DIRECT LINEARIZATION OF CONTINUOUS AND HYBRID DYNAMICAL SYSTEMS

A Thesis

by

JULIE MARIE JONES PARISH

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2007

Major Subject: Aerospace Engineering

DIRECT LINEARIZATION OF CONTINUOUS AND HYBRID DYNAMICAL SYSTEMS

A Thesis

by

JULIE MARIE JONES PARISH

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Approved by:

Chair of Committee,	John E. Hurtado
Committee Members,	John L. Junkins
	Guy Battle
Head of Department,	Helen L. Reed

December 2007

Major Subject: Aerospace Engineering

ABSTRACT

Direct Linearization of Continuous
and Hybrid Dynamical Systems. (December 2007)
Julie Marie Jones Parish, B.S., Texas A&M University
Chair of Advisory Committee: Dr. John E. Hurtado

Linearized equations of motion are important in engineering applications, especially with respect to stability analysis and control design. Traditionally, the full, nonlinear equations are formed and then linearized about the desired equilibrium configuration using methods such as Taylor series expansions.

However, it has been shown that the quadratic form of the Lagrangian function can be used to directly linearize the equations of motion for discrete dynamical systems. Here, this development is extended to directly generate linearized equations of motion for both continuous and hybrid dynamical systems, where a hybrid system is described with both discrete and continuous generalized coordinates. The results presented require only velocity level kinematics to form the Lagrangian and find equilibrium configuration(s) for the system. A set of partial derivatives of the Lagrangian are then computed and used to directly construct the linearized equations of motion about the equilibrium configuration of interest. This study shows that the entire nonlinear equations of motion do not have to be generated in order to construct the linearized equations of motion. Several examples are presented to illustrate application of these results to both continuous and hybrid system problems.

To my husband and dearest friend, aspen.

I thank my Father in Heaven.

For patiently guiding me through the plans He has laid.

For providing me a wonderful mentor and friend.

For giving me a father and family that supports me with love and laughter.

For blessing me with an excellent husband who loves me dearly.

For being my Rock when the rain pours

and my Restoration when the sun again shines.

For being my Joy and Purpose.

For loving me.

"For I know the plans I have for you," declares the Lord, "plans to prosper you and not to harm you, plans to give you hope and a future. Then you will call upon me and come and pray to me, and I will listen to you. You will seek me and find me when you seek me with all your heart."

Jeremiah 29:11-13

ACKNOWLEDGMENTS

I humbly owe many thanks to many people. To my advisor, Dr. John E. Hurtado, I owe many, many thanks for your guidance, patience, and enthusiasm. You have been a wonderful mentor and have continued to help me mature over the past three years. What a blessing to have walked into your office without quite knowing what I was getting myself into!

To my graduate committee members, Dr. John L. Junkins and Dr. Guy Battle, I am thankful for past research contributions, recent insights to my work, and lively entertainment. To my co-workers at Sandia and the Department of Defense, I thank you for helping me better understand how to apply my problem solving skills to real-world problems and encouraging my desire to continue my education. I am very grateful to the Department of Defense and the National Science Foundation for their financial support throughout my graduate studies.

My family has continued to love and support me and all my decisions, and I thank them for believing in me and lifting me up when times are tough. I am grateful to my friends and colleagues for providing a welcome distraction, a needed hug, or a hearty laugh when stress runs high and a little perspective goes a long way. I would like to especially thank Stefanie Beaver for generously offering her time and insight through reading and helping edit my thesis.

Finally, I thank Mr. Allen Parish for first challenging me to think beyond graduation and consider pursuing more advanced degrees. Little did I know that one day you would promise to be there for me through all of life's challenges.

TABLE OF CONTENTS

CHAPTER Page

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Linearized equations of motion do not embody the full nature of a dynamical system, but they are useful for studying the behavior of a system and designing feedback controls. The most common method to produce linearized equations is to reduce the fully nonlinear equations of motion, the motivation being that one is likely interested in the nonlinear equations anyway. This approach is called an indirect approach because the path to the linearized equations starts from a first principle of motion (i.e., Newton's law or Lagrange's fundamental equation) and passes through the fully nonlinear equations. Alternatively, a direct approach to linearization would produce the desired equations directly from a first principle of motion.

Several texts present a method for direct linearization of equations of motion for systems described only by discrete generalized coordinates [1][2]. What is missing, however, is a method to handle a large class of systems that include an elastic domain, such as a bendable arm. The elastic part of such a system is typically described with continuous, or infinite-dimensional coordinates.

In this thesis, a method for directly constructing linearized equations of motion for continuous and hybrid dynamical systems is developed. A hybrid system is a system described by a combination of discrete and continuous coordinates. Attention is given to systems composed of single or multiple elastic domains, the distinction being the number of independent infinite-dimensional coordinates necessary to describe the system configuration. This method builds on concepts utilized in direct linearization for discrete systems as well as formulations of Lagrange's equations for continuous

---

The journal model is *IEEE Transactions on Automatic Control.*

and hybrid systems and only requires the kinetic and potential energy functions for the system of interest [3]. Given these functional expressions, the equilibrium configuration(s) about which the equations of motion are to be linearized can be found. The method identifies partial derivatives of the Lagrangian that contribute to the linearized equations of motion. When evaluated at an equilibrium configuration, these partial derivatives are the coefficients for the directly-generated linearized equations of motion.

The thesis begins with a brief review of direct linearization derived for Lagrange's equations for discrete systems. Direct linearization of continuous and hybrid systems is then explored, as well as approaches for determining equilibrium solutions for continuous and hybrid dynamical systems. The procedure is presented using Einstein summation convention for brevity, and numerous examples are presented throughout this thesis to help clarify the main ideas and methodology.

CHAPTER II

BACKGROUND

The ideas upon which the research in this thesis is based are over 200 years old [4]. The primary building block, Lagrange's Equations, will be reviewed first, followed by a brief overview of equilibrium for dynamic systems. These results are utilized in the existing direct linearization method for discrete systems, which will be discussed in detail and later applied to an example in this chapter.

A. Lagrange's Equations

The principles behind Lagrangian mechanics are powerful because they allow one to "develop a universal form of the differential equations of motion, as a function of the system kinetic energy and unspecified generalized coordinates" [5]. In Newtonian mechanics, the traditional alternative to the Lagrangian approach, the sum of the forces is equated to the time rate change of the momentum along the coordinate axes. These expressions are then used to solve for the governing equations and constraint forces. However, in Lagrangian mechanics, the full equations of motion are derived in a more straightforward manner using partial differentials of a single scalar function. Furthermore, the Lagrangian approach uses velocity-level, as opposed to acceleration-level, kinematics. In this section, the formulation of Lagrange's equations for finite-dimensional systems will be presented.

1. Generalized Coordinates

The scope of the developments in this thesis is restricted to holonomic systems, or systems that can be described with a minimal set of independent coordinates called generalized coordinates. The number of coordinates in the minimal set, n, is equal to the number of degrees of freedom of the system of interest. Given an arbitrary choice of coordinates that exceed the number in the minimal set, holonomic constraints can be used to solve for the excess coordinates as a function of the minimal set of coordinates. There are an infinite number of choices for these independent coordinates, but one will find that certain selections will often result in more elegant results for the equations of motion.

2. D'Alembert's Principle

Given a set of independent generalized coordinates, $\{q_1, q_2, \ldots, q_n\}$, the position vector, $\mathbf{r}_i(q_1, q_2, \ldots, q_n, t) = \mathbf{r}_i(q_i, t)$ for the ith particle in a system of N particles can be constructed. Virtual displacements, $\delta\mathbf{r}$, are instantaneous differential displacements, and can be written in the following form.

$$\delta\mathbf{r}_i = \frac{\partial\mathbf{r}_i}{\partial q_k}\delta q_k = \frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_k}\delta q_k = \boldsymbol{\tau}_{i_k}\delta q_k \tag{2.1}$$

The vector $\boldsymbol{\tau}_{i_k}$ is called the Lagrangian vector [6]. The virtual displacements can also be used to define the virtual work of the ith particle [5].

$$\delta W_i \equiv \mathbf{F}_i \cdot \delta\mathbf{r}_i \tag{2.2}$$

Here, the total forces, $\mathbf{F}_i$, are the sum of the holonomic constraint forces, $\mathbf{f}_{c_i}$, and the given forces, $\mathbf{f}_i$. The constraint forces are normal to the plane that contains the virtual displacements, so the dot product $\boldsymbol{f}_{c_i} \cdot \delta\boldsymbol{r}_i$ is zero. Summing over the N particles, we then have the total virtual work.

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta\mathbf{r}_i \tag{2.3}$$

Now consider the dot product between Newton's second law, $\mathbf{F}_i = m_i\ddot{\mathbf{r}}_i$, and an arbitrary virtual displacement. This is the general form of d'Alembert's equations.

$$\begin{aligned}\delta W &= \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i \\ &= \sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i \cdot \sum_{k=1}^{n} \boldsymbol{\tau}_{i_k}\delta q_k = \sum_{k=1}^{n}\sum_{i=1}^{N} \mathbf{f}_i \cdot \boldsymbol{\tau}_{i_k}\delta q_k \qquad (2.4) \\ &\equiv \sum_{k=1}^{n} Q_k\delta q_k \qquad (2.5)\end{aligned}$$

Here, $Q_k \equiv \sum_{i=1}^{N} \mathbf{f}_i \cdot \boldsymbol{\tau}_{i_k}$ are called the generalized forces. We can then write the following.

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i \cdot \boldsymbol{\tau}_{i_k} - Q_k\right)\delta q_k = 0 \qquad (2.6)$$

For holonomic systems, the variations δq_k are arbitrary and independent, and Eq. (2.5) can be written as follows.

$$\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i \cdot \boldsymbol{\tau}_{i_k} = Q_k \qquad (2.7)$$

This version of d'Alembert's principle is also called the "fundamental equation" [4].

3. The Time Rate Change of the Lagrangian Vectors

Consider again the Lagrangian vectors, $\boldsymbol{\tau}_{i_k}$. The time rate change of these vectors can be written in the following manner.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_k}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\tau}_{i_k} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{r}_i}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right) = \frac{\partial\dot{\mathbf{r}}_i}{\partial q_k} \qquad (2.8)$$

Using "cancelation of the overdots," we can then write the following [4].

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_k}\right) - \frac{\partial(\dot{\mathbf{r}}_i)}{\partial q_k} = 0 \qquad (2.9)$$

This result can now be combined with the definition of kinetic energy to arrive at Lagrange's equations.

4. Lagrange's Equations

The kinetic energy is defined as follows.

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \tag{2.10}$$

Consider the partial derivative of the kinetic energy with respect to the generalized coordinates and velocities.

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \tag{2.11}$$

$$\frac{\partial T}{\partial q_k} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} \tag{2.12}$$

Expanding the left hand side of d'Alembert's principle, Eq. (2.7), we can write the following.

$$\begin{aligned}\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \boldsymbol{\tau}_{i_k} &= \sum_{i=1}^{N} m_i \frac{\mathrm{d}\dot{\mathbf{r}}_i}{\mathrm{d}t} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}\right) - \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}\right) - \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} = Q_k \end{aligned} \tag{2.13}$$

Substituting the expressions from Eqs. (2.11) and (2.12) into the above result, we arrive at Lagrange's equations.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \tag{2.14}$$

This result is a valid form of Lagrange's equations, but a different form of Eq. (2.14) is desired for the direct linearization development. This motivates a second look at the generalized forces on the right hand side of the equation.

5. The Lagrangian Function

The generalized forces, Q_k, can be divided into potential forces, Q_{k_p}, and non-potential forces, $Q_{k_{np}}$, the difference being that the former are derivable from a scalar potential function, $V(t, q_k)$.

$$Q_{k_p} = f_{i_p} \cdot \boldsymbol{\tau}_{i_k} = -\nabla V \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\frac{\partial V}{\partial \mathbf{r}} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\frac{\partial V}{\partial q_k} \tag{2.15}$$

A new function called the Lagrangian, $L = T - V$, can now be defined and used to construct the familiar form of Lagrange's Equations.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_{k_{np}} \tag{2.16}$$

The application of this equation results in n governing equations of motion for the n generalized coordinates. This formulation of Lagrange's Equations for finite-dimensional systems may also be developed through the extended Hamilton's principle. Using this approach and applying calculus of variations, Lee and Junkins extend the ideas to systems with both finite- and infinite-dimensional generalized coordinates [3].

B. Equilibrium Properties

A dynamic system is said to be in a state of equilibrium when all the generalized velocities and accelerations are zero. Consider the following system.

$$\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}, t) \tag{2.17}$$

Because all generalized velocities and accelerations are zero, the left hand side of this equation is zero. An equilibrium point, $\boldsymbol{x}^*$ is then defined when the following is satisfied for all time t.

$$\mathbf{f}(\boldsymbol{x}^*, t) \equiv 0 \tag{2.18}$$

When we apply these equilibrium properties in conjunction with Lagrange's Equations, the resulting equation provides a more direct way to calculate the equilibrium configuration(s) for the class of systems of interest. These equations will be presented in this thesis with their respective class of systems.

C. Direct Linearization of Discrete Systems

In this section, an existing method for directly linearizing equations of motion for discrete systems is reviewed and illustrated in a two degree of freedom example. The goal of direct linearization is to produce linearized equations of motion from a first principle of motion. Throughout this thesis, Lagrange's equations are taken as the starting point.

Recall that the Lagrangian energy approach uses partial derivatives of the Lagrangian function, L, to generate the governing equations of motion of a system, $\mathrm{d}/\mathrm{d}t(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = Q_i$. Here, $L = T - V$ where T and V are the kinetic and potential energy respectively, q_i are the generalized coordinates, $\dot{q}_i$ are the generalized velocities, and Q_i are the generalized nonconservative forces. The kinetic energy can be partitioned into terms that are quadratic in the generalized velocities, T_2, linear in the generalized velocities, T_1, or with no dependence on the generalized velocities, T_0. The dynamic potential, which has no dependence on the generalized velocities, can then be defined as $U = V - T_0$. The kinetic energy function can then be written as $T = T_2 + T_1 + T_0$, and the Lagrangian can be written as $L = T_2 + T_1 - U$.

1. Equilibrium Configuration Solutions

After applying the equilibrium properties to Lagrange's equations, the equilibrium configurations are identified from the following equation.

$$\frac{\partial U}{\partial q_i} = 0 \tag{2.19}$$

This equation allows one to solve for an equilibrium configuration, a step necessary regardless of the linearization method employed. Note that, in general, there may be several solutions to this equation, and therefore several possible equilibrium configurations. If this is the case, one "target" equilibrium configuration of interest can be chosen for linearization purposes. Throughout this thesis, it is assumed that a single equilibrium configuration of interest is chosen even if several are found. If desired, one could apply the results presented in this thesis to each of the equilibrium configurations separately.

2. Direct Linearization

Perhaps the most important aspect of the direct linearization approach is the quadratic Taylor series expansion of the Lagrangian function about the chosen equilibrium configuration, $\boldsymbol{q}^*$, determined from Eq. (2.19) [1][2]. With no loss of generality, we use a change of variables, $\boldsymbol{q}_{new} = \boldsymbol{q}_{original} - \boldsymbol{q}^*$, to write the Taylor series expansion for perturbations from the equilibrium state.

$$\begin{aligned} L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = L\Big|_{(\mathrm{eq})} + \frac{\partial L}{\partial q_i}\Big|_{(\mathrm{eq})} q_i + \frac{\partial L}{\partial \dot{q}_i}\Big|_{(\mathrm{eq})} \dot{q}_i + \frac{1}{2}\frac{\partial^2 L}{\partial q_i \partial q_j}\Big|_{(\mathrm{eq})} q_i q_j \\ + \frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\Big|_{(\mathrm{eq})} \dot{q}_i \dot{q}_j + \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}\Big|_{(\mathrm{eq})} q_i \dot{q}_j + \dots \end{aligned} \tag{2.20}$$

The above equation involves partial derivatives of L evaluated at equilibrium, so the dynamic potential is studied within the same context.

$$U(\boldsymbol{q}) = U\Big|_{(\mathrm{eq})} + \frac{\partial U}{\partial q_i}\Big|_{(\mathrm{eq})} q_i + \frac{1}{2}\frac{\partial^2 U}{\partial q_i \partial q_j}\Big|_{(\mathrm{eq})} q_i q_j + \ldots \tag{2.21}$$

The first term is constant and has no effect on the equations of motion. The second term is identically zero. Consequently, a second order approximation of U contains a single term.

$$U(\boldsymbol{q}) \approx \frac{1}{2}\frac{\partial^2 U}{\partial q_i \partial q_j}\Big|_{(\mathrm{eq})} q_i q_j \tag{2.22}$$

The partial derivatives of T can be studied in a similar manner and coefficients evaluated about an equilibrium point can be defined and used to write the Lagrangian in quadratic form.

$$m_{ij} = \frac{\partial^2 T_2}{\partial \dot{q}_i \partial \dot{q}_j}\Big|_{(\mathrm{eq})} \quad ; \quad f_{ij} = \frac{\partial^2 T_1}{\partial q_i \partial \dot{q}_j}\Big|_{(\mathrm{eq})} \quad ; \quad k_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j}\Big|_{(\mathrm{eq})} \tag{2.23}$$

$$L^*(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j + f_{ij} q_i \dot{q}_j - \frac{1}{2} k_{ij} q_i q_j \tag{2.24}$$

Applying Lagrange's equations to L^* yields the linearized equations of motion for perturbations about the equilibrium point.

$$m_{ij}\ddot{q}_j + f_{ji}\dot{q}_j - f_{ij}\dot{q}_j + k_{ij} q_j = Q_i \tag{2.25}$$

One should note that, given the potential and kinetic energy of a discrete dynamical system, only a select number of partial derivatives must be computed a priori to form the linearized equations of motion. There is no need to first construct and then reduce the full nonlinear equations of motion.

3. Example: Rotating Hub with Two-Link Rigid Arm

As an example, consider the two degree of freedom discrete problem of a massless hub of radius R with two identical linked rigid arms of length l. A point mass, m, is attached to the end of the second arm. The hub rotates at a constant angular velocity, Ω. A spring of stiffness k attaches the first arm to the hub and a second identical spring attaches the second arm to the first. The angular displacement for each arm is ϕ_1 and ϕ_2, respectively, where each angle is measured relative to the position of the inboard body when the associated spring is undeformed. Figure (1) shows an illustration of this system.

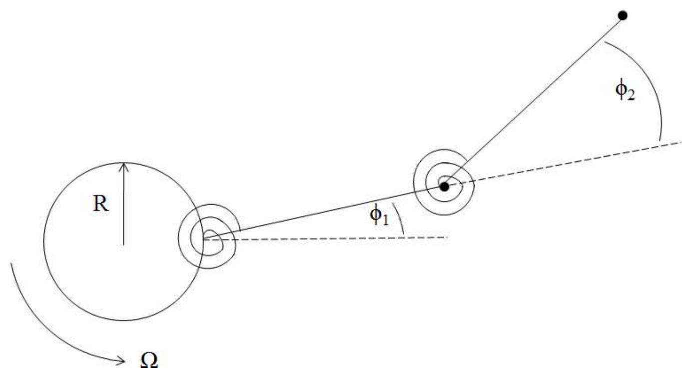

Fig. 1. Rotating Two-Link Arm Structure

The portions T_2 and T_1 of the kinetic energy are as follows.

$$T_2 = \frac{1}{2}m\left(2l^2\dot{\phi}_1^2 + l^2\dot{\phi}_2^2 + l^2\dot{\phi}_1\left(\dot{\phi}_1 + \dot{\phi}_2\right)\cos\phi_2\right) \tag{2.26}$$

$$T_1 = \frac{1}{2}m\Big(4l^2\Omega\dot{\phi}_1 + 2l^2\Omega\dot{\phi}_2 + 2Rl\Omega\dot{\phi}_1\cos\phi_1 + l^2\Omega\left(2\dot{\phi}_1 + \dot{\phi}_2\right)\cos\phi_2$$
$$+ Rl\Omega\left(\dot{\phi}_1 + \dot{\phi}_2\right)\cos\left(\phi_1 + \phi_2\right)\Big) \tag{2.27}$$

The dynamic potential is constructed from T_0 and the potential energy.

$$U = \frac{1}{2}k(\phi_1^2 + \phi_2^2) - \frac{1}{2}m\Big(2l^2\Omega^2 + R^2\Omega^2 + 2Rl\Omega^2\cos\phi_1 + l^2\Omega^2\cos\phi_2$$
$$+ Rl\Omega^2\cos\left(\phi_1 + \phi_2\right)\Big) \tag{2.28}$$

Using Eq. (2.19), an equilibrium solution $\phi_1 = \phi_2 = 0$ is found, and the following partial derivatives from Eq. (2.23) can be evaluated.

$$m_{11} = 3ml^2 \quad ; \quad m_{12} = m_{21} = \tfrac{1}{2}ml^2 \quad ; \quad m_{22} = ml^2$$
$$k_{11} = k + \tfrac{3}{2}mRl\Omega^2 \quad ; \quad k_{12} = k_{21} = \tfrac{1}{2}mRl\Omega^2 \quad ; \quad k_{22} = k + \tfrac{1}{2}ml(R+l)\Omega^2$$
$$f_{11} = f_{12} = f_{21} = f_{22} = 0 \tag{2.29}$$

Substituting these coefficients into Eq. (2.25) directly produces the linearized equations of motion.

$$3ml^2\ddot{\phi}_1 + \frac{1}{2}ml^2\ddot{\phi}_2 + \left(k + \frac{3}{2}mRl\Omega^2\right)\phi_1 + \frac{1}{2}mRl\Omega^2\phi_2 = 0 \tag{2.30}$$

$$\frac{1}{2}ml^2\ddot{\phi}_1 + ml^2\ddot{\phi}_2 + \frac{1}{2}mRl\Omega^2\phi_1 + \left(k + \frac{1}{2}ml(R+l)\Omega^2\right)\phi_2 = 0 \tag{2.31}$$

Again note that the linearized equations of motion were found directly; the full nonlinear governing equations were never constructed.

CHAPTER III

DIRECT LINEARIZATION OF DISCRETE RHEONOMIC SYSTEMS

In this chapter, the existing direct linearization results for discrete systems are extended to include rheonomic systems, or systems that have explicit time dependence. The addition of this system characteristic affects the formulation of equilibrium configurations as well as the directly linearized equations of motion. Note that the results are also valid for schleronomic systems which have no time dependence; terms related to explicit time dependence are zero for such systems.

A. The Kinetic Energy Function for Rheonomic Systems

The finite-dimensional direct linearization method can be generalized to include rheonomic systems. In order to construct the kinetic energy function for rheonomic systems, let us first define the position vector as an explicit function of generalized coordinates, $\boldsymbol{q}(t)$, and time.

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{q}, t) \tag{3.1}$$

We then have the following form for the velocity vector.

$$\dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial \boldsymbol{r}}{\partial q_i}\dot{q}_i + \frac{\partial \boldsymbol{r}}{\partial t} = \boldsymbol{\tau}_i \dot{q}_i + \boldsymbol{\tau}_0 \tag{3.2}$$

Here, $\boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_0$ are known as the Lagrangian vectors. The kinetic energy is then constructed as follows.

$$T = \frac{1}{2}m\left(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}\right) = \frac{1}{2}m\left(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j \dot{q}_i \dot{q}_j + 2\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_0 \dot{q}_i + \boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_0\right) \tag{3.3}$$

Recall that this function can be divided into three categories, T_2, T_1, and T_0. Terms that are quadratic in the generalized velocities are collected in T_2, terms linear in the generalized velocities comprise T_1, and terms independent of the generalized velocities are included in T_0.

$$T_2 = \frac{1}{2}m\left(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j \dot{q}_i \dot{q}_j\right) \quad ; \quad T_1 = m\left(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_0 \dot{q}_i\right) \quad ; \quad T_0 = \frac{1}{2}m\left(\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_0\right) \tag{3.4}$$

Note that the T_1 and T_0 terms contain components that are consistent only with rheonomic systems. That is, T_1 and T_0 do not exist for systems that are schleronomic. With this form for the kinetic energy, we can subtract the potential energy, V, to form Lagrangian, $L = L(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$.

B. Equilibrium Configuration Solutions

Let $U = V - T_0$ be the dynamic potential. Assuming only potential forces act on the system, the equilibrium configuration for the system is that which satisfies the following for all time t.

$$\frac{\partial^2 T_1}{\partial t \partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = 0 \tag{3.5}$$

Here, partial differentiation with respect to time indicates explicit differentiation only. That is, if $f = f(y_i(x,t), x, t)$, then $\partial f/\partial t = \partial f/\partial t$ only, and $\partial f/\partial t \neq (\partial f/\partial y_i)(\partial y_i/\partial t) + \partial f/\partial t$. For clarity, we adopt the Junkins and Kim notation df/dt $=(\partial f/\partial y_i)(\partial y_i/\partial t)$ $+\partial f/\partial t$, though this is technically also a partial derivative because both x and t are independent variables [7][8].

C. Direct Linearization

A change of variables is again chosen to simplify our development. We then apply Lagrange's equations to a quadratic Taylor series expansion of the Lagrangian in the generalized coordinates and velocities, and the following equations of motion result.

$$m_{ij}\ddot{q}_i + (\dot{m}_{ij} + f_{ij} - f_{ji})\dot{q}_i + (\dot{f}_{ij} + k_{ij})q_i = Q_i \tag{3.6}$$

Here, m_{ij}, f_{ij}, and k_{ij} are defined as before but may now also explicitly depend on time. Here and throughout this thesis, an overdot on the linearization coefficients is used only for notational compactness and indicates explicit partial differentiation with respect to time, $\partial/\partial t$. However, overdots on all other kinematic coordinates denote total time derivatives, $\dot{q}_i = \mathrm{d}q_i/\mathrm{d}t$.

D. Example: Accelerating, Rotating Rigid Arm

Consider the following simple example of a rheonomic system. A rigid arm of length l with a tip mass, m, is attached by a spring of stiffness k to an infinitesimally small hub with a prescribed angular velocity of $\dot{\theta} = \Omega t$. The angular displacement of the arm, ϕ, is measured with respect to a frame rotating with the hub as shown in Figure (2). The kinetic and potential energies for this system are the following.

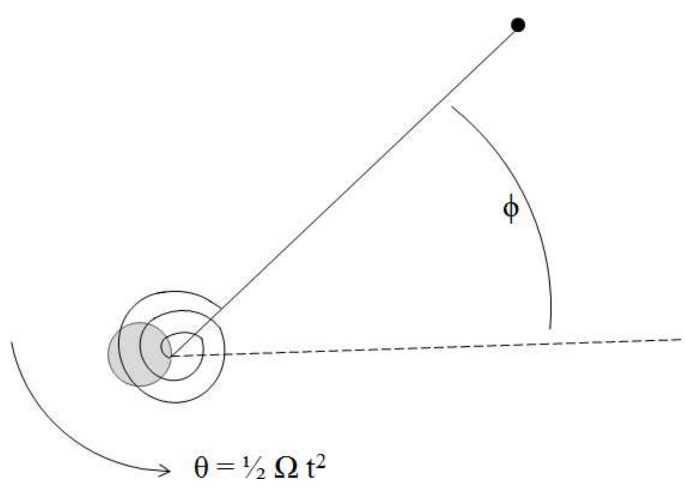

Fig. 2. Rotating One-Link Structure

$$T = \frac{1}{2} m l^2 \left(\dot{\theta} + \dot{\phi} \right)^2 \quad ; \quad V = \frac{1}{2} k \phi^2 \tag{3.7}$$

The equilibrium configuration can then be found for the system.

$$\frac{\partial^2 T_1}{\partial t \partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = m l^2 \Omega + k \phi = 0 \Rightarrow \phi^* = -\frac{m l^2 \Omega}{k} \tag{3.8}$$

A change of variables is employed such that $q = \phi - \phi^*$.

$$T = \frac{1}{2} m l^2 \left(\dot{\theta} + \dot{q} \right)^2 \quad ; \quad V = \frac{1}{2} k \left(q + \phi^* \right)^2 \tag{3.9}$$

The nonzero coefficients and resulting equation of motion about the equilibrium point are then the following.

$$m_{ij} = m l^2 \quad ; \quad k_{ij} = k \tag{3.10}$$

$$m l^2 \ddot{q} + k q = 0 \tag{3.11}$$

If desired, the change in variables can be reversed to obtain the equation of motion in terms of ϕ and its derivatives.

$$m l^2 \ddot{\phi} + k \phi + m l^2 \Omega = 0 \tag{3.12}$$

Note that incorrect equations would have resulted, regardless of the linearization method utilized, if the equilibrium condition had not been redefined to include rheonomic systems.

CHAPTER IV

DIRECT LINEARIZATION OF CONTINUOUS SYSTEMS

In this chapter, the direct linearization method is extended to infinite-dimensional systems. Lee and Junkins formulated a Lagrangian approach to produce the governing equations of motion for continuous and hybrid dynamical systems [3]. Here, the class of systems of interest are assumed to have a Lagrangian that can be written in the general form $L = L(\mathbf{w}_i, \dot{\mathbf{w}}_i, \mathbf{w}_i', \mathbf{w}_i'', x_i, t)$, where the strain energy terms, $\mathbf{w}_i'(x_i, t), \mathbf{w}_i''(x_i, t)$, belong only to the potential energy function. A hat over the Lagrangian indicates terms in the integrand. Note that the overdot represents the operator $\mathrm{d}/\mathrm{d}t$ acting on the variable, whereas the prime represents the operator $\mathrm{d}/\mathrm{d}x$ acting on the variable. The Lagrangian is constructed with the infinite-dimensional coordinate(s) $\mathbf{w}_i(x_i, t)$, its derivatives, boundary terms (L_B), and boundary conditions, where $i = 1$ for the single-body case (a), and $i = 1, \ldots, n$, for the $n > 1$ multi-body case (b).

$$(a) \quad L = \int_{l_0}^{l} \widehat{L} dx + L_B \quad ; \quad L_B = L_B(\mathbf{w}(l), \dot{\mathbf{w}}(l), \mathbf{w}'(l), \dot{\mathbf{w}}'(l), t)$$

$$\widehat{L} = \widehat{T} - \widehat{V} = \widehat{L}(\mathbf{w}, \dot{\mathbf{w}}, \mathbf{w}', \mathbf{w}'', x, t) \tag{4.1}$$

$$(b) \quad L = \sum_{i=1}^{n} \int_{l_{0i}}^{l_i} \widehat{L}^i \mathrm{d}x_i + L_B \quad ; \quad L_B = L_B(\underline{\mathbf{w}}(\underline{\mathbf{l}}), \underline{\dot{\mathbf{w}}}(\underline{\mathbf{l}}), \underline{\mathbf{w}'}(\underline{\mathbf{l}}), \underline{\dot{\mathbf{w}}}'(\underline{\mathbf{l}}), t)$$

$$\widehat{L}^i = \widehat{T}^i - \widehat{V}^i = \widehat{L}(\mathbf{w}_i, \dot{\mathbf{w}}_i, {\mathbf{w}_i}', {\mathbf{w}_i}'', \underline{\mathbf{w}}(\underline{\boldsymbol{l}}), \underline{\dot{\mathbf{w}}}(\underline{\boldsymbol{l}}), \underline{\mathbf{w}}'(\underline{\boldsymbol{l}}), \underline{\dot{\mathbf{w}}}'(\underline{\boldsymbol{l}}), x_i, t) \tag{4.2}$$

This distinction is necessary because boundary terms associated to multiple elastic domains must be accounted for in the multi-body Lagrangian. The underlined terms represent a vector of boundary terms, i.e., $\underline{\mathbf{w}}(\underline{\boldsymbol{l}}) = \mathbf{w}_i(l_i)$. For these boundary terms, l_i indicates a location at which $\mathbf{w}_i$ is evaluated, and the repeated index does not indicate summation. Also note the shorthand notation $\mathbf{w}(l) = \mathbf{w}(l, t)$, etc. Lagrange's equations for continuous systems are then the following [7].

$$(a)\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{L}}{\partial\dot{\mathbf{w}}}\right)-\frac{\partial\widehat{L}}{\partial\mathbf{w}}+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\widehat{L}}{\partial\mathbf{w}'}\right)-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial\widehat{L}}{\partial\mathbf{w}''}\right)=\widehat{\mathbf{f}}^T \tag{4.3}$$

$$(b)\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{L}^i}{\partial\dot{\mathbf{w}}_i}\right)-\frac{\partial\widehat{L}^i}{\partial\mathbf{w}_i}+\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\widehat{L}^i}{\partial\mathbf{w}_i'}\right)-\frac{\mathrm{d}^2}{\mathrm{d}x_i^2}\left(\frac{\partial\widehat{L}^i}{\partial\mathbf{w}_i''}\right)=\widehat{\mathbf{f}}^{iT} \tag{4.4}$$

Here, $\hat{\mathbf{f}}^{iT}$ is the nonconservative generalized force density vector related to $\boldsymbol{w}_i$, whereas $\mathbf{f}_1^{iT}$ and $\mathbf{f}_2^{iT}$ below are respectively the nonconservative force and torque vectors applied at the boundary, l_i.

When considering systems with an elastic domain, boundary conditions must be taken into account.

$$(a)\quad \left\{\frac{\partial\widehat{L}}{\partial\mathbf{w}'}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\widehat{L}}{\partial\mathbf{w}''}\right)\right\}\delta\mathbf{w}\Big|_{l_0}^{l}+\left\{\frac{\partial L_B}{\partial\mathbf{w}(l)}-\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L_B}{\partial\dot{\mathbf{w}}(l)}\right)\right\}\delta\mathbf{w}(l) \tag{4.5}$$

$$+\,\mathbf{f}_1^T\delta\mathbf{w}(l)=0 \tag{4.6}$$

$$\frac{\partial\widehat{L}}{\partial\mathbf{w}''}\delta\mathbf{w}'\Big|_{l_0}^{l}+\left\{\frac{\partial L_B}{\partial\mathbf{w}'(l)}-\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L_B}{\partial\dot{\mathbf{w}}'(l)}\right)\right\}\delta\mathbf{w}'(l)+\mathbf{f}_2^T\delta\mathbf{w}'(l)=0 \tag{4.7}$$

$$(b)\quad \left\{\frac{\partial\widehat{L}^i}{\partial\mathbf{w}_i'}-\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\widehat{L}^i}{\partial\mathbf{w}_i''}\right)\right\}\delta\mathbf{w}_i\Big|_{l_{0i}}^{l_i}+\left\{\frac{\partial L}{\partial\mathbf{w}_i(l_i)}-\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial\dot{\mathbf{w}}_i(l_i)}\right)\right\}\delta\mathbf{w}_i(l_i)$$

$$+\,\mathbf{f}_1^{iT}\delta\mathbf{w}_i(l_i)=0 \tag{4.8}$$

$$\frac{\partial\widehat{L}^i}{\partial\mathbf{w}_i''}\delta\mathbf{w}_i'\Big|_{l_{0i}}^{l_i}+\left\{\frac{\partial L}{\partial\mathbf{w}_i'(l_i)}-\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial\dot{\mathbf{w}}_i'(l_i)}\right)\right\}\delta\mathbf{w}_i'(l_i)+\mathbf{f}_2^{iT}\delta\mathbf{w}_i'(l_i)=0 \tag{4.9}$$

Note that we can again partition the kinetic energy in terms of the order of the generalized velocities, $\dot{\mathbf{w}}_i$.

$$\widehat{L}=\widehat{T}-\widehat{V}=\widehat{T}_2+\widehat{T}_1+\widehat{T}_0-\widehat{V}=\widehat{T}_2+\widehat{T}_1-\widehat{U}$$

$$L_B=T_{2B}+T_{1B}+T_{0B}-V_B=T_{2B}+T_{1B}-U_B \tag{4.10}$$

This notation allows the equilibrium configuration solutions to be clearly defined.

A. Equilibrium Configuration Solutions

In order to evaluate the partial derivatives at the equilibrium configuration, an understanding of equilibrium solutions for continuous systems must be formed. Equations for describing these configurations are obtained in the same manner as those for a discrete system. Applying equilibrium properties to Lagrange's Equations, Eqs. (4.3) and (4.4), one can define equilibrium configurations using the following equations.

$$(a)\quad \frac{\partial}{\partial t}\left(\frac{\partial \widehat{T}_1}{\partial \dot{\mathbf{w}}}\right) + \frac{\partial \widehat{U}}{\partial \mathbf{w}} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \widehat{U}}{\partial \mathbf{w}'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \widehat{U}}{\partial \mathbf{w}''}\right) = 0 \tag{4.11}$$

$$(b)\quad \frac{\partial}{\partial t}\left(\frac{\partial \widehat{T}_1^i}{\partial \dot{\mathbf{w}}_i}\right) + \frac{\partial \widehat{U}^i}{\partial \mathbf{w}_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \widehat{U}^i}{\partial \mathbf{w}_i'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \widehat{U}^i}{\partial \mathbf{w}_i''}\right) = 0 \tag{4.12}$$

Even at equilibrium we must consider both geometric and natural boundary conditions. The spatial boundary conditions simplify to the following at equlibrium.

$$(a)\quad \left\{\frac{\partial}{\partial x}\left(\frac{\partial \widehat{U}}{\partial \mathbf{w}''}\right) - \frac{\partial \widehat{U}}{\partial \mathbf{w}'}\right\}\delta \mathbf{w}\Big|_{l_0}^{l} + \left\{\frac{\partial^2 U_B}{\partial t \partial \dot{\mathbf{w}}(l)} - \frac{\partial U_B}{\partial \mathbf{w}(l)}\right\}\delta \mathbf{w}(l) = 0 \tag{4.13}$$

$$-\frac{\partial \widehat{U}}{\partial \mathbf{w}''}\delta \mathbf{w}'\Big|_{l_0}^{l} + \left\{\frac{\partial^2 U_B}{\partial t \partial \dot{\mathbf{w}}'(l)} - \frac{\partial U_B}{\partial \mathbf{w}'(l)}\right\}\delta \mathbf{w}'(l) = 0 \tag{4.14}$$

$$(b)\quad \left\{\frac{\partial}{\partial x_i}\left(\frac{\partial \widehat{U}^i}{\partial {\mathbf{w}_i}''}\right) - \frac{\partial \widehat{U}^i}{\partial {\mathbf{w}_i}'}\right\}\delta \mathbf{w}_i\Big|_{l_{0i}}^{l_i} + \left\{\frac{\partial^2 U}{\partial t \partial \dot{\mathbf{w}}_i(l_i)} - \frac{\partial U}{\partial \mathbf{w}_i(l_i)}\right\}\delta \mathbf{w}_i(l_i) = 0 \tag{4.15}$$

$$-\frac{\partial \widehat{U}^i}{\partial {\mathbf{w}_i}''}\delta {\mathbf{w}_i}'\Big|_{l_{0i}}^{l_i} + \left\{\frac{\partial^2 U}{\partial t \partial {\dot{\mathbf{w}}_i}'(l_i)} - \frac{\partial U}{\partial \mathbf{w}_i'(l_i)}\right\}\delta \mathbf{w}_i'(l_i) = 0 \tag{4.16}$$

In order to find an equilibrium solution for a continuous system with a single [multiple] elastic domain[s], the set of equations resulting from applying Eq. (4.11) [Eq. (4.12)] must be solved using the boundary conditions from Eqs. (4.13) and (4.14) [Eqs. (4.15) and (4.16)]. Recall that several equilibrium configurations which satisfy these equations may exist. In the following development, it is assumed that a single "target" equilibrium configuration is chosen about which to linearize the equations of motion.

B. Direct Linearization: Single Elastic Body

Consider only the continuous Lagrangian, $\widehat{L}$. The quadratic Taylor series form of this function, $\widehat{L}^*$, can be formed in the same manner as L^* for discrete systems: the Lagrangian is expressed in terms of partial derivatives evaluated at the target equilibrium configuration, and only terms of second order and lower are retained. Lagrange's equations are then applied to $\widehat{L}^*$ to construct the equations of motion. A change of variables is again chosen for simplification such that $\mathbf{w}_{new} = \mathbf{w}_{original} - \mathbf{w}^*$, where $\mathbf{w}^*$ is the equilibrium solution. If this expression is differentiated with respect to x, the result reveals that $\mathbf{w}'$, $\mathbf{w}''$, etc. are also equal to zero at equilibrium. This change of variables will eventually result in linearized equations of motion that describe the *departure* motion from the target equilibrium, or the motion of the system relative to this equilibrium configuration. Expanding $\widehat{L}$ using a Taylor series expansion gives the following.

$$\begin{aligned}
\widehat{L}(\mathbf{w}, \dot{\mathbf{w}}, \mathbf{w}', \mathbf{w}'', x, t) = \widehat{L}|_{(\mathrm{eq})} &+ \frac{\partial \widehat{L}}{\partial \mathbf{w}}\Big|_{(\mathrm{eq})} \mathbf{w} + \frac{\partial \widehat{L}}{\partial \dot{\mathbf{w}}}\Big|_{(\mathrm{eq})} \dot{\mathbf{w}} + \frac{\partial \widehat{L}}{\partial \mathbf{w}'}\Big|_{(\mathrm{eq})} \mathbf{w}' + \frac{\partial \widehat{L}}{\partial \mathbf{w}''}\Big|_{(\mathrm{eq})} \mathbf{w}'' \\
&+ \frac{1}{2}\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}^2}\Big|_{(\mathrm{eq})} \mathbf{w}^2 + \frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \dot{\mathbf{w}}}\Big|_{(\mathrm{eq})} \mathbf{w}\dot{\mathbf{w}} + \frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \mathbf{w}'}\Big|_{(\mathrm{eq})} \mathbf{w}\mathbf{w}' \\
&+ \frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \mathbf{w}''}\Big|_{(\mathrm{eq})} \mathbf{w}\mathbf{w}'' + \frac{1}{2}\frac{\partial^2 \widehat{L}}{\partial \dot{\mathbf{w}}^2}\Big|_{(\mathrm{eq})} \dot{\mathbf{w}}^2 + \frac{\partial^2 \widehat{L}}{\partial \dot{\mathbf{w}}\partial \mathbf{w}'}\Big|_{(\mathrm{eq})} \dot{\mathbf{w}}\mathbf{w}' \\
&+ \frac{\partial^2 \widehat{L}}{\partial \dot{\mathbf{w}}\partial \mathbf{w}''}\Big|_{(\mathrm{eq})} \dot{\mathbf{w}}\mathbf{w}'' + \frac{1}{2}\frac{\partial^2 \widehat{L}}{\partial (\mathbf{w}')^2}\Big|_{(\mathrm{eq})} (\mathbf{w}')^2 \\
&+ \frac{\partial^2 \widehat{L}}{\partial \mathbf{w}'\partial \mathbf{w}''}\Big|_{(\mathrm{eq})} \mathbf{w}'\mathbf{w}'' + \frac{1}{2}\frac{\partial^2 \widehat{L}}{\partial (\mathbf{w}'')^2}\Big|_{(\mathrm{eq})} (\mathbf{w}'')^2 + \ldots
\end{aligned} \tag{4.17}$$

Key coefficients in the above expansion can be defined.

$$
\begin{array}{lllll}
L_0 = \widehat{L}|_{(\mathrm{eq})} & ; & L_1 = \left.\frac{\partial \widehat{L}}{\partial \mathbf{w}}\right|_{(\mathrm{eq})} & ; & L_2 = \left.\frac{\partial \widehat{L}}{\partial \dot{\mathbf{w}}}\right|_{(\mathrm{eq})} \\
L_3 = \left.\frac{\partial \widehat{L}}{\partial \mathbf{w}'}\right|_{(\mathrm{eq})} & ; & L_4 = \left.\frac{\partial \widehat{L}}{\partial \mathbf{w}''}\right|_{(\mathrm{eq})} & ; & L_5 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}^2}\right|_{(\mathrm{eq})} \\
L_6 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \dot{\mathbf{w}}}\right|_{(\mathrm{eq})} & ; & L_7 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \mathbf{w}'}\right|_{(\mathrm{eq})} & ; & L_8 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \mathbf{w}''}\right|_{(\mathrm{eq})} \\
L_9 = \left.\frac{\partial^2 \widehat{L}}{\partial \dot{\mathbf{w}}^2}\right|_{(\mathrm{eq})} & ; & L_{10} = \left.\frac{\partial^2 \widehat{L}}{\partial \dot{\mathbf{w}}\partial \mathbf{w}'}\right|_{(\mathrm{eq})} & ; & L_{11} = \left.\frac{\partial^2 \widehat{L}}{\partial \dot{\mathbf{w}}\partial \mathbf{w}''}\right|_{(\mathrm{eq})} \\
L_{12} = \left.\frac{\partial^2 \widehat{L}}{\partial (\mathbf{w}')^2}\right|_{(\mathrm{eq})} & ; & L_{13} = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}'\partial \mathbf{w}''}\right|_{(\mathrm{eq})} & ; & L_{14} = \left.\frac{\partial^2 \widehat{L}}{\partial (\mathbf{w}'')^2}\right|_{(\mathrm{eq})}
\end{array}
\tag{4.18}
$$

The quadratic Taylor series version of $\widehat{L}$ is formed by neglecting terms higher than second order.

$$
\begin{aligned}
\widehat{L}^* = {} & L_0 + L_1\mathbf{w} + L_2\dot{\mathbf{w}} + L_3\mathbf{w}' + L_4\mathbf{w}'' + \frac{1}{2}L_5\mathbf{w}^2 + L_6\mathbf{w}\dot{\mathbf{w}} \\
& + L_7\mathbf{w}\mathbf{w}' + L_8\mathbf{w}\mathbf{w}'' + \frac{1}{2}L_9\dot{\mathbf{w}}^2 + L_{10}\dot{\mathbf{w}}\mathbf{w}' \\
& + L_{11}\dot{\mathbf{w}}\mathbf{w}'' + \frac{1}{2}L_{12}(\mathbf{w}')^2 + L_{13}\mathbf{w}'\mathbf{w}'' + \frac{1}{2}L_{14}(\mathbf{w}'')^2
\end{aligned}
\tag{4.19}
$$

The generalized velocity, $\dot{\mathbf{w}}$, is assumed to be found only in the kinetic energy, whereas the strain energy variables, $\mathbf{w}'$ and $\mathbf{w}''$, are assumed to be found only in the potential energy; thus coefficients involving both, L_{10} and L_{11}, are zero. Applying Lagrange's equations (4.3) to $\widehat{L}^*$, the continuous system linearized equation of motion is obtained.

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\left(L_2 + L_6\mathbf{w} + L_9\dot{\mathbf{w}}\right) - \left(L_1 + L_5\mathbf{w} + L_6\dot{\mathbf{w}} + L_7\mathbf{w}' + L_8\mathbf{w}''\right) \\
& \quad + \frac{\mathrm{d}}{\mathrm{d}x}\left(L_3 + L_7\mathbf{w} + L_{12}\mathbf{w}' + L_{13}\mathbf{w}''\right) \\
& - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(L_4 + L_8\mathbf{w} + L_{13}\mathbf{w}' + L_{14}\mathbf{w}''\right) = 0
\end{aligned}
\tag{4.20}
$$

Performing the implied partial derivatives gives the explicit linearized equation of motion.

$$L_9\ddot{\mathbf{w}} + \dot{L}_9\dot{\mathbf{w}} - (L_5 + \dot{L}_6 + L_7' + L_8'')\mathbf{w} - (2L_8' - L_{12}' + L_{13}'')\mathbf{w}'$$
$$- (2L_8 - L_{12} + L_{13}' + L_{14}'')\mathbf{w}'' - 2L_{14}'\mathbf{w}''' - L_{14}\mathbf{w}'''' = \widehat{\mathbf{f}} \tag{4.21}$$

By simply computing the partial derivatives associated with eight coefficients (L_5, L_6, L_7, L_8, L_9, L_{12}, L_{13}, and L_{14}) and substituting the results into the above equation, the linearized equation of motion for this class of continuous systems is directly constructed. That is, we can form the linearized equations directly from the Lagrangian with the above expression and an equilibrium configuration. The overdot and the prime over the linearization coefficients indicate explicit partial differentiation by t and x respectively. Note that the coefficient L_0 does not contribute when Lagrange's Equations are applied. Also, terms involving L_1, L_2, L_3, and L_4 sum to zero due to the equilibrium condition resulting from Lagrange's Equation.

C. Direct Linearization: Multiple Elastic Bodies

An analogous treatment can be constructed for the multiple deformable domains. However, for this class of systems it is assumed that the domains can interact at the boundaries, so the contribution of boundary terms, $\underline{\mathbf{w}}(\underline{l}), \underline{\dot{\mathbf{w}}}(\underline{l}), \underline{\mathbf{w}}'(\underline{l}), \underline{\dot{\mathbf{w}}}'(\underline{l})$, must also be taken to account. Note that the change of variables also affects these terms. The contributing coefficients from the resulting linearized equations of motion are the following.

$$
\begin{aligned}
&L_1^i = \frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i^2}\bigg|_{(\mathrm{eq})} \quad ; \quad L_2^i = \frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i \partial \dot{\mathbf{w}}_i}\bigg|_{(\mathrm{eq})} \quad ; \quad L_3^i = \frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i \partial \mathbf{w}_i'}\bigg|_{(\mathrm{eq})} \\
&L_4^i = \frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i \partial \mathbf{w}_i''}\bigg|_{(\mathrm{eq})} \quad ; \quad L_5^i = \frac{\partial^2 \widehat{L}^i}{\partial \dot{\mathbf{w}}_i^2}\bigg|_{(\mathrm{eq})} \quad ; \quad L_6^i = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}')^2}\bigg|_{(\mathrm{eq})} \\
&L_7^i = \frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i' \partial \mathbf{w}_i''}\bigg|_{(\mathrm{eq})} \quad ; \quad L_8^i = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i'')^2}\bigg|_{(\mathrm{eq})} \quad ; \quad L_9^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i) \partial \mathbf{w}_j(l_j)}\bigg|_{(\mathrm{eq})} \\
&L_{10}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i) \partial \dot{\mathbf{w}}_j(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{11}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i) \partial \mathbf{w}_j'(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{12}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i) \partial \dot{\mathbf{w}}_j'(l_j)}\bigg|_{(\mathrm{eq})} \\
&L_{13}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\dot{\mathbf{w}}_i) \partial \mathbf{w}_j(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{14}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\dot{\mathbf{w}}_i) \partial \dot{\mathbf{w}}_j(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{15}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\dot{\mathbf{w}}_i) \partial \mathbf{w}_j'(l_j)}\bigg|_{(\mathrm{eq})} \\
&L_{16}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\dot{\mathbf{w}}_i) \partial \dot{\mathbf{w}}_j'(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{17}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}'_i) \partial \mathbf{w}_j(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{18}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}'_i) \partial \dot{\mathbf{w}}_j(l_j)}\bigg|_{(\mathrm{eq})} \\
&L_{19}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}'_i) \partial \mathbf{w}_j'(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{20}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}'_i) \partial \dot{\mathbf{w}}_j'(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{21}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''_i) \partial \mathbf{w}_j(l_j)}\bigg|_{(\mathrm{eq})} \\
&L_{22}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''_i) \partial \dot{\mathbf{w}}_j(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{23}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''_i) \partial \mathbf{w}_j'(l_j)}\bigg|_{(\mathrm{eq})} \quad ; \quad L_{24}^{ij} = \frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''_i) \partial \dot{\mathbf{w}}_j'(l_j)}\bigg|_{(\mathrm{eq})}
\end{aligned}
\tag{4.22}
$$

A linearized equation of motion is then constructed for each of the $i = 1, \ldots, n$ elastic domains.

$$
\begin{aligned}
&L_5^i \ddot{\mathbf{w}}_i + \dot{L}_5^i \dot{\mathbf{w}}_i - (L_1^i + \dot{L}_2^i + {L_3^i}' + {L_4^i}'')\mathbf{w}_i - (2{L_4^i}' - {L_6^i}' + {L_7^i}'')\mathbf{w}_i' \\
&\quad - (2L_4^i - L_6^i + {L_7^i}' + {L_8^i}'')\mathbf{w}_i'' - 2{L_8^i}'\mathbf{w}_i''' - L_8^i \mathbf{w}_i'''' \\
&\quad - (L_9^{ij} - \dot{L}_{13}^{ij} - {L_{17}^{ij}}' + {L_{21}^{ij}}'')\mathbf{w}_j(l_j) - (L_{10}^{ij} - L_{13}^{ij} - \dot{L}_{14}^{ij} - {L_{18}^{ij}}' + {L_{22}^{ij}}'')\dot{\mathbf{w}}_j(l_j) \\
&\quad - (L_{11}^{ij} - \dot{L}_{15}^{ij} - {L_{19}^{ij}}' + {L_{23}^{ij}}'')\mathbf{w}_j'(l_j) - (L_{12}^{ij} - L_{15}^{ij} - \dot{L}_{16}^{ij} - {L_{20}^{ij}}' + {L_{24}^{ij}}'')\dot{\mathbf{w}}_j'(l_j) \\
&\quad + L_{14}^{ij}\ddot{\mathbf{w}}_j(l_j) + L_{16}^{ij}\ddot{\mathbf{w}}_j'(l_j) = \widehat{\mathbf{f}}^i
\end{aligned}
\tag{4.23}
$$

The interactions between the multiple bodies have clearly complicated the linearization process, but this direct result still provides a more straightforward means of generating the desired equations.

D. Example: Rotating Hub with Flexible T-shaped Arm

Figure (3) illustrates a three-beam system, where two beams are connected perpendicularly to the end of the first in a "T" shape. The system is rotating at a prescribed constant angular velocity, Ω, and the material properties, ρ_i, E_i, and I_i, and lengths, l_i, of the two perpendicular beams are identical.

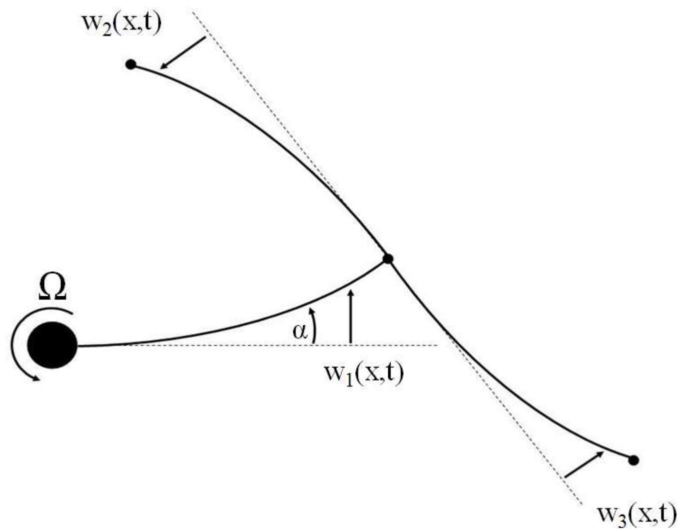

Fig. 3. Rotating Three-Beam "T" Structure

The Lagrangian for the system is first constructed and is as follows.

$$
\begin{aligned}
L =& \frac{1}{2}\rho_1 \int_0^{l_1} \left(\mathbf{w}_1^2\Omega^2 + \dot{\mathbf{w}}_1^2 + 2\dot{\mathbf{w}}_1 x_1\Omega + x_1^2\Omega^2\right) \mathrm{d}x_1 \\
&+ \frac{1}{2}\rho_2 \int_0^{l_2} \Bigg(\left[\mathrm{w}_1(l_1)\right]^2 \Omega^2 + 2\dot{\mathrm{w}}_1(l_1) l_1\Omega + l_1^2\Omega^2 + \left[\dot{\mathrm{w}}_1(l_1)\right]^2 + \mathbf{w}_2^2\left(\Omega + \dot{\alpha}\right)^2 \\
&+ \dot{\mathbf{w}}_2^2 + x_2^2\left(\Omega + \dot{\alpha}\right)^2 + 2\dot{\mathbf{w}}_2 x_2\left(\Omega + \dot{\alpha}\right) - 2\mathrm{w}_1(l_1)\Omega\mathbf{w}_2\left(\Omega + \dot{\alpha}\right)\sin\alpha \\
&+ 2\mathrm{w}_1(l_1)\Omega\left(\dot{\mathbf{w}}_2 + x_2\left(\Omega + \dot{\alpha}\right)\right)\cos\alpha - 2\left(\dot{\mathrm{w}}_1(l_1) + l_1\Omega\right)\mathbf{w}_2\left(\Omega + \dot{\alpha}\right)\cos\alpha \\
&- 2\left(\dot{\mathrm{w}}_1(l_1) + l_1\Omega\right)\left(\dot{\mathbf{w}}_2 + \left(\Omega + \dot{\alpha}\right)\right)\sin\alpha \Bigg) \mathrm{d}x_2 \\
&+ \frac{1}{2}\rho_3 \int_0^{l_3} \Bigg(\left[\mathrm{w}_1(l_1)\right]^2 \Omega^2 + 2\dot{\mathrm{w}}_1(l_1) l_1\Omega + l_1^2\Omega^2 + \left[\dot{\mathrm{w}}_1(l_1)\right]^2 + \mathbf{w}_3^2\left(\Omega + \dot{\alpha}\right)^2 \\
&+ \dot{\mathbf{w}}_3^2 + x_3^2\left(\Omega + \dot{\alpha}\right)^2 + 2\dot{\mathbf{w}}_3 x_3\left(\Omega + \dot{\alpha}\right) + 2\mathrm{w}_1(l_1)\Omega\mathbf{w}_3\left(\Omega + \dot{\alpha}\right)\sin\alpha \\
&- 2\mathrm{w}_1(l_1)\Omega\left(\dot{\mathbf{w}}_3 + x_3\left(\Omega + \dot{\alpha}\right)\right)\cos\alpha + 2\left(\dot{\mathrm{w}}_1(l_1) + l_1\Omega\right)\mathbf{w}_3\left(\Omega + \dot{\alpha}\right)\cos\alpha \\
&+ 2\left(\dot{\mathrm{w}}_1(l_1) + l_1\Omega\right)\left(\dot{\mathbf{w}}_3 + \left(\Omega + \dot{\alpha}\right)\right)\sin\alpha \Bigg) \mathrm{d}x_3 \\
&- \frac{1}{2}\sum_{i=1}^{3}\left(\int_0^{l_i} E_i I_i \left(\frac{\partial^2 \mathbf{w}_i}{\partial x_i^2}\right)^2 \mathrm{d}x_i\right)
\end{aligned}
\tag{4.24}
$$

Here, $\alpha = \mathrm{w}_1'(l_1)$ for clarity. The equilibrium solution is computed by applying Eqs. (4.11) with (4.15) and (4.16) to each elastic domain to obtain the following partial differential equations and boundary conditions.

- First Beam:

$$
\begin{aligned}
&\rho_1 \mathbf{w}_1 \Omega^2 - E_1 I_1 \mathbf{w}_1'''' = 0 \\
&\mathrm{w}_1(0) = 0 \quad ; \quad \mathrm{w}_1'(0) = 0 \\
&E_1 I_1 \mathrm{w}_1'''(l_1) + \rho_2 \left(\mathrm{w}_1(l_1) l_2 \Omega^2 - \Omega^2 \sin\alpha \int_0^{l_2} \mathbf{w}_2 \mathrm{d}x_2 + \tfrac{1}{2} l_2^2 \Omega^2 \cos\alpha \right) \\
&\quad + \rho_3 \left(\mathrm{w}_1(l_1) l_3 \Omega^2 + \Omega^2 \sin\alpha \int_0^{l_3} \mathbf{w}_3 \mathrm{d}x_3 - \tfrac{1}{2} l_3^2 \Omega^2 \cos\alpha \right) = 0 \\
&E_1 I_1 \mathrm{w}_1''(l_1) + \rho_2 \Bigg(\mathrm{w}_1(l_1) \Omega^2 \cos\alpha \int_0^{l_2} \mathbf{w}_2 \mathrm{d}x_2 + \tfrac{1}{2} \mathrm{w}_1(l_1) l_2^2 \Omega^2 \sin\alpha \\
&\quad - l_1 \Omega^2 \sin\alpha \int_0^{l_2} \mathbf{w}_2 \mathrm{d}x_2 + \tfrac{1}{2} l_1 l_2^2 \Omega^2 \cos\alpha \Bigg) \\
&\quad + \rho_3 \Bigg(- \mathrm{w}_1(l_1) \Omega^2 \cos\alpha \int_0^{l_3} \mathbf{w}_3 \mathrm{d}x_3 - \tfrac{1}{2} \mathrm{w}_1(l_1) l_3^2 \Omega^2 \sin\alpha \\
&\quad + l_1 \Omega^2 \sin\alpha \int_0^{l_3} \mathbf{w}_3 \mathrm{d}x_3 - \tfrac{1}{2} l_1 l_3^2 \Omega^2 \cos\alpha \Bigg) = 0
\end{aligned}
\tag{4.25}
$$

- Second Beam:

$$
\begin{aligned}
&\rho_2 \mathbf{w}_2 \Omega^2 - \rho_2 l_1 \Omega^2 \cos\alpha - \rho_2 \mathrm{w}_1(l_1) \Omega^2 \sin\alpha - E_2 I_2 \mathbf{w}_2'''' = 0 \\
&\mathrm{w}_2(0) = 0 \quad ; \quad \mathrm{w}_2'(0) = 0 \quad ; \quad \mathrm{w}_2''(l_2) = 0 \quad ; \quad \mathrm{w}_2'''(l_2) = 0 \\
&\rho_3 \mathbf{w}_3 \Omega^2 + \rho_3 l_1 \Omega^2 \cos\alpha + \rho_3 \mathrm{w}_1(l_1) \Omega^2 \sin\alpha - E_3 I_2 \mathbf{w}_3'''' = 0
\end{aligned}
\tag{4.26}
$$

- Third Beam:

$$
\mathrm{w}_3(0) = 0 \quad ; \quad \mathrm{w}_3'(0) = 0 \quad ; \quad \mathrm{w}_3''(l_3) = 0 \quad ; \quad \mathrm{w}_3'''(l_3) = 0 \tag{4.27}
$$

One equilibrium solution is given by $\mathbf{w}_1 = \mathbf{w}_1' = \mathbf{w}_1'' = \mathbf{w}_1''' = \mathbf{w}_1'''' = 0$. That is, the first beam is undeformed. This leaves the following equations, which must be solved to completely specify the equilibrium configuration.

$$
\rho_2 \mathbf{w}_2 \Omega^2 - \rho_2 l_1 \Omega^2 - E_2 I_2 \mathbf{w}_2'''' = 0 \quad ; \quad \rho_3 \mathbf{w}_3 \Omega^2 + \rho_3 l_1 \Omega^2 - E_3 I_2 \mathbf{w}_3'''' = 0 \tag{4.28}
$$

These linear, fourth-order equations can be solved by assuming a homogeneous and a particular solution before solving for the particular solution and the coefficients of the homogenous solution [9]. For the second and third beams, the particular solutions are $\mathbf{w}_{2_p} = l_1$ and $\mathbf{w}_{3_p} = -l_1$, respectively. The subscript i_p indicates a particular solution. Note that these solutions suggest axis-symmetry in the system and are dependent on the length of the first beam. Next, a homogeneous solution can be found by assuming a solution of the form $\mathbf{w}(x) = c_1 \sin(\beta x) + c_2 \cos(\beta x) + c_3 \cosh(\beta x) + c_4 \sinh(\beta x)$ and using the four boundary conditions to determine the constants c_i and d_i for second and third beams.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -\sin(\beta l_2) & -\cos(\beta l_2) & \cosh(\beta l_2) & \sinh(\beta l_2) \\ -\cos(\beta l_2) & \sin(\beta l_2) & \sinh(\beta l_2) & \cosh(\beta l_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -l_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.29}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -\sin(\beta l_3) & -\cos(\beta l_3) & \cosh(\beta l_3) & \sinh(\beta l_3) \\ -\cos(\beta l_3) & \sin(\beta l_3) & \sinh(\beta l_3) & \sinh(\beta l_3) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.30}$$

Each row of the matrix equations represents a boundary condition. The variable β is defined as $\beta^4 \equiv \Omega^2 \rho_i / E_i I_i$. This boundary value problem is nonhomogeneous, so it has a unique solution only if the determinant of the related homogeneous problem is nonzero [9]. That is, the value of $\beta_i l_j$ cannot be an eigenvalue of the homogeneous problem if one wishes to find a unique solution. Otherwise, there are an infinite number of solutions to the homogenous problem. Assuming that each $\beta_i l_j$ is not an eigenvalue of the homogeneous problem, both of the remaining beams have a unique solution.

By defining the coefficients for the solution to be c_1, c_2, c_3, and c_4 for the second beam and d_1, d_2, d_3, and d_4 for the third beam, the following solutions for $\mathbf{w}_2$ and

$\mathbf{w}_3$ at the specified equilibrium configuration can be constructed.

$$\mathbf{w}_2^*(x_2) = c_1 \sin(\beta x_2) + c_2 \cos(\beta x_2) + c_3 \cosh(\beta x_2) + c_4 \sinh(\beta x_2) + l_1 \tag{4.31}$$

$$\mathbf{w}_3^*(x_3) = d_1 \sin(\beta x_3) + d_2 \cos(\beta x_3) + d_3 \cosh(\beta x_3) + d_4 \sinh(\beta x_3) - l_1 \tag{4.32}$$

Here, $\mathbf{w}_i^*$ indicates the equilibrium solution for $\mathbf{w}_i$. The direct linearization method can now be applied by calculating the partial derivative coefficients for each elastic domain. The following nonzero coefficients for each of the three beams can then be computed.

- First Beam:

$$L_1^1 = \rho_1 \Omega^2 \quad ; \quad L_5^1 = \rho_1 \quad ; \quad L_8^1 = -E_1 I_1 \tag{4.33}$$

- Second Beam:

$$\begin{aligned} &L_1^2 = \rho_2 \Omega^2 &&; \quad L_5^2 = \rho_2 &&; \quad L_8^2 = -E_2 I_2 &&; \quad L_{10}^{21} = -\rho_2 \Omega \\ &L_{12}^{21} = -\rho_2 l_1 \Omega + 2\Omega \rho_2 \mathbf{w}_2^* &&; \quad L_{13}^{21} = \rho_2 \Omega &&; \quad L_{15}^{21} = -\rho_2 l_1 \Omega &&; \quad L_{16}^{21} = \rho_2 x_2 \end{aligned} \tag{4.34}$$

- Third Beam:

$$\begin{aligned} &L_1^3 = \rho_3 \Omega^2 &&; \quad L_5^3 = \rho_3 &&; \quad L_8^3 = -E_3 I_3 &&; \quad L_{10}^{31} = \rho_3 \Omega \\ &L_{12}^{31} = \rho_3 l_1 \Omega + 2\rho_3 \Omega \mathbf{w}_3^* &&; \quad L_{13}^{31} = -\rho_3 \Omega &&; \quad L_{15}^{31} = \rho_3 l_1 \Omega &&; \quad L_{16}^{31} = \rho_3 x_3 \end{aligned} \tag{4.35}$$

These coefficients can be directly substituted into Eq. (4.23) to find the three equations of motion. The equations of motion are then the following.

$$\rho_1 \ddot{\mathbf{w}}_1 - \rho_1 \Omega^2 \mathbf{w}_1 + E_1 I_1 \mathbf{w}_1'''' = 0 \tag{4.36}$$

$$\rho_2 \ddot{\mathbf{w}}_2 - \rho_2 \Omega^2 \mathbf{w}_2 + E_2 I_2 \mathbf{w}_2'''' + 2\rho_2 \Omega \dot{\mathrm{w}}_1(l_1) - 2\rho_2 \Omega \mathbf{w}_2^* \dot{\mathrm{w}}'(l_1) + \rho_2 x_2 \ddot{\mathrm{w}}'(l_1) = 0 \tag{4.37}$$

$$\rho_3 \ddot{\mathbf{w}}_3 - \rho_3 \Omega^2 \mathbf{w}_3 + E_3 I_3 \mathbf{w}_3'''' - 2\rho_3 \Omega \dot{\mathrm{w}}_1(l_1) - 2\rho_3 \Omega \mathbf{w}_3^* \dot{\mathrm{w}}'(l_1) + \rho_3 x_3 \ddot{\mathrm{w}}'(l_1) = 0 \tag{4.38}$$

In addition to highlighting the interaction between the multiple elastic domains, this example showcases how the direct linearization method presented here can reduce the effort required to obtain the linearized equations of motion for more complex continuous systems. The alternative would require one to first generate the complete set of nonlinear equations of motion for the system and then linearize, which can be difficult and prone to error.

CHAPTER V

DIRECT LINEARIZATION OF HYBRID SYSTEMS

The Lee and Junkins extension of Lagrange's equations to hybrid systems now follows [3]. First note that the Lagrangian is constructed in three parts grouped by the type(s) of coordinates present (finite, infinite, and/or boundary). Here, the class of systems of interest are assumed to have a Lagrangian with the general form $L = L(q_i, \dot{q}_i, \mathbf{w}_j, \dot{\mathbf{w}}_j, \mathbf{w}'_j, \mathbf{w}''_j, x_j, t)$, and the strain energy terms, $\mathbf{w}'_j(x,t), \mathbf{w}''_j(x,t)$, again belong only to the potential energy function.

$$(a) \quad \mathcal{L} = L_D + \int_{l_0}^{l} \widehat{L} \mathrm{d}x + L_B \qquad \text{Single Elastic Domain} \tag{5.1}$$

$$(b) \quad \mathcal{L} = L_D + \mathcal{L}_B \qquad \text{Multiple Elastic Domains} \tag{5.2}$$

Unlike in the discrete and continuous cases, equations of motion for the hybrid case are governed by two distinct expressions. Whereas one expression involves the full hybrid Lagrangian, $\mathcal{L}$, the other uses only the integrand, $\widehat{L}$. The components of the full hybrid Lagrangian are defined using the following notation and argument lists.

$L_D = T_D - V_D = L_D(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$	Discrete Lagrangian
$\widehat{L} = \widehat{T} - \widehat{V} = \widehat{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mathbf{w}, \dot{\mathbf{w}}, \mathbf{w}', \mathbf{w}'', x, t)$	Single Elastic Domain Continuous Lagrangian
$\widehat{L}^i = \widehat{T}^i - \widehat{V}^i = \widehat{L}(\mathbf{w}_i, \dot{\mathbf{w}}_i, {\mathbf{w}_i}', {\mathbf{w}_i}'', \underline{\mathbf{w}}(\underline{\boldsymbol{l}}), \underline{\dot{\mathbf{w}}}(\underline{\boldsymbol{l}}), \underline{\mathbf{w}}'(\underline{\boldsymbol{l}}), \underline{\dot{\mathbf{w}}}'(\underline{\boldsymbol{l}}), x_i, t)$	Multiple Elastic Domain Continuous Lagrangian
$L_B = T_B - V_B = L_B(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mathbf{w}(l), \dot{\mathbf{w}}(l), \mathbf{w}'(l), \dot{\mathbf{w}}'(l), t)$	Single Elastic Domain Boundary Lagrangian
$\mathcal{L}_B = L_B(\boldsymbol{q}, \dot{\boldsymbol{q}}, \underline{\mathbf{w}}(\underline{\boldsymbol{l}}), \underline{\dot{\mathbf{w}}}(\underline{\boldsymbol{l}}), \underline{\mathbf{w}}'(\underline{\boldsymbol{l}}), \underline{\dot{\mathbf{w}}}'(\underline{\boldsymbol{l}}), t) + \sum_{i=1}^{n} \int_{l_{0i}}^{l_i} \widehat{L}^i \mathrm{d}x_i$	Multiple Elastic Domains Boundary Lagrangian

The first expression of Lagrange's equations for hybrid systems is the following.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i \tag{5.3}$$

This expression is very similar to the familiar discrete case version of Lagrange's equations. Likewise, the second expression is akin to the continuous case of Lagrange's equations.

$$(a)\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widehat{L}}{\partial \dot{\mathbf{w}}}\right) - \frac{\partial \widehat{L}}{\partial \mathbf{w}} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \widehat{L}}{\partial \mathbf{w}'}\right) - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \widehat{L}}{\partial \mathbf{w}''}\right) = \widehat{\mathbf{f}}^T \tag{5.4}$$

$$(b)\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widehat{L}^i}{\partial \dot{\mathbf{w}}_i}\right) - \frac{\partial \widehat{L}^i}{\partial \mathbf{w}_i} + \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial \widehat{L}^i}{\partial \mathbf{w}_i'}\right) - \frac{\mathrm{d}^2}{\mathrm{d}x_i^2}\left(\frac{\partial \widehat{L}^i}{\partial \mathbf{w}_i''}\right) = \widehat{\mathbf{f}}^{iT} \tag{5.5}$$

Again, boundary conditions must be considered.

$$(a)\quad \left\{\frac{\partial \widehat{L}}{\partial \mathbf{w}'} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \widehat{L}}{\partial \mathbf{w}''}\right)\right\}\delta\mathbf{w}\Big|_{l_0}^{l} + \left\{\frac{\partial L_B}{\partial \mathbf{w}(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_B}{\partial \dot{\mathbf{w}}(l)}\right)\right\}\delta\mathbf{w}(l)$$

$$+\,\mathbf{f}_1^T\delta\mathbf{w}(l) = 0 \tag{5.6}$$

$$\frac{\partial \widehat{L}}{\partial \mathbf{w}''}\delta\mathbf{w}'\Big|_{l_0}^{l} + \left\{\frac{\partial L_B}{\partial \mathbf{w}'(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_B}{\partial \dot{\mathbf{w}}'(l)}\right)\right\}\delta\mathbf{w}'(l) + \mathbf{f}_2^T\delta\mathbf{w}'(l) = 0 \tag{5.7}$$

$$(b)\quad \left\{\frac{\partial \widehat{L}^i}{\partial \mathbf{w}_i'} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial \widehat{L}^i}{\partial \mathbf{w}_i''}\right)\right\}\delta\mathbf{w}_i\Big|_{l_{0i}}^{l_i} + \left\{\frac{\partial \mathcal{L}_B}{\partial \mathbf{w}_i(l_i)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}_B}{\partial \dot{\mathbf{w}}_i(l_i)}\right)\right\}\delta\mathbf{w}_i(l_i)$$

$$+\,\mathbf{f}_1^{iT}\delta\mathbf{w}_i(l_i) = 0 \tag{5.8}$$

$$\frac{\partial \widehat{L}^i}{\partial \mathbf{w}_i''}\delta\mathbf{w}_i'\Big|_{l_{0i}}^{l_i} + \left\{\frac{\partial \mathcal{L}_B}{\partial \mathbf{w}_i'(l_i)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}_B}{\partial \dot{\mathbf{w}}_i'(l_i)}\right)\right\}\delta\mathbf{w}_i'(l_i) + \mathbf{f}_2^{iT}\delta\mathbf{w}_i'(l_i) = 0 \tag{5.9}$$

Note the distinction here between systems with a single elastic domain (a) or multiple elastic domains (b). The applicable form of this second expression, Eq. (5.4) or (5.5) and the related boundary conditions, together with Eq. (5.3) are Lagrange's equations for a hybrid system.

A. Equilibrium Configuration Solutions

For the hybrid class of systems, the equilibrium results from both expressions of Lagrange's equations must be considered. Again the Lagrangian, $\mathcal{L}$, is partitioned into kinetic and potential energy components $\mathcal{T}_2$, $\mathcal{T}_1$, $\mathcal{T}_0$, and $\mathcal{V}$, with the dynamic potential defined as $\mathcal{U} = \mathcal{V} - \mathcal{T}_0$. Likewise, the continuous part of the Lagrangian, $\widehat{L}$, and boundary part of the Lagrangian, L_B, can be expressed in the same manner. We can then write the equilibrium conditions as follows.

$$\frac{\partial^2 \mathcal{T}_1}{\partial t \partial \dot{q}_i} + \frac{\partial \mathcal{U}}{\partial q_i} = 0 \tag{5.10}$$

$$(a) \quad \frac{\partial}{\partial t}\left(\frac{\partial \widehat{T}_1}{\partial \dot{\mathbf{w}}}\right) + \frac{\partial \widehat{U}}{\partial \mathbf{w}} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \widehat{U}}{\partial \mathbf{w}'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \widehat{U}}{\partial \mathbf{w}''}\right) = 0 \tag{5.11}$$

$$(b) \quad \frac{\partial}{\partial t}\left(\frac{\partial \widehat{T}_1^i}{\partial \dot{\mathbf{w}}_i}\right) + \frac{\partial \widehat{U}^i}{\partial \mathbf{w}_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \widehat{U}^i}{\partial \mathbf{w}_i'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \widehat{U}^i}{\partial \mathbf{w}_i''}\right) = 0 \tag{5.12}$$

The partial differential equations also have additional boundary equations that must be satisfied.

$$(a) \quad \left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \widehat{U}}{\partial \mathbf{w}''}\right) - \frac{\partial \widehat{U}}{\partial \mathbf{w}'}\right\}\delta\mathbf{w}\Big|_{l_0}^{l} + \left\{\frac{\partial^2 U_B}{\partial t \partial \dot{\mathbf{w}}(l)} - \frac{\partial U_B}{\partial \mathbf{w}(l)}\right\}\delta\mathbf{w}(l) = 0 \tag{5.13}$$

$$-\frac{\partial \widehat{U}}{\partial \mathbf{w}''}\delta\mathbf{w}'\Big|_{l_0}^{l} + \left\{\frac{\partial^2 U_B}{\partial t \partial \dot{\mathbf{w}}'(l)} - \frac{\partial U_B}{\partial \mathbf{w}'(l)}\right\}\delta\mathbf{w}'(l) = 0 \tag{5.14}$$

$$(b) \quad \left\{\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial \widehat{U}^i}{\partial {\mathbf{w}_i}''}\right) - \frac{\partial \widehat{U}^i}{\partial {\mathbf{w}_i}'}\right\}\delta\mathbf{w}_i\Big|_{l_{0i}}^{l_i} + \left\{\frac{\partial^2 \mathcal{U}_B}{\partial t \partial \dot{\mathbf{w}}_i(l_i)} - \frac{\partial \mathcal{U}_B}{\partial \mathbf{w}_i(l_i)}\right\}\delta\mathbf{w}_i(l_i) = 0 \tag{5.15}$$

$$-\frac{\partial \widehat{U}^i}{\partial {\mathbf{w}_i}''}\delta{\mathbf{w}_i}'\Big|_{l_{0i}}^{l_i} + \left\{\frac{\partial^2 \mathcal{U}_B}{\partial t \partial {\dot{\mathbf{w}}_i}'(l_i)} - \frac{\partial \mathcal{U}_B}{\partial \mathbf{w}_i'(l_i)}\right\}\delta\mathbf{w}_i'(l_i) = 0 \tag{5.16}$$

Both expressions of the equilibrium equations, Eq. (5.10) and Eq. (5.11) or (5.12), must be satisfied simultaneously in order to find an equilibrium configuration. If several solutions exist, one must choose a target equilibrium configuration about which to linearize the equations of motion.

B. Direct Linearization: Single Elastic Body

As one might expect, the direct linearization method for hybrid systems must be developed in a hybrid manner. First, Lagrange's Equations for infinite-dimensional systems will be considered, Eq. (5.4). Taking a second order Taylor series expansion, additional contributing terms must be added to Eq. (4.17) to account for the presence of finite-dimensional coordinates. The coefficients for the resulting set of terms will be defined as follows, renumbered for clarity.

$$\begin{aligned}
&L_1 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}^2}\right|_{(\mathrm{eq})} \quad ; \quad L_2 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \dot{\mathbf{w}}}\right|_{(\mathrm{eq})} \quad ; \quad L_3 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \mathbf{w}'}\right|_{(\mathrm{eq})} \\
&L_4 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}\partial \mathbf{w}''}\right|_{(\mathrm{eq})} \quad ; \quad L_5 = \left.\frac{\partial^2 \widehat{L}}{\partial \dot{\mathbf{w}}^2}\right|_{(\mathrm{eq})} \quad ; \quad L_6 = \left.\frac{\partial^2 \widehat{L}}{\partial (\mathbf{w}')^2}\right|_{(\mathrm{eq})} \\
&L_7 = \left.\frac{\partial^2 \widehat{L}}{\partial \mathbf{w}'\partial \mathbf{w}''}\right|_{(\mathrm{eq})} \quad ; \quad L_8 = \left.\frac{\partial^2 \widehat{L}}{\partial (\mathbf{w}'')^2}\right|_{(\mathrm{eq})} \quad ; \quad L_9^i = \left.\frac{\partial^2 \widehat{L}}{\partial q_i\partial \mathbf{w}}\right|_{(\mathrm{eq})} \\
&L_{10}^i = \left.\frac{\partial^2 \widehat{L}}{\partial q_i\partial \dot{\mathbf{w}}}\right|_{(\mathrm{eq})} \quad ; \quad L_{11}^i = \left.\frac{\partial^2 \widehat{L}}{\partial q_i\partial \mathbf{w}'}\right|_{(\mathrm{eq})} \quad ; \quad L_{12}^i = \left.\frac{\partial^2 \widehat{L}}{\partial q_i\partial \mathbf{w}''}\right|_{(\mathrm{eq})} \\
&L_{13}^i = \left.\frac{\partial^2 \widehat{L}}{\partial \dot{q}_i\partial \mathbf{w}}\right|_{(\mathrm{eq})} \quad ; \quad L_{14}^i = \left.\frac{\partial^2 \widehat{L}}{\partial \dot{q}_i\partial \dot{\mathbf{w}}}\right|_{(\mathrm{eq})}
\end{aligned} \tag{5.17}$$

Applying Lagrange's equations (5.4) to the resulting Taylor series yields the following linearized equation.

$$\begin{aligned}
&L_5\ddot{\mathbf{w}} + \dot{L}_5\dot{\mathbf{w}} - (L_1 + \dot{L}_2 + L_3{}' + L_4{}'')\mathbf{w} - (2L_4{}' - L_6{}' + L_7{}'')\mathbf{w}' \\
&\quad - (2L_4 - L_6 + L_7{}' + L_8{}'')\mathbf{w}'' - 2L_8{}'\mathbf{w}''' - L_8\mathbf{w}'''' \\
&\quad + (L_{10}^i - L_{13}^i + \dot{L}_{14}^i)\dot{q}_i - (L_9^i - \dot{L}_{10}^i - L_{11}^i{}' + L_{12}^i{}'')q_i = \widehat{\mathbf{f}}
\end{aligned} \tag{5.18}$$

Again, thinking in terms of a hybrid approach, we now consider the expression of Lagrange's equations akin to those for discrete systems, Eq. (5.3). As before, we construct a Taylor series using the following coefficients. Those that are eventually found to be non-contributing are omitted for brevity.

$$\begin{aligned}
&\mathcal{L}_1^{ij} = \left.\frac{\partial^2\mathcal{L}}{\partial q_j \partial q_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_2^{ij} = \left.\frac{\partial^2\mathcal{L}}{\partial q_j \partial \dot{q}_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_3^{ij} = \left.\frac{\partial^2\mathcal{L}}{\partial \dot{q}_j \partial \dot{q}_i}\right|_{(\text{eq})} \\
&\mathcal{L}_4^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \mathbf{w} \partial \dot{q}_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_5^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \mathbf{w} \partial q_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_6^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \dot{\mathbf{w}} \partial \dot{q}_i}\right|_{(\text{eq})} \\
&\mathcal{L}_7^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \dot{\mathbf{w}} \partial q_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_8^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \mathbf{w}' \partial q_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_9^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \mathbf{w}'' \partial q_i}\right|_{(\text{eq})} \\
&\mathcal{L}_{10}^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \mathbf{w}(l) \partial \dot{q}_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_{11}^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \mathbf{w}(l) \partial q_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_{12}^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \dot{\mathbf{w}}(l) \partial \dot{q}_i}\right|_{(\text{eq})} \\
&\mathcal{L}_{13}^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \dot{\mathbf{w}}(l) \partial \dot{q}_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_{14}^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \mathbf{w}'(l) \partial \dot{q}_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_{15}^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \mathbf{w}'(l) \partial q_i}\right|_{(\text{eq})} \\
&\mathcal{L}_{16}^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \dot{\mathbf{w}}'(l) \partial \dot{q}_i}\right|_{(\text{eq})} \quad ; \quad \mathcal{L}_{17}^{i} = \left.\frac{\partial^2\mathcal{L}}{\partial \dot{\mathbf{w}}'(l) \partial q_i}\right|_{(\text{eq})}
\end{aligned} \tag{5.19}$$

The following linearized equations of motion result when Lagrange's equations (5.3) are applied.

$$\begin{aligned}
&(\dot{\mathcal{L}}_2^{ij} - \mathcal{L}_1^{ij})q_j + (\dot{\mathcal{L}}_3^{ij} + \mathcal{L}_2^{ij} - \mathcal{L}_2^{ji})\dot{q}_j + (\mathcal{L}_3^{ij})\ddot{q}_j + (\dot{\mathcal{L}}_4^{i} - \mathcal{L}_5^{i})\mathbf{w} + (\dot{\mathcal{L}}_6^{i} + \mathcal{L}_4^{i} - \mathcal{L}_7^{i})\dot{\mathbf{w}} \\
&\quad + (\mathcal{L}_6^{i})\ddot{\mathbf{w}} - (\mathcal{L}_8^{i})\mathbf{w}' - (\mathcal{L}_9^{i})\mathbf{w}'' + (\dot{\mathcal{L}}_{10}^{i} - \mathcal{L}_{11}^{i})\mathbf{w}(l) + (\dot{\mathcal{L}}_{12}^{i} + \mathcal{L}_{10}^{i} - \mathcal{L}_{13}^{i})\dot{\mathbf{w}}(l) \\
&\quad + (\dot{\mathcal{L}}_{14}^{i} - \mathcal{L}_{15}^{i})\mathbf{w}'(l) + (\dot{\mathcal{L}}_{16}^{i} + \mathcal{L}_{14}^{i} - \mathcal{L}_{17}^{i})\dot{\mathbf{w}}'(l) + (\mathcal{L}_{12}^{i})\ddot{\mathbf{w}}(l) + (\mathcal{L}_{16}^{i})\ddot{\mathbf{w}}'(l) = Q_i
\end{aligned} \tag{5.20}$$

Eqs. (5.18) and (5.20) together are the linearized equations of motion for a hybrid system. Note that coefficients containing an integral operator also act on the succeeding variable if the related partial differentiation occurs in the integrand. This concept is applied in the example at the end of the chapter.

C. Direct Linearization: Multiple Elastic Bodies

It is straightforward to extend these results to hybrid systems with multiple elastic domains. Lagrange's Equations for infinite-dimensional systems with multiple elastic domains will now be considered, Eq. (5.5). Again, terms must be added to the Taylor series expansion to account for the presence of finite-dimensional coordinates. The coefficients for these terms can be defined as follows.

$$
\begin{aligned}
&L_1^i = \left.\frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i^2}\right|_{(\mathrm{eq})} \quad ; \quad L_2^i = \left.\frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i \partial \dot{\mathbf{w}}_i}\right|_{(\mathrm{eq})} \quad ; \quad L_3^i = \left.\frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i \partial \mathbf{w}'_i}\right|_{(\mathrm{eq})} \\
&L_4^i = \left.\frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}_i \partial \mathbf{w}''_i}\right|_{(\mathrm{eq})} \quad ; \quad L_5^i = \left.\frac{\partial^2 \widehat{L}^i}{\partial \dot{\mathbf{w}}_i^2}\right|_{(\mathrm{eq})} \quad ; \quad L_6^i = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}')^2}\right|_{(\mathrm{eq})} \\
&L_7^i = \left.\frac{\partial^2 \widehat{L}^i}{\partial \mathbf{w}'_i \partial \mathbf{w}''_i}\right|_{(\mathrm{eq})} \quad ; \quad L_8^i = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''_i)^2}\right|_{(\mathrm{eq})} \quad ; \quad L_9^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i) \partial \mathbf{w}_j(l_j)}\right|_{(\mathrm{eq})} \\
&L_{10}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i) \partial \dot{\mathbf{w}}_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{11}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i) \partial \mathbf{w}'_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{12}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}_i) \partial \dot{\mathbf{w}}'_j(l_j)}\right|_{(\mathrm{eq})} \\
&L_{13}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\dot{\mathbf{w}}_i) \partial \mathbf{w}_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{14}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\dot{\mathbf{w}}_i) \partial \dot{\mathbf{w}}_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{15}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\dot{\mathbf{w}}_i) \partial \mathbf{w}'_j(l_j)}\right|_{(\mathrm{eq})} \\
&L_{16}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\dot{\mathbf{w}}_i) \partial \dot{\mathbf{w}}'_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{17}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}'{}_i) \partial \mathbf{w}_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{18}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}'{}_i) \partial \dot{\mathbf{w}}_j(l_j)}\right|_{(\mathrm{eq})} \\
&L_{19}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}'{}_i) \partial \mathbf{w}'_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{20}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}'{}_i) \partial \dot{\mathbf{w}}'_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{21}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''{}_i) \partial \mathbf{w}_j(l_j)}\right|_{(\mathrm{eq})} \\
&L_{22}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''{}_i) \partial \dot{\mathbf{w}}_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{23}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''{}_i) \partial \mathbf{w}'_j(l_j)}\right|_{(\mathrm{eq})} \quad ; \quad L_{24}^{ij} = \left.\frac{\partial^2 \widehat{L}^i}{\partial (\mathbf{w}''{}_i) \partial \dot{\mathbf{w}}'_j(l_j)}\right|_{(\mathrm{eq})} \\
&L_{25}^{ki} = \left.\frac{\partial^2 \widehat{L}}{\partial q_k \partial \mathbf{w}_i}\right|_{(\mathrm{eq})} \quad ; \quad L_{26}^{ki} = \left.\frac{\partial^2 \widehat{L}}{\partial q_k \partial \dot{\mathbf{w}}_i}\right|_{(\mathrm{eq})} \quad ; \quad L_{27}^{ki} = \left.\frac{\partial^2 \widehat{L}}{\partial q_k \partial \mathbf{w}'_i}\right|_{(\mathrm{eq})} \\
&L_{28}^{ki} = \left.\frac{\partial^2 \widehat{L}}{\partial q_k \partial \mathbf{w}''_i}\right|_{(\mathrm{eq})} \quad ; \quad L_{29}^{ki} = \left.\frac{\partial^2 \widehat{L}}{\partial \dot{q}_k \partial \mathbf{w}_i}\right|_{(\mathrm{eq})} \quad ; \quad L_{30}^{ki} = \left.\frac{\partial^2 \widehat{L}}{\partial \dot{q}_k \partial \dot{\mathbf{w}}_i}\right|_{(\mathrm{eq})}
\end{aligned}
\tag{5.21}
$$

When Lagrange's equations (5.5) are applied, the following governing equations of motion result.

$$
\begin{aligned}
&L_5^i \ddot{\mathbf{w}}_i + \dot{L}_5^i \dot{\mathbf{w}}_i - (L_1^i + \dot{L}_2^i + L_3^{i\prime} + L_4^{i\prime\prime})\mathbf{w}_i - (2L_4^{i\prime} - L_6^{i\prime} + L_7^{i\prime\prime})\mathbf{w}'_i \\
&\quad - (2L_4^i - L_6^i + L_7^{i\prime} + L_8^{i\prime\prime})\mathbf{w}''_i - 2L_8^{i\prime}\mathbf{w}'''_i - L_8^i \mathbf{w}''''_i \\
&\quad - (L_9^{ij} - \dot{L}_{13}^{ij} - L_{17}^{ij\prime} + L_{21}^{ij\prime\prime})\mathbf{w}_j(l_j) - (L_{10}^{ij} - L_{13}^{ij} - \dot{L}_{14}^{ij} - L_{18}^{ij\prime} + L_{22}^{ij\prime\prime})\dot{\mathbf{w}}_j(l_j) \\
&\quad - (L_{11}^{ij} - \dot{L}_{15}^{ij} - L_{19}^{ij\prime} + L_{23}^{ij\prime\prime})\mathbf{w}'_j(l_j) - (L_{12}^{ij} - L_{15}^{ij} - \dot{L}_{16}^{ij} - L_{20}^{ij\prime} + L_{24}^{ij\prime\prime})\dot{\mathbf{w}}'_j(l_j) \\
&\quad + L_{14}^{ij}\ddot{\mathbf{w}}_j(l_j) + L_{16}^{ij}\ddot{\mathbf{w}}'_j(l_j) + (L_{28}^{ki} - L_{29}^{ki} + \dot{L}_{30}^{ki})\dot{q}_k \\
&\quad - (L_{25}^{ki} - \dot{L}_{26}^{ki} - L_{27}^{ki\prime} + L_{28}^{ki\prime\prime})q_k = \widehat{\mathbf{f}}^i
\end{aligned}
\tag{5.22}
$$

The expression of Lagrange's equations similar to those for discrete systems, Eq. (5.3), must also be examined. We again construct a Taylor series and apply Lagrange's equations, resulting in the following contributing coefficients.

$$
\begin{aligned}
&\mathcal{L}_1^{ij} = \frac{\partial^2 \mathcal{L}}{\partial q_j \partial q_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_2^{ij} = \frac{\partial^2 \mathcal{L}}{\partial q_j \partial \dot{q}_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_3^{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_j \partial \dot{q}_i}\bigg|_{(\text{eq})} \\
&\mathcal{L}_4^{ik} = \frac{\partial^2 \mathcal{L}}{\partial \mathbf{w}_k \partial \dot{q}_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_5^{ik} = \frac{\partial^2 \mathcal{L}}{\partial \mathbf{w}_k \partial q_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_6^{ik} = \frac{\partial^2 \mathcal{L}}{\partial \dot{\mathbf{w}}_k \partial \dot{q}_i}\bigg|_{(\text{eq})} \\
&\mathcal{L}_7^{ik} = \frac{\partial^2 \mathcal{L}}{\partial \dot{\mathbf{w}}_k \partial q_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_8^{ik} = \frac{\partial^2 \mathcal{L}}{\partial \mathbf{w}'_k \partial q_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_9^{ik} = \frac{\partial^2 \mathcal{L}}{\partial {\mathbf{w}_k}'' \partial q_i}\bigg|_{(\text{eq})} \\
&\mathcal{L}_{10}^{im} = \frac{\partial^2 \mathcal{L}}{\partial \mathbf{w}_m(l_m) \partial \dot{q}_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_{11}^{im} = \frac{\partial^2 \mathcal{L}}{\partial \mathbf{w}_m(l_m) \partial q_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_{12}^{im} = \frac{\partial^2 \mathcal{L}}{\partial \dot{\mathbf{w}}_m(l_m) \partial \dot{q}_i}\bigg|_{(\text{eq})} \\
&\mathcal{L}_{13}^{im} = \frac{\partial^2 \mathcal{L}}{\partial \dot{\mathbf{w}}_m(l_m) \partial q_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_{14}^{im} = \frac{\partial^2 \mathcal{L}}{\partial \mathbf{w}'_m(l_m) \partial \dot{q}_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_{15}^{im} = \frac{\partial^2 \mathcal{L}}{\partial {\mathbf{w}_m}'(l_m) \partial q_i}\bigg|_{(\text{eq})} \\
&\mathcal{L}_{16}^{im} = \frac{\partial^2 \mathcal{L}}{\partial \dot{\mathbf{w}}'_m(l_m) \partial \dot{q}_i}\bigg|_{(\text{eq})} \quad ; \quad \mathcal{L}_{17}^{im} = \frac{\partial^2 \mathcal{L}}{\partial \dot{\mathbf{w}}'_m(l_m) \partial q_i}\bigg|_{(\text{eq})}
\end{aligned}
\tag{5.23}
$$

The following linearized equations of motion result when Lagrange's equations (5.3) are applied.

$$
\begin{aligned}
&(\dot{\mathcal{L}}_2^{ij} - \mathcal{L}_1^{ij})q_j + (\dot{\mathcal{L}}_3^{ij} + \mathcal{L}_2^{ij} - \mathcal{L}_2^{ji})\dot{q}_j + (\mathcal{L}_3^{ij})\ddot{q}_j + (\dot{\mathcal{L}}_4^{ik} - \mathcal{L}_5^{ik})\mathbf{w}_k \\
&\quad + (\dot{\mathcal{L}}_6^{ik} + \mathcal{L}_4^{ik} - \mathcal{L}_7^{ik})\dot{\mathbf{w}}_k + (\mathcal{L}_6^{ik})\ddot{\mathbf{w}}_k - (\mathcal{L}_8^{ik})\mathbf{w}'_k - (\mathcal{L}_9^{ik})\mathbf{w}''_k + (\dot{\mathcal{L}}_{10}^{im} - \mathcal{L}_{11}^{im})\mathbf{w}_m(l_m) \\
&\quad + (\dot{\mathcal{L}}_{12}^{im} + \mathcal{L}_{10}^{im} - \mathcal{L}_{13}^{im})\dot{\mathbf{w}}_m(l_m) + (\dot{\mathcal{L}}_{14}^{im} - \mathcal{L}_{15}^{im})\mathbf{w}'_m(l_m) + (\dot{\mathcal{L}}_{16}^{im} + \mathcal{L}_{14}^{im} - \mathcal{L}_{17}^{im})\dot{\mathbf{w}}'_m(l_m) \\
&\quad + (\mathcal{L}_{12}^{im})\ddot{\mathbf{w}}_m(l_m) + (\mathcal{L}_{16}^{im})\ddot{\mathbf{w}}'_m(l_m) = Q_i
\end{aligned}
\tag{5.24}
$$

Again, note that coefficients containing an integral operator act on the succeeding variable. Together, Eqs. (5.22) and (5.24) are the linearized equations of motion for a hybrid system with multiple elastic domains.

D. Example: Rotating Hub with Flexible Arm

Consider a simple hybrid system with a flexible beam fixed to a rotating hub with radius R and inertia I_{hub} driven by a control torque u as shown in Figure (4). The angular position of the hub is described by the discrete generalized coordinate $\theta(t)$. The beam position is described with the coordinates x and $\mathbf{w}(x,t)$.

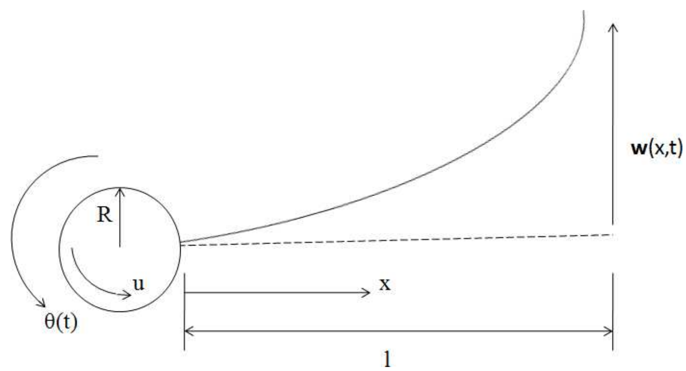

Fig. 4. Rotating Hybrid Structure

The Lagrangian for this system is constructed from the kinetic and potential energy functions.

$$\mathcal{L} = \frac{1}{2} I_{hub} \theta^2 + \frac{1}{2}\rho \int_0^l \left(\dot{\mathbf{w}}^2 + (R+x)^2\dot{\theta}^2 + 2(R+x)\dot{\mathbf{w}}\dot{\theta} + \mathbf{w}^2\dot{\theta}^2 \right) \mathrm{d}x - \frac{1}{2}\int_0^l EI \left(\mathbf{w}''\right)^2 \mathrm{d}x \tag{5.25}$$

The equilibrium equations can be determined by applying Eq. (5.10) and Eq. (5.11) with (5.13) and (5.14) to the proceeding equation.

$$\partial \mathcal{U}/\partial\theta = 0 \quad \text{(Satisfied for all values of } \theta) \tag{5.26}$$

$$EI\mathbf{w}'''' = 0 \quad ; \quad \mathrm{w}(0) = 0 \quad ; \quad \mathrm{w}'(0) = 0 \quad ; \quad \mathrm{w}''(l) = 0 \quad ; \quad \mathrm{w}'''(l) = 0$$

A solution to the infinite dimensional partial differential equation can be found by assuming a solution of the form $\mathbf{w}(x) = c_1x^3 + c_2x^2 + c_3x + c_4$ and using the four boundary conditions to determine the constants c_i.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 6l & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.27}$$

Each row of the matrix equation represents a boundary condition. Here, equation (5.27) is satisfied with $c_i = 0$. Using this equilibrium configuration, we can find the nonzero direct linearization coefficients.

$$L_1 = \rho\dot{\theta}^2 \quad ; \quad L_5 = \rho \quad ; \quad L_8 = -EI \quad ; \quad L_{14} = \rho(R+x) \tag{5.28}$$

$$\mathcal{L}_3 = I_{hub} + \frac{1}{2}\rho \int_0^l \left(2(R+x)^2\right) dx \quad ; \quad \mathcal{L}_6 = \frac{1}{2}\int_0^l \left(2(R+x)\right) dx \tag{5.29}$$

The coefficients $\mathcal{L}_3$ and $\mathcal{L}_6$ are examples of partial differentiation in the integrand. When placed in the directly linearized equations, the integrals in these coefficients will also act on the variable succeeding the coefficients as shown below. Substituting all of the coefficients into Eqs. (5.18) and (5.20), we have the following linearized equations of motion.

$$\rho\ddot{\mathbf{w}} + EI\mathbf{w}'''' + \rho(R+x)\ddot{\theta} = 0 \tag{5.30}$$

$$I_{hub}\ddot{\theta} + \rho\int_0^l \left((R+x)\ddot{\mathbf{w}} + (R+x)^2\ddot{\theta}\right) dx = 0 \tag{5.31}$$

If one were to find the equations of motion by applying Lagrange's equations directly to the full Lagrangian, the same equations would actually result. That is, the full nonlinear equations of motion for this system were, in fact, already linear. This example then provides a nice "sanity check" for our results.

CHAPTER VI

DISCUSSION

The direct linearization method presented shows how the linearized equations of motion for continuous and hybrid systems can be constructed in a straightforward manner. The development first utilizes a quadratic Taylor series expansion of the Lagrangian. Partial differentiation of the resulting expression via Lagrange's equations is then used to identify a contributing set of partial derivative coefficients. These terms are important for directly constructing the linearized equations describing the departure motion from the target equilibrium configuration. Note that the result of this development, not the development itself, is direct linearization. That is, depending on the number and types of dependent variables present, one of these four equations/equation sets, Eq. (4.21), (4.23), (5.18) and (5.20), or (5.22) and (5.24), is employed to directly construct the equations of motion for a continuous or hybrid system.

Because the method relies on partial derivatives, it is a prime application candidate for software programs that utilize processes such as operator overloading and automatic differentiation. The Object Oriented Coordinate Embedding Algorithm (OCEA) program is one such software implementation [10]. A software solution could provide a means for automatically generating the partial derivative coefficients and resulting linearized equations of motion from a given Lagrangian function and target equilibrium configuration. Moreover, the development presented in this thesis could be further generalized to allow for the automatic generation of equations of motion with quadratic, cubic, and even higher-order terms. That is, it could be extended to find the equations of motion of a higher-order about a point of interest.

One idea central to linearization, whether direct or indirect, is the calculation of equilibrium configuration solutions. It is noted that systems with continuous generalized coordinates may have an infinite number of possible equilibrium solutions or have an eigenvalue-related constraint for finding a unique solution. Numerical approaches are also viable methods for calculating equilibrium configurations, regardless of the linearization method utilized. However, the generation of the linearized equations of motion are not limited to motion about an equilibrium configuration. As an alternative, one might be interested in departure motion from a reference trajectory and could use these developments to directly construct the linearized equations of motion to approximate this departure motion.

Thus far, the direct linearization approach has only been discussed within the Lagrangian framework. In the Lagrangian view, the governing equations are the result of the Euler-Lagrange differential operator acting on a first principle function. But the governing equations for a finite-dimensional system may be generated by other operators acting on other first principle functions: for example, an appropriate operator acting on the Hamiltonian function; or an appropriate operator acting on the Gibbsian-Appellian function. A full discussion of these functions is outside the scope of this thesis. However, a brief digression to look at the linearization of Hamilton's equations is of interest.

The Hamiltonian, $H(q_i, p_i, t)$, is a scalar function closely related to the Lagrangian [4]. Whereas the Lagrangian is explicitly a function of the generalized velocities, coordinates, and time, the Hamiltonian is a function of a variable set consisting of the conjugate momenta, $p_i(q_i, \dot{q}_i, t)$, the generalized coordinates, $q_i(t)$, and time, t. Hamilton's equations are similar to Lagrange's equations in that they both employ partial derivatives of their respective scalar functions with respect to the dependent variables in their argument lists. Rather than produce $i = 1, \ldots, n$ second-order differential governing equations, Hamilton's equations are used to construct $2n$

first-order differential governing equations of the following form.

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad ; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i \tag{6.1}$$

This partial differential form allows us to follow the direct linearization development previously discussed: perform a Taylor series expansion about the equilibrium point, retain terms second order or lower, and then apply Hamilton's equations to produce the directly linearized equations of motion. The contributing terms and resulting equations of motion follow.

$$H_1^{ij} = \frac{\partial^2 H}{\partial q_i q_j} \quad ; \quad H_2^{ij} = \frac{\partial^2 H}{\partial p_i p_j} \quad ; \quad H_3^{ij} = \frac{\partial^2 H}{\partial q_i p_j} \tag{6.2}$$

$$\dot{p}_i = -H_1^{ij} q_j - H_3^{ij} p_j \quad ; \quad \dot{q}_i = H_2^{ij} p_j + H_3^{ij} q_j \tag{6.3}$$

Here, $\partial U/\partial q_i = 0$ again defines the equilibrium point.

That this result is derived from a non-Lagrangian framework begs the question: can one construct a generalized framework for understanding linearization outside of classical mechanics? This idea encompasses developing a mathematical process that could be applied to any system whose evolution is captured by a differential operator acting on a nonlinear function of system variables. An encouraging response to this question is that the governing equations for simple electrical circuits can be generated using Lagrangian and Hamiltonian methods [11].

One final question that arises is this: how does one know that this method generates the correct linearized equations of motion? That is, are the directly linearized equations of motion the same that one would obtain by first forming the full nonlinear equations of motion and then linearizing them with a Taylor series expansion that retains terms first order and lower? The answer is yes, and the reason traces back to partial derivatives. The commutative property of partial derivatives allows one to either first apply the 'dynamic' differential operator and then linearize via a Taylor series expansion, or first take a Taylor series expansion, instead retaining second order terms, and then apply the operator. Both operations involve partial differentiation at their core, and a change of variables eases comparison between the two methods. In fact, one can arrive at the final results presented in this thesis by first applying Lagrange's equations and then linearizing about the equilibrium, but as previously noted, this approach can be a long, arduous process. This verification is left to the reader, should he decide to get *direct*ly involved.

REFERENCES

[1] H. Baruh, *Analytical Dynamics.* New York, New York: McGraw-Hill Companies, Inc., 1999, §4.7, 5.1-5.3.

[2] L. Meirovitch, *Computational Methods in Structural Dynamics.* Alphen aan den Rijn, The Netherlands: Sijthoff and Noordhoff International Publishers B.V., 1980, §2.3.

[3] S. Lee and J. Junkins, "Explicit generalization of lagranges equations for hybrid coordinate dynamical systems," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 6, 1992, pp. 1443-1452.

[4] J. Hurtado, *Kinematic and Kinetic Principles.* College Station, Texas: John E. Hurtado/Published by Lulu, 2007, §8-9.

[5] H. Schaub and J. Junkins, *Analytical Mechanics of Space Systems.* Reston, Virginia: American Institute of Aeronautics and Astronautics, Inc., 2003, §5.2-5.3.

[6] J. Papastavridis, *Analytical Mechanics.* New York, New York: Oxford University Press, 2002, §2.5.

[7] J. Junkins and Y. Kim, *Introduction to Dynamics and Control of Flexible Structures.* Washington, District of Columbia: American Institute of Aeronautics and Astronautics, Inc., 1993, §2.1, 4.2.

[8] W. Kaplan, *Advanced Calculus, Fourth Edition.* Redwood City, California: Addison-Wesley Publishing Company, 1991, §2.15-2.16.

[9] R. Bronson, *Modern Introductory Differential Equations.* New York, New York: McGraw-Hill Companies, Inc., 1973, §37.1-37.2.

[10] D. Griffith, J. Turner, and J. Junkins, "Automatic generation and integration of equations of motion for flexible multibody dynamical systems," *Journal of the Astronautical Sciences*, Vol. 53, No. 3, 2005, pp. 251-279.

[11] S. Crandall, D. Karnopp, E. Kurtz Jr., and D. Pridemore-Brown, *Dynamics of Mechanical and Electromechanical Systems.* Malabar, Florida: Krieger Publishing Company, 1968, §5.6-5.7.

VITA

Julie Marie Jones Parish was born in Arizona and raised throughout the United States with her military family. She began her undergraduate studies at Texas A&M University in August 2001 and received her Bachelor of Science degree in Aerospace Engineering from Texas A&M University on May 13, 2005. Her undergraduate achievements include receiving a National Merit Scholarship and the Craig Brown Outstanding Senior Engineer Award. In August 2005, Julie began her graduate studies under the advisement of Dr. John E. Hurtado at Texas A&M. During her studies, she was awarded the NDSEG and NSF graduate fellowships. This thesis is the culmination of her graduate research and has been submitted in partial fulfillment of the requirements for graduation with a Master of Science degree in Aerospace Engineering. Professionally, Julie has interned as a graduate fellow at the Department of Defense and Sandia National Labs.

Mrs. Parish may be reached at julieparish@tamu.edu or by contacting Dr. John E. Hurtado, Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843.