the setup cost in the next period. The problem is how much to order in each period and how much of the stochastic demand to satisfy. They developed a (s; k; S) policy: where s and S determine when and how much to order, while k (rationing policy) specifies how much of the stochastic demand to satisfy. Balakrishnan et al. [1] considered the problem of allocating capacity to two product classes differentiated by their profit contribution, specifically for seasonal and short-life-cycle products. They do not consider decisions of ordering or producing. In particular, the problem applies to a make-to-order manufacturing setting, with demand in excess of the limited capacity available. They used a decision-theory-based approach and developed a capacity rationing policy. Their numerical analysis indicate the effectiveness of their capacity allocation policy under different conditions of parameters such as the available capacity and different unit revenues from each of the two demand classes.

Similarly, some papers consider the problem of production decisions under stochastic demand with capacity constraints, but without the flexibility of selling the product in a secondary market to avoid excess inventory carrying costs. The modeling approach followed by these papers is similar to the one we consider in this research. Federgruen and Zipkin [2, 3] characterized the optimal base-stock inventory policies for multi-period capacitated problems, for finite as well as infinite horizon problems. Tayur [10] provided an algorithm to compute the optimal policy parameters and the associated costs for the same problem and extended the work of Federgruen and Zipkin.

In this research, we integrate the decisions on production and economic retention quantities in the context of a finite-horizon production/inventory management model for firms serving two different market segments under non-stationary and stochastic demand. The joint decision of these two quantities for two market segments, and the production capacity considerations for each period differentiate our research from all previous works. To the best of our knowledge, no previous attempt has been made to jointly investigate production and retention decisions for a capacity constrained system serving two markets.

CHAPTER III

THE MODEL

A. Assumptions

We consider a production/inventory management problem with following assumptions:

1. Two market segments:
   (a) The primary market segment consists of customers with high priority and has stochastic demand.
   (b) The secondary market segment consists of customers with lesser priority and had ample demand.
2. No back-orders: Any demand that is not satisfied in the current period is lost.
3. Limited production capacity.
4. Instantaneous production: The production lead time is very short compared to the length each period.
5. Finite planning horizon.

B. Notation and Formulation

We introduce the following notation and develop a stochastic dynamic programming model to find the maximum total expected profit :

- T number of periods
- X_t continuous non-negative demand random variable for the primary market in period t
- $F_t(x_t)$ cumulative distribution function for demand X_t
- r_1 unit revenue earned from the primary market
- r_2 unit revenue earned from the secondary market
- p unit cost of production
- b unit penalty of lost sales
- h holding cost per unit, per period
- C_t production capacity of the firm in period t
- I_t onhand inventory at the beginning of period t
- y_t inventory in period t after production but before the demand from the primary is realized

$Y_t(I_t, C_t) \equiv \{y_t : I_t \leq y_t \leq C_t + I_t\}, y_t \ \epsilon \ Y_t(I_t, C_t)$

- z_t the maximum number of units to be carried from the period t to period $t-1$

In the following model, we index each period with the number of periods remaining until the end of the horizon. For $t = 1, 2...T$, the one period profit is given by

$$
\begin{aligned}
&\text{Expected profit earned in period } t \text{ when the supply is } y \text{ units and a maximum} \\
&\quad \text{of } z \text{ units carried over to the next period} \\
&= E[r_1 \min\{X_t, y\} - p(y - I) + r_2(y - X_t - z)1(y - X_t - z \geq 0) \\
&\quad - hz1(y - X_t - z \geq 0) - h(y - X_t)1((y - X_t)^+ \leq z) - b(X_t - y)^+]
\end{aligned}
$$

where $x^+ = \max\{x, 0\}$ and the expectation is taken over the distribution of the demand random variable X_t. Define

$$G_t(u) \equiv \int_u^\infty (x - u) dF_t(x) \tag{3.1}$$

$$
\begin{aligned}
P_t(y, z) \equiv\ & (r_1 - r_2)E[X_t] - (r_1 + b + h)G_t(y) + (r_2 + h)G_t(y - z) \\
& + (r_2 - p)y - (r_2 + h)z
\end{aligned} \tag{3.2}
$$

Then, the expected profit earned in period t is given by $pI + P_t(y, z)$. The objective is to determine the optimal policy parameters (y_t^*, z_t^*) when there are t periods to go until the end of the planning horizon that achieve the maximum expected profit over t periods. We define the following auxiliary functions:

$$
\begin{aligned}
J_t(y, z) =\ & P_t(y, z) + E[V_{t-1}(z)1(y - X_t \geq z)] \\
& + E[V_{t-1}(y - X_t)1(0 \leq y - X_t \leq z)] + E[V_{t-1}(0)1(y - X_t \leq 0)] \\
H_t(I) =\ & \max\{J_t(y, z) : y \ \epsilon \ Y(I, C), \ \ 0 \leq z \leq y\}
\end{aligned}
$$

$V_t(I)$ = Maximum expected profit earned from period t to end of

the planning horizon with I as the initial invetory

$V_t(I) = pI + H_t(I)$

Since no profits are earned after the end of the horizon, i.e., for $t = 0$,

$$V_0(\cdot) = J_0(\cdot, \cdot) = H_0(\cdot) = 0$$

The objective is to compute $V_T(I)$.

C. One Period Problem

To begin with, we investigate the one period problem, which is represented as:

$$V_1(I) = pI + \ \max\{J_1(y, z) : y_1 \ \epsilon \ Y_1(I_1, C_1), \ \ 0 \leq z \leq y\}$$

Since $V_0(\cdot) = 0$, we can write $J_1(y, z) = P_1(y, z)$. Taking the first order derivative with respect to z, we obtain:

$$\begin{aligned} \frac{\partial J_1(y, z)}{\partial z} &= \frac{\partial P_1(y, z)}{\partial z} \\ &= -(r_2 + h)F_1(y - z) \\ &\leq 0 \end{aligned}$$

For $y_1 \ \epsilon \ [I, C_1 + I]$, $J_1(y, z)$ is decreasing in z. Therefore, $z_1^* = 0$ and it is optimal not to retain any units at the end of the planning horizon. This is very intuitive, as in the one-period problem, there is a single opportunity to earn revenue, all inventory should be completely used. The one-period problem now simplifies to the classical newsvendor problem in which a single product is to be ordered at the beginning of a period and can be used only to satisfy the demand during that period. The only variation is that in our problem we may have a initial non-negative inventory that is carried from the previous period. The first and second order partial derivatives of $J_1(y,0)$ with respect to y are,

$$\frac{\partial J_1(y,0)}{\partial y} = (r_1 + b - r_2)[1 - F_1(y)] + r_2 - p \tag{3.3}$$

$$\frac{\partial^2 J_1(y,0)}{\partial y^2} = -(r_1 + b - r_2) f_1(y)$$

$$\leq 0 \tag{3.4}$$

Inequality (3.4) implies that $J_1(y,0)$ is concave in y. The optimal value of y for one-period problem depends on the relationship between r_2 and p. We consider the following two cases:

1. Profit from the Secondary Market: $r_2 \geq p$

If the unit revenue earned from the secondary market is greater than or equal to the unit production cost, i.e., if $r_2 \geq p$, selling in both the primary and the secondary markets is profitable. Subsequently,

$$\frac{\partial J_1(y,0)}{\partial y} = (r_1 + b - r_2)[1 - F(y)] + r_2 - p \geq 0$$

Hence, $J_1(y,0)$ is increasing and concave in y. There is ample demand in the secondary market and every unit sold earns a profit. Therefore, it is optimal to produce up to capacity and hence, $y_1^* = C_1 + I$. In addition, the maximum expected one period profit is, $V_1(I) = pI + J_1(C_1 + I, 0)$.

2. Salvaging in the Secondary Market: $r_2 < p$

If the revenue earned from the secondary market is less than the unit production cost, selling to these customers is equivalent to salvaging. We know that $J_1(y,0)$ is continuous and concave in y. The unconstrained maxima is then given by equating the first derivative of $J_1(y,0)$ with respect to y to zero. Let s_1 be this unconstrained optimal value of y, then from (3.3)

$$F_1(s_1) = \frac{r_1 + b - p}{r_1 + b - r_2}.$$

The optimal policy is given by $(y_1^*, z_1 = 0)$:

$$y_1^* = \begin{cases} C_1 + I & I \leq s_1 - C_1 \\ s_1 & s_1 - C_1 \leq I \leq s_1 \\ I & s_1 \leq I \end{cases} \tag{3.5}$$

and the maximum expected one period profit is:

$$V_1(I) = pI + J_1(y_1^*, 0)$$

$$V_1(I) = pI + \begin{cases} J_1(C_1 + I, 0) & I \leq s_1 - C_1 \\ J_1(s_1) & s_1 - C_1 \leq I \\ J_1(I) & s_1 \leq I \end{cases}$$

Define C_o as unit overage cost, or the cost of ordering one excess unit and C_u as the unit underage cost, i.e., the cost of ordering one unit less than the optimal order-up-to level. For our one-period problem, $C_o = p - r_2$ and $C_u = r_1 + b - p$. Then, the unconstrained optimal order-up-to level is also given by,

$$F_1(s_1) = \frac{C_u}{C_u + C_o},$$

which is similar to the optimal order level of a newsvendor problem. $\dfrac{C_u}{C_u + C_o}$ is called as the critical ratio and is equal to the probability of satisfying all the demand from the primary market if we produce up-to s_1 at the beginning of the period.

With this understanding of the one period problem, in the following chapter, we characterize the structure of the optimal policy for the multi-period problem.

CHAPTER IV

CHARACTERIZATION OF THE STRUCTURE OF THE OPTIMAL POLICY

For the multi-period problem the optimal policy is characterized by the pairs (y_t^*, z_t^*) for $t = 1, ..., T$. The structure of the optimal policy depends on the relationship between r_2 and p. The following theorems completely characterize the structure of the optimal policy for the two cases: a) $r_2 \geq p$ and b) $r_2 < p$ respectively.

Theorem 1. *If $r_2 \geq p$, for $t = 1, 2, ..T$*

(a) $J_t(y, z)$ has a finite maximizer denoted by (y_t^*, z_t^*) such that

$$z_t^* = \begin{cases} 0 & \text{if} \quad \left.\dfrac{dV_{t-1}(z)}{dz}\right|_{z=0} < r_2 + h \\ \\ \bar{z}_t & \text{if} \quad \left.\dfrac{dV_{t-1}(z)}{dz}\right|_{z=0} \geq r_2 + h \end{cases}$$

where $\bar{z}_t$ satisfies

$$\frac{dV_{t-1}(z)}{dz} = r_2 + h,$$
$$\text{and} \quad y_t^* = C_t + I.$$

(b) $V_t(I)$ is a concave increasing function of I.

(c) $\dfrac{dV_t(I)}{dI} \leq r_1 + b$

Proof. The first order derivatives of $J_t(y, z)$ with respect to y and z are given as follows:

$$\frac{\partial J_t(y,z)}{\partial y} = \frac{\partial P_t(y,z)}{\partial y} + \int_{y-z}^{y} \frac{dV_{t-1}(y-x)}{dy} dF_t(x)$$
$$= (r_1+b+h)[1-F_t(y)] - (r_2+h)[1-F_t(y-z)] + r_2 - p$$
$$+ \int_{y-z}^{y} \frac{dV_{t-1}(y-x)}{dy} dF_t(x), \text{and} \tag{4.1}$$

$$\frac{\partial J_t(y,z)}{\partial z} = \left[\frac{dV_{t-1}(z)}{dz} - (r_2+h)\right] F(y-z) \tag{4.2}$$

The proof follows by induction. We will first show that the properties (a)-(c) are true for $t = 1$. From §1, when $t = 1$, $\frac{dV_{t-1}(z)}{dz} = 0$, since $V_0(\cdot) = 0$. Then,

$$\frac{\partial J_1(y,z)}{\partial z} = -(r_2+h)F_1(y-z) < 0$$

and for $y \ \epsilon \ [I, C_1+I]$, $J_1(y,z)$ is decreasing in z. Therefore, $z_1^* = 0$. In addition,

$$\frac{\partial J_1(y,0)}{\partial y} = (r_1+b-r_2)[1-F_1(y)] + r_2 - p > 0$$

Since $J_1(y,0)$ is increasing in y, $y_1^* = C_1 + I$. It follows that

$$V_1(I) = pI + P_1(C_1+I, 0)$$
$$= pI + (r_1-r_2)E[X_1] - (r_1+b-r_2)G_1(C_1+I) + (r_2-p)(C_1+I)$$
$$\frac{dV_1(I)}{dI} = r_2 + (r_1+b-r_2)(1-F_1(C_1+I)) \geq 0, \qquad \text{and}$$
$$\frac{d^2V_1(I)}{dI^2} = -(r_1+b-r_2)f_1(C_1+I) \leq 0$$

Therefore, $V_1(I)$ is increasing and concave in I. Since $\frac{dV_1(I)}{dI} = r_2 + (r_1+b-r_2)(1 - F_1(C_1+I)) \leq r_1 + b$, property (c) holds.

Suppose that properties (a)-(c) are true for $t = 1, ..., n-1$, we will show that they

are true for $t = n$. By the induction hypothesis, $\frac{dV_{n-1}(z)}{dz}$ is a decreasing function of z. If $\left.\frac{dV_{n-1}(z)}{dz}\right|_{z=0} < r_2 + h$, then $\frac{dV_{t-1}(z)}{dz} < r_2 + h$ for $z > 0$. From equation(4.2) $\frac{\partial J_n(y,z)}{\partial z} < 0$ for $y \ \epsilon \ [I, C_n + I]$. Therefore, $z_n^* = 0$.
On the other hand, if $\left.\frac{dV_{n-1}(z)}{dz}\right|_{z=0} \geq r_2 + h$, then $\frac{dV_{n-1}(z)}{dz} = r_2 + h$ for some $\bar{z}_n \ \epsilon \ \Re^+$. $J_n(y,z)$ is increasing in z for $z < \bar{z}_n$, and decreasing in z for $z > \bar{z}_n$. Therefore, $J_n(y,z)$ attains its maximum at $z = \bar{z}_n$ for $y \ \epsilon \ [I, C_n + I]$. If $z_n^* = 0$,

$$\frac{\partial J_n(y,0)}{\partial y} = (r_1 + b - r_2)[1 - F_n(y)] + r_2 - p > 0$$

If $z_n^* = \bar{z}_n$,

$$\begin{aligned}
\frac{\partial J_n(y,\bar{z}_n)}{\partial y} &= (r_1 + b + h)[1 - F_n(y)] - (r_2 + h)[1 - F_n(y - \bar{z}_n)] + r_2 - p \\
&\quad + \int_{y-\bar{z}_n}^{y} \frac{dV_{n-1}(y-x)}{dy} dF_n(x) \\
&\geq (r_1 + b + h)[1 - F_n(y)] - (r_2 + h)[1 - F_n(y - \bar{z}_n)] + r_2 - p \\
&\quad + (r_2 + h)[F_n(y) - F_n(y - \bar{z}_n)] \\
&= (r_1 + b - r_2)[1 - F_n(y)] + r_2 - p > 0,
\end{aligned}$$

where the first inequality follows from the fact that $\frac{dV_{n-1}(y-x)}{dy} > r_2 + h$ when $x \ \epsilon \ [y - \bar{z}_n, y]$. Since $J_n(y, z_n^*)$ is increasing in y, $y_n^* = C_n + I$, which shows that property (a) holds.

Then, we can write $V_n(I) = pI + J_n(C_n + I, z_n^*)$. Taking the first derivative with respect to I we obtain:

$$\begin{aligned}\frac{dV_n(I)}{dI} &= p + \frac{dJ_n(C_n+I, z_n^*)}{dI} \\ &= r_2 + (r_1+b+h)[1-F_n(C_n+I)] - (r_2+h)[1-F_n(C_n+I-z_n^*)] \\ &\quad + \int_{C_n+I-z_n^*}^{C_n+I} \frac{dV_{n-1}(C_n+I-x)}{dI} dF_n(x)\end{aligned}$$

If $z_n^* = 0$,

$$\begin{aligned}\frac{dV_n(I)}{dI} &= r_2 + (r_1+b-r_2)[1-F_n(C_n+I)] > 0 \qquad (4.3) \\ \frac{d^2V_n(I)}{dI^2} &= -(r_1+b-r_2)f_n(C_n+I) < 0.\end{aligned}$$

From equation(4.3), it is also straight forward to see that $\frac{dV_n(I)}{dI} < r_1 + b$. As a result, properties (b) and (c) hold if $z_n^* = 0$. If $z_n^* = \bar{z}_n$,

$$\begin{aligned}\frac{dV_n(I)}{dI} &= r_2 + (r_1+b+h)[1-F_n(C_n+I)] - (r_2+h)[1-F_n(C_n+I-z_n^*)] \\ &\quad + \int_{C_n+I-z_n^*}^{C_n+I} \frac{dV_{n-1}(C_n+I-x)}{dI} dF_n(x) \qquad (4.4) \\ &\geq r_2 + (r_1+b-r_2)[1-F_n(C_n+I)] > 0,\end{aligned}$$

where the first inequality follows from the fact that $\frac{dV_{n-1}(C_n+I-x)}{dz} \geq r_2 + h$ for $x \ \epsilon \ [C_n+I-\bar{z}_n, C_n+I]$.

$$\begin{aligned}\frac{d^2V_n(I)}{dI^2} &= -(r_1+b+h)f_n(C_n+I) + (r_2+h)f_n(C_n+I-\bar{z}_n) \\ &\quad + \int_{C_n+I-\bar{z}_n}^{C_n+I} \frac{d^2V_{n-1}(C_n+I-x)}{dI^2} dF_n(x) \\ &\quad + \left.\frac{dV_{n-1}(I)}{dI}\right|_{I=0} f_n(C_n+I) - \left.\frac{dV_{t-1}(y-x)}{dy}\right|_{I=\bar{z}_n} f_n(C_n+I-\bar{z}_n) \\ &< \int_{C_n+I-\bar{z}_n}^{C_n+I} \frac{d^2V_{n-1}(C_n+I-x)}{dI^2} dF_n(x) < 0\end{aligned}$$

because $V_{n-1}(I)$ is concave in I,

$$\left.\frac{dV_{n-1}(I)}{dI}\right|_{I=0} < r_1 + b + h,$$

and by the induction hypothesis

$$\left.\frac{dV_{t-1}(y-X)}{dy}\right|_{I=\bar{z}_n} = r_2 + h.$$

From equation (4.4), it can be easily shown that $\frac{dV_n(I)}{dI} < r_1 + b$. As a result, properties (b) and (c) hold if $z_n^* = \bar{z}_n$ □

Interpretation of Theorem 1: Property (a) in Theorem 1 characterizes the effect of the cost structure on the economic retention quantity. If unit revenue earned from the secondary market is greater than the unit production cost, i.e. $r_2 \geq p$, then every unit sold in the secondary market results in a profit. Therefore, it is optimal to produce up-to the capacity at the beginning of every period. In addition, if the expected increase in profit in the future periods with respect to the number of units retained, evaluated at $z = 0$, and represented by $\frac{dV_{t-1}(z)}{dz}\big|_{z=0}$ is higher than the profit that can be earned by selling a unit in the secondary market now, given by $r_2 + h$, then the profit function $J_t(y, z)$ is unimodal in z. That is for some $\bar{z}$, $J_t(y, z)$ is monotonically increasing for $z \leq \bar{z}$ and monotonically decreasing for $z > \bar{z}$. The economic retention quantity is then equal to $\bar{z}$, such that, at $\bar{z}$ the expected increase in profit in future periods with respect to z is equal to the profit gained by selling in the secondary market in the current period. On the other hand if the expected increase in profit in the future periods with respect to the number of units retained, evaluated at $z = 0$ is less than the profit that can be earned by selling a unit in the secondary market now, then $J_t(y, z)$ decreases in z. And it is optimal not to carry any inventory to the next period.

For every unit of on-hand inventory available at the beginning of a period, the maximum increase in expected profit is equal to the sum of the unit revenue earned from the primary market and the savings in terms of lost sales penalty, i.e. $r_2 + h$. This is illustrated by part (c) of the Theorem 1.

Theorem 2. *If $r_2 \leq p$, for $t = 1, 2, ..T$*

(a) $J_t(y, z)$ has a finite maximizer denoted by (y_t^*, z_t^*) such that

$$z_t^* = \begin{cases} 0 & \text{if } \left.\frac{dV_{t-1}(z)}{dz}\right|_{z=0} < r_2 + h \\ \\ \bar{z}_t & \text{if } \left.\frac{dV_{t-1}(z)}{dz}\right|_{z=0} \geq r_2 + h \end{cases}$$

where $\bar{z}_t$ satisfies

$$\frac{dV_{t-1}(z)}{dz} = r_2 + h, \text{ and}$$

$$y_t^* = \begin{cases} I & s_t \leq I \\ s_t & I \leq s_t \leq C_t + I \\ C_t + I & s_t \geq C_t + I \end{cases}$$

where s_t is determined by, $J_t(s_t, z_t^*) = max\{J_t(y, z_t^*) : y \ \epsilon \ \Re\}$

(b) $V_t(I)$ is a concave increasing function of I.

(c) $\frac{dV_t(I)}{dI} \leq r_1 + b$

Proof. The proof follows by induction. From §2, for $t = 1$,

$$\frac{\partial J_1(y, z)}{\partial z} = -(r_2 + h)F_1(y - z) < 0$$

and for $y \ \epsilon \ [I, C_1 + I]$, $J_1(y, z)$ is decreasing in z. Therefore, $z_1^* = 0$. With $z_1^* = 0$, the first and second order derivatives of $J_1(y, 0)$ are given as follows:

$$\frac{\partial J_1(y,0)}{\partial y} = (r_1 + b - r_2)[1 - F_1(y)] + r_2 - p, \text{ and}$$
$$\frac{\partial^2 J_1(y,0)}{\partial y^2} = -(r_1 + b - r_2) f_1(y) < 0$$

Hence, $J_1(y,0)$ is concave in y. Solving for the first order condition, we have

$$F_1(s_1) = \frac{r_1 + b - p}{r_1 + b - r_2}.$$

Then, y_1^* can be expressed as:

$$y_1^* = \begin{cases} I & s_1 \leq I \\ s_1 & I \leq s_1 \leq C_1 + I \\ C_1 + I & s_1 \geq C_1 + I \end{cases}$$

Consequently, $V_1(I) = pI + J_1(y_1^*(I), 0)$. The first and second order derivatives of $V_1(I)$ are given as follows:

$$\frac{dV_1(I)}{dI} = \begin{cases} r_2 + (r_1 + b - r_2)(1 - F_1(I)) & s_1 \leq I \\ p & I \leq s_1 \leq C_1 + I \\ r_2 + (r_1 + b - r_2)(1 - F_1(C_1 + I)) & s_1 \geq C_1 + I \end{cases} \quad (4.5)$$

$$\frac{d^2V_1(I)}{dI^2} = \begin{cases} -(r_1 + b - r_2) f_1(I) & s_1 \leq I \\ 0 & I \leq s_1 \leq C_1 + I \\ -(r_1 + b - r_2) f_1(C_1 + I)) & s_1 \geq C_1 + I \end{cases}$$

Since $\frac{dV_1(I)}{dI} > 0$ and $\frac{d^2V_1(I)}{dI^2} \leq 0$, $V_1(I)$ is concave and increasing in I. From equation (4.5), it is straight forward to observe that $\frac{dV_t(I)}{dI} \leq r_1 + b$. As a result, properties (a) -(c) are satisfies for $t = 1$.

Suppose that properties (a)-(c) are true for $t = 1, ..., n-1$, we will show that they are true for $t = n$. By the induction hypothesis, $\frac{dV_{n-1}(z)}{dz}$ is a decreasing function of z. If $\left.\frac{dV_{n-1}(z)}{dz}\right|_{z=0} \leq r_2 + h$, then $\frac{dV_{t-1}(z)}{dz} < r_2 + h$ for $z > 0$, which implies that $\frac{\partial J_n(y,z)}{\partial z} < 0$ for $y \ \epsilon \ [I, C_n + I]$. Therefore, $z_n^* = 0$.

On the other hand, if $\left.\frac{dV_{n-1}(z)}{dz}\right|_{z=0} > r_2 + h$, then $\frac{dV_{n-1}(z)}{dz} = r_2 + h$ for some $\bar{z}_n \ \epsilon \ \Re$. $J_n(y,z)$ is increasing in z for $z < \bar{z}_n$, and decreasing in z for $z > \bar{z}_n$. Therefore, $J_n(y,z)$ attains its maximum at $z = \bar{z}_n$ for $y \ \epsilon \ [I, C_n + I]$. The first and second order derivatives of $J_n(y,z)$ evaluated at $z = z_n^*$ are as follows:

$$\begin{aligned}
\frac{\partial J_n(y, z_n^*)}{\partial y} &= (r_1 + b + h)[1 - F_n(y)] - (r_2 + h)[1 - F_n(y - z_n^*)] + r_2 - p \\
&\quad + \int_{y - z_n^*}^{y} \frac{dV_{n-1}(y - x)}{dy} dF_n(x) \\
\frac{\partial J_n^2(y, z_n^*)}{\partial y^2} &= -(r_1 + b + h) f_n(y) + (r_2 + h) f_n(y - z_n^*) \\
&\quad + \int_{y - z_n^*}^{y} \frac{d^2 V_{n-1}(C_n + I - x)}{dI^2} dF_n(x) \\
&\quad + \left.\frac{dV_{n-1}(I)}{dI}\right|_{I=0} f_n(y) - \left.\frac{dV_{n-1}(y - x)}{dy}\right|_{I=z_n^*} f_n(y - z_n^*) < 0
\end{aligned}$$

because $V_{n-1}(I)$ is concave in I, $\left.\frac{dV_{n-1}(I)}{dI}\right|_{I=0} < r_1 + b$, and $\left.\frac{dV_{n-1}(I)}{dI}\right|_{I=z_n^*} = r_2 + h$ for $z_n^* > 0$ by the induction hypothesis. Let us define $s_n \ \epsilon \ \Re$, such that

$\left.\frac{\partial J_n(y, z_n^*)}{\partial y}\right|_{y=s_n} = 0$. Then

$$y_n^* = \begin{cases} I & s_n \leq I \\ s_n & I \leq s_n \leq C_n + I \\ C_n + I & s_n \geq C_n + I \end{cases}$$

As a result, property (a) holds. Then we can write $V_n(I) = pI + J_n(y_n^*(I), z_n^*)$. Taking the first derivative with respect to I we obtain:

$$\frac{dV_n(I)}{dI} = p + \frac{dJ_n(y_n^*(I), z_n^*)}{dI}$$

$$\frac{dV_n(I)}{dI} = p + \begin{cases} (r_1+b+h)[1-F_n(I)] - (r_2+h)[1-F_n(I-z_n^*)] \\ \quad +(r_2-p) + \int_{I-z_n^*}^{I} \frac{dV_{n-1}(I-x)}{dI} dF_n(x) & s_n \leq I \\ 0 & I \leq s_n \leq C_n + I \\ (r_1+b+h)[1-F_n(C_n+I)] \\ \quad -(r_2+h)[1-F_n(C_n+I-z_n^*)] + (r_2-p) \\ \quad + \int_{C_n+I-z_n^*}^{C_n+I} \frac{dV_{n-1}(C_n+I-x)}{dI} dF_n(x) & s_n \geq C_n + I \end{cases}$$

$$\frac{d^2V_n(I)}{dI^2} = \begin{cases} -(r_1+b+h)f_n(I) + (r_2+h)f_n(I-z_n^*) \\ \quad + \int_{I-z_n^*}^{I} \frac{d^2V_{n-1}(I-X)}{dI^2} dF_n(x) \\ \quad + \frac{dV_{n-1}(I)}{dI}\Big|_{I=0} f_n(I) - \frac{dV_{n-1}(y-X)}{dy}\Big|_{I=z_n^*} f_n(I-z_n^*) & s_n \leq I \\ 0 & I \leq s_n \leq C_n + I \\ -(r_1+b+h)f_n(C_n+I) + (r_2+h)f_n(C_n+I-z_n^*) \\ \quad + \int_{C_n+I-z_n^*}^{C_n+I} \frac{d^2V_{n-1}(C_n+I-X)}{dI^2} dF_n(x) \\ \quad + \frac{dV_{n-1}(I)}{dI}\Big|_{I=0} f_n(C_n+I) \\ \quad - \frac{dV_{n-1}(y-X)}{dy}\Big|_{I=z_n^*} f_n(C_n+I-z_n^*) & s_n \geq C_n + I \end{cases}$$

Therefore, $\frac{dV_n(I)}{dI} \geq 0$ and $\frac{d^2V_n(I)}{dI^2} \leq 0$ and $V_n(I)$ is concave and increasing in I. It is straightforward to show that $\frac{dV_n(I)}{dI} < r_1 + b$. □

Interpretation of Theorem 2: Theorem 2 shows that the structure of the optimal policy for $r_2 < p$ case is similar to that for the $r_2 \geq p$ case. In addition, it illustrates the fact that since selling in the secondary market is equivalent to salvaging, it may not always be optimal to produce up-to the capacity.

A. Bounds for the Optimal Policy Parameters

Detailed analysis of two and three-period problems show that we can obtain a closed form solution for the optimal economic retention quantity for the one and two period problems only. For problems having more than two periods, closed form solution does not exist, and the following theorems establish lower and upper bounds for this class of problems. In order to establish these bounds, we again consider the cases where $r_2 \geq p$ and $r_2 < p$, separately.

Theorem 3. *If $r_2 \geq p$, for $t = 2, ..T$*

$$z'_t \leq \bar{z}_t \leq z''_t$$

$$\textit{such that} \qquad z'_t = F_{t-1}^{-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) - C_{t-1}$$

$$z''_t = F_{t-1}^{-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) - C_{t-1} + z^*_{t-1}$$

Proof. For $t = 2, 3, ..., T$, we can write $V_{t-1}(I) = pI + J_{t-1}(C_{t-1} + I, z^*_{t-1})$. Taking the first order derivative with respect to I we obtain:

$$\begin{aligned}\frac{dV_{t-1}(I)}{dI} &= p + \frac{dJ_{t-1}(C_{t-1} + I, z^*_{t-1})}{dI} \\ &= r_1 + b - (r_1 + b + h)F_{t-1}(C_{t-1} + I) + (r_2 + h)F_{t-1}(C_{t-1} + I - z^*_{t-1}) \\ &\quad + \int_{C_{t-1}+I-z^*_{t-1}}^{C_{t-1}+I} \frac{dV_{t-2}(C_{t-1} + I - x)}{dI} dF_{t-1}(x) \qquad (4.6)\end{aligned}$$

Referring to Theorem 1, properties (a) and (b),

$$\left.\frac{dV_{t-2}(z)}{dz}\right|_{z \leq z^*_{t-1}} \geq r_2 + h, \qquad (4.7)$$

Therefore,

$$\begin{aligned}\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=z'_t} &\geq r_1 + b - (r_1 + b - r_2)F_{t-1}(C_{t-1} + z'_t) \\ &= r_1 + b - (r_1 + b - r_2)\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right), \text{and} \\ \left.\frac{dV_{t-1}(z)}{dz}\right|_{z=z'_t} &\geq r_2 + h.\end{aligned}$$

Since $V_{t-1}(z)$ is concave and $\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=\bar{z}_t} = r_2 + h$, we have $z_t' \leq \bar{z}_t$. Similarly, referring to Theorem 1, property (c), we know that $\frac{dV_t(I)}{dI} \leq r_1 + b + h$. Using the above inequality in equation (4.6), it is easy to see that

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=z_t''} \leq r_2 + h,$$

and hence $\bar{z}_t \leq z_t''$. □

Interpretation of Theorem 3: The economic retention quantity for any period t depends on: (a) the maximum available capacity in the next period, C_{t-1}, (b) the demand distribution (c.d.f) in the next period, $F_{t-1}(x_{t-1})$, and (c) the number of units to be retained at the end of next period, z_{t-1}^*. We define C_u as the cost of selling a unit in the secondary market now, when it is needed in the primary market in the next period. And C_o as the cost of carrying a unit when it is not needed in primary market in the next period. Then, $C_u = r_1 + b - r_2 - h$ and $C_o = h$. When the unit revenue earned from the secondary market is greater than or equal to the unit production cost, i.e. $r_2 \geq p$, then it is optimal to produce up to capacity in every period. Hence, $y_{t-1}^* = C_{t-1} + z_t^*$. The economic retention quantity should at least be equal to the number of units needed in the next period, in excess of the maximum available capacity C_{t-1}, to satisfy the demand from the primary market in the next period. Therefore, the lower bound for z_t^* is such that, if z_t^* units are carried and we produce up-to the capacity in the next period, then the probability of satisfying all the demand from the primary market in the next period is equal to the critical ratio given by $\frac{C_u}{C_u + C_o}$. Similarly, the economic retention quantity should not be greater than the sum of the units needed in the next period, in excess of the maximum available capacity C_{t-1}, to satisfy the demand from the primary market and the units to be retained at the end of the next period. This is illustrated by the upper bound on z_t^*. The bounds are represented in terms of the critical ratio as follows:

$$F_{t-1}(z_t + C_{t-1}) \geq \frac{C_u}{C_u + C_o}$$
$$F_{t-1}(z_t + C_{t-1}) \leq \frac{C_u}{C_u + C_o} + z_{t-1}^* \tag{4.8}$$

Theorem 4. *If $r_2 < p$, for $t = 2, ..T$*

(a) If $p \leq r_2 + h$,

$$F_{t-1}^{-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) - C_{t-1} \leq \bar{z}_t \leq F_{t-1}^{-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) - C_{t-1} + z_{t-1}^*$$

If $p > r_2 + h$

$$F_{t-1}^{-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) \leq \bar{z}_t \leq F_{t-1}^{-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) + z_{t-1}^*$$

(b) s_t is bounded as follows:

$$F_t^{-1}\left(\frac{r_1 + b - p}{r_1 + b - r_2}\right) \leq s_t \leq F_t^{-1}\left(\frac{r_1 + b - p}{r_1 + b - r_2}\right) + z_t^*$$

Proof. For any $t = 2, 3, ..., T-1$, we can write $V_t(I) = pI + J_t(y_t^*(I), z_t^*)$. Taking the first order derivative with respect to I we obtain:

$$\frac{dV_t(I)}{dI} = p + \frac{dJ_t(y_t^*(I), z_t^*)}{dI}$$

$$\frac{dV_t(I)}{dI} = p + \begin{cases} (r_1+b+h)[1-F_t(I)] - (r_2+h)[1-F_t(I-z_t^*)] & \\ \qquad +(r_2-p) + \int_{I-z_t^*}^{I} \frac{dV_{t-1}(I-x)}{dI} dF_t(x) & s_t \le I \\ 0 & I \le s_t \le C_t + I \\ (r_1+b+h)[1-F_t(C_t+I)] & \\ \qquad -(r_2+h)[1-F_t(C_t+I-z_t^*)] & \\ \qquad +(r_2-p) + \int_{C_t+I-z_t^*}^{C_t+I} \frac{dV_{t-1}(C_t+I-x)}{dI} dF_t(x) & s_t \ge C_t + I \end{cases}$$

Referring to Theorem 2, we know that $V_t(I)$ is concave and

$$\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=\bar{z}_t} = r_2 + h \tag{4.9}$$

Therefore,

$$\frac{dV_t(I)}{dI} \begin{cases} \le p & s_t \le I \\ = p & I \le s_t \le C_t + I \\ \ge p & s_t \ge C_t + I \end{cases}$$

If $p \le r_2 + h$, then for all $z \ge s_{t-1} - C_{t-1}$,

$$\frac{dV_{t-1}(z)}{dz} \le p \le r_2 + h.$$

Hence, in order that $\bar{z}_t$ satisfies (4.9), $\bar{z}_t \le s_{t-1} - C_{t-1}$.

$$\begin{aligned} \left.\frac{dV_{t-1}(z)}{dz}\right|_{z=\bar{z}_t} &= r_2 + h \\ &= r_1 + b - (r_1+b+h)F_{t-1}(C_{t-1}+\bar{z}_t) + (r_2+h)F_{t-1}(C_{t-1}+\bar{z}_t - z_{t-1}^*) \\ &\quad + \int_{C_{t-1}+\bar{z}_t - z_{t-1}^*}^{C_{t-1}+\bar{z}_t} \frac{dV_{t-2}(C_{t-1}+\bar{z}_t - x)}{dI} dF_{t-1}(x) \end{aligned} \tag{4.10}$$

Using the property that for all $z \leq z^*_{t-1}$, $\dfrac{dV_{t-2}(z)}{dz} \geq r_2 + h$ we get,

$$\bar{z}_t \geq F^{-1}_{t-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) - C_{t-1}$$

Similarly, using the following inequality in equation (4.10),

$$\frac{dV_{t-2}(z)}{dz} \leq r_1 + b + h.$$

We observe that

$$\bar{z}_t \leq F^{-1}_{t-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) - C_{t-1} + z^*_{t-1}$$

If $p > r_2 + h$ then for all $z \leq s_{t-1}$,

$$\frac{dV_{t-1}(z)}{dz} \geq p \geq r_2 + h.$$

Hence, in order that $\bar{z}_t$ satisfies (4.9), $\bar{z}_t \leq s_{t-1}$ and

$$\begin{aligned}\left.\frac{dV_{t-1}(z)}{dz}\right|_{z=\bar{z}_t} &= r_2 + h \\ &= r_2 + (r_1 + b + h)[1 - F_{t-1}(\bar{z}_t)] - (r_2 + h)[1 - F_{t-1}(\bar{z}_t - z^*_{t-1})] \\ &\quad + \int_{\bar{z}_t - z^*_{t-1}}^{\bar{z}_t} \frac{dV_{t-2}(\bar{z}_t - x)}{dI} dF_{t-1}(x) \end{aligned} \tag{4.11}$$

From this, similar to the $p \leq r_2 + h$ case it is easy to see that,

$$F^{-1}_{t-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) \leq \bar{z}_t \leq F^{-1}_{t-1}\left(\frac{r_1 + b - r_2 - h}{r_1 + b - r_2}\right) + z^*_{t-1}$$

$s_t \in \Re$, is such that $\left.\dfrac{\partial J_t(y, z^*_t)}{\partial y}\right|_{y=s_t} = 0$. Therefore,

$$r_1 + b - p - (r_1 + b + h)F_t(s_t) + (r_2 + h)F_t(s_t - z^*_t) + \int_{\bar{s}_t - z^*_t}^{\bar{s}_t} \left.\frac{dV_{t-1}(y - x)}{dI}\right|_{y=s_t} dF_t(x) = 0$$

Again, using properties (a)-(c) of theorem 2, it is easy to observe that,

$$F_t^{-1}\left(\frac{r_1+b-p}{r_1+b-r_2}\right) \leq s_t \leq F_t^{-1}\left(\frac{r_1+b-p}{r_1+b-r_2}\right) + z_t^*$$

$\square$

Interpretation of Theorem 4**:** If the unit production cost is less than or equal to the sum of unit revenue from the secondary market and the unit holding cost, i.e. $p \leq r_2 + h$, then in the next period it is more profitable to produce for the primary market rather than carrying a unit from the current period. Therefore, if we carry inventory equal to the economic retention quantity from period t to period $t-1$, we would produce-to-capacity in the next period. Hence the bounds for this case are similar to the $r_2 \geq p$ case. On the other hand if the unit production cost is greater than the sum of the unit revenue from the secondary market and the unit holding cost, i.e. $p > r_2 + h$, then if we carry inventory equal to the economic retention quantity from period t to period $t-1$, we would not produce-to-capacity. And hence the bounds for the economic retention quantity when $p > r_2 + h$, are independent of C_{t-1}, the maximum available capacity in the next period. Again, $C_u = r_1 + b - r_2 - h$ and $C_o = h$. and the bounds can be represented in terms of the critical ratio as follows:

$$\begin{aligned} F_{t-1}(z_t) &\geq \frac{C_u}{C_u + C_o} \\ F_{t-1}(z_t) &\leq \frac{C_u}{C_u + C_o} + z_{t-1}^* \end{aligned} \tag{4.12}$$

The produce up-to level for a period t depends on: (a) the demand distribution in the current period $F_t(x_t)$, and (b) the number of units to be carried to the next period $t-1$. Again, similar to the one-period problem we define C_o as unit overage cost, or the cost of ordering one excess unit and C_u as the unit underage cost, i.e., the cost of ordering one unit less than the optimal order-up-to level. Then, $C_o = p - r_2$ and $C_u = r_1 + b - p$. The optimal produce up-to level should at-least be equal to the quantity needed to satisfy the demand from the primary market. On the other hand, the produce up-to level should not be more than that needed to satisfy the demand from the primary market and the amount to be retained at the end of the current period. Hence, the optimal produce up-to level is bounded on the lower side by the critical ratio given by $\dfrac{C_u}{C_u + C_o}$ and on the upper side by the sum of the critical ratio and the number of units to be carried to the next period z^*_{t-1}.

CHAPTER V

COMPUTATIONAL ANALYSIS AND RESULTS

This chapter discusses the computational study conducted to obtain insights into the problem from a managerial perspective. Our goal is to examine the sensitivity of the optimal policy parameters and the expected profit with respect to the changes in cost parameters, namely the unit penalty of lost sales and the unit holding cost. We also present and compare the performance of our policy with policies where either no units are sold in the secondary market or all units are sold in the secondary market with no units carried to the next period.

The experimental details are as follows: We analyze three-period problems, where the demand is assumed to be an exponential random variable with rate λ_t in period t and hence the expected demand in period t is $E[X_t] = \lambda_t^{-1}$. For each experiment, the unit revenue earned from the primary market is \$100 and the unit cost of production is \$50. Several experiments were conducted to compute the optimal policy parameters for different values of r_2, h, b , λ_t and production capacities for each period. For every experiment and the corresponding cost parameters, expected profit for $(y^*, z = 0)$ policy and the policy where no units are sold in the secondary market is also evaluated.

A. Sensitivity Analysis

Figure 1 shows that the economic retention quantity increases, almost linearly, with increase in the lost sales penalty. The reason is that as the lost sales penalty increases, the marginal value of an item in future period increases, and hence, it is more profitable to sell less number of units in the secondary market and retain more units for sale in the future periods. Similarly, it is more profitable to retain more units if the unit revenue earned in the secondary market is lower. In addition, given that all the parameters are constant, the economic retention quantity is higher if there are more number of periods to go until the end of the planning horizon. Furthermore, it is optimal not to retain any units at the end of the last period. This is very intuitive since the number of opportunities for sale are less uncertainty as we approach the end of the planning horizon, and hence, it is better to keep less inventory.

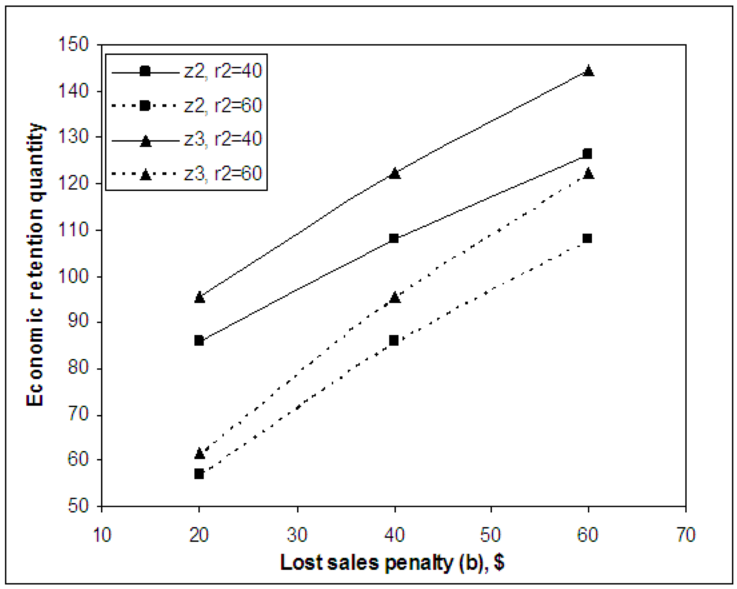

Fig. 1. Sensitivity of the economic retention quantity with respect to the lost sales penalty when $r_1 = 100$, $p = 50$, $h = 12.5$, $E[X_1] = E[X_2] = E[X_3] = 100$, $C_1 = C_2 = C_2 = 100$

Similar to the economic retention quantity, in Figure 2, we observe that the optimal produce-up-to level also shows a linear increase with the increase in the lost sales penalty. It is more profitable to produce more units when there are more periods to go until the end of the planning horizon.

As the unit holding cost increases, there is a decrease in the marginal value of an item in the future periods, because it is more costly to carry a unit to the next period. Hence, if the holding cost increases, it is more profitable to sell more units in the secondary market, and carry less units into future periods. Figure 3 shows that

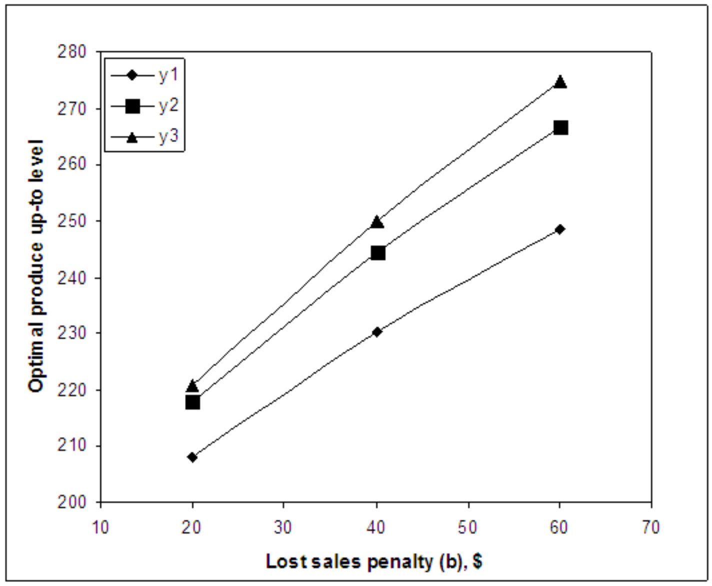

Fig. 2. Sensitivity of the optimal-produce up-to level with respect to the lost sales penalty when $r_1 = 100$, $p = 50$, $r_2 = 40$, $h = 12.5$, $E[X_1] = E[X_2] = E[X_3] = 100$, $C_1 = C_2 = C_2 = 100$

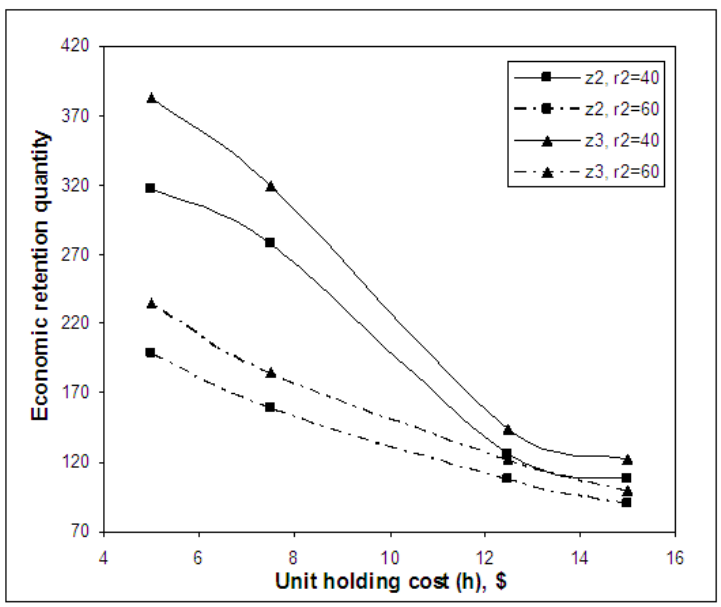

Fig. 3. Sensitivity of the economic retention quantity with respect to the unit holding cost when $r_1 = 100$, $p = 50$, $b = 60$, $E[X_1] = E[X_2] = E[X_3] = 100$, $C_1 = C_2 = C_2 = 100$

the optimal economic retention quantity (i.e., z^*) decreases as the unit holding cost increases. In contrast to the approximate linear variation of z^* with the changes in lost sales penalty, we note that the rate of decrease in z^* is higher for lower values of h. We also observe that z^* is more sensitive to the changes in unit revenue from the secondary market for lower values of h. On the other hand, for higher values of h, the variation in z^* due to changes in r_2 is negligible.

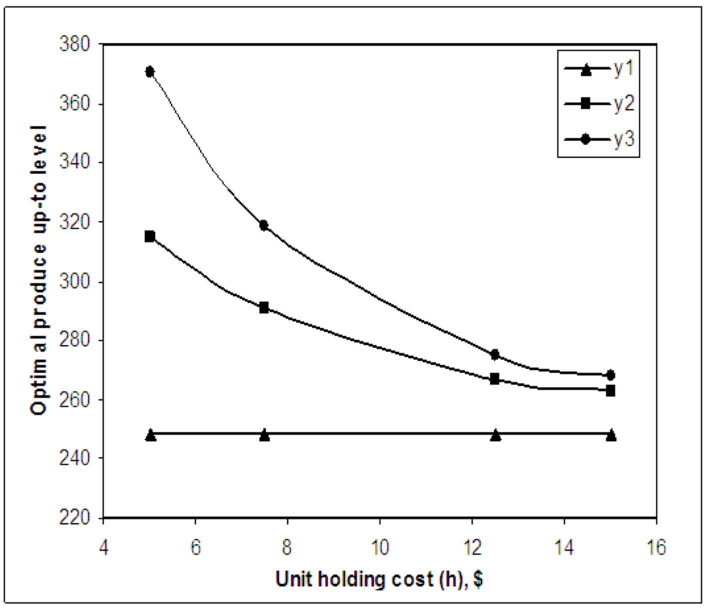

Fig. 4. Sensitivity of the optimal produce-up-to level with respect to the unit holding cost $r_1 = 100$, $p = 50$, $r_2 = 40$, $b = 60$, $E[X_1] = E[X_2] = E[X_3] = 100$, $C_1 = C_2 = C_2 = 100$

For the last period, $z_1^* = 0$ and the optimal produce-up-to level (i.e., y^*) is independent of the unit holding cost. For $t = 2$ and $t = 3$, the optimal produce-up-to level also decreases with the increase in the unit holding cost, and the rate of change decreases for higher values of h. If there are more periods to go until the end of the planning horizon, it is more profitable to produce more, and the rate of change in y^* with respect to changes in h is also higher, as shown in Figure 4.

As a result, we can conclude that higher the unit holding cost, the less sensitive are the optimal policy parameters with respect to the changes in the holding cost. In addition, the optimal policy parameters are more sensitive to the changes in h when there are more periods to go until the end of the planning horizon.

Figures 5 and 6 show the sensitivity of the maximum total expected profit for a two-period problem when the starting on-hand inventory is zero units. We observe that, the maximum total expected profit decreases, almost linearly, as the unit holding cost or the unit lost sales penalty increases. We also note that the maximum total expected profit is more sensitive to the changes in the lost sales penalty as compared to the changes in the unit holding cost.

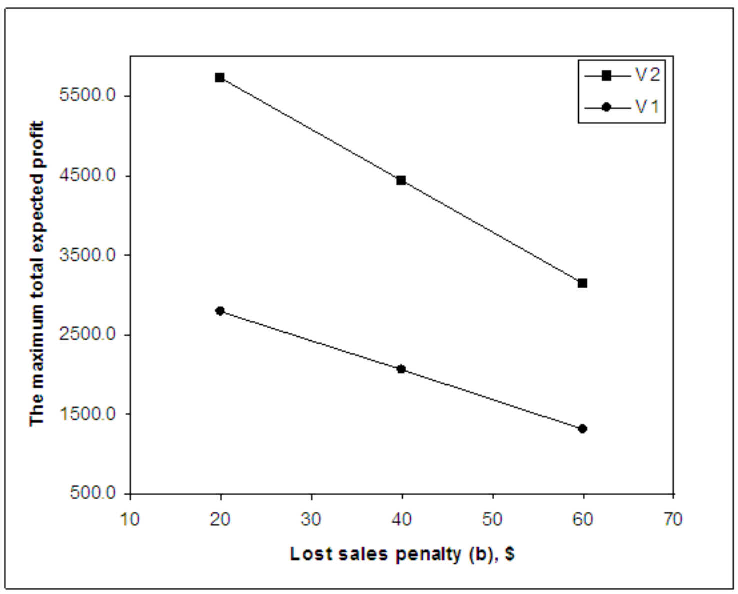

Fig. 5. Sensitivity of the maximum total expected profit with respect to the lost sales penalty when $r_1 = 100$, $r_2 = 50$, $p = 50$, $h = 12.5$, $E[X_1] = E[X_2] = E[X_3] = 100$, $C_1 = C_2 = C_2 = 100$

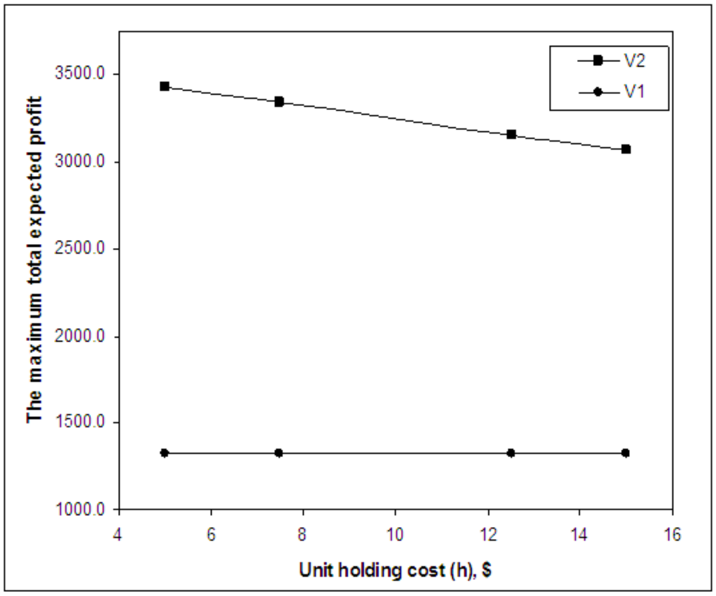

Fig. 6. Sensitivity of the maximum total expected profit with respect to the holding cost when $r_1 = 100$, $r_2 = 60$, $p = 50$, $b = 60$, $E[X_1] = E[X_2] = E[X_3] = 100$, $C_1 = C_2 = C_2 = 100$

B. Comparison with Other Policies

In this section, we compare the maximum total expected profit of our model with two other models. In the first model, no units are carried to the next period, i.e., $(y^*, z^* = 0)$. In the second model, no units are sold in the secondary market, i.e., $(y^*, z^* = y^*)$. Table I presents the maximum total expected profits for these three models for different sets of cost parameters for a two-period problem. We set the unit revenue earned from the primary market to \$100, unit revenue earned from the secondary market to \$60 and the unit cost of production to \$50. In all the experiments, we assume that the starting inventory is 100 units.

For parameter sets 1 and 2, the lost sales penalty is very high with respect to the holding cost. Hence, it is optimal to retain all the units, and we see no improvement in the maximum total profit with respect to the $(y^*, z^* = y^*)$ case but there is a

Table I. Optimal expected profits for the three models when $T = 2$

								(y^*, z^*)	$(y^*, z^* = 0)$		$(y^*, z^* = y)$	
No.	b	h	$E[X_2]$	$E[X_3]$	C_1	C_2	C_3	V_2	V_2	% Inc.	V_2	% Inc.
1	60	7.5	100	100	80	80	80	11143	9454	15.16	11143	0.0
2	60	5.0	100	100	100	100	100	12585	10968	12.85	12585	0.0
3	20	15.0	150	200	120	180	240	15824	15801	0.15	14692	7.2
4	20	15.0	100	100	120	120	120	13952	13928	0.18	13128	5.9
5	40	15.0	150	200	100	150	200	13628	13290	2.48	13124	3.7
6	60	15.0	100	100	120	120	120	12681	12280	3.16	12258	3.3
7	60	12.5	150	200	120	180	240	14207	13668	3.79	13735	3.3
8	60	15.0	150	200	100	150	200	12630	12630	5.08	12261	2.9
9	40	12.5	100	100	120	120	120	13396	13396	2.18	13021	2.8
10	40	15.0	100	100	100	100	100	12333	12333	2.91	12019	2.5
11	40	12.5	150	200	100	150	200	13772	13772	3.49	13445	2.4

significant improvement ($> 10\%$) with respect to the $(y^*, z^* = 0)$ case. Similarly, when the holding cost is high and the lost sales penalty is low, as in parameter sets 2 and 3, we observe about 6 to 7% improvement over the $(y^*, z^* = y^*)$ case, but there is no improvement when compared to the $(y^*, z^* = 0)$ case. When both the unit holding cost and the lost sales penalty are high, the proposed model shows about 3 to 4% improvement over both models. The improvement in maximum total expected profit as a result of using the proposed policy will be higher, if there are more periods to go before the end of the planning horizon.

We conclude that the proposed policy is more useful when the unit holding cost and the unit lost sales penalty are comparable to each other. In cases when one of these costs is very high and the other is very low, a simpler model such that $(y^*, z^* = y^*)$ or $(y^*, z^* = 0))$ can be used.

CHAPTER VI

CONCLUSIONS AND FURTHER RESEARCH

In this research, we considered production and inventory management decisions of a firm that sells its product in two markets during a finite planning horizon. We assumed that demand from the primary market is stochastic and there is ample demand in the secondary market. We addressed the problem by formulating it as a stochastic dynamic program to maximize the total expected profit, and characterized the structure of the optimal policy. For the one-period problem, we showed that it is optimal not to retain any units at the end of the period. The one-period problem then simplified to the classical newsvendor problem with non-negative starting inventory. The analysis of the multi-period problem was more complex owing to the fact that the joint concavity of one-period profit function no longer holds in the multi-period case. We showed that the total expected profit is a unimodal function of the produce-up-to level and the economic retention quantity. Furthermore, we established bounds for these optimal policy parameters. We performed a computational study to analyze the sensitivity of the optimal policy parameters with respect to the unit holding cost and the unit penalty of lost sales.

Clearly, the analysis in this paper makes assumptions to simplify the problem, such as the ample demand assumption in the secondary market. The assumptions allow the development of the optimal policy that is easy to characterize. The simple structure of this policy has the potential to provide insights into the optimal policies for more complicated problems.

Further research is necessary to fully characterize the structure of the optimal policy when demand from both primary and secondary markets are stochastic. It would be also interesting to characterize the structure of the optimal policy for situations with positive set-up cost as well as for situations with positive production lead time. Similarly, this could be extended to models which consider more than two market segments.

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VITA

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