$RO(G)$-GRADED EQUIVARIANT COHOMOLOGY THEORY AND SHEAVES

A Dissertation

by

HAIBO YANG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2008

Major Subject: Mathematics

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ABSTRACT

$RO(G)$-graded Equivariant Cohomology Theory

and Sheaves. (December 2008)

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Chair of Advisory Committee: Dr. Paulo Lima-Filho

If G is a finite group and if X is a G-space, then a Bredon $RO(G)$-graded equivariant cohomology theory is defined on X. Furthermore, if X is a G-manifold, there exists a natural Čech hypercohomology theory on X. While Bredon $RO(G)$-graded cohomology is important in the theoretical aspects, the Čech cohomology is indispensable when computing the cohomology groups. The purpose of this dissertation is to construct an isomorphism between these two types of cohomology theories so that the interplay becomes deeper between the theory and concretely computing cohomology groups of classical objects. Also, with the aid of Čech cohomology, we can naturally extend the Bredon cohomology to the more generalized Deligne cohomology.

In order to construct such isomorphism, on one hand, we give a new construction of Bredon $RO(G)$-graded equivariant cohomology theory from the sheaf-theoretic viewpoint. On the other hand, with Illman's theorem of smooth G-triangulation of a G-manifold, we extend the existence of good covers from the nonequivariant to the equivariant case. It follows that, associated to an equivariant good cover of a G-manifold X, there is a bounded spectral sequence converging to Čech hypercohomology whose E_1 page is isomorphic to the E_1 page of a Segal spectral sequence which converges to the Bredon $RO(G)$-graded equivariant cohomology. Furthermore, This isomorphism is compatible with the structure maps in the two spectral sequences. So

there is an induced isomorphism between two limiting objects, which are exactly the Čech hypercohomology and the Bredon $RO(G)$-graded equivariant cohomology.

We also apply the above results to real varieties and obtain a quasi-isomorphism between two commonly used complexes of presheaves.

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CHAPTER I

INTRODUCTION

For a finite group G, Illman [Ill78] showed that every smooth G-manifold admits a smooth equivariant triangulation onto a regular simplicial G-complex. With this result we extend to the equivariant context a well-known theorem [BT82, p. 42] about the existence of a good cover on a smooth manifold.

Theorem I.1 (Theorem III.25)**.** *Every smooth G-manifold has an equivariant good cover. Moreover, the equivariant good covers are cofinal in the set of all open covers of a G-manifold X.*

On the other hand, an $RO(G)$-graded cohomology theory is defined on any G-space X [May96]. It is a cohomology theory on X with coefficients in a Mackey functor M and is one that is graded by the real orthogonal representation ring $RO(G)$ of G. Since the origin of this theory dates back to Bredon [Bre72], We call it $RO(G)$-graded Bredon cohomology theory. We may apply this theory to a G-manifold X with a coefficient system $\underline{M}$ which is associated to a discrete $\mathbb{Z}[G]$-module M. For any finite representation V of G we define a cochain complex of presheaves $M(V)$ and show that, for any equivariant good cover $\mathfrak{U}$ of X and for any $n \in \mathbb{Z}$, there is a natural isomorphism

$$\check{\mathbb{H}}^n_{\mathrm{eq}}\left(\mathfrak{U}, M(V)\right) \cong H^{V+n-\dim(V)}_{\mathrm{Br}}\left(X, \underline{M}\right),$$

and since equivariant good covers are cofinal, we have the following main theorem

This dissertation follows the style of Transactions of the American Mathematical Society.

Theorem I.2 (Theorem V.5)**.** *There is a natural isomorphism*

$$\check{\mathbb{H}}^n_{eq}\left(X, M(V)\right) \cong H^{V+n-\dim(V)}_{Br}\left(X, \underline{M}\right).$$

This paper is organized as follows. Chapter II is a brief overview of the $RO(G)$-graded cohomology theory and a homotopy theoretic result about this theory. A consequence of the latter is that we get a simple model of the Eilenberg-MacLane space $K(M, V)$. Proposition II.18 further identifies the homotopy groups of the G-fixed subset of this model space with some $RO(G)$-graded Bredon cohomology groups of a point.

In Chapter III we first review the necessary results about nonequivariant simplicial complexes and then delve into the equivariant case. We define the terminology of an equivariant good cover of a smooth G-manifold and prove that every G-manifold has an equivariant good cover.

In Chapter IV for any presheaf on a G-manifold we construct an associated complex of presheaves for which we call singular cochain complex. This complex has homotopy invariant cohomology presheaves as stated in the same section.

Chapter V comes to the main theorem we present. A G-manifold, together with an equivariant good cover, carries both the $RO(G)$-graded Bredon cohomology and the Čech hypercohomology. The theorem says there is natural isomorphism between them. Note that Deligne cohomology on a smooth or holomorphic complex manifolds is defined as hypercohomology of some complexes. If we link Deligne cohomology with the Čech hypercohomology, there is a bridge from Bredon to Deligne cohomology.

CHAPTER II

EQUIVARIANT HOMOTOPY THEORY AND $RO(G)$-GRADED COHOMOLOGY THEORY

A. The equivariant homotopy category

Let G be a topological group with unit e and let X be a topological space. A left action of G on X is a map

$$\tau : G \times X \to X$$

such that

(i) $\tau(e, x) = x$,

(ii) $\tau(g, \tau(h, x)) = \tau(gh, x)$,

for all $g, h \in G$ and $x \in X$. Here we always use 'map' to refer to 'continuous function'. When such action exists, X is called a left G-space. For simplicity, we denote $\tau(g, x)$ by gx.

A right G-action is a map $X \times G$ to X, $(x, g) \mapsto xg$ such that $xe = x$ and $(xg)h = x(gh)$. If X carries a right G-action, X is called a right G-space. When X is a right G-space, the definition $(g, x) \mapsto xg^{-1}$ gives a left G-action on X. So we usually work with left G-spaces and simply call them G-spaces.

A map $f : X \to Y$ between G-spaces X and Y is called a G-*map* or an *equivariant map* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. With the usual composition of maps there is a category G-*Top* whose objects are G-spaces and the morphisms are G-maps between objects. We write $\mathrm{Hom}_{G\text{-}Top}(X, Y)$ or $\mathrm{Hom}_G(X, Y)$ for the set of G-maps $X \to Y$. The set $\mathrm{Hom}_{G\text{-}Top}(X, Y)$ is a topological subspace of the space $\mathrm{Hom}_{Top}(X, Y)$ with compact-open topology ([Bre93, VII, sec 2], [Dug66, XII, sec 1]).

Moreover, we can put a conjugate G-action on the space $\mathrm{Hom}_{Top}(X,Y)$ by $(g,f)\mapsto gf$ and $(g\cdot f)(x)=gf(g^{-1}x)$. With this G-action the set $\mathrm{Hom}_{Top}(X,Y)$ is an object in G-*Top*. It is called the *internal Hom* and we denote it by $\underline{Hom}_{Top}(X,Y)$. In addition, the internal Hom $\underline{Hom}_{Top}(X,Y)$ also denotes the topological space with compact-open topology.

For any closed subgroup H of G, define the H*-fixed point set* of X as

$$X^H=\{x\in X\mid hx=x \text{ for all } h\in H\}.$$

For $x\in X$, $G_x=\{g\mid gx=x\}$ is called the *isotropy group* of x. The equivariant homotopy type of X is completely determined by the system of fixed point sets $\{X^H\mid H\leqslant G\}$. For the details see [Elm83].

Given G-spaces X and Y, consider the G-space $\underline{Hom}_{Top}(X,Y)$ with the conjugate G-action. Then a G-fixed point f in $\underline{Hom}_{Top}(X,Y)$ has the property $gf(g^{-1}x)=f(x)$ or equivalently, $gf(x)=f(gx)$ for all $g\in G$ and $x\in X$. That is, f is a G-map. Reversely, a G-map $f:X\to Y$, when viewed as an element in $\underline{Hom}_{Top}(X,Y)$, is clearly fixed by G. Hence we have the following equivalence

$$\underline{Hom}_{G\text{-}Top}(X,Y)=\left(\underline{Hom}_{Top}(X,Y)\right)^G. \tag{II.1}$$

Next we prove some properties about the function space.

Proposition II.1. *Let X, Z be Hausdorff G-spaces and Y locally compact Hausdorff G-space. Then there is a natural G-homeomorphism*

$$\underline{Hom}_{Top}(X\times Y,Z)\xrightarrow[\cong]{\Phi}\underline{Hom}_{Top}(X,\underline{Hom}_{Top}(Y,Z)),$$

where Φ is defined by $(\Phi(f))(x)(y)=f(x,y)$ for all $f:X\times Y\to Z$, $x\in X$ and $y\in Y$.

Proof. Forget the G-actions on X, Y and Z. Then we claim that Φ is a natural nonequivariant homeomorphism. The proof can be found in [Bre93, VII, Theorem 2.5]. Add G-actions on the spaces and Hom sets. The map Φ becomes equivariant since

$$\Phi(g\cdot f)(x)(y) = (g\cdot f)(x,y) = gf(g^{-1}x, g^{-1}y)$$

and

$$\begin{aligned}(g\cdot\Phi(f))(x)(y) &= [g\cdot\left(\Phi(f)(g^{-1}x)\right)](y)\\ &= g\left(\Phi(f)(g^{-1}x)(g^{-1}y)\right) = gf(g^{-1}x, g^{-1}y).\end{aligned}$$

Hence Φ is a G-homeomorphism. □

Corollary II.2. *Let X, Y, Z and Φ be the same as in Proposition* II.1. *Then there is a natural homeomorphism*

$$\underline{Hom}_{G\text{-}Top}(X\times Y, Z) \underset{\cong}{\xrightarrow{\Phi}} \underline{Hom}_{G\text{-}Top}(X, \underline{Hom}_{Top}(Y,Z)).$$

Proof. If $f : X \to Y$ is a G-homeomorphism, then $f|_{X^G} : X^G \to Y^G$ is a homeomorphism. Now the result comes from Proposition II.1 and the equation (II.1). □

Proposition II.3. *Let X be a G-space and let A be a space with trivial G-action. Then there are the following natural homeomorphisms*

(i) $\underline{Hom}_{G\text{-}Top}(A, X) \cong \underline{Hom}_{Top}(A, X^G)$.

(ii) $\underline{Hom}_{G\text{-}Top}(X, A) \cong \underline{Hom}_{Top}(X/G, A)$.

Proof. See [May96, pp. 11-12]. □

Proposition II.4. *If G is a compact group and $H\subset G$ a closed subgroup then for any G-space X there is a natural homeomorphism*

$$X^H \underset{\cong}{\xrightarrow{\Phi}} \underline{Hom}_{G\text{-}Top}(G/H, X),$$

where Φ *sends* $a \in X^H$ *to* $f_a : G/H \to X$ *with* $f_a(gH) = ga$. *The inverse of* Φ *sends* $f \in \underline{Hom}_{G\text{-}Top}(G/H, X)$ *to* $f(H)$.

Proof. See [tD87, p. 25, Proposition 3.8]. $\square$

If $(X_\alpha \mid \alpha \in J)$ is a collection of G-spaces then the product $\prod_{\alpha \in J} X_\alpha$ is again a G-space under the diagonal action

$$(g, (x_\alpha)_{\alpha \in J}) \mapsto (gx_\alpha)_{\alpha \in J}.$$

Let $I = [0, 1]$ be the unit interval with trivial left G-action. Two G-maps $f_0, f_1 : X \to Y$ are called *G-homotopic* if there is a continuous G-map (where $X \times I$ carries the diagonal G-action.)

$$F : X \times I \to Y$$

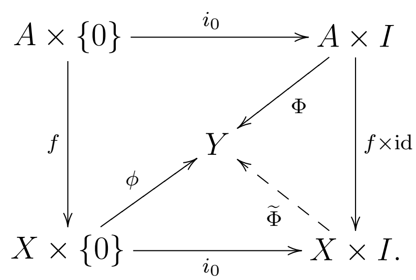
such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. The map F is called a *G-homotopy* from f_0 to f_1. Since the action on I is trivial, the map $f_t : X \to Y, x \mapsto F(x, t)$ is equivariant for all $t \in I$. As in the nonequivariant case, the relation of being G-homotopic is an equivalence relation. We denote by $[X, Y]_G$ the G-homotopy classes of G-maps $X \to Y$ and use $[f]$ to denote the G-homotopy class represented by a map $f : X \to Y$. The homotopy category *hG-Top* is then the category whose objects are same as those in *G-Top* and the morphisms are G-homotopy classes of G-maps between G-spaces.

Recall that the *cone* CX of a topological space X is the space $X \times I/X \times \{1\}$. If there is a G-action on X then CX is also a G-space with the diagonal G-action. For a G-map $f : A \to X$, the cone C_f of f is the quotient space

$$C_f = X \cup_f CA = X \amalg CA/\sim, \tag{II.2}$$

where $\sim$ is the equivalence relation generated by $(a, 0) \sim f(a)$ for all $a \in A$.

Definition II.5. A G-map $f : A \to X$ is an *equivariant cofibration* or *G-cofibration* if it satisfies the equivariant homotopy extension property (G-HEP). That is, for any G-space Y and G-maps $\phi : X \to Y$, $\Phi : A \times I \to Y$ satisfying $\Phi \circ i_0 = \phi \circ f$ there exists G-map $\widetilde{\Phi} : X \times I \to Y$ that makes the following diagram commute



Here $i_0(x) = (x, 0)$.

In many situations we would like to work with based G-spaces. A based G-space $(X, *)$ is a G-space with a basepoint $* \in X$ and the basepoint is assumed to be fixed by G, i.e. $g* = *$ for all $g \in G$. If no confusion arises we simply call X a based G-space whenever a basepoint exists on X. If Z is any (unbased) G-space, let Z_+ denotes the disjoint union of Z and a G-fixed basepoint $+$ so that Z_+ becomes a based G-space. If (X, x_0) and (Y, y_0) are based G-spaces, then a G-map $f : X \to Y$ is called a based G-map if f maps basepoint x_0 to basepoint y_0. The collection of based G-spaces and based G-maps form a category G-Top^0.

The *wedge* or *one-point union* $X \vee Y$ of based G-space (X, x_0) and (Y, y_0) is the quotient space of the disjoint union $X \coprod Y$ by the equivalence relation identifying x_0 with y_0. It is a G-space and is G-homeomorphic to the G-subspace $X \times \{y_0\} \cup Y \times \{x_0\}$ of the product $X \times Y$. The *smash product* $X \wedge Y$ is then defined to be the quotient $X \times Y / X \vee Y$. We can think of $X \wedge Y$ as the reduced version of $X \times Y$ which collapses $X \vee Y$ to a basepoint.

Let X and Y be based G-spaces. A *based G-homotopy* F between based G-maps $f_0, f_1 : X \to Y$ is just a continuous based G-map $F : X \wedge I_+ \to Y$ such that $F(x, 0) = f_0(x),\ F(x, 1) = f_1(x)$. Here $X \wedge I_+$ is a based G-space obtained from $X \times I$ by collapsing $\{x_0\} \times I$ to the basepoint. We write $[X, Y]^0_G$ the based G-homotopy classes of based G-maps $X \to Y$. Then the homotopy category $hG\text{-}Top^0$ is defined similar to the unbased counterpart $hG\text{-}Top$. The objects of $hG\text{-}Top^0$ are the based G-spaces and the morphisms are the based G-homotopy classes of based G-maps.

Let (X, x_0) be a based G-space. Then the *reduced cone* CX is the based quotient G-space

$$CX = X \times I/(\{x_0\} \times I \cup X \times \{1\}),$$

where $\{x_0\} \times I \cup X \times \{1\}$ collapses to the basepoint.

The reduced cone C_f of a based G-map $f : A \to X$ is defined the same as (II.2) except CA is replaced by the reduced cone of A.

The definition of G-cofibration has an obvious based version, in which all maps and spaces in the diagram in Definition II.5 are required to be based.

B. Equivariant CW-complexes

The theory of nonequivariant CW-complexes ([May99], [Whi49], [Whi78]) can be extended to the equivariant case accordingly.

Let $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ be the unit disk in $\mathbb{R}^n$ with boundary the unit sphere S^{n-1}. For convenience we set D^0 as a point space and $S^{-1} = \varnothing$. The spaces D^n and S^{n-1} have the trivial G-action on them. We also set $\mathring{D}^n = D^n - S^{n-1}$ the open disk. For a closed subgroup H of G, an *equivariant n-cell* of type G/H is just the product G-space $G/H \times D^n$.

An *equivariant CW-complex* or *G-CW-complex* X is a G-space X together with a filtration $\{X^n \mid n \geq 0\}$ of X satisfying:

(i) X^0 is a disjoint union of orbits G/H and by induction, X^n is obtained from X^{n-1} by attaching equivariant n-cells $G/H_\alpha \times D^n$ via *attaching G-maps* $\phi_\alpha : G/H_\alpha \times S^{n-1} \to X^{n-1}$. That is, X^n is the quotient space

$$X^{n-1} \bigcup_{\coprod \phi_\alpha} \left(\coprod_\alpha G/H_\alpha \times D^n \right).$$

(ii) $X = \cup_n X^n$ and X has the weak topology with respect to $\{X^n\}$, i.e. a subset $A \subset X$ is closed (or open) if and only if $A \cap X^n$ is closed (or open) in X^n for all n.

Remark II.6. 1. For each n the space X^n is a closed G-subspace of X and is called the *n-skeleton* of X. If there exists some n such that $X = X^n$ and $X \neq X^{n-1}$ then we say that X has *dimension n*.

2. For each equivariant equivariant cell $G/H_\alpha \times D^n$ there is a *characteristic map* $\Phi_\alpha : G/H_\alpha \times D^n \to X$ which extends the attaching map $\phi_\alpha : G/H_\alpha \times S^{n-1} \to X^{n-1}$. That is, we can take Φ_α to be the composition

$$G/H_\alpha \times D^n \hookrightarrow X^{n-1} \coprod \left(\coprod_\alpha G/H_\alpha \times D^n \right) \xrightarrow{q} X^n \hookrightarrow X$$

where q is the quotient map defining X^n. The images $\Phi_\alpha(G/H_\alpha \times D^n)$ and $\Phi_\alpha(G/H_\alpha \times \mathring{D}^n)$ are called the *closed* and *open n-cell* in X, respectively. We denote $\Phi_\alpha(G/H_\alpha \times D^n)$ by $e^n_{H_\alpha}$ and $\Phi_\alpha(G/H_\alpha \times \mathring{D}^n)$ by $\mathrm{int}(e^n_{H_\alpha})$.

3. The topology on X is equivalently characterized by the following: A subset $B \subset X$ is closed if and only if the intersection of B and any closed n cell is closed in that n-cell. Namely, $A \cap e^n_{H_\alpha}$ is closed in $e^n_{H_\alpha}$ for each equivariant n-cell $e^n_{H_\alpha}$. For the details see [FP90] and [McC01].

A G-subspace A of a G-CW-complex X is called a *G-subcomplex* if A is a union of some of the closed cells in X. In other words, if an open cell $\operatorname{int}(e^n_{H_\alpha})$ intersects A nontrivially then the whole closed cell $e^n_{H_\alpha}$ is contained in A. From this definition easy to see that A is itself a G-CW-complex which justifies the word G-subcomplex. If A is a G-subcomplex we call (X, A) a pair of G-CW-complexes.

Proposition II.7. *Let X be a G-CW-complex. Then the orbit space X/G is a nonequivariant CW-complex with n-skeleton X^n/G.*

Proof. It involves a straightforward translation of the definition of X into that of the CW structure of X/G. For example, an equivariant attaching map $\phi_\alpha : G/H_\alpha \times S^{n-1} \to X^{n-1}$ becomes a nonequivariant attaching map $\phi_\alpha/G : S^{n-1} \to X^{n-1}/G$, and so on. □

The next property shows a connection between equivariant and nonequivariant CW-complexes when G is discrete. Let G be a discrete group. Let X be a G-space and an nonequivariant CW-complex. A *regular G-action* on X is a G-action satisfying:

(i) For each open cell $\operatorname{int}(e^n_{H_\alpha})$ and each $g \in G$ the left translation $g \cdot \operatorname{int}(e^n_{H_\alpha})$ is again an open cell in X.

(ii) If $\operatorname{int}(e^n_{H_\alpha}) \cap g \cdot \operatorname{int}(e^n_{H_\alpha}) \neq \varnothing$ then g fixes pointwised the set $\operatorname{int}(e^n_{H_\alpha}) \cap g \cdot \operatorname{int}(e^n_{H_\alpha})$.

Proposition II.8. *Let G be a discrete group and let X be a nonequivariant CW-complex. If there is a regular G-action on X then X is a G-CW-complex with n-skeleton X^n.*

Proof. See [tD87, Proposition II.1.15]. □

C. $RO(G)$-graded cohomology theories

Let G be a finite group and let $RO(G)$ be the real orthogonal representation ring of G. For a representation space V of G, let $S^V = V \cup \{\infty\}$ be the one point compactification of V with ∞ as basepoint. Let $S^V X := S^V \wedge X$ be the smash product of S^V and X for any based G-space X. Then for representation spaces V and W, there is a natural isomorphism $S^W S^V X \cong S^{W+V} X$.

An $RO(G)$-graded equivariant cohomology theory consists of the following data:

(a) A family $\{h_G^\alpha \mid \alpha \in RO(G)\}$ of contravariant functors $G\text{-}Top^0 \to Ab$, where Ab is the category of abelian groups;

(b) For each representation V of G, a family of natural transformations $\sigma^V : h_G^\alpha(X) \to h_G^{\alpha+V}(S^V X)$.

These data satisfy the following axioms:

(i) If f_0 is G-homotopic to a map $f_1\colon X \to Y$, then $h_G^\alpha(f_0) = h_G^\alpha(f_1) : h_G^\alpha(X) \to h_G^\alpha(Y)$;

(ii) σ^V is an isomorphism for all V;

(iii) The sequence

$$h_G^\alpha(C_f) \to h_G^\alpha(X) \to h_G^\alpha(A)$$

is exact for each G-map $f : A \to X$;

(iv) The suspension $\sigma^0 = \mathrm{id}$ and for any pair (V, W) the following diagram commutes:

$$\begin{array}{ccc} h_G^\alpha(X) & \xrightarrow{\sigma^V} & h_G^{\alpha+V}(S^V X) \\ & \searrow^{\sigma^{V+W}} \quad \swarrow_{\sigma^W} & \\ & h_G^{\alpha+V+W}(S^{V+W} X). & \end{array}$$

It is called the *transitivity* of the suspension functors.

Next let us sketch Bredon's construction [Bre67] of a $\mathbb{Z}$-graded equivariant cohomology theory $\{H^n_G(X, M) \mid n \in \mathbb{Z}\}$.

Definition II.9. The orbit category $Or(G)$ is the category whose objects are the homogeneous spaces G/H and whose morphisms are the equivariant maps between them. A *contravariant coefficient system* M is just a contravariant functor $Or(G) \to Ab$, and a *covariant coefficient system* M is a covariant functor $Or(G) \to Ab$. The collection of contravariant coefficient systems for G and the natural transformations between them form an abelian category $\mathcal{C}_G$.

Example II.10. Let $C_*(-, \mathbb{Z})$ be the usual singular chain complex. Pick a G-space X. For each $n \in \mathbb{Z}$ define an element $\underline{C}_n(X, \mathbb{Z}) \in \mathcal{C}_G$ by $\underline{C}_n(X, \mathbb{Z})(G/H) = C_n(X^H, \mathbb{Z})$ together with the obvious values on morphisms of $Or(G)$.

For each n, there is an induced morphism $d : \underline{C}_n(X, \mathbb{Z}) \to \underline{C}_{n-1}(X, \mathbb{Z})$ with $d^2 = 0$, hence the collection $\{\underline{C}_n(X, \mathbb{Z}) \mid n \in \mathbb{Z}\}$ forms a chain complex in the category $\mathcal{C}_G$. For each contravariant coefficient system M let $C^*_G(X, M)$ be the cochain complex $\mathrm{Hom}_{\mathcal{C}_G}(\underline{C}_*(X, \mathbb{Z}), M)$ with differentials $\delta = \mathrm{Hom}_{\mathcal{C}_G}(d, \mathrm{id})$. Then the Bredon cohomology $H^n_G(X, M)$ is defined to be the n-th cohomology group of this cochain complex, i.e.

$$H^n_G(X, M) := H^n(\mathrm{Hom}_{\mathcal{C}_G}(\underline{C}_*(X, \mathbb{Z}), M)).$$

On the other hand, in order to define *Bredon homology* with coefficients in a covariant coefficient system, we need the following definition.

Definition II.11. Let $M : \mathcal{C}_G \to Ab$ be a contravariant functor and $N : \mathcal{C}_G \to Ab$ covariant. Then the *coend*, or *categorical tensor product* is the following quotient abelian group

$$M \otimes_{\mathcal{C}_G} N = \bigoplus_{H<G} M(G/H) \otimes N(G/H)/\sim,$$

where the equivalence relation is generated by $(f^*m, n) \sim (m, f_*n)$ for a G-map $f : G/H \to G/K$ and elements $m \in M(G/K)$ and $n \in M(G/H)$.

If N is a covariant coefficient system then for each $n \in \mathbb{Z}$ we define the abelian group

$$C_n^G(X, N) = \underline{C}_n(X, \mathbb{Z}) \otimes_{\mathcal{C}_G} N.$$

The boundary $\partial : C_n^G(X, N) \to C_{n-1}^G(X, N)$ is $\partial = d \oplus 1$. It has property $\partial^2 = 0$, so $C_*^G(X, N)$ is a chain complex of abelian groups. The Bredon homology $H_n^G(X, N)$ is just the n-th homology group of the chain complex $C_*^G(X, N)$. For the details, see [Bre67] and [tD87, II.9].

Definition II.12. Let G-*Fin* be the category of finite G-sets and G-maps between them. A *Mackey functor* $M = (M^*, M_*) : G\text{-}Fin \to Ab$ consists of a contravariant functor $M^* : G\text{-}Fin \to Ab$ and a covariant functor $M_* : G\text{-}Fin \to Ab$ satisfying:

1. $M^*(S) = M_*(S)$ for all finite G-sets S;
2. For a pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

in G-*Fin*, we have the following commutative diagram

$$\begin{array}{ccc} M(A) & \xrightarrow{f_*} & M(B) \\ {\scriptstyle g^*}\uparrow & & \uparrow{\scriptstyle h^*} \\ M(C) & \xrightarrow[k_*]{} & M(D) \end{array}$$

in Ab, where we use the notations $f_* = M_*(f)$, $f^* = M^*(f)$;

3. The two inclusions $S \to S \coprod T \leftarrow T$ into disjoint union define an isomorphism $M(S \coprod T) \cong M(S) \oplus M(T)$.

Remark II.13. A Mackey functor defines both a contravariant coefficient system and a covariant coefficient system.

Example II.14. A discrete $\mathbb{Z}[G]$-module M defines an associated Mackey functor $\underline{M}$ by $\underline{M}(G/H) := M^H$ and the value of the contravariant part on the projection $G/H \to G/K$, for $H \leq K \leq G$, is the inclusion of M^K into M^H while that of the covariant part is $M^H \to M^K : x \mapsto \sum k_i x$, where $\{k_i\}$ are a set of representatives of left cosets K/H.

In [LMM81] J. P. May et al. showed that, when M is a Mackey functor, the Bredon cohomology theory can be uniquely extended to an $RO(G)$-graded cohomology theory $\{H^{\alpha}_{\mathrm{Br}}(X, M) \mid \alpha \in RO(G)\}$, which is called *$RO(G)$-graded ordinary equivariant Bredon cohomology theory.* We also use the notation $H^{\alpha}_{\mathrm{Br},G}(X, M)$ for this theory to emphasize the group G. We refer the reader to [LMS86, May96, tD87] for exhaustive study of this theory.

For any topological space X, let $\mathbb{Z}X$ denote the free abelian group on X with induced topology from that of X. If X is a based space with base point x_0, then define $\mathbb{Z}_0(X) := \mathbb{Z}X/\mathbb{Z}x_0$. Or equivalently, let $\mathbb{Z}_0(X)$ be the kernel of the augmentation homomorphism $\epsilon : \mathbb{Z}X \to \mathbb{Z}$. One can show that, if X is path connected then $\mathbb{Z}_0(X)$ is the connected component of $0 \in \mathbb{Z}X$.

When X is a based G-space, the topological abelian group $\mathbb{Z}X$ has a naturally induced G-action by

$$g(\sum n_x x) = \sum n_x gx.$$

In [LF97] Lima-Filho proved that when X is a based G-CW complex then

$$\pi_n((\mathbb{Z}_0(X))^G) \cong \tilde{H}_n^{\mathrm{Br},G}(X, \underline{\mathbb{Z}}),$$

where $\tilde{H}_n^{\mathrm{Br},G}(X, \underline{\mathbb{Z}})$ denote the reduced Bredon homology of X with coefficients in the Mackey functor $\underline{\mathbb{Z}}$. This is just an equivariant version of the classical Dold-Thom theorem [DT58].

In [dS03b] dos Santos further generalizes the above result as follows. Let M be a discrete $\mathbb{Z}[G]$-module, i.e. a discrete abelian group M with a (left) G-action. Given a based G-space X (with base point x_0), let $\mathbb{Z}_0(X) \otimes M$ be the $\mathbb{Z}[G]$-module[1] $\sum_{x \in X - \{x_0\}} M$. Each element in $\mathbb{Z}_0(X) \otimes M$ has a unique representation

$$\sum_{x \in X - \{x_0\}} m_x x,$$

in which only finitely many $m_x \neq 0$. The G-action is

$$g(\sum m_x x) = \sum (g m_x)(g x).$$

Let $\pi_V^G(\mathbb{Z}_0(X) \otimes M)$ denotes the set of equivariant homotopy classes of maps $[S^V, \mathbb{Z}_0(X) \otimes M]_G^0$.

Theorem II.15. *[dS03b] If X is a based G-CW complex then $\mathbb{Z}_0(X) \otimes M$ is an equivariant infinite loop space and there is a natural isomorphism*

$$\pi_V^G(\mathbb{Z}_0(X) \otimes M) \cong \tilde{H}_V^{Br,G}(X, \underline{M}), \tag{II.3}$$

where $\tilde{H}_V^{Br,G}(X, \underline{M})$ is the reduced $RO(G)$-graded Bredon homology group.

Remark II.16. In [Nie07] Nie unified the above results.

[1] In [dS03b] it is denoted by $M \otimes X$.

Definition II.17. Let $V \in RO(G)$. A $K(\underline{M}, V)$ space is a classifying space for the functor $\tilde{H}^V_{\mathrm{Br}}(-, \underline{M})$. In other words, a based G-space Z is a $K(\underline{M}, V)$ space if for every based G-space X there is a natural isomorphism

$$\tilde{H}^V_{\mathrm{Br}}(X, \underline{M}) \cong [X, Z]^0_G.$$

An essential consequence of (II.3) is that $\mathbb{Z}_0(S^V) \otimes M$ is a $K(\underline{M}, V)$ space. So there is a natural isomorphism

$$\tilde{H}^V_{\mathrm{Br}}(X, \underline{M}) \cong [X, \mathbb{Z}_0(S^V) \otimes M]^0_G \tag{II.4}$$

for every based G-space X.

Proposition II.18. *Given a finite-dimensional representation V of G and a discrete $\mathbb{Z}[G]$-module M, there is a natural isomorphism*

$$\pi_n\left((\mathbb{Z}_0(S^V) \otimes M)^G\right) \cong \tilde{H}^{V-n}_{Br}(*, \underline{M}), \tag{II.5}$$

where $*$ is a G-fixed point.

Proof. One has the following isomorphisms

$$\begin{aligned} \pi_n\left((\mathbb{Z}_0(S^V) \otimes M)^G\right) &= \left[S^n, (\mathbb{Z}_0(S^V) \otimes M)^G\right]^0_{Top} && \text{(definition)} \\ &\cong [S^n, \mathbb{Z}_0(S^V) \otimes M]^0_G && \text{(trivial } G\text{-action on } S^n) \\ &\cong \tilde{H}^V_{\mathrm{Br}}(S^n, \underline{M}) && \text{(II.4)} \\ &\cong \tilde{H}^{V-n}_{\mathrm{Br}}(*, \underline{M}). && \text{(suspension axiom)} \end{aligned}$$

□

This result will be used in Chapter IV.

CHAPTER III

EQUIVARIANT GOOD COVER OF A G-MANIFOLD

A. Simplicial G-complex

Recall that an open cover $\mathfrak{U} = \{U_\alpha\}$ of a smooth manifold M is called a *good cover* if all nonempty finite intersections $U_{\alpha_0 \cdots \alpha_n} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$ are contractible. There is a classical theorem (c.f. [BT82, Theorem 5.1]) stating that every smooth manifold M has a good cover when considering the geodesic convex balls for a Riemannian metric on M ([GHL04, Corollary 2.89], [dC92, p. 70]). We extend this theorem to the equivariant case for a finite group G.

Let us start with some notations and terminology about simplicial complex and G-complex. The first section of this chapter is to give some important results about simplicial complexes needed for the latter chapters. It is by no means a thorough overview of the theory of simplicial complexes. All of the contents in this section are elementary and can be found in almost any textbook about algebraic or combinatorial topology. Among them we list the following: [Spa81], [Dol80], [Mac67], [Pra06], [Bre72] and [Rot88].

If $\{v_0, \ldots, v_n\}$ is an affine independent set of some euclidean space, then the subspace

$$s = \langle v_0, \ldots, v_n \rangle := \left\{ \sum_{i=0}^{n} \lambda_i v_i \mid \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0 \right\}$$

is called the *n-simplex* spanned by $\{v_i\}$. We denote its *vertex set* by $\mathrm{Vert}(s) = \{v_0, \ldots, v_n\}$. A face s' of s is a simplex s' with $\mathrm{Vert}(s') \subset \mathrm{Vert}(s)$.

Definition III.1. A *(geometric) simplicial complex* K is a collection of simplices in some euclidean space such that:

(i) if $s \in K$, then every face of s is also in K;

(ii) if $s, t \in K$, then $s \cap t$ is either empty or is a face of both s and t.

If K is a simplicial complex, we write $\mathrm{Vert}(K)$ for the vertex set of K, i.e. the set of 0-simplices in K. The dimension of K is defined by $\dim K = \sup\{\dim(s) \mid s \in K\}$. K is *locally finite* if every point $x \in |K|$ has a neighborhood intersecting only finite many simplices of K ([Mun66, p. 69]). K is *finite* if it contains only a finite number of simplices. Here for our purpose it suffices to assume that all simplicial complexes are locally finite.

A simplicial *subcomplex* L of K is a subset of K such that L is itself a simplicial complex. It is clear that a subset L of K is a subcomplex if and only if any simplex in K that is a face of a simplex in L is a simplex in L.

Examples III.2. We give some examples of simplicial complexes.

1. Given an n-simplex s, let $\bar{s}$ denote the set of all faces of s and $\dot{s}$ the set of all proper faces of s (If $n = 0$, let $\dot{s} = \varnothing$). Then $\bar{s}$ and $\dot{s}$ are both simplicial complexes with dimension n and $n-1$, respectively. The set $s^\circ = s - \dot{s}$ is called an open n-simplex.

2. If K is a simplicial complex, then its *n-skeleton* K^n is a simplicial complex consisting of all simplices in K with dimension less than or equal to n.

We put $|K| := \bigcup_s \{s \mid s \in K\} \subset \mathbb{R}^N$, a subset of some ambient euclidean space. We call $|K|$ the associated *polyhedron* or the *underlying space* of K.

Definition III.3. A topological space X is called a *polyhedron* if there exists a homeomorphism $h : |K| \xrightarrow{\cong} X$ for some simplicial complex K. The pair (K, h) is called a *triangulation* of X.

Definition III.4. 1. Let K be a simplicial complex. Pick $x \in |K|$. The *carrier* carr(x) of x is defined to be the (unique) smallest simplex of K containing x. In some cases we write $\mathrm{carr}_K(x)$ for $\mathrm{carr}(x)$ to emphasize K.

2. If v is a vertex of K then the *open star* of v is

$$\mathrm{st}_K(v) = \{x \in |K| \mid v \in \mathrm{carr}(x)\}.$$

Proposition III.5. *Let K be a simplicial complex. Then $x \in \mathrm{st}_K(v)$ if and only if $v \in \mathrm{carr}(x)$ and for $x, y \in |K|$, $y \in \mathrm{carr}(x)$ implies $\mathrm{carr}(y) \subset \mathrm{carr}(x)$.*

Proof. The conclusions are clear from the definitions. □

Proposition III.6. *If $v_0, \ldots, v_n$ are vertices of a simplicial complex K then $\bigcap_i \mathrm{st}_K(v_i) \neq \varnothing$ if and only if $\langle v_0, \ldots, v_n \rangle$ is a simplex of K.*

Proof. A point $x \in \bigcap_i \mathrm{st}_K(v_i)$ iff $v_i \in \mathrm{carr}(x)$ for all i iff $\langle v_0, \ldots, v_n \rangle$ is a face of $\mathrm{carr}(x)$, and the result follows. □

The following theorem shows that the set of open stars of vertices is a good cover of a simplicial complex. This gives an alternative proof of the existence of a good cover for a smooth manifold since it is known that every manifold admits a triangulation [Whi40].

Theorem III.7. *Let K be a locally finite simplicial complex. Then for each vertex v of K,*

$$\mathrm{st}_K(v) = \bigcup_{\substack{s \in K \\ v \in \mathrm{Vert}(s)}} s^\circ. \tag{III.1}$$

Furthermore, the set

$$\mathfrak{U} = \{\mathrm{st}_K(v) \mid v \in \mathrm{Vert}(K)\}$$

is a good cover of K.

Before we prove this theorem we need the following lemma.

Lemma III.8. *If K is a simplicial complex, then $|K|$ is the disjoint union of all the open simplices s° with $s \in K$. Hence each $x \in |K|$ lies in a unique open simplex. In fact, $x \in (\mathrm{carr}(x))^\circ$.*

Proof. It follows by induction on the dimension of K. □

Remark III.9. By Lemma III.8, for any two simplices s, t in a simplicial complex K, the intersection $s^\circ \cap t^\circ$ is empty whenever s is not identically equal to the simplex t. In particular, even if s is a face of t, the open simplex s° is still disjoint from t°.

Proof of Theorem III.7. Denote $\bigcup_{\substack{s \in K \\ v \in \mathrm{Vert}(s)}} s^\circ$ by A. If $x \in \mathrm{st}_K(v)$ then $v \in \mathrm{carr}(x)$ by definition hence $v \in \mathrm{Vert}(\mathrm{carr}(x))$. Since $x \in (\mathrm{carr}(x))^\circ$ by Lemma III.8, $x \in A$. On the other hand, pick $x \in s^\circ$ for any simplex s containing v as a vertex. Then $s = \mathrm{carr}(x)$ by the uniqueness statement in Lemma III.8. Hence the condition $v \in \mathrm{Vert}(s)$ implies $v \in \mathrm{Vert}(\mathrm{carr}(x)) \subset \mathrm{carr}(x)$. Thus by definition $x \in \mathrm{st}_K(v)$. By arbitrarity of x and s, $A \subset \mathrm{st}_K(v)$.

So $\mathfrak{U}$ is a set of open subsets of $|K|$ and it is clear that $\mathfrak{U}$ covers $|K|$. Furthermore, given any finite set of elements $U_i = \mathrm{st}_K(v_i)$, $i = 0, \ldots, n$ in $\mathfrak{U}$, one has

$$U_0 \cap \cdots \cap U_n = \bigcup_{\substack{v_i \in \mathrm{Vert}(s) \\ \text{for } all\ i = 0, \ldots, n}} s^\circ$$

by (III.1) and Remark III.9, and this set is clearly contractible. □

Definition III.10. Let K and L be simplicial complexes.

1. A *simplicial map* $f : K \to L$ is a map $f : |K| \to |L|$ which sends $\mathrm{Vert}(K)$ into $\mathrm{Vert}(L)$ and is linear on each simplex of K, i.e. $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. Given a continuous map $\varphi : |K| \to |L|$, if $f(x) \in \mathrm{carr}(\varphi(x))$ for each $x \in |K|$ then f is called a *simplicial approximation* to φ.

2. A *subdivision* of K is a simplicial complex K' such that $|K'| = |K|$ and each simplex s' of K' lies in some simplex of K. A *barycentric subdivision* of K is a simplicial complex K' whose vertices are the simplices of K and whose simplices are the sets $\langle s_0, \ldots, s_n \rangle$ such that s_i are simplices of K with

$$s_0 \subset s_1 \subset \cdots \subset s_n.$$

That is, s_i is a face of s_{i+1} for all i.

From the definition it is easy to see that if $f : K \to L$ is simplicial then whenever $\{w_0, \ldots, w_n\}$ spans a simplex in K, $\{f(w_0), \ldots, f(w_n)\}$ spans a simplex in L.

Proposition III.11. *Let K and L be simplicial complexes.*

(1) If $f : K \to L$ is a simplicial map then $f(\mathrm{carr}(x)) = \mathrm{carr}(f(x))$ and $f(\mathrm{st}_K(v)) \subset \mathrm{st}_L(f(v))$ for any $x \in |K|$ and $v \in \mathrm{Vert}(K)$.

(2) A simplicial map $f : K \to L$ is a simplicial approximation to $\varphi : |K| \to |L|$ if and only if $\varphi(\mathrm{st}_K(v)) \subset \mathrm{st}_L(f(v))$ for any vertex v of K.

Proof. For (1), let $s = \langle v_0, \ldots, v_n \rangle$ be a simplex of K and $x = \sum_{i=0}^{n} \lambda_i v_i$ with all $\lambda_i > 0$. Then $\mathrm{carr}(x) = s$. Since f is simplicial, $f(x) = \sum_{i=0}^{n} \lambda_i f(v_i)$ (some of the $f(v_i)$ may be equal). Thus $\mathrm{carr}(f(x)) =$ convex hull of $\{f(v_0), \ldots, f(v_n)\}$ which is just $f(\mathrm{carr}(x))$. Now pick $x \in \mathrm{st}_K(v)$. Then $v \in \mathrm{carr}(x)$. Applying f yields $f(v) \in f(\mathrm{carr}(x)) = \mathrm{carr}(f(x))$. This gives $f(x) \in \mathrm{st}_L(f(v))$. Since x is arbitrary, we have $f(\mathrm{st}_K(v)) \subset \mathrm{st}_L(f(v))$.

The proof of (2) is similar to (1). If f is a simplicial approximation to φ, then by definition $f(x) \in \mathrm{carr}(\varphi(x))$ for all $x \in |K|$. Pick $x \in \mathrm{st}_K(v)$. The inclusion $v \in \mathrm{carr}(x)$ yields $f(v) \in f(\mathrm{carr}(x)) = \mathrm{carr}(f(x)) \subset \mathrm{carr}(\varphi(x))$. The last inclusion comes from Proposition III.5. So $\varphi(x) \in \mathrm{st}_L(f(v))$. For the 'if' part, pick $x \in |K|$

and let $\mathrm{carr}_K(x) = \langle v_0, \ldots, v_n$. Then $x \in \mathrm{st}_K(v_i)$ for all i by Proposition III.5. By the assumption, $\varphi(x) \in \varphi(\mathrm{st}_K(v_i)) \subset \mathrm{st}_L(f(v_i))$ for all i. Then $f(v_i) \in \mathrm{carr}_L(\varphi(x))$ for all i and hence one has $f(x) \in \langle f(v_0), \ldots, f(v_n)\rangle \subset \mathrm{carr}_L(\varphi(x))$. This shows that f is a simplicial approximation to φ. □

Given a simplicial complex K, let $K^{(0)} = K$, $K^{(1)} = K'$, the barycentric subdivision of K, and by induction let $K^{(n)}$ be the barycentric subdivision of $K^{(n-1)}$. Then we have the following theorem.

Theorem III.12 (The Simplicial Approximation Theorem)**.** *Let K and L be finite simplicial complexes and let $\varphi : |K| \to |L|$. Then there exist an integer $q \geq 0$ and a simplicial approximation $f : K^{(q)} \to L$ to φ.*

Proof. See [Rot88, p. 139] or [Bre93, p. 252]. □

Next we consider an action of a group G on the simplicial complexes.

Definition III.13. Let G be a finite group.

1. A *simplicial G-complex* consists of a simplicial complex K together with a G-action on K such that for every $g \in G$ the map $g : K \to K$ is a simplicial homeomorphism.

2. A simplicial G-complex K is a *regular G-complex* if the following conditions are satisfied.

   (R1) If vertices v and gv belong to the same simplex then $v = gv$.

   (R2) If $s = \langle v_0, \ldots, v_n\rangle$ is a simplex of K and $s' = \langle g_0v_0, \ldots, g_nv_n\rangle$, where $g_i \in G$, $i = 0, \ldots, n$, also is a simplex of K then there exists $g \in G$ such that $gv_i = g_iv_i$, for $i = 0, \ldots, n$.

Remark III.14. (a) If K is a simplicial G-complex, then the underlying space $|K|$ carries a natural G-action so that $|K|$ is a G-space.

(b) In fact the condition (R2) implies (R1) since if v and gv belong to some simplex, then $\langle v, v\rangle$ and $\langle v, gv\rangle$ are simplices of K, so for some g', $v = g'v = gv$.

Proposition III.15. *Let K be a simplicial G-complex. Then for any vertex v of K and any $g \in G$, we have*

$$\mathrm{st}_K(gv) = g(\mathrm{st}_K(v)).$$

Proof. Pick $x \in \mathrm{st}_K(gv)$. By Proposition III.5, $gv \in \mathrm{carr}(x)$. So $v \in g^{-1}(\mathrm{carr}(x)) = \mathrm{carr}(g^{-1}x)$ by Proposition III.11. By Proposition III.5 again, $g^{-1}x \in \mathrm{st}_K(v)$. Thus $\mathrm{st}_K(gv) \subset g\,\mathrm{st}_K(v)$. On the other hand, If $x \in \mathrm{st}_K(v)$, then $v \in \mathrm{carr}(x)$ implies $gv \in g(\mathrm{carr}(x)) = \mathrm{carr}(gx)$ which is equivalent to say $gx \in \mathrm{st}_K(gv)$, and this yields the desired result. □

The following proposition shows that any simplicial G-complex becomes regular after passing to the second barycentric subdivision. So restricting to regular G-complexes is not seriously harmful. The following proposition comes from [Bre72].

Proposition III.16 ([Bre72])**.** *If K is a simplicial G-complex, then the induced action on the barycentric subdivision K' satisfies (R1). If (R1) is satisfied for K, then (R2) is satisfied for K'.*

Proof. Pick a vertex s of K'. s is a simplex of K. If s and gs belong to a same simplex of K', then either s is a face of gs or vice versa. But s and gs have the same dimension, so $s = gs$.

Now suppose K satisfies (R1). We will prove (R2) for K' by induction on n. Suppose that $\langle s_0, s_1, \ldots, s_n\rangle$ is a simplex of K' and after some reordering we may assume that $s_0 \subset s_1 \subset \cdots \subset s_{n-1} \subset s_n$ as simplices of K. Suppose that $\langle g_0 s_0, \ldots, g_n s_n\rangle$ is also a simplex of K'. By the inductive assumption, there is a g in G with $gs_i = g_i s_i$

for $0 \leq i < n$. Left multiplicating g^{-1} shows that

$$\langle s_0, s_1, \dots, s_{n-1}, g^{-1} g_n s_n \rangle$$

is a simplex of K'. Since s_i's are ordered by dimension, we have

$$s_0 \subset s_1 \subset \cdots \subset s_{n-1} \subset g^{-1} g_n s_n$$

Then $s_{n-1} \subset (s_n \cap g^{-1} g_n s_n)$. But by (R1), g acts trivially on simplex $s \cap g(s)$ for any $g \in G$ since for any vertex v of $s \cap g(s)$, $\langle v, gv \rangle \subset g(s)$ so that $v = gv$. Thus $g^{-1} g_n$ acts trivially on s_{n-1} hence trivially on s_i for all $i < n$. So we have $g_n s_i = g s_i = g_i s_i$ for $i < n$ and therefore $g_n s_i = g_i s_i$ for all i. □

For a subgroup H of G, we define $K^H := |K|^H$, the fixed point set of $|K|$ by H. The next proposition shows that when K is a regular G-complex, K^H a subcomplex of K.

Proposition III.17. *Let K be a regular G-complex.*

(1) For any subgroup $H \leq G$, K^H is a nonequivariant subcomplex of K.

(2) For $x \in K^H$, $\operatorname{carr}_{K^H}(x) = \operatorname{carr}_K(x)$. Moreover, if v is a vertex of K^H, then

$$\operatorname{st}_{K^H}(v) = \operatorname{st}_K(v) \cap K^H.$$

Proof. (1) $K^H = \cap_{h \in H} |K|^h$, so it suffices to prove that for any $h \in H$, $|K|^h$ is a subcomplex. Thus we pick $x \in |K|^h$ and consider the carrier $\operatorname{carr}(x) = \operatorname{carr}_K(x)$. Since $h(x) = x$, then $h \operatorname{carr}(x) = \operatorname{carr}(h(x)) = \operatorname{carr}(x)$. In particular, it implies that for any vertex v of $\operatorname{carr}(x)$, hv and v are in the same simplex $\operatorname{carr}(x)$. By the regularity of K, $hv = v$. Then the linearity of the action of h on $\operatorname{carr}(x)$ shows that each point in $\operatorname{carr}(x)$ is fixed by h. That is, $\operatorname{carr}(x) \subset |K|^h$. So $|K|^h$ is a collection of simplices of K. It is obvious that $|K|^h$ satisfies the conditions (i) and (ii) in Definition III.1. Hence $|K|^h$ is indeed a simplicial subcomplex of K.

(2) The proof of (1) shows that if $x \in K^H$ then $\mathrm{carr}_K(x) \subset K^H$. Since $\mathrm{carr}_{K^H}(x)$ is the smallest simplex of K^H containing x, there is the inclusion $\mathrm{carr}_{K^H}(x) \subset \mathrm{carr}_K(x)$. On the other hand, the set $\mathrm{carr}_{K^H}(x)$, as a simplex of K^H, is also a simplex of K containing x. Then the definition of $\mathrm{carr}_K(x)$ yields the reverse inclusion $\mathrm{carr}_K(x) \subset \mathrm{carr}_{K^H}(x)$. Thus $\mathrm{carr}_{K^H}(x) = \mathrm{carr}_K(x)$ if $x \in K^H$. It follows that the following equivalent relations hold.

$$\begin{aligned} x \in \mathrm{st}_{K^H}(v) &\Leftrightarrow x \in K^H \text{ and } v \in \mathrm{carr}_{K^H}(x) \\ &\Leftrightarrow x \in K^H \text{ and } v \in \mathrm{carr}_K(x) \\ &\Leftrightarrow x \in \mathrm{st}_K(v) \cap K^H. \end{aligned}$$

□

B. The equivariant good covers

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of a paracompact G-space X. Then for any $g \in G$, the set

$$g\mathfrak{U} := \{gU_\alpha \mid U_\alpha \in \mathfrak{U}\}$$

is still an open cover of X. If $g\mathfrak{U} = \mathfrak{U}$ for all g, we say $\mathfrak{U}$ is *G-invariant* or just *invariant* for simplicity. In this case there is an induced action of G on the index set I defined by $g\alpha$ being the unique index with $U_{g\alpha} = gU_\alpha$.

If $\mathfrak{U}$ and $\mathfrak{V} = \{V_\beta\}_{\beta \in J}$ are covers, then

$$\mathfrak{U} \cap \mathfrak{V} = \{U_\alpha \cap V_\beta \mid U_\alpha \in \mathfrak{U}, V_\beta \in \mathfrak{V}\}$$

is an open cover which refines both $\mathfrak{U}$ and $\mathfrak{V}$. Clearly (note that G is finite)

$$\bigcap_{g\in G} g\mathfrak{U}$$

is an invariant cover refining $\mathfrak{U}$. Moreover this is locally finite if $\mathfrak{U}$ is. Thus, for X paracompact, the locally finite invariant covers are cofinal in the set of all covers of X.

Now let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be a locally finite invariant cover of X and let

$$f = \{f_\alpha\}_{\alpha\in I}$$

be a partition of unity subordinate to $\mathfrak{U}$ (in particular, $\mathrm{supp}(f_\alpha) \subset U_\alpha$). Then f is called a *G-partition of unity* if $f_{g\alpha}(gx) = f_\alpha(x)$ for all g, x and α.

If $f = \{f_\alpha\}$ is any partition of unity subordinate to the invariant cover $\mathfrak{U}$, we define

$$\tilde{f}_\alpha(x) = \frac{1}{|G|}\sum_g f_{g\alpha}(gx)$$

Then

$$\sum_\alpha \tilde{f}_\alpha(x) = \frac{1}{|G|}\sum_\alpha\sum_g f_{g\alpha}(gx) = \frac{1}{|G|}\sum_g\left(\sum_\alpha f_{g\alpha}(gx)\right) = \frac{1}{|G|}\sum_g 1 = 1$$

and

$$\tilde{f}_{g'\alpha}(g'x) = \frac{1}{|G|}\sum_g f_{gg'\alpha}(gg'x) = \frac{1}{|G|}\sum_{gg'} f_{(gg')\alpha}((gg')x) = \tilde{f}_\alpha(x),$$

so $\tilde{f} = \{\tilde{f}_\alpha\}_{\alpha\in I}$ is a G-partition of unity.

Let Δ be the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$, and whose morphisms are nondecreasing functions. If $\mathcal{A}$ is any category, a *simplicial object* A in $\mathcal{A}$ is a contravariant functor from Δ to $\mathcal{A}$, i.e. $A \in \mathcal{A}^{\Delta^{op}}$. For simplicity, we write A_n for $A([n])$. Similarly, a *cosimplicial object* B in $\mathcal{A}$ is a covariant functor from Δ to $\mathcal{A}$, i.e. $B \in \mathcal{A}^\Delta$. We write B^n for $B([n])$. There is a category $S\mathcal{A}$ whose objects are simplicial objects in $\mathcal{A}$ and whose morphisms are the natural transformations.

Example III.18. Let $\mathcal{A}$ be the category *Set* of sets. Then there is a category *SSet* whose objects are called *simplicial sets*. Similarly, for $\mathcal{A} = Ab$ we have the category *SAb* of simplicial abelian groups.

Definition III.19. Let A be a simplicial set. For each $n \geq 0$, let A_n have the discrete topology. The *geometric realization* $|A|$ of A is the topological space

$$|A| := \left(\coprod_{n \geq 0} A_n \times \Delta^n \right) / \sim$$

where Δ^n is the standard n-simplex and the equivalence relation is generated by

$$(f^*(x), t) \sim (x, f_*(t)).$$

Here f is a nondecreasing function $[m] \to [n]$ and $x \in A_n$, $t \in \Delta^m$.

Definition III.20. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of a topological space X. If the index set $\{\alpha\}_{\alpha \in I}$ is ordered we associate a simplicial set $\mathcal{N}(\mathfrak{U})$ as follows. For any nonnegative integer n, let $\mathcal{N}(\mathfrak{U})_n$ consist of all ordered $(n+1)$-tuples $(\alpha_0, \ldots, \alpha_n)$ of indices, possibly including repetition, such that $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \varnothing$. For each nondecreasing function $f : [m] \to [n]$, define $f^* = \mathcal{N}(\mathfrak{U})(f) : \mathcal{N}(\mathfrak{U})_n \to \mathcal{N}(\mathfrak{U})_m$ by

$$f^*(\alpha_0, \ldots, \alpha_n) = (\alpha_{f(0)}, \ldots, \alpha_{f(m)}).$$

Easy to check that $\mathcal{N}(\mathfrak{U})$ is indeed a simplicial set and we call it the *nerve* of the cover $\mathfrak{U}$. If $\sigma^n = (\alpha_0, \ldots, \alpha_n) \in \mathcal{N}(\mathfrak{U})_n$, we denote by U_{σ^n} or $U_{\alpha_0 \ldots \alpha_n}$ the nonempty finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$.

There is a simplicial complex $\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))$ associated to the nerve $\mathcal{N}(\mathfrak{U})$ whose vertices $\{v_\alpha\}$ are in one-to-one correspondence with the index set $\{\alpha \mid \alpha \in I\}$. A set

$\{v_{\alpha_0}, \ldots, v_{\alpha_n}\}$ is a simplex of $\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))$ if and only if $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \varnothing$, that is, if and only if $(\alpha_0, \ldots, \alpha_n) \in \mathcal{N}(\mathfrak{U})_n$.

Similarly, we define a simplicial space $\mathcal{N}^{Top}(\mathfrak{U})$ as follows. Let

$$\mathcal{N}^{Top}(\mathfrak{U})_n = \coprod_{(\alpha_0,\ldots,\alpha_n)\in\mathcal{N}(\mathfrak{U})_n} U_{\alpha_0\ldots\alpha_n}$$

with disjoint union topology. For each nondecreasing function $f : [m] \to [n]$, the induced map $f^* : \mathcal{N}^{Top}(\mathfrak{U})_n \to \mathcal{N}^{Top}(\mathfrak{U})_m$ is defined by

$$f^*|_{U_{\alpha_0\ldots\alpha_n}} : U_{\alpha_0\ldots\alpha_n} \to U_{\alpha_{f(0)}\ldots\alpha_{f(m)}},$$

where the latter is either an inclusion or the identity.

Definition III.21. An invariant open cover $\mathfrak{U}$ of a G-space X is a *regular G-cover* if the complex associated to its nerve $\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))$ is a regular G-complex, that is, if it satisfies the following two conditions.

(RC1) For $U_\alpha \in \mathfrak{U}$ and $g \in G$, if $U_\alpha \cap gU_\alpha \neq \varnothing$ then $U_\alpha = gU_\alpha$.

(RC2) If $U_0, \ldots, U_n$ are members of $\mathfrak{U}$ and $g_0, \ldots, g_n$ are elements in G, and if the intersections $U_0 \cap \cdots \cap U_n$ and $g_0U_0 \cap \cdots \cap g_nU_n$ are nonempty, then there exists $g \in G$ such that $gU_i = g_iU_i$ for all $i \leq n$.

Theorem III.22. *Let X be a paracompact G-space, where G is finite. Then the locally finite, regular G-covers of X are cofinal in the set of open covers of X.*

Proof. Pick an invariant cover $\mathfrak{U}$ of X. Let $\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))$ be the simplicial complex associated to the nerve of $\mathfrak{U}$. Then $\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))$ is a simplicial G-complex. Let $f = \{f_\alpha\}$ be a G-partition of unity subordinate to $\mathfrak{U}$ and let $\bar{f} : X \to |\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))|$ be the associated map with

$$\bar{f}(x) = \sum_\alpha f_\alpha(x) v_\alpha$$

Then $\bar{f}$ is a well-defined G-map since all but finite $f_\alpha = 0$ and

$$\bar{f}(gx) = \sum_\alpha f_\alpha(gx)v_\alpha = \sum_\alpha f_{g^{-1}\alpha}(x)v_\alpha$$
$$= g\sum_\alpha f_{g^{-1}\alpha}(x)g^{-1}v_\alpha = g\sum_\alpha f_\alpha(x)v_\alpha = g\bar{f}(x).$$

For any map $f : X \to |K|$ to a polyhedron, let $f^{-1}(\mathrm{st}_K)$ denote the open cover of X by inverse images of open stars of vertices of K. Suppose that K is a G-complex and that f is equivariant. Then $f^{-1}(\mathrm{st}_K)$ is an invariant cover by Proposition III.15. Moreover, if K is a regular G-complex then $f^{-1}(\mathrm{st}_K)$ is a regular G-cover. This is from the fact that if $U_0 \cap \cdots \cap U_n \neq \varnothing \neq g_0U_0 \cap \cdots \cap g_nU_n$, where $U_i = f^{-1}(\mathrm{st}_K(v_i))$, then by Proposition III.6 $\langle v_0, \ldots, v_n\rangle$ and $\langle g_0v_0, \ldots, g_nv_n\rangle$ are simplices of K. Now the regularity of K implies that $f^{-1}(\mathrm{st}_K)$ is regular.

Back to the equivariant map $\bar{f} : X \to |\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))|$, note that $\bar{f}^{-1}(\mathrm{st}_{\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))})$ is a refinement of $\mathfrak{U}$. Actually, for any α, $\bar{f}^{-1}(\mathrm{st}_{\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))}(v_\alpha)) = f_\alpha^{-1}((0,1]) \subset U_\alpha$. Let L be the second barycentric subdivision of $\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))$ such that $|L| = |\mathrm{Comp}(\mathcal{N}(\mathfrak{U}))|$ and L is a regular G-complex by Proposition III.16. So $\mathfrak{V} = \bar{f}^{-1}(\mathrm{st}_L)$ is a regular G-cover which refines $\mathfrak{U}$. □

Theorem III.23. *Let X be a smooth G-manifold. Then*

(1) There exists a regular simplicial G-complex K and a smooth equivariant triangulation $h : K \to X$.

(2) If $h : K \to X$ and $h_1 : L \to X$ are smooth equivariant triangulations of X there exist equivariant subdivisions K' and L' of K and L, respectively, such that K' and L' are G-isomorphic.

Proof. See [Ill78]. □

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of G-space X. For any subgroup H of G and

$\alpha \in I$, let $U_\alpha^H = U_\alpha \cap X^H = \{x \in U_\alpha \mid hx = x \text{ for all } h \in H\}$. Denote by $\mathfrak{U}^H$ the collection of $\{U_\alpha^H\}_{\alpha \in I}$. It is clear that $\mathfrak{U}^H$ is an open cover of X^H.

Definition III.24. $\mathfrak{U}$ is called an *equivariant good cover* of X if it is a regular G-cover (see Definition III.21) and $\mathfrak{U}^H$ is a good cover of X^H for all subgroups $H \leq G$.

Theorem III.25. *Every smooth G-manifold has an equivariant good cover. Moreover, the equivariant good covers are cofinal in the set of open covers of a G-manifold X.*

Proof. By Theorem III.23 it is no loss to assume X is a realization of a *regular* simplicial G-complex K. Consider the open cover

$$\mathfrak{W} = \{\mathrm{st}_K(v) \mid v \in \mathrm{Vert}(K)\}.$$

By Proposition III.15 $\mathfrak{W}$ is G-invariant. Moreover, We claim that $\mathfrak{W}$ is a regular G-cover. The proof is as follows. Let $U = \mathrm{st}_K(v) \in \mathfrak{W}$ and $g \in G$ with $\varnothing \neq U \cap gU = \mathrm{st}_K(v) \cap \mathrm{st}_K(gv)$. It follows that $\langle v, gv\rangle$ is a simplex in K by Proposition III.6. The regularity of K yields $v = gv$ and hence $U = gU$. If for $i = 0, \ldots, n$, $U_i = \mathrm{st}_K(v_i)$ are members of $\mathfrak{W}$ and g_i are members of G such that $U_0 \cap \cdots \cap U_n$ and $g_0U_0 \cap \cdots \cap g_nU_n = \mathrm{st}_K(g_0v_0) \cap \cdots \cap \mathrm{st}_K(g_nv_n)$ are nonempty, then again by Proposition III.6 there are two simplices in K: $\langle v_0, \ldots, v_n\rangle$ and $\langle g_0v_0, \ldots, g_nv_n\rangle$. Since K is regular, there exists $g \in G$ such that $gv_i = g_iv_i$ for all i which is equivalent to $gU_i = g_iU_i$ for all i. So by Definition III.21 $\mathfrak{W}$ is a regular G-cover.

For any subgroup H of G, the Proposition III.17 (1) shows that K^H is a simplicial subcomplex of K and X^H is homeomorphic to K^H. Pick an element $U = \mathrm{st}_K(v)$ of $\mathfrak{W}$. Consider the intersection $U \cap K^H = \mathrm{st}_K(v) \cap K^H$. If $v \in \mathrm{Vert}(K^H)$ then by Proposition III.17 (2), $U \cap K^H = \mathrm{st}_{K^H}(v)$. If $v \notin K^H$, we claim that $U \cap K^H = \varnothing$. To justify this, assume $U \cap K^H \neq \varnothing$. Pick $x \in U \cap K^H$. Then $x \in U = \mathrm{st}_K(v)$

implies $v \in \mathrm{carr}_K(x)$ by Proposition III.5 and $x \in K^H$ yields $\mathrm{carr}_K(x) \subset K^H$ by Proposition III.17 (2). Hence $v \in K^H$, contradicting the assumption $v \notin K^H$. So

$$\mathfrak{W}^H = \{U \cap K^H \mid U \in \mathfrak{W}\} = \{\mathrm{st}_{K^H}(v) \mid v \in \mathrm{Vert}(K^H)\},$$

and hence $\mathfrak{W}^H$ is a good cover of K^H by Theorem III.7.

Note that a barycentric subdivision a regular G-complex is still regular. Then for any given open cover $\mathfrak{U}$ of X, there exists an integer m such that the m-th barycentric subdivision $K^{(m)}$ of the above K has the properties that $\mathfrak{V} = \{\mathrm{st}_{K^{(m)}}(v) \mid v \in \mathrm{Vert}(K^{(m)})\}$ refines $\mathfrak{U}$ and that $\mathfrak{V}$ is still an equivariant good cover since $\mathfrak{V}$ is again the set of open stars of the regular G-complex $K^{(m)}$, which shows the cofinality of equivariant good covers in the set of open covers of X. □

If K is a G-complex, then the orbit space K/G has the structure of an ordinary simplicial complex if we define the vertices of K/G to be the orbits $\bar{v} = G(v)$ of the action of G on the vertices v of K and $\bar{s} = \langle \overline{v}_0, \ldots, \overline{v}_n \rangle$ is a simplex of K/G if and only if there exist representatives v_i of $\overline{v}_i$ such that $s = \langle v_0, \ldots, v_n \rangle$ is a simplex of K. In this case s is called to be over $\bar{s}$.

By the above definition of K/G the natural projection $\pi : K \to K/G, v \mapsto \overline{v} = G(v)$ is simplicial and π maps each simplex of K homeomorphically onto the corresponding image simplex of K/G. By regularity, if $s = \langle v_0, \ldots, v_n \rangle$ and $s' = \langle v'_0, \ldots, v'_n \rangle$ are simplices of K over the same simplex $\bar{s}$ of K/G then $\langle v_0, \ldots, v_n \rangle = g\langle v'_0, \ldots, v'_n \rangle$ for some $g \in G$. So the set of all simplices over a given simplex $\bar{s}$ of K/G form an orbit of a simplex s of K which is over $\bar{s}$.

Now consider the good cover $\mathrm{st}_{K/G}$ of K/G. The set $\mathfrak{U}' = \pi^{-1}(\mathrm{st}_{K/G})$ is an open cover of K. We claim that for each vertex v of K,

$$\pi^{-1}(\mathrm{st}_{K/G}(\overline{v})) = \coprod_{g\in G} \mathrm{st}_K(gv). \tag{III.2}$$

First, pick $g, g' \in G$. If $\mathrm{st}_K(gv) \cap \mathrm{st}_K(g'v) \neq \varnothing$, then $\langle gv, g'v\rangle$ is a simplex in K by Proposition III.6 and hence $gv = g'v$ by Definition III.13 (R1). Therefore either $\mathrm{st}_K(gv) = \mathrm{st}_K(g'v)$ or they are disjoint for $g, g' \in G$. Thus the right hand side of (III.2) is indeed a disjoint union of open stars. For the equality, pick $x \in \mathrm{st}_K(gv)$ for some g. Then $gv \in \mathrm{carr}_K(x)$ and $\overline{v} = \pi(gv) \in \pi(\mathrm{carr}_K(x)) = \mathrm{carr}_{K/G}(\pi(x))$ by Proposition III.11. So $\pi(x) \in \mathrm{st}_{K/G}(\overline{v})$ which shows $\coprod_{g\in G} \mathrm{st}_K(gv) \subset \pi^{-1}(\mathrm{st}_{K/G}(\overline{v}))$. On the other hand, pick $x \in \pi^{-1}(\mathrm{st}_{K/G}(\overline{v}))$ and let $\mathrm{carr}_K(x) = \langle w_0, \ldots, w_n\rangle$. By Lemma III.8 there is a unique expression $x = \sum_{i=0}^n \lambda_i w_i$ with all $\lambda_i > 0$. Applying π we have $\pi(x) = \sum_{i=0}^n \lambda_i \overline{w}_i$. This shows $\mathrm{carr}_{K/G}(\pi(x)) = \langle \overline{w}_0, \ldots, \overline{w}_n\rangle$. Since $x \in \pi^{-1}(\mathrm{st}_{K/G}(\overline{v}))$, $\pi(x) \in \mathrm{st}_{K/G}(\overline{v})$ and hence $\overline{v} \in \mathrm{carr}_{K/G}(\pi(x)) = \langle \overline{w}_0, \ldots, \overline{w}_n\rangle$. This yields that $\overline{v} = \overline{w}_i$, i.e. $w_i = gv$ for some $g \in G$. So $gv \in \mathrm{carr}_K(x)$ and $x \in \mathrm{st}_K(gv)$. This shows the reverse inclusion.

Corollary III.26. *Let X be a G-manifold. Then there is an open cover consisting of G-invariant subspaces such that every finite intersection of the elements in this open cover is homeomorphic to the orbit of a contractible space, i.e., a space of the form $G/H \times D$, where H is a subgroup of G and D is contractible.*

Proof. By the proof of Theorem III.25, the G-manifold X has an equivariant good cover $\mathfrak{U} = \{U_\alpha\}$ such that each U_α is the star of a vertex $v_\alpha \in \mathrm{Vert}(K)$. Here K is a regular G-complex. Now define a new open cover $\mathfrak{V} = \{V_\alpha\}$ by letting $V_\alpha = \bigcup_{g\in G} g(\mathrm{st}_K(v_\alpha))$. Then V_α is G-invariant and every finite intersection $V_{\alpha_0} \cap \cdots \cap V_{\alpha_p}$ is homeomorphic to $G/H \times D$ where D is the contractible space $\mathrm{st}_{K/G}(\overline{v}_{\alpha_0}) \cap \cdots \cap \mathrm{st}_{K/G}(\overline{v}_{\alpha_p})$. □

The nerve of an equivariant good cover carries a great deal of information on the G-homotopy structure of X. Let us first review the ideas of "fat realization" of a simplicial space introduced by Segal.

Definition III.27. Let A be a simplicial space. The *fat geometric realization* of A is the topological space

$$\|A\| := \left(\coprod_{n\geq 0} A_n \times \Delta^n\right) / \sim$$

where Δ^n is the standard n-simplex and the relation is $(\partial_i(x), t) \sim (x, \partial^i(t))$, for $\partial^i : \Delta^n \to \Delta^{n+1}$ the inclusion as the ith face and $\partial_i : A_{n+1} \to A_n$ the face map for A.

If A is a simplicial G-space, then the fat realization $\|A\|$ naturally carries a G-action so that $\|A\|$ is a G-space.

A simplicial map f between simplicial spaces A and A' induces a map $\|f\| : \|A\| \to \|A'\|$. If f is a simplicial G-map between simplicial G-spaces then $\|f\|$ is a G-map between topological G-spaces.

Proposition III.28. *Let A and A' be simplicial spaces and let $f : A \to A'$ be a simplicial map.*

(1) If $f_n : A_n \to A'_n$ is a homotopy equivalence for all n then $\|f\| : \|A\| \to \|A'\|$ is a homotopy equivalence.

(2) $\|A \times A'\|$ is homotopy equivalent to $\|A\| \times \|A'\|$.

(3) The ith degeneracy map $\eta_i : [n] \to [n-1]$ induces a map $s_i : A_{n-1} \to A_n$ and s_i maps A_{n-1} into A_n as a retraction. If the inclusion $s_i(A_{n-1}) \hookrightarrow A_n$ is a closed cofibration for all i and n, then $\|A\| \to |A|$ is a homotopy equivalence.

Proof. See [Seg74]. □

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of a topological space X. Recall that If $\sigma^n = (\alpha_0, \ldots, \alpha_n) \in \mathcal{N}(\mathfrak{U})_n$, we denote by U_{σ^n} the nonempty finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$. Let $X_{\mathfrak{U}}$ be the fat realization $\|\mathcal{N}^{Top}(\mathfrak{U})\|$, i.e.

$$X_{\mathfrak{U}} = \left(\coprod_{\substack{n \geqslant 0 \\ \sigma^n \in \mathcal{N}(\mathfrak{U})_n}} U_{\sigma^n} \times \Delta^n_{\sigma^n} \right) / \sim$$

where $\Delta^n_{\sigma^n}$ is the standard n-simplex with vertices $v_{\alpha_0}, \ldots, v_{\alpha_n}$ and the equivalence relation is $(\partial_i(x), t) \sim (x, \partial^i(t))$, where $\partial^i : \Delta^{n-1} \to \Delta^n$ is the ith face map and ∂_i is the inclusion $U_{\alpha_0 \ldots \alpha_n} \to U_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_n}$.

Let $\pi : \coprod_{\sigma^n} (U_{\sigma^n} \times n_{\sigma^n}) \to X_{\mathfrak{U}}$ be the quotient map.

Proposition III.29. *If $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ is a locally finite open cover of a paracompact space X, then the fat realization $X_{\mathfrak{U}} = \|\mathcal{N}^{Top}(\mathfrak{U})\|$ is homotopy equivalent to X.*

Proof. For each $\sigma^n = (\alpha_0 \ldots \alpha_n) \in \mathcal{N}(\mathfrak{U})$ let p_{σ^n} be the composite of maps $U_{\sigma^n} \times \Delta^n_{\sigma^n} \xrightarrow{p_1} U_{\sigma^n} \hookrightarrow X$, where p_1 is the first coordinate projection. The set of maps p_{σ^n} induces a map

$$p : \coprod_{\substack{n \geqslant 0 \\ \sigma^n \in \mathcal{N}(\mathfrak{U})_n}} (U_{\sigma^n} \times \Delta^n_{\sigma^n}) \to X.$$

Easy to verify p preserves the equivalence relation, so there is a unique map $q : X_{\mathfrak{U}} \to X$ such that $q\pi = p$.

$$\begin{array}{ccc} \coprod_{\sigma^n} (U_{\sigma^n} \times \Delta^n_{\sigma^n}) & \xrightarrow{p} & X \\ \downarrow \pi & \nearrow_{q} & \\ X_{\mathfrak{U}} & & \end{array}$$

For every point $x \in X$, let $\{\alpha_0, \ldots, \alpha_n\}$ be the set of *all* the indices such that $x \in U_{\alpha_i}$. That is, $x \notin U_\alpha$ for all $\alpha \neq \alpha_0, \ldots, \alpha_n$. This set is finite since the cover is locally finite. Then the preimage $q^{-1}(x)$ is just the n-simplex $\{x\} \times \Delta^n_{\alpha_0 \ldots \alpha_n}$. Hence every point in $q^{-1}(x)$ can be represented as $\{x\} \times \sum_{i=1}^n t_i v_{\alpha_i}$, where $t_i \geq 0$, $\sum t_i = 1$.

Since X is paracompact, there exists a partition of unity $\{f_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. In particular, $\mathrm{supp}(f_\alpha) \subset U_\alpha$ for each α. Pick $x \in X$ and let $\{\alpha_0, \ldots, \alpha_n\}$ be the set of *all* the indices such that $x \in U_{\alpha_i}$. Then the set of α's such that $f_\alpha(x) > 0$ is a subset of $\{\alpha_0, \ldots, \alpha_n\}$ and hence is finite. Now consider a map $s : X \to \coprod_{\sigma^n} U_{\sigma^n} \times \Delta^n_{\sigma^n}$, $x \mapsto \{x\} \times \sum f_{\alpha_i}(x) v_{\alpha_i} \in U_{\alpha_0 \ldots \alpha_n} \times \Delta^n_{\alpha_0 \ldots \alpha_n} \subset \coprod_{\sigma^n} U_{\sigma^n} \times \Delta^n_{\sigma^n}$, and let $r : X \to X_{\mathfrak{U}}$ be the composite πs. Clearly, $qr = \mathrm{id}_X$. We need to verify that $rq \simeq \mathrm{id}_{X_{\mathfrak{U}}}$. Suppose that a point x belongs to sets $U_{\alpha_0}, \ldots, U_{\alpha_n}$ and does not belong to any other U_α. Then the points $y = \{x\} \times \sum t_i v_{\alpha_i}$ and $r(q(y)) = \{x\} \times \sum f_{\alpha_i}(x) v_{\alpha_i}$ belong to the simplex with vertices $v_{\alpha_0}, \ldots, v_{\alpha_n}$. The required homotopy uniformly moves $r(q(y))$ to y along the segment joining these points. □

Proposition III.30. *If $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ is a good cover of a topological space X, then the fat realization $X_{\mathfrak{U}} = \|\mathcal{N}^{Top}(\mathfrak{U})\|$ is homotopy equivalent to the normal realization $|\mathcal{N}(\mathfrak{U})|$ of the nerve $\mathcal{N}(\mathfrak{U})$.*

Proof. Proposition III.28 (1) implies that if $\mathfrak{U}$ is a good cover then $X_{\mathfrak{U}}$ is homotopy equivalent to the fat realization $\|\mathcal{N}(\mathfrak{U})\|$ of the nerve $\mathcal{N}(\mathfrak{U})$. Here we identify simplicial sets with discrete simplicial spaces. On the other hand, as simplicial sets $\|\mathcal{N}(\mathfrak{U})\|$ is homotopy equivalent to $|\mathcal{N}(\mathfrak{U})|$ by Proposition III.28 (3). □

Corollary III.31. *If $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ is a locally finite good cover of a paracompact space X, then the normal realization $|\mathcal{N}(\mathfrak{U})|$ of the nerve $\mathcal{N}(\mathfrak{U})$ is homotopy equivalent to X.*

Now let us turn to the equivariant case.

Lemma III.32. *If A is a simplicial G-space, then the realizations $|A|$ and $\|A\|$ inherit G-actions such that*

$$|A^H| = |A|^H \quad \textit{and} \quad \|A^H\| = \|A\|^H$$

for all subgroups H of G.

Proof. The inclusion $A^H \hookrightarrow A$ induces a well-defined map

$$\coprod_n A_n^H \times \Delta^n / \sim \to \coprod_n A_n \times \Delta^n / \sim$$

whose image is fixed by H. On the other hand, if $a \in |A|^H$ (or $a \in \|A\|^H$), then a is the equivalent class of an element $(x, t) \in A_i^H \times \Delta^i$ for some i which indicates $a \in |A^H|$ (or $a \in \|A^H\|$). □

Theorem III.33. *If $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ is a locally finite equivariant good cover of a G-CW complex X, then the normal realization $|\mathcal{N}(\mathfrak{U})|$ of the nerve $\mathcal{N}(\mathfrak{U})$ is G-homotopy equivalent to X.*

Proof. The realization $|\mathcal{N}(\mathfrak{U})|$ is a G-space since $\mathfrak{U}$ is G-invariant. With the natural CW complex structure on a realization, $|\mathcal{N}(\mathfrak{U})|$ becomes a G-CW complex. So it is sufficient to show that $|\mathcal{N}(\mathfrak{U})|$ is weakly G-homotopy equivalent to X. We prove this by showing that for any subgroup H of G, $|\mathcal{N}(\mathfrak{U})|^H$ is homotopy equivalent to X^H.

By definition, if $\mathfrak{U}$ is an equivariant good cover of X then $\mathfrak{U}^H$ is a good cover of X^H, so by Corollary III.31 $|\mathcal{N}(\mathfrak{U}^H)|$ is homotopy equivalent to X^H. But $\mathcal{N}(\mathfrak{U}^H) = \mathcal{N}(\mathfrak{U})^H$. Hence, together with Lemma III.32, we have

$$|\mathcal{N}(\mathfrak{U})|^H = |\mathcal{N}(\mathfrak{U})^H| = |\mathcal{N}(\mathfrak{U}^H)| \simeq X^H.$$

□

Remark III.34. By the conclusions of the above theorem and Theorem III.25, for every smooth G-manifold, there exists an equivariant good cover such that the normal realization of the nerve of this cover is G-homotopy equivalent to the G-manifold.

CHAPTER IV

PRESHEAVES ON G-MANIFOLDS

Given a finite group G, let *G-Man* denote the category of smooth manifolds with smooth G-action and equivariant smooth morphisms. Given $U \in$ *G-Man*, let $\widehat{U}$ denote the full subcategory of *G-Man* $\downarrow U$ consisting of equivariant finite covering maps $p_X : X \to U$. The morphisms in $\widehat{U}$ from $X \xrightarrow{p_X} U$ to $X' \xrightarrow{p_{X'}} U$ are G-maps $\phi : X \to X'$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle p_{X'}} & \\ & U. & \end{array}$$

In particular, when $U = pt$, a one-point space, $\widehat{U}$ is the category *G-Fin* of finite G-sets.

Proposition IV.1. *Let $f : V \to U$ be a morphism in G-Man. Given $p_X : X \to U$ in $\widehat{U}$, let $f^*X = V \times_U X$ be the pull-back of X along f. Then the morphism $p_{f^*X} : f^*X \to V$ is in $\widehat{V}$.*

$$\begin{array}{ccc} f^*X & \xrightarrow{q_X} & X \\ {\scriptstyle p_{f^*X}}\downarrow & & \downarrow{\scriptstyle p_X} \\ V & \xrightarrow[f]{} & U \end{array}$$

Proof. The fiber $(p_{f^*X})^{-1}(b)$ on b is homeomorphic to $(p_X)^{-1}(f(b))$. □

Definition IV.2. A *Mackey presheaf* on *G-Man* is a contravariant functor M : *G-Man* $\to$ *Ab* which is covariant for morphisms in $\widehat{U}$ for all $U \in$ *G-Man*. Furthermore, if

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p_X} \\ V & \xrightarrow[f]{} & U \end{array}$$

is a pull-back diagram with $p \in \widehat{U}$, then

$$\begin{array}{ccc} M(Y) & \xleftarrow{q^*} & M(X) \\ {\scriptstyle (p_Y)_*}\downarrow & & \downarrow{\scriptstyle (p_X)_*} \\ M(V) & \xleftarrow[f^*]{} & M(U) \end{array}$$

commutes.

A topological G-module M represents an abelian Mackey presheaf $\underline{M}$ on G-*Man* by sending a G-manifold X to $\underline{M}(X) := \mathrm{Hom}_{G\text{-}Top}(X, M)$. If ϕ:

$$\begin{array}{ccccc} X' & & \xrightarrow{\phi} & & X \\ & {\scriptstyle p_{X'}}\searrow & & \swarrow{\scriptstyle p_X} & \\ & & U & & \end{array}$$

is a morphism in $\widehat{U}$, the covariant part of $\underline{M}$ on ϕ is $\phi_* : \mathrm{Hom}_{G\text{-}Top}(X', M) \to \mathrm{Hom}_{G\text{-}Top}(X, M)$ with

$$\phi_*(f)(a) = \sum_{a' \in \phi^{-1}(a)} f(a'),$$

where $f \in \mathrm{Hom}_{G\text{-}Top}(X', M)$ and $a \in X$.

Recall that an abelian presheaf on G-*Man* is just a contravariant functor G-*Man* $\to$ *Ab*. Let $\mathcal{F}$ be an abelian presheaf on G-*Man* and let M be a Mackey presheaf. Given $U \in G$-*Man*, we denote by $\mathcal{F} \otimes_{\widehat{U}} M$ the coend

$$\left(\bigoplus_{\{X \xrightarrow{p_X} U\} \in \widehat{U}} \mathcal{F}(X) \otimes M(X) \right) / K_{\mathcal{F},M}(U)$$

in the category *Ab*, where $K_{\mathcal{F},M}(U)$ is the subgroup generated by elements of the form

$$(\phi^*_{\mathcal{F}} a) \otimes m' - a \otimes (\phi_M)_*(m')$$

where $\phi : X' \to X$ is a morphism in $\widehat{U}$, $a \in \mathcal{F}(X)$ and $m' \in M(X')$.

Given an abelian presheaf $\mathcal{F}$ on *G-Man*, for any nonnegative integer n, the presheaf $C^{-n}(\mathcal{F})$ is defined by $C^{-n}(\mathcal{F})(U) = \mathcal{F}(\Delta^n \times U)$, where Δ^n is the standard topological n-simplex with the trivial G-action. The natural cosimplicial structure (see [Wei94, Chapter 8]) of $\{\Delta^n \mid n \geq 0\}$ induce a simplicial abelian presheaf $C^\bullet(\mathcal{F})$ on *G-Man*. Denote the associated complex of presheaves by $C^*(\mathcal{F})$. For the convenience, let $C^i(\mathcal{F}) = 0$ for $i > 0$.

Proposition IV.3. *Let $\mathcal{F}$ be an abelian presheaf and let M be a Mackey presheaf on G-Man. Then the assignment $U \mapsto \mathcal{F} \otimes_{\widehat{U}} M$ is a contravariant functor G-Man $\to$ Ab. We denote by $\mathcal{F} \int M$ the resulting abelian presheaf on G-Man, i.e. $\mathcal{F} \int M(U) := \mathcal{F} \otimes_{\widehat{U}} M$.*

Proof. Let $f : V \to U$ be a morphism in *G-Man*. Given $p_X : X \to U$ in $\widehat{U}$, then by Proposition IV.1 $p_{f^*X} : f^*X \to V$ is an element in $\widehat{V}$, and the pull-back square

$$\begin{array}{ccc} f^*X & \xrightarrow{q_X} & X \\ \downarrow{\scriptstyle p_{f^*X}} & & \downarrow{\scriptstyle p_X} \\ V & \xrightarrow[f]{} & U \end{array} \tag{IV.1}$$

implies there is a functor $f^* : \widehat{U} \to \widehat{V}$. This functor in turn, induces a morphism

$$\widehat{f^*} : \bigoplus_{\{X \xrightarrow{p_X} U\} \in \widehat{U}} \mathcal{F}(X) \otimes M(X) \to \bigoplus_{\{Y \xrightarrow{p_Y} V\} \in \widehat{V}} \mathcal{F}(Y) \otimes M(Y)$$

sending $a \otimes m \in \mathcal{F}(X) \otimes M(X)$ to $q^*_{X,\mathcal{F}}(a) \otimes q^*_{X,M}(m) \in \mathcal{F}(f^*X) \otimes M(f^*X)$, where the morphism $q^*_{X,\mathcal{F}}$ is just $\mathcal{F}(q_X)$ obtained by applying $\mathcal{F}$ to the pull-back diagram (IV.1):

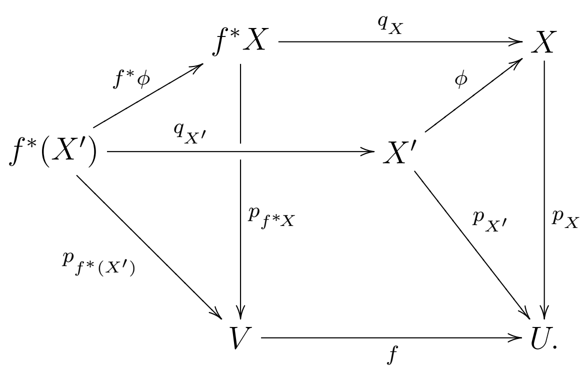
$$\begin{array}{ccc} \mathcal{F}(f^*X) & \xleftarrow{q^*_{X,\mathcal{F}}} & \mathcal{F}(X) \\ \uparrow{\scriptstyle \mathcal{F}(p_{f^*X})} & & \uparrow{\scriptstyle \mathcal{F}(p_X)} \\ \mathcal{F}(V) & \xleftarrow[\mathcal{F}(f)]{} & \mathcal{F}(U). \end{array}$$

Similarly, $q^*_{X,M} = M(q_X)$.

We claim that the homomorphism $\widehat{f^*}$ sends $K_{\mathcal{F},M}(U)$ to $K_{\mathcal{F},M}(V)$. The proof is as follows. Let ϕ

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ & {\scriptstyle p_{X'}}\searrow \quad \swarrow{\scriptstyle p_X} & \\ & U & \end{array}$$

be a morphism in $\widehat{U}$. Consider the following diagram:



Since all of the squares and triangles are commutative except the top square, the top one is also commutative, i.e. $q_X \circ f^*\phi = \phi \circ q_{X'}$. After applying $\mathcal{F}$ and M to the top square, we have the following commutative diagrams.

$$\begin{array}{ccc} \mathcal{F}(f^*X') & \xleftarrow{q^*_{X',\mathcal{F}}} & \mathcal{F}(X') \\ {\scriptstyle (f^*\phi)^*_{\mathcal{F}}}\uparrow & & \uparrow{\scriptstyle \phi^*_{\mathcal{F}}} \\ \mathcal{F}(f^*X) & \xleftarrow[q^*_{X,\mathcal{F}}]{} & \mathcal{F}(X), \end{array}$$

and

$$\begin{array}{ccc} M(f^*X') & \xleftarrow{q^*_{X',M}} & M(X') \\ {\scriptstyle ((f^*\phi)_M)_*}\downarrow & & \downarrow{\scriptstyle (\phi_M)_*} \\ M(f^*X) & \xleftarrow[q^*_{X,M}]{} & M(X), \end{array}$$

where $\phi^*_{\mathcal{F}} = \mathcal{F}(\phi)$ and $(\phi_M)_* = M(\phi)$, etc.

Now given $a \in \mathcal{F}(X)$ and $m' \in M(X')$, we have

$$\begin{aligned}
&\widehat{f}^*(\phi^*_{\mathcal{F}}(a), m')\\
=&q^*_{X',\mathcal{F}} \circ \phi^*_{\mathcal{F}}(a) \otimes q^*_{X',M}(m')\\
=&(f^*\phi)^*_{\mathcal{F}} \circ q^*_{X,\mathcal{F}}(a) \otimes q^*_{X',M}(m')
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{f}^*(a, (\phi_M)_*(m'))\\
=&q^*_{X,\mathcal{F}}(a) \otimes q^*_{X,M} \circ (\phi_M)_*(m')\\
=&q^*_{X,\mathcal{F}}(a) \otimes ((f^*\phi)_M)_* \circ q^*_{X',M}(m'),
\end{aligned}$$

where the latter equality follows from the fact that M is a Mackey presheaf and the top square in diagram (IV.1) is commutative. Now the claim follows.

Using the claim we obtain a homomorphism

$$\mathbf{f}^* : \mathcal{F} \otimes_{\widehat{U}} M \to \mathcal{F} \otimes_{\widehat{V}} M.$$

It is easy to check that $\mathbf{f}^*$ makes the assignment $U \mapsto \mathcal{F} \otimes_{\widehat{U}} M$ a contravariant functor. $\square$

Given a G-manifold X, let $\mathcal{Z}X$ be the abelian presheaf on G-*Man* defined by $\mathcal{Z}X(U) := \mathbb{Z}\mathrm{Hom}_{G\text{-}Man}(U, X)$.

Let $\mathcal{F}$ be an abelian presheaf on G-*Man* and $\underline{M}$ the Mackey presheaf associated to a G-module M. The *singular cochain complex* $C^*(\mathcal{F}, \underline{M})$ of $\mathcal{F}$ with coefficients in $\underline{M}$ is defined by

$$C^*(\mathcal{F}, \underline{M}) = C^*(\mathcal{F}) \int \underline{M}.$$

In particular, for a finite-dimensional representation space V of G let $\mathcal{F} = \mathcal{Z}S^V$. Denote by $M(V)$ the shifted complex $C^*(\mathcal{Z}S^V, \underline{M})[-\dim(V)]$. Here we recall that, for an integer $q \in \mathbb{Z}$, the shifted complex $C^*[q]$ of a cochain complex (C^*, δ) is still a cochain complex defined by $C^*[q]^n := C^{n+q}$ and the differential $\delta'^n = (-1)^q \delta^{n+q} : C^*[q]^n \to C^*[q]^{n+1}$ for each $n \in \mathbb{Z}$.

Lemma IV.4. *As abelian presheaves on G-Man, $C^{-n}(\mathcal{Z}X)$ is naturally isomorphic to $C^{-n}(\underline{Hom}_{G\text{-}Top}(-, X))$ for any $n \geq 0$.*

For the definition of $\underline{Hom}$, see page 4.

Proof. Pick a G-manifold U. We have

$$\begin{aligned}
&C^{-n}(\mathcal{Z}X)(U) \\
&= \mathbb{Z}\mathrm{Hom}_{G\text{-}Top}(\Delta^n \times U, X) && \text{(definition)} \\
&\cong \mathbb{Z}\mathrm{Hom}_{G\text{-}Top}(\Delta^n, \underline{Hom}_{Top}(U, X)) && \text{(Corollary II.2)} \\
&\cong \mathbb{Z}\mathrm{Hom}_{Top}(\Delta^n, (\underline{Hom}_{Top}(U, X))^G) && \text{(Proposition II.3 (i))} \\
&\cong \mathbb{Z}\mathrm{Hom}_{Top}(\Delta^n, \underline{Hom}_{G\text{-}Top}(U, X)) && \text{(II.1)} \\
&= C^{-n}(\underline{Hom}_{G\text{-}Top}(U, X)). && \text{(definition)}
\end{aligned}$$

The naturality is clear. □

Lemma IV.5. *Let $*$ be the one point set G/G. Then for any G-manifold X, we have*

$$\begin{aligned}
C^{-n}(\mathcal{Z}X, \underline{M})(*) &= C^{-n}(\mathcal{Z}X) \otimes_{\hat{*}} \underline{M} \\
&\cong C^{-n}(\underline{Hom}_{G\text{-}Top}(-, X)) \otimes_{G\text{-}Fin} \underline{M}
\end{aligned}$$

for every $n \geq 0$. Hence

$$H^{-n}(C^*(\mathcal{Z}X, \underline{M})(*)) \cong H_n^{Br,G}(X, \underline{M}) \tag{IV.2}$$

where the right hand side is the ordinary n-th Bredon homology groups.

Proof. Since the category $\hat{*}$ is exactly the category G-Fin, there is an isomorphism $C^*(\mathcal{Z}X)\otimes_{\hat{*}}\underline{M} \cong C^*(\mathcal{Z}X)\otimes_{G\text{-}Fin}\underline{M}$. By Lemma IV.4, the right hand side is isomorphic to $C^*(\underline{Hom}_{G\text{-}Top}(-,X))\otimes_{G\text{-}Fin}\underline{M}$. The cohomology groups of this complex are exactly by definition the Bredon homology groups. $\square$

In many cases we need to consider actions of various groups at the same time. This leads important functors of restriction and induction.

Let H be a subgroup of G. Restricting the group action from G to H induces the functor

$$\mathrm{Res}_H^G : G\text{-}Top \to H\text{-}Top$$

which is called *restriction*. There is also a functor

$$\mathrm{Ind}_H^G : H\text{-}Top \to G\text{-}Top$$

defined as follows. Pick any H-space A. The cartesian product $G \times A$ carries an H-action

$$(h, (g, a)) \mapsto (gh^{-1}, ha),$$

and define $\mathrm{Ind}_H^G(A)$ to be the H-orbit space $G \times_H A := G \times A/H$. The G-action $(g', (g, a)) \mapsto (g'g, a)$ on $G \times A$ induces a G-action on $\mathrm{Ind}_H^G(A) = G \times_H A$. For an H-map $f : A \to B$, there is an induced map

$$\mathrm{Ind}_H^G(f) = G \times_H f : G \times_H A \to G \times_H B,\ (g, a) \mapsto (g, f(a)).$$

The functors Res and Ind are adjoint pairs as shown in the next proposition.

Proposition IV.6. *Pick a G-space Y and an H-space A. Then there is a natural bijection*

$$\mathrm{Hom}_{G\text{-}Top}(G \times_H A, Y) \cong \mathrm{Hom}_{H\text{-}Top}(A, \mathrm{Res}_H^G(Y)).$$

Proof. See [tD87, p. 32, Proposition 4.3]. □

Apply this to the category of presheaves. For a presheaf $\mathcal{F}$ on G-*Man* and a subgroup K of G, let $\operatorname{Res}_K^G \mathcal{F}$ be the presheaf on the K-*Man* defined on $U \in K$-*Man* by

$$(\operatorname{Res}_K^G \mathcal{F})(U) = \mathcal{F}(G \times_K U) = \mathcal{F}(\operatorname{Ind}_K^G U). \tag{IV.3}$$

Lemma IV.7. *(1)* $\operatorname{Res}_K^G(\underline{M})$ *is a Mackey presheaf on* K*-Man.*

(2) Let $\mathcal{F}$ *be an abelian presheaf on* G*-Man. Then for any* K*-manifold* U *there is a natural isomorphism*

$$(\operatorname{Res}_K^G \mathcal{F}) \otimes_{\widehat{U}} (\operatorname{Res}_K^G \underline{M}) \cong \mathcal{F} \otimes_{\widehat{G \times_K U}} \underline{M}. \tag{IV.4}$$

Proof. It is easy to check that there is a natural G-homeomorphism $G \times_K X \cong G/K \times X$ for any G-space X. For a K-map $f : X \to Y$ of K-manifold X and Y, the induced map $\operatorname{Ind}_K^G f : G \times_K X \to G \times_K Y$ is just the map $1_{G/K} \times f$.

(1) For any K-map $f : X \to Y$, the contravariant part $f^* : \operatorname{Res}_K^G(\underline{M})(Y) \to \operatorname{Res}_K^G(\underline{M})(X)$ is just the induced map $(1 \times f)^* : \operatorname{Hom}_{G\text{-}Top}(G/K \times Y, M) \to \operatorname{Hom}_{G\text{-}Top}(G/K \times X, M)$. The covariant part ϕ_* for a map

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Y \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle p_Y} & \\ & U & \end{array}$$

in K-*Man* $\downarrow U$ is just the covariant part $(1 \times \phi)_*$ of the Mackey presheaf $\underline{M}$ for the map $1 \times \phi$:

$$\begin{array}{ccc} G/K \times X & \xrightarrow{\ 1\times\phi\ } & G/K \times Y \\ & {\scriptstyle 1\times p_X}\searrow \quad \swarrow{\scriptstyle 1\times p_Y} & \\ & G/K \times U. & \end{array}$$

Furthermore, a pullback diagram

$$\begin{array}{ccc} Z & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[m]{} & U \end{array}$$

in K-*Man* induces a pullback diagram in G-*Man*

$$\begin{array}{ccc} G/K \times Z & \xrightarrow{1\times q} & G/K \times Y \\ {\scriptstyle 1\times p}\downarrow & & \downarrow{\scriptstyle 1\times f} \\ G/K \times Y' & \xrightarrow[1\times m]{} & G/K \times U. \end{array}$$

Hence the pullback condition for $\mathrm{Res}_K^G(\underline{M})$ comes from that of $\underline{M}$.

(2) For any G-manifold X with (surjective) structure map $p : X \to G/K \times U$, let $Y = Y_1 = p^{-1}(eK \times U)$. If $\{g_1 = e, g_2, \ldots, g_l\}$ is a set of representatives of left cosets G/K, then $X = \amalg Y_i$ where $Y_i = p^{-1}(g_iK \times U)$ and $X = G/K \times Y$. This implies each $p : X \to G \times_K U \in \widehat{G \times_K U}$ is one to one correspondence to $\tilde{p} : Y \to U \in \widehat{U}$. Then easy to show, by definition, $(\mathrm{Res}_K^G \mathcal{F}) \otimes_{\widehat{U}} (\mathrm{Res}_K^G \underline{M}) \cong \mathcal{F} \otimes_{\widehat{G\times_K U}} \underline{M}$.

□

Corollary IV.8. *Let K be a subgroup of G. Then for any G-manifold X, there is an isomorphism*

$$C^{-n}(\mathcal{Z}X, \underline{M})(G/K) \cong C^{-n}(\underline{Hom}_{K\text{-}Top}(-, X)) \otimes_{K\text{-}Fin} \underline{M}$$

for every $n \geq 0$. Hence

$$H^{-n}(C^*(\mathcal{Z}X, \underline{M})(G/K)) \cong H_n^{Br,K}(X, \underline{M}). \qquad \text{(IV.5)}$$

Proof. For every $U \in K\text{-}Man$,

$$\begin{aligned}
&\operatorname{Res}^G_K(C^{-n}(\mathcal{Z}X))(U) \\
&= C^{-n}(\mathcal{Z}X)(\operatorname{Ind}^G_K U) && \text{(IV.3)} \\
&= \mathbb{Z}\mathrm{Hom}_{G\text{-}Top}(\Delta^n \times \operatorname{Ind}^G_K U, X) && \text{(definition)} \\
&\cong \mathbb{Z}\mathrm{Hom}_{G\text{-}Top}(\Delta^n, \underline{Hom}_{Top}(\operatorname{Ind}^G_K U, X)) && \text{(Corollary II.2)} \\
&\cong \mathbb{Z}\mathrm{Hom}_{Top}(\Delta^n, (\underline{Hom}_{Top}(\operatorname{Ind}^G_K U, X))^G) && \text{(Proposition II.3 (i))} \\
&\cong \mathbb{Z}\mathrm{Hom}_{Top}(\Delta^n, \underline{Hom}_{G\text{-}Top}(\operatorname{Ind}^G_K U, X)) && \text{(II.1)} \\
&\cong \mathbb{Z}\mathrm{Hom}_{Top}(\Delta^n, \underline{Hom}_{K\text{-}Top}(U, X)) && \text{(Proposition IV.6)} \\
&\cong C^{-n}(\underline{Hom}_{K\text{-}Top}(U, X)). && \text{(definition)}
\end{aligned}$$

So

$$\begin{aligned}
&C^{-n}(\mathcal{Z}X, \underline{M})(G/K) \\
&= C^{-n}(\mathcal{Z}X) \otimes_{\widehat{G/K}} \underline{M} && \text{(definition)} \\
&\cong \operatorname{Res}^G_K(C^{-n}(\mathcal{Z}X)) \otimes_{\widehat{K/K}} \operatorname{Res}^G_K \underline{M} && \text{(IV.4)} \\
&= C^{-n}(\underline{Hom}_{K\text{-}Top}(-, X)) \otimes_{K\text{-}Fin} \operatorname{Res}^G_K \underline{M}.
\end{aligned}$$

Then the isomorphism (IV.5) is again from the definition of the Bredon homology groups. □

Pick a G-manifold X and let $\mathfrak{U}$ be an open G-cover of X. For any complex of presheaves $\mathcal{F}^\bullet$ on $G\text{-}Man$, denote by $\check{\mathbb{H}}^n_G(\mathfrak{U}, \mathcal{F}^\bullet)$ or $\check{\mathbb{H}}^n_{eq}(\mathfrak{U}, \mathcal{F}^\bullet)$ the n-th Čech equivariant hypercohomology of $\mathfrak{U}$ with coefficients in $\mathcal{F}^\bullet$. Let $\check{\mathbb{H}}^n_G(X, \mathcal{F}^\bullet)$(or $\check{\mathbb{H}}^n_{eq}(X, \mathcal{F}^\bullet)$) $= \varinjlim_{\mathfrak{U}} \check{\mathbb{H}}^n_G(\mathfrak{U}, \mathcal{F}^\bullet)$.

Recall the definition of the complex $\underline{M}(V)$. Given a Mackey presheaf $\underline{M}$ associated to a discrete G-module M, let $C^*(S^V, \underline{M}) := C^*(\mathcal{Z}S^V) \int \underline{M}$ be the singular cochain complex of presheaves on G-*Man* and denote by $M(V)$ the shifted complex $C^*(S^V, \underline{M})[-\dim(V)]$.

Definition IV.9. A presheaf $\mathcal{F}$ is *homotopy invariant* if for every space X the induced map $p^* : \mathcal{F}(X) \to \mathcal{F}(X \times I)$ of projection $p : X \times I \to X$ is an isomorphism.

Remark IV.10. As $p : X \times I \to X$ has a section, p^* is always split injective. Thus homotopy invariance of $\mathcal{F}$ is equivalent to p^* being onto.

Lemma IV.11. *Let $i_t : X \to X \times I, x \mapsto (x,t)$ be the inclusion map. A presheaf $\mathcal{F}$ is homotopy invariant if and only if $i_0^* = i_1^* : \mathcal{F}(X \times I) \to \mathcal{F}(X)$ for all X.*

Proof. One way is obvious. Now suppose $i_0^* = i_1^*$ for all X. Applying $\mathcal{F}$ to the multiplication map $m : I \times I \to I, (s,t) \mapsto st$, yields the following diagram

$$\begin{array}{ccccc} & & \mathcal{F}(X\times I) & \xrightarrow{i_0^*} & \mathcal{F}(X) \\ & \swarrow^{1} & \downarrow{\scriptstyle (1_X\times m)^*} & & \downarrow{\scriptstyle p^*} \\ \mathcal{F}(X\times I) & \xleftarrow[(i_1\times 1_I)^*]{} & \mathcal{F}(X\times I\times I) & \xrightarrow[(i_0\times 1_I)^*]{} & \mathcal{F}(X\times I)\end{array}$$

Hence $p^* i_0^* = (i_0 \times 1_I)^*(1_X \times m)^* = (i_1 \times 1_I)^*(1_X \times m)^* = \mathrm{id}$. Since $i_0^* p^* = \mathrm{id}$, p^* is an isomorphism. □

Lemma IV.12. *Let $\mathcal{F}$ be a presheaf. Then the maps $i_0^\#, i_1^\# : C^*\mathcal{F}(X \times I) \to C^*\mathcal{F}(X)$ are chain homotopic for all X.*

Proof. For all $i = 0, \ldots, n$, define $\theta_i : \Delta^{n+1} \to \Delta^n \times I$ to be the map that sends the vertex v_j to $v_j \times \{0\}$ for $j \le i$ and to $v_{j-1} \times \{1\}$ otherwise. The maps θ_i induce maps

$$h_i = (1_X \times \theta_i)^* : C^{-n}\mathcal{F}(X \times I) \to C^{-n-1}\mathcal{F}(X)$$

The h_i form a simplicial homotopy from $i_1^\# = \partial_0 h_0$ to $i_0^\# = \partial_{n+1} h_n$, so the alternating sum $s = \sum(-1)^i h_i$ is a chain homotopy from $i_1^{\#}$ to $i_0^{\#}$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C^{-n-1}\mathcal{F}(X \times I) & \xrightarrow{d} & C^{-n}\mathcal{F}(X \times I) & \xrightarrow{d} & C^{-n+1}\mathcal{F}(X \times I) & \longrightarrow \cdots \\
& \downarrow {\scriptstyle i_1^{\#}-i_0^{\#}} & \swarrow {\scriptstyle s} & \downarrow {\scriptstyle i_1^{\#}-i_0^{\#}} & \swarrow {\scriptstyle s} & \downarrow {\scriptstyle i_1^{\#}-i_0^{\#}} & \\
\cdots \longrightarrow & C^{-n-1}\mathcal{F}(X) & \xrightarrow[d]{} & C^{-n}\mathcal{F}(X) & \xrightarrow[d]{} & C^{-n+1}\mathcal{F}(X) & \longrightarrow \cdots
\end{array}$$

□

Corollary IV.13. *For any presheaf $\mathcal{F}$ the complex $C^*\mathcal{F}$ has homotopy invariant cohomology presheaves. That is, for every p, $\mathcal{H}^p(C^*\mathcal{F})$ is homotopy invariant. In particular, $M(V)$ has homotopy invariant cohomology presheaves.*

We apply this corollary to some suitable open covers of a G-manifold in the following chapters.

CHAPTER V

$RO(G)$-GRADED BREDON COHOMOLOGY AND ČECH HYPERCOHOMOLOGY

Given a group G, let $\mathfrak{h}_G^*$ be a generalized reduced $RO(G)$-graded equivariant cohomology theory which is defined by a G-spectrum $\{E_V | V \in RO(G)\}$. That is, for any G-space X,

$$\mathfrak{h}_G^V(X) := \varinjlim_{W \supset V} [S^{W-V} \wedge X_+, E_W]_G^0.$$

As a special case, fix a finite dimensional representation V of G and a Mackey functor $\underline{M}$ associated to a discrete $\mathbb{Z}[G]$-module M. Define the functors h^p $(p \in \mathbb{Z})$ on G-CW complexes X graded by $\mathbb{Z}$ by $h^p(X) := H_{\mathrm{Br}}^{p+V-\dim(V)}(X, \underline{M})$, which is just isomorphic to the homotopy classes of maps $[S^{\dim(V)} \wedge X_+, K(\underline{M}, p+V)]_G^0$.

The functors h^* satisfy the following cohomology axioms:

(i) *Homotopy invariance.* If $f, g : X \to Y$ are G-homotopic, then $f^* = g^* : h^*(Y) \to h^*(X)$.

(ii) *Exact sequence* for G-CW pairs (X, A). This is from the standard G-cofibration sequence associated to (X, A).

(iii) *Suspension.* Clear from the homotopy representation.

Let $A_\bullet$ be a simplicial G-space. We denote by A_p^d the degenerate part of A_p, i.e. the union of the images of all maps $A_r \to A_p$ with $r < p$, and by A_p^{nd} the non-degenerate part of A_p. The geometric realization $|A_\bullet|$ has a natural skeleta filtration:

$$|A_\bullet| \supset \cdots \supset |A_\bullet|^{(p)} \supset |A_\bullet|^{(p-1)} \supset \cdots \supset |A_\bullet|^{(0)} \supset \{*\}$$

and it gives rise to an associated spectral sequence which is first formulated in [Seg68]. Here we apply it to the equivariant case.

Lemma V.1. *The filtration of $|A_\bullet|$ induces a natural spectral sequence converging to $h^*(|A_\bullet|)$ with $E_1^{pq} = h^q(A_p/A_p^d) = h^q(A_p^{nd})$. Moreover, under the natural map $E_1^{pq} \to h^q(A_p)$ the differential $d_1^{pq} : E_1^{pq} \to E_1^{p+1,q}$ is compatible with the differential of the cochain complex $h^q(A_\bullet)$.*

Proof. The filtration on $|A_\bullet|$ yields a spectral sequence converging to $h^*(|A_\bullet|)$ with $E_1^{pq} = h^{p+q}(|A_\bullet|^{(p)}/|A_\bullet|^{(p-1)})$. There is a homeomorphism

$$A_p \times \Delta^p/((A_p^d \times \Delta^p) \cup (A_p \times \partial\Delta^p)) \to |A_\bullet|^{(p)}/|A_\bullet|^{(p-1)}$$

where $\partial\Delta^p$ is the $(p-1)$-skeleton of the simplex Δ^p. Thus the space $|A_\bullet|^{(p)}/|A_\bullet|^{(p-1)}$ can be identified with the p-fold suspension of A_p/A_p^d, and accordingly $E_1^{pq} \cong h^q(A_p/A_p^d)$.

Next, the compatibility of the differentials, i.e. the commutativity of the diagram

$$\begin{CD}
E_1^{pq} @>>> h^q(A_p)\\
@V{d_1}VV @VV{d}V\\
E_1^{p+1,q} @>>> h^q(A_{p+1})
\end{CD}$$

follows from the commutativity of the following diagram

$$\begin{tikzcd}
h^n(|A_\bullet|^{(p)}/|A_\bullet|^{(p-1)}) \arrow[rr, "d_1"] \arrow[d] & & h^n(|A_\bullet|^{(p+1)}/|A_\bullet|^{(p)}) \arrow[ddddd] \\
h^n(A_p \times \Delta^p/A_p \times \partial\Delta^p) \arrow[d, "\theta\times 1"'] & h^{n-p}(A_p) \arrow[l, "\cong", "S^p\wedge -"'] \arrow[d, "\theta"] \arrow[dd, bend left, "d"] & \\
\prod_p h^n(A_{p+1} \times \Delta^p/A_{p+1} \times \partial\Delta^p) & \prod_p h^{n-p}(A_{p+1}) \arrow[l, "\cong", "S^p\wedge -"'] \arrow[d, "\Sigma"] & \\
h^n(A_{p+1} \times \partial\Delta^{p+1}/A_{p+1} \times \partial^2\Delta^{p+1}) \arrow[u, "\cong"] \arrow[drr, "d_1"'] & h^{n-p}(A_{p+1}) \arrow[dr, "S^{p+1}\wedge -", "\cong"'] & \\
& & h^{n+1}(A_{p+1} \times \Delta^{p+1}/A_{p+1} \times \partial\Delta^{p+1})
\end{tikzcd}$$

where $\partial^2\Delta^p$ means the $(p-2)$-skeleton of Δ^p. The maps θ are induced by the $p+2$

face maps $[p] \to [p+1]$, and Σ denotes the alternative sum, so that the composite $\Sigma \circ \theta$ is the differential d. □

Remark V.2. By the the proof of the above Lemma, the E_2-term of the spectral sequence is $E_2^{p,q} \cong H^p(h^q(A_\bullet^{nd}))$.

Pick an equivariant good cover $\mathfrak{U}$ of a smooth G-manifold X. Then for every $p \geq 0$ the nonempty finite intersection $U_{\sigma^p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ has the form $(G/J_{\sigma^p}) \times D$ where D is a contractible space and J_{σ^p} is a subgroup of G. Applying Theorem III.33 and Lemma V.1 to h^* and $\mathcal{N}(\mathfrak{U})$ yields

Lemma V.3. *Given X and $\mathfrak{U}$ as above, there is a spectral sequence converging to $H_{Br}^{V+*-\dim(V)}(X, \underline{M})$ whose E_1-term is*

$$E_1^{pq} = \prod_{\sigma^p \in \mathcal{N}(\mathfrak{U})_p^{nd}} H_{Br, J_{\sigma^p}}^{q+V-\dim(V)}(*, \underline{M})$$

where $*$ denotes the trivial coset J/J for subgroups J of G. Moreover, the differential of E_1 is compatible with the differential of the complex $\prod_{\sigma^* \in \mathcal{N}(\mathfrak{U})_*} H_{Br}^{q+V-\dim(V)}(U_{\sigma^*}, \underline{M})$.

Lemma V.4. *Given X and $\mathfrak{U}$ as above, there is a spectral sequence converging to $\check{\mathbb{H}}_{eq}^*(\mathfrak{U}, M(V))$ whose E_1-term is*

$$\begin{aligned} E_1^{pq} &= \prod_{\sigma^p \in \mathcal{N}(\mathfrak{U})_p} \mathcal{H}^q(M(V))(U_{\sigma^p}) \\ &\cong \prod_{\sigma^p \in \mathcal{N}(\mathfrak{U})_p} H_{Br, J_{\sigma^p}}^{q+V-\dim(V)}(*, \underline{M}). \end{aligned}$$

Proof. The standard filtration on the double complex $\check{C}^*(\mathfrak{U}, M(V))$ yields

$$\begin{aligned} E_1^{pq} &= H^q(\check{C}^p(\mathfrak{U}, M(V))) \\ &= \check{C}^p(\mathfrak{U}, \mathcal{H}^q(M(V))) \\ &= \prod_{\sigma^p \in \mathcal{N}(\mathfrak{U})_p} \mathcal{H}^q(M(V))(U_{\sigma^p}). \end{aligned}$$

For any homogeneous space G/J we proved in Corollary IV.8 that

$$\begin{aligned} &(\mathcal{H}^q(M(V)))(G/J) \\ &= H^q(M(V)(G/J)) \\ &\cong H^{q+V-\dim(V)}_{\mathrm{Br},J}(*, \underline{M}). \end{aligned}$$

Now since U_{σ^p} is homeomorphic to $(G/J_{\sigma^p}) \times D$, the homotopy invariant property of $\mathcal{H}^q(M(V))$ implies $\mathcal{H}^q(M(V))(U_{\sigma^p}) \cong H^{q+V-\dim(V)}_{\mathrm{Br},J_{\sigma^p}}(*, \underline{M})$. So

$$\begin{aligned} E_1^{pq} &= \prod_{\sigma^p \in \mathcal{N}(\mathfrak{U})_p} \mathcal{H}^q(M(V))(U_{\sigma^p}) \\ &\cong \prod_{\sigma^p \in \mathcal{N}(\mathfrak{U})_p} H^{q+V-\dim(V)}_{\mathrm{Br},J_{\sigma^p}}(*, \underline{M}). \end{aligned}$$

□

Theorem V.5. *There is a natural isomorphism*

$$\check{\mathbb{H}}^n_{eq}(X, M(V)) \cong H^{V+n-\dim(V)}_{Br}(X, \underline{M}).$$

Proof. If denote by E_1 and E_1' the E_1 terms in Lemma V.3 and Lemma V.4, respectively, the natural map $f_1 : E_1 \to E_1'$ induces a morphism f of spectral sequences (by the compatibility stated in Lemma V.3 and $f^{pq} : E_1^{pq} \cong E'^{pq}_1$. Hence f induces an isomorphism on E_∞ pages. □

CHAPTER VI

EXAMPLES AND APPLICATIONS

One of the applications to algebraic geometry of our results is to develop a version of bigraded cohomology and Deligne cohomology for real varieties. The reader can find the work of dos Santos and Lima-Filho on this topic in [dS03a, dSLF07, dSLF08].

Definition VI.1. A real algebraic variety X is a complex algebraic variety endowed with an anti-holomorphic involution $\sigma : X \to X$. A morphism of real varieties $(X, \sigma) \to (X', \sigma')$ is a morphism of complex varieties $f : X \to X'$ such that f is compatible with the involution, i.e. $f \circ \sigma = \sigma' \circ f$.

Let $\mathfrak{G} := Gal(\mathbb{C}/\mathbb{R})$, the Galois group of $\mathbb{C}$ over $\mathbb{R}$. It is isomorphic to the group $\mathbb{Z}/2$. If (X, σ) is a real variety, the anti-holomorphic involution σ induces a $\mathfrak{G}$-action on X. The fixed point set $X^{\mathfrak{G}}$ of this action is called the set of *real points* of X and denoted by $X(\mathbb{R})$. On the other hand, we use $X(\mathbb{C})$ to denote the set of *complex-valued points* of X.

One of the example of real algebraic variety is the complex projective space $\mathbb{P}^n$ with involution $\sigma(z_0, \ldots, z_n) = (\bar{z}_0, \ldots, \bar{z}_n)$. The $\mathfrak{G}$ action induced by σ is just by taking complex conjugation.

In this chapter we mainly consider the case $G = \mathfrak{G} \cong \mathbb{Z}/2$. The real orthogonal representation ring of $\mathfrak{G}$ is $RO(\mathfrak{G}) = \mathbb{Z} \cdot \mathbf{1} \oplus \mathbb{Z} \cdot \xi$, where $\mathbf{1}$ is the trivial representation and ξ is the sign representation. Furthermore, we use the bigraded cohomology notation $H^{r,s}_{\mathrm{Br}}(X, \underline{M})$ for the $\mathfrak{G}$-equivariant Bredon cohomology with coefficients $\underline{M}$ in dimension $(r-s) \cdot \mathbf{1} + s \cdot \xi$, i.e.

$$H^{r,s}_{\mathrm{Br}}(X, \underline{M}) := H^{(r-s)\cdot\mathbf{1}+s\cdot\xi}_{\mathfrak{G}}(X, \underline{M}).$$

One of the interesting problem is the computation of the bigraded cohomology groups of a point $H^{*,*}_{\mathrm{Br}}(\mathrm{pt}, \underline{M})$. The following result shows that even this simplest case is far from trivial.

Proposition VI.2. *The bigraded cohomology groups of a point are as follows:*

$$H^{r,s}_{Br}(\mathrm{pt}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2, & \text{if } r-s \text{ is even, } 0 < r \leq s \text{ or if } r-s \text{ is odd, } 1+s < r \leq 0; \\ \mathbb{Z}, & \text{if } s \text{ is even and } r = 0; \\ 0, & \text{otherwise.} \end{cases}$$

There is also a cup product $\smile$: $H^{r,s}_{\mathrm{Br}}(\mathrm{pt}, \underline{\mathbb{Z}}) \otimes H^{r',s'}_{\mathrm{Br}}(\mathrm{pt}, \underline{\mathbb{Z}}) \to H^{r+r',s+s'}_{\mathrm{Br}}(\mathrm{pt}, \underline{\mathbb{Z}})$ which gives a ring structure to $\mathcal{B} := \sum_{r,s} H^{r,s}_{\mathrm{Br}}(\mathrm{pt}, \underline{\mathbb{Z}})$. In [dSLF07] the ring $\mathcal{B}$ is explicitly formulated as follows.

In order to describe $\mathcal{B}$, first consider indeterminates $\varepsilon, \varepsilon^{-1}, \tau, \tau^{-1}$ satisfying $\deg \varepsilon = (1,1)$, $\deg \varepsilon^{-1} = (-1,-1)$, $\deg \tau = (0,2)$ and $\deg \tau^{-1} = (0,-2)$. Henceforth, ε and ε^{-1} will always satisfy $2\varepsilon = 0 = 2\varepsilon^{-1}$.

As an abelian group, $\mathcal{B}$ can be written as a direct sum

$$\mathcal{B} := \mathbb{Z}[\varepsilon, \tau] \cdot 1 \oplus \mathbb{Z}[\tau^{-1}] \cdot \alpha \oplus \mathbb{F}_2[\varepsilon^{-1}, \tau^{-1}] \cdot \theta$$

where each summand is a free bigraded module over the indicated ring. The bidegrees of the generators 1, α and θ are, respectively, $(0,0)$, $(0,-2)$ and $(0,-3)$.

The product structure on $\mathcal{B}$ is completely determined by the following relations

$$\alpha \cdot \tau = 2, \qquad \alpha \cdot \theta = \alpha \cdot \varepsilon = \theta \cdot \tau = \theta \cdot \varepsilon = 0.$$

Recall that in Chapter IV we defined the complex of presheaves $C^*(\mathcal{F})$ for any presheaf $\mathcal{F}$ on *G-Man* whose $(-n)$-th term is

$$C^{-n}(\mathcal{F}) : U \mapsto \mathcal{F}(\Delta^n \times U), \ n \geq 0.$$

Also, given a G-manifold X, the abelian presheaf $\mathcal{Z}X$ on *G-Man* was defined by $\mathcal{Z}X(U) := \mathbb{Z}\mathrm{Hom}_{G\text{-}Man}(U, X)$. We then defined the *singular cochain complex* $C^*(\mathcal{F}, \underline{M})$ of $\mathcal{F}$ with coefficients in $\underline{M}$ by

$$C^*(\mathcal{F}, \underline{M}) = C^*(\mathcal{F}) \int \underline{M}.$$

In particular, for a finite-dimensional representation space V of $G = \mathfrak{S}$, let $\mathcal{F} = \mathcal{Z}S^V$. We denoted by $\mathbb{Z}(V)$ the shifted complex $C^*(\mathcal{Z}S^V, \underline{\mathbb{Z}})[-\dim(V)]$.

In [dSLF08], a complex of presheaves called *Bredon complex* is defined as follows. First denote

$$(\mathbb{C}^\times)_i^{p-1} := \mathbb{C}^\times \times \cdots \times 1 \times \cdots \times \mathbb{C}^\times \subset \mathbb{C}^{\times p},$$

where 1 appears in the i-th coordinate.

Definition VI.3 ([dSLF08])**.** Given a $\mathfrak{S}$-manifold X, let

$$J_{X,p} : \bigoplus_{i=1}^{p} C^*(\mathcal{Z}((\mathbb{C}^\times)_i^{p-1} \times X)) \longrightarrow C^*(\mathcal{Z}(\mathbb{C}^{\times p} \times X))$$

be the map induced by the inclusions and denote

$$C^*(\mathcal{Z}_0(S^{p,p} \wedge X_+)) := \mathrm{cone}(J_{X,p}).$$

We denote $\mathrm{cone}(J_{X,p})$ by $C^*(\mathcal{Z}_0(S^{p,p}))$ when $X = \varnothing$. The *p-th Bredon complex with coefficients in* $\mathbb{Z}$ is the complex of presheaves

$$\mathbb{Z}(p)_{\mathcal{B}r} := C^*(\mathcal{Z}_0(S^{p,p})) \int \underline{\mathbb{Z}}\ [-p].$$

Proposition VI.4. *Pick an integer $p \geq 0$. Let V be the representation space $p \cdot \xi$ of $\mathfrak{S}$. Then there is a natural quasi-isomorphism $f : \mathbb{Z}(p)_{\mathcal{B}r} \to \mathbb{Z}(V)$.*

We proceed the proof by two lemmas. First we define a complex of presheaves similar to the Bredon complex.

Definition VI.5. Let $S^\xi \subset \mathbb{C}$ be the unit circle. Denote

$$(S^\xi)_i^{p-1} := S^\xi \times \cdots \times 1 \times \cdots \times S^\xi \subset (S^\xi)^p,$$

where 1 appears in the i-th coordinate. Let

$$K_p \; : \; \bigoplus_{i=1}^{p} C^*(\mathcal{Z}((S^\xi)_i^{p-1})) \;\longrightarrow\; C^*(\mathcal{Z}((S^\xi)^p))$$

be the map induced by the inclusions. Define $L(p)$ to be the complex of presheaves

$$L(p) := \mathrm{cone}(K_p) \int \underline{\mathbb{Z}}\;[-p].$$

Lemma VI.6. *The map of complexes*

$$\varphi : \mathbb{Z}(p)_{\mathcal{B}r} \to L(p)$$

induced by the retraction $r : \mathbb{C}^\times \to S^\xi$ *is a quasi-isomorphism of complexes of presheaves.*

Proof. Given $U \in G\text{-}Man$, for each $j,\ 0 \le j \le p$, we have

$$\begin{aligned}
\mathbb{Z}(p)^j_{\mathcal{B}r}(U) &= C^j(\mathcal{Z}_0(S^{p,p}))\int \underline{\mathbb{Z}}\;[-p]\,(U)\\
&= \bigoplus_{\{T\xrightarrow{\pi}U\}\in\widehat{U}} \Big[\Big(\oplus_{i=1}^p C^{j+1-p}(\mathcal{Z}((\mathbb{C}^\times)_i^{p-1}))(T) \oplus C^{j-p}(\mathcal{Z}(\mathbb{C}^{\times p}))(T)\Big)\otimes \mathbb{Z}(T)\Big]/K\\
&= \bigoplus_{\{T\xrightarrow{\pi}U\}\in\widehat{U}} \Big[\Big(\oplus_{i=1}^p \mathbb{Z}\mathrm{Hom}_{G\text{-}Man}(\Delta^{p-j-1}\times T, (\mathbb{C}^\times)_i^{p-1})\\
&\qquad\qquad \oplus \mathbb{Z}\mathrm{Hom}_{G\text{-}Man}(\Delta^{p-j}\times T, \mathbb{C}^{\times p})\Big)\otimes \mathrm{Hom}_{G\text{-}Top}(T,\mathbb{Z})\Big]/K.
\end{aligned}$$

So elements in $\mathbb{Z}(p)^j_{\mathcal{B}r}(U)$ are represented by sums of pairs of the form $\alpha \otimes m = (a, f)\otimes m$ where a, f and m are equivariant maps satisfying

1. $a\colon \Delta^{p-j-1}\times T \to (\mathbb{C}^\times)_i^{p-1} \subset \mathbb{C}^{\times p}$ is smooth and $\pi\colon T \to U$ is a map in $\widehat{U}$;
2. $f\colon\ \Delta^{p-j}\times T \to (\mathbb{C}^\times)^p$ is a smooth map;

3. $m\colon\ T \to \mathbb{Z} \in \underline{\mathbb{Z}}(T)$ is locally constant (since $\mathbb{Z}$ has discrete topology).

With the same argument each element in $L(p)^j(U)$ is represented by sums of pairs of the form $\alpha' \otimes m' = (a', f') \otimes m'$ where equivariant maps a', f' and m' satisfy

1. $a'\colon \Delta^{p-j-1} \times T \to (S^\xi)_i^{p-1} \subset \mathbb{C}^{\times p}$ is smooth and $\pi\colon T \to U$ is a map in $\widehat{U}$;
2. $f'\colon\ \Delta^{p-j} \times T \to (S^\xi)^p$ is a smooth map;
3. $m'\colon\ T \to \mathbb{Z} \in \underline{\mathbb{Z}}(T)$ is locally constant.

The map $\varphi : \mathbb{Z}(p)_{\mathcal{B}r} \to L(p)$ induced by the retraction $r : \mathbb{C}^\times \to S^\xi$ is defined as follows. If $j < 0$ or $j > p$, let $\varphi = 0 : \mathbb{Z}(p)^j_{\mathcal{B}r} \to L(p)^j$. If $0 \leq j \leq p$, let $\varphi : \mathbb{Z}(p)^j_{\mathcal{B}r}(U) \to L(p)^j(U)$ be the map sending a representative element $(a, f) \otimes m$ to

$$(r_1 \circ a, r_2 \circ f) \otimes m$$

where $r_1 : (\mathbb{C}^\times)_i^{p-1} \to (S^\xi)_i^{p-1}$ and $r_2 : (\mathbb{C}^\times)^p \to (S^\xi)^p$ are maps both induced by r. It is easy to check φ is a map of complexes.

Since both $\mathbb{Z}(p)_{\mathcal{B}r}$ and $L(p)$ have homotopy invariant cohomology presheaves by Corollary IV.13 and G-manifolds are locally contractible, in order to show φ induces an isomorphism of cohomology presheaves, it suffices to check $\varphi : \mathbb{Z}(p)^*_{\mathcal{B}r}(\mathrm{pt}) \to L(p)^*(\mathrm{pt})$ induces an isomorphism of cohomology groups. But in this case the map of complexes $\psi : L(p)^*(\mathrm{pt}) \to \mathbb{Z}(p)^*_{\mathcal{B}r}(\mathrm{pt})$ induced by the inclusion $\iota : S^\xi \to \mathbb{C}^\times$ serves as inverse of φ in the cohomology level. □

Lemma VI.7. *There is a quasi-isomorphism*

$$\varphi : L(p) \to \mathbb{Z}(V).$$

Proof. Similar to the proof of Lemma VI.6. □

Corollary VI.8. *Let X be a $\mathfrak{S}$-manifold. Then for all $n, p \geq 0$ there is a natural* *isomorphism*

$$\check{\mathbb{H}}^n_{eq}(X, \mathbb{Z}(p)_{\mathcal{B}r}) \cong H^{n,p}_{Br}(X, \underline{\mathbb{Z}}).$$

Proof. This comes from Theorem V.5 and Proposition VI.4. Let $V = p \cdot \xi$ be a representation space of $\mathfrak{S}$. Then we have

$$\begin{aligned}
&\check{\mathbb{H}}^n_{\mathrm{eq}}(X, \mathbb{Z}(p)_{\mathcal{B}r}) \\
&\cong \check{\mathbb{H}}^n_{\mathrm{eq}}(X, \mathbb{Z}(V)) && \text{(Proposition VI.4)} \\
&\cong H^{V+n-\dim(V)}_{\mathrm{Br}}(X, \underline{\mathbb{Z}}) && \text{(Theorem V.5)} \\
&= H^{(n-p)\cdot\mathbf{1}+p\cdot\xi}_{\mathrm{Br}}(X, \underline{\mathbb{Z}}) \\
&= H^{n,p}_{\mathrm{Br}}(X, \underline{\mathbb{Z}}).
\end{aligned}$$

□

CHAPTER VII

SUMMARY

In this dissertation we investigate a theoretic approach to generalized $RO(G)$-graded equivariant cohomology theory. When X is an equivariant smooth manifold, using equivariant good covers on X we construct complexes of sheaves $M(V)$ on the site *G-Man* of smooth G-manifolds associated to a representation space V of G and a discrete G-module M such that

$$\check{\mathbb{H}}^n_{\mathrm{eq}}(X, M(V)) \cong H^{V+n-\dim(V)}_{\mathrm{Br},G}(X, \underline{M}),$$

where the later denotes $RO(G)$-graded equivariant cohomology. The proof relies on the existence of good covers and on comparisons of various spectral sequences.

The results naturally apply to the Deligne cohomology for a real variety X. For a real holomorphic proper manifold X, let $p \geq 0$ and define the *Deligne cohomology* of X as the Čech hypercohomology groups

$$H^i_{\mathcal{D}/\mathbb{R}}(X, \mathbb{Z}(p)) := \check{\mathbb{H}}^i_{\mathrm{eq}}(X, \mathbb{Z}(p)_{\mathcal{D}/\mathbb{R}}),$$

where $\mathbb{Z}(p)_{\mathcal{D}/\mathbb{R}}$ is some equivariant Deligne complex. If $p < 0$, then define Deligne cohomology such that it coincides with equivariant Bredon cohomology. The author will address in the near future the generalization of our results to Deligne cohomology.

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