PARTIALLY AVERAGED NAVIER-STOKES TURBULENCE MODELING: INVESTIGATION OF COMPUTATIONAL AND PHYSICAL CLOSURE ISSUES IN FLOW PAST A CIRCULAR CYLINDER

A Thesis
by
DASIA A. REYES

Submitted to the Office of Graduate Studies of Texas A&M University in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE

August 2008

Major Subject: Aerospace Engineering
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Approved by:
Chair of Committee, Sharath Girimaji
Committee Members, Rodney Bowersox
Prabir Daripa
Head of Department, Helen Reed

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ABSTRACT


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Dasia A. Reyes, B.S., Texas A&M University

Chair of Advisory Committee: Dr. Sharath Girimaji

Partially Averaged Navier-Stokes (PANS) turbulence method provides a closure model for any degree of velocity field filtering - ranging from completely resolved Direct Numerical Simulation (DNS) to completely averaged Reynolds Averaged Navier-Stokes (RANS) method. Preliminary investigations of PANS show promising results but there still exist computational and physical issues that must be addressed.

This study investigates the performance of the PANS method for flow past a cylinder at a Reynolds number of 140,000. The cylinder flow is a benchmark flow problem for which there are significant experimental results available for validation of PANS. First, we examine if RANS convergence criteria and discretization schemes - which are meant for robust, nearly steady-state calculations - are adequate for PANS, which is inherently unsteady and may contain delicate flow features. For the range of $f_k$ values tested here, it is determined that the standard RANS residual value and the $2^{nd}$ order spatial discretization scheme are appropriate for PANS. The physical closure investigations begin with the validation of turbulent transport models: the Zero Transport Model, the Maximum Transport Model and the Boundary Layer Transport Model. The implementation of the PANS $k_u - \omega_u$ model is also performed and compared against the standard PANS $k_u - \epsilon_u$ model. All these studies yield interesting insights into the PANS models. This study concludes with an investigation of a low Reynolds number correction for the PANS $k_u - \omega_u$ model which yields excellent
improvement.
To my mother and father, I could not have done this without you.
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CHAPTER I

INTRODUCTION

One of the most challenging fields in aerodynamics is the study and treatment of turbulent flows. The governing Navier Stokes equations for turbulence are well known but the difficulty arises in computing them at high Reynolds numbers. The difficulty is due to rapid increase, with Reynolds numbers, of the time and length scales of motion. The scales can be broadly classified as large (dynamically active), intermediate (inertial) and small (dissipative) scales. Directly solving the Navier-Stokes equations is only possible for relatively simple problems at low Reynolds numbers with the use of Direct Numerical Simulations (DNS). Meaningful problems with complex geometries and at high Reynolds numbers are not computationally feasible with DNS. Appropriate turbulence models that significantly reduce the number of scales of motion must be employed for these practical flow computations.

Turbulence computational approaches can be classified on the basis of the scales they resolve. In Large Eddy Simulations (LES) all of the large and most of the inertial scales are resolved. Typically, even LES is too computationally expensive for practical problems. Other turbulence computational approaches further limit the scales they resolve. In Reynolds Averaged Navier-Stokes (RANS) all fluctuating scales are modeled and this results in a computationally inexpensive approach that solves only for the mean flow. For many years these approaches, LES and RANS, were the only two possible computational approaches available. In recent years a need has surfaced for treatments that can overcome the inadequacies of LES and RANS approaches and provide increased resolution of the flow field over RANS without a

The journal model for this thesis is *Journal of Fluid Mechanics.*
dramatic increase in computation time as with LES. Such models are called hybrid or bridging methods. The Partially Averaged Navier-Stokes (PANS) is a bridging model that is the subject of the present study.

In LES methodology, the flow field is separated into resolved and unresolved parts. The resolved flow includes all of the dynamically active scales and most of the inertial scales. The direct computation of a large portion of the fluctuations enhances the fidelity of the approach (Pope 2000). The grid must be fine enough to capture the smallest scale the user desires to be resolved directly. This is a limiting factor that makes solving problems utilizing LES computationally expensive. The remaining unresolved portion of the flow is typically modeled using a simple (e.g. Smagorinsky) model.

The most widely used turbulence approach within industry is the RANS method. In the RANS methodology, the entire flow field is decomposed into mean and fluctuating terms. By applying an averaging operator to the Navier-Stokes equations, the RANS equations are derived. The effect of the fluctuating field on the mean velocity evolutions manifests via the Reynolds stress term. The evolution equation for the Reynolds stresses can then be developed from the full Navier-Stokes equations. The closure of the RANS equation is possible at several levels. Two-equation RANS models that are quite popular are the $k - \varepsilon$ and $k - \omega$ models which are discussed in detail in subsequent chapters. RANS models are quite popular since they can be implemented inexpensively. These RANS models, however, are inherently limited in physical fidelity as they do not possess a sufficiently detailed description of the fluctuating field. There is a desire for more sophisticated approaches that can accurately predict turbulence for more complex problems without a severe increase in computation time.

Recent developments in the field of turbulence methods involve hybrid turbu-
lence models. The models in this field include Detached Eddy Simulation (DES) and hybrid RANS/LES methods. A hybrid model can invoke different models for different regions of the flow. For example, DES uses the RANS method in the boundary layer and LES in the separated regions of the flow (Travin, Shur, Strelets & Spalart 1999). These methods show improvement over RANS models but are not as computationally expensive as LES. The bridging approach can be considered to be an important subset of the hybrid methods. The bridging methods allow for gradual change in the resolution, going from RANS at one extreme to DNS at the other. The implementation and validation of the Partially Averaged Navier-Stokes (PANS) bridging method is the focus of this study.

A. PANS Theory

The PANS bridging method (Grimaji 2006) provides a smooth transition from DNS to RANS based upon a user specified filter parameter. The flow field is decomposed into resolved and residual terms as opposed to the mean and fluctuating terms of RANS methodology. The difference between filtering and averaging the flow field is more apparent in Figure 1. The blue line represents the PANS resolution of this variable and the green line is the RANS resolution of the variable. It is obvious that a PANS simulation can be expected to be more accurate as key fluctuations are considered in its resolved part of the flow. The difference between PANS and LES methodologies is also apparent in Figure 1. PANS is purported for resolving significantly lesser number of scales. The control parameters of the PANS model are resolved-to-unresolved kinetic energy and resolved-to-unresolved dissipation. The closure equations are developed in a manner similar to the RANS model equations for an arbitrary filter (Germano 1992). The PANS model are derived systematically
from corresponding parent RANS models. In this study, the parent RANS models used for simulations are $k - \varepsilon$ and $k - \omega$.

B. Proposal of Research

The purpose of this study is to continue evaluation and testing of PANS model development by investigating the numerical robustness and physical closure issues associated with the model. The simulations will be performed on flow past a circular cylinder at a Reynolds number of 140,000 using FLUENT CFD software. The cylinder problem is a benchmark flow with significant complexities for which substantial experimental data is available to be used to validate the PANS model.

The issues to be considered in this study are:

- Convergence criterion effects on PANS accuracy.
• Dependence of PANS accuracy on numerical scheme.

• The PANS turbulence transport closure model validation.

• $k_u - \varepsilon_u$ vs. $k_u - \omega_u$ implementation of PANS.

• Investigation of low Reynolds number $C_\mu$ correction.

C. Thesis Outline

The following chapter develops the PANS methodology beginning with the derivation of the PANS governing equations. The chapter then follows with the development of the PANS $k_u - \varepsilon_u$ and $k_u - \omega_u$ models developed from the parent RANS models. Chapter III develops the motivation for the issues addressed in the present study and the anticipated impact of the work. The results of these investigations are presented in Chapter IV. The study concludes with a summary of observations and recommendations for future work in Chapter V and VI, respectively.
CHAPTER II

PANS METHODOLOGY

The PANS governing equations are derived in detail in Girimaji (2006), Frendi & Girimaji (2005) and Girimaji & Abdol-Hamid (2005). The physical motivation and closure assumption are clearly discussed in these references. In this chapter, the key mathematical steps are presented with explanations as necessary. The development of the PANS $k_u - \varepsilon_u$ and $k_u - \omega_u$ models are presented here.

A. PANS Governing Equations

The governing PANS equations are derived from the incompressible Navier-Stokes conservation equations:

\[
\frac{\partial V_k}{\partial x_k} = 0 \quad \text{(mass conservation)} \tag{2.1}
\]

\[
\frac{\partial V_i}{\partial t} + V_k \frac{\partial V_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 V_i}{\partial x_k \partial x_k} \quad \text{(momentum conservation)} \tag{2.2}
\]

For incompressible flow the conservation of mass can be restated as Poisson’s equation of pressure:

\[
\frac{\partial^2 p}{\partial x_i \partial x_i} = -\frac{\partial V_i}{\partial x_j} \frac{\partial V_j}{\partial x_i} \tag{2.3}
\]

Similar to the development of the RANS and LES methodology, the variables of the flow field are decomposed into resolved and residual portions as follows.

\[
V_i = U_i + v_i' \tag{2.4}
\]

\[
p = P + p_u
\]

The cut-off can be arbitrary but the filter must commute with temporal and spatial differentiation (Germano 1992). When the filter is applied, we get $\langle V_i \rangle = U_i$ and
$\langle p \rangle = P$. However, this decomposition is unlike that of the statistical one of RANS, in that the filter is not an average. For example, with an arbitrary cut-off, the filtered value of the unresolved portion need not vanish.

$$\langle v'_i \rangle \neq 0 \quad (2.5)$$

When the filter is applied to Equations 2.1 - 2.3, the filtered Navier-Stokes (PANS) equations are obtained:

$$\frac{\partial U_k}{\partial x_k} = 0 \quad (2.6)$$

$$\frac{\partial U_i}{\partial t} + \frac{\partial \langle V_k V_i \rangle}{\partial x_k} = - \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} \quad (2.7)$$

$$\frac{\partial^2 P}{\partial x_i \partial x_i} = - \langle \frac{\partial V_i}{\partial x_j} \frac{\partial V_i}{\partial x_i} \rangle \quad (2.8)$$

Since $\langle V_i V_j \rangle$ is not yet known, a new term is now introduced into the Navier-Stokes equations. This term can be represented by the generalized central moment which is similar to the Reynolds stress and is defined as follows.

$$\tau(f, g) = \langle fg \rangle - \langle f \rangle \langle g \rangle \quad (2.9)$$

Applying this definition to Equation 2.7 and 2.8 results in the following:

$$\frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} = - \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} - \frac{\partial \tau(V_i, V_k)}{\partial x_k}; \quad (2.10)$$

$$- \frac{\partial^2 P}{\partial x_i \partial x_i} = \frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} + \frac{\partial^2 \tau(V_i, V_j)}{\partial x_j \partial x_i} \quad (2.11)$$

To develop a closure equation for the generalized central moment, also called sub-filter-stress (SFS), the moment of Equation 2.2 is taken by multiplying it by $V_j$. A similar equation can be obtained by transposing $i$ and $j$ indices. Then adding this
equation with its transposed counterpart we can write:

\[ V_j \frac{\partial V_i}{\partial t} + V_i \frac{\partial V_j}{\partial t} + V_j \frac{\partial V_i V_k}{\partial x_k} + V_i \frac{\partial V_j V_k}{\partial x_k} = -V_j \frac{\partial p}{\partial x_i} - V_i \frac{\partial p}{\partial x_j} + \nu V_j \frac{\partial^2 V_i}{\partial x_k \partial x_k} \]

\[ + \nu V_i \frac{\partial^2 V_j}{\partial x_j \partial x_i} \]  \hspace{1cm} (2.12)

By combining terms, Equation 2.12 is then simplified to the following expression.

\[ \frac{\partial V_i V_j}{\partial t} + \frac{\partial V_i V_j V_k}{\partial x_k} = -V_j \frac{\partial p}{\partial x_i} - V_i \frac{\partial p}{\partial x_j} + \nu V_i \frac{\partial^2 V_j}{\partial x_j \partial x_i} + \nu V_i \frac{\partial^2 V_j}{\partial x_j \partial x_i} \]  \hspace{1cm} (2.13)

Equation 2.13 is restated in index notation to simplify the subsequent steps of the derivation.

\[ (V_i V_j)_t + (V_i V_j V_k)_k = -[V_j p \delta_{ik} + V_i p \delta_{jk} - \nu (V_i V_j)_k]_k + 2p S_{ij} - 2\nu V_{i,k} V_{j,k} \]  \hspace{1cm} (2.14)

The rate of strain is defined as \( S_{ij} = \frac{1}{2} (V_i, V_j) \).

Applying the filter to the above equation will initiate the development of the SFS term.

\[ (V_i V_j)_t + (V_i V_j V_k)_k = -[(V_j p) \delta_{ik} + (V_i p) \delta_{jk} - \nu (V_i V_j)_k]_k \]
\[ + 2\langle p S_{ij} \rangle - 2\nu \langle V_{i,k} V_{j,k} \rangle \]  \hspace{1cm} (2.15)

Introducing the definitions of Equation 2.9 to Equation 2.15 results in the modifications of the equation’s terms.

\[ \langle V_i V_j \rangle_t = \tau (V_i, V_j)_t + (U_i U_j)_t = \tau (V_i, V_j)_t + U_{i,t} U_j + U_i U_{j,t} \]
\[ \langle V_i V_j V_k \rangle_k = [U_i \tau (V_j, V_k) + U_j \tau (V_i, V_k) + U_k \tau (V_i, V_j) + U_i U_j U_k]_k \]
\[ + \tau (V_i, V_j, V_k)_k \]
\[ \langle V_j, p \rangle \delta_{ik} = \tau (V_j, p) \delta_{ik} + U_j P \delta_{ik} \]
\[ \langle V_i, p \rangle \delta_{jk} = \tau (V_i, p) \delta_{jk} + U_i P \delta_{jk} \]
\[
\langle V_i V_j \rangle_k = \tau(V_i V_j)_k + (U_i U_j)_k = \tau(V_i, V_j)_k + U_{i,k} U_j + U_i U_{j,k}
\]

Substituting these above terms back into Equation 2.15 yields the following expansion of the equation.

\[
\begin{align*}
\tau(V_i, V_j)_t + U_{i,t} U_j + U_{i,j,t} &+ \tau(V_i, V_j)_k \\
&+ [U_i \tau(V_j, V_k) - U_j \tau(V_i, V_k) - U_k \tau(V_i, V_j) - U_{i,j} U_k]_k \\
&= -\tau(V_j, p)\delta_{ik} + U_{j} P\delta_{ik} - \tau(V_i, p)\delta_{jk} \\
&+ U_{i} P\delta_{j,k} + \nu(\tau(V_i, V_j)_k + U_{i,k} U_j + U_i U_{j,k})
\end{align*}
\]

(2.16)

Simplifying the above equation results in the evolution equation for the SFS.

\[
\begin{align*}
\tau(V_i, V_j)_t + U_k \tau(V_i, V_j) &= -[\tau(V_i, V_j)_k - \nu \tau(V_i, V_j)_k + \tau(V_i, p)\delta_{jk} + \tau(V_i, p)\delta_{jk}]_k \\
&- 2\nu \tau(V_{i,k} V_{j,k}) + 2\tau(p, S_{ij}) - U_{j,k} \tau(V_i, V_k) \\
&- U_{i,k} \tau(V_j, V_k)
\end{align*}
\]

(2.17)

The final form of the evolution equation for the SFS term is represented as follows (Girimaji 2006).

\[
\frac{\partial \tau(V_i, V_j)}{\partial t} + U_k \frac{\partial \tau(V_i, V_j)}{\partial x_k} = P_{ij} + \phi_{ij} - D_{ij} + T_{ij}
\]

(2.18)

with,

\[
\begin{align*}
P_{ij} &= -U_{j,k} \tau(V_i, V_k) - U_{i,k} \tau(V_j, V_k) \\
\phi_{ij} &= 2 \tau(p, S_{ij}) \\
D_{ij} &= 2 \nu \tau(V_{i,k}, V_{j,k}) \\
T_{ij} &= -[\tau(V_i, V_j, V_k)_k - \nu \tau(V_i, V_j)_k + \tau(V_i, p)\delta_{ik} + \tau(V_i, p)\delta_{jk}]_k
\end{align*}
\]
The terms of Equation 2.18 are defined as follows; $P_{ij}$ is production, $\phi_{ij}$ is pressure-correlation, $D_{ij}$ is dissipation, and $T_{ij}$ is transport. The transport term is responsible for transporting energy to different locations. The production term generates new scales while the $D_{ij}$ term dissipates energy at the smallest scales. The pressure correlation term redistributes energy between the various SFS components (Frendi & Girimaji 2005).

If the trace of the subfilter stress term is taken, the evolution equation of the unresolved kinetic energy equation emerges.

$$\frac{\partial k_u}{\partial t} + U_j \frac{\partial k_u}{\partial x_j} = P_u - \epsilon_u + T_{ku}$$ (2.19)

This is the generalized form of the unresolved kinetic energy evolution equation with $k_u = \frac{1}{2} \tau (V_i, V_j)$ where $P_u = \frac{1}{2} \tau (V_i, V_j) \frac{\partial U_i}{\partial x_j}$. In PANS closure at the two-equation level, the Boussinesq approximation is utilized to relate the SFS term to the resolved velocity field. This one point closure is defined as follows.

$$\tau (V_i V_j) = -\nu_u S_{ij} + \frac{2}{3} k_u S_{ij}$$ (2.20)

Subject to the Boussinesq constitutive relation we need closure expressions for $v_u$ and $k_u$ to close the PANS equations. These closures are obtained from the parents RANS models.

B. PANS $k_u - \varepsilon_u$ Model

Important details of the derivation of the PANS $k_u - \varepsilon_u$ model are found here. A full description can be found in Girimaji (2006). We start by utilizing the parent two-equation RANS $k - \varepsilon$ model:

$$\nu_t = C_{\mu} \frac{k^2}{\varepsilon}$$ (2.21)
\[
\frac{\partial k}{\partial t} + \mathbf{U}_j \frac{\partial k}{\partial x_j} = P - \epsilon + \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_j} \right)
\]  
(2.22)

\[
\frac{\partial \epsilon}{\partial t} + \mathbf{U}_j \frac{\partial \epsilon}{\partial x_j} = C_{\epsilon 1} \frac{P \epsilon}{k} - C_{\epsilon 2} \frac{\epsilon^2}{k} + \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right)
\]  
(2.23)

The variables of the RANS equations are listed below in Table I.

<table>
<thead>
<tr>
<th></th>
<th>Table I. RANS Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean kinetic Energy:</td>
<td>( k )</td>
</tr>
<tr>
<td>Mean Dissipation:</td>
<td>( \epsilon )</td>
</tr>
<tr>
<td>Mean Velocity:</td>
<td>( \overline{U}_j )</td>
</tr>
<tr>
<td>Mean Pressure:</td>
<td>( P )</td>
</tr>
<tr>
<td>Turbulent Prandtl Number:</td>
<td>( \sigma_k )</td>
</tr>
<tr>
<td>Turbulent Viscosity:</td>
<td>( \nu_t )</td>
</tr>
<tr>
<td>Turbulent Prandtl Number:</td>
<td>( \sigma_\epsilon )</td>
</tr>
</tbody>
</table>

The purpose of the PANS modeling is to provide as SFS closure for any resolution between RANS and DNS. This is accomplished by correcting the model coefficients of the parent RANS model. The corrections must be sensitive to the user defined control filter parameters. These parameters are now introduced as the ratio of resolved-to-unresolved kinetic energy and resolved-to-unresolved dissipation or \( f_k \) and \( f_\epsilon \), respectively (Girimaji, Jeong & Srinivasan 2006):

\[
f_k = \frac{k_u}{k}
\]  
(2.24)

\[
f_\epsilon = \frac{\epsilon_u}{\epsilon}
\]  
(2.25)

If \( f_k \) and \( f_\epsilon \) are constant in space and time (commuting filter) we can write:

\[
\frac{\partial k_u}{\partial t} + \mathbf{U}_j \frac{\partial k_u}{\partial x_j} = f_k \left( \frac{\partial k}{\partial t} + \mathbf{U}_j \frac{\partial k}{\partial x_j} \right)
\]  
(2.26)
The following equations provide a step by step derivation for determining the relationship between the RANS model and the unresolved kinetic energy equation.

\[ \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = f_k \left( \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} \right) + (U_j - U_j) \frac{\partial k}{\partial x_j} \tag{2.27} \]

\[ \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = f_k \left( P - \epsilon + \frac{\nu}{\sigma_k} \frac{\partial k}{\partial x_j} \right) + (U_j - U_j) \frac{\partial k}{\partial x_j} \tag{2.28} \]

\[ P_u - \epsilon_u + T_{ku} = f_k \left( P - \epsilon + \frac{\nu}{\sigma_k} \frac{\partial k}{\partial x_j} \right) + (U_j - U_j) \frac{\partial k}{\partial x_j} \tag{2.29} \]

It is now possible to relate the terms of Equation 2.17 to the RANS model.

\[ P_u - \epsilon_u = f_k (P - \epsilon) \tag{2.30} \]

\[ T_{ku} = f_k \left( \frac{\partial}{\partial x_j} \left( \frac{\nu}{\sigma_k} \frac{\partial k}{\partial x_j} \right) \right) + (U_j - U_j) \frac{\partial k}{\partial x_j} \tag{2.31} \]

\( T_{ku} \) is the transport of unresolved kinetic energy by the unresolved velocity fluctuations. The development of a transport model will be discussed in a later section of this chapter.

Similar to the development of the PANS kinetic energy equation, the unresolved dissipation can be related to the RANS dissipation as follows.

\[ \frac{\partial \epsilon_u}{\partial t} + U_j \frac{\partial \epsilon_u}{\partial x_j} = f_\epsilon \left( \frac{\partial \epsilon}{\partial t} + U_j \frac{\partial \epsilon}{\partial x_j} \right) \]

\[ = f_\epsilon \left( C_{e1} \frac{P \epsilon}{k} - C_{e2} \frac{\epsilon^2}{k} + \frac{\nu}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right) \tag{2.32} \]

Further manipulation of Equation 2.35 results in the following expressions.

\[ \frac{\partial \epsilon_u}{\partial t} + U_j \frac{\partial \epsilon_u}{\partial x_j} = C_{e1} f_k \frac{P \epsilon_u}{k_u} - C_{e2} \frac{f_k \epsilon_u^2}{k_u} + \frac{\nu}{\sigma_\epsilon} \frac{\partial \epsilon_u}{\partial x_j} + (U_j - U_j) \frac{\partial \epsilon_u}{\partial x_j} \tag{2.33} \]

\[ \frac{\partial \epsilon_u}{\partial t} + U_j \frac{\partial \epsilon_u}{\partial x_j} = C_{e1} f_k \left( \frac{P_u - \epsilon_u (f_\epsilon - f_k)}{f_k} \right) \frac{\epsilon_u}{k_u} - C_{e2} \frac{f_k \epsilon_u^2}{f_\epsilon \epsilon_k} + T_{\epsilon u} \tag{2.34} \]
\[
\frac{\partial \epsilon_u}{\partial t} + U_j \frac{\partial \epsilon_u}{\partial x_j} = C_{\epsilon 1} \frac{P_u \epsilon_u}{k_u} - C_{\epsilon 2} \frac{\epsilon_u^2}{k_u} + T_{\epsilon u} \tag{2.35}
\]

where:

\[
C_{\epsilon 2} = C_{\epsilon 1} + \frac{f_k}{f_\epsilon} (C_{\epsilon 2} - C_{\epsilon 1})
\]

\(T_{\epsilon u}\) is the transportation of the unresolved dissipation by the unresolved scales of motion. The closure of \(T_{\epsilon u}\) will be discussed later along with \(T_{ku}\).

C. PANS \(k_u - \omega_u\) Model

This section details the derivation of the PANS \(k_u - \omega_u\) from the parent RANS model of Wilcox (1991). Important details of the derivation of the PANS \(k_u - \omega_u\) model are found here but a full description can be found in Lakshmipathy & Girimaji (2006).

The parent RANS \(k - \omega\) model is given below.

\[
\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = P - \beta^* k \omega + \frac{\partial}{\partial x_j} \left( \nu_t \frac{k}{\sigma_k \partial x_j} \right) \tag{2.36}
\]

\[
\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} = \alpha \frac{P \omega}{k} - \beta \omega^2 + \frac{\partial}{\partial x_j} \left( \nu_t \frac{\partial \omega}{\sigma_\omega \partial x_j} \right) \tag{2.37}
\]

where \(\omega\) is the turbulent frequency and is defined as follows:

\[
\omega = \frac{\varepsilon}{\beta^* k} \tag{2.38}
\]

In the above equation \(\beta^*\), \(\beta\) and \(\alpha\) are model coefficients.

As in the \(k_u - \varepsilon_u\) case, the unresolved kinetic energy can be related to the resolved kinetic energy with the use of the filter parameters of Equations 2.24 and 2.25.

\[
\frac{\partial k_{ku}}{\partial t} + U_j \frac{\partial k_{ku}}{\partial x_j} = f_k \left( \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} \right) \tag{2.39}
\]
The following derivations relate the evolution equation for the unresolved kinetic energy to the parent RANS model.

\[
\frac{\partial k_u}{\partial t} + U_j \frac{\partial k_u}{\partial x_j} = f_k \left( \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} \right) + (U_j - U_j) \frac{\partial k_u}{\partial x_j} \tag{2.40}
\]

\[
\frac{\partial k_u}{\partial t} + U_j \frac{\partial k_u}{\partial x_j} = f_k \left( P - \beta^* k \omega + \frac{\partial}{\partial x_j} \left( \frac{\nu_t k}{\sigma_k} \frac{\partial x_j}{\partial x_j} \right) \right) + (U_j - U_j) \frac{\partial k_u}{\partial x_j} \tag{2.41}
\]

\[
P_u - \beta^* k_u \omega_u + T_{ku} = f_k \left( P - \beta^* k \omega + \frac{\partial}{\partial x_j} \left( \frac{\nu_t k}{\sigma_k} \frac{\partial x_j}{\partial x_j} \right) \right) + (U_j - U_j) \frac{\partial k_u}{\partial x_j} \tag{2.42}
\]

From the above derivation, the following derivations can be developed.

\[
P_u - \beta^* k_u \omega_u = f_k \left( P - \beta^* k \omega \right) \tag{2.43}
\]

\[
P = \frac{1}{f_k} \left( P_u - \beta^* k_u \omega_u \right) + \frac{\beta^* k_u \omega_u}{f_k f_k} \tag{2.44}
\]

Similar to the development of the model equations of the previous section, it is necessary to find a closure for the \( T_{ku} \) term.

\[
T_{ku} = f_k \left( \frac{\partial}{\partial x_j} \left( \frac{\nu_t k}{\sigma_k} \frac{\partial x_j}{\partial x_j} \right) \right) + (U_j - U_j) \frac{\partial k_u}{\partial x_j} \tag{2.45}
\]

In Equation 2.45 the \( (U_j - U_j) \frac{\partial k_u}{\partial x_j} \) is the only term that still requires closure. For PANS \( k_u - \omega_u \) it is necessary to introduce a new filter parameter which is the unresolved-to-resolved turbulent frequency which is shown in equation form below

\[
\omega_u = \frac{\omega_u}{\omega} \tag{2.46}
\]

The unresolved turbulent frequency can be defined as follows.

\[
\omega_u = \frac{\varepsilon_u}{\beta^* k_u} \tag{2.47}
\]
Equation 2.46 can be used to define the LHS of Equation 2.37, in terms of the filter parameters.

\[
\frac{\partial \omega_u}{\partial t} + U_j \frac{\partial \omega_u}{\partial x_j} = f_\omega \left( \frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} \right)
\]

\[
= f_\omega \left( \frac{P_\omega}{k} - \frac{\beta \omega^2}{f_\omega} + \frac{\partial}{\partial x_j} \left( \frac{\nu_t \partial \omega}{\sigma_\omega \partial x_j} \right) \right)
\]

(2.48)

Further manipulation results in the following expressions.

\[
\frac{\partial \omega_u}{\partial t} + U_j \frac{\partial \omega_u}{\partial x_j} = \alpha \frac{P_\omega u}{k_u} f_k - \frac{\beta \omega^2}{f_\omega} + \frac{\partial}{\partial x_j} \left( \frac{\nu_t \partial \omega_u}{\sigma_\omega \partial x_j} \right) + (U_j - U_j) \frac{\partial \omega_u}{\partial x_j}
\]

(2.49)

\[
\frac{\partial \omega_u}{\partial t} + U_j \frac{\partial \omega_u}{\partial x_j} = \alpha \left( P_u - \beta^* k_u \omega_u + \beta^* k_u \omega_u \right) \frac{\omega_u}{k_u}
\]

\[- \frac{\beta \omega^2}{f_\omega} + \frac{\partial}{\partial x_j} \left( \frac{\nu_t \partial \omega_u}{\sigma_\omega \partial x_j} \right) + (U_j - U_j) \frac{\partial \omega_u}{\partial x_j}
\]

(2.50)

\[
\frac{\partial \omega_u}{\partial t} + U_j \frac{\partial \omega_u}{\partial x_j} = \alpha \frac{P_\omega u}{k_u} - \alpha \beta^* \omega^2_u + \alpha \beta^* \frac{\omega^2_u}{f_\omega}
\]

\[- \frac{\beta \omega^2}{f_\omega} + \frac{\partial}{\partial x_j} \left( \frac{\nu_t \partial \omega_u}{\sigma_\omega \partial x_j} \right) + (U_j - U_j) \frac{\partial \omega_u}{\partial x_j}
\]

(2.51)

Now we need to a closure model for the \( T_{\omega u} \), the transport of unresolved turbulent frequency due to the unresolved velocity fluctuations. The model of this term will be discussed in the next section.

D. Transport Modeling of \( T_{ku} \), \( T_{\varepsilon u} \) and \( T_{\omega u} \)

As seen in the previous sections, the closure modeling of turbulent transport is one of the key unsettled issues of PANS. Currently, there are three postulates for closing this term. We will describe each one of these models now. Of the three models two were first developed in Murthi (2004) and the third one is original to this work. We
choose to describe the closure of the transport term for a variable, \( Q \). The validity of each transport model is then discussed for \( k, \varepsilon \) and \( \omega \). Further details can be found in Murthi (2004). The transport of the unresolved variable \( Q \) due to the unresolved velocity fluctuations or \( T_{qu} \) is defined as follows:

\[
T_{qu} = \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_q} \frac{\partial Q_u}{\partial x_j} \right) + \left( U_j - \overline{U}_j \right) \frac{\partial Q_u}{\partial x_j} \tag{2.52}
\]

In Equation 2.52 the transport of the unresolved variable due to the resolved velocity fluctuations or, \( \left( U_j - \overline{U}_j \right) \frac{\partial Q_u}{\partial x_j} \), first requires closure.

1. Zero Transport Model

The starting point for this model is the postulate that the resolved fluctuating field does not contribute at all toward the turbulent transport of the unresolved field. This is due to the argument that the resolved and unresolved fluctuating parts are of different length-scales and hence poorly correlated. If this postulate is correct, then we must have:

\[
\left( U_j - \overline{U}_j \right) \frac{\partial Q_u}{\partial x_j} = 0 \tag{2.53}
\]

This leads to:

\[
T_{qu} = \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_q} \frac{\partial Q_u}{\partial x_j} \right) \tag{2.54}
\]

This approach is the Zero Transport Model where the transport due to unresolved velocity fluctuations is zero.

2. Maximum Transport Model

The underlying postulate for this model is that the resolved and unresolved fluctuating fields contribute proportionally to the turbulent transport of \( Q_u \). The turbulent transport due to resolved fluctuations is taken to be proportional to the eddy viscosity
\[ \nu_r = \nu_t - \nu_u \]  \hspace{1cm} (2.55)

The turbulent transport due to resolved fluctuations is given by:

\[ \left( \overline{U_j} - U_j \right) \frac{\partial Q_u}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\nu_r \partial Q_u}{\sigma_q \partial x_j} \right) \]  \hspace{1cm} (2.56)

The assumption from Equation 2.56 and the definition from Equation 2.55 combine to reduce Equation 2.52 to the following:

\[ T_{qu} = \frac{\partial}{\partial x_j} \left( \frac{\nu_t \partial Q_u}{\sigma_q \partial x_j} \right) - \frac{\partial}{\partial x_j} \left( \frac{\nu_t - \nu_u \partial Q_u}{\sigma_q \partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial Q_u}{\sigma_q \partial x_j} \right) \]  \hspace{1cm} (2.57)

From this relation we can relate the unresolved to resolved turbulent Prandtl numbers as shown below.

\[ \sigma_{qu} = \sigma_q \]  \hspace{1cm} (2.58)

This approach to closing the transport term is the Maximum Transport Model (MTM).

Now we apply these models to the transport of the kinetic energy defined as follows:

\[ T_{ku} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u f_\epsilon \partial k_u}{\sigma_k f_k^2 \partial x_j} \right) + \left( \overline{U_j} - \overline{U_j} \right) \frac{\partial k_u}{\partial x_j} \]  \hspace{1cm} (2.59)

If we utilize the Zero Transport Model then the transport reduces to:

\[ T_{ku} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u f_\epsilon \partial k_u}{\sigma_k f_k^2 \partial x_j} \right) \]  \hspace{1cm} (2.60)

\[ T_{ku} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u f_\epsilon \partial k_u}{\sigma_k f_k^2 \partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial k_u}{\sigma_{ku} \partial x_j} \right) \]  \hspace{1cm} (2.61)

From this approximation, the kinetic energy Prandtl number is defined as follows.

\[ \sigma_{ku} = \sigma_k \frac{f_k^2}{f_\epsilon} \]  \hspace{1cm} (2.62)
If we approach the closure of the dissipation equation using the ZTM model, we find the following:

\[
T_{\varepsilon u} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u f_k \partial \varepsilon_u}{\sigma_k f_k^2 \partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \varepsilon_u}{\sigma_{ku} \partial x_j} \right) \tag{2.63}
\]

From this approximation, the dissipation Prandtl number is defined as follows.

\[
\sigma_{\varepsilon u} = \sigma_{\varepsilon} \frac{f_k^2}{f_\varepsilon} \tag{2.64}
\]

If we approach the closure of the turbulent frequency equation using the ZTM model, we find the following:

\[
T_\omega = \frac{\partial}{\partial x_j} \left( \frac{\nu_u f_\omega \partial \omega_u}{\sigma_\omega f_k \partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \varepsilon_u}{\sigma_{\omega u} \partial x_j} \right) \tag{2.65}
\]

From this approximation, the turbulent frequency Prandtl number is defined as follows.

\[
\sigma_{\omega u} = \sigma_\omega \frac{f_k}{f_\omega} \tag{2.66}
\]

If the MTM is applied to the kinetic energy equation, we get the following expression.

\[
(U_j - U_j) \frac{\partial k_u}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial k_u}{\sigma_{ku} \partial x_j} \right) \tag{2.67}
\]

The unresolved kinetic energy Prandtl number will therefore be equal to its RANS counterpart.

\[
\sigma_{ku} = \sigma_k \tag{2.68}
\]

Again, if the MTM is applied to the dissipation equation we get the following expression:

\[
(U_j - U_j) \frac{\partial \varepsilon_u}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \varepsilon_u}{\sigma_{\varepsilon u} \partial x_j} \right) \tag{2.69}
\]

The unresolved dissipation Prandtl number will therefore be equal to its RANS counterpart.

\[
\sigma_{\varepsilon u} = \sigma_\varepsilon \tag{2.70}
\]
Finally, if the MTM is applied to the turbulent frequency equation we get the following expression:

\[
(U_j - U_j) \frac{\partial \omega_u}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \omega_u}{\sigma_\omega \partial x_j} \right)
\]  

(2.71)

The unresolved turbulent frequency Prandtl number will therefore be equal to its RANS counterpart.

\[ \sigma_{\omega u} = \sigma_\omega \]  

(2.72)

3. Boundary Layer Transport Model

This model is meant exclusively for \( \sigma_{\varepsilon u} \) and is motivated by the requirement of logarithmic scaling of the resolved flow in the log-layer of a boundary layer. It is well known that within the log-layer RANS production and dissipation are nearly identical scaling as

\[ P = \varepsilon = \frac{u_r^2}{y}, \]  

(2.73)

where \( u_r \) is friction velocity and \( y \) is the distance from the wall. This leads to kinetic energy being nearly constant in the log-layer. The dissipation equation reduces to:

\[ 0 = \frac{d}{dy} \left( \frac{\nu_t}{\sigma_k} \frac{dk}{dy} \right) + P - \varepsilon \]  

(2.74)

\[ 0 = \frac{d}{dy} \left( \frac{\nu_t}{\sigma_\varepsilon} \frac{d\varepsilon}{dy} \right) + C_{\varepsilon 1} \frac{P\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \]  

(2.75)

These equations can reduce to the following relationship between \( k \) and \( \varepsilon \):

\[ \varepsilon = \frac{C_{\varepsilon 1}^{1/2} k u_r}{\kappa y} \]  

(2.76)

with

\[ \kappa^2 = \sigma_\varepsilon C_{\mu}^{1/2} k (C_{\varepsilon 2} - C_{\varepsilon 1}) \]  

(2.77)
and $u_r$ is a velocity length scale.

If Equation 2.77 is put in terms of PANS variables we have:

$$
\kappa^2 = \sigma_\varepsilon \mu C_{\mu}^{1/2} k f_k \f (C_{\varepsilon 2} - C_{\varepsilon 1})
$$

The dissipation turbulent Prandtl number can now be found to be:

$$
\sigma_{\varepsilon u} = f_{\varepsilon} \sigma_\varepsilon
$$

As mentioned previously, the validity of this model is only within the log-layer and is not expected to capture the physics of the flow outside of the boundary layer.

E. Final PANS Closure Equations

Now that the approaches to closing the transport term have been introduced the PANS $k_u - \varepsilon_u$ and $k_u - \omega_u$ models are now summarized below.

The final form of the PANS $k_u - \varepsilon_u$ model is revealed below.

$$
\frac{\partial k_u}{\partial t} + U_j \frac{\partial k_u}{\partial x_j} = P_u - \varepsilon_u \frac{\partial}{\partial x_j} \left( \frac{\nu_u}{\sigma_{k_u}} \frac{\partial k_u}{\partial x_j} \right)
$$

$$
\frac{\partial \varepsilon_u}{\partial t} + U_j \frac{\partial \varepsilon_u}{\partial x_j} = C_{\varepsilon 1} P_u \varepsilon_u k_u - C_{\varepsilon 2} \varepsilon_u^2 k_u + \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_{\varepsilon u}} \frac{\partial \varepsilon_u}{\partial x_j} \right)
$$

$$
C_{\varepsilon 2} = C_{\varepsilon 1} + \frac{f_k}{f_\varepsilon} (C_{\varepsilon 2} - C_{\varepsilon 1})
$$

and

$$
\sigma_{k_u} = \sigma_k, \sigma_{\varepsilon u} = \sigma_\varepsilon \text{ for MTM};
$$

$$
\sigma_{k_u} = \sigma_k \frac{f_k^2}{f_\varepsilon}, \sigma_{\varepsilon u} = \sigma_\varepsilon \frac{f_k^2}{f_\varepsilon} \text{ for ZTM};
$$
\[ \sigma_{\varepsilon u} = \sigma_{\varepsilon} \frac{f_k}{f} \text{ for BLTM} \]

The final form of the PANS \( k_u - \omega_u \) model.

\[
\frac{\partial k_u}{\partial t} + U_j \frac{\partial k_u}{\partial x_j} = P_u - \beta * k_u \omega + \frac{\partial}{\partial x_j} \left( \nu_u \frac{k_u}{\sigma_{k_u}} \right) \tag{2.83}
\]

\[
\frac{\partial \omega_u}{\partial t} + U_j \frac{\partial \omega_u}{\partial x_j} = \alpha \frac{P_u \omega_u}{k_u} - \beta' \omega_u^2 + \frac{\partial}{\partial x_j} \left( \nu_u \frac{\partial \omega_u}{\sigma_{\omega u}} \right) \tag{2.84}
\]

and

\[
\beta' = \alpha \beta' - \beta' \frac{\alpha}{f} + \frac{\beta}{f_w}
\]

\[ \sigma_{k_u} = \sigma_k, \ \sigma_{\omega u} = \sigma_\omega \] for MTM;

\[ \sigma_{k_u} = \sigma_k \frac{f_k}{f}, \ \sigma_{\omega u} = \sigma_\omega \frac{f_k}{f_w} \] for ZTM

These are the model equations that are used in this study.
CHAPTER III

COMPUTATIONAL AND PHYSICAL ISSUES ADDRESSED

The objective of this thesis is to continue the development and validation of the PANS bridging method. While PANS shows great promise in preliminary computations, some key issues still need to be validated and established. The purpose of this chapter is to detail the underlying computational and physical issues that are the focus of this study, with results following in Chapter IV.

A. Flow Simulation for Investigations

The PANS simulations were conducted for flow past a circular cylinder at a Reynolds number of 140,000. The results were then compared to experimental work for validation. The standard case that is used as a basis for comparison in each study is the PANS $k - \varepsilon$ model computed at $f_k$ values of 0.4, 0.6, and 1.0 and a $f_\varepsilon$ value of 1.0. The In previous work a $f_k$ sensitivity study was performed for a square cylinder at a Reynolds number of a 22,000. The results of that study showed that with decreasing $f_k$ values the performance of PANS improved. The reason for this improvement is that as the $f_k$ value decreases more scales of motion are liberated and, therefore, more of the flow is being solved for directly (Jeong 2003). The results of this study also show the same behavior as the $f_k$ value decreases and will be presented in subsequent discussions.

The geometry of the standard grid is shown in detail in Figure 2. The grid size is 320 x 240 x 32. The simulations are performed using FLUENT software. FLUENT is a widely available commercial CFD software. Within the computational setup of the problem, there are several options for numerical and computational settings. A summary of the settings chosen in this study are presented in Table II. The boundary
The problem of flow past a cylinder at a Reynolds number of 140,000 is a good choice to use to continue testing and validation of the PANS approach. At this Reynolds number the flow is considered to be in the sub-critical range. The flow is still laminar when it separates and transition to turbulence occurs in the free shear layer in the wake. A Reynolds number of $2 \times 10^5 - 3.5 \times 10^6$ is in critical range. A Reynolds number that is larger than this range is considered to be super-critical. At super-critical Reynolds numbers, the flow turns turbulent prior to separation. The
<table>
<thead>
<tr>
<th>Settings</th>
<th>Choice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation type</td>
<td>3d, unsteady</td>
</tr>
<tr>
<td>Solver</td>
<td>Implicit</td>
</tr>
<tr>
<td>Temporal discretization</td>
<td>$2^{nd}$ order implicit</td>
</tr>
<tr>
<td>Turbulence model</td>
<td>$k - \varepsilon$ with PANS parameters</td>
</tr>
<tr>
<td>Pressure</td>
<td>PRESTO</td>
</tr>
<tr>
<td>Pressure-velocity coupling</td>
<td>SIMPLE</td>
</tr>
<tr>
<td>Momentum</td>
<td>$2^{nd}$ order upwind</td>
</tr>
<tr>
<td>Turbulent kinetic energy</td>
<td>$2^{nd}$ order upwind</td>
</tr>
<tr>
<td>Turbulent dissipation rate</td>
<td>$2^{nd}$ order upwind</td>
</tr>
</tbody>
</table>

Table II. FLUENT Computational Settings

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Condition</th>
</tr>
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<tbody>
<tr>
<td>Inlet</td>
<td>Velocity inlet</td>
</tr>
<tr>
<td>Outlet</td>
<td>Outflow</td>
</tr>
<tr>
<td>Top</td>
<td>Wall</td>
</tr>
<tr>
<td>Bottom</td>
<td>Wall</td>
</tr>
<tr>
<td>Lateral (sides)</td>
<td>Periodic</td>
</tr>
</tbody>
</table>

Table III. Simulation Boundary Conditions
flow stays attached for a longer distance as the turbulent boundary layer is more energetic.

The types of validation that will be used in this work are internal and external validation. External validation will be accomplished by comparing the results of this simulation to that of experimental results. Specifically, the $C_p$ distribution over the surface of the cylinder will be compared to experimental results from Roshko (1961) and from Cantwell & Coles (1983). The statistical plots of mean x-velocity profiles in the wake region will be compared to Cantwell & Coles (1983). The experimental results performed by Cantwell were performed at a Reynolds number of a 140,000. The Roshko experimental results are performed for the cylinder problem at a higher Reynolds number. The reason for including the experimental data of Roshko is explained as follows. The computational setup in FLUENT considers the entire flow as turbulent, however, we know from the previous discussion that at this Reynolds number flow separation is laminar. This results in an unphysical representation of the flow physics in the simulation. Specifically, turbulent kinetic energy will be generated forward of the cylinder which is not present in the actual flow. The result of having turbulent kinetic energy in the simulation is that turbulent viscosity is generated. In turn the stagnation pressure is no longer constant as defined by Bernoulli’s law. The final result is that the $C_p$ value at the stagnation point will no longer be equal to one. Since the $C_p$ distribution is difficult to capture for laminar separation it may be useful to compare results to data for a Reynolds number where flow is turbulent before separation occurs. Internal validation will be performed by comparing the distribution of the ratio of unresolved to resolved turbulent viscosity in the flow. This will be accomplished by comparing the ratio of the PANS turbulent viscosity at each point in the domain to that of the RANS turbulent viscosity at that same point. The pdf of this plot will show how much of the flow is actually consistent with the specified
f_k value. The ratio of the turbulent viscosity is equal to the following:

\[
\frac{\nu_u}{\nu_t} = \frac{f_k^2}{f_\varepsilon}
\]  

(3.1)

The purpose of performing internal validation is to determine if the PANS simulation is internally consistent with the specified f_k value.

In previous work, grid and temporal refinement studies were performed with PANS for the cylinder flow problem at a Reynolds number of 140,000 with the same dimensions of the domain studied here. Grid refinement was assumed to be sufficient when the mean statistical results were within five percent of the previous grid. The grid refinement study showed that the results did not change for grids finer than 140 x 120 x 32. This study was performed for f_k values of 0.5 and 0.7. As more scales are resolved, a finer grid is needed. Since the simulations in this study include PANS cases for f_k = 0.4, a finer grid is needed. It is assumed that the grid used in this simulation is sufficient, since it is significantly finer than the one found to be appropriate for a PANS simulation performed at f_k = 0.5. Similarly, a temporal refinement study was performed for this cylinder problem. It was found that a \( \delta t \) value of 0.0525 was sufficient for both f_k values. Using the previous argument, the \( \delta t \) must be decreased for a f_k value of 0.4. In all the simulations a time step of 0.025 is used (Lakshmipathy 2004).

B. Computational Issues

1. Convergence

Since PANS is an inherently unsteady calculation, it is important to examine if the convergence criteria should be different than that of RANS. In other words, it must be determined how sensitive PANS performance is to the convergence criteria in the
range of RANS values. The test of the convergence issue is examined by changing
the residual values in the simulation. Table IV details the computational set up of
the simulations for this study. Simulation 1 represents the standard RANS values
of the residuals used in the test case and all subsequent studies. This is then tested
against the reduced residual values given in column 3 of Table IV. The new values
represent a reduction of the RANS value by a factor of five. If there is negligible
change between the results of Simulation 1 and Simulation 2, then it can be assumed
that the standard RANS residual values are sufficient and that a smaller values are
not needed. The advantage of using a larger value for the residual is that it requires
less computational effort to complete the simulation (Blazek 2001).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Simulation 1</th>
<th>Simulation 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
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<td>0.0002</td>
</tr>
<tr>
<td>X-Velocity</td>
<td>0.001</td>
<td>0.0002</td>
</tr>
<tr>
<td>Y-Velocity</td>
<td>0.001</td>
<td>0.0002</td>
</tr>
<tr>
<td>Z-Velocity</td>
<td>0.001</td>
<td>0.0002</td>
</tr>
<tr>
<td>Kinetic Energy</td>
<td>0.001</td>
<td>0.0002</td>
</tr>
<tr>
<td>Dissipation</td>
<td>0.001</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Table IV. Residual Values
2. Order of Numerical Scheme

The second computational issue considered in this study is the choice of the order of the spatial discretization scheme. Most standard RANS commercial software use a 2\textsuperscript{nd} order scheme. Hybrids methods pose a special challenge to the order of discretization. While they must be refine enough to capture fine resolved scales, they must also be robust for RANS calculations. While a 2\textsuperscript{nd} order scheme is quite appropriate for RANS simulations, it is important to investigate if this low order scheme is appropriate for PANS. Here we compare the FLUENT 2\textsuperscript{nd} order scheme against the 3\textsuperscript{rd} order Monotone Upstream Centered Schemes for Conservation Laws (MUSCL) scheme. The MUSCL scheme falls into the class of Total Variation Diminishing (TVD) finite volume schemes (Harten 1983). TVD schemes were developed to reduce the extrema in the solution (Blazek 2001). Originally the TVD schemes were only 1\textsuperscript{st} order accurate. The MUSCL scheme was developed to provide 2\textsuperscript{nd} order accuracy to the TVD method (van Leer 1978). Currently, a third order MUSCL scheme is available in FLUENT. The purpose of this study is to determine if the increase in accuracy of the simulation out weighs the computational expense.

C. Physics Issues

1. Transport Model

As mentioned in the previous section, the issue of turbulent transport closure in hybrid/bridging models is not settled. There are two past proposals (ZTM and MTM) and we propose a new closure (BLTM) as described in the previous section. Qualitative investigations of ZTM and MTM have been performed in Murthi (2004) for the test case of flow over a 3-D cavity. It was found that ZTM for $T_{ku}$ and $T_{\varepsilon u}$ yielded the best results. While MTM for $T_{\varepsilon u}$ alone yielded reasonable agreement with ex-
periments, applying MTM for $T_{ku}$ lead to a completely unphysical solution. Various considerations also suggest that ZTM is appropriate for reasonably high Reynolds number flows and MTM is more accurate for lower Reynolds number flows. To date no qualitative investigations for ZTM and MTM have been performed for the flow past a cylinder test case.

The physical issue of transport model selection is detailed in this section. In the past, the choice of transport model is made between two extrema. The first extrema is the Zero Transport Model (ZTM) and the second is the Maximum Transport Model (MTM). The details of the assumptions and physical issues associated with each model are presented here. For simplicity, this discussion is limited to using MTM and ZTM for the transport of unresolved kinetic energy.

The transport term of the unresolved kinetic energy equation is given below.

$$T_{ku} = \frac{\partial}{\partial x_j} \left( \nu_k f_\epsilon \frac{\partial K_u}{\sigma_k f_k^2 \partial x_j} \right) + \left( U_j - \bar{U}_j \right) \frac{\partial K_u}{\partial x_j} \quad (3.2)$$

If we apply the ZTM Model then the transport due to resolved velocity fluctuations disappears and the following expression is left.

$$T_{ku} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u}{\sigma_{ku}} \frac{\partial K_u}{\partial x_j} \right) \quad (3.3)$$

where

$$\sigma_{ku} = \frac{f_k^2}{f_\epsilon} \sigma_k$$

The equation above closes the unresolved transport term and the relation for the PANS kinetic energy Prandtl number is found in terms of its RANS counterpart and filter parameters.

When MTM is applied to the unresolved kinetic energy equation, then Equation
3.2 reduces to the following expression.

\[ T_{ku} = \frac{\partial}{\partial x_j} \left( \frac{\nu_u}{\sigma_{ku}} \frac{\partial K_u}{\partial x_j} \right) \]  \hspace{1cm} (3.4)

where

\[ \left( U_j - \bar{U}_j \right) \frac{\partial K_u}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\nu_r}{\sigma_k} \frac{\partial K_u}{\partial x_j} \right) \]

and

\[ \sigma_k = \sigma_{ku} \]

The equation reveals the PANS kinetic energy Prandtl number is equal to its RANS counterpart.

The BLTM is the last transport model to be discussed in this section. The theory of the BLTM model is presented in Chapter II. It is assumed that this model will only hold validity in the boundary layer and therefore the results of this study are expected to be poor since the entire flow domain is considered. The BLTM is only appropriate to be applied to the dissipation equation. The details of the derivation for this transport model is found in Chapter II. Using the BLTM, the dissipation Prandtl number reduces to the following.

\[ \sigma_{\varepsilon u} = \frac{f_\varepsilon}{f_k} \sigma_\varepsilon \]

2. Length Scale Model: \( k_u - \omega_u \) vs \( k_u - \varepsilon_u \)

The RANS \( k - \varepsilon \) model is the easiest full closure model to implement for turbulence simulations and, subsequently, is also the most widely used. Another popular two equation model is the \( k - \omega \) model introduced by Wilcox (1991). The \( k - \omega \) model has been found to be more accurate in the near wall region and in the presence of
stream-wise pressure gradients than the standard $k - \varepsilon$ model (Pope 2000). However, the $k - \varepsilon$ model is more accurate in the free stream in comparison to the $k - \omega$ model. It is important to investigate the implementation of the PANS $k_u - \varepsilon_u$ model and the PANS $k_u - \omega_u$ model. Specifically, this study will be considering if the PANS length scale models behave similarly to their RANS counterparts and how the respective PANS models perform in comparison to each other. Previous investigations of the PANS $k_u - \omega_u$ model have been performed for different $f_k$ values than those presented in this work (Lakshmipathy & Girimaji 2006).

3. Low Reynolds Number $C_\mu$ Correction

The standard two equation turbulence models such as the $k - \varepsilon$ and $k - \omega$ models over predict the turbulent kinetic energy in the region upstream of a bluff body, such as a cylinder. This phenomenon results in unrealistic recovery of the $C_p$ distribution since Bernoulli’s equation for pressure is no longer valid. This behavior is most apparent at the stagnation point where the $C_p$ value will not be equal to one due to the presence of the turbulent kinetic energy. If we examine the dissipation equation the reason for this anomaly is more apparent.

$$\frac{\partial \varepsilon_u}{\partial t} + U_j \frac{\partial \varepsilon_u}{\partial x_j} = C_{e1} \frac{P_u}{T} - C_{e2} \frac{\varepsilon_u}{T} + \frac{\partial}{\partial x_j} \left( \frac{\nu_t}{\sigma_{\varepsilon u}} \frac{\partial \varepsilon_u}{\partial x_j} \right)$$

(3.5)

where

$$T = \frac{K_u}{\varepsilon_u}$$

(3.6)

In the region near the stagnation point $T$ becomes very large and subsequently the dissipation will become very small. It can be seen that the kinetic energy will become very large from:

$$\frac{\partial K_u}{\partial t} + U_j \frac{\partial K_u}{\partial x_j} = P_u - \varepsilon_u \frac{\partial}{\partial x_j} \left( \frac{\nu_u}{\sigma_{ku}} \frac{\partial K_u}{\partial x_j} \right)$$

(3.7)
A correction must be made to the definition of the generalized central moment defined below to account for the anomaly.

\[
\tau(V_i, V_j) = -\nu_u S_{ij} + \frac{2}{3} K_u S_{ij}
\]  

(3.8)

This work utilizes the "realizability" condition that limits this term to be \(2K \geq \tau(V_i, V_j) \geq 0\) as presented in Durbin (1996). Durbin places a limit on the limit on \(T\) and here the same constraint is defined in terms of \(C_\mu\) instead.

\[
C_\mu = \frac{\sqrt{2}}{6} \left( \frac{\varepsilon}{KS} \right)
\]  

(3.9)

In work currently underway within our group, we are developing a new variable low Reynolds number \(C_\mu\) correction. One of the proposed corrections is

\[
C_\mu = MIN \left( 1.125 \left( \frac{\beta^* \omega_u}{S} \right)^2, 0.09 \right)
\]  

(3.10)

for the \(k_u - \omega_u\) model (personal communication, April 2008). We will examine the effects of this correction in the hopes in improving the recovery of the \(C_\rho\) distribution.
CHAPTER IV

RESULTS
Several RANS and PANS simulations were performed to address the various computational and physical closure issues addressed in this study. The results presented in this chapter are organized into separate cases for each specific study. A list of descriptions of these cases can be found below.

- Case 1: Standard $k - \varepsilon$ PANS model
- Case 2: $k - \varepsilon$ PANS model implemented with new convergence criteria
- Case 3: $k - \varepsilon$ PANS model implemented with MUSCL scheme
- Case 4 (a): $k - \varepsilon$ PANS model implemented with Maximum Transport Model
- Case 4 (b): $k - \varepsilon$ PANS model implemented with Boundary Layer Transport Model
- Case 5: $k - \omega$ PANS model
- Case 6: $k - \omega$ PANS model implemented with low Reynolds number $C_\mu$ correction

As described in the previous section, two types of validation were performed in this study. The first was a comparison against experimental data. This was accomplished by comparing the simulation results to experimental results of $C_p$ distribution on the surface of the cylinder and mean x-velocity profiles in the wake. These experimental results were taken from Cantwell & Coles (1983) and Roshko (1961). The second validation was an internal validation whereby the pdf plots of the ratio of PANS to
RANS turbulent viscosity were generated. The ratio of the turbulent viscosity is equal to the following:

\[
\frac{\nu_u}{\nu_t} = \frac{f_k^2}{f_\varepsilon}
\]  

(4.1)

This ratio is equal to 0.16 and 0.36 for \(f_k=0.4\) and \(f_k=0.6\), respectively. The purpose of performing internal validation is to determine whether the PANS simulation is internally consistent with the specified \(f_k\) value. Simulations, unless otherwise specified, will be performed for \(f_k\) values of 0.4, 0.6 and 1.0. It is important to note that the \(f_k = 1\) case is equivalent to a RANS simulation since all fluctuations were averaged.

A. Case 1: Baseline PANS \(k_u - \varepsilon_u\) Simulation

The results from this baseline case will be used to infer improvement or lack thereof for each model modification. The results for the standard case were shown below in figure 3 through figure 8.
Fig. 3. Case 1: $C_p$ Distribution

Fig. 4. Case 1: Mean Statistics Near Wake Region at $x/D = 1$
Mean $V_x$ Near Wake Region at $x/D = 3$

![Graph showing mean $V_x$ near wake region at $x/D = 3$.]

Fig. 5. Case 1: Mean Statistics Near Wake Region at $x/D = 3$

Mean $V_x$ at the Wake Centerline

![Graph showing mean $V_x$ at the wake centerline.]

Fig. 6. Case 1: Mean Statistics at the Wake Centerline
Fig. 7. Case 1: $f_k = 0.4$ Recovery Plot

Fig. 8. Case 1: $f_k = 0.6$ Recovery Plot
There is not a significant difference between the PANS $f_k = 0.4$ and $f_k = 0.6$ simulations in capturing the $C_p$ distribution as shown in figure 3. However, these simulations do show improvement over the $f_k = 1$ RANS simulation, which is to be expected since more scales were solved for directly. It appears that the PANS simulations show the most improvement before separation occurs. After the laminar separation occurs at approximately 77 degrees, both the PANS and RANS simulations perform poorly. The baseline PANS simulations were poor at capturing the mean x-velocity profile of the near wake region at $x/D=1.0$, as shown in figure 4. It also appears that the RANS simulation performs slightly better in the region directly aft of the cylinder. The PANS simulations do provide significant improvement over the RANS simulation in capturing this profile in the near wake region at $x/D = 3$ seen in figure 5 and at the wake centerline seen in figure 6. From these observations, it can be concluded that the PANS simulations performs better in the far wake region than they does in the near wake region. Also, another important observation is that there is not a distinct difference between the ability of the PANS $f_k = 0.4$ and $f_k = 0.6$ simulations to capture the mean statistics. In other words, the physics of circular cylinder flow can be captured with the $f_k = 0.6$ resolution at nearly the same quality of agreement as the $f_k = 0.4$ simulation. Another observation made in each plot is that the PANS simulations do provide improvement over the RANS simulation. The internal validation test show that the $f_k$ recovery is good for the baseline case since the peaks were in line with the correct ratio shown in figures 7 and 8 for the $f_k$ values of 0.4 and 0.6, respectively. Therefore, the baseline case is determined to be internally consistant.
B. Case 2: Convergence

Case 2 represents the implementation of a new convergence criterion. The convergence criterion is changed from the standard value of 0.001 to 0.0002 as described in Chapter III. The purpose of this case is to determine the effect of the convergence criterion on the accuracy of the simulation and if the RANS convergence criterion is adequate for PANS simulations for this range of $f_k$ values. Case 2 was only implemented for $f_k$ values of 0.6 and 1. A simulation for $f_k = 0.4$ was not necessary because the results of the $f_k = 0.6$ simulation were adequate to make meaningful conclusions for this case.

![Fig. 9. Case 2: $C_p$ Distribution](image-url)
Fig. 10. Case 2: Mean Statistics Near Wake Region at x/D = 1

Fig. 11. Case 2: Mean Statistics Near Wake Region at x/D = 3
Fig. 12. Case 2: Mean Statistics at the Wake Centerline

Fig. 13. Case 2: $f_k = 0.6$ Recovery Plot
The following can be concluded from the convergence study. It can be observed from figure 9 that the results for \( f_k = 0.6 \) and \( f_k = 1.0 \) were very close to the results found in Case 1: The PANS simulation performs better before separation and both PANS and RANS perform poorly after the separation occurs. The implementation of a tighter convergence criterion does not provide significant improvement in the recovery of the statistics. As shown in figures 10 - 12 the results were nearly identical to those in Case 1, which was implemented with the standard residual value. Once again, the PANS simulation for \( f_k = 0.6 \) shows significant improvement over the \( f_k = 1.0 \) case in the far wake region. The \( f_k \) recovery plot shows nearly identical plots for the viscosity ratio in figure 13. In summary, increasing the residual by a factor of five does not contribute to any significant improvements over the simulation in Case 1. It can be assumed that the accuracy of the solution does not change beyond an increase of 0.1% of the residual. The RANS convergence criterion is adequate for PANS fat this range of \( f_k \) values.

C. Case 3: Order of Numerical Scheme

The results presented here are for Case 3, which uses the 3\textsuperscript{rd} order MUSCL spatial discretization scheme with the \( k_u - \varepsilon_u \) model. The aim of this study is to compare the accuracy of the results of this case to the results in Case 1, which implements a 2\textsuperscript{nd} order scheme. It is important to determine, for the range of \( f_k \) values tested here, if the 2\textsuperscript{nd} order scheme is adequate. The concern is that with the PANS simulation more scales of motion are liberated and the order of the standard scheme may not be adequate.
Fig. 14. Case 3: $C_p$ Distribution

Fig. 15. Case 3: Mean Statistics Near Wake Region at $x/D = 1$
Fig. 16. Case 3: Mean Statistics Near Wake Region at $x/D = 3$

Fig. 17. Case 3: Mean Statistics at the Wake Centerline
Fig. 18. Case 3: $f_k = 0.4$ Recovery Plot

Fig. 19. Case 3: $f_k = 0.6$ Recovery Plot
As was observed with the results of Study 1, the \( f_k < 1 \) values do show improvement over the \( f_k = 1 \) results for capturing the \( C_p \) distribution and mean statistics in the wake. The same trend of the PANS simulations providing better results before separation occurs is seen in figure 14. However, once again it is harder to distinguish between the improvements of the \( f_k = 0.4 \) vs. \( f_k = 0.6 \) results. Small improvements can be seen in figures 15 and 16 as the \( f_k = 1.0 \) simulation is better at capturing the statistics than the \( f_k = 1.0 \) simulation in Case 1. In figure 17 the difference between \( f_k = 0.4 \) vs. \( f_k = 0.6 \) is more apparent than it was for Case 1. The recovery of the turbulent viscosity ratio improves slightly with the implementation of the MUSCL scheme for the \( f_k = 0.4 \) case seen in figure 18. However, in figure 19 the recovery of the turbulent viscosity ratio of the \( f_k = 0.6 \) simulation does not show improvement, since the peak is less well-defined for the MUSCL case. Overall, the implementation of a MUSCL scheme does show small improvements for a RANS simulation but not PANS. It appears that the MUSCL scheme is not worth the added computational cost as the 2\(^{nd}\) order scheme performs equally well in this \( f_k \) range.

D. Case 4: Transport Model

1. Maximum Transport Model

The effects of the MTM is examined first. As discussed previously, the Maximum Transport Model is theorized to be more accurate for lower Reynolds number flows. Recall, as this is a high Reynolds number flow, that the baseline model in Case 1 employs ZTM. Comparing the results of Case 4 with those of Case 1 will enable us to evaluate the MTM vs. ZTM closures.
Fig. 20. Case 4 (a): $C_p$ Distribution

Fig. 21. Case 4 (a): Mean Statistics Near Wake Region at $x/D = 1$
Fig. 22. Case 4 (a): Mean Statistics Near Wake Region at $x/D = 3$

Fig. 23. Case 4 (a): Mean Statistics at the Wake Centerline
Fig. 24. Case 4 (a): $f_k = 0.4$ Recovery Plot

Fig. 25. Case 4 (a): $f_k = 0.6$ Recovery Plot
There is little difference between RANS simulation of $f_k = 1$ and the PANS simulations of $f_k = 0.4$ and 0.6 for capturing the experimental $C_p$ data as seen in figure 20. This is also the case for the mean statistics in the near wake region at $x/D=1.0$ seen in figure 21. In this near wake region, it can be concluded that the PANS simulations behave like RANS. This seems to be accurate since in the near wake region the Reynolds number is higher and ZTM should be more appropriate. Further in the wake, the Reynolds number decreases and the MTM model should be more appropriate. This behavior is seen in the plots of the near wake region at $x/D=3.0$ and at the wake centerline. In figure 22, the $f_k = 0.4$ and $f_k = 0.6$ show vast improvement over the RANS case. The $f_k = 0.4$ case shows the best recovery. The PANS simulations also show improvement over the RANS case in capturing the mean velocity profile at the wake centerline as can be seen in figure 23. The viscosity recovery plots for the transport model study were especially interesting. The MTM does not show good recovery of the viscosity ratio and fails the internal validation shown in figures 24 and 25. Those experimental validation tests for which MTM does show good results have to be approached with caution since the internal validation of MTM failed. The conclusion made from this study of the results between the ZTM and MTM approaches is summarized as follows. The ZTM model is more accurate for this problem at this Reynolds number because it passes internal validation and provides more accurate results in the near wake region. This result is expected since this problem is for a relatively high Reynolds number.

2. Boundary Layer Transport Model

This subsection presents the results of the Boundary Layer Transport Model simulations. This is the first time this model has been presented and tested.
Fig. 26. Case 4 (b): $C_p$ Distribution

Fig. 27. Case 4 (b): Mean Statistics Near Wake Region at $x/D = 1$
Fig. 28. Case 4 (b): Mean Statistics Near Wake Region at $x/D = 3$

Fig. 29. Case 4 (b): Mean Statistics at the Wake Centerline
Fig. 30. Case 4 (b): $f_k = 0.4$ Recovery Plot

Fig. 31. Case 4 (b): $f_k = 0.6$ Recovery Plot
As expected, the BLTM model does not perform well in experimental or internal validation as it is only valid inside the log-layer of the boundary layer. The recovery of the $C_p$ distribution and statistical profiles in the wake region is very poor as shown in figures 26 - 29. The viscosity ratio plots, seen in figure 30 and figure 31 show no recovery of the original $f_k$ value and the BLTM fails internal validation. This model should be further investigated in log-layer of the boundary layer.

E. Case 5: $k_u - \omega_u$ Closure

The purpose of Case 5 is to present the results for the PANS $k_u - \omega_u$ model. In general, the RANS $k - \omega$ model is more accurate in the boundary layer and in the presence of free stream pressure gradients in comparison to the RANS $k - \epsilon$ model (Pope 2000). Therefore, it is important to implement the PANS $k_u - \omega_u$ model and compare it to the PANS $k_u - \varepsilon_u$ model presented in Case 1.
Fig. 32. Case 5: Mean Statistics Near Wake Region at x/D = 1

Fig. 33. Case 5: Mean Statistics Near Wake Region at x/D = 3
Fig. 34. Case 5: Mean Statistics at the Wake Centerline

Fig. 35. Case 5: $f_k = 0.4$ Recovery Plot
Fig. 36. Case 5: $f_k = 0.6$ Recovery Plot
The plots of statistics reveal two important features of the performance of the $k_u - \omega_u$ model. First, like that in Case 1, the $f_k=0.4$ and $f_k=0.6$ show better recovery of the mean velocity profiles than does the RANS $f_k=1.0$ simulations. Second, unlike that in Case 1, there is a clear distinction between the $f_k=0.4$ and $f_k=0.6$ results. In each plot, it is apparent that the $f_k=0.4$ simulation provided more accurate results and this behavior is to be expected as the $f_k=0.4$ simulation resolves more scales. The ability of the PANS to capture the mean statistics is also improved in the far wake. These two behaviors can be seen in figures 32 - 33. The PANS $k_u - \omega_u$ model performs poorer in capturing the wake centerline distribution shown in figure 34 in comparison to the PANS $k_u - \varepsilon_u$ in Case 1. This is consistent with the ability of the RANS $k - \varepsilon$ model to perform better outside of the boundary layer so we expect the $k_u - \varepsilon_u$ model to perform better in this region as well. The pdf $f_k$ recovery plots show much improvement for both $f_k=0.4$ seen in figure 35 and $f_k=0.6$ seen in figure 36. The $k_u - \omega_u$ model performs the best in the internal validation test in comparison to all other cases in this study. Overall, the PANS $k_u - \omega_u$ model is much more effective than the PANS $k_u - \varepsilon_u$ model. The main reason for this is that the near wall behavior is very critical for separation prediction as well as other statistics examined here.

F. Case 6: Low Reynolds Number $C_\mu$ Correction

Case 6 is the implementation of the $C_\mu$ correction described in Chapter III. The purpose of this study is to correct the large amounts of kinetic energy that is generated in the region upstream of the stagnation point. In previous studies, the $C_p$ distribution plots show a $C_p$ value that is not exactly equal to one at the stagnation point. The aim of this study is to improve the ability to capture the $C_p$ distribution (Durbin 1996).
Fig. 37. Case 6: $C_p$ Distribution

Fig. 38. Case 6: Mean Statistics Near Wake Region at $x/D = 1$
Fig. 39. Case 6: Mean Statistics Near Wake Region at $x/D = 3$

Fig. 40. Case 6: Mean Statistics at the Wake Centerline
Fig. 41. Case 6: $f_k = 0.5$ Recovery Plot

Fig. 42. Case 6: $f_k = 0.7$ Recovery Plot
The results for figure 37 show that Case 6 provides the best results for capturing the $C_p$ profile. The $C_\mu$ correction produces the correct value of one for $C_p$ at the stagnation point. In previous cases, the stagnation point did not have a value of one. This was due to the fact that the simulation in FLUENT considers the entire domain as a turbulent domain and generates turbulent kinetic energy in the region in front of the cylinder. In actuality, the region forward of the cylinder is laminar and the flow separation over the cylinder is laminar at this Reynolds number. The transition to turbulence occurs later in the wake. The $f_k=0.5$ case appears to have the best recovery of the $C_p$ profile before and after flow separation occurs but fails in the separation region. Generally, it is very difficult to accurately predict this complex region. The results of this simulation are very promising. The capturing of the statistical results in figure 38 is somewhat poor. However, in figure 39 and figure 40 the PANS results are good especially for the $f_k=0.5$ case. It can be concluded that the PANS simulation performs better in the far wake then it does in the near wake. Case 6 does not perform well in internal validation as shown if figures 41 and 42. This must be further investigated and improvements should be made $C_\mu$ correction simulation such that it performs better in internal validation.
CHAPTER V

CONCLUSIONS

The conclusions from the study are presented in this chapter. This discussion begins with an overview of the results of the standard test case of the $k_{u} - \varepsilon_{u}$ model. The discussion is then continued with a summary of the results from the computational and physical issue studies.

The results of Case 1 show that the standard $k_{u} - \varepsilon_{u}$ model performed well in internal validation. The plot of the $C_{p}$ distribution had some expected errors due to the phenomenon of the unrealistic presence of turbulent kinetic energy ahead of the cylinder. This case captured the mean statistics reasonable well in the near wake and wake centerline plots. From this case, it appears that the physics of the flow is captured with nearly the same quality of agreement with the $f_{k} = 0.6$ simulation as the $f_{k} = 0.4$ simulation. Overall the PANS simulations shows improvement over the RANS simulation.

A. Computational Issues Conclusions

The purpose of Case 2 was to implement a new convergence criterion of 0.0002 for the residual values in place of the standard value of 0.001 in Case 1. The simulations of Case 2 were implemented for $f_{k}$ values of 0.6 and 1 only. The results from this case showed very little change in the ability of the model to capture the $C_{p}$ distribution or the mean statistics from the model of Case 1. The ability to recapture the viscosity ratio was nearly identical to Case 1 as well. From this study, it can be concluded that decreasing the residual by a factor of five does not contribute to any significant improvements over the simulation of Case 1. It can be assumed that the accuracy of the solution does not change beyond an increase 0.1% of the residual. The RANS
value for the residuals is adequate for this range of $f_k$ values.

In Case 3 a 3$^{rd}$ order MUSCL spatial discretization scheme was used instead of the standard 2$^{nd}$ order scheme of Case 1. The aim of this study was to determine if the accuracy of the results changed with the implementation of the new scheme. Overall, the implementation of a MUSCL scheme does show small improvements for the RANS $f_k = 1$ simulation but not the PANS $f_k = 0.4$ and $f_k = 0.6$ simulations. It can be concluded that the MUSCL scheme is not worth the added computational cost as the 2$^{nd}$ order scheme performs equally well in this $f_k$ range. This conclusion is only valid for the range of $f_k$ values tested here as a lower range of $f_k$ values may require a different discretiation scheme as more scales are solved for directly.

B. Physical Issues Conclusions

The purpose of Case 4 (a) and (b) was to test the choice of transport closure model. The standard transport closure model used in Case 1 was the Zero Transport model. In Case 4 (a) the Maximum Transport model was tested. The MTM did not past internal validation but did show some improvement over Case 1 in capturing the statistical descriptions in the far wake. This behavior is likely due to the fact that this model is more appropriate for lower Reynolds number flows and in the wake the flow will have a lower Reynolds number. The results of Case 1 reveal that the ZTM provided excellent recovery of the viscosity ratios and therefore it is more appropriate for this Reynolds number. In Case 4 (b) the BLTM model was implemented in the dissipation equation. It was found that this model does not perform well in either internal or external validation. This result is to be expected as BLTM is only appropriate for the log-layer of the boundary layer.

In Case 5 the $k_u - \omega_u$ model was implemented instead of the $k_u - \varepsilon_u$ model of Case
1. It was found that the $k_u - \omega_u$ model is more sensitive to $f_k$ values. Specifically, the results of the simulation for a $f_k$ value of 0.4 was significantly improved over the 0.6 simulation. The $k_u - \omega_u$ model also performed much better in recovering the viscosity ratios then the $k_u - \varepsilon_u$ model. Outside the boundary layer the $k_u - \omega_u$ model performs poorer than the $k_u - \varepsilon_u$ model of Case 1. This result is to be expected as the parent $k - \omega$ model does not perform well in this region. Overall, the PANS $k_u - \omega_u$ model is much more effective than the PANS $k_u - \varepsilon_u$ model.

In Case 6 a variable $C_{\mu}$ correction was implemented in the PANS $k_u - \omega_u$ model. The $C_{\mu}$ correction allowed for the correct value of the $C_p$ to be captured at the stagnation point which was lacking in all previous cases. In Figure 39 and Figure 40 the PANS results are good especially for the $f_k=0.5$ case. It also can be concluded that the PANS simulation performs better in the far wake then it does in the near wake and provides improvement over the $k_u - \varepsilon_u$ model in the far wake. The results of this simulation are very promising. Case 6 did not perform well in internal validation which must be further investigated.
CHAPTER VI

SUMMARY OF RECOMMENDATIONS

The recommendations for future work are presented in this chapter. The computational issues studies should be further investigated. Specifically, it is recommended to determine the range of $f_k$ values for which the convergence criteria and spatial discretization scheme are appropriate. This study can be accomplished by testing additional $f_k$ values smaller than 0.4. As more scales of motion are liberated with smaller $f_k$ values the standard RANS convergence criteria and schemes will no longer be adequate.

The choice of closure model for the cylinder problem showed interesting results and should be inquired into further. The MTM model should be implemented for lower Reynolds number flows as it appeared to perform better in regions in the wake where the flow had retarded from its original state. The BLTM should be investigated more thoroughly in the boundary layer region as this is the region for which the physics of the model was developed. The results of the PANS $k_u - \omega_u$ model showed promising results and it may be useful to perform similar numerical studies for this model as was performed here for the $k_u - \varepsilon_u$ model. The $C_\mu$ correction provides excellent recovery of the $C_p$ distribution and should be implemented in future PANS simulations for which the flow is laminar when separation occurs.

New studies may include implementing the SST PANS model. This model integrates the strengths of the RANS $k - \varepsilon$ and $k - \omega$ models and may perform well with the PANS methodology (Menter 1993). In addition, another important validation of PANS would be to implement these studies on the backward facing step problem. The backward facing step is another benchmark flow for which there exists many experimental results that could be used for validation and testing of the PANS
approach.
REFERENCES


VITA

Dasia A. Reyes was born in Lubbock, TX, USA. She obtained a B.S. degree in aerospace engineering from Texas A&M University in December 2005. She began graduate studies in aerospace engineering at Texas A&M University in January 2006. Upon completion of her master’s work she plans to pursue a Ph.D. degree at Texas A&M University, researching in the area of turbulence modeling and validation. She can be contacted through Dr. Girimaji at Texas A&M University, Department of Aerospace, H.R. Bright Building, Rm. 701, Ross Street - TAMU 3141, College Station TX 77843-3141.

The typist for this thesis was Dasia Reyes.